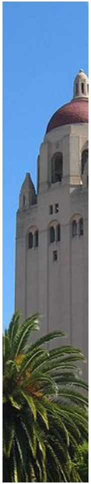
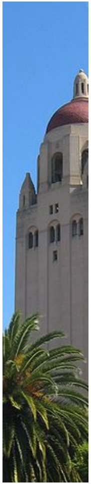


Recipes and Economic Growth: A Combinatorial March Down an Exponential Tail

Chad Jones

NBER Growth Meeting, July 2022

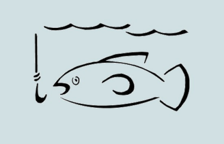


How I Work and Other Random Points

Chad Jones

Stanford GSB

NBER Innovation Bootcamp, July 2023



As for myself, I only like basic problems and could characterize my own research by telling you that when I settled in Woods Hole and took up fishing, I always used an enormous hook. I was convinced that I would catch nothing anyway, and I thought it much more exciting not to catch a big fish than not to catch a small one.

— Albert Szent-Gyorgi, 1893-1986

Nobel Prize, 1937 (discovered Vitamin C)

How I Work

- Find a question that excites you (and others)
- Document the basic facts
- Build a model to try to generate those facts (Lucas, Feynman)
- See what else pops out

The Role of Models

If we understand the process of economic growth — or of anything else — we ought to be capable of demonstrating this knowledge by creating it in these pen and paper (and computer-equipped) laboratories of ours. If we know what an economic miracle is, we ought to be able to make one.

— Robert E. Lucas, Jr.

What I cannot create, I do not understand.

— Richard P. Feynman

Thoughts on Research

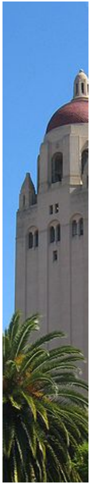
- PPF for economics (macro vs. micro)
- Motivate research by simple, indisputable facts. (cf estimation)
- Build models to explain the facts.
- Keep a “notebook”
- On reading papers
- Try to have research be the thing you think about when sleeping/bathing/etc.

On Writing Papers with Models

- Start as simple as possible (or at least get there eventually!)
- Show entire economic environment (preferences + technology) in one slide and in Table 1 of paper
- Allocating resources: always count equations and unknowns
  - Rule of thumb easiest (Solow)
  - Optimal allocation / social planner: pretty easy and where we'd like to begin
  - Equilibrium: most complicated, and details matter (is there an NSF?). Define it fully and carefully.

Research Questions

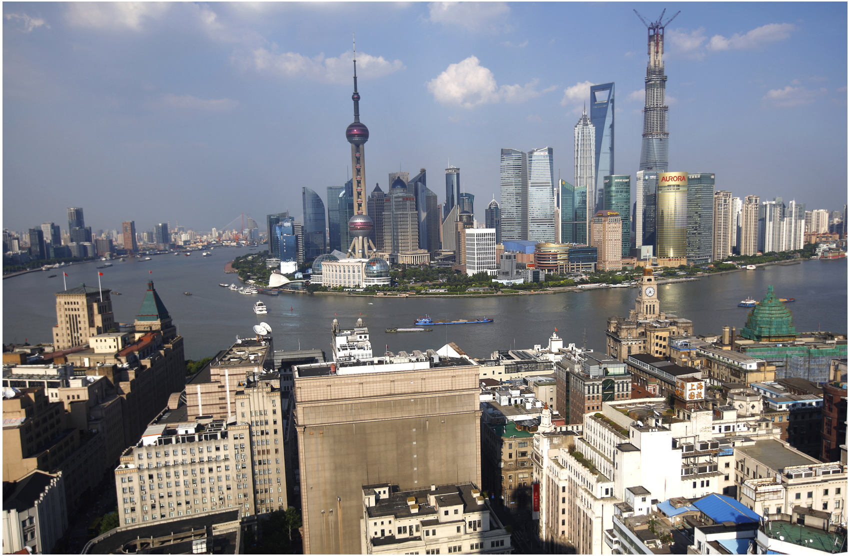
- How do we understand economic growth?
- Why is health spending / GDP rising everywhere?
- A Schumpeterian Model of Top Income Inequality
- The Allocation of Talent and U.S. Economic Growth
- Artificial Intelligence and Economic Growth
- Taxing Top Incomes in a World of Ideas



Other Specific Points

Shanghai 1987

Shanghai 2013



Growth Theory

- Conclusion of any growth theory:

$$\frac{\dot{y}_t}{y_t} = g \quad \text{and a story about g}$$

- Key to this result is (essentially) a linear differential equation somewhere in the model:

$$\dot{X}_t = \_\_ \, X_t$$

- Growth models differ according to what they call the X_t variable and how they fill in the blank.

