

The Social Cost of Near-Rational Investment:^{*}

Why we should worry about volatile stock markets

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Abstract

Excess volatility in stock returns may arise and drastically reduce welfare even if the stock market appears to be efficient and disconnected from the real economy. We solve a macroeconomic model in which information about fundamentals is dispersed and agents make small, correlated errors around their optimal investment policies. As information aggregates in the market, these errors amplify and result in large amounts of excess volatility in stock returns. The increase in volatility makes holding stocks unattractive and distorts the long-run level of capital accumulation. Through its effect on capital accumulation excess volatility causes costly (first-order) distortions in the long-run level of consumption.

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1 Introduction

An important function of financial markets is to aggregate information that is dispersed across market participants. Market prices reflect the information held by countless investors and direct resources to their most efficient use. If stock prices reflect information, investors have an incentive to learn from equilibrium prices and to update their expectations accordingly. But if investors learn from equilibrium prices, *anything* that moves prices has an impact on the expectations held by *all* market participants. We explore the general equilibrium implications of this basic dynamic in a world in which people are less than perfect - they make small mistakes when investing their wealth.

We solve a real business cycle model in which information is dispersed across market participants. Households observe the equilibrium stock price as well as a private signal about aggregate productivity in the next period. Based on this information they trade in stocks and bonds. As households place their trades, the equilibrium stock price aggregates the information in the market and becomes informative about future productivity. Because households optimize when they decide how to allocate their portfolios, small (potentially infinitesimal) deviations from their optimal policy have little impact on their individual welfare. However, if these deviations are correlated across households (say households are on average just a little bit too optimistic in some states of the world and a little bit too pessimistic in others), they affect the equilibrium price and hence may have a large external effect on the equilibrium expectations held by all market participants.

The first main insight from our model is that if information is dispersed, small errors in households' investment decisions may result in large amounts of excess volatility in equilibrium stock returns. Consider a state of the world in which households are on average just a little bit too optimistic about future productivity. If the average investor is slightly too optimistic, the stock price must rise. Households who observe this rise in the stock price may interpret it in one of two ways. It may either be due to errors made by their peers or, with some probability, it may reflect more positive information about future productivity received by the other market participants. Rational households should thus revise their expectations of future productivity upwards whenever they see a rise in the stock price. As households revise their expectations upwards, the stock price must rise further, triggering yet another revision in expectations, and so on. Small errors in the investment decision of the average household may thus lead to large deviations in equilibrium stock prices and large amounts of excess volatility in stock returns.

The second main insight from our model is that financial risk determines the amount of capital that is accumulated in the economy. If the equilibrium variance of stock returns rises, stocks become a riskier asset to hold and households demand a higher risk premium for holding stocks rather than bonds. This risk premium determines the marginal product of capital in the

long run (at the stochastic steady state). Changes in the (conditional) variance of stock returns thus change the *level* of capital accumulation, output and consumption. Excess volatility in stock returns may therefore cause large aggregate welfare losses by distorting the level of consumption in the stochastic steady state. Interestingly, this is true even if the capital stock responds very little to any given change in stock returns and there is an observed disconnect between the stock market and the real economy.

The combination of these two insights produces a surprising result: A model in which excess volatility in stock returns causes large aggregate welfare losses although there are no opportunities for earning abnormal returns in financial markets and all households are arbitrarily close to their rational behavior.

The Model Our model is a standard real business cycle model in which a consumption good is produced from capital and labor. Households supply labor to a representative firm and invest their wealth by trading claims to capital ('stocks') and bonds. The consumption good can be transformed into capital, and vice versa, by incurring a convex adjustment cost. The accumulation of capital is thus governed by its price relative to the consumption good (Tobin's Q). The only source of real risk in the economy are shocks to total factor productivity.

We extend this standard setup by assuming that each household receives a private signal about productivity in the next period and solve for equilibrium expectations. We first analyze the case in which all households are perfectly accurate in making their investment decisions (the rational expectations equilibrium). If the private signal is an unbiased predictor of productivity, the equilibrium price becomes perfectly revealing about productivity in the next period as in Grossman (1976). If, on the other hand, the private signal is completely uninformative, the rational expectations equilibrium coincides with the standard real business cycle model, in which productivity shocks are unpredictable.

We then show that the rational expectations equilibrium is unstable in the sense that the economy behaves very differently if households make small, correlated errors around their optimal investment policy. We refer to this as the "near-rational expectations equilibrium" to emphasize that the expected utility cost accruing to an individual household due to deviations from its optimal policy must be economically small.

When households form their expectations about tomorrow's productivity they inform on the equilibrium stock price. However, they cannot infer whether a given change in the stock price is attributable to information about productivity or to near-rational errors made by their peers. The average near-rational error thus feeds from the stock price into households' expectations and back into the stock price. The more dispersed information is across households the stronger is this feedback effect, because households rely more heavily on the stock price when they have less to learn from their private signal. In particular, we show that below some upper bound, a

given level of excess volatility in stock returns can be sustained by arbitrarily small near-rational errors if information is sufficiently dispersed.¹

We remain agnostic about the exact mechanism prompting households to make small, correlated errors in their investment decisions. We may think of some form of behavioral bias as in Dumas et al. (2006), where households falsely believe that an uninformative public signal contains a tiny amount of information about future productivity.² Alternatively, we may think of "animal spirits" or of a world in which investors must incur a small menu cost in order to eliminate small correlated errors from their investment decisions (Mankiw (1985)). For example, think of a world in which there are two computer programs for pricing stocks; a free program which prices stocks with a small error and another version which is available at a menu cost and prices stocks accurately. The point is that the private gain from avoiding near-rational errors is low, while the social gain from avoiding the resulting excess volatility in stock returns may be large.

This is easiest to see for the example of a small open economy in which households can borrow and lend at an exogenous international interest rate. Risk-averse investors demand a higher risk premium for holding stocks when returns are excessively volatile. The marginal unit of capital installed must therefore yield a higher expected return in order to compensate investors for the additional risk they are bearing. It follows that excess volatility depresses the equilibrium level of capital installed at the stochastic steady state and consequently lowers the level of output and consumption in the long run.³ Moreover, returns to capital rise while wages fall.⁴

Because welfare losses are driven mainly by a distortion in the stochastic steady state rather than by an intertemporal misallocation of capital, excess volatility in stock returns may cause large welfare losses even if the capital stock responds little to any given change in stock returns. In our model, the elasticity of the capital stock with respect to stock returns is therefore uninformative about the welfare consequences of excess volatility in stock returns. This contrasts with a widely held view among macroeconomists that pathologies in the stock market may not matter for the real economy if there is an observed disconnect between stock returns and changes in the capital stock (Morck et al. (1990)).

¹We define excess volatility as the difference in the conditional standard deviation of stock returns in the rational vs the near-rational expectations equilibrium.

²A large literature in behavioral finance has developed psychologically founded mechanisms that prompt households to make correlated mistakes in their investment decisions. Some examples are Odean (1998); Odean (1999); Daniel, Hirshleifer, and Subrahmanyam (2001); Barberis, Shleifer, and Vishny (1998); Bikhchandani, Hirshleifer, and Welch (1998); Hong and Stein (1999) and Allen and Gale (2001). Note these biases are not necessarily "near-rational" in the sense that they may entail an economically large cost to the individual.

³The stochastic steady-state is the vector of capital, bonds, and prices at which those quantities do not change in unconditional expectation.

⁴In a closed economy the fact remains that any distortion in the level of output and consumption is associated with first-order welfare losses. However, the effects are slightly more complicated (due to the precautionary savings motive), such that excess volatility in stock returns may drive consumption at the stochastic steady state up or down.

Calibration We quantify the aggregate welfare losses attributable to excess volatility in stock returns as the percentage rise in consumption that would make households indifferent between remaining in an equilibrium in which stock prices are excessively volatile (the near-rational expectations equilibrium) and transitioning to the stochastic steady state of an economy in which all households behave fully rationally until the end of time (the rational expectations equilibrium). We calibrate our model to match the standard deviation of stock returns observed in the data. Our baseline results are for the case of a small open economy. In our standard specification the conditional variance of stock returns in the rational expectations equilibrium is 1/3rd lower than in the near-rational expectations equilibrium. Aggregate welfare losses due to excess volatility amount to 2.53% of consumption. Most of this loss is attributable to lower capital accumulation due to higher risk premia. The results for a closed economy are quantitatively and qualitatively similar.

Related Literature This paper is to our knowledge the first to address welfare effects of excess volatility in stock returns within a full-fledged dynamic stochastic general equilibrium model. In a related paper, Mertens (2009) derives policies which mitigate the welfare cost of excess volatility. He shows that the stabilization of asset prices enhances welfare and that history-dependent policies may improve the information content of asset prices.

Our work relates to a literature that studies the welfare cost of excess volatility in stock returns, including Stein (1987) and Lansing (2008). Most closely related are DeLong, Shleifer, Summers, and Waldmann (1989) who analyze the general equilibrium effects of noise-trader risk in an overlapping generations model with endogenous capital accumulation. A large literature in macroeconomics and in corporate finance focuses on the sensitivity of firms' investment to a given mispricing in the stock market. Some representative papers in this area are Morck, Shleifer, and Vishny (1990); Blanchard, Rhee, and Summers (1993); Baker, Stein, and Wurgler (2003); Gilchrist, Himmelberg, and Huberman (2005); and Farhi and Panageas (2006).⁵ While most of these papers find that investment responds moderately to mispricings in the stock market, our model suggests that welfare losses due to excess volatility in stock returns may be large regardless of how responsive investment is to the stock market.

Moreover, this paper relates to a large literature on the costs of business cycles in two ways:⁶ First, we demonstrate that macroeconomic fluctuations affect the *level* of consumption if they create financial risk. This level effect is to our knowledge new to the literature and is not captured in standard cost-of-business cycles calculations in the spirit of Lucas (1987). Second, our model suggests that this level effect may cause economically large welfare losses if near-rational investor behavior causes a substantial amount of financial risk.

⁵Also see Galeotti and Schiantarelli (1994); Polk and Sapienza (2003); Panageas (2005); and Chirinko and Schaller (2006)

⁶See Barlevy (2004) for an excellent survey.

The notion of near-rationality is due to Akerlof and Yellen (1985) and Mankiw (1985). In their models near-rational behavior amplifies business cycles. Our application is closest to Cochrane (1989) and Chetty (2009) who use the utility cost of small deviations around an optimal policy to derive "economic standard errors".⁷

In our application, we argue that stock prices may deviate far from their fundamental values, because households have little incentive to avoid small correlated errors in their investment decisions: the lack of incentives to individuals adversely affects the quality of public information. In this sense our paper relates closely to an emerging literature which is concerned with the social value of public information. Recent work in this area includes Morris and Shin (2002), Amador and Weill (2007), Angeletos et al. (2007), and Angeletos and La'O (2008).

