A. Log P/D Ratios in the General Long Run Risk Model

The methodology here closely follows that of Bansal and Yaron (2004) and Bansal, Yaron, and Kiku (2007) as there are only two cases in the literature where solutions are available for models with Epstein-Zin preferences. The first case, which we are interested in here, is when the returns are loglinear in the state variables and the second is when $\psi = 1$.

Let c, $X_i, 1 \leq i \leq n$ and $V_j, 1 \leq j \leq m$ be the log consumption process, n processes that determine it's conditional growth rate and m processes that determine it's conditional growth rate volatility respectively. Let $d_l, l \leq 1 \leq L$ be the log dividend processes of L assets (in general, the lower case variables correspond to the logarithm of the upper case variables). We assume that these quantities follow the processes

$$\begin{aligned} c_{t+\Delta t} =& c_t + \left(\mu + \sum_{i=1}^{n} X_{i,t}\right)\Delta t + \sqrt{\sum_{j=1}^{m} \delta_{c,j}^2 V_{j,t}}\,(W_{t+\Delta t} - W_t) \\ & - \sum_{k=1}^{m} \varphi_{w,k}\sqrt{V_{k,t}}\,(Z_{k,t+\Delta t} - Z_{k,t}) \end{aligned} \tag{28}$$

$$X_{i,t+\Delta t} = X_{i,t}(1 - \alpha_i \Delta t) + \varphi_{i,x}\sqrt{\sum_{j=1}^{m} \delta_{x,i,j}^2 V_{j,t}}\,(Y_{i,t+\Delta t} - Y_{i,t}), 1 \leq i \leq n \tag{29}$$

$$V_{i,t+\Delta t} = V_{i,t} - \kappa_i(V_{i,t} - \bar{V}_i)\Delta t + \sigma_i\sqrt{V_{i,t}}\,(Z_{i,t+\Delta t} - Z_{i,t}), 1 \leq i \leq m \tag{30}$$

$$\begin{aligned} d_{l,t+\Delta t} =& d_{l,t} + \left(\mu_l + \sum_{i=1}^{n} \phi_{l,i} X_{i,t}\right)\Delta t + \pi_{l,d}\left(\Delta c_{t+\Delta t} - \left(\mu + \sum_{i=1}^{n} X_{i,t}\right)\Delta t\right) \\ & + \sum_{i=1}^{n} \pi_{i,l,x}(X_{i,t+\Delta t} - X_{i,t}(1 - \alpha_i \Delta t)) \\ & + \sum_{j=1}^{m} \pi_{j,l,w}\sigma_j\sqrt{V_{j,t}}\,(Z_{j,t+\Delta t} - Z_{j,t}) \\ & + \sqrt{\sum_{k=1}^{m} \delta_{l,d,k}^2 V_{k,t}}\,(B_{t+\Delta t} - B_t) \end{aligned} \tag{31}$$

where W, $Y_i, 1 \leq i \leq n$, $Z_j, 1 \leq j \leq m$ and B are independent Brownian processes and $\sum_{i=1}^{m} \delta_{c,i}^2 = \sum_{j=1}^{m} \delta_{x,i,j}^2 = \sum_{k=1}^{m} \delta_{l,d,k}^2 = 1$. We have written the equations in this form (with the time step being Δt rather than 1) to make the time scale dependence of the parameters explicit so that the connection with the continuous time solution can be made in a straightforward manner. We also define the consumption and dividend variables as rates since they are flow variables. This means, for example, that consumption from time t to $t+\Delta t$ is given by $C_{t+\Delta t}\Delta t$.

Since the consumer preferences are of the Epstein-Zin type

$$U_t = ((1-\delta)(C_t \Delta t)^{\frac{1-\gamma}{\theta}} + \delta E_t[U_{t+\Delta t}^{1-\gamma}]^{\frac{1}{\theta}})^{\frac{\theta}{1-\gamma}} \tag{32}$$

where

$$\theta = \frac{1-\gamma}{1-1/\psi} \tag{33}$$

the log stochastic discount factor in discrete time can be written as

$$m_{t+\Delta t} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+\Delta t} + (\theta - 1) r_{c,t+\Delta t} \tag{34}$$

where $r_{c,t+\Delta t}$ is the continuously compounded rate of return on the wealth W (which is the asset that delivers a dividend of per capita consumption at every time period) from t to $t+\Delta t$. Since we assume complete markets,

$$E_t[\exp(m_{t+\Delta t} + r_{c,t+\Delta t})] = 1 \tag{35}$$

must hold.

The loglinear approximation pioneered by Campbell and Shiller (1988) allows us to write

$$r_{c,t+\Delta t} = \nu_0 + \nu_1 (w_{t+\Delta t} - c_{t+\Delta t}) - (w_t - c_t) + \Delta c_{t+\Delta t} \tag{36}$$

where

$$\nu_0 = \log(\Delta t + \exp(\overline{w-c})) - \nu_1(\overline{w-c}) \approx \exp(\overline{c-w})(1 + (\overline{c-w}))\Delta t \tag{37}$$

$$\nu_1 = \frac{1}{1 + \exp(\overline{c-w})\Delta t} \approx 1 - \exp(\overline{c-w})\Delta t \tag{38}$$

(the approximation holds when Δt is small) where the bar stands for the mean value. We further assume that the log wealth to consumption ratio can be written as

$$w_t - c_t = A_0 + \sum_{i=1}^{n} A_{1,i} X_{i,t} + \sum_{j=1}^{m} A_{2,j} V_{j,t} \tag{39}$$

and justify this below. (This approach is standard and followed by Bansal and Yaron (2004), Bansal, Yaron, and Kiku (2007) and Zhou and Zhu (2009) as the only non-trivial models with Epstein-Zin preferences which can be solved are those where the consumption to wealth ratio is loglinear in the state variables as above or where $\psi = 1$, as in the model of Hansen, Heaton, and Li (2008)).

