Online Technical Appendix

This Online Appendix accompanies the paper "Inference for VARs Identified with Sign Restrictions" by H.R. Moon, F. Schorfheide, E. Granziera, and M. Lee. The Appendix has three sections. In Section A we provide proofs of the lemmas stated in the main text. Section B states and proves lemmas that are needed to prove Theorems 1 and 2 in the main text. Finally, Section C provides analytical derivations for the Monte Carlo experiment presented in Section 5 of the main text.

A Proofs of Lemmas stated in the Main Text

Proof of Lemma 1. To simplify the notation in the proof we omit tildes and write $S_\theta(q)$, $S_R(q)$ instead of $\tilde{S}_\theta(q)$, $\tilde{S}_R(q)$. *Convexity:* Suppose $\theta_i \in \Theta(\phi)$, $i = 1, 2$, and $\theta_1 < \theta_2$. Then there exist q_i with $\|q_i\| = 1$ and $\mu_i \geq 0$ such that

$$S_\theta(q_i)\phi - \theta_i = 0, \quad S_R(q_i)\phi - \mu_i = 0. \tag{A.1}$$

Convexity-Case (i): Suppose that $q_1 \neq -q_2$. We now verify that for any $\lambda \in [0, 1]$ $\theta = \lambda\theta_1 + (1 - \lambda)(\theta_2) \in \Theta(\phi)$. For $\tau \in [0, 1]$ define

$$q(\tau) = \frac{\tau q_1 + (1-\tau)q_2}{\|\tau q_1 + (1-\tau)q_2\|}, \quad H(\tau) = S_\theta(q(\tau))\phi - \theta.$$

The linearity of $S_\theta(q)$ with respect to q and (A.1) imply that

$$\begin{aligned} H(\tau) &= \frac{\tau S_\theta(q_1)\phi}{\|\tau q_1 + (1-\tau)q_2\|} + \frac{(1-\tau)S_\theta(q_2)\phi}{\|\tau q_1 + (1-\tau)q_2\|} - \lambda\theta_1 - (1-\lambda)\theta_2 \\ &= \frac{\tau\theta_1}{\|\tau q_1 + (1-\tau)q_2\|} + \frac{(1-\tau)\theta_2}{\|\tau q_1 + (1-\tau)q_2\|} - \lambda\theta_1 - (1-\lambda)\theta_2. \end{aligned}$$

Using $\|q_i\| = 1$ we obtain

$$\begin{aligned} H(0) &= \theta_2 - \lambda\theta_1 - (1-\lambda)\theta_2 = \lambda(\theta_2 - \theta_1) \geq 0 \\ H(1) &= \theta_1 - \lambda\theta_1 - (1-\lambda)\theta_2 = -(1-\lambda)(\theta_2 - \theta_1) \leq 0 \end{aligned}$$

Since $H(\tau)$ is continuous we deduce that there exists a τ^* such that $H(\tau^*) = 0$. Now consider

$$\begin{aligned} S_R(q(\tau^*)) &= \frac{\tau^* S_R(q_1)\phi}{\|\tau^* q_1 + (1-\tau^*)q_2\|} + \frac{(1-\tau^*)S_R(q_2)\phi}{\|\tau^* q_1 + (1-\tau^*)q_2\|} \\ &= \frac{\tau^*\mu_1}{\|\tau^* q_1 + (1-\tau^*)q_2\|} + \frac{(1-\tau^*)\mu_2}{\|\tau^* q_1 + (1-\tau^*)q_2\|} \\ &\geq 0. \end{aligned}$$

The first equality follows from the linearity of $S_R(q)$, the second equality is implied by (A.1), and the inequality follows from $\mu_i \geq 0$. Thus, $\theta \in \Theta(\phi)$.

Convexity-Case (ii): Suppose that $q_1 = -q_2$. The linearity of $S_\theta(q)$ implies that $\theta_1 = -\theta_2$. By assumption there exists a $q_3 \neq q_1, -q_1$ with the property that $S_R(q_3)\phi \geq 0$. Let $\theta_3 = S_\theta(q_3)\phi$. By construction, $\theta_3 \in \Theta(\phi)$. If $\theta_3 = \theta_1$ ($\theta_3 = \theta_2$) we simply replace q_1 (q_2) by q_3 and follow the steps outlined for Case (i). If $\theta_1 < \theta_3 < \theta_2$, then the Case (i) argument implies that any θ in the intervals $[\theta_1, \theta_3]$ and $[\theta_3, \theta_2]$ and thereby any $\theta = \lambda\theta_1 + (1-\lambda)\theta_2$ is included in the identified set. Finally, if $\theta_3 < \theta_1$ ($\theta_2 < \theta_3$), we deduce from Case (i) that the interval $[\theta_3, \theta_2]$ ($[\theta_1, \theta_3]$) is included in the identified set.

Boundedness: We shall prove a slightly more general result. Let

$$\mathcal{S}_c^\theta = \left\{ \theta \;\middle|\; Q(\theta; \phi, I) \leq c \right\}$$

For $c = 0$ $\mathcal{S}_c^\theta = \Theta(\phi)$. Suppose that $\tilde{\theta} \in \mathcal{S}_c^\theta$. We will assume that $\tilde{\theta} > 0$ and show by contradiction that $\mathcal{S}_c^\theta$ must have an upper bound. Consider a sequence $a_n > 0$ with $a_n \uparrow \infty$ with the property that $a_n\tilde{\theta} \in \mathcal{S}_c^\theta$ for each n. The unboundedness of $\Theta(\phi)$ guarantees the existence of such a series. Then

$$\begin{aligned} Q(a_n\tilde{\theta}; \phi, I) &= \min_{q=\|1\|,\, \mu \geq 0} \|S_\theta(q)\phi - a_n\tilde{\theta}\|^2 + \|S_R(q)\phi - \mu\|^2 \\ &\geq \min_{q=\|1\|} \|S_\theta(q)\phi - a_n\tilde{\theta}\|^2 \end{aligned}$$

The definition of $S_\theta(q)$ implies that

$$\|S_\theta(q)\| = \sqrt{tr[(S_\theta \otimes q')S_\phi S_\phi'(S_\theta \otimes q')']} = \sqrt{tr[(S_\theta S_\theta')} = 1.$$

Since ϕ and $\tilde{\theta}$ are fixed, we deduce that as $n \longrightarrow \infty$

$$\min_{q=\|1\|} \|S_\theta(q)\phi - a_n\tilde{\theta}\|^2 \longrightarrow \infty.$$

Thus, $Q(a_n\tilde{\theta}; \phi, I) > c$ eventually, which contradicts the assumption that $a_n\tilde{\theta} \in S_c^\theta$ for all n. The existence of a lower bound can be established by considering a sequence $-a_n$. Moreover, $\theta < 0$ can be handled by a straightforward modification of the argument. $\square$