A technical complication is that our model requires solving for equilibrium expectations under dispersed information in a non-linear (general equilibrium) framework. We are able to do so due to recent advances in computational economics. We follow the solution method in Mertens (2009) to solve for the equilibrium. This method builds on Judd (1998) and Judd and Guu (2000) in using a higher-order expansion in all state variables around the deterministic steady state of the model with a nonlinear change of variables (Judd (2002)).⁸

In the main part of the paper we concentrate on the slightly more tractable small open economy version of the model (alternatively we may think of it as a closed economy in which households have access to a certain type of storage technology). After setting up the model we discuss equilibrium expectations and how excess volatility endogenously arises in the model. In section 4 we build intuition for the macroeconomic implications of excess volatility by presenting a simplified version of the model which allows us to show all the main results with pen and paper. In this simplified version of the model households consist of two specialized agents: a "capitalist" who has access to the stock and bond markets and a "worker" who provides labor services but is excluded from trading in the stock market. We then solve the full model computationally in section 6 and also give results for a closed economy version of the model.

2 Setup of the Model

The model is a de-centralization of the standard Mendoza (1991) framework: A continuum of households work and trade in stocks and bonds. A representative firm produces a homogenous consumption good by renting capital and labor services from households. Total factor productivity is random in every period and the firm adjusts factor demand accordingly. An investment goods sector has the ability to transform units of the consumption good into units of capital,

⁷Other recent applications include Woodford (2005) and Dupor (2005).

⁸See Devereux and Sutherland (2006) and Tille and van Wincoop (2007) for other recent applications based on perturbation.

while incurring convex adjustment costs.⁹ All households and the representative firm are price takers and plan for infinite horizons.

At the beginning of each period, households receive a private signal about productivity in the next period. Given this signal and their knowledge of prices and the state of the economy, they form expectations of future returns. Households make correlated near-rational mistakes when forming expectations about future productivity.

2.1 Economic Environment

Technology is characterized by a linear homogenous production function that uses capital, K_t , and labor, L as inputs

$$Y_t = e^{\eta_t} F(K_t, L), \quad (1)$$

where Y_t stands for output of the consumption good. Total factor productivity, η_t , is normally distributed with a mean of $-\frac{1}{2}\sigma_\eta^2$ and a variance of σ_η^2 . The equation of motion of the capital stock is

$$K_{t+1} = K_t(1 - \delta) + I_t, \quad (2)$$

where I_t denotes aggregate investment and δ is the rate of depreciation. Furthermore, there are convex adjustment costs to capital,

$$AC = \frac{1}{2}\chi \frac{I_t^2}{K_t}, \quad (3)$$

where χ is a positive constant. There is costless trade in the consumption good at the world price, which we normalize to one. All households can borrow and lend abroad at rate r . Foreign direct investment and international contracts contingent on η are not permitted.

2.2 Households

There is continuum of identical households indexed by $i \in [0, 1]$. At the beginning of every period each household receives a private signal about tomorrow's productivity:

$$s_t(i) = \eta_{t+1} + \nu_t(i), \quad (4)$$

where $\nu_t(i)$ represents *i.i.d.* draws from a normal distribution with zero mean and variance σ_ν^2 . Given this information and their knowledge about the economy, households maximize lifetime utility by choosing an intertemporal allocation of consumption, $\{C_t(i)\}_{t=0}^\infty$, and by weighting their portfolios between stocks and bonds at every point in time, $\{\omega_t(i)\}_{t=0}^\infty$, where ω represents

⁹The alternative to introducing an investment goods sector is to incorporate the investment decision into the firm's problem. The two modeling devices are equivalent as long as there are no frictions in contracting between management and shareholders.

the share of equity in their portfolio. Formally, an individual household's problem is

$$\max_{\{C_t(i)\}_{t=0}^{\infty}, \{\omega_t(i)\}_{t=0}^{\infty}} U_t(i) = \mathcal{E}_{it} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \log(C_s(i)) \right\} \quad (5)$$

subject to

$$W_{t+1}(i) = [(1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1})](W_t(i) + w_t L - C_t(i)) \quad \forall t, \quad (6)$$

where \mathcal{E}_{it} stands for household i 's conditional expectations operator, $W_t(i)$ stands for financial wealth of household i at time t and \tilde{r}_{t+1} is the equilibrium return on stocks. We denote the market price of capital with Q_t and dividends with D_t .¹⁰

$$1 + \tilde{r}_{t+1} = \frac{Q_{t+1}(1 - \delta) + D_{t+1}}{Q_t}. \quad (7)$$

Finally, E_{it} denotes the rational expectations operator, conditional on all information available to household i at time t :

$$E_{it}(\cdot) = E(\cdot | Q_t, s_t(i), K_t, B_t, \eta_t). \quad (8)$$

The expectations operator \mathcal{E} allows households to make small "mistakes" when forecasting future productivity. In particular, we assume that their conditional expectation of tomorrow's productivity deviates from the rational expectation by a small error $\tilde{\epsilon}_t$:

$$\mathcal{E}_{it}(\eta_{t+1}) \equiv E_{it}(\eta_{t+1}) + \tilde{\epsilon}_t. \quad (9)$$

Households thus have rational expectations, but their conditional probability density function of η_{t+1} has been shifted by $\tilde{\epsilon}_t$.¹¹ For simplicity we assume that all households make the same small mistake. Alternatively, we may think of $\tilde{\epsilon}_t$ as the average mistake made by households trading in the stock market. The deviation caused by $\tilde{\epsilon}_t$ is zero in expectation and its variance, $\sigma_{\tilde{\epsilon}}^2$, is small enough such that the expected utility loss from making this mistake is below some threshold level.¹² Our favorite interpretation of this error is that households observe an

¹⁰Note that we implicitly assume here that stocks split proportionally to the percentage change in aggregate capital stock at the end of each period. The stock price is then always equal to the price of a claim to one unit of capital.

¹¹More formally, \mathcal{E} is a rational expectations operator with the only the only exception that the conditional probability density function of η_{t+1} has been shifted by $\tilde{\epsilon}_t$: $\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] = E[\eta_{t+1} | s_t(i), Q_t] + \tilde{\epsilon}_t$. As a consequence households have the correct perception of all higher moments of the conditional distribution of η_{t+1} :

$$\mathcal{E} \left[(\eta_{t+1} - \mathcal{E}(\eta_{t+1} | s_t(i), Q_t))^k | s_t(i), Q_t \right] = E \left[(\eta_{t+1} - E(\eta_{t+1} | s_t(i), Q_t))^k | s_t(i), Q_t \right] \text{ for all } k \neq 1.$$

¹²More precisely, $\tilde{\epsilon}_t(i)$ has a mean of $-\frac{1}{2}\sigma_{\tilde{\epsilon}}^2$ such that agents hold the correct expectation of log returns in expectation.

uninformative public signal and falsely believe that it contains a small amount of information about η_{t+1} (Dumas et al. (2006)). However, we may think of a number of other interpretations involving animal spirits, menu costs, behavioral biases, or even an evolutionary regime under which households invest by rules of thumb and change their rules only if they expect a significant utility gain from doing so.

For convenience, we assume that households can insure against idiosyncratic risk due to their private signal. They can buy contingent claims at the beginning of the period that pay off at the beginning of the next period. Contingent claims trading thus completes markets between periods and leads all households, in equilibrium, to hold the same amount of wealth.¹³

2.3 Firms

A representative firm purchases capital and labor services from households. As it rents services from an existing capital stock, its maximization collapses to a period-by-period problem.¹⁴ The firm's problem is to

$$\max_{K_t^d, L_t^d} e^{\eta_t} F(K_t^d, L_t^d) - w_t L_t^d - D_t K_t^d, \quad (10)$$

where K_t^d and L_t^d denote factor demands for capital and labor respectively. First order conditions with respect to capital and labor pin down the fair wage and the dividend. Both factors receive their marginal product:

$$e^{\eta_t} F_K(K_t^d, L_t^d) = D_t \quad (11)$$

and

$$e^{\eta_t} F_L(K_t^d, L_t^d) = w_t. \quad (12)$$

As the production function is linear homogenous, the representative firm makes zero economic profits.

2.4 Investment Goods Sector

The representative firm owns an investment goods sector which converts the consumption good into units of capital, while incurring adjustment costs. It takes the price of capital as given and then performs instant arbitrage:

$$\max_{I_t} Q_t I_t - I_t - \frac{1}{2} \chi \frac{I_t^2}{K_t}, \quad (13)$$

¹³We have suppressed arguments relating to contingent claims in equation (6).

¹⁴Note that by choosing a structure in which firms rent capital services from households, we abstract from all principal agent problems between managers and stockholders. Managers therefore cannot prevent errors in stock prices from impacting investment decisions, as in Blanchard, Rhee, and Summers (1993). On the other hand, they do not amplify shocks or overinvest as in Albuquerque and Wang (2005).

where the first term is the revenue from selling I_t units of capital and the second and third terms are the cost of acquiring the necessary units of consumption goods (recall the price of the consumption good is normalized to one) and the adjustment costs respectively. Since there are decreasing returns to scale in converting consumption goods to capital, the investment goods sector makes positive profits in each period. Profits are paid to shareholders as a part of dividends.¹⁵

Taking the first order condition of (13), gives us equilibrium investment as a function of the market price of capital:

$$I_t = \frac{K_t}{\chi} (Q_t - 1) \quad (14)$$

Whenever the market price of capital is above one, investment is positive, raising the capital stock in the following period. When it is below one the investment goods sector buys units of capital and transforms them back into the consumption good. Note that the parameter χ scales the adjustment costs and can be used to calibrate the sensitivity of capital investment with respect to the stock price.

2.5 Definition of Equilibrium

Definition 2.1

Given a time path of shocks $\{\eta_t, \tilde{\epsilon}_t, \{\tilde{\nu}_t(i) : i \in [0, 1]\}\}_{t=0}^{\infty}$ an equilibrium in this economy is a time path of quantities $\{\{C_t(i), B_t(i), W_t(i), \omega_t(i) : i \in [0, 1]\}, C_t, B_t, W_t, \omega_t, K_t^d, L_t^d, Y_t, K_t, I_t\}_{t=0}^{\infty}$, signals $\{s_t(i) : i \in [0, 1]\}_{t=0}^{\infty}$ and prices $\{Q_t, r, D_t, w_t\}_{t=0}^{\infty}$ with the following properties:

1. $\{\{C_t(i), \omega_t(i)\}\}_{t=0}^{\infty}$ solve the households' maximization problem (5) given the vector of prices, initial wealth, and the random sequences $\{\tilde{\epsilon}_t, \{\tilde{\nu}_t(i)\}\}_{t=0}^{\infty}$;
2. $\{K_t^d, L_t^d\}_{t=0}^{\infty}$ solve the representative firm's maximization problem (10) given the vector of prices;
3. $\{I_t\}_{t=0}^{\infty}$ is the investment goods sector's optimal policy (14) given the vector of prices;
4. $\{w_t\}_{t=0}^{\infty}$ clears the labor market, $\{Q_t\}_{t=0}^{\infty}$ clears the stock market, and $\{D_t\}_{t=0}^{\infty}$ clears the market for capital services;
5. There is a perfectly elastic supply of the consumption good and of bonds in world markets. Bonds pay the rate r and the price of the consumption is normalized to one;
6. $\{Y_t\}_{t=0}^{\infty}$ is determined by the production function (1), $\{K_t\}_{t=0}^{\infty}$ evolves according to (2), $\{\{W_t(i)\}\}_{t=0}^{\infty}$ evolve according to the budget constraints (6), and $\{s_t(i)\}_{t=0}^{\infty}$ is determined by (4);

¹⁵ Alternatively, profits may be paid to individuals as a lump-sum transfer; this assumption matters little for the results of the model.