Catalog of Growth Models: What is X_t?

Solow	$\dot{k}_t = sk_t^{\alpha}$
Solow	$\dot{A}_t = \bar{g}A_t$
AK model	$\dot{K}_t = sAK_t$
Lucas	$\dot{h}_t = uh_t$
Romer/AH	$\dot{A}_t = RA_t$
Semi-endogenous growth	$\dot{L}_t = nL_t$

Why did I write "Are Ideas Getting Harder to Find?" (BJVW 2020 AER)

- In response to the "scale effects" critique:
  - Howitt (1999), Peretto (1998), Young (1998) and others
  - Composition bias: perhaps research productivity *within* every quality ladder is constant, e.g. if number of products N_t grows at the right rate:

$$\frac{\dot{A}_{it}}{A_{it}} = \alpha \, S_{it} \qquad (*)$$

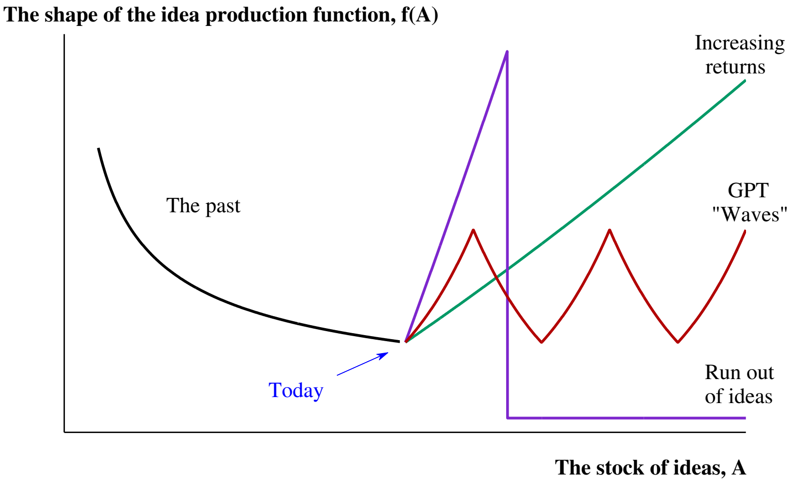
$\Rightarrow S_{it} = \frac{S_t}{N_t}$ invariant to scale, but responds to subsidies

  - Aggregate evidence would then be misleading
  - Permanent subsidies would still have growth effects.

- Key to addressing this concern:

Study (*) directly $\Rightarrow$ research productivity within a variety!

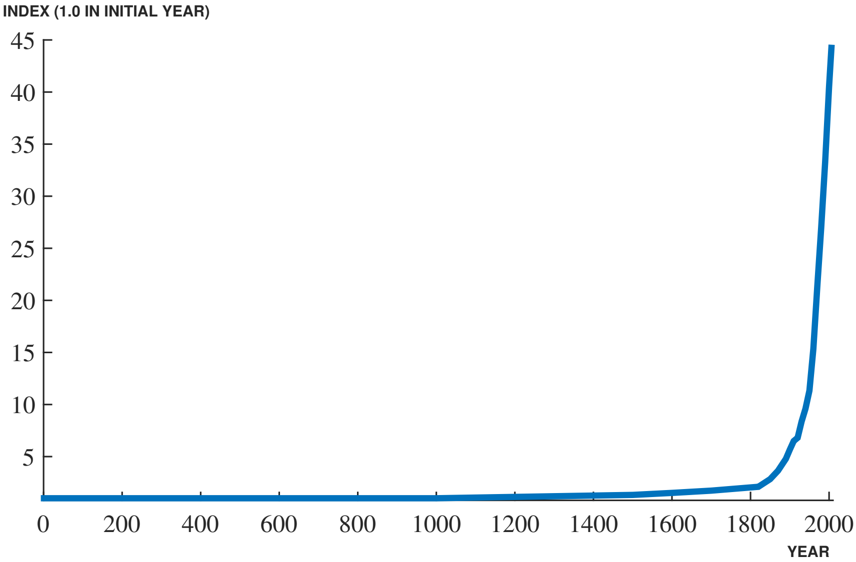
Alternative Futures?