Substituting (34), (36) and (39) into (35), using the fact that

$$\log E_t[\exp A(W_{t+\Delta t} - W_t)] = \frac{A^2 \Delta t}{2} \tag{40}$$

for any $A \in \mathbb{R}$ and Wiener process W, and that (35) should hold for any possible attainable combination of state variables (X_i, V_j), we obtain a set of equations which enable us to solve for A_0, $A_{1,i}, 1 \le i \le n$ and $A_{2,j}, 1 \le j \le m$. The fact that such a set of equations with non-vacuous solutions exist justifies the assumption (39).

The set of equations for $A_{1,i}$ are

$$(1-\gamma)\Delta t + \theta A_{1,i}(\nu_1(1 - \alpha_i \Delta t) - 1) = 0 \tag{41}$$

so that

$$A_{1,i} = \frac{(1 - \frac{1}{\psi})\Delta t}{1 - \nu_1(1 - \alpha_i \Delta t)} \tag{42}$$

which, in the limit $\Delta t \to 0$, becomes $A_{1,i} = \frac{1-1/\psi}{\exp(\overline{c-w})+\alpha_i}$. This is the same result as that obtained by Zhou and Zhu (2009), where there is only one X variable, once we relate his notation of g_1 for $\exp(\overline{c-w})$ and allow for the negative sign which arises from his definition of A_1 in terms of the consumption to wealth ratio. Once we set $\Delta t = 1$ and relabel ν_1 as κ_1 and α_i as $1-\rho$ (again, there being only one X state variable) to match the notation of Bansal and Yaron (2004), we find that our result also matches their's.

The analogous set of equations which enables us to solve for $A_{2,j}, 1 \leq j \leq m$ is

$$\begin{aligned} &\frac{(1-\gamma)^2\delta_{c,j}^2\Delta t}{2} + \theta A_{2,j}(\nu_1(1-\kappa_j\Delta t)-1) \\ &+ \frac{\Delta t}{2}\left(\left(\theta\nu_1\sum_{i=1}^{n}A_{1,i}\varphi_{x,i}\delta_{x,i,j}\right)^2 + (\theta\nu_1 A_{2,j}\sigma_j - (1-\gamma)\varphi_{w,j})^2\right) = 0 \end{aligned} \tag{43}$$

Since these equations are quadratic, there are two solutions for each $A_{2,j}$. However, one of them diverges when $\sigma_j \to 0$. Hence, the other solution is the one which is relevant to the model. The final equation, which allows us to solve for A_0, is

$$\theta\left(\log\delta + \nu_0 + (\nu_1 - 1)A_0 + \nu_1\sum_{j=1}^{m}A_{2,j}\kappa_j\Delta t\bar{V}_j\right) + (1-\gamma)\mu\Delta t = 0 \tag{44}$$

Putting the values for A_0, $A_{1,i}, 1 \leq i \leq n$ and $A_{2,j}, 1 \leq j \leq m$ into (39) and using (36) and (34), we obtain the log stochastic discount factor

$$\begin{aligned} m_{t+\Delta t} =& \Delta t\left(\Gamma_0 + \sum_{i=1}^{n}\Gamma_{1,i}X_{i,t} + \sum_{j=1}^{m}\Gamma_{2,j}V_{j,t}\right) \\ &- \alpha_c\sqrt{\sum_{j=1}^{m}\delta_{c,j}^2V_{j,t}}(W_{t+\Delta t} - W_t) \\ &- \sum_{i=1}^{n}\alpha_{x,i}\sqrt{\sum_{j=1}^{m}\delta_{x,j}^2V_{j,t}}(Y_{i,t+\Delta t} - Y_{i,t}) \\ &- \sum_{j=1}^{m}\alpha_{v,j}\sqrt{V_{j,t}}(Z_{j,t+\Delta t} - Z_{j,t}) \end{aligned} \tag{45}$$

where $\Gamma_{1,i} = 1/\psi$, $\alpha_c = \gamma$ and $\alpha_{x,i} = \frac{\gamma - 1/\psi}{1-\nu_1(1-\alpha_i \Delta t)}$. The expression for $\alpha_{v,j}$ is complicated and does not directly concern us here.

Using the process for dividend growth (31), we can use a similar loglinear approximation to write the return for asset l as

$$r_{l,t+\Delta t} = \nu_{0,l} + \nu_{1,l}(p_{l,t+\Delta t} - d_{l,t+\Delta t}) - (p_{l,t} - d_{l,t}) + \Delta d_{l,t+\Delta t} \tag{46}$$

where

$$\begin{aligned} \nu_{0,l} &= \log(\Delta t + \exp(\overline{d_l - p_l})) - \nu_{1,l}(\overline{p_l - d_l}) \\ &\approx \exp(\overline{d_l - p_l})(1 + \overline{d_l - p_l})\Delta t \end{aligned} \tag{47}$$

$$\nu_{1,l} = \frac{1}{1 + \exp(\overline{d_l - p_l})\Delta t} \approx 1 - \exp(\overline{d_l - p_l})\Delta t \tag{48}$$

As before, we assume that $\log\left(\frac{P_t}{D_t}\right)$ can be written as

$$\log\left(\frac{P_{l,t}}{D_{l,t}}\right) = p_{l,t} - d_{l,t} = A_{0,l} + \sum_{i=1}^{n} A_{1,l,i} X_{i,t} + \sum_{j=1}^{m} A_{2,l,j} V_{j,t} \tag{49}$$

We put (49) into (46) and use the fact that (35) must hold for any possible attainable combination of state variables (X_i, V_j) to obtain a set of equations which enables us to solve for $A_{0,l}$, $A_{1,l,i}, 1 \le i \le n$ and $A_{2,l,j}, 1 \le j \le m$. The fact that such a set of equations with non-vacuous solutions exist justifies the assumption (49).

The equations for $A_{1,l,i}, 1 \le i \le n, 1 \le l \le L$ are

$$(\phi_{l,i} - 1/\psi)\Delta t - A_{1,l,i}(1 - \nu_{1,l}(1 - \alpha_i \Delta t)) = 0 \tag{50}$$

which give

$$A_{1,l,i} = \frac{(\phi_{l,i} - 1/\psi)\Delta t}{1 - \nu_{1,l}(1 - \alpha_i \Delta t)} \tag{51}$$

As with the solution for $A_{1,i}, 1 \leq i \leq n$, this solution agrees with the continuous time one (with $n = 1, m = 2$) of Zhou and Zhu (2009) and the discrete time one (with $n = m = 1$) of Bansal and Yaron (2004) and (Bansal, Yaron, and Kiku 2007).

