# Theoretical and Computational Appendix to: "Public versus Private Risk Sharing"


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**Abstract**

In this appendix we provide more extensive proofs of results in the main paper. In particular we explicitly prove results that are direct adaptations of arguments from Atkeson and Lucas (1995) and hence omitted in the main paper

We also provide a detailed discussion of the algorithm used in the quantitative exercises of the main papers.


## 1 Proofs[1]

### 1.1 Optimal Policies Induce Efficient Allocations

In this subsection we prove our claim in Section 3.1 of the main text that a stationary allocation $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ is efficient if it is induced by an optimal policy $(h, \{g_{y'}\})$ from the functional equation (16) with $R > 1$ and satisfies the resource constraint with equality.

The first step is to find an operational way to solve for efficient allocations, which will then lead us to the recursive problem (16). Consider the problem of a social planner faced with a sequence of intertemporal shadow prices $\{R_t\}_{t=0}^{\infty}$ to minimize the value of resources needed to deliver expected discounted utility of $w_0$ to an individual with initial endowment given by $y_0$. The planner chooses

*We would like to thank Andrew Atkeson for many helpful discussions. All errors are our own

[1]Equation numbers refer to equations in the main paper, unless they are preceded by an $A$, in which case they refer to equations in this appendix.

$\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ to solve the problem

$$
\begin{aligned}
& W(w_0, y_0) && \text{(A1)} \\
= \;& \min \left(1 - \frac{1}{R_0}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0, y^t))\pi(y^t|y_0)
\end{aligned}
$$

subject to (11), (12), and (13). One obtains the following

**Theorem 1** *(Atkeson and Lucas (1995)) If there exist allocations $\{h_t(w_0, y^t)\}$, shadow prices $\{R_t\}$ and distribution $\Phi_0$ such that:*

1. *Given $\{R_t\}_{t=0}^{\infty}$ , for each $(w_0, y_0) \in supp(\Phi_0)$, $\{h_t(w_0, y^t)\}$ solves the planners' problem*
2. *Feasibility (Equation (14) ) holds with equality for every $t$*
3. $$1 - \frac{1}{R_0} + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) < \infty \qquad \text{(A2)}$$

   *Then the allocation is efficient with respect to $\Phi_0$.*

**Proof.** To show efficiency we first need to show that $\{h_t(w_0, y^t)\}$ is constrained feasible with respect to $\Phi_0$. By assumption the allocation satisfies feasibility, equation (14), and since it solves the planners' problem it also satisfies $(11) - (13)$. It is therefore constrained feasible. Now we need to show that there does not exist another allocation $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ that is constrained feasible with respect to $\Phi_0$ and such that

$$\sum_{y^t} \int C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) d\Phi_0 < \sum_{y^t} \int C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) d\Phi_0 \text{ for some } t \qquad \text{(A3)}$$

Suppose this is the case. Since $\{h_t(w_0, y^t)\}$ solves $CPP$ for all $w_0, y_0$ we have

$$
\begin{aligned}
& \left(1 - \frac{1}{R_0}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0, y^t))\pi(y^t|y_0) \\
\leq \;& \left(1 - \frac{1}{R_0}\right) C(\hat{h}_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) \\
& && \text{(A4)}
\end{aligned}
$$

where the left hand side of equation (A4) is finite[2]. Integrating both sides of

[2] This is guaranteed since we can always pick a constant $h_t(w_0, y^t) = \max(w_0, \max_y V^{Aut}(y))$. Such a policy satisfies all the constraints of the planners' problem and since $\max(w_0, \max_y V^{Aut}(y)) \in D$ and by assumption $(A2)$ is satisfied the value of the minimization problem is finite.

$(A4)$ with respect to $\Phi_0$ one finds

$$\int \left\{ \left(1-\frac{1}{R_0}\right) C(h_0(w_0,y_0)) + \sum_{t=1}^{\infty} \left(1-\frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0,y^t))\pi(y^t|y_0) \right\} d\Phi_0$$
$$\leq \int \left\{ \left(1-\frac{1}{R_0}\right) C(\hat{h}_0(w_0,y_0)) + \sum_{t=1}^{\infty} \left(1-\frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(\hat{h}_t(w_0,y^t))\pi(y^t|y_0) \right\} d\Phi_0 \tag{A5}$$

From the fact that for $\{h_t(w_0,y^t)\}$ feasibility holds with equality for all $t$, that $\left\{\hat{h}_t(w_0,y^t)\right\}$ is constrained feasible and from $(A3)$ one obtains:

$$\sum_{y^t|y_0} \int C(\hat{h}_t(w_0,y^t))\pi(y^t|y_0)d\Phi_0 \leq \sum_{y^t|y_0} \int y_t\pi(y^t|y_0)d\Phi_0 = \sum_{y^t|y_0} \int C(h_t(w_0,y^t))\pi(y^t|y_0)d\Phi_0 \tag{A6}$$

for all $t$ with the inequality being strict for some $t$. Multiplying each inequality by the appropriate term $\left(1-\frac{1}{R_t}\right)\prod_{s=0}^{t-1}\left(\frac{1}{R_s}\right) > 0$ and summing over all $t$ we obtain $(A5)$, but with the inequality reversed and strict, a contradiction ■

This theorem states that if we can find an allocation that solves the planners problem, for a sequence of sufficiently high intertemporal prices $\{R_t\}_{t=0}^{\infty}$ and satisfies the resource constraint with equality, then that allocation is efficient. In particular note that condition $(A2)$ is satisfied for a sequence of intertemporal prices satisfying $R_t = R > 1$ for all $t$.

Now we show that an allocation $\hat{\sigma} = \{\hat{h}_t(w_0,y^t)\}_{t=0}^{\infty}$ induced by the optimal policies $(h, \{g_{y'}\})$ from the recursive problem (16), with the additional constraints $g_{y'} \leq \bar{w}$ solve the planners' problem for interest rates $\{R_t\}_{t=0}^{\infty}$ constant at $R$. For this allocation be efficient we then only have to demonstrate that it satisfies the resource constraint with equality

So for given $(w_0,y_0)$ define

$$\begin{aligned} \hat{h}_0(w_0,y_0) &= h(w_0) \\ \hat{w}_1(w_0,y_0) &= g_{y_1}(w_0) \end{aligned} \tag{A7}$$

and in general recursively

$$\begin{aligned} \hat{w}_t(w_0,y^t) &= g_{y_t}(\hat{w}_{t-1}(w_0,y^{t-1})) \\ \hat{h}_t(w_0,y^t) &= h(\hat{w}_t(w_0,y^t)) \end{aligned} \tag{A8}$$

be the allocation induced by the recursive policy rules. The constraints $g_{y'} \leq \bar{w}$ assure that

$$\hat{w}_t(w_0,y^t) \leq \bar{w}. \tag{A9}$$

Let $W = [\underline{w}, \bar{w}]$ as in Section 3.2 in the main text. We have

**Theorem 2** *Suppose that the sequence* $\{R_t\}_{t=0}^{\infty}$ *is constant at* $R \in (1, \frac{1}{\beta})$*. Then the allocation* $\hat{\sigma}$ *constructed from the optimal policies of the functional equation solves the component planning problem, for every* $(w_0, y_0) \in W \times Y$ *with* $w_0 \geq U^{Aut}(y_0)$*.*

**Proof.** For any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ define

$$U_t(w_0, y^t, \sigma) = (1-\beta)\left(h_t(w_0, y^t) + \sum_{s>t}^{\infty}\sum_{y^s}\beta^{s-t}\pi(y^s)h_t(w_0, y^s)\right) \tag{A10}$$

and $U_t^{Aut}(y_t)$ correspondingly. By Theorem 4.3 in Stokey et al. (1989)[3], for all $w_0 \in W$ and all $y_0 \in Y$, the solution to the functional equation, $V_R$ satisfies

$$V_R(w_0) = \inf_{\{h_t(w_0,y^t), w_t(w_0,y^t)\}} \left(1-\frac{1}{R}\right)C(h_0(w_0,y_0)) + \sum_{t=1}^{\infty}\left(1-\frac{1}{R}\right)\frac{1}{R^t}\sum_{y^t}C(h_t(w_0,y^t))\pi(y^t)$$

$$\text{s.t.} \tag{A11}$$

$$w_t(w_0, y^t) = (1-\beta)h_t(w_0, y^t) + \beta\sum_{y^{t+1}}\pi(y_{t+1})w_{t+1}(w_0, y^{t+1}) \qquad \text{all } t \tag{A12}$$

$$U_t^{Aut}(y_t) \leq w_t(w_0, y^t) \leq \bar{w} \qquad \text{all } t \geq 1 \tag{A13}$$