Proof of Lemma 4: In the proof of Lemma 1 we showed that sets of the form

$$\mathcal{S}_c^\theta = \left\{ \theta \;\middle|\; Q(\theta; \phi, I) \leq c \right\} \tag{A.2}$$

are bounded. The finite-sample confidence sets take the form

$$CS^{\theta}_{(i)} = \left\{ \theta \,\middle|\, Q\big(\theta; \hat{\phi}, \hat{W}^*(\cdot)\big) \leq c_{(i)} \right\}.$$

Recall that

$$Q\big(\theta; \hat{\phi}, \hat{W}^*(\cdot)\big) = \min_{\|q\|=1, \mu \geq 0} T \left\| S(q)\hat{\phi} - V(q)\begin{pmatrix} \theta \\ \mu \end{pmatrix} \right\|^2_{\hat{\Sigma}^{-1}(q)},$$

where

$$\hat{\Sigma}^{-1} = (S(q)\hat{\Lambda}S'(q))^{-1}.$$

Now let

$$\hat{W}_{min}(q) = \frac{1}{\lambda_{max}(\hat{\Lambda})\lambda_{max}(S(q)S(q)')} I.$$

By construction $\hat{W}^*(q) \geq \hat{W}_{min}(q) > 0$ for all q and

$$Q\big(\theta; \hat{\phi}, \hat{W}^*(\cdot)\big) \geq Q\big(\theta; \hat{\phi}, \hat{W}_{min}(\cdot)\big) = \frac{1}{\lambda_{max}(\hat{\Lambda})\lambda_{max}(S(q)S(q)')} Q\big(\theta; \hat{\phi}, I\big).$$

The statement of the lemma follows from setting $c = c_i \lambda_{max}(\hat{\Lambda})\lambda_{max}(S(q)S(q)')$ in (A.2). □

Proof of Lemma 5: We need to verify that the confidence set constructed by taking unions of the identified sets can be represented according to (42). Let

$$CS^*_U = \bigcup_{\phi \in CS^{\phi}} \Theta(\phi),$$

where

$$CS^{\phi} = \left\{ \phi \in \mathcal{P} \,\middle|\, T\|\hat{\phi} - \phi\|^2_{\hat{\Lambda}^{-1}} \leq c(\chi^2_m) \right\}.$$

(i) Show that $CS^*_U \subseteq CS^{\theta}_U$. Suppose $\theta \in CS^*_U$. Thus, there exists a $\phi \in CS^{\phi}$ such that $\theta \in \Theta(\phi)$. So, $T\|\hat{\phi} - \phi\|^2_{\hat{\Lambda}^{-1}} \leq c(\chi^2_m)$. This implies that there exist a q_* with $\|q_*\| = 1$ and a $\mu_* \geq 0$ such that $\tilde{S}(q_*)\phi = [\theta', \mu_*']'$. In turn,

$$\begin{aligned}
Q(\theta; \hat{\phi}, \hat{W}^*) &= \min_{\|q\|=1,\, \mu \geq 0} \left\| \tilde{S}(q)\hat{\phi} - \begin{pmatrix} \theta \\ \mu \end{pmatrix} \right\|^2_{\hat{W}^*(q)} \\
&\leq \left\| \tilde{S}(q_*)\hat{\phi} - \begin{pmatrix} \theta \\ \mu_* \end{pmatrix} \right\|^2_{\hat{W}^*(q_*)} \\
&= \big\| \tilde{S}(q_*)(\hat{\phi} - \phi) \big\|^2_{\hat{W}^*(q_*)} \\
&= T(\hat{\phi} - \phi)'S'(S\hat{\Lambda}S')^{-1}S(\hat{\phi} - \phi) \\
&\leq T\|\hat{\phi} - \phi\|^2_{\hat{\Lambda}^{-1}} \leq c(\chi^2_m).
\end{aligned}$$

The third equality uses the definition of $\hat{W}^*$ in (23) and $S(q) = V(q)\tilde{S}(q)$. The second inequality can be obtained as follows. Factorize $\Lambda = \hat{L}\hat{L}'$ and define $\hat{A} = \hat{L}'S'$ as well as the projection matrix $P_{\hat{A}} = \hat{A}(\hat{A}'\hat{A})^{-1}\hat{A}'$. Then,

$$\begin{aligned} T(\hat{\phi}-\phi)'S'(S\hat{\Lambda}S')^{-1}S(\hat{\phi}-\phi) &= T(\hat{\phi}-\phi)'(\hat{L}')^{-1}\hat{L}'S'(S\hat{L}\hat{L}'S')^{-1}S\hat{L}\hat{L}^{-1}(\hat{\phi}-\phi) \\ &= T(\hat{\phi}-\phi)'(\hat{L}')^{-1}P_{\hat{A}}\hat{L}^{-1}(\hat{\phi}-\phi) \\ &\leq T(\hat{\phi}-\phi)'(\hat{L}')^{-1}\hat{L}^{-1}(\hat{\phi}-\phi) \\ &= T\|\hat{\phi}-\phi\|^2_{\hat{\Lambda}^{-1}} \end{aligned}$$

Thus, we deduce that $\theta \in CS^{\theta}_U$.

(ii) Show that $CS^{\theta}_U \subseteq CS^*_U$. Suppose to the contrary that there exists a $\theta \in CS^{\theta}_U$ and $\theta \notin CS^*_U$. Thus, $\theta \in (CS^*_U)^c = \bigcap_{\phi \in CS^{\phi}} (\Theta(\phi))^c$. Hence, for any ϕ such that $\theta \in \Theta(\phi)$, it has to be the case that $\phi \notin CS^{\phi}$. Define $\psi(\mu) = [\theta', \mu']'$. Thus,

$$\begin{aligned} c(\chi^2_m) &< \left(\min_{\phi \in \mathcal{P},\, \|q\|=1,\, \mu \geq 0} T\|\hat{\phi}-\phi\|^2_{\hat{\Lambda}^{-1}} \quad \text{s.t.} \quad 0 = \tilde{S}(q)\phi - \psi(\mu) \right) \\ &= \min_{\|q\|=1,\, \mu\geq 0} \left(\min_{\phi\in\mathcal{P}} T\|\hat{\phi}-\phi\|^2_{\hat{\Lambda}^{-1}} \quad \text{s.t.} \quad 0 = \tilde{S}(q)\phi - \psi(\mu) \right) \end{aligned} \tag{A.3}$$

Since the matrix $V(b)$ eliminates rows of zeros from $\tilde{S}(q)$, we can rewrite the constraint in (A.3) as

$$0 = V(q)\tilde{S}(q) - V(q)\psi(\mu) = S(q) - V(q)\psi(\mu).$$

Conditional on q and μ the Lagrangian associated with the minimization over ϕ is given by

$$\mathcal{L} = T\|\hat{\phi}-\phi\|^2_{\hat{\Lambda}^{-1}} - \lambda'(S(q)\phi - V(q)\psi(\mu)).$$

The first-order conditions are

$$0 = T\hat{\Lambda}^{-1}(\phi_* - \hat{\phi}) - S'\lambda_*, \quad 0 = S(\phi_* - \hat{\phi}) + S\hat{\phi} - V\psi(\mu).$$