The equations for $A_{2,l,j}, 1 \leq j \leq m, 1 \leq l \leq L$ are quadratic in nature and fairly complex (as for $A_{2,j}$, the solutions which do not diverge as $\sigma_j \to 0$ are chosen). Since their precise structure does not concern us here, we do not include them for brevity. Similarly, we do not include the equation for $A_{0,l}, 1 \leq l \leq L$.[35]

It must be noted that, as the equations for $A_{2,j}, 1 \leq j \leq m$ and $A_{2,l,j}, 1 \leq j \leq m, 1 \leq l \leq L$ are quadratic in nature, real solutions are not guaranteed. Our numerical experiments indicate that this is not a serious concern as several sets of reasonable parameter values do not give rise to this problem (this is also shown by Zhou and Zhu (2009)). If this is a concern, we can replace the volatility processes by Ornstein-Uhlenbeck ones as done by Bansal and Yaron (2004) and Bansal, Yaron, and Kiku (2007). However, such volatility processes suffer from the problem of admitting negative values even in continuous time. This can be quite serious, even for some common parameter values, as pointed out by Beeler and Campbell (2009). The square root processes used here can also give rise to negative values in discrete time but the probability of this occurring for reasonable parameter values is minuscule and our numerical experiments confirm this. Since both ways of modeling volatility have issues but have received wide attention in the literature and there is no known alternative for which analytical solutions can be derived, we use results which hold for both of them.

[35] They are available upon request from the authors.

B. Testing Long Run Risk Models : Monte Carlo Evidence

A. The Model

For the purpose of analyzing the performance of the asset pricing tests, we use the long run risk model of Bansal and Yaron (2004). In this model, the per capita consumption and dividend growth rates Δc and Δd (for M assets indexed by l) and their common persistent component x are assumed to follow the processes (see Bansal and Yaron (2004))

$$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1} \quad (52)$$

$$x_{t+1} = \rho x_t + \varphi_x \sigma_t e_{t+1} \quad (53)$$

$$\Delta d_{l,t+1} = \mu_{l,d} + \phi_l x_t + \varphi_{l,d} \sigma u_{l,t+1}, 1 \leq l \leq M \quad (54)$$

$$\sigma^2_{t+1} = \sigma^2 + \nu(\sigma^2_t - \sigma^2) + \sigma_w w_{t+1} \quad (55)$$

where the shocks e_{t+1}, η_{t+1} and w_{t+1} are taken to be independent standard normals for parsimony. $u_{l,t+1}$ is a vector of normally distributed shocks with covariance V_u which is independent of e, η and w. In the simulations, V_u is set so as to fit the factor structure of returns. (Note that we follow the convention that lowercase characters stand for the logarithm of quantities denoted by the corresponding uppercase characters.)

Consumers in the model have Epstein-Zin preferences (as defined by Epstein and Zin (1989))

$$U_t = ((1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{\theta}})^{\frac{\theta}{1-\gamma}} \quad (56)$$

with $\gamma > 1/\psi$. This implies that they prefer early resolution of uncertainty and that persistent consumption and volatility shocks have a positive market price of risk. With these preferences, asset returns satisfy

$$E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\theta/\psi} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}\right] = 1 \quad (57)$$

where C is per capita consumption, R_a is the gross return on an asset that pays a dividend of per capita consumption, R_i is the asset return, $0 < \delta < 1$ is the time discount factor, γ is the relative risk aversion, ψ is the intertemporal elasticity of substitution (IES) and θ is defined to be

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} \tag{58}$$

The log P/D ratios of assets in this economy have a factor structure (within the loglinear approximation) with the factors being x_t and σ_t^2. In other words, if $z_{i,t}$ is the log P/D ratio of asset i, we have

$$z_{i,t} = A_{0,i} + A_{1,i}x_t + A_{2,i}\sigma_t^2 \tag{59}$$

This is shown for this particular model by Bansal and Yaron (2004) and similar results for related models are shown by Bansal, Yaron, and Kiku (2007), Drechsler and Yaron (2011), Zhou and Zhu (2009), Ferson, Nallareddy, and Xie (2009) and in appendix A of this paper. Since the dividend processes of the assets are specified in this model, the relation above gives the time series of their prices for a given realization of the random variables. Hence, the prices and other quantities of interest in this economy are readily simulated.

B. Monte Carlo Simulation of the Model

We use the global and asset specific parameters summarized in tables (XIX) and (XX) for the simulations below. We first note that these parameters generate economic moments (calculated from 500 simulations of the long run risk economy) which are roughly in line with the values observed in post-1942 (to account for the structural break identified by Marakani (2009)) US consumption and return data as shown in table (XXI). When realistic noise is added to the log P/D ratios as described below, they are also compatible with the predictability of real time consumption growth in the data as seen from the numbers in table (XXII). One moment which does not match well is the standard deviation of the real risk free rate which is *much* smaller in the simulations than in the data. This, however, as argued by Beeler and Campbell (2009), points to a strength rather than a weakness of the long run risk model as most models struggle to make this quantity low enough. Further, as we argue in the next section, this quantity is very noisily measured which means that the reported standard deviation would be significantly larger than the actual one.[36]

Table XIX
Global parameters for the simulation

Global parameters for the simulation (the time unit is one year). μ represents the unconditional mean of consumption growth, σ it's conditional volatility, ρ the first order autocorrelation of the long run risk state variable x, φ_x the conditional volatility of x in relation to that of consumption growth, ν the first order autocorrelation of volatility, σ_w the volatility of volatility, γ the relative risk aversion, ψ the elasticity of intertemporal substitution and δ the time preference.

