Foreign Firms, Domestic Entrepreneurial Skills and Development *

ALEXANDER MONGE-NARANJO
Pennsylvania State University

September 2008.

Abstract

Foreign firms may enhance a developing country’s formation of know-how by exposing or directly transferring local entrepreneurs the productive ideas of developed countries. However, foreign firms may also reduce the domestic entrepreneurs’ incentive to accumulate know-how by increasing their competition and reducing the returns to entrepreneurial skills. It is shown that if externalities drive the formation of skills, after openness, initial conditions determine if a country converges to one of two steady states or to exhibit non-monotone dynamics. If instead, the costs and benefits of skill formation are fully internalized, openness gradually removes the pre-existing sector, generates a new sector of domestic firms, and the country catches up with developed countries. In both models, convergence requires the destruction of pre-existent firms. The implications for empirical work are also discussed.

*alexmonge@gmail.com. I am thankful to Pol Antras, Gadi Barlevi, Paco Buera, Hugo Hopenhayn and Boyan Jovanovic and Andrés Rodríguez-Clare for conversations that help me improve the paper. Comments welcome.
1. Introduction

Would hosting foreign firms lead a developing country to catch up with developed countries? Can this form of openness instead lead the country to lag further behind? Does the presence of foreign firms enhance or impair the accumulation of productive know-how by domestic firms? What is the impact of this openness on the overall welfare of a country? Should the governments of developing nations promote the entry of foreign firms? In this paper I use simple general equilibrium growth models to answer these questions.

Entrepreneurial and managerial know-how can be the limiting factor in a country’s aggregate productivity. They determine the technologies and market opportunities that local firms can efficiently operate and access. Broadly defined, these “skills” can be as simple as knowing key individuals and conventions of a particular market and as sophisticated as the scientific and technological training to coordinate the development, selection and marketing of a new slow-release drug. Like other skills, entrepreneurial know-how is self-productive. Countries with an initial poor supply may never, by themselves, accumulate the amounts required to access the world’s technological frontier.

A country can import entrepreneurial and managerial skills from abroad by allowing foreign firms to operate in the country. Foreign firms can combine their control of domestic labor with know-how that is only available elsewhere in the world. Indeed, this form international trade of skills seems to have gained importance with the increasing multinational activity and foreign direct investment of recent years. Burstein and Monge-Naranjo (2009) quantify significant output and consumption gains for developing countries that host firms from developed countries. Moreover, Antras, Garicano and Rossi-Hansberg (2006) show that the distributional impact of foreign skills go well beyond those implied by standard factor endowment models because they alter the organization of production and the within occupations distribution of income. Yet, by taking skills as fixed endowments, these papers are silent about their accumulation over time and across countries.

Aside from static output gains, the presence of foreign firms can impact the acquisition of the country’s own entrepreneurial and managerial know-how. Diffusion of productive know-how can take place via externalities, as the exposition to new and possibly more advanced ideas from abroad facilitates domestic entrepreneurs in their accumulation of know-how. Diffusion of productive ideas can also take place via implicit or explicit fully internalized transactions as a foreign expert trains a local future firm manager. However, the presence

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1 There is an extensive literature that links the productivity of firms to the quality of their management. See, for example, Kaldor (1934), Lucas (1978), Oi (1983), Prescott and Visscher (1980), and Rosen (1982).
2 Ramondo (2008) and McGrattan and Prescott (2008) show that the gains would be even larger if, instead of skills, productivity is driven by non-rival factors, i.e. that can be used simultaneously in multiple locations.
of foreign firms increase the competition faced by local managers and reduce their total and marginal returns to entrepreneurial skills. This paper studies the aggregate dynamics and welfare consequences for a developing country that opens up to foreign firms.

In the model, production is carried out by teams of workers led by an entrepreneur. The skills of the manager determines the productivity of the team. The model is an OLG economy in which some of the young individuals invest in skills to become the manager—and residual claimant—of a firm when old. Over time, the equilibrium skill formation of young future managers is determined by the set of productive ideas currently implemented by active managers operating in the country. Workers are fixed in their country of origin but managers can move across countries. In the context of the model, a “closed” country is one in which only national managers can lead firms while an “open” country allows free entry of foreign firm leaders. Entry of foreign managers impacts they country by increasing the domestic price of labor, reducing the marginal return to entrepreneurial skills and increasing the set of productive ideas upon which the young can form their future skills.

The presence of externalities is a standard assumption in models of human capital formation, e.g. Romer (1986), Lucas (1988) and Stokey (1991). I use a variant of Stokey (1991) in which the aggregate skills level of old managers impacts the cost of accumulating skills of young potential managers. In this variant of the model, the set of ideas that surround individuals during their formative years is a national public good shaped by all skills effectively used within the country. There is an externality because firm leaders only receive returns for their production activities, and not for their contribution to the ideas circulating in the country. Equally, the presence of foreign firms has an externality that impacts the national public good in an open economy. Hence, the framework integrates quite naturally the mechanism in Findlay (1978), in which foreign firms have positive technology spillovers on domestic firms.

Regardless of initial conditions, closed economies always follow a balanced-growth-path (BGP). A small open economy exhibits a significantly more complex dynamics. Openness to foreign firms has countervailing forces as it reduces both the marginal costs and the marginal returns for the accumulation of skills of domestic entrepreneurs. On one hand, the current entry of foreign firms enhances the set of productive ideas surrounding the forming crop of

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3 In the model entrepreneurial and managerial know-how are equivalent. For models in which they are different see Holmes and Schmitz (1991) and Chari, Golosov, and Tsyvinski (2004).
4 See Klein and Ventura (2004) for an analysis of cross country labor mobility.
5 The emphasis on the cross-border reallocation of management conforms with the observation that multinational firms heavily rely on home expatriates—and home trained individuals—to manage their operations, specially in developing countries (see Chapters 5 and 6 of UNCTAD 1994). It also conforms with the emphasis of the existent literature on firm specific intangible assets for multinational activity (e.g. Barba-Navarretti 2004 and Markusen 2004).
young managers. On the other hand, the foreseen future entry of foreign firms bids up the domestic cost of labor and squeeze total and marginal returns to entrepreneurial skills. As a result, open economies exhibit a form of predatory-prey dynamics because entry is an inverse function of the relative level of domestic entrepreneurial skills. Indeed, an open country can exhibit non-monotone dynamics and there might be two different steady states to which the country can converge depending on initial conditions. In one steady state, the country converges to the skill levels of developed countries (and foreign firms no longer enter). In the other steady state, the country remains forever behind (and foreign firms are always present). It is shown that, regardless of the path followed by the country, it catches up with developed countries if and only if for one period domestic firms are shut-down and the entire production of the country is led by foreign firms. The model also predicts leapfrogging, as the most backward countries not only are more likely to converge to the high steady state, but also to do it more quickly. Finally, it is shown that the extent in which a country gains with openness depends on initial conditions.

Another leading hypothesis for the formation of know-how is that it results from transactions that fully internalize the costs and benefits of all parties involved, e.g. Boyd and Prescott (1987a,b), Chari and Hopenhayn (1991) and Jovanovic and Nyarko (1995).\footnote{Agarwal, R. et al (2004) and Filson and Franco (2006) extend the Chari-Hopenhayn model.} I consider a variant of the Boyd-Prescott model in which the skill formation of a young manager depends on the skills and actions of the leader for whom he works. Both trainers and apprentices must be purposely involved and a well functioning market for skill formation operates.\footnote{Abstracting from private information frictions that lead to inefficiency as in Jovanovic and Nyarko (1995).} No externalities are present, since the skill formation of each young future manager depends solely on the skills of his manager/trainer. Instead of an externality, foreign firms disseminate skills to the host country by directly training the workers under their control.

The internalization of the returns of skills in future production of skills can easily lead to dynamic increasing returns and degenerate solutions.\footnote{Recall that the profit function in a Lucas (1978) is strictly convex. The self-productivity of skills adds to the convexity of the value function.} I provide sufficient conditions for existence, uniqueness and efficiency of a BGP in a closed economy.\footnote{The conditions in Boyd and Prescott (1987) and Prescott and Boyd (1987) are not enough for this result.} In an open economy, the entry of more advanced foreign firms introduces heterogeneity in the population of trainers, which in turn creates heterogeneity of domestic firms. Specifically, openness leads to the emergence of a sector of new-domestic entrepreneurs all of which have the same skill levels of foreign managers. This new sector grows over time and eventually overtakes the entire economy. Along the transition, the progeny of the pre-existing domestic managers shrinks in size and level of skills and then disappears. Regardless of its initial relative backwardness,
an open country fully catches up with developed countries a finite number of periods after opening. Also, regardless of initial conditions, openness is always welfare improving.

The implications of both models complement the negative empirical results on the positive effect of foreign firms on the productivity of existing domestic firms. As in the model with externalities, with internalized diffusion a developing country catches up with developed countries if and only when domestic pre-existent firms (or their progeny) are entirely removed and replaced by a new sector of domestic firms. In both models, the country as a whole can catch up, even if pre-existing firms respond reducing, not increasing, the investments that drive their productivities. This implication is in line with ample evidence (e.g. Aitken and Harrison [1999]) on the absence of positive effects of foreign firms on the productivity of existing domestic firms. But even if spillovers drive aggregate productivity, it is shown that it is quite likely that the negative impact of foreign competition overdoes the positive effect of spillovers.