7. $\{B_t(i), C_t, B_t, W_t, \omega_t\}_{t=0}^{\infty}$ are given by the identities

$$B_t(i) = (1 - \omega_t(i)) (W_t(i) - C_t(i)), \quad (15)$$

$$X_t = \int_0^1 X_t(i) di, \quad X = C, B, W \quad (16)$$

and

$$\omega_t = \frac{Q_t K_{t+1}}{W_t - C_t}. \quad (17)$$

The *rational expectations equilibrium* is the equilibrium in which $\sigma_{\tilde{\epsilon}} = 0$, such that the expectations operator \mathcal{E} in equation (5) coincides with the rational expectation in (8). The *near-rational expectations equilibrium* posits that $\sigma_{\tilde{\epsilon}} > 0$; households make small errors around their rational expectation, as given in (9). The idea behind the near-rational expectations equilibrium is that small errors in households' policies result in minor welfare losses for the individual household. The following definition formalizes what it means for near-rational households to suffer only "economically small" losses:

Definition 2.2

A near-rational expectations equilibrium is k-percent stable if the welfare gain to an individual household of obtaining rational expectations is less than k% of consumption.

3 Equilibrium Expectations

In this section we explore how small correlated mistakes in households' investment behavior may result in large errors in market expectations and in excess volatility in stock returns. To fix ideas, let us define the error in market expectations of η_{t+1} as the difference between the average expectation held by households in the near-rational expectations equilibrium and the average expectation they would hold if $\tilde{\epsilon}_t$ happened to be zero in this period. We call the error in the market expectations

$$\epsilon_t = \gamma \tilde{\epsilon}_t$$

and solve for γ below.¹⁶ The main insight is that the multiplier γ may be very large. This amplification of errors is a result of households learning from equilibrium prices: a rise in prices causes households to revise their expectations upwards; and when households act on their revised expectations, the price rises further. Trades that are correlated with the average error made by investors thus represent an externality on other households' expectations.

¹⁶More formally, $\epsilon_t = \int (E_{it}(\eta_{t+1}) + \tilde{\epsilon}_t) di|_{\tilde{\epsilon}_t > 0, \sigma_{\tilde{\epsilon}} > 0} - [\int E_{it}(\eta_{t+1}) di|_{\tilde{\epsilon}_t = 0, \sigma_{\tilde{\epsilon}} > 0}]$.

3.1 Solving for Expectations in General Equilibrium

In order to say more about the relationship between $\tilde{\epsilon}_t$ and ϵ_t we need to solve for equilibrium expectations. This is a challenge because our model is non-linear, and in particular because the market price of capital is a non-linear function of η_{t+1} .

If the market for stocks is to clear, the amount of capital demanded by households must equal the amount of capital supplied in the economy. The supply of capital, however, rises with Q (see equation (14)). At the same time, the demand for capital depends on the payoff the average investor expects to receive from holding stocks; and investors in turn learn from Q when forming their expectations about this payoff. These complexities are reflected in equilibrium conditions which take the form:

$$g_{S_t^k}^L(Q_t, C_t) = \int \mathcal{E} \left(g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1}) | Q_t, s_t(i), K_t, B_{t-1}, \eta_t \right) di, \quad (18)$$

where $g_{S_t^k}^L(Q_t, C_t)$ and $g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})$ are non-linear functions of Q , C , and of the state variables and shocks known at time t and $t+1$, respectively.¹⁷ This equilibrium condition poses two difficulties. First, households care about variables reflected in $g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})$, such as the payoff they receive from stocks and their future consumption, but they receive information about η_{t+1} , and there is a complicated non-linear relationship between the two. Second, households learn from Q_t about η_{t+1} , but Q_t is again a non-linear function of η_{t+1} .

We use two tricks developed in Mertens (2009) to transform (18) into a form which we can solve with standard techniques: First, we use perturbation methods to show that given the households' information sets, the conditional expectation of η_{t+1} is a sufficient statistic for their expectation of $g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})$; i.e. there is a deterministic relationship between households' expectations of tomorrow's productivity and what they expect to happen in the future more generally. Moreover, K_t, B_{t-1} and η_t have no predictive power over and above the information contained in Q_t and $s_t(i)$. This reduces the problem to solving for $\mathcal{E}(\eta_{t+1} | Q_t, s_t(i))$. Second, we use a nonlinear change of variables to obtain a linear transformation of the equilibrium stock price. This linear transformation, we call it \hat{q}_t , is a linear function of η_{t+1} , but has the same information content as Q_t (i.e. both variables span the same σ -algebra). The basic

¹⁷In the simplified version of our model in which households consist of specialized capitalists and workers we can solve for the consumption policy in closed form and therefore only get one condition of the form (18) which can be written as:

$$\underbrace{\left((1+r)Q_t + \frac{Q_t^2 K_t \left(1 - \delta + \frac{1}{\chi} (Q_t - 1)\right)}{\beta(B_{t-1}(1+r) + Q_t K_t (1 - \delta) + e^{\eta_t} F_K(K_t, L) K_t)} \sigma^2 \right)}_{\equiv g_{S_t^k}^L(Q_t, C_t)} = \int \mathcal{E}_{it} \underbrace{\left(Q_{t+1}(1 - \delta) + D_{t+1} \right)}_{= g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})} di. \quad (19)$$

intuition is that Q_t is a monotonic function of η_{t+1} , such that learning from Q_t is just as good as learning from its linear transformation. Framed in terms of this \hat{q}_t , the equilibrium boils down to computing prices and expectations such that the following equation is satisfied.

$$\hat{q}_t = \int E(\eta_{t+1} | \hat{q}_t, s_t(i)) di + \tilde{\epsilon}_t, \quad (20)$$

where \hat{q}_t is a function of the state variables and shocks known at time t . Equation (20) is the familiar linear equilibrium condition of a standard noisy rational expectations model. We can now apply standard methods to solve for equilibrium expectations in terms of \hat{q}_t (Hellwig (1980)) and then transform the system back to recover the equilibrium Q_t . Technical details are given in Appendix A.1.

3.2 Amplification of Small Errors

We now obtain equilibrium expectations by solving for \hat{q} . As it turns out we are able to show all the main qualitative results on the aggregation of information in this linear form. In section 6 we map the solution back into its non-linear form to show the quantitative implications for the equilibrium stock price and for stock returns.

Since \hat{q}_t equals the market expectation of η_{t+1} in (20), we may guess that the solution for \hat{q}_t is some linear function of η_{t+1} and $\tilde{\epsilon}_t$:

$$\hat{q}_t = \pi_0 + \pi_1 \eta_{t+1} + \gamma \tilde{\epsilon}_t. \quad (21)$$

This guess formally defines the multiplier γ . Our task is to solve for the coefficients in this equation. Assuming that our guess for \hat{q}_t is correct, the rational expectation of η_{t+1} given the private signal and \hat{q}_t is

$$E_{it}(\eta_{t+1}) = A_0 + A_1 s_t(i) + A_2 \hat{q}_t, \quad (22)$$

where the constants A_0 , A_1 and A_2 are the weights that households give to the prior, the private signal and the market price of capital respectively. We get market expectations by adding the near-rational error and summing up across households. Combining this expression with our guess (21) yields

$$\int E_{it}(\eta_{t+1}) di + \tilde{\epsilon}_t = (A_0 + A_2 \pi_0) + (A_1 + A_2 \pi_1) \eta_{t+1} + A_2 \gamma \tilde{\epsilon}_t + \tilde{\epsilon}_t, \quad (23)$$

where we have used the fact that $\int s_t(i) di = \eta_{t+1}$. This expression reflects all the different ways in which $\tilde{\epsilon}_t$ affects market expectations: The last term on the right hand side is the direct effect of the near-rational error on individual expectations. If we introduced a fully rational household into the economy and gave it the same private signal as one of the near-rational households, the

two households' expectations of η_{t+1} would differ exactly by $\tilde{\epsilon}_t$. The third term on the right hand side represents the deviation in market expectations that results from the fact that the market price transmits the average error as well as information about future fundamentals. The extent of this amplification depends on how much weight the market price has in the rational expectation (22) and on how sensitive \hat{q}_t is to $\tilde{\epsilon}_t$ in (21). Finally, the second term on the right hand side tells us that the mere fact that households make near-rational errors may reduce the extent to which the market can predict η_{t+1} by changing the coefficients A_1 and A_2 .

Plugging (23) into (20) and matching coefficients with (20) allows us to solve for the amplification of $\tilde{\epsilon}_t$:

Proposition 3.1

Through its effect on the market price of capital, the near-rational error, $\tilde{\epsilon}_t$, feeds back into the rational expectation of η_{t+1} . The more weight households place on the market price of capital when forming their expectations about η_{t+1} , the larger is the error in market expectations relative to $\tilde{\epsilon}_t$. We have that

$$\gamma = \frac{1}{1 - A_2}. \quad (24)$$

Proof. See appendix A. ■

It follows that the larger the weight on the market price of capital in the rational expectation, A_2 , the larger is the variance in ϵ_t relative to the variance in $\tilde{\epsilon}$. We can solve for this weight and the other coefficients in (21) and (22) by applying the projection theorem. With explicit solutions in hand, we can show the following result:

Proposition 3.2

For any given level of $\sigma_{\tilde{\epsilon}}$, the noise to signal ratio in the market price of capital goes to infinity as the precision of the private signal goes to zero,

$$\lim_{\sigma_{\nu} \rightarrow \infty} \frac{\sqrt{\text{var}(\gamma\tilde{\epsilon}_t)}}{\sqrt{\text{var}(\pi_1\eta_{t+1})}} = \infty.$$

Proof. See appendix A. ■

As information becomes more dispersed across households, the private signal becomes less informative relative to the stock price. Households adjust by paying relatively more attention to the public signal. This has two effects. First, if households put less weight on their private signal, less information enters the equilibrium price. Second, the more attention they pay to the market price, the larger is the amplification of $\tilde{\epsilon}_t$. Both effects result in a rising noise to signal ratio in equilibrium stock prices. The implication of this finding is that if the private signal received by households is sufficiently noisy, *arbitrarily small* correlated errors in investor behavior may completely destroy the market's ability to aggregate information.

While the model can generate an arbitrarily large noise to signal ratio with a given $\sigma_{\bar{\epsilon}}$, there is an upper bound for the absolute amount of non-fundamental volatility in market expectations: As the noise in the private signal increases, households put more and more weight on the stock price when forming their expectations. This leads to a larger and larger amplification of a given amount of near-rational errors and hence a larger and larger amount of noise in the stock price. As both the private signal and the stock price become less informative, households begin to rely more on their priors, the overall amount of volatility in market expectations peaks and then eventually decreases.