Parameter	Value
μ	0.02
σ	0.012
ρ	0.85
φ_x	0.45
ν	0.99
σ_w	10^{-5}
γ	25
ψ	1.5
δ	0.994

The scaled eigenvalues of the covariance matrix of the post-1942 continuously compounded excess returns of the 25 Fama-French portfolios formed on the basis of size and book to market ratio are tabulated in table (XXIII) together with the mean, 5th and 95th percentiles of the corresponding values obtained in 500 simulations of the economy for the same time period (65 years).[37] Since the first few eigenvalues, which are of principal interest, are very similar to those in the data, the model replicates the observed factor structure of excess returns quite well.

The model also replicates the observed factor structure of log P/D ratios fairly well. This is best seen from the normalized eigenvalues for the covariance matrix of the log P/D ratios

[36] Measurement error (in either inflation or dividends) can also account for the somewhat low standard deviation of real dividend growth of the portfolios in the simulations.

[37] The model was actually simulated for 165 years with the data for the first 100 years being discarded so as to minimize the effect of the assumed initial values of the dynamic quantities.

of the assets, both from the data as well as the simulations, which are tabulated in table (XXIV). The model's two factor structure is highly evident here as all the eigenvalues after the second one are zero. To better reflect the data and investigate the possible consequences of the inclusion of small, irrelevant factors into the long run risk model, we added white noise with a variance of 20% of the simulated values to the log P/D ratios. The introduction of this noise can also be thought of as representing measurement error in the prices or dividends brought about due to liquidity issues or other market imperfections. The normalized eigenvalues after adding this noise are summarized in table (XXV). From it, we see that the model is able to replicate the key elements of this factor structure after adding the noise.[38]

We thus see that the long run risk model being simulated here is compatible not only with many of the important observed moments of macroeconomic quantities but also with the observed factor structure of excess returns and P/D ratios. Given this, it is interesting to examine the performance of different asset pricing tests for long run risk models within the context of these simulations. This will enable the study of the effect of finite sample size and measurement noise on the efficacy of these tests and will point to the choice of test to be used in this paper. Since we are particularly interested in examining the impact of measurement noise on these tests, we first turn to the task of establishing a reasonable estimate for the size of this noise for two important quantities in long run risk models, the consumption growth and the real risk free rate.

[38]Note that it is not necessary to replicate the features of the small factors as these represent a very small fraction of the variance and are not economically interesting.

Table XX
Asset-specific parameters for the simulation.

Asset-specific parameters for the simulation. The assets are indexed by l. $\mu_{l,d}$ represents the unconditional mean of the dividend growth for asset l, ϕ_l the dependence of predictable dividend growth on the long run risk state variable x and $\varphi_{l,d}$ the idiosyncratic volatility of dividend growth.

Parameters for the asset dividend growths

l	$\mu_{l,d}$	ϕ_l	$\varphi_{l,d}$
1	−0.0286	1.7834	19.1677
2	0.0889	3.7689	21.7081
3	0.0160	3.2545	19.4655
4	0.0456	3.4405	23.5766
5	0.0471	2.6758	24.0000
6	0.0907	4.6342	16.6065
7	0.0778	5.8088	16.3543
8	0.0457	2.4918	8.5237
9	0.0928	9.5089	24.0000
10	−0.0145	5.5979	24.0000
11	−0.0012	4.8912	24.0000
12	0.0821	8.5459	22.0032
13	0.0556	10.9271	8.9635
14	0.0272	6.0810	21.8607
15	0.0926	5.1230	24.0000
16	0.0454	5.1540	6.0000
17	0.0327	3.0965	21.1709
18	0.0317	3.3548	16.4485
19	0.0147	3.5232	23.0091
20	0.0619	3.3028	6.6980
21	0.0167	2.5690	12.5081
22	0.0421	10.8271	6.0000
23	0.0901	3.7845	11.6097
24	0.0436	2.5953	24.0000
25	0.0788	3.7323	11.0877

Table XXI
Model implied moments for important economic quantities compared with the data

The model implied moments are obtained from 500 simulations.

Moment	Data	Simulation mean	5th percentile	95th percentile
$E[\Delta c_t]$	0.0199	0.0200	0.0153	0.0246
$\text{Std}[\Delta c_t]$	0.0136	0.0151	0.0105	0.0194
$AC(1)[\Delta c_t]$	0.243	0.320	0.148	0.488
$E[r_{f,t}]$	0.0059	0.0035	-0.0012	0.0079
$\text{Std}[r_{f,t}]$	0.0343	0.0067	0.0045	0.0089
$\text{Min}[r_{l,t} - r_{f,t}]$	0.010	0.018	-0.012	0.049
$\text{Max}[r_{l,t} - r_{f,t}]$	0.133	0.209	0.131	0.292
$\text{Min}\, E[\Delta d_{l,t}]$	-0.023	-0.030	-0.062	0.002
$\text{Max}\, E[\Delta d_{l,t}]$	0.105	0.104	0.070	0.149
$\text{Min}\,\text{Std}[\Delta d_{l,t}]$	0.087	0.085	0.075	0.095
$\text{Max}\,\text{Std}[\Delta d_{l,t}]$	0.385	0.306	0.279	0.333

Table XXII
Predictability of consumption growth in the model and in the data.

For the data, we use real time consumption growth as the measure of consumption growth. The results for the model are derived from 1000 simulations over 165 years with the data for the first 100 years being dropped so as to limit the impact of initial values on the numbers.

Data	Simulation mean	5th percentile	95th percentile
17.4%	32.6%	10.6%	55.2%

Table XXIII
Factor structure of excess returns in the model and in the data

Eigenvalues of the covariance matrix of the continuously compounded excess returns of the 25 Fama-French portfolios as well as those obtained by simulating the model.