The remarkable differences in the implications of the two models highlight the importance of distinguishing between externalities and fully internalized transfers of know-how. Some authors (e.g. Javorcik [2004] and Kugler [2005]) have argued for the existence of inter-industry spillovers, specifically, from foreign firms to local suppliers. However, productivity gains might be driven by internalized transfers, not spillovers, since as Javorcik herself reports, foreign firms in her sample were directly involved providing training, equipment and know-how to the local suppliers. At the level of domestic industries, skill formation at the interior of the firm seems to be a major mechanism for aggregate skill formation and dissemination, as indicated by the empirical evidence that links the characteristics and the outcomes of parent firms with their spin-offs. Unfortunately, with the exception of the recent work of Malchow-Møller et. al (2007) and Poole (2006) on multinational firms in Denmark and Brazil, respectively, there has been little empirical work on internalized skill formation by multinational firms.

The paper proceeds as follows. Section 2 lays out the model and other preliminaries. Sections 3 and 4 respectively, study the accumulation of skills via externalities and internalized diffusion, and Section 5 concludes. An appendix contains the proofs.

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10 See Xu (2000) and Alfaro et al. (2006) for a discussion of the empirical findings. While there is some evidence of positive spillovers for developed countries (e.g. Griffith et al 2002), for developing countries most authors come back empty-handed when trying to revert the negative results of Aitken and Harrison (1999).


12 There is however, ample anecdotal. The best known case may be the emergence of a textile sector in Bangladesh after the seed planted by a Korean firm (see Easterly [2001]). Also, some multinationals spend significant resources in training their workers (see UNCTAD [1994]).
2. The Model

This section lays out the environment and some static conditions for the equilibria studied in subsequent sections.

Consider a discrete time, infinite horizon OLG economy, with two-period lived individuals and a single consumption good. The utility $U^t$ of an agent born in period $t$ depends linearly on the consumption $c^t_\tau$ in periods $\tau = t$ and $\tau = t + 1$:

$$U^t = c_t^t + \beta c_{t+1}^t,$$

with $0 < \beta < 1$.

Cohorts are ex-ante identical and represent a continuum of size one. A fraction $\omega \in (0, 1]$ are “potential firm leaders” and a fraction of $1 - \omega$ are “laborers.” Laborers are workers in both periods of life, providing one unit of labor in each period. Potential firm leaders are also workers in their first period of life, but in their maturity they have the alternative of leading a group of workers. Thus, every period there is a minimum mass $2 - \omega$ of workers (the young plus the old laborers) and a maximum mass of $\omega$ of active firm leaders. I shall define $\eta \equiv (2 - \omega)/\omega \geq 1$, the aggregate ratio of workers-per-firm if all old potential managers opt to be active managers.

Output is produce in teams of one leader and $n$ units of labor. If the firm leader has entrepreneurial skills $z$ the team produces

$$y = zn^\alpha,$$

units of the good. As in Lucas (1978) the person specific skills $z$ of the leader determines the productivity of the firm. The span-of-control parameter $\alpha \in (0, 1)$ is the degree of decreasing returns to the amount of labor $n$. I also follow Lucas (1978) in calling these teams “firms” even if they can equally be seen as parts of a conglomerate of teams within the boundaries of the same firm.\(^\text{13}\)

2.1. Notation, Equilibrium Preliminaries and Static Conditions

The leader of a firm is also its residual claimant. As firm leaders set up firms and also control them, I will interchangeably call them “entrepreneurs” or “managers.” Each potential firm leader chooses between remaining a worker or running a firm. In the latter case, he must choose the amount of labor to hire. Foreseeing this decision, a young, potential firm leader must also choose his investment in skills.

Labor and financial markets are frictionless and competitive. The discount factor $\beta$ pins down the one period interest, i.e. $R_t = \beta^{-1}$ for all $t$. Then, the equilibrium is determined by the wage sequence $\{w_t\}_{t=0}^{\infty}$, where $w_t$ is the price of one unit of labor in period $t$. I consider two alternative market clearing conditions: (1) A single country that is closed and initially populated by homogeneous firm leaders, and (2) a small country that is open to entry of managers from a large country with more advanced firm leaders.

Before tackling the accumulation of skills over time, it is convenient to examine the static hiring and occupation decisions involved. Consider a potential firm leader with a level $z > 0$ of skills and facing a wage $w$ for labor. Should he decide to be a firm leader, he would earn a payoff

$$\pi (z, w) = \max_{\{n\}} \{zn^\alpha - wn\}$$

$$= \theta z \frac{1}{1-\alpha} w^{1-\alpha} , \quad (2.1)$$

and hire workers in the amount

$$n^* (z, w) = \left[ \frac{\alpha z}{w} \right] \frac{1}{1-\alpha} , \quad (2.2)$$

where $\theta \equiv \alpha \frac{z}{w} (1-\alpha) > 0$. Notice that, given $w$, $\pi (z, w)$ increases more than proportionally with $z$.

The potential firm leader becomes an active entrepreneur iff $\pi (z, w) \geq w$. This condition is equivalent to $z/w > 1/(\alpha^\alpha [1-\alpha]^{1-\alpha})$, i.e. only if his skills are high enough relative to the wage rate, he opts to be a firm leader.

In what follows, I use the subscript $n$ to indicate a national variable of the “home” country, the subscript $f$ to indicate a variable of the “foreign” country and the subscript $g$ to indicate a “geographic” variable in the home country. Lower cases are used for individual variables and upper cases for aggregate variables.

### 2.1.1. A Closed Economy

Consider an economy in which all potential firm leaders have a uniform level of skills $Z > 0$. There can be two types of equilibria: (1) All potential firm leaders and strictly better-off than workers; (2) potential firm leaders are indifferent between being workers or firm leaders.

In a type 1 equilibrium, the supply of workers is equal to $2 - \omega$ and the supply of managers is equal to $\omega$. Each firm leader demands labor in the amount $[\alpha Z/w]^{1-\alpha}$ which in equilibrium must equal $\eta$. Then, the market-clearing $w$ is

$$w = \alpha Z \eta^{\alpha-1} . \quad (2.3)$$
and each firm leader earns

$$\pi = (1 - \alpha) Z \eta^\alpha. \quad (2.4)$$

In a type 2 equilibrium, wages and entrepreneurial rents are $$w = \pi = \theta^{1-\alpha} Z$$ and each active manager hires $$n = \alpha / (1 - \alpha)$$ workers.$^{14}$

For the rest of the paper I will impose the following condition, which warrants focusing on type 1 equilibrium.

**Condition 1:** \( \eta \geq \frac{\alpha}{1-\alpha}. \)

Under Condition 1, the country’s geographic aggregate output \( Y_g \) is

$$Y_g = Z \omega^{1-\alpha} (2 - \omega)^\alpha, \quad (2.5)$$

which in this context is also equal to aggregate national consumption \( C_n \).

### 2.1.2. A Small Open Economy

Consider now a economy that is open to hosting foreign firms. In this environment, a foreign firm is a production team that combines foreign know-how with domestic labor. I focus on the case of a small developing economy. I assume managers within each country are identical. Finally, “small” means that the country does not affect the equilibrium of the foreign country, and “developing” means that \( Z_n \) is strictly below \( Z_f \).

In equilibrium foreign managers must be indifferent between leading teams in the foreign country or in the home country. In either case they earn a payoff equal to

$$\pi_f = (1 - \alpha) \eta^\alpha Z_f. \quad (2.6)$$

For this to be the case, the entry of foreign managers pushes the domestic wage rate \( w_n \) to catch-up with the foreign wage rate

$$w_f = \alpha \eta^{\alpha-1} Z_f. \quad (2.7)$$

Facing the same prices, at home as in the foreign country, foreign managers operating in the developing country hire \( n_f = \eta \) units of domestic labor. Higher wage rates reduce the payoffs \( \pi_n \) and employment \( n_n \) of national firm leaders. Expressions (2.1) and (2.2) imply that

$$\pi_n = (1 - \alpha) \eta^\alpha Z_n \left[ \frac{Z_n}{Z_f} \right]^{\alpha \over 1-\alpha}, \quad \text{and}$$

$$n_n = \eta \left[ \frac{Z_n}{Z_f} \right]^{1-\alpha}. \quad (2.9)$$

$^{14}$In either case, \( Z \) shifts \( \pi \) proportionally, because own skills impact more than proportionally the payoffs to a firm leader but a higher economy wide \( Z \) increases \( w \) proportionally.
Lastly, recall that domestic firm leaders also have the option of being workers. This option becomes more relevant the lower is $Z_n/Z_f$. Comparing $\pi_n$ with $w_f$ implies that if $Z_n/Z_f$ is below the threshold

$$R_S \equiv \left[ \frac{\alpha}{\eta (1 - \alpha)} \right]^{1-\alpha},$$

then, domestic firm leaders become workers.

Let $m \in [0,1]$ be the fraction of aggregate domestic labor working under foreign management. Since each foreign manager hires $\eta$ local workers, a value $m$ implies a there is an entry of $m/\eta$ of foreign firms. The equilibrium value of $m$ is pinned down by the labor market-clearing condition of the country. First, if $Z_n/Z_f < R_S$, the domestic supply of labor is equal to 2 (the entire population) and $m = 1$ (all domestic workers are controlled by foreign managers). If instead $Z_n/Z_f \geq R_S$, the domestic supply of labor is only $2 - \omega$, since all domestic potential managers are active. In this case, using expression (2.9) and $n_f = \eta$, the domestic labor market-clearing condition is $m (2 - \omega) + \omega \eta \left[ \frac{Z_n}{Z_f} \right]^{\frac{1}{1-\alpha}} = (2 - \omega)$. The resulting equilibrium $m$ is

$$m = \begin{cases} 
1 & \text{if } Z_n/Z_f < R_S \\
1 - \left[ \frac{Z_n}{Z_f} \right]^{\frac{1}{1-\alpha}} & \text{if } Z_n/Z_f \geq R_S.
\end{cases} \quad (2.10)$$

Clearly, when $m < 1$, the entry of foreign management skills is a decreasing of $Z_n/Z_f$, the relative endowment of competing domestic skills.