$$w_0 \geq U_0^{Aut}(y_0) \text{ given} \tag{A14}$$

By Theorem 4.4 and 4.5 of Stokey et al. (1989), which are applicable as $V_R$ is bounded on $W$ and the sequence $\{\hat{w}_t(w_0, y^t)\}_{t=1}^{\infty}$ defined above never leaves $W$, the allocation $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ together with $\{\hat{w}_t(w_0, y^t)\}_{t=1}^{\infty}$ defined above uniquely attains the minimum of the problem above. In order to argue that $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ solves the planners problem we have to show that any allocation $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ together with some $\{w_t(w_0, y^t)\}_{t=1}^{\infty}$ satisfies $(A12)$ and $(A13)$ if and only if $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ satisfies $(11)-(13)$, i.e. if

$$w_0 = U_0(w_0, y_0, \sigma) \tag{A15}$$

$$U_t^{Aut}(y_t) \leq U_t(w_0, y^t, \sigma) \leq \bar{w} \qquad \text{all } t \tag{A16}$$

$$\lim_{t\to\infty}\beta^t \sup_{y^t} U_t(w_0, y^t, \sigma) = 0 \tag{A17}$$

**Step 1:** Pick any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ that satisfies $(A15)-A(17)$. Define $w_t(w_0, y^t) = U_t(w_0, y^t, \sigma)$. It is immediate from $(A16)$ that $(A13)$ is satisfied. From the definition of $U_t(w_0, y^t, \sigma)$ it follows that $(A12)$ is satisfied as well.

**Step 2:** Pick any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ and $\{w_t(w_0, y^t)\}_{t=1}^{\infty}$ that satisfies $(A12)$ and $(A13)$. Since for all $t$, $w_t(w_0, y^t) \leq \bar{w}$ from $(A13)$, by using

[3] The assumption of which are satisfied as $C(w) \geq 0$, all $w$.

$(A12)$ we see that the allocation satisfies $(A17)$. Now for all allocations satisfying (17), and for all $t$

$$\begin{aligned}\left|w_t(w_0,y^t)-U_t(w_0,y^t,\sigma)\right| &= \beta\left|\sum_{y^{t+1}}\pi(y_{t+1})\left(w_{t+1}(w_0,y^{t+1})-U_t(w_0,y^t,\sigma)\right)\right| \\ &\leq \beta\sup_{y^{t+1}}\left|w_{t+1}(w_0,y^{t+1})-U_t(w_0,y^t,\sigma)\right| \\ &\leq \beta^s\sup_{y^{t+s}}\left|w_{t+s}(w_0,y^{t+s})-U_{t+s}(w_0,y^{t+s},\sigma)\right| \\ &\leq \beta^s\sup_{y^{t+s}}\left(\left|w_{t+s}(w_0,y^{t+s})\right|+\left|U_{t+s}(w_0,y^{t+s},\sigma)\right|\right)\end{aligned} \tag{A18}$$

This inequality is valid for all $t$ and all $s$. Taking the limit with respect to $s$ one obtains (by $(A17)$ and $(A13)$) that $w_t(w_0,y^t)=U_t(w_0,y^t,\sigma)$ for all $t$. Hence $(A12)$ implies that

$$\begin{aligned} w_0 &= (1-\beta)h_0(w_0,y_0)+\beta\sum_{y^1}\pi(y_1)w_1(w_0,y^1) \\ &= (1-\beta)h_0(w_0,y_0)+\beta\sum_{y^1}\pi(y_1)U_1(w_0,y^1,\sigma) \\ &= U_0(w_0,y_0,\sigma)\end{aligned} \tag{A19}$$

and hence $(A15)$ is satisfied. For $t\geq 1$ $(A16)$ is obviously satisfied, and it is satisfied for $t=0$ by the assumption that $w_0\geq U_t^{Aut}(y_t)$. This proves that the allocation constructed from the policies of the functional equation solves the component planning problem with the additional constraint $U_t(w_0,y^t,\sigma)\leq\bar{w}$. By Theorem 5 in the main text $g_{y'}(w)<\bar{w}$ for all $w\in W$. By construction $(A8)$ this implies that $\hat{w}_t(w_0,y^t)<\bar{w}$ and hence, as $\hat{w}_t(w_0,y^t)=U_t(w_0,y^t,\hat{\sigma})$, the constraint is never binding. Since the constraint set associated with the social planners problem is convex, this implies that the allocation $\hat{\sigma}$ indeed solves the original planning problem for constant interest rates. ■

The previous result shows that allocations induced by optimal policies from the recursive problem (16) solve the social planners problem, for a constant sequence of intertemporal prices $R_t=R>1$. To show efficiency of the allocation it remains to be shown that, for the appropriate $R$, the allocation indeed satisfies the resource constraint with equality. This is the content of Sections 3.3 and 3.4 of the main text.

## 1.2 Theoretical Properties of the Recursive Problem

In this subsection we prove theoretical properties of the operator $T_R$ defined in equation (16). These properties are important to establish the main results of Section 3.2 of the main text, Lemmas 3 and 4 and Theorem 5.

**Lemma 3** *$T_R$ maps $C(W)$ into itself and is a contraction.*

**Proof.** For every $w \in W$ the objective function in (16) is continuous in $h, g_{y'}$ and the constraint set is compact and non-empty; therefore the minimum exists. $V$ is bounded and since $\underline{h} \leq h \leq \bar{h}$ , $C(h)$ is bounded as well. It follows that $T_R V$ is a bounded function. The fact that $T_R V$ is continuous follows from the Theorem of the maximum (note that the constraint set is continuous in $w$). It is also easy to show that since $R > 1$ the operator $T_R$ satisfies the hypotheses of Blackwell's theorem and thus is a contraction with modulus $\frac{1}{R}$ ■

**Corollary 4** *For $R > 1$, the operator $T_R$ has a unique fixed point $V_R \in C(W)$ (i.e. $V_R$ is continuous and bounded) and for all $v_0 \in C(A)$, $||T_R^n v_0 - V_R|| \leq \frac{1}{R^n}||v_0 - V_R||$, with the norm being the sup-norm.*

**Proof.** Follows directly from the fact that $T_R$ is a contraction mapping with modulus $\frac{1}{R}$ ■

**Lemma 5** *$V_R$ is strictly increasing and strictly convex.*

**Proof.** For the first part we note that $C(W)$ (together with the sup-norm) is a complete metric space and that the set of bounded continuous nondecreasing functions on $W$, $C'(W)$, is a closed subset of $C(W)$ and that the set of bounded continuous strictly increasing functions, $C''(W)$, satisfies $C''(W) \subset C'(W)$. By Lemma 3 of this appendix $T_R$ is a contraction mapping. Hence by Corollary 1 of Stokey et al., p. 52, it is sufficient to show that, whenever $V_R \in C'(W)$, then $T_R V_R \in C''(W)$. Fix $w, \hat{w}$ with $\underline{w} \leq w < \hat{w} \leq \bar{w}$. We need to show that $(T_R V_R)(w) < (T_R V_R)(\hat{w})$. Let $\hat{h}, \hat{g}_{y'}$ be the optimal choices for $\hat{w}$. The choices $g_{y'} = \hat{g}_{y'}$ and $h = \hat{h} - \hat{w} + w < \hat{h}$ are feasible for $w$ and therefore

$$\begin{aligned}(T_R V_R)(\hat{w}) &= \left(\frac{R-1}{R}\right) C(\hat{h}) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V_R(\hat{g}_{y'}) \\ &> \left(\frac{R-1}{R}\right) C(h) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V_R(g_{y'}) \\ &\geq (T_R V_R)(w) \qquad \text{(A20)}\end{aligned}$$

To prove that $V_R$ is convex we note that the set of bounded continuous convex functions, $C'''(W)$ is a closed subset of $C(W)$. Again by Corollary 1 of Stokey et al., p. 52, it is sufficient to show that if $V_R \in C'''(W)$, then $(T_R V_R)$ is convex. So we have to show that for all $w, \hat{w} \in W$ with $w \neq \hat{w}$, and all $\lambda \in (0, 1)$, $(T_R V)(\lambda w + (1 - \lambda)\hat{w}) \leq \lambda(T_R V)(w) + (1 - \lambda)(T_R V)(\hat{w})$. Let $\hat{h}, \hat{g}_y$ be the optimal choices for $\hat{w}$ and $h, g_y$ be the optimal choices for $w$ and define $h^\lambda = \lambda h + (1 - \lambda)\hat{h}$, $g_y^\lambda = \lambda g_y + (1 - \lambda)\hat{g}_y$. Since $h^\lambda, g_y^\lambda$ are feasible for

$(\lambda w + (1-\lambda)\hat{w}, y)$, and

$$\begin{aligned}
& (T_R V_R)\left(\lambda w + (1-\lambda)\hat{w}\right) \\
\leq\ & \left(\frac{R-1}{R}\right) C(h^{\lambda}) + \frac{1}{R}\sum_{y' \in Y} \pi(y') V(g^{\lambda}_{y'}) \\
\leq\ & \left(\frac{R-1}{R}\right)\left(\lambda C(h) + (1-\lambda) C(\hat{h})\right) + \frac{1}{R}\sum_{y' \in Y} \pi(y')\left(\lambda V(g_{y'}) + (1-\lambda) V(\hat{g}_{y'})\right) \\
=\ & \lambda \left(T_R V_R\right)(w) + (1-\lambda)\left(T_R V_R\right)(\hat{w}) \qquad \text{(A21)}
\end{aligned}$$

by convexity of $V$ and strict convexity of $C$.