Eigenvalues of the covariance matrix of excess returns

Data	Simulation mean	5th percentile	95th percentile
1.00000	1.00000	1.00000	1.00000
0.06052	0.06171	0.04705	0.07889
0.04741	0.03926	0.02994	0.04970
0.01280	0.01135	0.00871	0.01403
0.00823	0.00807	0.00637	0.01010
0.00626	0.00667	0.00531	0.00824
0.00535	0.00573	0.00455	0.00711
0.00389	0.00497	0.00399	0.00613
0.00339	0.00433	0.00351	0.00541
0.00316	0.00375	0.00302	0.00460
0.00288	0.00331	0.00267	0.00403
0.00231	0.00294	0.00240	0.00359
0.00207	0.00263	0.00214	0.00324
0.00200	0.00236	0.00191	0.00289
0.00149	0.00213	0.00170	0.00265
0.00142	0.00191	0.00152	0.00234
0.00132	0.00171	0.00136	0.00212
0.00108	0.00151	0.00118	0.00186
0.00099	0.00132	0.00106	0.00164
0.00097	0.00112	0.00087	0.00139
0.00074	0.00076	0.00059	0.00095
0.00067	0.00058	0.00045	0.00075
0.00056	0.00040	0.00030	0.00050
0.00045	0.00030	0.00023	0.00038
0.00043	0.00023	0.00017	0.00029

Table XXIV
Factor structure of log P/D ratios in the model and in the data.

Eigenvalues of the covariance matrix of the log P/D ratios of the 25 Fama-French portfolios as well as those obtained by simulating the model.

Eigenvalues of the covariance matrix of log P/D ratios

Data	Simulation mean	5th percentile	95th percentile
1.00000	1.00000	1.00000	1.00000
0.06041	0.03598	0.01272	0.07067
0.01669	0.00000	0.00000	0.00000
0.01169	0.00000	0.00000	0.00000
0.00627	0.00000	0.00000	0.00000
0.00522	0.00000	0.00000	0.00000
0.00494	0.00000	0.00000	0.00000
0.00318	0.00000	0.00000	0.00000
0.00245	0.00000	0.00000	0.00000
0.00238	0.00000	0.00000	0.00000
0.00215	0.00000	0.00000	0.00000
0.00168	0.00000	0.00000	0.00000
0.00137	0.00000	0.00000	0.00000
0.00101	0.00000	0.00000	0.00000
0.00094	0.00000	0.00000	0.00000
0.00085	0.00000	0.00000	0.00000
0.00072	0.00000	0.00000	0.00000
0.00063	0.00000	0.00000	0.00000
0.00052	0.00000	0.00000	0.00000
0.00049	0.00000	0.00000	0.00000
0.00046	0.00000	0.00000	0.00000
0.00040	0.00000	0.00000	0.00000
0.00028	0.00000	0.00000	0.00000
0.00022	0.00000	0.00000	0.00000
0.00018	0.00000	0.00000	0.00000

Table XXV
Factor structure of log P/D ratios in the model with noise and in the data.

Eigenvalues of the covariance matrix of the log P/D ratios of the 25 Fama-French portfolios as well as those obtained by simulating the model and adding some noise to the result.

Eigenvalues of the covariance matrix of noisy log P/D ratios

Data	Simulation mean	5th percentile	95th percentile
1.00000	1.00000	1.00000	1.00000
0.06041	0.04536	0.02144	0.08128
0.01669	0.01451	0.01323	0.01577
0.01169	0.01337	0.01234	0.01442
0.00627	0.01251	0.01160	0.01352
0.00522	0.01179	0.01095	0.01269
0.00494	0.01114	0.01046	0.01183
0.00318	0.01057	0.00989	0.01132
0.00245	0.01003	0.00936	0.01071
0.00238	0.00952	0.00877	0.01022
0.00215	0.00904	0.00842	0.00972
0.00168	0.00859	0.00800	0.00918
0.00137	0.00813	0.00757	0.00872
0.00101	0.00771	0.00712	0.00837
0.00094	0.00730	0.00677	0.00786
0.00085	0.00692	0.00639	0.00751
0.00072	0.00652	0.00603	0.00706
0.00063	0.00615	0.00567	0.00664
0.00052	0.00579	0.00533	0.00627
0.00049	0.00541	0.00496	0.00588
0.00046	0.00506	0.00464	0.00554
0.00040	0.00470	0.00429	0.00514
0.00028	0.00433	0.00388	0.00477
0.00022	0.00394	0.00350	0.00437
0.00018	0.00345	0.00295	0.00390

C. Measurement Error

We do so by analyzing the degree of correlation between different measures for the same fundamental macroeconomic quantities. For consumption growth, we use the estimates of consumption growth derived from the continuously revised NIPA tables as well as those from the real time database maintained by the Federal Reserve Bank of St. Louis (described in detail by Croushore and Stark (2001)). Regressing these estimates against each other leads to the results in table (XXVI). The R^2 of 67% or about $\frac{2}{3}$ indicates that the variance of measurement noise in consumption growth is about half of the variance of actual consumption growth. We thus simulate measured consumption growth as actual consumption growth plus iid noise with half it's realized variance in that simulation.

Table XXVI
Measurment error in consumption growth

Regression of the conventional revised measure of consumption growth Δc on the corresponding real time measure Δc^{RT}.

	Intercept	Δc^{RT}	R^2
Δc	0.0060 (0.0019)	0.838 (0.092)	67.0%

Similarly, we regress three measures of the real risk free rate on each other to estimate the amount of measurement noise in it. We use the three measures considered by Marakani (2009), i.e. estimates constructed with the use of lagged, realized and expected inflation. From the results tabulated in table (XXVII), we see that the R^2 of each of the regressions is quite low with the average being under 33%. This indicates that the measurement noise in the reported real risk free rate has about twice the variance of the underlying quantity. Hence, for the simulations, we model the measured real risk free rate as the actual risk free rate plus iid noise with twice it's realized variance.

Table XXVII
Measurement error in the real risk free rate

Regression of three measures of the real risk free rate on each other. The three measures are computed using the lagged, realized and expected inflation. The regressions are restricted to the post-1946 period as expected inflation data is only available for it.