The country’s geographic aggregate output is

$$Y_g = \begin{cases} 
2\eta^{\alpha-1}Z_f & \text{if } Z_n/Z_f < R_S \\
\omega^{1-\alpha} \left( \frac{2-\omega}{\omega} \right)^{\alpha} Z_f & \text{if } Z_n/Z_f \geq R_S.
\end{cases} \quad (2.11)$$

If $Z_n/Z_f < R_S$, aggregate output in the developing country is independent of $Z_n$, and interestingly, it is higher than when $Z_n/Z_f \geq R_S$. Subtracting foreign profits, aggregate national consumption is

$$C_n = \begin{cases} 
\alpha Y_g & \text{if } Z_n/Z_f < R_S \\
Y_g \left[ \alpha + (1 - \alpha) \left( \frac{Z_n}{Z_f} \right)^{\frac{1}{1-\alpha}} \right] & \text{if } Z_n/Z_f \geq R_S.
\end{cases} \quad (2.12)$$

It can be shown that $C_n$ is higher when $Z_n/Z_f \geq R_S$ than when $Z_n/Z_f < R_S$. Notice also that in the second branch, $C_n$ is increasing in $Z_n$, since output remains the same, but a higher fraction of the profits remain in the country.

2.1.3. Static Gains from Openness

Openness to hosting foreign entrepreneurial skills unambiguously improves the utility of workers. Potential firm leaders are also better off if their earnings as workers are higher than
the entrepreneurial earnings in a closed economy. Thus, openness is a Pareto improvement if $Z_n/Z_f < (R_S)^{1/\omega}$. If no compensation is implemented when $Z_n/Z_f > (R_S)^{1/\omega}$, then openness reduces the utility of domestic firm leaders.

The ability of winners to compensate losers is assessed by comparing aggregate consumption. Using (2.5) and (2.12), the aggregate welfare gains from openness are

$$\frac{C_{n,\text{Open}}}{C_{n,\text{Closed}}} = \begin{cases} \left[ \frac{Z_f}{Z_n} \right] \left[ \frac{2\alpha}{(2-\omega)} \right] & \text{if } Z_n/Z_f < R_S \\ \left[ \frac{Z_f}{Z_n} \right] \left[ \alpha + (1-\alpha) \left( Z_n/Z_f \right)^{1/\omega} \right] & \text{if } Z_n/Z_f \geq R_S. \end{cases} \quad (2.13)$$

The gains are always positive whenever if $Z_n < Z_f$ and can be substantial. 15 For instance, if $\alpha = 0.85$ and $\omega = 0.1$ the aggregate welfare gains $(C_{n,\text{Open}}/C_{n,\text{Closed}} - 1)$ are 23% if $Z_n/Z_f = 75\%$, 269% if $Z_n/Z_f = 25\%$ and 823% if $Z_n/Z_f = 10\%$.

3. Externalities and the Accumulation of Know-How

Consider now that to be able to master $z'$ units of productive know-how at maturity, a young individual must incur a cost in the current period of

$$Z_g \phi \left( \frac{z'}{Z_g} \right)$$

units of consumption goods. Here $Z_g$ is the productive knowledge in the environment where the young agent live. I adapt the idea of Stokey (1991) the level of skills of older generations impact the cost of younger generations to accumulate skills. As in Stokey (1991) there is an externality since firm leaders are not compensated from their contribution of the set of ideas $Z_g$. In what follows I explore economies closed economies in which $Z_g = Z_n$ and open economies in which $Z_g$ is a function of $Z_n$ and $Z_f$.16

The function $\phi(\cdot)$ has the standard properties of an adjustment cost function: twice continuously differentiable, strictly increasing, strictly convex. I assume that $\phi(0) = 0$. For the rest of the paper I will use the functional form

$$\phi(x) = v_0 \frac{x^{1+v}}{1+v}, \quad (3.1)$$

where both $v_0 > 0$ and $v > 0$. Yet, I keep using the short-hand $\phi(\cdot)$ to condense some of the formulas.

15If $Z_n > Z_f$, the country would instead export entrepreneurial skills to the rest of the world. With openness the country as a whole is better off. since the loss in the utility of workers, are more than compensated by the gains of firm leaders.

16It might be helpful to think of this model as a happy hour diffusion: Imagine that everyday, after work, all the young workers go to a bar for a happy hour. Among more other things –and with objectives different than training– they talk about their work and the ideas they confront every day. After the many happy hours of a typical young person, the set of ideas in the brain of each and every one is $Z_g$. 

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In a perfect-foresight equilibrium, each potential manager can foresee the payoffs \( \pi (z', w') \) attainable with skills \( z' \) with wages \( w' \). Therefore, surrounded by ideas \( Z_g \), the optimal accumulation of a young agent that next period will become and active manager solves

\[
\max_{\{z'\}} \left\{ \beta \pi (z', w') - Z_g \phi \left( \frac{z'}{Z_g} \right) \right\}.
\]

(3.2)

This optimization consist of the difference of two convex functions. In all what follows, I impose the following restriction on parameters:

**Condition 2:** \( v > \frac{a}{1-a} \).

Under this condition, the cost \( \phi (\cdot) \) is “more convex” than the rents \( \pi (\cdot, w') \) as functions of \( z' \). Therefore, the first order conditions are necessary and sufficient. Then, if condition 2 holds, the optimal level of skills for a future active manager is given by the first order condition

\[
\beta \alpha^{\frac{1}{1-a}} \left[ \frac{z'}{w'} \right]^{\frac{a}{1-a}} = v_0 \left[ \frac{z'}{Z_g} \right]^v.
\]

(3.3)

Otherwise, the optimal acquisition of skills is \( z' = 0 \).

**3.1. A Closed Economy**

Assume that everyone in the cohort of mature potential managers command a level of skills \( Z_n \). Since the economy is closed, \( Z_g = Z_n \). In this case, the market clearing wages and rents are given by (2.3 ) and ( 2.4) respectively. Using these expression in (3.3) I obtain that \( G \), the gross growth in the level of skills is:

\[
G \equiv \frac{Z_n'}{Z_n} = \left[ \frac{\beta v_0^a}{v_0} \right]^\frac{1}{a}.
\]

(3.4)

Hereafter I impose the following restriction on parameters:

**Condition 3:** \( \beta \eta^a > v_0 \).

Under Condition 3, \( G > 1 \), and the economy exhibits constant and sustained growth. Notice that the economy does not exhibit transition dynamics. After one period, any pre-existing heterogeneity across managers disappears and the economy locates itself the unique BGP.

However, for these expressions to define an equilibrium, I also need to verify that occupation choices are optimal. Under Condition 1, the potential managers that are old prefer to be active managers. What is left is to verify that the young crop of potential managers will find it optimal to invest in skills and become active managers, obtaining a net present value of income of \( Z_n [\alpha \eta^{a-1} - \phi (G) + \beta (1-\alpha) \eta^a G] \) over remaining a worker in both periods and obtaining \( Z_n [\alpha \eta^{a-1} + \beta \eta^{a-1} G] \).
I impose now the following restriction on parameters:

**Condition 4:** \( v/(1 + v) > \alpha (1 + 1/\eta) \).

Under this condition, the following result can be directly verified:

**Lemma 3.1.** If Condition 4 holds then all potential managers invest in skills according to (3.4) and become active managers in the second period of their life.

The accumulation of skills requires real resources. In the closed economy, gross domestic \( Y_g \) and national \( Y_n \) output levels are equal and given by expression (2.5). Aggregate consumption is by \( C_n = Y_n - Z_n \omega \phi(G) \) as the aggregate cost of skill accumulation is subtracted.

### 3.2. A Small Open Economy

The intratemporal equilibrium conditions are the same as in the exogenous growth model. The free entry of foreign firms pins the wage of local young workers to the international level \( w_f \), each foreign managers hires \( n_f = \eta \) young workers. As before, if \( Z_n/Z_f < R_S \), then, national old potential managers become workers of multinational firms and \( m = 1 \). Otherwise, both national and multinational firms coexist and \( m = 1 - [Z_n/Z_f]^{\frac{1}{1-\eta}} \).

#### 3.2.1. Two Simple Extreme Cases

Before considering the general case, it is instructive to consider two extreme simple cases.

**No Competition Effect:** \( Z_g = \max\{Z_f, Z_n\} \) This optimistic case would be the relevant one if only the best of all the ideas surrounding the youth in the country is the relevant for their future skills. The equilibrium in this case is as follows: in the period it opens up, expression (2.10) determines a positive fraction \( m \) all domestic labor controlled by foreign firms. Any \( m > 0 \) suffices for \( Z^*_n = GZ_f \) since it is the unique solution to (3.3). After one period, the country reaches the level of develop countries and stop hosting foreign firms.

**No Diffusion Effect:** \( Z_g = \min\{Z_f, Z_n\} \) This is a much somber case, and it would if the development of ideas remains in the source country, and the foundation of productive know-how is never disseminated in the host country. In such case, domestic managers only face the negative impact of higher wages and lower returns to their own know-how and do not perceive any benefits in terms of improve ideas in their environment.