Figure 1 illustrates this point. It plots σ_{ϵ} , the standard deviation of the error in market expectations of η_{t+1} over the standard deviation of near-rational errors and the noise in the private signal. All standard deviations are normalized with the standard deviation of productivity shocks.¹⁸ As σ_{ν} rises, the standard deviation of errors in market expectations rises, peaks and then slowly begins to decrease. However, the absolute amount of information in market expectations falls at a faster rate than the absolute amount of noise, such that the noise to signal ratio in Figure 2 continues to rise monotonically. A parallel logic holds for the standard deviation of the near-rational error itself: At very low levels of $\sigma_{\bar{\epsilon}}$ the amplification of errors is very large. In Figure 1, small errors in investor behavior get amplified most. As these errors cause more and more noise in the market price of capital, the amplification peaks and then drops as investors begin relying more on their priors. The plot in Figure 1 ends when $\sigma_{\bar{\epsilon}}$ amounts to 1% of the standard deviation of the productivity shock. Were we to continue the plot to the left, σ_{ϵ} would continue falling to a certain point and then begin to increase again as investors start making economically large errors (in this case they would become more like noise traders).

This pattern highlights a second channel through which near-rational errors affect the aggregation of information in the stock market:

Proposition 3.3

The absolute amount of information aggregated in the stock price decreases with $\sigma_{\bar{\epsilon}}$,

$$\frac{\partial \pi_1}{\partial \sigma_{\bar{\epsilon}}} < 0$$

Proof. See appendix A. ■

While near-rational errors amplify and lead to large amounts of noise in the stock price, they simultaneously hamper the capacity of the stock market to transmit and aggregate information. The conditional variance of η_{t+1} in the near-rational expectations equilibrium therefore exceeds the conditional variance in the rational expectations equilibrium for two reasons: First, because

¹⁸The relationship depicted in the Figures 1 and 2 is independent scale and independent of the parameters of the model.

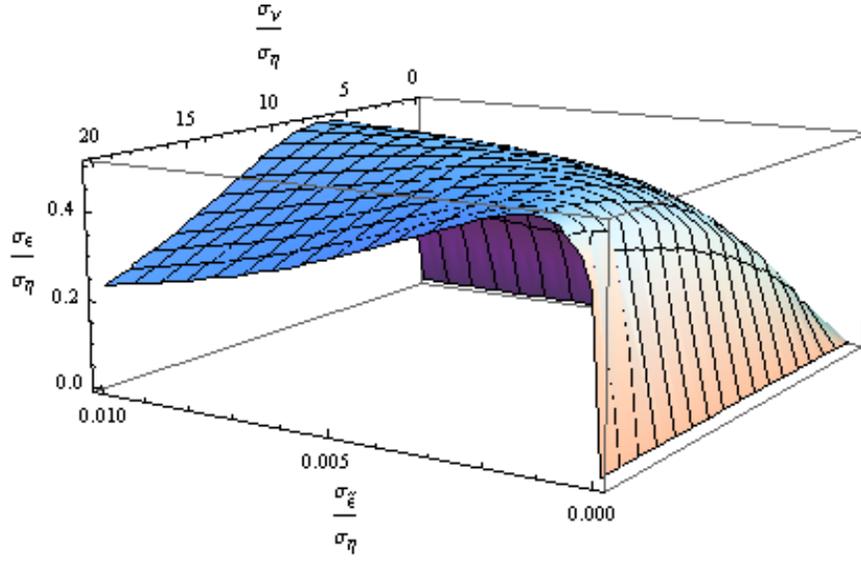


Figure 1: Standard deviation of the error in market expectations of η_{t+1} plotted over the standard deviation of the near-rational error, $\tilde{\epsilon}$, and the standard deviation of noise in the private signal, σ_ν . All values are normalized with the standard deviation of the productivity shock.

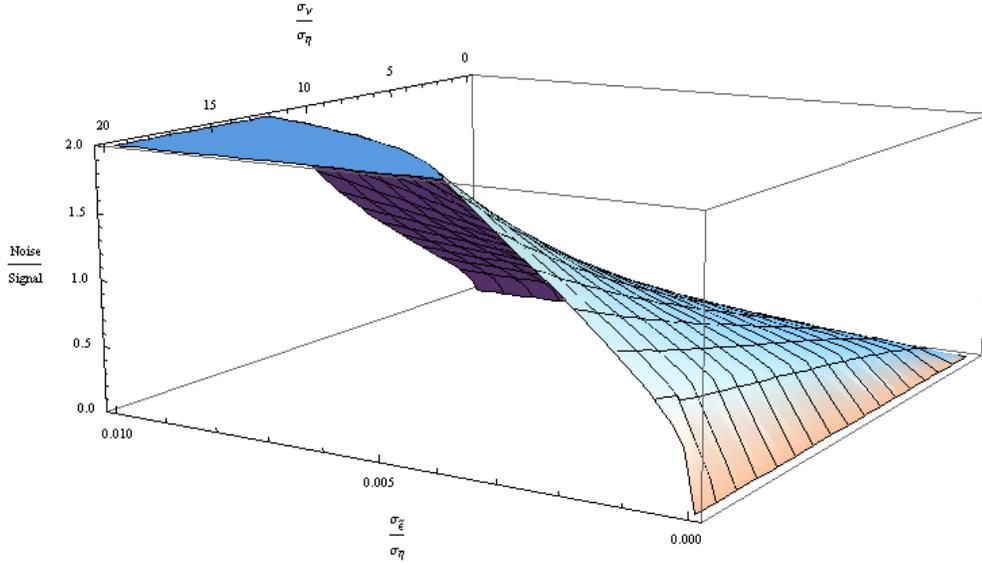


Figure 2: Noise to signal ratio in market expectations, $\sqrt{\text{var}(\gamma\tilde{\epsilon}_t)}/\sqrt{\text{var}(\pi_1\eta_{t+1})}$, plotted over the standard deviation of the near-rational error and the standard deviation of noise in the private signal. All values are normalized with the standard deviation of the productivity shock.

the stock price becomes noisy and second because it contains less information about the future.¹⁹

Since we know that \hat{q} and Q have the same information content, we can make a parallel statement about the conditional variance of stock returns, which leads us to the following definition:

Definition 3.4

Excess volatility in stock returns is the percentage amount by which the conditional standard deviation of stock returns in the near-rational expectations equilibrium, σ , exceeds the conditional standard deviation of stock returns in the rational expectations equilibrium, σ^ ,*

$$\frac{\sigma - \sigma^*}{\sigma} 100.$$

The amount of excess volatility in stock returns that may arise due to near-rational errors depends on the non-linearities of the model. Before we turn to quantifying these effects we first build some intuition for the impact that this particular pathology in financial markets may have on the macroeconomy.

4 Intuition: The Macroeconomic Effects of Financial Risk

In this section we turn to the effect that excess volatility in stock returns has on the macroeconomic equilibrium. In order to provide a maximum of intuition for the mechanisms at work, this section focuses on a simplified version of the model for which we are able to derive the main results analytically. In section 6 we show computationally that the relevant implications of the simplified model carry over to the full model.

Assume that households consist of two specialized agents, a "capitalist" who trades in the stock and bond markets and a "worker" who provides labor services, receives wages and the profits from the investment goods sector, but is excluded from trading in the stock market. This division eliminates non-tradable income from the capitalist's portfolio problem such that we can solve it with pen and paper. A capitalist's budget constraint is

$$W_{t+1}(i) = ((1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1}))(W_t(i) - C_t(i)) \quad \forall t. \quad (25)$$

Taking as given that the distribution of equilibrium asset returns is approximately log-normal (this is true to a first-order approximation), we can solve for the capitalist's optimal consumption and portfolio allocation:²⁰

¹⁹See Appendix A.4 for an analytical solution for the conditional variance of η_{t+1} .

²⁰We require approximate log-normality for the analytical solution below but not for the computational results.

Lemma 4.1

Capitalists' optimal consumption is a constant fraction of financial wealth

$$C_t(i) = (1 - \beta)W_t(i) \quad (26)$$

and the optimal portfolio share of stocks is the expected excess return divided by the conditional variance of stock returns, σ^2

$$\omega_t(i) = \frac{\mathcal{E}_{it}(1 + \tilde{r}_{t+1}) - (1 + r)}{\sigma^2}. \quad (27)$$

Proof. Appendix B gives a detailed derivation which proceeds analogous to Samuelson (1969).

■

Where of course the capitalist, rather than the entire household makes small mistakes as in (9) when investing in the stock market. The stock market clears when the value of shares demanded equals the value of shares in circulation:

$$\int_0^1 \beta \frac{\mathcal{E}_{it}(1 + \tilde{r}_{t+1}) - (1 + r)}{\sigma^2} W_t(i) di = Q_t K_{t+1}. \quad (28)$$

It is this condition that links the stock market to the real economy. We can apply the definition (7), as well as (26) and use the fact that all capitalists will hold the same wealth in equilibrium to get

$$\int_0^1 \mathcal{E}_{it} \left(\frac{Q_{t+1}(1 - \delta) + D_{t+1}}{Q_t} \right) di = 1 + r + \omega_t \sigma^2, \quad (29)$$

where ω_t is defined in equation (17) and represents the aggregate degree of leverage required in order to finance the domestic capital stock. In equilibrium, the average capitalist holds a share ω_t of her wealth in stocks. The left hand side of (29) is the market expectation of stock returns; the right hand side is the required return that investors demand given the risk that they are exposed to. The equity premium, $\omega_t \sigma^2$, rises with the conditional variance of stock returns and with the amount of leverage required to hold the domestic capital stock.

Any error in aggregate expectations has two important channels through which it affects the real side of the model. First, it causes a temporary misallocation of capital by distorting Q_t and aggregate investment (14). Second, the rise in the conditional variance of returns implied by excess volatility raises the equity premium and with it the expected dividend demanded by capitalists in general equilibrium. While the former channel mainly influences the dynamics of the model, the latter channel has a direct effect on the stochastic steady state. We discuss each in turn.

4.1 Distortion of Capital Accumulation

Definition 4.2

The stochastic steady-state is the level of capital, bonds, and prices at which those quantities do not change in unconditional expectation.

In the simplified version of the model we are able to obtain a closed form solution for the stochastic steady state and thus analytically show the following result:

Proposition 4.3

The equilibrium has a unique stochastic steady state iff $\beta \leq \frac{1}{1+r}$. At the stochastic steady state the aggregate degree of leverage is

$$\omega_o = \sqrt{\frac{1}{\sigma^2} \left(\frac{1-\beta}{\beta} - r \right)}; \quad (30)$$

and the stochastic steady state capital stock is characterized by

$$(1 + \delta\chi) (r + \omega_o\sigma^2 + \delta) = F_K(K_o, L). \quad (31)$$

Proof. See Appendix C. ■

The intuition for the first result is simple: If the time discount factor is larger than $\frac{1}{1+r}$, investors are so patient that even those holding a perfectly riskless portfolio containing only bonds would accumulate wealth indefinitely. In that case, no stochastic steady state can exist. However, if $\beta \leq \frac{1}{1+r}$, there exists a unique value ω_o at which the average capitalist has an expected portfolio return that exactly matches his time discount factor: $\beta = (1 + r + \omega_o^2\sigma^2)^{-1}$. At this value, there is no expected growth in consumption and the economy is at its stochastic steady state.²¹

The second result, (31), follows directly from applying the steady state to equation (29). On the left hand side, $1 + \delta\chi$ is the market price of a unit of capital at the stochastic steady state. This is multiplied with the required return to capital: the risk free rate plus the risk premium and the rate of depreciation. At the stochastic steady state, the required return on one unit of capital must equal the expected dividend, which is precisely the expected marginal product of capital (on the right hand side of the equation). This brings us to the one of the main results of this paper:

Proposition 4.4

A rise in the conditional variance of stock returns unambiguously depresses the stochastic steady

²¹Conversely we can determine the stochastic steady state wealth of our economy relative to the value of its capital stock by choosing an appropriate time discount factor. We shall make use of this feature when we calibrate the model in section 5.

state level of capital stock and output.

$$\frac{\partial K_o}{\partial \sigma} < 0$$

Proof. We use (30) to eliminate ω_o in (31) and take the total differential, see Appendix C for details. ■

The higher the risk of investing in stocks, the higher is the risk premium demanded by capitalists. A higher risk premium implies higher dividends at the stochastic steady state and, with a neoclassical production function, a lower level of capital stock. Less installed capital in turn implies lower production. The conditional variance of stock returns thus has a *level* effect on the amount of capital accumulated at the stochastic steady state. It follows immediately that excess volatility in the stock market depresses output at the stochastic steady state.