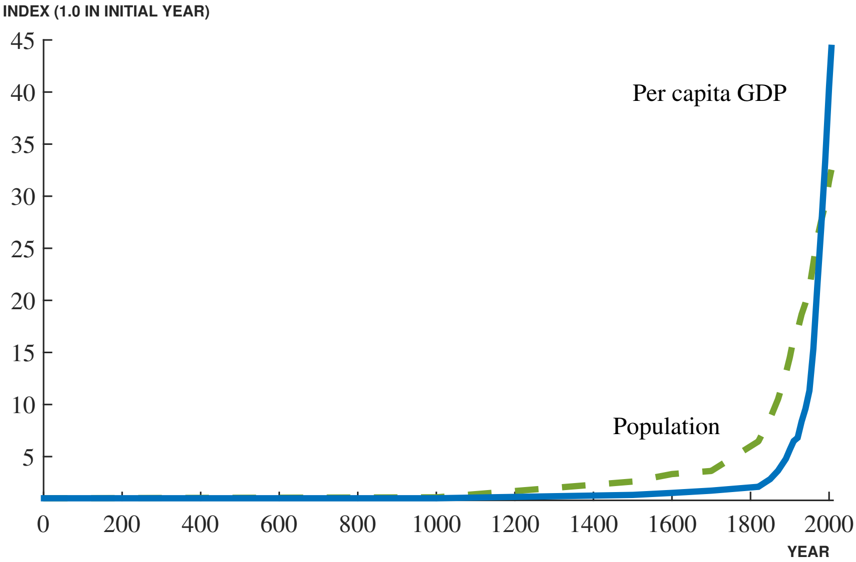
Taxing Top Incomes in a World of Ideas (*JPE* 2022)

- Large literature but interaction with ideas underappreciated.
- Consider raising the top marginal income tax rate from 50% to 75%
  - $\approx 10\%$ of GDP faces the top rate, so mechanically +2.5% GDP in revenue
  - Halving the "keep rate" from 50% to 25% $\Rightarrow$ entrepreneurs may create fewer ideas
  - Akcigit et al (2022 QJE) suggest a behavioral elasticity η of ideas wrt $1-\tau \geq 0.2$
  - Suppose degree of IRS is $\gamma = 1/2$
  - Then lower effort reduces GDP by a factor of $2^{\gamma\eta} = 2^{0.5\times 0.2} = 2^{0.1} \approx 1.07$
- Everyone's income falls by 7%, while tax raises 2.5% of GDP in revenue. Not worth it!
- **Question:** Is the 7% number large or small?

What is graphed here?



Population and Per Capita GDP: the Very Long Run



Growth over the Very Long Run

- Malthus: $c = y = AL^{\alpha}, \quad \alpha < 1$
  - Fixed supply of land: $\uparrow L \;\Rightarrow\; \downarrow c$ holding A fixed
- Story:
  - 100,000 BC: small population $\Rightarrow$ ideas come very slowly
  - New ideas $\Rightarrow$ temporary blip in consumption, but permanently higher population
  - This means ideas come more frequently
  - Eventually, ideas arrive faster than Malthus can reduce consumption!
- People produce ideas and Ideas produce people
  - If nonrivalry $>$ Malthus, this leads to the hockey stick

What is this?

North versus South Korea: Institutions Matter!