Foreseeing that the cost of labor next period next period will be \( w'_f = \alpha \eta^{a-1} GZ_f \) (where \( G \) is the BGP growth rate for a closed economy) and being surrounded by ideas \( Z_g = Z_n \),
the solution for the optimal accumulation of skills (3.3) implies
\[
\frac{Z_n'}{Z_f'} = \left( \frac{Z_n}{Z_f} \right)^\mu, \tag{3.5}
\]
. Here and I have defined \( \mu \equiv v(1 - \alpha) / [v(1 - \alpha) - \alpha] \). Because of condition 1, it is the case that \( \mu > 1 \) and since \( (Z_n/Z_f) < 1 \), then \( (Z_n'/Z_f') < (Z_n/Z_f) \). Since the home country’s managers are surrounded by an inferior set of ideas than the foreigners, their optimal response to the presence of foreign competition in the future is to reduce their relative productivity.

The young potential managers opts for the previous accumulation of skills only if it he becomes an active manager. This is, if
\[
w_f - Z_n \varphi \left( \frac{Z_n'}{Z_n} \right) + \beta \pi \left( Z_n', w_f' \right) > w_f + \beta w_f'. \tag{3.6}
\]
Define the function \( \Xi : [0, 1] \rightarrow R \) as
\[
\Xi(R) \equiv (1 - \alpha)(R)^\frac{\mu}{1 + v} - \frac{1}{1 + v} (R)^{\mu(1 + v) - 1}.
\]
Clearly, \( \Xi(0) = 0 \), and, under Condition 4, \( \Xi(1) > \alpha/\eta \). With this function, I can characterize the optimal occupation choice of young potential managers.

**Lemma 3.2.** The inequality in (3.6) holds if and only if \( \Xi \left( \frac{Z_n}{Z_f} \right) > \alpha/\eta \). Moreover, if Condition 4 holds, there exists a unique threshold \( R_N \), such that \( \Xi(R_N) = \alpha/\eta \), and for \( \Xi(R_0) < \alpha/\eta < \Xi(R_1) \) for all \( R_0 < R_N < R_1 \).

I omit a formal proof since the first part only entails the use (2.7), (2.6) and (3.5) in (3.6) and simplification; the second results does not depend on the monotonicity of \( \Xi(\cdot) \), i.e. the relative size \( \mu / (1 - \alpha) \) vs. \( \mu (1 + v) - 1 \), and follows solely from Condition 4, the fact that \( \Xi(0) = 0 \), and the continuity \( \Xi \).

Recall that if national potential managers become active, they would choose to have relatively less skills than the current generation, i.e. \( (Z_n'/Z_f') < (Z_n/Z_f) \). Thus, if the current crop of domestic managers become workers for multinational firms, then the young cohort of potential managers will also choose to be workers. Therefore, the transition function \( \Gamma : [0, 1] \rightarrow [0, 1] \) of the relative skills of home managers is the following:
\[
\begin{cases}
\left( \frac{Z_n'}{Z_f'} \right) = \Gamma \left( \frac{Z_n}{Z_f} \right) = \left( \frac{Z_n}{Z_f} \right)^\mu & \text{if } \left( \frac{Z_n}{Z_f} \right) > R_N \\
0 & \text{otherwise}.
\end{cases}
\]

With this transition function, it is also immediate to verify the following result:
Proposition 3.3. For any initial condition $R_N < (Z^0_n/Z^0_f) < 1$, and let $T(Z^0_n/Z^0_f) = \min \left\{ s \in \mathbb{N} : s \geq \frac{\ln(R_N)}{\mu \ln(Z^0_n/Z^0_f)} \right\}$. Then, for any $t \geq T(Z^0_n/Z^0_f)$, $Z'_n/Z'_f = 0$ and $m^t = 1$. The sequence $\{Z'_n/Z'_f : t \geq 0\}$ is strictly decreasing and $\{m^t : t \geq 0\}$ is strictly increasing for $t < T(Z^0_n/Z^0_f)$.

Regardless of how close the initial productivity of home country is with respect to the foreign country, as long as it is below, opening to foreign competition without somehow conveying their superior ability to endogenously produce skills will eventually lead the country to a “colonial” limiting point in which $Y_g = 2n^{a-1}Z_f$ because the output is entirely generated by multinational firms and national households consume the returns to their labor $C_n = \alpha Y_g$. The result not only implies that this is the only limiting point of the economy but also that it will be reached in finite time.

3.2.2. The Leading Case: $Z_g = (Z_f)^m (Z_n)^{1-m}$

For the leading case of this model, I will assume that the set of ideas $Z_g$ within the geographic boundaries of the country is a local public good and is given by a geometric average of the know-how of local and foreign firms:

$$Z_g = (Z_f)^m (Z_n)^{1-m},$$

(3.7)

where, as before, $m$ is the fraction of all young agents that work in multinational firms. I assume that $m$ also represents the fraction of potential managers that work for multinational firms. This formulation is consistent with Findlay (1978) since the growth of domestic ideas is directly related to the relative domestic gap and to the relative importance of foreign firms in the country. This formulation has three desirable properties: (a) $Z_g$ is increasing in both $Z_n$ and $Z_f$ (b) the relative importance of $Z_f$ increases with $m$ reflecting the intensity in which the country is exposed to foreign ideas; and (c) $Z_g$ is homogeneous of degree one in $Z_n$ and $Z_f$ and there are no scale effects driven by the total mass of managers operating in the country.

Following the same steps as before but using $Z_g = Z_f^m Z_n^{1-m}$ in (3.3) the relative skills of that active national managers would accumulate –relative to their foreign counterpart– are given by:

$$\left( \frac{Z'_n}{Z'_f} \right) = \left( \frac{Z_n}{Z_f} \right)^{\mu(1-m)},$$

where $\mu$ is as defined above. Notice that now the presence of foreign firms ($m > 0$) helps closing the gap between national managers with the foreign managers. Indeed, the closer is
is $m$ to 1, the closer the ratio $Z'_n/Z'_f$ gets to one. However, the value of $m$ is an equilibrium outcome that depend on $Z_n$ and $Z_f$. Define $\Gamma_0 : [0, 1] \rightarrow [0, 1]$ as the transition function of the skills conditional on potential managers becoming active managers (i.e. ignoring occupation choice). Using (2.10), $\Gamma_0(\cdot)$ is given by

$$
\left( \frac{Z'_n}{Z'_f} \right) = \Gamma_0 \left( \frac{Z_n}{Z_f} \right) \equiv \left( \frac{Z_n}{Z_f} \right)^{\mu [Z_n/Z_f]^{1-\alpha}} .
$$

(3.8)

[Insert Here Figure 1: $\Gamma_0$]

Notice that as displayed in Figure 1, the function $\Gamma_0$ is non-monotone, $\Gamma_0(x) < 1$ for any $x \in (0, 1)$ and has two fixed points. The first one is:

$$
\frac{Z_n}{Z_f} = 1.
$$

Here, the country is at par with the rest of the world, $m = 1$ and the country is neither subject nor in need of spillovers from the rest of the world. The second fixed point is

$$
\frac{Z_n}{Z_f} = R_L \equiv \left[ \frac{1}{\mu} \right]^{1-\alpha} < 1,
$$

where the superindex $L$ indicates that this is a “laggard” BGP, the country never fully catches up with the rest of the world, and a fraction $m^L = \frac{\alpha}{(1-\alpha)v} \in (0, 1)$ of the labor force works for multinational firms.

To determine the dynamics of the skills in the country, the occupation choices of both, young and old potential managers must be consider. As before, even if they already had invested in skills $Z_n$, old potential managers would rather become workers for a multinational firm if their relative productivity is low enough: $Z_n/Z_f < R_S$. In this case, $m = 1$, and $Z_g = Z_n$ since only foreign knowledge surrounds the young cohort. In this case we have the extreme opposite implication with respect to the model without spillovers. Now, the cost of accumulating for the national young cohort is the same as for foreign youth. Therefore, they both would choose the same level of skills, $Z'_n = Z'_f$ and, in the next period, the country fully catches up.

On the other hand, before investing in skills, each agent compares the alternative not investing in skills and being a worker in both periods versus the alternative of optimally investing in skills and being a manager next period. Following the same steps as in the previous section, I define the function $\Phi(\cdot)$ to characterize the occupation/investment decision of young potential managers. Let
\[ \Phi(x) \equiv (1 - \alpha) \Gamma_0(x)^{1-v} - \frac{1}{1+v} \Gamma_0(x)^{\frac{1+v}{\nu}} (x)^{-(1+v)}. \]

Similarly as with a Lemma in the previous section, it is easy to verify that young potential managers will invest and become active managers if and only if \( \Phi(x) > \alpha/\eta \). However, contrarily to the function \( \Xi(\cdot) \) of the model with no spillovers, even if Condition 4, the presence of spillovers induce non-monotonicities that can lead \( \Phi(\cdot) \) to cross multiple times \( \alpha/\eta \) in the interval \([0, 1]\). Therefore, the transition function \( \Gamma(\cdot) \) has to be defined directly using the function \( \Phi(\cdot) \):

\[
\left( \frac{Z'_n}{Z'_f} \right) = \Gamma \left( \frac{Z_n}{Z_f} \right) \equiv \begin{cases} 
1 & \text{if } Z_n/Z_f < R_S, \\
0 & R_S < Z_n/Z_f, \text{ and } \Phi \left( \frac{Z_n}{Z_f} \right) < \alpha/\eta \\
\Gamma_0 \left( \frac{Z_n}{Z_f} \right) & \text{otherwise}.
\end{cases} \tag{3.9}
\]

In the first branch, as explained above, only foreign knowledge is active in the country and the national young potential managers become active and the economy converges in one period. The second branch indicates the possibility that old potential managers become active but young ones do not invest. In this case, the activation of the lower skills of national managers blocks the entry of foreign knowledge and this happens to the extreme of making it pointless for the youth to invest in skills to compete with the next period entry of foreign managers. But, since in the next period \( m' = 1 \), then \( Z'_n = Z'_f \), and the economy will converge in two periods. In the last case, both cohorts are active.