Interestingly, this level effect may operate even if the stock market seems to have little influence on the allocation of capital in the economy:

Corollary 4.5

A rise in the conditional variance of stock returns depresses the stochastic steady state level of output even if the sensitivity of the capital stock with respect to stock prices is low.

Proof. From (14) we have that $\frac{\partial(I_t/K_t)}{\partial Q_t} = \frac{1}{\chi}$. The sensitivity of physical investment as a share of the existing capital stock with respect to the stock price is fully determined by the adjustment cost parameter χ . From (31) and (30) we have that $\frac{\partial^2 F_K(K_o, L)}{\partial \sigma^2 \partial \chi} = \delta \sqrt{\frac{1}{\sigma^2} \left(\frac{1-\beta}{\beta} - r \right)} > 0$. ■

If the adjustment cost parameter χ is sufficiently large, the stock market in this economy may appear as a “sideshow” (Morck, Shleifer, and Vishny (1990)) in the sense that a given change in the stock price has little influence on investment. To the casual observer it may therefore seem as though excess volatility in stock returns has little influence on the real economy. However, a low responsiveness of physical investment to the stock price is uninformative about the impact that excess volatility has on the stochastic steady state. Excess volatility may cause a large depression of output at the stochastic steady state while leaving virtually no evidence to the econometrician. Since our model does not exempt replacement investments from capital adjustment costs, the impact of an incremental rise in stock market volatility on the stochastic steady state level of capital actually *rises* with χ , implying that excess volatility may actually have a larger effect on the stochastic steady state in economies in which the stock market appears to be a “sideshow”.

Finally, the volatility of stock returns has an important implication for the distribution of income in the economy:

Corollary 4.6

A rise in the conditional variance of stock returns unambiguously lowers wages and raises dividends at the stochastic steady state.

Proof. The result follows directly from (11), (12) and proposition 4.4. ■

Excess volatility may paradoxically raise the incomes of stock market investors: At lower levels of K , dividends rise relative to wages, increasing the return to each unit of capital. Over some range, such a rationing raises the total payments to capital. As the conditional variance of stock returns rises, it pushes the economy towards higher dividends, compensating capital for the loss of aggregate output at the expense of payments to labor. In the simplified version of our model, excess volatility in stock returns may thus work like a coordination device that allows capitalists to ration the capital stock and thereby earn monopoly rents on their assets.

4.2 Dynamics of the Model

We can best understand the dynamic effects of the aggregate error in market expectations, ϵ_t , by comparing the near-rational expectations equilibrium with the rational expectations equilibrium in which all investors behave fully rationally. There are two reasons why the conditional variance of stock returns is lower in the rational expectations equilibrium. First, because it has no noise in the equilibrium price; and second because the absence of noise in the equilibrium price enhances the stock market's ability to aggregate information,

$$\sigma^* < \sigma,$$

where we denote the variables pertaining to the rational expectations equilibrium with an asterisk. From Proposition 4.4, it follows that the stochastic steady state level of capital, output, and consumption is higher in the rational expectations equilibrium.

Solving the dynamics of the model requires a computational algorithm that we discuss in section 5. However, we can gain some intuition from the simplified version of our model. Equations (2), (11), (14), (29), and the standard transversality condition jointly determine the market price of capital. Every vector of state variables and shocks is therefore associated with a unique stock price. In the rational expectations equilibrium, the market price of capital equals its fundamental value. In the near-rational expectations equilibrium near-rational errors may cause large departures of the market price of capital from its fundamental value. Through the arbitrage performed by the investment goods sector, the error then passes into physical investment, causing a temporary misallocation of capital.

Regardless of initial conditions and of whether capitalists behave near-rationally or not, the economy transitions to a unique stochastic steady state in expectation. To understand this, imagine an economy that is at its stochastic steady state and receives a positive shock. Capitalists will save a fraction of the rise in dividends and are now on average richer than they were before. This implies that the aggregate portfolio share required to finance the domestic

capital stock in the following period falls, $\omega_{o+1} < \omega_o$. As capitalists are now less leveraged, they require a lower risk premium in the next period. Expected returns therefore tend to be lower following a positive shock and higher following a negative shock. Equilibrium returns thus exhibit negative autocorrelation and thereby generate stationary dynamics.²²

To summarize, the near-rational expectations equilibrium of the simplified version of our model exhibits a higher volatility of returns around a lower stochastic steady state level of capital and output. In unconditional expectation, the returns to capital are higher and wages are lower than in the rational expectations equilibrium. As we show below, all of these conclusions carry over to the full version of the model.

5 Quantifying Welfare Cost

In this section we return to the full version of our model and quantify the welfare cost of the near-rational errors made by households. To this end, we first derive a standard welfare metric, based on a simple experiment in which near-rational behavior is purged from financial markets and the economy transitions to the stochastic steady state of the rational expectations equilibrium. We then briefly describe the computational algorithm used to solve this problem and calibrate the model to the data.

5.1 Welfare Calculations

Consider an economy that is at the stochastic steady state of the near-rational expectations equilibrium and suppose that at time 0, there is a credible announcement that all households henceforth commit to fully rational behavior until the end of time. Immediately after the announcement, the conditional variance of stock returns falls and households require a lower risk-premium for holding stocks. The stochastic steady state levels of capital and output rise. Although the economy does not jump to the new stochastic steady state immediately, it accumulates capital over time and converges to it in expectation. Over the adjustment process, output rises, wages rise and returns to capital fall. Aggregate consumption increases not only due to the rise in output, but may increase further due to a fall in capital adjustment costs incurred. Finally, households may now enjoy smoother consumption due to the reduced volatility of the capital stock.

Formally, we ask by what fraction λ , we would have to raise the average household's consumption in order to make it indifferent between remaining in the near-rational expectations

²²There is a large body of literature discussing the non-stationarity of small open economy models (see for example Schmitt-Grohé and Uribe (2003)). The issue of non-stationarity is, however, a consequence of the linearization techniques typically employed to solve these models and *not* an inherent feature of the small open economy setup. Since we solve our model using higher order expansions we obtain stationary dynamics.

equilibrium and transitioning to the stochastic steady state of the rational expectations equilibrium. λ then indicates the magnitude of the welfare loss attributable to excess volatility as a fraction of total consumption. It is defined as follows:

$$E \int_0^1 \sum_{t=0}^{\infty} \beta^t \log((1 + \lambda)C_t(i)) di \equiv E \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t^*(i)) di. \quad (32)$$

From (32) we can see that welfare losses may result either from a lower *level* of consumption or from a higher *volatility* of consumption. Given our previous discussion, we can identify three channels through which excess volatility can affect welfare: (1) A change in the level of consumption due to a distortion in the stochastic steady state level of capital; (2) A change in the level of consumption due to excess capital adjustment costs; (3) A change in the volatility of consumption. In appendix D, we derive fractions λ^Δ , λ^x and λ^σ which quantify the relevance of each of these channels respectively. We have that $1 + \lambda = (1 + \lambda^\Delta)(1 + \lambda^x)(1 + \lambda^\sigma)$.

5.2 Numerical Solution

The numerical solution of our model employs perturbation methods in combination with a non-linear change of variables. It proceeds in three stages. First, we expand the conditions of optimality around the deterministic steady state. Second, we employ the non-linear change of variables described in section 3.1 in order to bring the equilibrium conditions of the model into a form which allows us to solve for conditional expectations in closed form. Finally, we make a natural guess for the equilibrium price function, solve for conditional expectations taking equilibrium prices as given, and verify the validity of the guess as described in section 3.2.

For the first step, we obtain two conditions of optimality from (5) and stack them:

$$\mathcal{E}_{it} \begin{pmatrix} C_t(i)^{-1} - \beta \left[C_{t+1}(i)^{-1} (1 + \tilde{r}_{t+1}) \right] \\ C_t(i)^{-1} - \beta \left[C_{t+1}(i)^{-1} \right] (1 + r) \end{pmatrix} = 0 \quad (33)$$

We then plug in for stock returns, individual's budget constraints, optimal investment, wages and dividends. Ultimately, we obtain two functions of known and unknown state variables and shocks which characterize the optimal behavior of the individual.

We then obtain two equilibrium conditions of the form given in (18) by solving the Euler equations for the optimal policies and imposing market clearing. We then solve for the deterministic steady state of the model and begin with a higher-order expansion in state variables and shocks around this point (We use a fourth order expansion to generate the results below). The crucial step which gets us back to a stochastic economy is to build at least a second-order

expansion in the standard deviation of η and in the standard deviation of the conditional expectation of η . Financial risk thus affects the economy through the second moments of shocks. For details on perturbation methods see Judd (2002).

5.3 Calibration

Our main objective in this paper is to explore the fragile interaction between the aggregation of information in financial markets and the macroeconomy. We have therefore refrained from complicating the analysis by adding state-of-the-art features of calibrated real business cycle models and of calibrated macro-finance models. For example, none of the features of our model are geared towards matching the equity premium puzzle or related puzzles in the data. The calibrations below should therefore not be viewed as a moment-matching exercise, although endogenous generation of excess volatility in stock returns may be an interesting avenue to explore in this regard.²³ Instead, we focus on the more modest goal of establishing conservative estimates of (1) the amount of excess volatility in stock returns that can plausibly be generated by near-rational behavior and (2) the order of magnitude of aggregate welfare losses they may cause.

In our standard specification we set the standard deviation of $\tilde{\epsilon}$ to a very low level as to ensure that the losses of individual households due to their near-rational errors remain economically small; we set $\frac{\sigma_{\tilde{\epsilon}}}{\sigma_{\eta}} = 0.01$. We choose an adjustment cost parameter of $\chi = 1$, a risk free rate of $r = 0.03$, and a rate of depreciation of $\delta = 0.3$. We pick the time discount factor β such that the entire capital stock is owned by domestic households at the stochastic steady state of the near-rational expectations equilibrium, $\omega_o = 1$. Finally, we choose a Cobb-Douglas production technology with a capital share of $\frac{1}{3}$. Since our economy is scale-independent, we can normalize labor supply to one without loss of generality. Finally, we set $\sigma_{\nu} = 15$ and choose σ_{η} to match the standard deviation of stock returns to 0.18, which conforms to long-run international data (Campbell (2003)).