Finally we want to show that the fixed point of $T_R$, $V$, is strictly convex on $W$. We know that $V$ is convex, continuous and strictly increasing. These facts imply that $V$ is differentiable almost everywhere on $W$ and that for the countable number of points at which $V$ is not differentiable, right hand derivatives $V'_+$ and left hand derivatives $V'_-$ exist (although need not coincide).

Now suppose that $V$ is not strictly convex on $W$. Then there exists an interval $I \subseteq W$ such that $V$ is linear on $I$. Take $w, w' \in I$ with $w < w'$. From the envelope theorem for any solution $\{g_{y'}(w)\}, \{g_{y'}(w')\}$

$$\begin{aligned}
a &= V'(w) = V'(w') \\
&= \frac{R-1}{R(1-\beta)} C'\left(\frac{w - \beta \sum_{y'} \pi(y') g_{y'}(w)}{1-\beta}\right) \\
&= \frac{R-1}{R(1-\beta)} C'\left(\frac{w' - \beta \sum_{y'} \pi(y') g_{y'}(w')}{1-\beta}\right) \qquad \text{(A22)}
\end{aligned}$$

for some $a > 0$. Hence there exists $\bar{y}$ such that $U^{Aut}(\bar{y}) \leq g_{\bar{y}}(w) < g_{\bar{y}}(w')$. From the first order conditions, combining with the envelope condition

$$\begin{aligned}
a\beta R &\leq V'_+(g_{\bar{y}}(w)) \\
V'_-(g_{\bar{y}}(w')) &\leq a\beta R \leq V'_+(g_{\bar{y}}(w')) \qquad \text{(A23)}
\end{aligned}$$

By convexity of $V$ and the fact that $g_{\bar{y}}(w) < g_{\bar{y}}(w')$ it follows that

$$a\beta R \leq V'_+(g_{\bar{y}}(w)) \leq V'_-(g_{\bar{y}}(w')) \leq a\beta R \qquad \text{(A24)}$$

Hence $V$ is linear on $I' = (g_{\bar{y}}(w), g_{\bar{y}}(w')) \subseteq A$ with slope $ag < a$. Repeating the above argument one shows that there exists interval $I^{(n)}$ such that $V$ is linear on $I^{(n)}$ with slope $ag^n$, for all $n > 1$.

Now let $d = \frac{R-1}{R(1-\beta)} C'(\underline{h}(\underline{w})) > 0$ and pick $n$ such that $ag^n < d$. Then for all $w \in I^{(n)}$, using the envelope condition

$$ag^n = V'(w) = \frac{R-1}{R(1-\beta)} C'(h(w)) < d = \frac{R-1}{R(1-\beta)} C'(\underline{h}(\underline{w})) \qquad \text{(A25)}$$

Therefore $h(w) < \underline{h}$, which is impossible. Hence $V$ cannot contain a linear segment on $W$. ■

**Lemma 6** *For any strictly increasing and strictly convex function $V \in C(A)$, $T_R V$ is continuous, strictly increasing and strictly convex. The optimal policies $h(w), g_{y'}(w)$ are continuous, single-valued functions.*

**Proof.** The fact that $T_R V$ is strictly increasing and strictly convex follows from the properties of $V$. The choice variables $h$ and $g_{y'}$ are constrained to lie in compact and convex intervals, and by assumption the objective function is strictly convex. Hence the minimizers are unique. Since the constraint set is continuous in $w$, the theorem of the maximum applies and $T_R V$ is continuous and $h(w), g_{y'}(w)$ are upper hemicontinuous correspondences. Since $h(w), g_{y'}(w)$ are functions, they are continuous. ■

**Lemma 7** *The unique fixed point of $T_R$ is continuously differentiable.*

**Proof.** Consider the following sequence of functions $\{V^n\}_{n=0}^{\infty}$, defined recursively as:

$$\begin{aligned} V^0(w) &= C(w) && \forall w \in W \\ V^{n+1}(w) &= \left(T^R V^n\right)(w) && \forall w \in W \end{aligned} \tag{A26}$$

From Corollary 4 of this appendix we know that this sequence converges uniformly to the unique fixed point $V_R$ of $T_R$. Also Lemma 6 of this appendix assures that each $V^n$ is continuous, strictly increasing and strictly convex (as by assumption $C$ possesses these properties) and that the associated policies $h^n(w)$ and $g^n_{y'}(w)$ are continuous functions. From the envelope condition (21) we have (as $C$ is continuously differentiable by assumption) that each $V^n$ is differentiable and that this derivative is continuous, since $h^{n-1}(w)$ is a continuous function. Now we will establish that $V_R$ is continuously differentiable.

From Lemmas 6, 5 and Corollary 4 of this appendix we know that each $V^n$ as well as $V_R$ are strictly convex and continuous and that the sequence $\{V^n\}_{n=0}^{\infty}$ converges to $V_R$ uniformly. Also $W$ is compact. Then by Theorem 3.8 of Stokey et al., p. 64, the sequences $\left\{h^n(w), g^n_{y'}(w)\right\}_{n=1}^{\infty}$ converge uniformly to the optimal policies associated with $V_R$, $h^R(w)$ and $g^R_{y'}(w)$, respectively. Therefore from equation (21) we conclude that $(T_R V^n)'$ converges to $\frac{(R-1)}{R(1-\beta)} C'(h^R(w))$ uniformly. Since $\{V^n\}_{n=0}^{\infty}$ converges to $V_R$ uniformly, we have that $V_R$ is differentiable, with

$$\left(V_R\right)'(w) = \frac{(R-1)}{R(1-\beta)} C'(h^R(w)) \tag{A27}$$

■

These results assure that $V_R$ and the associated policies $h, g_{y'}$ have the properties asserted in Section 3.2 of the main text.

## 1.3 Continuity and Monotonicity of the Excess Demand Function

In this subsection we show that the excess demand function $d(R)$, as defined in Section 3.4 of the main text, is continuous and increasing on $(1, \frac{1}{\beta})$. We start with continuity. First we show that the value functions, as functions of $R$, are continuous in $R$.

**Lemma 8** *(Atkeson and Lucas (1995)) Let $R \in (1, \frac{1}{\beta})$ and $\{R_n\}_{n=0}^{\infty}$ be a sequence satisfying $R_n \in (1, \frac{1}{\beta})$ and $\lim_{n\to\infty} R_n = R$. Then the sequence $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to $V_R$ on $[\underline{w}, \bar{w}]$.*

**Proof.** We have to show that

$$\lim_{n\to\infty} ||V_{R_n} - V_R|| = 0 \tag{A28}$$

where $||V_{R_n} - V_R|| = \sup_{[\underline{w},\bar{w}]} |V_{R_n} - V_R|$. By the triangle inequality

$$||V_{R_n} - V_R|| \leq ||V_{R_n} - T^n_{R_n} V_R|| + ||T^n_{R_n} V_R - V_R|| \tag{A29}$$

Now the operator $T_{R_n}$ is a contraction mapping on $[\underline{w}, \bar{w}]$ with unique fixed point $V_{R_n}$ (see Corollary 1 of this appendix). Hence

$$\lim_{n\to\infty} ||V_{R_n} - T^n_{R_n} V_R|| = 0 \tag{A30}$$

For the second term in the sum we note that

$$||T^n_{R_n} V_R - V_R|| \leq \sum_{k=1}^{n} ||T^k_{R_n} V_R - T^{k-1}_{R_n} V_R|| \leq \sum_{k=1}^{n} \frac{1}{(R_n)^k} ||T_{R_n} V_R - V_R|| \tag{A31}$$