Notice that the system exhibits predatory-prey dynamics: as foreign firms enter, local managers accelerate their skills formation and reduce their distance with the rest of the world. This is reinforced by the fact that as they increase their skills, next period the mass of foreign firms diminish, reducing the effective competition for workers. In turns, the stock of productive ideas that young agents are being exposed to is also diminished, the country increases its lag with respect of the world, implying a higher presence of foreign firms in the subsequent period and so on.

**Proposition 3.4.** The following results hold: (a) If \( \Phi(R_L) < \alpha/\eta \) or \( R_L < R_S \), then \( Z_n/Z_f = 1 \) is the unique resting point of \( \Gamma \) and is globally stable. (b) If instead \( \Phi(R_L) > \alpha/\eta \) and \( R_L > R_S \), then both \( Z_n/Z_f = 1 \) and \( Z_n/Z_f = R_L \) are resting points and \( R_L \) is locally stable. Moreover, if for all \( x \in (R_S, 1), \Gamma_0(x) > R_S \) and \( \Phi(x) > \alpha/\eta \), then,

\[
\lim_{t \to \infty} \left( \frac{Z'_n}{Z'_f} \right) = \begin{cases} 
R_L & \text{if } (Z'_n/Z'_f) > R_S \\
1 & \text{otherwise}.
\end{cases}
\]

**Proof.** See appendix.  

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Figure 2 displays several cases of the transition function. In all the panels, the dash line is the function $\Gamma_0$ with the thick line is for the function $\Gamma$. In all cases, if $\left(\frac{Z_0}{Z_1} / \frac{Z_0}{Z_2}\right) < R_S$, then the country converges in levels after one period in which all output was generated by multinational firms. On the other hand, for some initial conditions, the $R_L$ steady state is the limiting point in the cases in which $\Gamma$ crosses the 45° line at $R_L$ and the convergence can be cyclical.

[Insert Figure 2: different cases for $\Gamma_0, \Phi, \Gamma$]

In sum, even if we do not get the disastrous results of the model with competition without spillovers, the competition effect is still present and either completely dominates in one period and the country converges in levels, or remains operative forever blocking the level convergence of the country. The expectation that by opening to foreign firms the national productivity will increase may lead to disappointment since, foreseeing future competition, national firms may opt to reduce, not increase their efforts to increase productivity.\\[17\\]

4. Internalized Accumulation of Know-How

I now consider skills formation inside the firm. As in Boyd and Prescott (1987a,b), Chari and Hopenhayn (1991) and Jovanovic and Nyarko (1995), the skills that a manager commands depend on the skills and actions of the manager for whom he worked when young.\\[18\\] Since skill formation requires the direct involvement of the initial holder, and the costs and benefits of accumulating skills are fully foreseen by both parties involved, the equilibrium is efficient.\\[19\\]

4.1. Technology and the problem of the Firm

Besides consumption goods $y$, firms produce skills $z'$ that young individuals can command next period should they become active entrepreneurs. The production possibilities frontier for $(y, z')$ is as follows. To provide $z'$ units of skills to one worker, a manager with skills level $z$ incurs a cost $z\phi \left( \frac{z'}{z} \right)$. Therefore, a team of $n_1$ young potential managers, $n_2$ laborers working under the direction of a manager with skills $z$ produce $y$ units of consumption goods and $z'$ units of skills for each of the $n_1$ future managers according to

$$y = z \left[ (n_1 + n_2)^\alpha - n_1 \phi \left( \frac{z'}{z} \right) \right]. \quad (4.1)$$

\\[17\\]Aitken and Harrison (1999) discuss a similar “competition” effect driven by strategic interaction in the context of a static partial equilibrium model.


\\[19\\]I abstract from private information issues such as adverse selection that introduces inefficiencies in the equilibrium of Jovanovic and Nyarko (1995).
Here $\phi(\cdot)$ is the same function as defined above, implying that total and marginal training costs are increasing in $z'$ and decreasing $z$.

Each active manager is the residual claimant of the team and hires workers in competitive labor markets. Let $w_t$ be the wage rate for the labor supplied by workers and young potential managers. Let also $W_t$ be the the equilibrium discounted utility at time $t$ of a young potential managers. Old managers can deliver $W_t$ by different combinations of transfers of goods and skills. As in the previous section, the level of skills $z$ determines whether an old agent opts to be a firm leader, and if so, how many workers to hire.

For a laborer, the manager for whom he works at $t$ has no impact on the wage he earns at $t+1$. The labor market for laborers operate in the same way as in the previous model. The labor market for young potential managers is a bit more complicated. Both, current and future managers foresee that $\beta\pi_{t+1}(z')$ is the (discounted) value of the payoff that the young manager appropriates next period. If the equilibrium price is $W_t$ an old manager offering a skill transfer $z'$ only needs to transfer $[W_t - \beta\pi_{t+1}(z')]$ units of consumption goods to hire one young potential manager. However, the transfers $z'$ come at the cost of an output reduction of $z\phi\left(\frac{z'}{z}\right)$ for each trainee. Then, the effective net cost of hiring a future manager is $[W_t + z\phi\left(\frac{z'}{z}\right) - \beta\pi_{t+1}(z')]$. Therefore, a manager maximizes his net payoff $\pi_t(z)$ by choosing $n$ and that $z'$:

$$
\pi_t(z) = \max_{(n_1,n_2,z') \geq 0} \left\{ z(n_1 + n_2)^\alpha - n_1 \left[ W_t + z\phi\left(\frac{z'}{z}\right) - \beta\pi_{t+1}(z') \right] - wn_2 \right\}. \tag{4.2}
$$

The first result is that, in the margin, the costs and benefit of skill transmission are fully internalized. Assume the following condition:

**Condition 5:** The function $\pi_{t+1}(\cdot)$ is increasing, differentiable and for all $z > 0$, $z' \geq 0$, $[W_t + z\phi(z'/z)]/\beta > \pi_{t+1}(z')$.

This condition is a generalization of Condition 1 and implies that (4.2) is well defined and bounded since the effective cost of hiring a young potential manager is strictly positive.

**Proposition 4.1.** Assume that Condition 5 holds. Then, (1) the optimal transfer $z_{t^*}^i(z)$ of skills is independent of the number of either type of workers in the firm $n_1^*(z)$, $n_2^*(z)$; (2) the payoff to the old manager is

$$
\pi_t(z) = (1 - \alpha) z [n_1^*(z) + n_2^*(z)]^\alpha;
$$

and (3) The total amount of labor in control of the manager $[n_1^*(z) + n_2^*(z)]$ is strictly increasing in $z$. 

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Proof. See appendix. ■

Part (a) results from the linear homogeneity in the production of skills, i.e. the number of trainees does not affect the unitary cost of providing skills to each one. Part (b) implies that the costs and benefits of skill formation are fully internationalized by the trainee. Part (c) implies that the payoff of the manager \( \pi_t(z) \) increase more than proportional with \( z \) since \( n^*(z) \) is also strictly increasing function of \( z \). Notice that there are two forces that make \( n^*(z) \) is increasing. First, as in the Lucas’ span of control models of the previous sections, the marginal product of labor –in producing goods– is increasing in \( z \). Second, a higher \( z \) reduces the cost of any training \( z' \), reducing the effective cost of labor for the entrepreneur.

A direct implication of the previous proposition, which follows directly from the envelope condition, is the following:

**Corollary 4.2.** The optimal \( n^*_{t+1}(z') \) is strictly increasing in \( n^*_{1,t+1}(\cdot) \), the number of young workers that the current trainee will control in the next period.

The first order condition for the accumulation of skills \( z' \) of trainees under a manager with skills \( z \) is

\[
\beta \pi'_{t+1}(z') = \phi' \left( \frac{z'}{z} \right). \tag{4.3}
\]

The envelope condition, using the specific functional form \( \phi(\cdot) \), is

\[
\pi'_{t+1}(z') = (n'_1 + n'_2)^\alpha + n'_1 \left[ \frac{\nu n_0}{1 + \nu} \left( \frac{z''}{z'} \right)^{1+\nu} \right] \geq 0. \tag{4.4}
\]

The envelope condition indicates that if potential manager expects to be a worker then \( n'_1 = n'_2 = 0 \), the returns of investing in skills are zero. In general, the higher the amount of labor \( n'_1 + n'_2 \) under his control, the higher the returns to investing in skills. Moreover, the higher the number of trainees \( n'_1 \) and skill accumulation \( z'' \) the higher the returns of \( z' \). These forces will be important in determining the aggregate dynamics of a closed economy and the response of domestic firms to entry of foreign firms.