6 Results

For our standard specification we obtain an excess volatility of stock returns of 32%, and a compensating variation of 2.53% of consumption. Households would thus be willing to give up 2.53 % of their consumption if they could eliminate near-rational behavior from the economy and thereby lower the conditional standard deviation of stock returns by 32%.

Figure 3 plots excess volatility in stock returns for the parameters given above and a range of σ_{ν} . Excess volatility in stock returns rises monotonically with the dispersion of information until

²³For such an application it may be attractive to consider signals about future returns instead of signals about future productivity. Such a model could generate even larger variability in equilibrium stock returns.

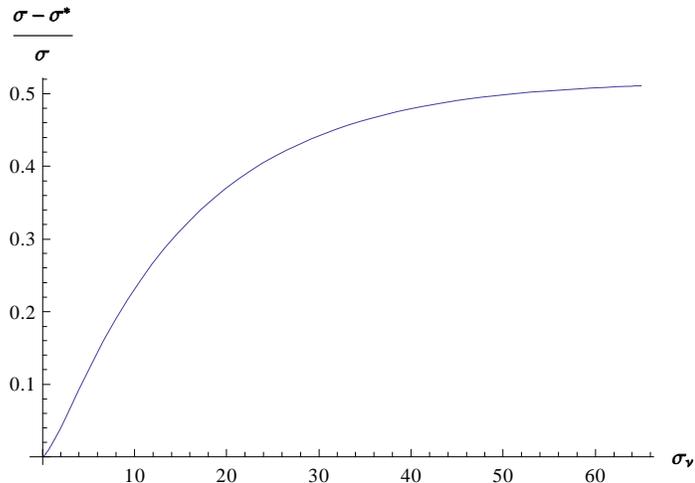


Figure 3: Excess volatility in stock returns plotted over a range of σ_ν .

it plateaus around 51% for very high values of σ_ν . The most striking result from our simulations is that the welfare cost of near-rational investor behavior is very large, even for moderate levels of excess volatility in stock returns. Using the same comparative static over σ_ν , Figure 4 plots the compensating variation for households over a range of levels of excess volatility. At the low end, when dispersion of information is relatively low and excess volatility accounts for 20% of the conditional standard deviation of stock returns, aggregate welfare losses amount to roughly 0.6% of consumption. At the high end, when excess volatility reaches 40%, the compensating variation is around 2.8% of consumption.

Our estimates for the welfare losses due to excess volatility in stock returns exceed even relatively high estimates of the costs of business cycles (see for example Alvarez and Jermann (2005)). We can gain some intuition for why this is the case from Figure 5. It plots the time path of two economies that start at the stochastic steady state of the near-rational expectations equilibrium. The solid line gives the evolution of the capital stock of an economy that remains in the near-rational expectations equilibrium. The dashed line does the same for an economy in which all households behave fully rationally from time 0 onwards. A standard cost of business cycles calculation as in Lucas (1987) is equivalent to calculating the gain from putting a straight line through the oscillations in the near-rational expectations equilibrium. However, the economy in the rational expectations equilibrium does not merely have a lower variance in its capital stock but it also converges to a higher steady state level of output and consumption.

Figure 6 plots the comparative static of the compensating variation with respect to the adjustment cost parameter χ . At our baseline specification with $\chi = 1$, the covariance of stock returns with investment is 0.022. At higher levels of χ , the sensitivity of investment to any given change in stock returns decreases and vice versa. At $\chi = 2$ the covariance is 0.011 and

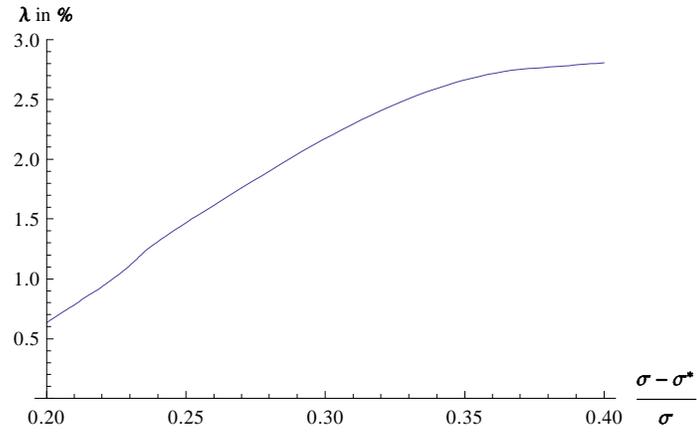


Figure 4: Compensating variation for eliminating all present and future near-rational behavior and transitioning to the rational expectations equilibrium.

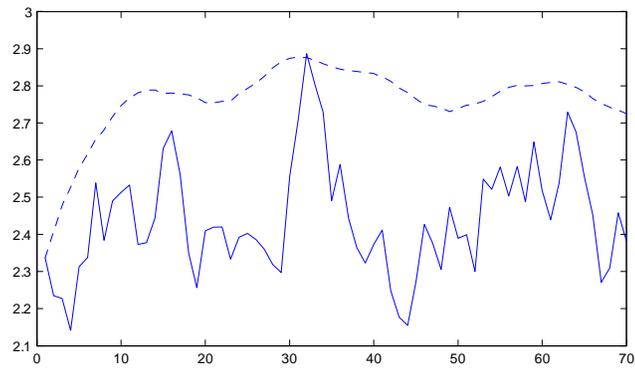


Figure 5: Time paths of the capital stock for an economy remaining in the near-rational expectations equilibrium (solid line) and an economy in which all households behave fully rationally from time 0 onwards (dashed line).

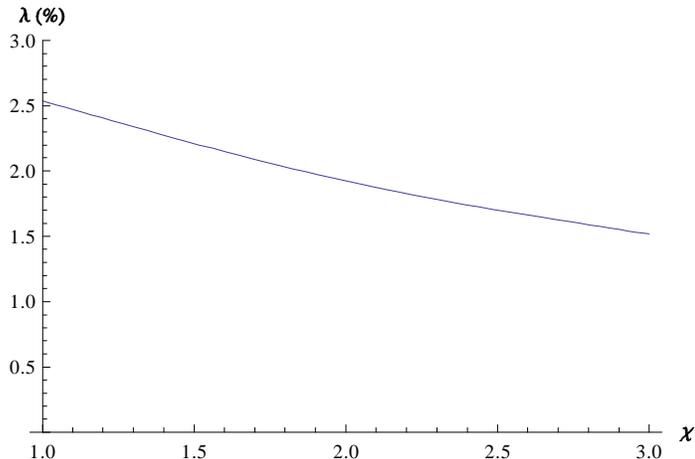


Figure 6: Compensating variation for eliminating all present and future near-rational behavior and transitioning to the steady state of the rational expectations equilibrium plotted over the adjustment cost parameter χ . At $\chi = 1$ the covariance between stock returns and capital investment is 0.022. It falls as χ rises. At $\chi = 3$ the covariance is 0.007.

at $\chi = 3$ it is 0.007. By varying the adjustment cost parameter we can therefore compare the welfare losses caused by excess volatility at different levels of observed co-movement between the stock market and investment. Strikingly, welfare losses change little when the covariance between stock returns and physical investment is lower. The extent to which the stock market and investment co-move thus gives very little indication of the magnitude of welfare losses caused by excess volatility. (As a case in point, there are parameter combinations for which λ actually rises with χ .)

In Table 1, we return to the simplified version of our model in order to assess the impact of excess volatility in stock returns on the welfare of agents that have access to the stock market (capitalists), and those that do not (workers). Column 1 of Table 1 decomposes overall welfare losses by type of agent. [The numbers in the table below are from an older calibration in which we chose σ_v to match a level of excess volatility of 50%, which is why total losses are substantially higher than those given in the standard specification above.] The first line gives the compensating variation for the entire household, assuming costless side payments between the members of the household. The second line gives the compensating variation for capitalists. The negative value indicates that capitalists as a group actually gain from excessively volatile stock returns. These gains are attributable to the fact that returns to capital are higher in the near-rational expectations equilibrium, as the market compensates capitalists for the additional risk they bear. However, the gains enjoyed by capitalists are by their very nature insufficient to make up for the losses suffered by workers. (Recall that we have chosen a capital share of $1/3$, which is why there are aggregate welfare losses although the compensating variation for

capitalists is far below zero.)

The remaining columns give the welfare losses attributable to the distortion in capital accumulation, λ^Δ , to excess adjustment costs, λ^x , and to changes in the volatility of consumption, λ^σ , respectively. Note that the latter two channels combined account for only 0.14 percentage points of the 3.89% overall welfare losses. The critical difference between the near-rational expectations equilibrium and the rational expectations equilibrium is clearly in the distortion of capital accumulation, which accounts for the lion share of the losses incurred by households. Compensating households for the losses suffered through this channel would require a rise in their consumption of 3.75%.

	(1)	(2)	(3)	(4)
	Total, λ	Distortion of capital accumulation λ^Δ	Excess capital adjustment costs λ^x	Volatility of consumption λ^σ
Aggregate	3.89	3.75	0.10	0.04
Capitalists	-13.45	-13.91	0.54	0.01
Workers	7.39	7.36	0.00	0.03

Table 1: Decomposition of welfare losses in percent of consumption by channel and type of agent.

6.1 Closed Economy

In the closed economy version of our model the interest rate r becomes an endogenous variable and bonds are in zero net supply, $B_t = 0$. The dynamics of the model are slightly more involved in the closed economy case as the capital stock at the stochastic steady state of the rational expectations equilibrium may be either higher or lower. This is due to the precautionary savings motive which may or may not dominate the effect of a higher risk-premium. Nevertheless the basic economic intuition holds: Any distortion in capital accumulation causes a distortion in the level of consumption; and any distortion in the level of consumption causes first-order welfare losses.

We calibrate the closed economy version to the parameters given in the standard specification above, and choose σ_η and σ_ν to match a standard deviation of stock returns of 0.18 with excess volatility of 32%. The compensating variation for eliminating near-rational behavior in this specification is 2.46% of consumption.

7 Conclusion

This paper showed that, excess volatility in stock returns may arise and drastically reduce welfare even if the stock market appears to be efficient and disconnected from the real economy. In our model, each household has some private information about future productivity. As individual investors trade in financial markets, prices come to reflect the information held by all market participants. If stock prices reflect information, investors have an incentive to learn from the equilibrium price and to update their expectations accordingly. But if investors watch the equilibrium price, then anything that moves the equilibrium price has an impact on the expectations held by all market participants.

In our model, stock market investors make small correlated errors when choosing their financial portfolio. These errors are amplified as households rationally inform on the equilibrium price when forming their expectations. If information is sufficiently disperse, arbitrarily small errors on the part of stock market investors may result in large amounts of excess volatility in stock returns. While individual investors suffer only small losses due to slight imbalances in their portfolios, the macroeconomic impact of the resulting excess volatility in stock returns may be large: Higher volatility in stock returns induces investors to demand higher risk premia for holding stocks. Higher risk premia in turn distort the level of capital installed at the stochastic steady state. Through its effect on capital accumulation, excess volatility in stock returns causes costly (first-order) distortions in the level of consumption and large aggregate welfare losses.