Here the first inequality again follows from the triangle inequality and the second from the fact that $T_{R_n}$ is a contraction mapping on $[\underline{w}, \bar{w}]$ with modulus $\frac{1}{(R_n)^k}$. Hence

$$\begin{aligned} \lim_{n\to\infty} ||T^n_{R_n} V_R - V_R|| &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(R_n)^k} ||T_{R_n} V_R - V_R|| \\ &\leq \lim_{n\to\infty} \frac{R_n}{R_n - 1} ||T_{R_n} V_R - V_R|| \\ &= \frac{R}{R-1} \lim_{n\to\infty} ||T_{R_n} V_R - T_R V_R|| \end{aligned} \tag{A32}$$

where we used the fact that $V_R$ is the unique fixed point of $T_R$. Hence $\lim_{n\to\infty} ||T^n_{R_n} V_R - V_R|| = 0$ if and only if $\lim_{n\to\infty} ||T_{R_n} V_R - T_R V_R|| = 0$, i.e. if the operator $T_{R_n}$ is continuous in $R_n$. To see that $T_{R_n}$ is in fact continuous in $R_n$ consider the following argument: for arbitrary $\hat{w} \in [\underline{w}, \bar{w}]$ by the theorem of the maximum

$$\lim_{n\to\infty} |T_{R_n} V_R(\hat{w}) - T_R V_R(\hat{w})| = 0 \tag{A33}$$

Since $[\underline{w}, \bar{w}]$ is a compact set and $T_{R_n} V_R$, $T_R V_R$ are continuous functions in $w$, we have

$$\lim_{n\to\infty} \max_{\hat{w}\in[\underline{w},\bar{w}]} |T_{R_n} V_R(\hat{w}) - T_R V_R(\hat{w})| = \lim_{n\to\infty} ||T_{R_n} V_R - T_R V_R|| = 0 \tag{A34}$$

Hence both terms on the right hand side of $(A29)$ converge to 0, which proves the result ■

This result proves that $V_R$ is continuous in $R$. Next we show the same result with respect to the optimal policies $g_{y'}^R$. This is crucial since these policies induce the Markov process to which $\Phi_R$ is the invariant measure, and to prove continuity of $\Phi_R$ with respect to $R$ one first has to show that $g_{y'}^R$ is continuous in $R$.

**Lemma 9** *(Atkeson and Lucas (1995)) Let a sequence* $\{R_n, w_n\}_{n=0}^{\infty}$ *with* $R_n \in (1, \frac{1}{\beta})$ *and* $w_n \in [\underline{w}, \bar{w}]$ *converge to* $(R, w) \in (1, \frac{1}{\beta}) \times [\underline{w}, \bar{w}]$*. Then for each* $y' \in Y$*, the sequence* $\left\{g_{y'}^{R_n}(w_n)\right\}_{n=0}^{\infty}$ *converges to* $g_{y'}^R(w)$.

**Proof.** We have to show that for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ we have $|g_{y'}^{R_n}(w_n) - g_{y'}^R(w)| < \varepsilon$. We note that by the triangle inequality

$$|g_{y'}^{R_n}(w_n) - g_{y'}^R(w)| \leq |g_{y'}^{R_n}(w_n) - g_{y'}^R(w_n)| + |g_{y'}^R(w_n) - g_{y'}^R(w)| \qquad \text{(A35)}$$

Since the function $g_{y'}^R$ is continuous, for each $\varepsilon_1 > 0$ there exists $N(\varepsilon_1)$ such that $|g_{y'}^R(w_n) - g_{y'}^R(w)| < \varepsilon_1$ for all $n \geq N(\varepsilon_1)$. By Lemma 5 of this appendix $V_R$ as well $V_{R_n}$ are strictly convex, for each $n \in N$. Also $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to $V_R$, by Lemma 8 of this appendix, on the compact set $[\underline{w}, \bar{w}]$. Then by Theorem 3.8, Stokey et al. (1989), for each $\varepsilon_2 > 0$ there exists $N(\varepsilon_2)$ such that $|g_{y'}^{R_n}(w) - g_{y'}^R(w)| < \varepsilon_2$ for all $m \geq N(\varepsilon_2)$ and *all* $w \in [\underline{w}, \bar{w}]$. So fix $\varepsilon > 0$ and choose $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ and $N(\varepsilon) = \max\{N(\varepsilon_1), N(\varepsilon_2)\}$. Then for all $n \geq N$

$$|g_{y'}^{R_n}(w_n) - g_{y'}^R(w)| \leq |g_{y'}^{R_n}(w_n) - g_{y'}^R(w_n)| + |g_{y'}^R(w_n) - g_{y'}^R(w)| < \varepsilon_2 + \varepsilon_1 = \varepsilon \qquad \text{(A36)}$$

■

The previous two lemmas can be used to prove our first main result about the excess demand function $d(.)$, namely continuity on $(1, \frac{1}{\beta})$.

**Theorem 10** *(Atkeson and Lucas (1995), Lemma 12):* $d(R)$ *is continuous on* $(1, \frac{1}{\beta})$.

**Proof.** Consider a sequence $\{R_n\}_{n=0}^{\infty}$ with $R_n \in (1, \frac{1}{\beta})$ converging to $R \in (1, \frac{1}{\beta})$. With each $R_n$ and with $R$ there is associated an operator $T_{R_n}^*$ and $T_R^*$, respectively. By Theorem 6 of the main text there exist a unique sequence of probability measures $\{\Phi_{R_n}\}_{n=0}^{\infty}$ such that $\Phi_{R_n} = T_{R_n}^* \Phi_{R_n}$ and a unique $\Phi_R$ such that $\Phi_R = T_R^* \Phi_R$. We will argue that the sequence $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to $\Phi_R$.

First, the state space $[\underline{w}, \bar{w}] \times Y$ is compact. Now consider the sequence of transition functions $\{Q_{R_n}\}_{n=0}^{\infty}$ associated with $\{R_n\}_{n=0}^{\infty}$. For any sequence $\{w_n\}_{n=0}^{\infty}$ in $[\underline{w}, \bar{w}]$ converging to $w \in [\underline{w}, \bar{w}]$, for all $y' \in Y$, $g_{y'}^{R_n}(w_n)$ converges to $g_{y'}(w)$ by Lemma 9 of this appendix. Now consider the sequence of probability measures $\{Q_{R_n}((w_n, y), .)\}_{n=0}^{\infty}$ and the probability measure $Q_R((w,y),.)$. If we can show that for each set $B \in \mathcal{B}(W) \times \mathcal{P}(Y)$ for which $Q_R((w,y), \partial B) = 0$,

$$\lim_{n\to\infty} Q_{R_n}((w_n, y), B) = Q_R((w,y), B) \tag{A37}$$

then the sequence $Q_{R_n}((w_n, y), .)$ converges weakly to $Q_R((w,y),.)$ by Theorem 12.3, Stokey et al. Here $\partial B$ denote the boundary of $B$, i.e.. the set of points that are limit points of $B$ as well as $B^C$.

Take an arbitrary such set $B$. By definition of $Q_R$, for all $w'$ such that $g^R_{y'}(w) = w'$ for some $y' \in Y$, we have that $w'$ is in the interior of $B$ (otherwise $Q_R((w,y), \partial B) > 0$). But then, since $g^{R_n}_{y'}(w_n)$ converges to $g_{y'}(w)$, $Q_{R_n}((w_n, y), B) = Q_R((w,y), B)$ for $n$ sufficiently big. Hence $(A37)$ is satisfied and the sequence $Q_{R_n}((w_n, y), .)$ converges weakly to $Q_R((w,y),.)$.

This result enables us to apply Theorem 12.13 of Stokey et al. to conclude that the sequence $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to $\Phi_R$. By Lemma 8 of this appendix $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to $V_R$. To show continuity of $d(.)$ we note that

$$\begin{aligned} |d(R_n) - d(R)| &= \left| \int V_{R_n}(w) d\Phi_{R_n} - \int V_R(w) d\Phi_R \right| \\ &\le \left| \int V_{R_n}(w) d\Phi_{R_n} - \int V_R(w) d\Phi_{R_n} \right| + \left| \int V_R(w) d\Phi_{R_n} - \int V_R(w) d\Phi_R \right| \end{aligned} \tag{A38}$$

by the triangle inequality. The first term converges to zero (as $n \to \infty$) as $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to $V_R$, the second term converges to zero as $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to $\Phi_R$ and $V_R$ is a continuous and bounded function ■

The previous result establishes that the excess demand function varies continuously with $R$. Now we want to establish a result about the slope of the excess demand function. In order to prove this we first establish that future utility promises are increasing in the interest rate $R$.