**4.2. A Closed Economy**

Consider a closed economy with initially identical managers, i.e. all old potential managers have the same level of know-how \( Z_n > 0 \). Decreasing returns to labor in the production of goods imply that all managers command equal numbers of workers \( \eta \). Ex-ante identical young potential managers attain the same expected utility levels. Absent randomizations, all young potential managers attain the same level of utility, which here imply the same level of skills \( Z''_n \), and hence, the homogeneity of entrepreneurs is preserved over time.
For now, assume (which I check below) that all potential managers become active managers. With homogeneous managers and all active, each one commands \( n_1 + n_2 = \eta \) units of labor, and trains \( n_1 = 1 \) young potential managers. Combining the first order condition (4.3) and the envelope condition (4.4) then, the equation that must be satisfied by \( G = z'/z \), the constant growth rate in a BGP is:

\[
v_0 G^v = \beta \left[ \eta^\alpha + \frac{v_0}{1 + v} G^{1+v} \right]. \tag{4.5}
\]

The left-hand side of this equation is the marginal cost and the right-hand side is the marginal benefit of skill accumulation. A growth rate \( G \) that solves this equation is an equilibrium BGP if it satisfies three conditions: (1) it is a “maximization”, i.e. the marginal cost crosses the marginal benefit from below; (2) net output is positive; and (3) all potential managers are better off training when and being active managers when old than being workers in both periods. In a BGP, condition (2) boils down to \( \eta^\alpha > \frac{v_0}{1+v} G^{1+v} \), and condition (3) boils down to \( \frac{v_0}{1+v} G^v \leq \beta \eta^\alpha \left[ 1 - \alpha (1 + 1/\eta) \right] \), i.e. the (present value) difference in the income of managers vs. workers compensate for the training costs.

The maximization condition is a bit more convoluted, since it requires to consider the returns of skills not only in terms of producing goods but also in terms of further producing skills in the future.

**Condition 6:** The parameters \((v_0, v, \beta)\) satisfy \( \beta < \left( \frac{\eta^\alpha}{\eta^\alpha (1 + v)} \right)^{1/v} \).

**Proposition 4.3.** (Existence and Uniqueness of a BGP.) Assume that Condition 6 holds. Then there exists a unique \( G \in (0, \beta^{-1}) \) that solves (4.5), and satisfies the maximization condition (1). Moreover, if \( \beta > 1 / [\eta^\alpha / v_0 + v / (1 + v)] \), then \( G > 1 \) (positive growth).

Therefore, under Condition 6, the lowest root of equation (4.5) is an equilibrium BGP if it also satisfies conditions (2) and (3) as laid out above. If so, managerial skills evolve accordingly to \( Z_n' = G Z_n \), and every period the wage rate of laborers, the payoff of active managers and the discounted utility of young potential managers are, respectively:

\[
\begin{align*}
w &= Z_n \alpha \eta^{\alpha-1}, \\
\pi &= Z_n (1 - \alpha) \eta^\alpha, \\
W &= Z_n \left[ \alpha \eta^{\alpha-1} - \phi (G) + \beta (1 - \alpha) \eta^\alpha \right].
\end{align*}
\]

It is instructive to consider the social planner’s allocation in this economy. Given a cohort of old managers, all with the same expertise \( Z_n \), the planner must decide the units of labor to assign to each manager and the skills \( Z_n' \) to invest in each of the young workers. Because of decreasing returns, each old manager will end up commanding the same amount of labor
and aggregate output of goods is $Z_n \omega \eta^\alpha$. On the other hand, forming $Z_n'$ skills for each of the young potential managers entails an aggregate cost of $\omega Z_n \phi (Z_n' / Z_n)$. The value function $S (Z_n)$ for the planner is defined by the Bellman Equation (BE):

$$S (Z_n) = \max_{\{Z_n' \geq 0\}} \{ Z_n [\omega \eta^\alpha - \omega \phi (Z_n') / Z_n] + \beta S (Z_n') \}. \tag{4.6}$$

Notice that the period return function $Z_n [\omega \eta^\alpha - \omega \phi (Z_n') / Z_n]$ is linearly homogeneous and jointly concave in $(Z_n, Z_n')$ and that the feasible set for $Z_n'$ does not depend on $Z_n$. These properties lead to the following result:

**Proposition 4.4.** Assume Condition 6 holds. Then, there is a unique value function that solves (4.6) and it has the form $S (Z_n) = S_0 Z_n$ with $0 < S_0 < \infty$ that solves $S_0 = \max_{G \in [0, \infty]} \{ \omega \eta^\alpha - \omega v_0 (G) (1^v) / (1 + v) + \beta G S_0 \}$. Moreover, the value $G$ that solves this maximization coincides with the $G$ in the previous proposition.

### 4.3. A Small Open Economy.

Consider now an initially closed country that is in a BGP but unexpectedly and permanently allows entry of foreign firms. As before, openness to foreign firms means that foreign managers can hire workers in the country. I assume that the technology frontier of both countries is the same, but that the foreign country is more advanced in the sense that its local managers have a higher level of skills, i.e. $Z_f > Z_n$. In particular, foreign managers can use also transfer skills to the workers under their control. I restrict attention to the case in which the home country is small and has not impact on the equilibrium of the foreign country, which I assume moves along a BGP.

In the absence of frictions, foreign managers must be indifferent between operating in the home country or staying abroad. Hence, if they enter the home country, it is because they earn also $\pi_f = Z_f (1 - \alpha) \eta^\alpha$. With openness, the equilibrium is achieved by a mass of foreign managers that pushes the domestic prices of labor $w_n$ and of young potential managers to be equal to the international prices, i.e. $w_n = w_f = Z_f \alpha \eta^{\alpha-1}$ and $W_n = W_f = Z_f [\alpha \eta^{\alpha-1} - \phi (G) + \beta (1 - \alpha) \eta^\alpha]$. Foreign managers face the same problem and market prices as in the foreign country and hence, choose they same values $n = \eta$ and $z' = GZ_f$ units of skills. Domestic managers, however, have lower levels of skills $z = Z_d$ and might instead opt to be a worker. Otherwise, they might hire only laborers.

For simplicity I look directly at the social planner’s problem, i.e. the maximization the present value of aggregate consumption of goods of the country by allocating domestic labor to domestic and foreign managers and choosing the skill formation of the country’s young potential managers. The country collects the entire output of domestic firms, i.e. the sum
of payments to managers, workers and young potential managers. However, for each foreign manager the country must pay out \( \pi_f \). In either case, the country gathers the future returns and takes on the costs of skill accumulation.

The initial heterogeneity between domestic and foreign managers in an open economy makes it necessary to determination of different firms. However, in our case, in each period there can be at most three ‘types’ of managers. The first are the foreign managers, whose skills evolve exogenously to the country. The second type is “new domestic sector,” which are those who were directly trained by a foreign manager, by someone who was or by some who was trained by someone who was, etc. The third type is “deep-rooted domestic sector” which is the progeny of the domestic managers that existed in the country before it opened.

Managers in the new domestic sector have the same skills as foreign managers. To see this, recall that after the country opens, in all period \( w_n = w_f \) and \( W_n = W_f \). Hence, a domestic trainee receives \( z' = GZ_f \), i.e. identical to foreign peers. Next period he will transfer \( z'' = G^2Z_f \) to the group of young trainees. And so on. Hence, the “state” for the optimal allocation problem is the triplet \((X, Z_n, Z_f)\) which indicates, respectively, the fraction \( X \in [0,1] \) of the mass \( \omega \) of domestic managers in the old-domestic sector and \( Z_n \) their level of skills. The variable \( Z_f \) is the skill level of foreign and new-domestic managers, which grows at the exogenous rate \( G \) of the BGP. Given \((X, Z_n, Z_f)\), a planner would first allocate labor optimally across sectors.

Given the state \((X, Z_n, Z_f)\), the country has two options: employ the deep-rooted managers as managers or as workers. In the first case, if all \( X \) are active managers, then foreign and new-domestic firms command \( \eta \) units of labor and old-domestic managers command \( n_n = \eta (Z_n/Z_f)^{1+\alpha} \). With this, it is easy to show that if next period the country chooses \( X' \leq X \) and \( Z_n' \geq 0 \), then aggregate output flows, net of training costs and foreign remittances, are as follows: \( Z_n \omega \left\{ X (Z_n/Z_f)^{1+\alpha} \eta^{\alpha} - X' \phi (Z_n'/Z_n) \right\} \) from old-domestic firms, \( Z_f \omega (1 - X) \left\{ \eta^{\alpha} - \phi (G) \right\} \) and \( Z_f \omega (X - X') \left\{ \alpha \eta^{\alpha} - \phi (G) \right\} \) from new-domestic is and foreign sector are, respectively. Adding the three sources, the flow of consumption goods for the home country is:

\[
C^M (Z_f, Z_n, X, Z_n', X')
= \omega Z_f [\eta^{\alpha} - \phi (G)] - \omega Z_f \left( X \eta^{\alpha} \left[ 1 - \alpha - \left( \frac{Z_n}{Z_f} \right)^{1+\alpha} \right] - X' \left[ \alpha \eta^{\alpha} + \left( \frac{Z_n}{Z_f} \right) \phi \left( \frac{Z_n}{Z_n} \right) - \phi (G) \right] \right) \]

(4.7)

where the superscript \( M \) indicates that the country maintains active the deep-rooted sector. Notice that \( C^M \) is equal to the consumption of a country with all domestic managers with the leading \( Z_f \) skill levels, minus the net-output gap of the deep-rooted sector, minus the payout of foreign profits.
Alternatively, the country can liquidate that sector and release the labor units of old potential managers. Notice that in this case, the country has a supply of 2 units of labor. In this case, the \( \omega (1 - X) \) new-domestic managers demand an aggregate amount of labor equal to \( \omega (1 - X) \eta \). The remaining \( [2 - \omega (1 - X) \eta] \) units of labor must be hired by foreign managers, and since each one of them hires \( \eta \), the country must pay a total \( [2/\eta - \omega (1 - X)] \pi_f \) of foreign profits. Using the value of \( \pi_f \) and simplifying, the aggregate consumption flow of the country is,

\[
C^L (Z_f, X) = Z_f \left\{ \eta^\alpha \left[ 2\alpha/\eta + \omega (1 - \alpha) (1 - X) \right] - \omega \phi (G) \right\},
\]

where the superscript \( L \) liquidates that the country’s deep-rooted sector is being liquidated. Observe that when the country scraps this sector, the country’s output might be higher than if it had all domestic managers with the leading level \( Z_f \). However, \( C^L (\cdot) \) indicates that domestic consumption might be significantly lower. On one hand, much of this output might be used to train the country’s future crop of managers. On the other hand, much of this output might flow out of the country as foreign profits.