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Appendix

A Equilibrium Expectations

A.1 Non-linear Change in Variables

Lemma A.1

The market expectation of $g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})$ on the right hand side of (18) can be written in the following form

$$\int \mathcal{E}_{it} \left(g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1}) \right) di = h_{S_t^k} \left(\int (E[\eta_{t+1}|s_t(i), Q_t] di + \tilde{\epsilon}_t) di \right)$$

where $h_{S_t^k}(\cdot)$ depends solely on a vector of known state variables and moments, S_t^k , as well as the market expectation of next period's productivity conditional on the information set at time t .

To see this result, take an infinite order Taylor series expansion of $g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})$ in K_{t+1} , B_t , η_{t+1} , $E[\eta_{t+2}|s_t(i), \log(Q_t)]$, σ_η , and take the expectation conditional on $s_t(i)$ and Q_t . This gives us a series of terms depending on K_{t+1} , B_t , and σ_η , which are known at time t . Moreover, we get a series of terms depending on the conditional expectation of η_{t+2} . Since η_{t+2} is unpredictable for an investor at time t , the first-order term is 0, and all the higher-order terms depending on $E[\eta_{t+2}|s_t(i), Q_t]$ are just cumulants of the unconditional distributions of η and $\tilde{\epsilon}$. The only interesting terms are then those depending on η_{t+1} . We can write

$$\mathcal{E}_{it}[g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})] = \sum_{j=0}^{\infty} c_j(K_{t+1}, B_t) \mathcal{E}[(\eta_{t+1} - E[\eta_{t+1}])^j | s_t(i), Q_t],$$

where the coefficients $c_j(K_{t+1}, B_t)$ involve all the terms depending on the K_{t+1} , B_t , σ_η , and the higher cumulants of η and $\tilde{\epsilon}$. Next, take the term in the expectations operator on the right hand side and expand it to get

$$\begin{aligned}
& \mathcal{E}[(\eta_{t+1} - E[\eta_{t+1}])^j | s_t(i), Q_t] \\
&= \mathcal{E}[(\eta_{t+1} - \mathcal{E}[\eta_{t+1} | s_t(i), Q_t]) + (\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] - E[\eta_{t+1}])]^j | s_t(i), Q_t] \\
&= \sum_{k=0}^j \binom{j}{k} \mathcal{E} [(\eta_{t+1} - \mathcal{E}[\eta_{t+1} | s_t(i), Q_t])^k (\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] - E[\eta_{t+1}])^{j-k} | s_t(i), Q_t] \\
&= \sum_{k=0}^j \binom{j}{k} \mathcal{E} [(\eta_{t+1} - \mathcal{E}[\eta_{t+1} | s_t(i), Q_t])^k | s_t(i), Q_t] (\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] - E[\eta_{t+1}])^{j-k} \\
&= \sum_{k=0}^j \binom{j}{k} m(k) (\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] - E[\eta_{t+1}])^{j-k},
\end{aligned}$$

where $m(k) = \mathcal{E} [(\eta_{t+1} - \mathcal{E}[\eta_{t+1} | s_t(i), Q_t])^k | s_t(i), Q_t]$. Now we can use the fact that the operator \mathcal{E} is a rational expectations operator in which the probability density function of η has been shifted by $\tilde{\epsilon}$. This means that we can replace

$$\mathcal{E} [(\eta_{t+1} - \mathcal{E}[\eta_{t+1} | s_t(i), Q_t])^k | s_t(i), Q_t] = E [(\eta_{t+1} - E[\eta_{t+1} | s_t(i), Q_t])^k | s_t(i), Q_t] \text{ for all } k,$$

where for $k = 1$, the expression collapses to zero. $m(k)$ is then just the k -th moment of the conditional distribution of η .

The conditional expectation that households hold of all higher moments of η_{t+1} is thus a non-linear function of their conditional expectation (the first moment) of η_{t+1} and all higher conditional moments, $m(k)$. However, since η_{t+1} is normally distributed, we know that its conditional distribution must also be normal. Therefore all the higher conditional moments depend only on the conditional variance and on known parameters. Moreover, the conditional variance is constant.

We can now collect terms in the expression above to get

$$\int \mathcal{E}_{it} (g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1})) di = \int \sum_{j=0}^{\infty} c_j(K_{t+1}, B_t) \left(\sum_{k=0}^j \binom{j}{k} m(k) (\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] - E[\eta_{t+1}])^{j-k} \right) di \tag{34}$$

The last step is to use (9) and (22) in combination with (4) and integrate over households to write

$$\mathcal{E}[\eta_{t+1} | s_t(i), Q_t] = A_1 \nu_t(i) + \int \mathcal{E}[\eta_{t+1} | s_t(i), Q_t] di,$$

where $A_1 \nu_t(i)$ is the weight households put on their private signal multiplied with the error they receive in their private signal. This term represents the only source of idiosyncratic variation

in household expectations. We then substitute this expression into (34) and expand the sum in its polynomial terms. We then integrate over households. In the resulting expression, all terms containing $\nu_t(i)$ give us the unconditional moments of the distribution of ν , which is known. Finally, we can define the resulting expression on the right hand side as $h_{S_t^k}(\int \mathcal{E}_{it}[\eta_{t+1}]di)$.

The only remaining piece of the puzzle is then to obtain the conditional expectation and the conditional variance of η_{t+1} , as well as the coefficient A_1 . See section 3.2 for a derivation of the conditional expectation and of A_1 . Appendix A.4 gives the conditional variance. ■

Moreover, we can show computationally that $h_{S_t^k}(\cdot)$ is invertible with

$$h_{S_t^k}(0) = 0 \quad h'_{S_t^k}(\cdot) > 0 \quad h_{S_t^k}(\infty) = \infty. \quad (35)$$

and that $g_{S_t^k}(Q_t)$ is an invertible function in the equilibrium stock price Q_t with the same properties as (35).

Using A.1, we can re-write equation(18) in the linear form

$$\hat{q} = \int E(\eta_{t+1}|\hat{q}_t, s_t(i), K_t, B_t, \eta_t) di + \tilde{\epsilon}_t,$$

where $\hat{q} \equiv h_{S_t^k}^{-1}(g_{S_{t+1}^k}^R(Q_{t+1}, C_{t+1}))$. See Mertens (2009) for a more detailed derivation of these results.

A.2 Proof of Proposition 3.1

Matching coefficients between (23) and (21) yields three equations: $A_0 + A_2\pi_0 = \pi_0$, $A_1 + A_2\pi_1 = \pi_1$, and $1 + A_2\gamma = \gamma$. Solving the three equations and three unknowns yields

$$\pi_0 = \frac{A_0}{1 - A_2}, \quad (36)$$

$$\pi_1 = \frac{A_1}{1 - A_2}, \quad (37)$$

and

$$\gamma = \frac{1}{1 - A_2}. \quad (38)$$

A.3 Proof of Proposition 3.2

The vector $(\eta_{t+1}, s_t(i), \hat{q}_t)$ has the following variance covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_\eta^2 & \sigma_\eta^2 & \pi_1 \sigma_\eta^2 \\ \sigma_\eta^2 & \sigma_\eta^2 + \sigma_\nu^2 & \pi_1 \sigma_\eta^2 \\ \pi_1 \sigma_\eta^2 & \pi_1 \sigma_\eta^2 & \pi_1^2 \sigma_\eta^2 + \gamma^2 \sigma_\epsilon^2 \end{pmatrix}$$

Applying the projection theorem yields the coefficients A_1 and A_2 that correspond to the rational expectation of η_{t+1} given $s_t(i)$ and \hat{q}_t in (22):

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \sigma_\eta^2 & \pi_1 \sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \pi_1 \sigma_\eta^2 \\ \pi_1 \sigma_\eta^2 & \pi_1^2 \sigma_\eta^2 + \gamma^2 \sigma_\epsilon^2 \end{pmatrix}^{-1},$$

yielding

$$A_1 = \frac{\gamma^2 \sigma_\eta^2 \sigma_\epsilon^2}{\gamma^2 \sigma_\nu^2 \sigma_\epsilon^2 + \sigma_\eta^2 (\pi_1^2 \sigma_\nu^2 + \gamma^2 \sigma_\epsilon^2)}, \quad A_2 = \frac{\pi_1 \sigma_\eta^2 \sigma_\nu^2}{\gamma^2 \sigma_\nu^2 \sigma_\epsilon^2 + \sigma_\eta^2 (\pi_1^2 \sigma_\nu^2 + \gamma^2 \sigma_\epsilon^2)}. \quad (39)$$

These coefficients are still functions of endogenous variables π_1 and γ . Combining them with equations (37) and (38) yields the following closed-form solutions:

$$\gamma = \frac{1}{6\sigma_\eta^4} \left[\begin{aligned} & 2\sigma_\eta^2 (\sigma_\eta^2 - 2\sigma_\nu^2) + \frac{22^{1/3} \sigma_\eta^4 (\sigma_\eta^2 + \sigma_\nu^2)^2 \sigma_\epsilon^2}{(27\sigma_\eta^{12} \sigma_\nu^2 \sigma_\epsilon^4 + 2\sigma_\eta^6 (\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^6 + 3\sqrt{3} \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)})^{1/3}} \\ & + \frac{2^{2/3} (27\sigma_\eta^{12} \sigma_\nu^2 \sigma_\epsilon^4 + 2\sigma_\eta^6 (\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^6 + 3\sqrt{3} \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)})^{1/3}}{\sigma_\epsilon^2} \end{aligned} \right]$$

and

$$\begin{aligned} \pi_1 &= (92^{2/3} \sigma_\eta^{16} \sigma_\nu^2 \sigma_\epsilon^6 + 92^{2/3} \sigma_\eta^{14} \sigma_\nu^4 \sigma_\epsilon^6 + 2^{2/3} \sqrt{3} \sigma_\eta^4 \sigma_\epsilon^2 \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)}) \\ &+ 2^{2/3} \sqrt{3} \sigma_\eta^2 \sigma_\nu^2 \sigma_\epsilon^2 \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)}) - 92^{1/3} \sigma_\eta^{12} \sigma_\nu^2 \sigma_\epsilon^4 (\Psi)^{1/3} \\ &- 2^{1/3} \sqrt{3} \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)} (\Psi)^{1/3} + 6\sigma_\eta^{10} \sigma_\epsilon^2 (\Psi)^{2/3} / (6\sigma_\eta^{10} \sigma_\epsilon^2 (\Psi)^{2/3}), \end{aligned}$$

where $\Psi = 27\sigma_\eta^{12} \sigma_\nu^2 \sigma_\epsilon^4 + 2\sigma_\eta^6 (\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^6 + 3\sqrt{3} \sqrt{\sigma_\eta^{18} \sigma_\nu^2 \sigma_\epsilon^8 (27\sigma_\eta^6 \sigma_\nu^2 + 4(\sigma_\eta^2 + \sigma_\nu^2)^3 \sigma_\epsilon^2)}$. Given these results

$$\lim_{\sigma_\nu \rightarrow \infty} \frac{\text{var}(\gamma \tilde{\epsilon}_t)}{\text{var}(\pi_1 \eta_{t+1})} = \infty$$

can easily be calculated using a mathematical software package.