**Lemma 11** *The optimal policies $g^R_{y'}(w)$ are increasing in $R$ and the optimal policy $h^R(w)$ is decreasing in $R$.*

**Proof.** Let $R > \hat{R}$. We want to show that $h^R(w) \le h^{\hat{R}}(w)$ and $g^R_{y'}(w) \ge g^{\hat{R}}_{y'}(w)$, for all $y' \in Y$ and all $w \in [\underline{w}, \bar{w}]$. Define the sequence $\{V^n\}_{n=1}^{\infty}$ by $V^n = \left(T_{\hat{R}}\right)^n V_R$. Note that as $V_R$ is strictly convex and differentiable (by the argument in the proof to Lemma 7 in this appendix), so are all $V^n$ (again by the argument in the proof to Lemma 7). Let $(h^n, g^n_{y'})$ be the optimal policies associated with $V^n$, i.e.

$$V^n(w) = \left(1 - \frac{1}{R}\right) C(h^n(w)) + \frac{1}{R} \sum_{y'} \pi(y') V^{n-1}(g^n_{y'}(w)). \tag{A39}$$

We prove by induction that for all $n \geq 1$,

$$g_{y'}^R(w) \geq g_{y'}^n(w) \tag{A40}$$

$$h^R(w) \leq h^n(w) \tag{A41}$$

$$\frac{V_R'(w)}{R-1} \leq \frac{(V^n)'(w)}{\hat{R}-1} \tag{A42}$$

for all $y' \in Y$ and all $w \in [\underline{w}, \bar{w}]$. Since $\{V^n\}_{n=1}^{\infty}$ converges to $V_{\hat{R}}$ uniformly (by corollary 4 of this appendix) and $\left\{h^n, g_{y'}^n\right\}_{n=1}^{\infty}$ converge uniformly to $(h^{\hat{R}}, g_{y'}^{\hat{R}})$ (again see Lemma 7), it then follows that $g_{y'}^R(w) \geq g_{y'}^{\hat{R}}(w)$ (and the other two relations also hold for $n$ replaced with $\hat{R}$).

**Step 1:** Let $n = 1$ and fix $w \in [\underline{w}, \bar{w}]$. Suppose, to obtain a contradiction, that there exists $y'$ such that $g_{y'}^1(w) > g_{y'}^R(w) \geq U^{Aut}(y')$. Then from the respective first order conditions (note that $V^1 = T_{\hat{R}} V_R$)

$$V_R'(g_{y'}^1(w)) = \frac{\beta(\hat{R}-1)}{1-\beta} C'(h^1(w)) \tag{A43}$$

$$V_R'(g_{y'}^R(w)) \geq \frac{\beta(R-1)}{1-\beta} C'(h^R(w)) \tag{A44}$$

Since $V_R$ is strictly convex $V_R'(g_{y'}^1(w)) > V_R'(g_{y'}^R(w))$ and hence (as $R > \hat{R}$), $h^1(w) > h^R(w)$. From the promise keeping constraint there must exist $\bar{y}'$ such that $g_{\bar{y}'}^R(w) > g_{\bar{y}'}^1(w) \geq U^{Aut}(\bar{y}')$. But then, using $(A43)$ and $A(44)$

$$V_R'(g_{\bar{y}'}^R(w)) = \frac{\beta(R-1)}{1-\beta} C'(h^R(w)) < \frac{\beta(\hat{R}-1)}{1-\beta} C'(h^1(w)) \leq V_R'(g_{\bar{y}'}^1(w)) \tag{A45}$$

which implies $g_{\bar{y}'}^R(w) < g_{\bar{y}'}^1(w)$, a contradiction. Hence $g_{y'}^1(w) \leq g_{y'}^R(w)$, for all $y' \in Y$. Then from the promise keeping constraint $h^1(w) \geq h^R(w)$. The envelope conditions are

$$\frac{\left(V^1\right)'(w)}{\hat{R}-1} = \frac{C'(h^1(w))}{\hat{R}(1-\beta)} \tag{A46}$$

$$\frac{V_R'(w)}{R-1} = \frac{C'(h^R(w))}{R(1-\beta)} \tag{A47}$$

It follows from the previous result that $\frac{\left(V^1\right)'(w)}{\hat{R}-1} \geq \frac{V_R'(w)}{R-1}$.

**Step 2:** Suppose that $(A40) - (A42)$ are true for $n-1$. We want to show that $(A40) - (A42)$ are true for $n$. Again suppose, to obtain a contradiction, that there exists $y'$ such that $g_{y'}^n(w) > g_{y'}^R(w) \geq U^{Aut}(y')$. From the first order conditions (equations (20) of the main text) we have

$$\frac{\left(V^{n-1}\right)'(g_{y'}^n(w))}{\hat{R}-1} = \frac{\beta}{1-\beta} C'(h^n(w))$$

$$\frac{V_R'(g_{y'}^R(w))}{R-1} \geq \frac{\beta}{1-\beta} C'(h^R(w)) \tag{A48}$$

Since $V_R$ and $V^{n-1}$ are convex, $g^n_{y'}(w) > g^R_{y'}(w)$ and $(A42)$ holds for $n-1$, we have that $h^n(w) > h^R(w)$. Again by the promise keeping constraints there exists $\bar{y}'$ such that $g^R_{\bar{y}'}(w) > g^n_{\bar{y}'}(w) \geq U^{Aut}(\bar{y}')$. But by Lemma 3 of the main text

$$\begin{aligned} g^n_{y'}(w) &\leq g^n_{\bar{y}'}(w) \\ g^R_{y'}(w) &\geq g^R_{\bar{y}'}(w) \end{aligned} \tag{A49}$$

and hence

$$g^n_{y'}(w) \leq g^n_{\bar{y}'}(w) < g^R_{\bar{y}'}(w) \leq g^R_{y'}(w) < g^n_{y'}(w) \tag{A50}$$

a contradiction. It follows that for all $y' \in Y$, $g^n_{y'}(w) \leq g^R_{y'}(w)$. From promise keeping we have $h^n(w) \geq h^R(w)$. As before the envelope conditions imply that $\frac{(V^n)'(w)}{\hat{R}-1} \geq \frac{V'_R(w)}{R-1}$ ■

This result enables us to draw conclusions about how the invariant measure over utilities and endowment shocks, $\Phi_R$ varies with $R$. The next result shows that for larger interest rates the invariant measure puts more mass on higher utility entitlements. For every $\Phi$ define the probability measures $\Phi^y$ on $(W, \mathcal{B}(W))$ by $\Phi^y(B) = \frac{\Phi(B,\{y\})}{\pi(y)}$, for every $B \in \mathcal{B}(W)$. Note that every such measure is well-defined as $\pi(y) > 0$ by assumption, and that $\Phi^y(W) = 1$.

**Lemma 12** *(Atkeson and Lucas (1995)) Let $R > \hat{R}$. Then for every $y \in Y$, $\Phi^y_R$ stochastically dominates $\Phi^y_{\hat{R}}$, i.e.. for every increasing and continuous function $f$ on $W$,*

$$\int f(w) d\Phi^y_R \geq \int f(w) d\Phi^y_{\hat{R}} \tag{A51}$$

**Proof.** Define the sequence of measures $\{\Phi_n\}_{n=1}^{\infty}$ by $\Phi_n = \left(T^*_{\hat{R}}\right)^n \Phi_R$. We shall prove by induction that for each $n \geq 1$, and each $y \in Y$, $\Phi^y_R$ stochastically dominates $\Phi^y_n$. Since by Theorem 6 of the main text $\{\Phi_n\}$ converges to $\Phi_{\hat{R}}$ in total variation norm, the result then follows.

It will be convenient to define the distribution function associated with any probability measure $\Phi^y_n$, $F^y_n : W \to [0,1]$, as $F^y_n(w) = \Phi^y_n([\underline{w}, w]) = \Phi_n([\underline{w}, w], \{y\})/\pi(y)$. Since the domain of these functions is a subset of $\Re^1$, in order to prove that $\Phi^y_R$ stochastically dominates $\Phi^y_n$ it is sufficient to prove that for all $w \in W$, $F^y_R(w) \leq F^y_n(w)$.

**Step 1:** Let $n = 1$

By definition $\Phi_1 = T^*_{\hat{R}}\Phi_R$ whereas $\Phi_R = T^*_R\Phi_R$. Fix an arbitrary $y \in Y$, $w \in W$. Then

$$\begin{aligned} F^y_R(w) &= \frac{\Phi_R([\underline{w}, w], \{y\})}{\pi(y)} \\ &= \int_{\{v \in W | g^R_y(v) \leq w\}} d\Phi^y_R \\ &\leq \int_{\{v \in W | g^{\hat{R}}_y(v) \leq w\}} d\Phi^y_R \\ &= \frac{\Phi_1([\underline{w}, w], \{y\})}{\pi(y)} \\ &= F^y_1(w) \end{aligned} \quad \text{(A52)}$$

where the inequality is due to the fact that $g^R_y(w) \geq g^{\hat{R}}_y(w)$, for all $w \in W$.