Solving for ϕ_* yields

$$\phi_* = \hat{\phi} + \hat{\Lambda}S'(S\hat{\Lambda}S')^{-1}(S\hat{\phi} - V\psi).$$

Thus, using the definition $\hat{\Sigma}(q) = S(q)\hat{\Lambda}S'(q)$ we can express the constrained minimization in (A.3) as

$$\min_{\|q\|=1,\, \mu\geq 0} T \left\| S(q)\hat{\phi} - V(q)\begin{pmatrix}\theta \\ \mu\end{pmatrix} \right\|^2_{\hat{\Sigma}^{-1}(q)}. \tag{A.4}$$

Thus, $\min_{\|q\|=1,\, \mu\geq 0}\; Q(\theta;\hat{\phi},\hat{W}^*) > c(\chi^2_m)$, which contradicts the initial assumption that $\theta \in CS^{\theta}_U$. $\square$

B Technical Lemmas Used in the Proofs of the Main Results

Lemma B 1 *Suppose that Assumptions 1 to 3 are satisfied. The sample estimate $\hat{\Lambda}$ that enters $\hat{W}^*(\cdot)$ in the bounding function $\bar{Q}\big(q;\hat{\phi},\hat{W}^*(\cdot)\big)$, defined in (28), can be replaced by the population covariance matrix Λ:*

$$\left|\bar{Q}\big(q;\hat{\phi},\hat{W}^*(\cdot)\big) - \bar{Q}\big(q;\hat{\phi},W^*(\cdot)\big)\right| = o_p(1)$$

uniformly in (ϕ,q).

Proof of Lemma B 1: Notice that the penalty term in (24) cancels and thus is omitted from the subsequent calculations. Let

$$\begin{aligned} v(\hat{\Lambda}) &= \arg\min_{v\geq 0}\left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v\right\|^2_{\hat{\Sigma}^{-1}(q)} \\ v(\Lambda) &= \arg\min_{v\geq 0}\left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v\right\|^2_{\Sigma^{-1}(q)}. \end{aligned}$$

First we show that uniformly in (ϕ,q)

$$\bar{Q}\big(q;\hat{\phi},\hat{W}^*(\cdot)\big) - \bar{Q}\big(q;\hat{\phi},W^*(\cdot)\big) \leq o_p(1). \tag{B.1}$$

To do so, consider the following inequalities: Notice that

$$\begin{aligned} &\bar{Q}\big(q;\hat{\phi},\hat{W}^*(\cdot)\big) - \bar{Q}\big(q;\hat{\phi},W^*(\cdot)\big) \\ &= \min_{v\geq 0}\left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v\right\|^2_{\hat{\Sigma}^{-1}(q)} - \min_{v\geq 0}\left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v\right\|^2_{\Sigma^{-1}(q)} \\ &= \left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\hat{\Lambda})\right\|^2_{\hat{\Sigma}^{-1}(q)} - \left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right\|^2_{\Sigma^{-1}(q)} \\ &\leq \left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right\|^2_{\hat{\Sigma}^{-1}(q)} - \left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right\|^2_{\Sigma^{-1}(q)} \\ &= \left[S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right]'\Sigma^{-1/2}(q) \\ &\quad\times\left[\Sigma^{1/2}(q)\hat{\Sigma}^{-1}(q)\Sigma^{1/2}(q) - I_{l(q)}\right]\Sigma^{-1/2}(q)\left[S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right] \\ &\leq \left\|S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v(\Lambda)\right\|^2_{\Sigma^{-1}(q)}\left\|\Sigma^{1/2}(q)\hat{\Sigma}^{-1}(q)\Sigma^{1/2}(q) - I_{l(q)}\right\| \\ &= I\times II,\ \text{say}. \end{aligned}$$

The first inequality is obtained by replacing the minimizer $\nu(\hat{\Lambda})$ with the inferior value $\nu(\Lambda)$. The third equality follows from writing out the two norms and rearranging the weight matrix differential $\hat{\Sigma}^{-1} - \Sigma^{-1}$.

We now bound the terms I and II. Recall that $\Sigma = S\Lambda S'$ and $\Lambda = LL'$. For term I we obtain

$$\begin{aligned}
I &= \min_{v\geq 0} \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - V(q)M_v v \right\|^2_{\Sigma^{-1}(q)} \\
&\leq \left\| S(q)LL^{-1}\sqrt{T}(\hat{\phi}-\phi) \right\|^2_{\Sigma^{-1}(q)} \\
&= [L^{-1}\sqrt{T}(\hat{\phi}-\phi)]' P_{L'S'(q)} [L^{-1}\sqrt{T}(\hat{\phi}-\phi)] \\
&\leq \left\| L^{-1}\sqrt{T}(\hat{\phi}-\phi) \right\|^2 \lambda_{\max}(P_{L'S'(q)}) \\
&= \left\| L^{-1}\sqrt{T}(\hat{\phi}-\phi) \right\|^2 \\
&= O_p(1)
\end{aligned}$$

uniformly in (ϕ, q). Here $P_{L'S'(q)}$ is the matrix that projects onto the column space of $L'S'(q)$. The second term can be bounded as follows:

$$\begin{aligned}
II &= \left\| \Sigma^{1/2}(q)(\hat{\Sigma}^{-1}(q) - \Sigma^{-1}(q))\Sigma^{1/2}(q) \right\| \\
&= \left\| \Sigma^{-1/2}(q)(\Sigma(q) - \hat{\Sigma}(q))\hat{\Sigma}^{-1}(q)\Sigma^{1/2}(q) \right\| \\
&= \left\| \Sigma^{-1/2}(q)S(q)(\Lambda - \hat{\Lambda})S(q)'\hat{\Sigma}^{-1/2}(q)\hat{\Sigma}^{-1/2}(q)\Sigma^{1/2}(q) \right\| \\
&= \left\| (\Sigma^{-1/2}(q)S(q)L)(L'(\hat{L}')^{-1} - L^{-1}\hat{L})\hat{L}'S(q)'\hat{\Sigma}^{-1/2}(q)\hat{\Sigma}^{-1/2}(q)\Sigma^{1/2}(q) \right\| \\
&\leq \left\| \Sigma^{-1/2}(q)S(q)L \right\| \left\| L'(\hat{L}')^{-1} - L^{-1}\hat{L} \right\| \left\| \hat{L}'S(q)'\hat{\Sigma}^{-1/2}(q) \right\| \left\| \hat{\Sigma}^{-1/2}(q)\Sigma^{1/2}(q) \right\| \\
&= II_1 \times II_2 \times II_3 \times II_4, \text{ say.}
\end{aligned}$$

Bounds for the four terms can be obtained as follows. First,

$$II_1^2 = \left\| \Sigma^{-1/2}(q)\, S(q)\, L \right\|^2 = \operatorname{tr}\left(L'S(q)' \left(S(q)\, LL'S(q)' \right)^{-1} S(q)\, L \right) = l(q) \leq \tilde{k} + \tilde{r}_2. \tag{B.2}$$