A.4 Conditional Variance

The projection theorem also gives us the conditional variance of η_{t+1} as

$$\text{var}(\eta_{t+1}|\hat{q}_t, s_t(i)) = \sigma_\eta^2 - \begin{pmatrix} \sigma_\eta^2 & \pi_1\sigma_\eta^2 \end{pmatrix} \begin{pmatrix} \sigma_\eta^2 + \sigma_\nu^2 & \pi_1\sigma_\eta^2 \\ \pi_1\sigma_\eta^2 & \pi_1^2\sigma_\eta^2 + \gamma^2\sigma_\varepsilon^2 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_\eta^2 \\ \pi_1\sigma_\eta^2 \end{pmatrix} \quad (40)$$

$$= \frac{\gamma^2\sigma_\eta^2\sigma_\nu^2\sigma_\varepsilon^2}{\gamma^2\sigma_\nu^2\sigma_\varepsilon^2 + \sigma_\eta^2(\pi_1^2\sigma_\nu^2 + \gamma^2\sigma_\varepsilon^2)}. \quad (41)$$

A closed form solution follows from combining this expression with equations (37) and (38).

A.5 Proof of Proposition 3.3

The derivative $\frac{\partial\pi_1}{\partial\sigma_\varepsilon}$ can easily be calculated from (37). However, the resulting expression is too complex to be reproduced here. The fact that $\frac{\partial\pi_1}{\partial\sigma_\varepsilon} < 0$ can be verified using a mathematical software package.

B Proof of Lemma 4.1

We can re-write (5) in Bellman form:

$$V(W_t(i), \pi_t(i)) = \max_{C_t(i), \omega_t(i)} \log(C_t(i)) + \beta\mathcal{E}_{it} [V(W_{t+1}(i), \pi_{t+1}(i))],$$

where we abbreviate $\pi_t(i) = E_{it}(1 + \tilde{r}_{t+1}) - (1 + r)$. The conditions of optimality are:

$$\frac{1}{C_t(i)} = \beta\mathcal{E}_{it} \left[R_{i,t+1}^p V'(W_{t+1}(i), \pi_{t+1}(i)) \right], \quad (42)$$

$$\mathcal{E}_{it} \left((\tilde{r}_{t+1} - r)(W_t(i) - C_t(i))V'(R_{i,t+1}^p(W_t(i) - C_t(i)), \pi_{t+1}(i)) \right) = 0, \quad (43)$$

and

$$V'(W_t(i), \pi_t(i)) = \beta\mathcal{E}_{it} \left(R_{i,t+1}^p V'(W_{t+1}(i), \pi_{t+1}(i)) \right),$$

where $R_{i,t+1}^p \equiv ((1 - \omega_t(i))(1 + r) + \omega_t(i)(1 + \tilde{r}_{t+1}))$ and V' denotes $\frac{\partial V}{\partial W}$. It follows immediately that

$$\frac{1}{C_t(i)} = V'(W_t(i)). \quad (44)$$

Guess the value function:

$$V_t(W_t(i)) = \kappa_1 \log(W_t(i)) + \kappa_2(\pi_t(i)) + \kappa_3 \quad (45)$$

Verification yields:

$$\begin{aligned}\kappa_1 &= \frac{1}{1-\beta} \\ \kappa_2 &= \frac{1}{1-\beta} \mathcal{E}_{it} \left\{ \sum_{s=1}^{\infty} \beta^s \log(R_{t+s}^{p*}(i)) \right\} \\ \kappa_3 &= \frac{1}{1-\beta} \log(1-\beta) + \frac{\beta}{(1-\beta)^2} \log(\beta),\end{aligned}$$

where R_t^{p*} is the optimized portfolio return. Furthermore, the transversality condition has to hold:

$$\lim_{s \rightarrow \infty} \beta^s \kappa_2(R_{t+s}^{p*}(i)) = 0$$

The first result in Proposition 4.1 follows directly from taking the derivative with respect to $W_t(i)$ in (45) and combining it with (44). For the second result, combine (43) with (45) to obtain

$$(1+r)\mathcal{E}_{it}(R_{t+1}^p(i))^{-1} = \mathcal{E}_{it}\left((1+\tilde{r}_{t+1})(R_{t+1}^p(i))^{-1}\right),$$

take logs on both sides, use the fact that

$$\log \mathcal{E}_{it}(\cdot) = \mathcal{E}_{it} \log(\cdot) + \frac{1}{2} \text{var}(\log(\cdot)),$$

and re-arrange the resulting expression to recover (27).

C Solving for the stochastic steady state

C.1 Proof of Proposition 4.3

If at any time o the economy is at its stochastic steady state, we can write $E_o B_{o+1} = B_o$, $E_o K_{o+1} = K_o$ and $I_o = \delta K_o$, where E_o is the unconditional expectations operator, which conditions only on public information available at time o , $E_o(\cdot) = E(\cdot | Q_o, K_o, B_o, \eta_o)$. From equation (14) it immediately follows that $Q_o = E_o Q_{o+1} = 1 + \delta\chi$. We first calculate the steady state dividend, from which we then back out the steady state capital stock. Finally we derive the steady state value of ω .

From equation (11),

$$D_{t+1} = e^{\eta_{t+1}} F_K(K_{t+1}, L),$$

At the steady state:

$$E_o D_{o+1} = F_K(K_o, L)$$

Taking the unconditional expectation of (29) and plugging in yields

$$r + \omega_o \sigma^2 = -\delta + \frac{1}{1 + \delta\chi} (F_K(K_o, L))$$

and

$$(1 + \delta\chi) (r + \omega_o \sigma^2 + \delta) = F_K(K_o, L).$$

This proves the second statement in Proposition 4.3.²⁴

We now turn to solving for ω_o . The first step is to derive the equilibrium resource constraint for capitalists from (2), (11), (14), (25) and (17): From (17) we get that $W_t - C_t = Q_t K_{t+1} + B_t$ plugging this into (25) yields

$$Q_t K_{t+1} + B_t + C_t = (1 + r)B_{t-1} + (Q_t(1 - \delta) + D_t) K_t.$$

Now we can use (2) to eliminate K_{t+1} :

$$Q_t(1 - \delta) K_t + Q_t I_t + B_t + C_t = (1 + r)B_{t-1} + (Q_t(1 - \delta) + D_t) K_t.$$

This simplifies to

$$Q_t I_t + B_t + C_t = (1 + r)B_{t-1} + D_t K_t. \quad (47)$$

The next step is to re-write (47) in terms of K_o and ω_o . For this purpose note that

$$C_o = (1 - \beta)W_o,$$

$$\beta W_o = K_o(1 + \delta\chi) + B_o,$$

$$B_o = \beta W_o(1 - \omega_o),$$

and

$$(1 + \delta\chi)K_o = \beta W_o \omega_o$$

$$\rightarrow B_o = \frac{1 - \omega_o}{\omega_o} (1 + \delta\chi) K_o$$

Plugging these conditions into (47) and simplifying yields

$$(1 + \delta\chi) \left(\delta + \frac{1 - \beta}{\beta} + \frac{1 - \omega_o}{\omega_o} \left(\frac{1 - \beta}{\beta} - r \right) \right) = F_K(K_o, L) \quad (48)$$

²⁴With a Cobb-Douglas specification and a capital share of α we can further write

$$\left(\frac{(1 + \delta\chi) (r + \omega_o \sigma^2 + \delta)}{\alpha L^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} = K_o. \quad (46)$$

We can eliminate K_o from this equation by substituting in (31). Some manipulations yield

$$\omega_o = \sqrt{\frac{1}{\sigma^2} \left(\frac{1-\beta}{\beta} - r \right)},$$

proving the first statement in Proposition 4.3.

C.2 Proof of Proposition 4.4

Combining (30) and (31) and taking the total differential gives

$$\frac{dK_o}{d\sigma} = \frac{1 + \delta\chi}{F_{KK}(K_o, L)} \left(\frac{1-\beta}{\beta} - r \right)^{.5}.$$

Proposition 4.3 states that a stochastic steady state exists iff $\beta \leq \frac{1}{1+r}$. Proposition 4.4 then follows directly from the fact that $F_{KK}(K_t, L) < 0$.

D Decomposition of welfare losses

This section decomposes households' total welfare loss into components attributable to additional variability of consumption, excess adjustment costs, and a distortion in the level of capital accumulation. Given the parameters of the model and initial conditions K_o, ω_o, B_o (see Appendix C), define the expected utility level of the average household in the near-rational competitive equilibrium U as

$$U = E_o \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t(i)) di,$$

where E_o is the unconditional expectations operator, which conditions only on public information available at time o , $E_o(\cdot) = E(\cdot | Q_o, K_o, B_o, \eta_o)$. Similarly, given the same parameters and initial conditions define the expected utility level U^* of transitioning to the stochastic steady state of the rational expectations equilibrium as

$$U^* = E_o \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t^*(i)) di.$$

We can solve (32) for λ to obtain

$$1 + \lambda = \exp[(E_o U^* - E_o U)(1 - \beta)]. \quad (49)$$

Now define two reference levels of utility: First, one at which households get compensated for the difference in the variability of their consumption in the rational versus the near-rational

expectations equilibrium

$$U^\sigma = E_o \int_0^1 \sum_{t=0}^{\infty} \beta^t \log(C_t(i)) di + \frac{1}{2(1-\beta)} [\text{var}(\log(C)) - \text{var}(\log(C^*))],$$

where we use unconditional variances and abstract from any predictable variation in the conditional variance of consumption. The second reference level of utility we introduce furthermore compensates for the higher average adjustment cost incurred in the near-rational competitive equilibrium; households receive the average excess adjustment costs between the current period and infinity:

$$U^{\sigma,\chi} = E_o \int_0^1 \sum_{t=0}^{\infty} \beta^t \log \left(C_t(i) + \left(\lim_{T \rightarrow \infty} \sum_{\tau=0}^T \frac{\chi}{2T} \left(\frac{I_\tau^2}{K_\tau} - \frac{I_\tau^{*2}}{K_\tau^*} \right) \right) \right) di \\ + \frac{1}{2(1-\beta)} [\text{var}(\log(C)) - \text{var}(\log(C^*))].$$

We know from the discussion in the text that the remainder of the difference in average expected utility between the rational expectations equilibrium and the near-rational expectations equilibrium must be due to a distortion in the stochastic steady state level of capital.²⁵ We can write

$$U^\Delta = U^* - U^{\sigma,\chi}$$

We can now apply these definitions in (32):

$$1 + \lambda = \exp[(U^* - U^{\sigma,\chi} + U^{\sigma,\chi} - U^\sigma + U^\sigma - U)(1 - \beta)]$$

and

$$1 + \lambda = \exp[(U^* - U^{\sigma,\chi})(1 - \beta)] \cdot \exp[(U^{\sigma,\chi} - U^\sigma)(1 - \beta)] \\ \cdot \exp[(U^\sigma - U)(1 - \beta)].$$

This implies that

$$1 + \lambda = (1 + \lambda^\Delta)(1 + \lambda^\chi)(1 + \lambda^\sigma).$$

For the simplified version of our model, all of the definitions above transfer to the case of workers analogously. However, since workers are not concerned with excess adjustment costs we have that $U^{w,\sigma} = U^{w,\sigma\chi}$.

²⁵We subsume the second order effect due to the variability of the capital stock in this category.