**Step 2:** Suppose $F^y_R(w) \leq F^y_{n-1}(w)$, for all $w \in W$, all $y \in Y$. We want to show that the same is true for $n$. Note that

$$\begin{aligned} F^y_n(w) &= \frac{\Phi_n([\underline{w}, w], \{y\})}{\pi(y)} \\ &= \int_{\{v \in W | g^{\hat{R}}_y(v) \leq w\}} d\Phi^y_{n-1} \\ &= \sum_{\bar{y} \in Y} \pi(\bar{y}) F^{\bar{y}}_{n-1}(v_n) \end{aligned} \quad \text{(A53)}$$

where $v_n := \max\{v \in W | g^{\hat{R}}_y(v) \leq w\}$. Note that the last equality requires $g^{\hat{R}}_y$ to be increasing in $v$ as shown in Lemma 3 of the main text. Continuity of $g^{\hat{R}}_y$ ensures that $v_n$ is well-defined. Similarly $F^y_R(w) = \sum_{\bar{y} \in Y} \pi(\bar{y}) F^{\bar{y}}_R(v_R)$ with $v_R := \max\{v \in W | g^R_y(v) \leq w\}$. Lemma 11 of this appendix implies that $v_R \leq v_n$. Then the induction hypothesis implies that for all $\bar{y} \in Y$, $F^{\bar{y}}_R(v_R) \leq F^{\bar{y}}_{n-1}(v_n)$, and hence $F^y_R(w) \leq F^y_n(w)$ ■

The previous two results can now be combined to show that the excess demand function is increasing in the interest rate.

**Theorem 13** *(Atkeson and Lucas (1995), Lemma 14) Let $R > \hat{R}$. Then $d(R) \geq d(\hat{R})$.*

**Proof.** By definition of $d(R)$

$$d(R) = \int V_R(w) d\Phi_R - \int y d\Phi_R \quad \text{(A54)}$$

Since for all $R$, $\int y d\Phi_R$ is a constant, we focus on the analysis of the first part of the excess demand function. From the functional equation

$$\int V_R(w) d\Phi_R = \left(1 - \frac{1}{R}\right) \int C(h^R(w)) d\Phi_R + \frac{1}{R} \sum_{y'} \pi(y') \int V(g^R_{y'}(w)) d\Phi_R \quad \text{(A55)}$$

we note that by stationarity and the definition of $\Phi_R^y$,

$$\int V_R(w)d\Phi_R = \sum_{y\in Y}\pi(y)\int V_R(w)d\Phi_R^y \tag{A56}$$

$$\int V(g_{y'}^R(w))d\Phi_R = \int V(w)d\Phi_R^{y'} \tag{A57}$$

so that

$$\int V_R(w)d\Phi_R = \sum_{y'}\pi(y')\int V(g_{y'}^R(w))d\Phi_R \tag{A58}$$

It follows that

$$\int V_R(w)d\Phi_R = \int C(h^R(w))d\Phi_R. \tag{A59}$$

We want to prove that

$$\int V_R(w)d\Phi_R \geq \int V_{\hat{R}}(w)d\Phi_{\hat{R}} \tag{A60}$$

By the previous lemma for all $y \in Y, \Phi_R^y$ stochastically dominates $\Phi_{\hat{R}}^y$, and since $V_{\hat{R}}$ is strictly increasing it follows, using $(A56)$ that

$$\int V_{\hat{R}}(w)d\Phi_R \geq \int V_{\hat{R}}(w)d\Phi_{\hat{R}} \tag{A61}$$

So if we can prove that

$$\int V_R(w)d\Phi_R \geq \int V_{\hat{R}}(w)d\Phi_R \tag{A62}$$

we are done. Define the sequence $\{V^n\}_{n=1}^\infty$ by $V^n = \left(T_{\hat{R}}\right)^n V_R$. We will prove by induction that for all $n \geq 1$

$$\int V_R(w)d\Phi_R \geq \int V^n(w)d\Phi_R \tag{A63}$$

Since the sequence $\{V^n\}_{n=1}^\infty$ converges uniformly to $V_{\hat{R}}$ (by Corollary 4 of this appendix), this proves $(A62)$. Let $\{h^n, g_{y'}^n\}_{n=1}^\infty$ be the optimal policies associated with $\{V^n\}_{n=1}^\infty$ and $(h^R, g_{y'}^R)$ be the optimal choices associated with $V_R$.

**Step 1:** Let $n = 1$. By definition $V^1 = T_{\hat{R}}V_R$. Hence

$$\begin{aligned} V^1(w) &= \left(1-\frac{1}{\hat{R}}\right)C(h^1(w)) + \frac{1}{\hat{R}}\sum\pi(y')V_R(g_{y'}^1(w)) \\ &\leq \left(1-\frac{1}{\hat{R}}\right)C(h^R(w)) + \frac{1}{\hat{R}}\sum\pi(y')V_R(g_{y'}^R(w)) \end{aligned} \tag{A64}$$

since $(h^1, g^1_{y'})$ are the minimizing choices associated with $V^1$. Integrating with respect to $\Phi_R$ and using $(A58)$ and $(A59)$ yields

$$\begin{aligned}\int V^1(w)d\Phi_R &= \left(1-\frac{1}{\hat{R}}\right)\int C(h^1(w))d\Phi_R + \frac{1}{\hat{R}}\sum \pi(y')\int V_R(g^1_{y'}(w))d\Phi_R \\ &\leq \left(1-\frac{1}{\hat{R}}\right)\int C(h^R(w))d\Phi_R + \frac{1}{\hat{R}}\sum \pi(y')\int V_R(g^R_{y'}(w))d\Phi_R \\ &= \left(1-\frac{1}{\hat{R}}\right)\int V_R(w)d\Phi_R + \frac{1}{\hat{R}}\sum \pi(y')\int V_R(g^R_{y'}(w))d\Phi_R \\ &= \int V_R(w)d\Phi_R \qquad \text{(A65)}\end{aligned}$$

**Step 2:** Suppose $\int V_R(w)d\Phi_R \geq \int V^{n-1}(w)d\Phi_R$. We want to show that the same is true for $n$. By definition $V^n = T_{\hat{R}}V^{n-1}$, hence

$$\begin{aligned}V^n(w) &= \left(1-\frac{1}{\hat{R}}\right)C(h^n(w)) + \frac{1}{\hat{R}}\sum \pi(y')V^{n-1}(g^n_{y'}(w)) \\ &\leq \left(1-\frac{1}{\hat{R}}\right)C(h^R(w)) + \frac{1}{\hat{R}}\sum \pi(y')V^{n-1}(g^R_{y'}(w)) \qquad \text{(A66)}\end{aligned}$$

by the same reason as in step 1. Again integrating with respect to $\Phi_R$ and using $(A58)$ and $(A59)$ we obtain

$$\begin{aligned}\int V^n(w)d\Phi_R &= \left(1-\frac{1}{\hat{R}}\right)\int C(h^n(w))d\Phi_R + \frac{1}{\hat{R}}\sum \pi(y')\int V^{n-1}(g^n_{y'}(w))d\Phi_R \\ &\leq \left(1-\frac{1}{\hat{R}}\right)\int C(h^R(w))d\Phi_R + \frac{1}{\hat{R}}\sum \pi(y')\int V^{n-1}(g^R_{y'}(w))d\Phi_R \\ &= \left(1-\frac{1}{\hat{R}}\right)\int V_R(w)d\Phi_R + \frac{1}{\hat{R}}\int V^{n-1}(w)d\Phi_R \\ &\leq \left(1-\frac{1}{\hat{R}}\right)\int V_R(w)d\Phi_R + \frac{1}{\hat{R}}\int V_R(w)d\Phi_R \\ &= \int V_R(w)d\Phi_R \qquad \text{(A67)}\end{aligned}$$

where the last inequality uses the induction hypothesis ■

## 1.4 Detailed Proof of Theorem 12 in Main Text

In this subsection we provide a detailed proof of Theorem 12 of the main paper.

**Proof.** It is obvious that the allocation satisfies the resource constraint (9) since the efficient allocation by construction satisfies the resource constraint, and $\Theta_0$ is derived from $\Phi_0$. Also the allocation satisfies the continuing participation constraints, and, by construction of $a_0(w_0, y_0)$, the budget constraint. So it remains to be shown that, for almost all $(a_0, y_0)$, $\{c_t(a_0, y^t)\}$ is utility maximizing among the allocations satisfying the budget and the continuing participation constraints.

The proof is in two steps. We first show that the first order conditions

$$(1-\beta)\beta^t \pi(y^t|y_0) u'\left(c_t(a_0,y^t)\right)\left(1+\sum_{y^\tau \in P(y^t)} \mu(a_0,y^\tau)\right) = \lambda(a_0,y_0)p(y^t) \tag{A68}$$

are sufficient for optimality and we then show that the allocation defined above indeed satisfies the first order conditions.