Similarly, the second term can be bounded by

$$II_2^2 \leq \left\| \hat{L}'S(q)'\, \hat{\Sigma}^{-1/2}(q) \right\|^2 = l(q) \leq \tilde{k} + \tilde{r}_2. \tag{B.3}$$

Third, since $\hat{\Lambda} - \Lambda(\phi) = o_p(1)$ uniformly in ϕ and $\Lambda(\phi) > \Lambda_{\min} > 0$, for some positive definite matrix $\Lambda_{\min}$, we have

$$II_3 = \left\| L'(\hat{L}')^{-1} - L^{-1}\hat{L} \right\| = o_p(1) \tag{B.4}$$

uniformly in (ϕ, q). Finally, we obtain

$$
\begin{aligned}
II_4^2 &= \left\| \hat{\Sigma}^{-1/2}(q)\,(S(q)\,L)\left(L'S(q)'\left(S(q)\,LL'S(q)'\right)^{-1}\right)\Sigma^{1/2}(q)\right\|^2 \\
&= \left\| \hat{\Sigma}^{-1/2}(q)\,(S(q)\,L)\left(L'S(q)'\,\Sigma^{-1}(q)\right)\Sigma^{1/2}(q)\right\|^2 \\
&= \left\| \hat{\Sigma}^{-1/2}(q)\,(S(q)\,L)\left(L'S(q)'\right)\Sigma^{-1/2}(q)\right\|^2 \\
&\leq \left\| \hat{\Sigma}^{-1/2}(q)(S(q)\hat{L})\right\|^2 \left\|(\hat{L}^{-1}L)\right\|^2 \left\|(L'S(q)')\Sigma^{-1/2}(q)\right\|^2 \\
&\leq (\tilde{k}+\tilde{r}_2)^2 \left\|(\hat{L}^{-1}L)\right\|^2 \text{ by } (B.2) \text{ and } (B.3) \\
&\leq (\tilde{k}+\tilde{r}_2)^2 O_p(1)
\end{aligned} \tag{B.5}
$$

where $O_p(1)$ is uniform in (ϕ, q). From the bounds $(B.2), (B.3), (B.4), (B.5)$, we have

$$
II = o_p(1)
$$

uniformly in (ϕ, q). The desired result in (B.1) is obtained by combining I and II. In a similar manner it can be shown that

$$
\bar{Q}\big(q; \hat{\phi}, W^*(\cdot)\big) - \bar{Q}\big(q; \hat{\phi}, \hat{W}^*(\cdot)\big) \leq o_p(1). \quad \square
$$

The sample estimate $\hat{\Lambda}$ that enters $\hat{W}^*(\cdot)$ in the bounding function $\bar{Q}\big(q; \hat{\phi}, \hat{W}^*(\cdot)\big)$ can be replaced by the population covariance matrix Λ:

$$
\left| \bar{Q}\big(q; \hat{\phi}, \hat{W}^*(\cdot)\big) - \bar{Q}\big(q; \hat{\phi}, W^*(\cdot)\big) \right| = o_p(1)
$$

uniformly in (ϕ, q).

Lemma B 2 *Suppose that Assumptions 1 to 3 are satisfied. The sample estimate $\hat{\Lambda}$ that enters $\hat{W}^*(\cdot)$ in the objective function $G\big(\theta, q; \hat{\phi}, \hat{W}^*(\cdot)\big)$, defined in (31), can be replaced by the population covariance matrix Λ:*

$$
\left| G\big(\theta, q; \hat{\phi}, \hat{W}^*(\cdot)\big) - G\big(\theta, q; \hat{\phi}, W^*(\cdot)\big) \right| = o_p(1)
$$

uniformly in (ϕ, θ, q) for $\phi \in \mathcal{P}$, $\theta \in \Theta(\phi)$, and $q \in \mathbb{Q}(\theta, q)$.

Proof of Lemma B 2: Notice that the penalty term in (24) cancels and thus is omitted from the subsequent calculations. By definition

$$
\begin{aligned}
G(\theta, q; \hat{\phi}, \hat{W}^*(\cdot)) &= \min_{v \geq -\sqrt{T}\hat{D}_R^{-1/2}\mu(q,\phi)} \left\| \hat{D}^{-1/2} S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v \right\|^2_{\hat{\Omega}^{-1}(q)} \\
&= \min_{v \geq -\sqrt{T}\mu(q,\phi)} \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v \right\|^2_{\hat{\Sigma}^{-1}(q)}.
\end{aligned}
$$

Moreover, define

$$\begin{aligned} v(\hat{\Lambda}) &= \arg \min_{v \geq -\sqrt{T}\mu(q,\phi)} \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v \right\|^2_{\hat{\Sigma}^{-1}(q)} \\ v(\Lambda) &= \arg \min_{v \geq -\sqrt{T}\mu(q,\phi)} \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v \right\|^2_{\Sigma^{-1}(q)}. \end{aligned}$$

By using similar arguments as in the proof of Lemma B 1, we have

$$\begin{aligned} & G(\theta, q; \hat{\phi}, \hat{W}^*(\cdot)) - G(\theta, q; \hat{\phi}, W^*(\cdot)) \\ \leq\ & \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v(\Lambda) \right\|^2_{\hat{\Sigma}^{-1}(q)} - \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v(\Lambda) \right\|^2_{\Sigma^{-1}(q)} \\ =\ & \left\{ S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v(\Lambda) \right\}' \Sigma^{-1/2}(q) \\ & \times \left\{ \Sigma^{1/2}(q)\hat{\Sigma}^{-1}(q)\Sigma^{1/2}(q) - I_{l(q)} \right\} \Sigma^{-1/2}(q) \left\{ S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v(\Lambda) \right\} \\ \leq\ & \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v(\Lambda) \right\|^2_{\Sigma^{-1}(q)} \left\| \Sigma^{1/2}(q)\hat{\Sigma}^{-1}(q)\Sigma^{1/2}(q) - I_{l(q)} \right\| \\ =\ & I \times II, \text{ say.} \end{aligned}$$

Notice that

$$\begin{aligned} I &= \min_{v \geq -\sqrt{T}\mu(q,\phi)} \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) - M_v v \right\|^2_{\Sigma^{-1}(q)} \\ &\leq \left\| S(q)\sqrt{T}(\hat{\phi}-\phi) \right\|^2_{\Sigma^{-1}(q)} \text{ since } \mu(q,\phi) \geq 0 \\ &\leq O_p(1) \end{aligned}$$

uniformly in (ϕ, θ, q) for $\phi \in \mathcal{P}$, $\theta \in \Theta(\phi)$, and $q \in \mathbb{Q}(\theta, q)$. The first inequality follows from $\mu(q, \phi) \geq 0$. The second inequality can be verified using the same steps as in the proof of Lemma B 1. Similarly, we can follow the proof of Lemma B 1 to establish that $II = o_p(1)$ uniformly in (ϕ, θ, q) for $\phi \in \mathcal{P}$, $\theta \in \Theta(\phi)$, and $q \in \mathbb{Q}(\theta, q)$. This leads to the required result. The bound

$$G(\theta, q; \hat{\phi}, W^*(\cdot)) - G(\theta, q; \hat{\phi}, \hat{W}^*(\cdot)) \leq o_p(1)$$

can be established in a similar fashion and the statement of the lemma follows. □