Regression of $r_{f,t}^{\text{lagged}}$ against $r_{f,t}^{\text{realized}}$

Coefficient	Estimate (Std. Err.)
Intercept	0.0046 (0.0028)
$r_{f,t}^{\text{realized}}$	0.454 (0.106)
R^2	23.6%

Regression of $r_{f,t}^{\text{lagged}}$ against $r_{f,t}^{\text{expected}}$

Coefficient	Estimate (Std. Err.)
Intercept	-0.0023 (0.0030)
$r_{f,t}^{\text{expected}}$	0.890 (0.145)
R^2	38.6%

Regression of $r_{f,t}^{\text{realized}}$ against $r_{f,t}^{\text{expected}}$

Coefficient	Estimate (Std. Err.)
Intercept	-0.0007 (0.0035)
$r_{f,t}^{\text{expected}}$	0.859 (0.169)
R^2	30.4%

D. Type I error of Asset Pricing Tests with Respect to the Long Run Risk Model

We now analyze the performance of tests of four different asset pricing restrictions of the long run risk model in order to determine which is the most reasonable one to use in the analysis in this paper. The first two asset pricing restrictions that we consider are related to the one analyzed by Ferson, Nallareddy, and Xie (2009).[39] Of these, the first is[40]

$$\begin{aligned} E[r_{i,t+\Delta t} - r_{f,t}] + \frac{1}{2}\text{Var}[r_{i,t+\Delta t} - r_{f,t}] \approx & \beta_{\tilde{x}}\lambda_{\tilde{x}} + \beta_{\tilde{\sigma^2}}\lambda_{\tilde{\sigma^2}} + \sum_{i=1}^{2}\beta_{\tilde{\epsilon}}\lambda_{\tilde{\epsilon}} \\ & + \sum_{j=1}^{2}\beta_{\tilde{w}}\lambda_{\tilde{w}} \end{aligned} \tag{60}$$

where the returns $r_{i,t}$ are continuously compounded, $\tilde{x}$ and $\tilde{\sigma^2}$ are the estimated values of x_t and σ_t^2 (note from the subscript that these are *lagged* values), and $\tilde{\epsilon}$ and $\tilde{w}$ are the estimated values of the innovations of these processes. x and σ^2 are estimated in the same manner as by Bansal, Yaron, and Kiku (2007) and Ferson, Nallareddy, and Xie (2009), i.e. by the use of the following regressions

$$\Delta c_{t+\Delta t} = \alpha_0 + \alpha_1 z_{m,t} + \alpha_2 r_{f,t} + \sigma_t \eta_{t+\Delta t}\sqrt{\Delta t} \tag{61}$$

$$\tilde{x}_t = \alpha_0 - \mu + \alpha_1 z_{m,t} + \alpha_2 r_{f,t} \tag{62}$$

$$\tilde{x}_{t+\Delta t} = \rho\tilde{x}_t + \tilde{\epsilon}_{t+\Delta t}\sqrt{\Delta t} \tag{63}$$

$$\sigma_t^2\eta_{t+\Delta t}^2\Delta t = \beta_0 + \beta_1 z_{m,t} + \beta_2 r_{f,t} + \omega_{t+\Delta t} \tag{64}$$

$$\tilde{\sigma}_t^2\Delta t = \beta_0 + \beta_1 z_{m,t} + \beta_2 r_{f,t} \tag{65}$$

$$\tilde{\sigma}_{t+\Delta t}^2 = \nu\tilde{\sigma}_t^2 + \tilde{w}_{t+\Delta t}\sqrt{\Delta t} \tag{66}$$

where $z_{m,t}$ is the log market P/D ratio (taken to be the log P/D ratio of the first asset in the simulations) and Δt is one year. The second asset pricing restriction that we consider comes from considering only the innovations to the stochastic discount factor as in Ferson, Nallareddy, and Xie (2009). This simplifies (60) to

$$E[r_{i,t+\Delta t} - r_{f,t}] + \frac{1}{2}\text{Var}[r_{i,t+\Delta t} - r_{f,t}] \approx \sum_{i=1}^{2}\beta_{\tilde{\epsilon}}\lambda_{\tilde{\epsilon}} + \sum_{j=1}^{2}\beta_{\tilde{w}}\lambda_{\tilde{w}} \tag{67}$$

[39] Ferson, Nallareddy, and Xie (2009) use GMM with the Euler moment restrictions in the SDF framework. We use the beta representation which is approximate but quite accurate when dealing with continuously compounded returns.

[40] Note that we don't need a $\beta_{\Delta c}$ term as there is no contemporaneous correlation between the dividend growth and consumption growth innovations

The third and fourth asset pricing restrictions that we consider are analogous but use the two largest estimated log P/D ratio factors instead of the log market P/D ratio and the real risk free rate as they should also span x and σ^2. The principal idea behind this approach is that given the null, they should be more accurately estimated in the presence of measurement error since they are estimated using multiple assets. The asset pricing restriction analogous to (60) is then given by

$$E[r_{i,t+\Delta t} - r_{f,t}] + \frac{1}{2}\text{Var}[r_{i,t+\Delta t} - r_{f,t}] \approx \sum_{i=1}^{2} \beta_{F_i}\lambda_{F_i} + \sum_{j=1}^{2} \beta_{IF_j}\lambda_{IF_j} \tag{68}$$

where F_i and IF_i are the ith principal components of the log P/D ratios of the assets and their estimated innovations respectively (the latter are estimated by fitting the former to an AR(1) process). The asset pricing restriction analogous to (67), which only uses the estimated innovations, is then

$$E[r_{i,t+\Delta t} - r_{f,t}] + \frac{1}{2}\text{Var}[r_{i,t+\Delta t} - r_{f,t}] \approx \sum_{j=1}^{2} \beta_{IF_j}\lambda_{IF_j} \tag{69}$$

We examine whether the hypothesis that the factors being considered are useless is rejected by the cross sectional regression methodology. This is done using the Wald test for the risk premia of the factors with their covariance matrix being estimated in the standard manner (see for eg., Shanken (1992) and Shanken and Zhou (2007)). The rejection frequencies for each of these tests in 1000 simulations are reported in table (XXVIII). The results show that the test of the asset pricing restriction involving the log P/D ratio factors (which also include noise calibrated to fit the observed factor structure of log P/D ratios) and/or their innovations display much greater power than those involving the estimated long run risk processes and their innovations. Hence, we use the former in our analysis in this paper.