**Step 1:** Define

$$U(a_0,y^t) = (1-\beta)u\left(c_t(a_0,y^t)\right) + \sum_{s>t}\sum_{y^s|y^t} \beta^{s-t}\pi(y^s|y^t)(1-\beta)u\left(c_s(a_0,y^s)\right) \tag{A69}$$

Suppose there exist Lagrange multipliers $\lambda(a_0,y_0), \{\mu(a_0,y^t)\} \geq 0$ that jointly with $\{c_t(a_0,y^t)\}$ satisfy $(A68)$, the budget constraint (7), at prices defined in (34), as well as the continuing participation constraints (8)

$$U(a_0,y^t) \geq U^{Aut}(y_t) \tag{A70}$$

together with

$$\mu(a_0,y^t)\left(U(a_0,y^t) - U^{Aut}(y_t)\right) = 0. \tag{A71}$$

Now suppose that there is a consumption allocation for individuals of type $(a_0,y_0)$, $\{\hat{c}_t(a_0,y^t)\}$, that satisfies (7) and (8), and that dominates $\{c_t(a_0,y^t)\}$, i.e. $\hat{U}(a_0,y_0) > U(a_0,y_0)$, where $\hat{U}(a_0,y_0)$ is defined analogously to $(A69)$.

Then

$$0 < \hat{U}(a_0, y_0) - U(a_0, y_0)$$

$$= (1-\beta)\left(\sum_{t=0}^{\infty}\sum_{y^t|y_0}\beta^t \pi(y^t|y_0)\left[u\left(\hat{c}_t(a_0, y^t)\right) - u\left(c_t(a_0, y^t)\right)\right]\right) \tag{A72}$$

$$\leq (1-\beta)\left(\sum_{t=0}^{\infty}\sum_{y^t|y_0}\beta^t \pi(y^t|y_0)\left[1 + \sum_{y^\tau \in P(y^t)}\mu(a_0, y^\tau)\right]\left[u\left(\hat{c}_t(a_0, y^t)\right) - u\left(c_t(a_0, y^t)\right)\right]\right) \tag{A73}$$

$$< (1-\beta)\left(\sum_{t=0}^{\infty}\sum_{y^t|y_0}\beta^t \pi(y^t|y_0)\left[1 + \sum_{y^\tau \in P(y^t)}\mu(a_0, y^\tau)\right]u'\left(c_t(a_0, y^t)\right)\left(\hat{c}_t(a_0, y^t) - c_t(a_0, y^t)\right)\right) \tag{A74}$$

$$= \lambda(a_0, y_0)\left(\sum_{t=0}^{\infty}\sum_{y^t|y_0}p(y^t)\left(\hat{c}_t(a_0, y^t) - c_t(a_0, y^t)\right)\right) \tag{A75}$$

$$= \lambda(a_0, y_0)\left(\sum_{t=0}^{\infty}\sum_{y^t|y_0}p(y^t)\hat{c}_t(a_0, y^t) - \sum_{t=0}^{\infty}\sum_{y^t|y_0}p(y^t)y_t - a_0\right) \tag{A76}$$

$$\leq 0 \tag{A77}$$

a contradiction. The several steps in the argument are justified as follows: $(A72)$ is by definition, $(A73)$ will be proved below, $(A74)$ follows from strict concavity of the utility function, $(A75)$ follows from $(A68)$, $(A76)$ from the budget constraint and the fact that $u$ is strictly increasing and prices are strictly positive, and finally $(A77)$ follows from the budget constraint. Hence there does not exist a consumption allocation $\{\hat{c}_t(a_0, y^t)\}$, that satisfies (7) and (8), and that dominates $\{c_t(a_0, y^t)\}$.

Now we prove that inequality $(A73)$ holds. For this we first note that for all $t, y^t$, we have

$$(1 + \mu(a_0, y^t))\left(\hat{U}(a_0, y^t) - U(a_0, y^t)\right) \geq \hat{U}(a_0, y^t) - U(a_0, y^t) \tag{A78}$$

If $U(a_0, y^t) > U^{Aut}(y_t)$, then from $(A71)$ it follows that $\mu(a_0, y^t) = 0$, so that $(A78)$ is satisfied. If $U(a_0, y^t) = U^{Aut}(y_t)$, then $\hat{U}(a_0, y^t) \leq U(a_0, y^t)$ and

$\mu(a_0, y^t) \geq 0$, and again $(A78)$ holds. Now

$$
\begin{aligned}
& \hat{U}(a_0, y_0) - U(a_0, y_0) \\
\leq \; & (1 + \mu(a_0, y_0)) \left( \hat{U}(a_0, y_0) - U(a_0, y_0) \right) \\
= \; & (1 + \mu(a_0, y_0)) * (1 - \beta) \left( \sum_{t=0}^{\infty} \sum_{y^t | y_0} \beta^t \pi(y^t | y_0) \left[ u\left( c_t(a_0, y^t) \right) - u\left( \hat{c}_t(a_0, y^t) \right) \right] \right) \\
= \; & (1 + \mu(a_0, y_0)) * (1 - \beta) \left[ u\left( \hat{c}_0(a_0, y_0) \right) - u\left( c_t(a_0, y^t) \right) \right] \\
& + (1 + \mu(a_0, y_0)) * \beta \sum_{y_1} \pi(y_1 | y_0) \left( \hat{U}(a_0, y^1) - U(a_0, y^1) \right) \\
\leq \; & (1 + \mu(a_0, y_0)) * (1 - \beta) \left[ u\left( \hat{c}_0(a_0, y_0) \right) - u\left( c_t(a_0, y^t) \right) \right] \\
& + \beta \sum_{y_1} \pi(y_1 | y_0) (1 + \mu(a_0, y_0) + \mu(a_0, y^1)) \left( \hat{U}(a_0, y^1) - U(a_0, y^1) \right) \\
& \vdots \\
\leq \; & \sum_{t=0}^{T} \sum_{y^t | y_0} \beta^t \pi(y^t | y_0) \left( 1 + \sum_{y^\tau \in P(y^t)} \mu(a_0, y^\tau) \right) \left[ u\left( c_t(a_0, y^t) \right) - u\left( \hat{c}_t(a_0, y^t) \right) \right] \\
& + \sum_{y^T | y_0} \sum_{y_{T+1}} \beta^{T+1} \pi(y^{T+1} | y_0) \left( 1 + \sum_{y^\tau \in P(y^{T+1})} \mu(a_0, y^\tau) \right) \left( \hat{U}(a_0, y^{T+1}) - U(a_0, y^{T+1}) \right)
\end{aligned}
\tag{A79}
$$

Taking limits yields

$$
\begin{aligned}
& \hat{U}(a_0, y_0) - U(a_0, y_0) \\
\leq \; & (1 - \beta) \left( \sum_{t=0}^{\infty} \sum_{y^t | y_0} \beta^t \pi(y^t | y_0) \left[ 1 + \sum_{y^\tau \in P(y^t)} \mu(a_0, y^\tau) \right] \left[ u\left( \hat{c}_t(a_0, y^t) \right) - u\left( c_t(a_0, y^t) \right) \right] \right) \\
& + \lim_{T \to \infty} \sum_{y^{T+1} | y_0} \beta^{T+1} \pi(y^{T+1} | y_0) \left( 1 + \sum_{y^\tau \in P(y^{T+1})} \mu(a_0, y^\tau) \right) \left( \hat{U}(a_0, y^{T+1}) - U(a_0, y^{T+1}) \right)
\end{aligned}
\tag{A80}
$$

We need to show that the last limit is nonpositive. Now note that from $(A68)$

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{y^{T+1}|y_0}\beta^{T+1}\pi(y^{T+1}|y_0)\left(1+\sum_{y^\tau\in P(y^{T+1})}\mu(a_0,y^\tau)\right)\left(\hat{U}(a_0,y^{T+1})-U(a_0,y^{T+1})\right)\\
=\;&\lim_{T\to\infty}\frac{\lambda(a_0,y_0)}{(1-\beta)R^{T+1}}\sum_{y^{T+1}|y_0}\frac{\pi(y^{T+1}|y_0)\left(\hat{U}(a_0,y^{T+1})-U(a_0,y^{T+1})\right)}{u'(c_{T+1}(a_0,y^{T+1}))}\\
=\;&\frac{\lambda(a_0,y_0)}{(1-\beta)}\lim_{T\to\infty}\sum_{y^{T+1}|y_0}\frac{\pi(y^{T+1}|y_0)\hat{U}(a_0,y^{T+1})}{R^{T+1}u'(c_{T+1}(a_0,y^{T+1}))} \qquad \text{(A81)}
\end{aligned}$$

because, since $\{c_t(a_0,y^t)\}$ is bounded, $\lim_{T\to\infty}\sum_{y^{T+1}|y_0}\frac{U(a_0,y^{T+1})}{R^{T+1}}=0$. Now

$$\begin{aligned}
&\frac{\lambda(a_0,y_0)}{(1-\beta)}\lim_{T\to\infty}\sum_{y^{T+1}|y_0}\frac{\pi(y^{T+1}|y_0)\hat{U}(a_0,y^{T+1})}{u'(c_{T+1}(a_0,y^{T+1}))R^{T+1}}\\
\leq\;&\frac{\lambda(a_0,y_0)}{(1-\beta)}\lim_{T\to\infty}\sum_{s\geq T+1}\sum_{y^s|y^{T+1}}\frac{\beta^{s-T-1}\pi(y^s|y_0)u\left(\hat{c}_s(a_0,y^s)\right)}{u'(c_{T+1}(a_0,y^{T+1}))R^{T+1}} \qquad \text{(A82)}
\end{aligned}$$