Lemma B 3 *Suppose Assumption 3 is satisfied. Denote* $\bar{S}^{M,0}(q) = \bar{S}(q)$ *and* $\bar{S}^{M,k}(q) = M^{S,k}\bar{S}^{M,k-1}(q)$ *for* $k = 1, \ldots, 4$. *Finally, let* $\tilde{S}(q) = \bar{S}^{M,4}(q)M^{S,5}$. *Suppose that* q_T *is a converging sequence. Then, there exists a subsequence* $q_{T'}$ *and index sets* $\mathcal{J}_k$ *that are constant over* T' *such that for all* $k = 0, 1, ..., 5$, *(i)* $\left\| \bar{S}_j^{M,k}(q_{T'}) \right\| > 0$ *if and only if* $j \in \mathcal{J}_k$ *and (ii)*

$$\left[\frac{\bar{S}_j^{M,k}(q_{T'})}{\left\| \bar{S}_j^{M,k}(q_{T'}) \right\|} \right]_{j \in \mathcal{J}_k} \longrightarrow \left[\bar{S}_{j,*}^{M,k} \right]_{j \in \mathcal{J}_k},$$

where $\left[\bar{S}_{j,*}^{M,k}\right]_{j\in\mathcal{J}_k}$ *has full row rank.*

Proof of Lemma B 3: Part (i): Since the singularity of $\bar{S}^{M,k}(q_T)$ is caused only by zero rows, we can choose a subsequence of $\{T\}$ such that the rank of $\bar{S}^{M,k}(q_T)$ is the same along the subsequence. Then, we can choose a further subsequence such that the index of the nonzero rows are the same. We set this subsubsequence as $\{T''\}$, and the index set contains nonzero rows along this sequence as $\mathcal{J}_k$.

Part (ii): Along the subsequence chosen in Part (i), $\frac{\bar{S}_j^{M,k}(q_T)}{\|\bar{S}_j^{M,k}(q_T)\|}$ is well defined for all $j \in \mathcal{J}_k$, $k = 0, ..., 5$. Since $\left\{\frac{\bar{S}_j^{M,k}(q_{T''})}{\|\bar{S}_j^{M,k}(q_{T''})\|}\right\} \subset \mathbb{S}^m$, the unit sphere in $\mathbb{R}^m$, which is a compact set, we can choose a further subsequence, denoted by $\{T'\}$ such that

$$\frac{\bar{S}_j^{M,k}(q_{T'})}{\left\|\bar{S}_j^{M,k}(q_{T'})\right\|} \longrightarrow \bar{S}_{j,*}^{M,k}.$$

For the required result in Part (ii), we show that the row vectors $\bar{S}_{j,*}^{M,k}$ over $j \in \mathcal{J}_k$ are linearly independent for $k = 0, ..., 5$. In what follows we show this required result for the cases $k = 0, 1, 2$. The cases of $k = 3, 4$ follow immediately from the case $k = 2$ because $\mathcal{J}_2 = \mathcal{J}_3 \supset \mathcal{J}_4$ and $\left\{\bar{S}_{j,*}^{M,2} : j \in \mathcal{J}_2\right\} = \left\{\bar{S}_{j,*}^{M,3} : j \in \mathcal{J}_3\right\} \supset \left\{\bar{S}_{j,*}^{M,4} : j \in \mathcal{J}_4\right\}$ by the definition of the $M^{S,3}$ and $M^{S,4}$. The case of $k = 5$ follows because $M^{S,5}$ deletes only the zero columns.

Before we start the proof, notice that since $M^{S,1}$ is a full rank diagonal matrix, $M^{S,2}$ is the quasi-lower triagular structure of $M^{S,2}$ with full rank, and both $M^{S,1}$ and $M^{S,2}$ do not depend on q, we have $\mathcal{J}_0 = \mathcal{J}_1 = \mathcal{J}_2$.

Case $k = 0$*:* First notice that the row vectors in $\left\{\bar{S}_j(q_T) : j \in \mathcal{J}_0\right\}$ are orthogonal to each other and so are the row vectors in $\left\{\bar{S}_{j,*}^{M,0} : j \in \mathcal{J}_0\right\}$. Therefore, the row vectors in $\left\{\bar{S}_{j,*}^{M,0} : j \in \mathcal{J}_0\right\}$ are linearly independent.

Case $k = 1$*:* Notice that $M^{S,1}$ is a full rank diagonal matrix that does not depend on q_T. Therefore, $\begin{bmatrix} \vdots \\ \bar{S}_{j,*}^{M,1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \left[M^{S,1}\bar{S}_*^{M,0}\right]_j \\ \vdots \end{bmatrix}$ has full row rank, and we have the required result for the case $k = 1$.

Case $k = 2$*:* Similarly, $M^{S,1}$ is a full rank matrix that does not depend on q_T, it follows that

$\begin{bmatrix} \vdots \\ \bar{S}^{M,2}_{j,*} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \left[M^{S,2}\bar{S}^{M,1}_{*}\right]_j \\ \vdots \end{bmatrix}$ has full row rank, and we have the required result for the case $k = 2$. □

Lemma B 4 *Suppose Assumption 3 is satisfied. For a converging sequence $\{q_T\}$ such that $V(q_T)$ is constant and the rank of $S(q_T)$ equals to l for all T, there exists a subsequence $\{T'\} \subseteq \{T\}$ such that the matrix defined as*

$$\lim_T \begin{bmatrix} \frac{S_1(q_{T'})}{\|S_1(q_{T'})\|} \\ \vdots \\ \frac{S_l(q_{T'})}{\|S_l(q_{T'})\|} \end{bmatrix}$$

has full row rank l.

Proof of Lemma B 4: Along the sequence q_T, $V(q_T)$ is a constant matrix such that each row of $V(q_T)$ has only one nonzero element, which is one. Choose the subsequence $\{T'\}$ in Lemma B 3. Then,

$$\left[\frac{S_j(q_{T'})}{\|S_j(q_{T'})\|}\right]_{j=1,\dots,l} = \left[\frac{\tilde{S}_j(q_{T'})}{\left\|\tilde{S}_j(q_{T'})\right\|}\right]_{j\in\mathcal{J}_5},$$

where $\mathcal{J}_5$ is defined in Lemma B 3. The required result follows by Lemma B 3. □

Lemma B 5 *Suppose Assumptions 1 to 3 are satisfied. For a converging sequence $\{\phi_T, \theta_T, q_T\}$ that satisfies the rank condition $l(q_T) = l$ for all T, there exists a subsequence $\{T''\} \subseteq \{T\}$*

$$\begin{aligned} [D^{-1/2}(q_{T''})S(q_{T''})L(\phi_{T''})]' &\longrightarrow A \\ \Omega(q_{T''}) &\longrightarrow A'A, \end{aligned}$$

where A is a full rank matrix.