Misallocation and TFP: A Simple Example

Production: $X_{steel} = L_{steel}, \quad X_{latte} = L_{latte}$

Resource constraint: $L_{steel} + L_{latte} = \bar{L}$

GDP (aggregation): $Y = X_{steel}^{1/2} X_{latte}^{1/2}$

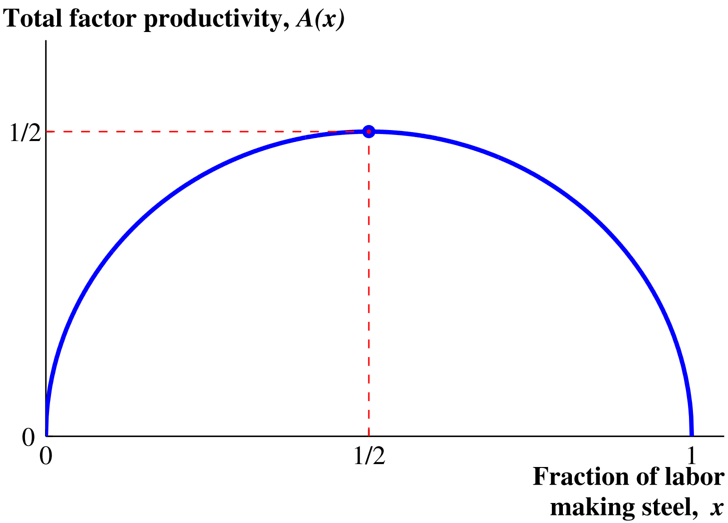
$x \equiv L_{steel}/\bar{L}$ denotes the allocation
(markets, distortions, central planner, etc).

Then GDP and TFP are

$$Y = A(x)\bar{L}$$

$$A(x) = \sqrt{x\,(1-x)}$$

Misallocation Reduces TFP



Misallocation in the United States (HHJK 2019 ECMA)

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- Sandra Day O'Connor, Supreme Court Justice (1981–2006)
  - Graduated 3rd in her class at Stanford Law School, 1952
  - Only job offer in the private sector: legal secretary

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- Consider white men in U.S. business:

  1960: **94%** of doctors, lawyers, and managers

  2010: **60%** of doctors, lawyers, and managers

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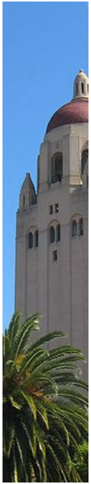
- Consider white men in U.S. business:

  1960: **94%** of doctors, lawyers, and managers

  2010: **60%** of doctors, lawyers, and managers
- Over the past 50 years, the U.S. allocation of talent has improved! Accounts for
  - **40%** of growth in GDP per person, and
  - **20%** of growth in GDP per worker



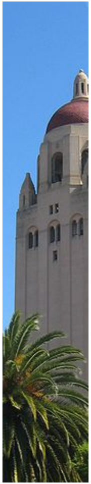
The Past and Future of Economic Growth: A Semi-Endogenous Perspective

Chad Jones

March 2023

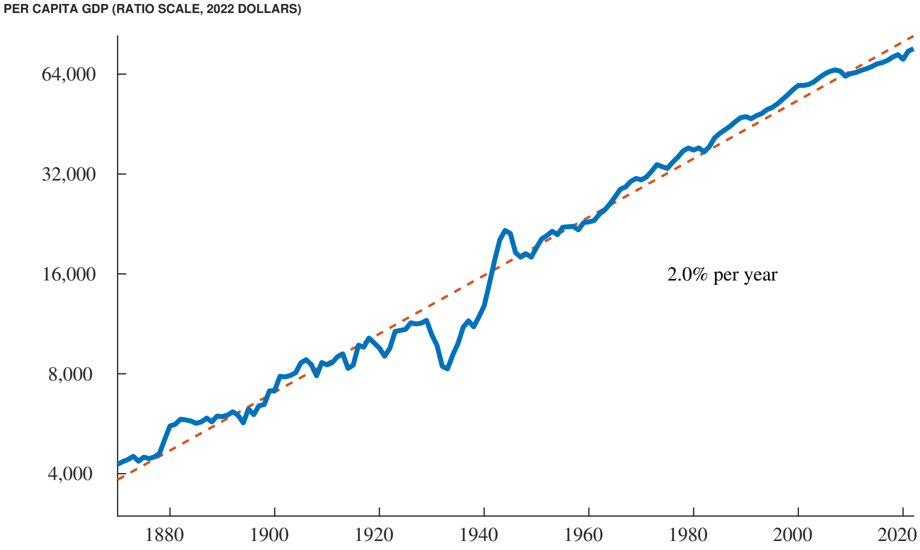
Outline: The Past and Future of Economic Growth