Without loss of generality we can sum only over those elements for which $u\left(\hat{c}_s(a_0,y^s)\right)>0$ (it makes the expression only bigger). Then

$$\begin{aligned}
&\frac{\lambda(a_0,y_0)}{(1-\beta)}\lim_{T\to\infty}\sum_{s\geq T+1}\sum_{y^s|y^{T+1}}\frac{\beta^{s-T-1}\pi(y^s|y_0)u\left(\hat{c}_s(a_0,y^s)\right)}{u'(c_{T+1}(a_0,y^{T+1}))R^{T+1}}\\
\leq\;&\frac{\lambda(a_0,y_0)}{(1-\beta)u'(\bar{c})}\lim_{T\to\infty}\sum_{s\geq T+1}\sum_{y^s|y^{T+1}}\frac{\pi(y^s|y_0)u\left(\hat{c}_s(a_0,y^s)\right)}{R^s} \qquad \text{(A83)}
\end{aligned}$$

where we used the facts that if we can show that $\{c_t(a_0,y^t)\}$ is bounded is bounded above by, say $\bar{c}$, and that $\beta<\frac{1}{R}$. From the budget constraint we know that (given the conjectured equilibrium prices)

$$\lim_{T\to\infty}\sum_{s\geq T+1}^{\infty}\sum_{y^s|y^{T+1}}\frac{\pi(y^s|y_0)\hat{c}_t(a_0,y^t)}{R^s}=0 \qquad \text{(A84)}$$

Since the utility function satisfies the INADA conditions, there exists $c^*>0$ such that $u'(c^*)=1$. By concavity $u\left(\hat{c}_t(a_0,y^t)\right)\leq u(c^*)+u'(c^*)\left(\hat{c}_t(a_0,y^t)-c^*\right)$.

Hence

$$
\begin{aligned}
& \lim_{T\to\infty} \sum_{s\geq T+1}^{\infty} \sum_{y^s|y^{T+1}} \frac{\pi(y^s|y_0)u\left(\hat{c}_t(a_0,y^t)\right)}{R^s} \\
\leq & \lim_{T\to\infty} \sum_{s\geq T+1}^{\infty} \sum_{y^s|y^{T+1}} \frac{\pi(y^s|y_0)\hat{c}_t(a_0,y^t)}{R^s} + \left(u(c^*)-c^*\right) \lim_{T\to\infty} \sum_{s\geq T+1}^{\infty} \frac{1}{R^s} \\
= & \left(u(c^*)-c^*\right) \lim_{T\to\infty} \sum_{s\geq T+1}^{\infty} \frac{1}{R^s} \\
= & \left(u(c^*)-c^*\right) \frac{R}{R-1} \lim_{T\to\infty} \frac{1}{R^{T+1}} \\
= & \ 0
\end{aligned}
\tag{A85}
$$

and we are done.

**Step 2:** We want to show that there exist Lagrange multipliers $\lambda(a_0, y_0)$, $\{\mu(a_0, y^t\} \geq 0$ that, together with the consumption allocation $\{c_t(a_0, y^t)\}$ satisfies the first order conditions. Let

$$
\begin{aligned}
\mu(a_0, y_0) &= 0 \\
\lambda(a_0, y_0) &= (1-\beta)u'\left(c_0(a_0, y_0)\right)
\end{aligned}
\tag{A86}
$$

and recursively

$$
1 + \sum_{y^\tau|y^t} \mu(a_0, y^\tau) = \frac{u'\left(c_0(a_0, y_0)\right)}{\left(\beta R\right)^t u'\left(c_t(a_0, y^t)\right)}
\tag{A87}
$$

Note that the allocation by construction (see equation (33) in the main text) satisfies

$$
\frac{u'\left(c_t(a_0, y_0)\right)}{\beta R u'\left(c_{t+1}(a_0, y^{t+1})\right)} \geq 1,
\tag{A88}
$$

with equality if the limited enforcement constraint is not binding in contingency $y^{t+1}$. Hence $\mu(a_0, y^{t+1}) \geq 0$, and $\mu(a_0, y^{t+1}) = 0$ if the constraint is not binding. Obviously the allocation and multipliers satisfy $(A68)$ ■

## 2 The Computational Procedure

In this subsection we describe how, for a parametric class of our economy, we compute a constant $R$, policy rules $h^R(w,y), g^R_{y'}(w,y)$ and a stationary distribution over utility entitlements and endowment shocks, $\Phi_R$.

Our computational method is an implementation of the policy function iteration algorithm proposed by Coleman (1990). For a fixed $R$ we search for the optimal policies $g_{y'}(w,y)$ and $h(w,y)$ within the class of piecewise-linear functions in $w$ . We start by specifying a $k$ point grid $G = \{w_0, ..., w_k\} \subseteq D$ and by guessing the values of a function $V'_0(.,.)$ on $G \times Y$ . Notice that this defines a function piecewise linear in $w$ for a fixed $y$. For a given $w, y \in G \times Y$ we then use the first order condition

$$\begin{aligned} C'(h(w,y)) &\leq \frac{1-\beta}{\beta(R-1)} V'(g_{y'}(w,y), y') \\ &= \quad \text{if } g_{y'}(w,y) > U^{Aut}(y') \end{aligned} \tag{A89}$$

together with the constraint

$$(1-\beta)h(w,y) + \beta \sum_{y' \in Y} \pi(y'|y) g_{y'}(w,y) = w \tag{A90}$$

to solve for solve $N+1$ equations[4] for the $N+1$ optimal policies $g^0_{y'}(w,y)$ and $h^0(w,y)$. Notice that $g^0_{y'}(w,y)$ and $h^0(w,y)$ are not constrained to lie in $G$. Carrying out this procedure for all $w, y \in G \times Y$ defines $g^0_{y'}(.,.)$ and $h^0(.,.)$ that are piecewise linear functions in $w$.

We then use envelope condition

$$V'_1(w,y) = \frac{(R-1)}{R(1-\beta)} C'(h^0(w,y)) \tag{A91}$$

to update our guess of $V'$ and repeat the procedure until convergence of $g^n_{y'}(.,.)$, $h^n(.,.)$ and $V'_n(w,y)$ is achieved. This yields policy functions that are piecewise linear in $w$.

To compute the stationary joint measure over $(w,y)$ we proceed as follows: for a given $(w,y)$ we find $w^l_{y'}(w,y), w^h_{y'}(w,y)$ and $\alpha_{y'}(w,y)$ such that

- $w^l_{y'}(w,y) = \max\{w \in G | w \leq g_{y'}(w,y)\}$
- $w^h_{y'}(w,y) = \min\{w \in G | w > g_{y'}(w,y)\}$
- $\alpha_{y'}(w,y)$ solves $\alpha_{y'}(w,y) w^l_{y'}(w,y) + (1-\alpha_{y'}(w,y)) w^h_{y'}(w,y) = g_{y'}(w,y)$.

[4] Note that whenever the first order condition does not hold with equality we know that $g_{y'}(w,y) = U^{Aut}(y')$ and we can drop the first order condition for the specific $y'$ as the number of unknowns is reduced by 1.

We then define the Markov transition matrix $Q : (G \times Y) \times (G \times Y) \to [0, 1]$ as

$$Q((w,y),(w',y')) = \begin{cases} \pi(y'|y)\alpha_{y'}(w,y) & \text{if} \quad w' = w^l_{y'}(w,y) \\ \pi(y'|y)(1-\alpha_{y'}(w,y)) & \text{if} \quad w' = w^h_{y'}(w,y) \\ 0 & \text{else} \end{cases} \tag{A92}$$

Note that the matrix $Q$ has dimension $(K \cdot N) \times (K \cdot N)$. We then solve the matrix equation

$$\Phi = Q^T \Phi \tag{A93}$$

for $\Phi$, where $\Phi$ has dimension $K \cdot N$ and $\Phi(w, y)$ gives the steady state probability of being in state $(w, y)$. In this way we can find, for a given $R \in (1, \frac{1}{\beta})$, $\Phi_R$, $h^R(w, y)$ and $g^R_{y'}(w, y)$. We then compute the excess demand function

$$d(R) = \sum_{(w,y) \in G \times Y} (C(h^R(w,y)) - y)\Phi_R(w,y) \tag{A94}$$

and use a Newton procedure to find $R$ such that $d(R) = 0$.

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