Proof of Lemma B 5: For notational convenience we denote $\Lambda(\phi_T) = \Lambda_T$ and $L(\phi_T) = L_T$. Consider the spectral decomposition $\Lambda_T = U_T diag(\lambda_{1,T}, \dots, \lambda_{m,T})U_T'$. Since $\Lambda_T > \Lambda_{min} > 0$, the smallest eigenvalue $\lambda_{min,T} > \delta > 0$. Then,

$$\Sigma_T = S(q_T)U_T diag\left(\lambda_{1,T}, \dots, \lambda_{m,T}\right) U_T' S'(q_T)$$

and the diagonal elements of Σ_T are given by

$$D_{jj}(q_T) = \lambda_{j,T} S_j(q_T)(U_T U_T') S_j'(q_T) = \lambda_{j,T} \|S_j(q_T)\|^2 > 0 \quad \forall j, T.$$

Now we can express

$$D^{-1/2}(q_T) S(q_T) L_T = diag\left(\lambda_{1,T}^{-1/2}, \ldots, \lambda_{l,T}^{-1/2}\right) \begin{bmatrix} \frac{S_1(q_T)}{\|S_1(q_T)\|} \\ \vdots \\ \frac{S_l(q_T)}{\|S_l(q_T)\|} \end{bmatrix}.$$

Since $\lambda_{j,T}^{-1/2} > 0$ for all j and $S_j(q_T)/\|S_j(q_T)\|$ exists on the unit hypersphere there exists a subsequence $\{T'\}$ such that

$$[D^{-1/2}(q_{T'}) S(q_{T'}) L(\phi_{T'})]' \longrightarrow A.$$

According to Lemma B 4, we can construct a further subsequence $\{T''\}$ along which the matrix

$$\begin{bmatrix} \frac{S_1(q_{T''})}{\|S_1(q_{T''})\|} \\ \vdots \\ \frac{S_l(q_{T''})}{\|S_l(q_{T''})\|} \end{bmatrix}$$

has full row rank. Since the limit of the matrix $diag\left(\lambda_{1,T}^{-1/2}, \ldots, \lambda_{l,T}^{-1/2}\right)$ is full rank, the limit matrices A and $A'A$ are also full rank. □

Lemma B 6 *If $CS_{(2)}^{\theta,q}$ in (32) is a valid $1-\tau$ confidence set, then $CS_{(2)}^{\theta}$ in (33) is a valid $1-\tau$ confidence set.*

Proof of Lemma B 6: The lemma follows since $(\theta, q) \in CS_{(2)}^{\theta,q}$ if and only if

$$M_\theta \theta \geq 0 \quad \text{and} \quad G(\theta, q; \hat{\phi}, \hat{W}^*) \leq c_{(2)}(q),$$

where $\|q\| = 1$. Thus,

$$M_\theta \theta \geq 0 \quad \text{and} \quad \min_{\tilde{q}=\|1\|} \left[G(\theta, \tilde{q}; \hat{\phi}, \hat{W}^*) - c_{(2)}(\tilde{q})\right] \leq 0$$

and therefore $\theta \in CS_{(2)}^{\theta}$. In turn,

$$\begin{aligned} 1-\tau &\leq \liminf_T \inf_{\phi\in\Phi} \inf_{\theta\in\Theta(\phi)} \inf_{q\in\mathbb{Q}(\theta,\phi)} P_\phi\left\{(\theta, q) \in CS_{(2)}^{\theta,q}\right\} \\ &\leq \liminf_T \inf_{\phi\in\Phi} \inf_{\theta\in\Theta(\phi)} P_\phi\left\{\theta \in CS_{(2)}^{\theta}\right\}. \quad \square \end{aligned}$$

Lemma B 7 *Suppose Assumptions 1 to 3 are satisfied. Then under Case (i) considered in the proof of Theorem 2, the contribution of the moment conditions deemed to be nonbinding to the objective function $G\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big)$ is asymptotically negligible:*

$$G\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) = G_1\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) + o_p(1),$$

Proof of Lemma B 7: Recall the definition

$$\begin{aligned} G_1\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) &= \min_{\nu_1 \geq -\sqrt{T}\mu_{1,T}} \left\| S_{1,T}\sqrt{T}(\hat{\phi} - \phi_T) - M_{\nu_1}\nu_1 \right\|^2_{\Sigma^{-1}_{11,T}} \\ &= \min_{\nu_1 \geq -\sqrt{T}D^{-1/2}_{R,1,T}\mu_{1,T}} \left\| D^{-1/2}_{1,T} S_{1,T} L_T L_T^{-1} \sqrt{T}(\hat{\phi} - \phi_T) - M_{\nu_1}\nu_1 \right\|^2_{\Omega^{-1}_{11,T}}, \end{aligned}$$

where $D_{1,T} = diag(D_{\theta,T},\, D_{R,1,T})$. Now define $h_{1,T} = \sqrt{T}D^{-1/2}_{R,1,T}\mu_{1,T}$ and $\zeta_T = L_T^{-1}\sqrt{T}(\hat{\phi} - \phi_T)$. Thus, we can write

$$G_1\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) = \min_{\nu_1 \geq -h_{1,T}} \left\| D^{-1/2}_{1,T} S_{1,T} L_T \zeta_T - M_{\nu_1}\nu_1 \right\|^2_{\Omega^{-1}_{11,T}}$$

and define

$$\nu^*_{1,T} = \text{argmin}_{\nu_1 \geq -h_{1,T}} \left\| D^{-1/2}_{1,T} S_{1,T} L_T \zeta_T - M_{\nu_1}\nu_1 \right\|^2_{\Omega^{-1}_{11,T}}.$$

Using the definitions $h_{2,T} = \sqrt{T}D^{-1/2}_{R,2,T}\mu_{2,T}$ and $h_T = [h_{1,T},\, h_{2,T}]'$, we can write

$$G\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) = \min_{\nu \geq -h_T} \left\| D^{-1/2}_{T} S_T L_T \zeta_T - M_{\nu}\nu \right\|^2_{\Omega^{-1}_{T}}.$$

The $G(\cdot)$ function can be decomposed as follows:

$$\begin{aligned} G\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) &= G_1\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) \\ &\quad + \min_{\nu_2 \geq -h_{2,T}} \Big\| \big(D^{-1/2}_{2,T} S_{2,T} L_T - \Omega_{21,T}\Omega^{-1}_{11,T} D^{-1/2}_{1,T} S_{1,T} L_T\big)\zeta_T \\ &\quad - \big(\nu_2 - \Omega_{21,T}\Omega^{-1}_{11,T} M_{\nu_1}\nu^*_{1,T}\big) \Big\|^2_{\Omega^{-1}_{2.11,T}}, \end{aligned}$$

where $\Omega_{2.11,T} = \Omega_{22,T} - \Omega_{21,T}\Omega^{-1}_{11,T}\Omega_{12,T}$. Now denote

$$\zeta^*_T = \big(D^{-1/2}_{2,T} S_{2,T} L_T - \Omega_{21,T}\Omega^{-1}_{11,T} D^{-1/2}_{1,T} S_{1,T} L_T\big)\zeta_T + \Omega_{21,T}\Omega^{-1}_{11,T} M_{\nu_1}\nu^*_{1,T}.$$

For any $\eta > 0$ it follows that

$$P\left\{ \left| G\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) - G_1\big(\theta_T, q_T; \hat{\phi}, W^*(\cdot)\big) \right| \leq \eta \right\} \geq P\{\zeta^*_T \geq -h_{2,T}\}.$$

In the remainder of the proof we will show that $\liminf_T \; P\{\zeta_T^* \geq -h_{2,T}\} \geq 1-\epsilon$ for any $\epsilon > 0$, which implies the desired result.