With the functions \( C^M (\cdot) \) and \( C^L (\cdot) \), it is straightforward to write the Bellman Equation for the social planner’s problem. First, notice that if the country liquidates the domestic deep-rooted sector, next period it would converge to the BGP of developed countries. The value \( V^L (Z_f, X) \) of this option is:

\[
V^L (Z_f, X) = C^L (Z_f, X) + \beta S (GZ_f),
\]

where \( S (\cdot) \) is the value function defined by (4.6). If the country opts to maintain the deep-rooted sector active this period, it has to choose its size and skill level for the next period. Since each foreign and new-domestic manager in the country trains one young potential manager, the value \( V^M (Z_f, Z_n, X) \) of this option is:

\[
V^M (Z_f, Z_n, X) = \max \left\{ C^M (Z_f, Z_n, X, Z'_n, X') + \beta V (GZ_f, Z'_n, X') \right\},
\]

where the value function \( V (Z_f, Z_n, X) \) is defined by

\[
V (Z_f, Z_n, X) = \max \left\{ V^M (Z_f, Z_n, X), V^L (Z_f, X) \right\}.
\]

Direct inspection reveals that \( C^M (\cdot) \) and \( C^L (\cdot) \) are linearly homogeneous in \((Z_f, Z_n, Z'_f, Z'_n)\). This property simplifies the proof for the following result:

**Proposition 4.5.** Assume Condition 6 holds. Then, there is a unique value function \( V \) that solves the Bellman Equation defined by (4.7), (4.8), (4.9), (4.10) and (4.11). Moreover: (a) If \( Z_n < Z_f \) then \( V (Z_f, Z_n, X) \) is strictly decreasing in \( X \); (b) if \( X > 0 \), there is a \( r_0 > 0 \) st. \( V (Z_f, Z_n, X) = V^L (Z_f, X) \) for \( 0 \leq Z_f, Z_n \leq r_0 \), while \( V (Z_f, Z_n, X) = V^M (Z_f, Z_n, X) \) for \( Z_f/Z_n > r_0 \). In the latter case, \( V (Z_f, Z_n, X) \) is strictly increasing in \( Z_n \).
The intuition of this result is straightforward. The farther behind is a country with respect to developed countries, i.e. the lower $Z_n$, the lower its welfare. Even if the country choose to scrap the old domestic sector and temporarily exhibits a large output level. Notice that the lower is $Z_n/Z_f$, the more likely a country is to liquidate the laggard deep-rooted sector and catch up next period with developed countries in terms of skills and income levels. On the other hand, given $Z_n/Z_f < 1$, the higher the fraction of managers $X$ that are lagging behind, the lower the welfare of the country.

[Insert Figure 5: opening up: fraction, relative productivity, aggregate output]

Figure 5 considers the evolution of an closed economy that opens up permanently to hosting foreign firms. In the initial period, $X = 1$ and $Z_n/Z_f < 1$. The optimal response is for the deep-rooted sector to shrink over time and fully disappear in finite time. The optimal response is to gradually reduce $Z_n/Z_f$. As can be expected, in the period before the country opts to scrap the deep-rooted sector, the social planner sets $Z_n' = 0$, since, obviously, it is pointless to form managerial skills in individuals that would be workers in the next period. The figure shows that the mass of new-domestic firms grows over time. Eventually this sector controls the supply of domestic workers and train the entire crop of young potential entrepreneurs. When a country reaches that point, the country has converged and foreign managers no longer step in.

It is important to notice that the transfer of skills from multinational firms materialize in a new sector of firms, not in the pre-existing sector of firms. This is indeed, somewhat in line with empirical findings. The model implies that the presence of foreign firms should hurt the productivity of pre-existing firms –because of the competition effect and the absence of spillovers. However, the economy as a whole fully catches up. When facing competition, the optimal behavior of foreign firms is to use their ability to form national skills in order to hire local workers.

As with the model with spillovers, with internalized diffusion, a developing country catches up with developed countries only when the less productive domestic know-how is fully replaced by more advanced foreign know-how in the formation of skills of future generations. Moreover, the two models are compatible with leapfrogging, since laggard countries converge more quickly than more advanced developing countries. However, there are important differences. First, with internalized diffusion all developing countries eventually catch up. This is far from being the case in the model with spillovers. Indeed, in the latter model, leapfrogging is not only in terms of speed of convergence, but also whether such convergence takes place at all. Second, the impact of openness is drastically different in the two models. With national spillovers, the ideas of foreign firms impact the formation of skills for all the
agents in the economy. The dynamics is governed by the skill ratio of domestic-to-foreign managers. With internalized diffusion, the presence of foreign firms lead to the emergence of a new sector of domestic firms. This new sector is at par with the firms in developed countries, and eventually overtakes the whole economy. Along the adjustment, the deep-rooted group of firms decline in size and in relative (and absolute) skills and eventually disappear. Third, by construction, with internalized diffusion the allocation are optimal. There is no room left for government policy. On the contrary, many interesting issues for government policy may arise in the model of spillovers. In particular, since the current (future) presence of foreign firms enhances (impairs) the accumulation of domestic skills, time-consistency might limit the ability to implement the optimal national policy. This issue should be examined further.

5. Concluding Remarks.

In this paper I used simple general equilibrium growth models to study the impact of multinational firms on the formation of skills and the long run behavior of output a small developing country. Within the context of a simple model environment, I examined three different growth models: (a) an exogenous growth model, (b) an endogenous growth model with an externality in the formation of skills and (c) an endogenous growth model in which skills are internally produced in the firm. The impact of multinational firms on the host country in models (a) and (b) is via spillovers. These two models are rather standard in the growth literature and the existence and measurement of spillovers have been the subject of a vast empirical literature. Spillovers are also the tenet underlying much debate and policy proposals and programs. I show that spillovers are not sufficient to propel the country to catch up with developed countries.

In model (c) there are no spillovers and any transfer of skills is the result of a market transaction. In a competitive environment, firms will optimally transfer a level of skills that gradually obliterate the existent sector and creates a new sector of domestic firms that are at par with the ones in developed countries. In finite time, the small country converges to the income level of developed countries. Spillovers are not necessary.

These results are very suggestive about the role of government policy. In model (a), a benevolent government would definitely want to subsidize foreign firms. In model (b) optimal policy can be quite kinky. For instance, if local skills are very low, a subsidy would be pointless since the country will converge next period. For higher initial level of skills, the government may want to subsidize foreign firms fully obliterating the local firms for one period and converge to the developed country level in the next. The competition effect in this model can also introduce interesting issues of time consistency.
In model (c) the equilibrium is efficient and the government must not subsidize. However, if there are obstacles to the transmission of skills from foreign firms to local agents, a government may want to undo the obstacle by providing a subsidy. But it is necessary to go beyond these somewhat speculative arguments. Optimal government policy in models with obstacles to the flow of firms and/or the transmission of skills deserve a rigorous and comprehensive consideration.

A. Proofs

Proof of Lemma XXX (Transition Function for Endogenous Growth with Spillovers)

First, notice that if the conditions for (a) hold, then necessarily the economy will reach the set \([0, R_S]\) in a finite number of steps and therefore converge to 1. On the other hand, if the conditions for (b) hold, given that the function \(\Gamma_0\) crosses the \(45^0\) line from above and that it is continuous, there is an \(\epsilon > 0\) such that the ball \(B (R_L, \epsilon)\) is such that \(B (R_L, \epsilon) \subset (R_L, 1)\), \(\Gamma_0 [B (R_L, \epsilon)] \subset B (R_L, \epsilon)\), and \(\Gamma_0 (x) \forall x \in B (R_L, \epsilon)\). Therefore, \(R_L\) is locally stable. Finally, if \(\Gamma_0 (x) > R_S\) and \(\Phi (x) > \alpha\), then if the economy starts in a position where old agents become managers, it will always remain there, and in this case the limiting point is \(R_L\).