Table XXVIII
Power of the two type of tests tested in the simulation

Rejection frequencies for the hypothesis that the λs of the relevant factors are zero.

Hypothesis	Non-rejection rate		
	p=0.10	p=0.05	p=0.01
$\lambda_{\tilde{x}}, \lambda_{\tilde{\sigma^2}}, \lambda_{\tilde{\epsilon}}, \lambda_{\tilde{w}} = 0$	14.9%	26.4%	48.8%
$\lambda_{\tilde{\epsilon}}, \lambda_{\tilde{w}} = 0$	12.3%	24.0%	50.4%
$\lambda_{F_1}, \lambda_{F_2}, \lambda_{IF_1}, \lambda_{IF_2} = 0$	0	0.2%	0.4%
$\lambda_{IF_1}, \lambda_{IF_2} = 0$	0.4%	0.6%	1.5%

E. Conclusion

In this appendix, we simulate a 25 asset long run risk economy with parameters chosen so as to match key economic and financial moments with those in U.S. economic and financial data. We analyze the type I error of different asset pricing tests within this economy and find, when realistic measurement noise is introduced into it, that tests using estimates of the long run risk components derived from projections of consumption growth onto the log market price dividend ratio and real risk free rate display high type I error while those estimating the same components using the principal components of the log price dividend ratios of the assets do not do so. This implies that the latter type of tests have a more desirable profile. Hence, we use such tests in this paper.

C. Out of Sample Tests

Table XXIX
Out of sample test for the relation between the first two principal components and consumption growth volatility

Results of regressing real annual market dividend growth against lagged $F_1^{a,os}$ and $F_2^{a,os}$, the out of sample estimates of the first and second log P/D factors. The standard errors are Newey-West corrected with the number of lags required estimated using the procedure of Newey and West (1994).

Regression of 24 quarter consumption growth volatility on $F_1^{a,os}$ and $F_2^{a,os}$

	Intercept	$F_1^{a,os}$	$F_2^{a,os}$	R^2
v_t^{24}	0.171^{***} (0.016)	-0.0050^{***} (0.0007)	0.0022 (0.0026)	74.1%

To check the robustness of the results, we estimated the rotation matrices relating the log price dividend ratios of the portfolios to their first two principal components only using data from 1943 to 1975 and used them to construct out of sample factors from 1975 to 2008. We found that these estimated out of sample factors also track consumption growth volatility and predict market dividend and real time consumption growth in a manner similar to that documented for the in sample factors.

The results of regressing 24 quarter consumption growth volatility on the estimated out of sample factors, summarized in table (XXIX), show that the relation found in the paper is robust. Specifically, consumption growth volatility is found to be very significantly negatively related to the first out of sample factor $F_1^{a,os}$ and to be unrelated to the second out of sample factor $F_2^{a,os}$.

The predictability of real time consumption and market dividend growth using the out of sample factors are summarized in table (XXX). As can be seen, only the second factor is relevant in predicting real time consumption growth and market dividend growth. The result for the three year market dividend growth seems marginal but that is because the number of data points is much smaller and the R^2 of the regression is still found to be quite high.

Table XXX
Out of sample test for the relation between the first two principal components and future market dividend and real time consumption growth

Results of regressing real annual market dividend growth and real time consumption growth (Δc^{RT}) against lagged $F_1^{a,os}$ and $F_2^{a,os}$, the out of sample estimates of the first and second log P/D factors. The standard errors are Newey-West corrected with the number of lags required estimated using the procedure of Newey and West (1994). The regressions using the log market price dividend ratio use data from 1976 onwards in order to be consistent with the others.

Regression of market dividend growth on $F_1^{a,os}$ and $F_2^{a,os}$ and the log market price dividend ratio

	$F_1^{a,os}$	$F_2^{a,os}$	$\log(P/D)_m$	R^2
1 yr. Market div. growth	-0.0066 (0.0055)	0.0491*** (0.0183)		20.8%
			0.012 (0.036)	0.8%
3 yr. Market div. growth	0.0026 (0.0263)	0.0593 (0.0453)		13.4%
			0.066 (0.138)	5.7%

Regression of real time annual consumption growth on lagged values of $F_1^{a,os}$ and $F_2^{a,os}$.

	$F_1^{a,os}$	$F_2^{a,os}$	R^2
Δc_{t+1}^{RT}	$4.1 \times 10^{-4}(0.0010)$	0.0063** (0.0031)	13.9%
Δc_{t+2}^{RT}	$5.5 \times 10^{-4}(6.8 \times 10^{-4})$	0.0045** (0.0016)	5.8%
$\Delta c_{t+1}^{RT} + \Delta c_{t+2}^{RT}$	$7.6 \times 10^{-4}(1.5 \times 10^{-3})$	0.0123** (0.0050)	18.9%

D. Robust Test Statistics

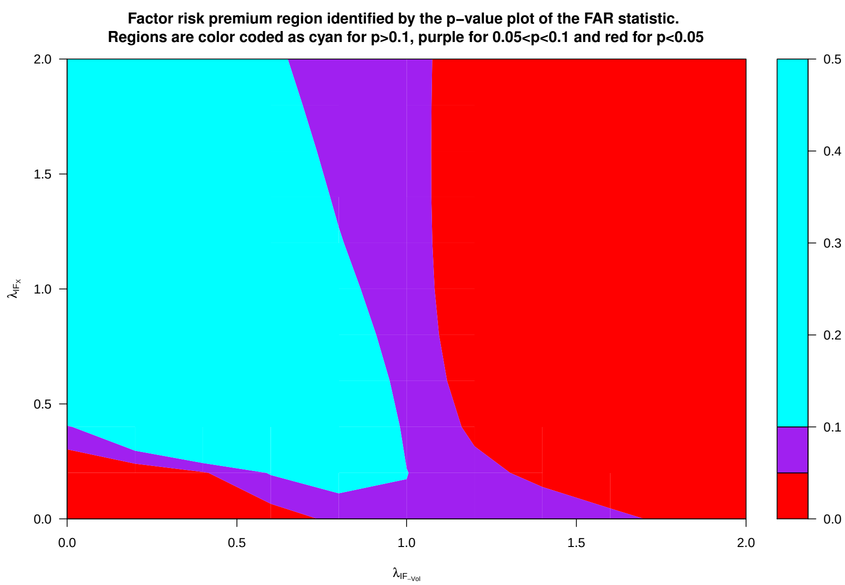

Figure 11

p-value plot of the test of the joint hypothesis of factor pricing together with $(\lambda_{IF_{-Vol}}, \lambda_{IF_X}) = (\hat{\lambda}_{IF_{-Vol}}, \hat{\lambda}_{IF_X})$ using the FAR statistic proposed by Kleibergen (2009). $\lambda_{IF_{-Vol}}$ and λ_{IF_X} are respectively the factor risk premia for the negative volatility and consumption/dividend growth factors.