We proceed by showing that $\nu_{1,T}^*$ is stochastically bounded. Notice that

$$\left\| D_{1,T}^{-1/2} S_{1,T} L_T \zeta_T \right\| \leq \| D_{1,T}^{-1/2} S_{1,T} L_T \| \cdot \left\| \zeta_T \right\| = O(1) O_p(1) = O_p(1).$$

Since our subsequence is constructed such that $\Omega_T \longrightarrow A'A > 0$ and Ω_T is a sequence of correlation matrices, we deduce that

$$\nu_{1,T}^* = O_p(1). \tag{B.6}$$

In turn, $\Omega_{21,T} \Omega_{11,T}^{-1} M_{\nu_1} \nu_{1,T}^* = O_p(1)$. Now consider

$$\begin{aligned} & \left\| \left(D_{2,T}^{-1/2} S_{2,T} L_T - \Omega_{21,T} \Omega_{11,T}^{-1} D_{1,T}^{-1/2} S_{1,T} L_T \right) \zeta_T \right\| \\ \leq & \left(\left\| D_{2,T}^{-1/2} S_{2,T} L_T \right\| + \| \Omega_{21,T} \| \cdot \left\| \Omega_{11,T}^{-1} \right\| \cdot \left\| D_{1,T}^{-1/2} S_{1,T} L_T \right\| \right) \| \zeta_T \| \\ = & \; O(1) O_p(1) = O_p(1) \end{aligned} \tag{B.7}$$

Combining the $O_p(1)$ results in B.6 and (B.7), we can deduce that

$$\zeta_T^* = O_p(1).$$

Thus, for any $\epsilon > 0$, there exists a constant $M > 0$ such that

$$\liminf_T \; P\left\{ \zeta_T^* \in [-M, \, M]^{r_{22}} \right\} \geq 1 - \epsilon.$$

Here $[-M, \, M]^{r_{22}}$ denotes the Cartesian power of the interval $[-M, \, M]$. Since $h_{2,T} \longrightarrow \infty$ there exists a T^* such that

$$P\{\zeta_T^* \geq -h_{2,T}\} \geq P\left\{ \zeta_T^* \in [-M, \, M]^{r_{22}} \right\}, \quad \text{for all} \quad T > T^*,$$

which completes the proof. □

Lemma B 8 *Suppose Assumptions 1 to 3 are satisfied. Consider the Case (i) in the proof of Theorem 2. Along the (ϕ_T, θ_T, q_T) sequence, the critical value based on the estimated number of potentially binding moment conditions is more conservative in the following sense:*

$$c^{(1)}_{k(q_T)+\hat{r}_{21}(q_T)} \geq c^{(1)}_{k+r_{21}}.$$

in probability approaching one.

Proof of Lemma B 8: From Section 4.3 recall the definition

$$\hat{r}_{21}(q_T) = \sum_{j=1}^{r_2} \mathcal{I}\left\{ \hat{\xi}_{j,T}(q_T) \leq \kappa_T \right\}, \quad \text{where} \quad \hat{\xi}_{j,T}(q_T) = \hat{D}_{jj,R}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \hat{\phi}).$$

Then,

$$\begin{aligned}
\kappa_T^{-1}\hat{\xi}_{j,T}(q_T) &= \kappa_T^{-1} D_{jj,R}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \phi_T) \\
&\quad + \left[\hat{D}_{jj,R}^{-1/2}(q_T) D_{jj,R}^{1/2}(q_T) - 1 \right] \kappa_T^{-1} D_{jj,R}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \phi_T) \\
&\quad + \kappa_T^{-1}\hat{D}_{jj,R}^{-1/2}(q_T)\sqrt{T}[\mu_j(q_T, \hat{\phi}) - \mu_j(q_T, \phi_T)] \\
&= I + II \times I + III, \text{ say.}
\end{aligned}$$

Term I: by definition we obtain

$$I = \kappa_T^{-1} D_{jj,R}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \phi_T) \longrightarrow \pi_j$$

Term II: can be bounded as follows:

$$\begin{aligned}
\left| \hat{D}_{jj,R}^{-1}(q_T) D_{jj,R}(q_T) - 1 \right| &= \frac{\left| D_{jj,R}(q_T) - \hat{D}_{jj,R}(q_T) \right|}{\hat{D}_{jj,R}(q_T)} \\
&\leq \frac{\left\| \hat{\Lambda} - \Lambda(\phi_T) \right\|}{\lambda_{\min}(\hat{\Lambda})} = o_p(1).
\end{aligned}$$

The $o_p(1)$ statement follows because $\|\hat{\Lambda} - \Lambda(\phi_T)\| \stackrel{p}{\longrightarrow} 0$ and $\Lambda(\phi_T) > 0$.

Term III: Let $S_{j,R}(q_T)$ be the j^{th} row of $S_R(q_T)$. Then, since

$$\begin{aligned}
&\left| \hat{D}_{jj,R}^{-1/2}(q_T)\sqrt{T}(\mu_j(q_T, \hat{\phi}) - \mu_j(q_T, \phi_T)) \right| \\
&= \left| \frac{S_{j,R}(q_T)\hat{L}}{\left\| S_{j,R}(q_T)\hat{L} \right\|} \hat{L}^{-1}\sqrt{T}(\hat{\phi} - \phi_T) \right| \leq \left\| \hat{L}^{-1}\sqrt{T}(\hat{\phi} - \phi_T) \right\| = O_p(1),
\end{aligned}$$

we have

$$III = O_p(\kappa_T^{-1}) = o_p(1).$$

Combining the results, we deduce that if $\pi_j < \infty$,

$$\kappa_T^{-1}\hat{\xi}_{j,T}(q_T) = \kappa_T^{-1}\hat{D}_{jj,R}^{-1/2}(q_T)\mu_j(q_T, \hat{\phi}) \longrightarrow_p \pi_j. \tag{B.8}$$

In particular, if $\pi_j = 0$, then

$$\mathcal{I}\left\{ \hat{\xi}_{j,T}(q_T) \leq \kappa_T \right\} \stackrel{p}{\longrightarrow} 1,$$

which leads to the desired result:

$$\text{plim}_T\ \hat{r}_{21}(q_T) = \text{plim}_T \sum_{j=1}^{r_2} \mathcal{I}\left\{\hat{\xi}_{j,T}(q_T) \leq \kappa_T\right\} \geq \sum_{j=1}^{r_2} 1\left\{\pi_j = 0\right\} = r_{21}. \quad \Box$$