Characterization of the Manager’s problem. To prove (1) notice that the maximization can be factorized, i.e.:

\[
\pi_t(z) = \max_{\{n_1, n_2, z'\geq 0\}} \left\{ z (n_1 + n_2)^\alpha - n_1 \left[ W_t + z \phi \left( \frac{z'}{z} \right) - \beta \pi_{t+1}(z') \right] - w n_2 \right\}
\]

\[
= \max_{\{n_1, n_2\geq 0\}} \left\{ z (n_1 + n_2)^\alpha - W_t n_1 - w n_2 + n_1 \max_{\{z'\geq 0\}} \left\{ \beta \pi_{t+1}(z') - \beta \phi \left( \frac{z'}{z} \right) \right\} \right\}.
\]

Since the inner maximization is independent of the choice of \(n_1\) and \(n_2\), the result follows. To prove (2), take the (or one of, if there are multiple ones) optimal choice of \(z^* (z)\). Then, the remaining problem is to choose \(n_1, n_2\) optimally. This is a maximization of a concave objective function with a convex feasible set, and under Condition 5, \(\pi_t(z)\) is bounded. The first order conditions with respect to \(n_1\) and \(n_2\) are sufficient and are respectively given by:

\[
z \alpha [n_1^* (z) + n_2^* (z)]^{\alpha-1} \leq W - \beta \pi_{t+1} [z^* (z)] + z \phi \left[ \frac{z^* (z)}{z} \right], \quad n_1 \geq 0,
\]

\[
z \alpha [n_1^* (z) + n_2^* (z)]^{\alpha-1} \leq w, \quad n_2 \geq 0,
\]

and in both cases at least one inequality holds with equality. Given \(z\), the manager can (a) only hire laborers, (b) only hire young potential managers and (c) hire both. Assume the latter is the case, i.e. \(n_1 > 0\) and \(n_2 > 0\). This can only happen if \(w = W - \beta \pi_{t+1} [z^* (z)] + z \phi \left[ z^* (z) / z \right]\). The net payoff of the manager is

\[
\pi_t(z) = z [n_1^* (z) + n_2^* (z)]^\alpha - w n_2^* (z) - [W - \beta \pi_{t+1} [z^* (z)] + z \phi \left[ z^* (z) / z \right]] n_1^* (z),
\]

\[
= z [n_1^* (z) + n_2^* (z)]^\alpha - z \alpha [n_1^* (z) + n_2^* (z)]^{\alpha-1} \left[ n_2^* (z) + n_1^* (z) \right],
\]

where the second lines uses the F.O.C.s with equality. From here, the result is straightforward. The same argument applies to cases (a) and (b). To prove part (3) notice that in case (a) \(n_1^* (z) + n_2^* (z) = n_2^* (z) = \alpha z / \omega [1/(1-\alpha)\), proving the statement for this case. Now, consider case (b).

Rearranging the first order condition,

\[
n_1^* (z) = \left\{ \alpha z / [W - \beta \pi_{t+1} [z^* (z)] + z \phi \left[ z^* (z) / z \right]] \right\}^{1/(1-\alpha)}.
\]
Using the envelope condition and the functional form of $\phi(\cdot)$ it can be shown that,

$$\frac{\partial \ln n_1^*(z)}{\partial z} = \left( \frac{1}{1+\alpha} \right) \left\{ \frac{1}{z} + \frac{(v \nu_0) [z^{\nu_*(z) / \nu} / z]^{1+\nu}}{W - \beta \pi_{t+1} [z^{\nu_*(z) / \nu} + z \phi(z)]} \right\} > 0,$$

which, of course, implies that $\partial n_1^*(z) / \partial z > 0$. Case (c) follows from cases (a) and (b).

Existence and Uniqueness of a BGP. As a short hand, let $L(G) \equiv v_0 G^v$ and $R(G) \equiv \beta \left[ \eta^\alpha + \frac{\alpha m}{1+\alpha} G^{1+\nu} \right]$. Obviously, $L(0) = 0 < R(0) = \beta \eta^\alpha$. Moreover, since the curvatures of $L$ and $R$ are $\nu$ and $1 + \nu$, respectively, and $L'(0) = R'(0) = 0$, then the derivatives cross only once, i.e. $\exists \bar{G} \in (0, \infty)$ implicitly defined by $L'(\bar{G}) = R'(\bar{G})$, $L'(G) > R'(G)$ for $G < \bar{G}$ and $L'(G) < R'(G)$ for $G > \bar{G}$. It so happens that $\bar{G} = \beta^{-1}$. Then, if $L(\beta^{-1}) > R(\beta^{-1})$, then $L$ crosses $R$ twice, once for below and once from above. If $L(\beta^{-1}) = R(\beta^{-1})$, then, the unique crossing is $G = \beta^{-1}$. Finally, $L(\beta^{-1}) < R(\beta^{-1})$, then $L(\cdot) < R(\cdot)$ in all the positive reals. It can be directly shown that $\beta < (v_0 / [\eta^\alpha (1 + v)])^{1+\nu}$ implies that $L(\beta^{-1}) > R(\beta^{-1})$, and hence that there exists a single $G < \beta^{-1}$ where $L$ crosses $R$ from below, i.e. satisfies the maximization condition. On the other hand, it can be directly verified that if $\beta > 1 / [\eta^\alpha / v_0 + v / (1 + v)]$, then $L(1) < R(1)$, and hence, the lowest crossing is above 1.

The Social Planner’s Problem in a Closed Economy. First, notice that the problem can be equivalently written as the optimal choice of the rate of growth $G = Z_n^0 / Z_n$, i.e.:

$$S(Z_n) = \max_{G \geq 0} \{ Z_n \left[ \omega \eta^\alpha - \omega \phi(G) + \beta S(GZ_n) \right] \}.$$ 

Since the return function is homogeneous of degree one in $Z_n$ and the feasible set is clearly convex, then, following Alvarez and Stokey (1998), it can be shown the functional operator defined by this BE maps linearly homogeneous functions into linearly homogeneous functions. Hence, the unique fixed point of this BE is of the form $S(Z_n) = S_0 Z_n$ for a constant $S_0 > 0$. Then, plugging $S(Z_n) = S_0 Z_n$, in the objective function, the constant $S_0$ must solve (4.5). This is a maximization of a concave function in a convex set, so the first order conditions are sufficient and given by

$$\omega v_0 G^v = \beta S_0,$$

which implies that $G = [\beta S_0 / (\omega v_0)]^{v}$ and, then the constant $S_0$ satisfies

$$S_0 = \omega \eta^\alpha - \omega v_0 \left( \frac{\beta S_0}{\omega v_0} \right)^{1+\nu} + \beta \left[ \frac{\beta S_0}{\omega v_0} \right]^v S_0$$

$$= \omega \eta^\alpha + \left( \frac{1}{\omega v_0} \right)^{v / (1+\nu)} \left( \frac{\nu}{1+\nu} \right) (\beta S_0)^{1+\nu}$$

Since the first order condition implies $S_0 = \omega v_0 G^v / \beta$, which plugged in the previous equation leads to (4.5). Therefore, under Condition 6, there is a unique fixed point and it coincides with the equilibrium $G$ defined in the previous proposition.

The Social Planner’s Problem in an Open Economy. Define $r \equiv Z_n / Z_f$, and since $C^M(\cdot)$ and $C^L(\cdot)$ are HD1 in $(Z_f,Z_n,Z_f',Z_n')$, then $C^M(\cdot) = Z_f c^M(r,X,r',X')$ and $C^L(\cdot) = Z_f c^L(X)$ where

$$c^L(X) = \eta^\alpha [2 \alpha / \eta + \omega (1 - \alpha)(1 - X)] - \omega \phi(G),$$

$$c^M(r,X,r',X') = \omega \left\{ \eta^\alpha \phi(G) - X \eta^\alpha [1 - \alpha - (r)^{1-\alpha}] - X' \left[ \alpha \eta^\alpha + r \phi \left( \frac{r'}{r} G \right) - \phi(G) \right] \right\}.$$
With these functions, the BE can be written in normalized terms as:

\[
\begin{align*}
  v^L(X) &= c^L(X) + \beta S_0 G, \\
  v^M(r, X) &= \max_{r' \geq 0, 0 \leq X' \leq X} \left\{ c^M(r, X, r', X') + \beta G v(r', X') \right\}, \\
  v(r, X) &= \max \left\{ v^M(r, X), v^L(X) \right\},
\end{align*}
\]

where \( S_0 > 0 \) is as defined for the closed economy. The first equation uniquely defines the function \( v^L(X) \) and since Condition 6 implies that \( \beta G < 1 \), this Bellman Equation defines a contraction mapping \( T : Y \rightarrow Y \) on the set of bounded and continuous functions \( Y \) defined over \([0, 1] \times [0, 1]\). This establishes existence and uniqueness of \( v \). Notice that \( c^M(r, X, r', X') \) is strictly increasing in \( r \) and the feasible set for \( \{L, M\} \) and, if \( M \), for \( \{r', X'\} \) are independent of \( r \), then, if \( v \) is weakly increasing in \( r \), \( v^M \) is strictly increasing in \( r \). Notice that for any \( X \in (0, 1) \), \( v^M(0, X) < v^L(X) \). Because of the theorem of the maximum, both \( v^M \) and \( v^L \) are continuous functions, then, for all \( X \) there must exist an \( r_0 \) st. \( v^M(r_0, X) = v^L(X) \). This proves (b). Part (a) follows directly from the fact that \( c^L(\cdot) \) and \( c^M(\cdot) \) are strictly decreasing in \( X \). ■

References


Transtion Function of Relative Productivities

Stable BGP

Unstable BGP

Transition Function of Relative Productivities

\[ \frac{Z_d}{Z_f} \]

Stable BGP

Unstable BGP

Transition

45°
Transition Function: 

$$\frac{Z'}{Z_{l}'} = \Gamma\left(\frac{Z_d}{Z_f}\right)$$

$$\Gamma\left(\frac{Z_d}{Z_f}\right) = \Gamma(0.60571, 0.91446, 45^\circ)$$

$$\Phi\left(\frac{Z_d}{Z_f}\right)$$
Transition Function: \[ \frac{Z_d'}{Z_f'} = \Gamma \left( \frac{Z_d}{Z_f} \right) \]

\[ R_1 = 0.60571 \]

\[ R_L = 0.90728 \]

\[ \alpha \]

\[ \frac{Z_d'}{Z_f'} \]

\[ 45^\circ \]
Transition Function: \( \frac{Z_d}{Z_f} = \Gamma \left( \frac{Z_d}{Z_l} \right) \)