Since the excess returns of the 25 Fama-French portfolios formed on the basis of size and book to market ratio have a strong factor structure, it is important to use robust test statistics to eliminate the problem of useless factors being identified as useful (a problem forcefully brought out by Kleibergen (2009) and Kleibergen (2010)). Hence, we use the robust test statistics suggested by Kleibergen (2009) to ensure that the factors here are not useless.

We find that these robust test statistics reject the joint hypothesis that $\lambda_{IF_{-Vol}} = \lambda_{IF_X} = 0$ (non-rejection of the hypothesis would indicate that the pricing factors are useless) and do not reject either the hypothesis of factor pricing or that of $\lambda_{IF_{-Vol}} = \hat{\lambda}_{IF_{-Vol}}, \lambda_{IF_X} = \hat{\lambda}_{IF_X}$ for many values of $(\hat{\lambda}_{IF_{-Vol}}, \hat{\lambda}_{IF_X})$ including those estimated using the cross sectional regressions (rejection of this would indicate that the model is rejected by the data). Figure 11 contains the plot of the p-values of the FAR test statistic for many different values of $(\hat{\lambda}_{IF_{-Vol}}, \hat{\lambda}_{IF_X})$. This statistic tests the joint hypothesis of factor pricing and of $\lambda_{IF_{-Vol}} = \hat{\lambda}_{IF_{-Vol}}, \lambda_{F_X} = \hat{\lambda}_{IF_X}$.

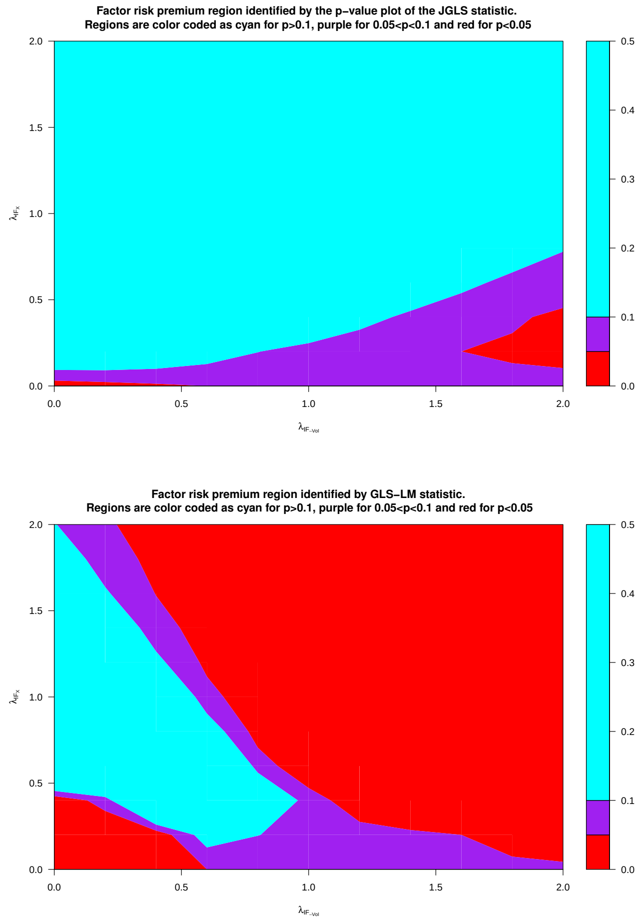

Figure 12

p-value plot of the test of the hypothesis of factor pricing given $(\lambda_{IF_{-Vol}}, \lambda_{IF_X}) = (\hat{\lambda}_{IF_{-Vol}}, \hat{\lambda}_{IF_X})$ using the JGLS and GLS-LM statistics proposed by Kleibergen (2009). $\lambda_{IF_{-Vol}}$ and λ_{IF_X} are respectively the factor risk premia for the negative volatility and consumption/dividend growth factors.

It shows that the joint hypothesis is rejected at $\hat{\lambda}_{IF_{-Vol}} = \hat{\lambda}_{IF_X} = 0$ and also that it is not rejected for many other values of $\hat{\lambda}_{IF_{-Vol}}$ and $\hat{\lambda}_{IF_X}$ including those in table VIII. Further, the region identified by $p > 0.1$ excludes $\lambda_{IF_X} = 0$ but not $\lambda_{IF_{-Vol}} = 0$. This is consistent with the findings using GMM which are analyzed in the next subsection.

The JGLS statistic which tests the hypothesis of factor pricing for a given value of $\lambda_{IF_{-Vol}}$ and λ_{IF_X} is plotted in figure 12. Since it tests a weaker hypothesis, it is not surprising that it rejects fewer values of $\lambda_{IF_{-Vol}}$ and λ_{IF_X} than the FAR statistic. When combined with the GLS-LM statistic, also plotted in figure 12, which tests the hypothesis that $\lambda_{IF_{-Vol}} = \hat{\lambda}_{IF_{-Vol}}, \lambda_{F_X} = \hat{\lambda}_{IF_X}$ given that factor pricing is correct, it gives very similar results to those given by the FAR statistic.

Hence, we can conclude that the robust test statistics show that (21) cannot be rejected. However, they, together with the findings made using GMM, do cast some doubt on the significance of $\lambda_{IF_{-Vol}}$.