Lemma B 9 *Suppose Assumptions 1 to 3 are satisfied. Consider the Case (i) in the proof of Theorem 2. Along the* $\{T\}$ *sequence,* $\hat{c}^*_{(22)}(q_T) \xrightarrow{p} c^*_{(22)}$. *The two critical values are defined in (53) and (54).*

Proof of Lemma B 9: The proof proceeds in three steps. First, show

$$\left(\hat{\xi}_T, \hat{\Omega}(q_T)\right) \xrightarrow{p} (\pi, A'A) \quad \text{and} \quad \hat{\varphi}^*_T(q_T) \xrightarrow{p} \pi^*.$$

Second, show

$$\begin{aligned} & P\left\{\min_{v \geq -\hat{\varphi}^*_T(q_T)} \left\|(\hat{D}^{-1/2}(q_T)S(q_T)\hat{L})Z_m - M_v v\right\|^2_{\hat{\Omega}^{-1}(q_T)} \leq x\right\} \\ & \xrightarrow{p} \quad P\left\{\min_{v \geq -\pi^*}\left\{\left\|A'Z_m - M_v v\right\|^2_{(A'A)^{-1}}\right\} \leq x\right\}. \end{aligned}$$

Third, deduce $\hat{c}^*_{(22)}(q_T) \xrightarrow{p} c^*_{(22)}$, as required for Part (b).

Proof of Step 1: By the choice of the sequence $\{T\}$ and the limit result in (B.8) and $\hat{\Lambda} \xrightarrow{p} \Lambda$,

$$\left(\hat{\xi}_T(q_T), \hat{\Omega}(q_T)\right) \xrightarrow{p} (\pi, A'A).$$

Notice that if $\pi_j = 0$, then $\hat{\xi}_T(q_T) < \kappa_T$ and $\hat{\varphi}^*_{j,T}(q_T) = \hat{\varphi}_{j,T}(q_T) = 0 = \pi^*_j$ with probability one. On the other hand, if $\pi_j > 0$, then $\hat{\varphi}^*_{j,T}(q_T) = \infty = \pi^*_j$.

Proof of Step 2: The desired result can be obtained by the same argument used in the proof of (S1.17) of Andrews and Soares (2010b).

Proof of Step 3: It is immediate from Step 2 and the fact that the distribution of

$$\min_{v \geq -\pi^*}\left\{\left\|A'Z_m - M_v v\right\|^2_{(A'A)^{-1}}\right\}$$

is continuous if $k \geq 1$, and continuous near the $(1-\tau)'s$ quantile, where $\tau < 1/2$, if $k = 0$. $\Box$

C Derivations for Bivariate VAR(1)

Consider a bivariate ($n = 2$) VAR(1) of the form $y_t = \Phi_1 y_{t-1} + u_t$ and focus on the response at horizon $h = 1$, which can be constructed from $R_1^v = \Phi\Sigma_{tr}$. Hence, let $\phi = vec((R_1^v)')$. The object of interest is $\theta = \partial y_{1,t+1}/\partial\epsilon_{1,t}$, and we impose the sign restriction that both θ as well as $\partial y_{2,t+1}/\partial\epsilon_{1,t}$ are nonnegative. Let $q = [q_1, q_2]'$. Then

$$\begin{aligned} \tilde{S}_\theta(q) &= \begin{bmatrix} q_1 & q_2 & 0 & 0 \end{bmatrix} \\ \tilde{S}_R(q) &= \begin{bmatrix} 0 & 0 & q_1 & q_2 \end{bmatrix}. \end{aligned}$$

Notice that in this example $\tilde{S}(q) = [\tilde{S}'_\theta(q), \tilde{S}'_R(q)]'$ is of full row rank for all values of q. Thus, we can set $V(q) = I$, replace $\tilde{S}(q)$ by $S(q)$, and write the objective function as

$$Q\big(\theta; \phi, W(\cdot)\big) = \min_{\|q\|=1} G\big(\theta, q; \phi, W(\cdot)\big)$$

where

$$G\big(\theta, q; \phi, W(\cdot)\big) = \min_{\mu \geq 0} \left\| S(q)\phi - \begin{pmatrix} \theta \\ \mu \end{pmatrix} \right\|_W^2 .$$

An analytical expression for $G\big(\theta, q; \phi, W(\cdot)\big)$ can be obtained as follows. Decompose

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} 1 & W_{12}W_{22}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_{11.22} & 0 \\ 0 & W_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ W_{22}^{-1}W_{21} & 1 \end{bmatrix},$$

where

$$W_{11.22} = W_{11} - W_{12}W_{22}^{-1}W_{21}.$$

Thus, we can write

$$\left\| S(q)\phi - \begin{pmatrix} \theta \\ \mu \end{pmatrix} \right\|_W^2 = W_{11.22}(\theta - S_\theta\phi)^2 + W_{22}\Big(\mu - \Big[S_R\phi - W_{12}W_{22}^{-1}(\theta - S_\theta\phi)\Big]\Big)^2.$$

Now let

$$\begin{aligned} \hat{\mu}(\theta, q) &= \text{argmin}_{\mu \geq 0} \left\| S_(q)\phi - \begin{pmatrix} \theta \\ \mu \end{pmatrix} \right\|_W^2 \\ &= \begin{cases} S_R\phi - W_{12}W_{22}^{-1}(\theta - S_\theta\phi) & \text{if } S_R\phi - W_{12}W_{22}^{-1}(\theta - S_\theta\phi) \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$
\begin{aligned}
& G(\theta, q; \phi, W) \\
& \quad = \quad W_{11.22}(\theta - S_\theta \phi)^2 \\
& \qquad + \begin{cases} 0 & \text{if } S_R\phi - W_{12}W_{22}^{-1}(\theta - S_\theta\phi) \geq 0 \\ W_{22}\Big(S_R\phi - W_{12}W_{22}^{-1}(\theta - S_\theta\phi)\Big)^2 & \text{otherwise} \end{cases}
\end{aligned}
$$

Let $\hat{\Lambda}$ be the bootstrap estimator of the covariance matrix of $\sqrt{T}(\hat{\phi} - \phi)$. The weight matrix $\hat{W}_T^*$ is given by $\hat{W}^*(q) = T(S(q)\hat{\Lambda}S'(q))^{-1}$, and the unit length vector q can be parameterized in spherical coordinates as $q(\alpha) = [\cos\alpha,\ \sin\alpha]'$. Thus the objective function for the construction of the confidence set is given by

$$
Q\big(\theta; \hat{\phi}, \hat{W}^*(\cdot)\big) = \min_{\alpha \in [-\pi, \pi]} G\big(\theta, q(\alpha); \hat{\phi}, \hat{W}^*(\cdot)\big).
$$