- A simple semi-endogenous growth model
- Historical growth accounting
- Why future growth could slowdown
- Why future growth might not slow and could speed up



A Simple Model of Semi-Endogenous Growth

U.S. GDP per Person



The "Infinite Usability" of Ideas (Paul Romer, 1990)

- Objects: Almost everything in the world
  - Examples: iphones, airplane seats, and surgeons
  - Rival: If I'm using it, you cannot at the same time
  - The fundamental scarcity at the heart of most economics
- Ideas: They are different — nonrival = infinitely useable
  - Can be used by any number of people simultaneously
  - Examples: calculus, HTML, chemical formula of new drug

The Essence of Romer's Insight

- **Question:** In generalizing from the neoclassical model to incorporate ideas (A), why do we write the PF as

$$Y = AK^{\alpha}L^{1-\alpha} \quad (*)$$

instead of

$$Y = A^{\alpha}K^{\beta}L^{1-\alpha-\beta}$$

- Does A go inside the CRS or outside?
  - The "default" (*) is sometimes used, e.g. 1960s
  - 1980s: Griliches et al. put knowledge capital inside CRS

The Nonrivalry of Ideas ⇒ Increasing Returns

- Familiar notation, but now let A_t denote the "stock of knowledge" or ideas:

$$Y_t = F(K_t, L_t, A_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Constant returns to scale in K and L holding knowledge fixed. Why?

$$F(\lambda K, \lambda L, A) = \lambda \times F(K, L, A)$$

- But therefore increasing returns in K, L, and A together!

$$F(\lambda K, \lambda L, \lambda A) > F(\lambda K, \lambda L, A)$$

  - Replication argument + Nonrivalry ⇒ CRS to objects
  - Therefore there must be IRS to objects and ideas

A Simple Model

Final good $Y_t = A_t^{\sigma} L_{yt}$

Ideas $\dot{A}_t = R_t A_t^{\phi} \quad \Rightarrow \quad \frac{\dot{A}_t}{A_t} = R_t A_t^{-\beta}$

Resource constraint $R_t + L_{yt} = L_t = L_0 e^{nt}$

Allocation $R_t = \bar{s} L_t, \quad 0 < \bar{s} < 1$

ϕ captures knowledge spillovers.

$$\beta \equiv 1 - \phi > 0$$

A Simple Model

Final good $Y_t = A_t^{\sigma} L_{yt}$

$$y_t \equiv \frac{Y_t}{L_t} = A_t^{\sigma}(1-\bar{s})$$

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On BGP, $\dot{A}/A = \text{Constant} \Rightarrow$

$$A_t^* = \text{Constant} \cdot R_t^{\frac{1}{\beta}}$$

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On BGP, $\dot{A}/A = \text{Constant} \;\Rightarrow$

$$A_t^* = \text{Constant} \cdot R_t^{\frac{1}{\beta}}$$

Combine these two equations...

Steady State of the Simple Model

- Level of income on the BGP (where $\gamma \equiv \frac{\sigma}{\beta}$)

$$y_t^* = \text{Constant} \cdot R_t^{\gamma}$$

$\Rightarrow$ BGP growth rate:

$$g_y = \frac{\sigma n}{\beta} = \gamma n$$

Long-Run Growth	=	Degree of IRS, $\gamma \equiv \frac{\sigma}{\beta}$	$\times$	Rate at which scale grows

What's the difference between these two equations?

Romer	$y_t = A_t^{\sigma}$
Solow	$y_t = k_t^{\alpha}$

Hint: It's not the exponent: $\sigma = \alpha = 1/3$ is possible

What’s the difference between these two equations?

Romer	$y_t = A_t^\sigma$
Solow	$y_t = k_t^\alpha$

Hint: It’s not the exponent: $\sigma = \alpha = 1/3$ is possible

A_t is an aggregate, while k_t is per capita

But easy to make aggregates grow: population growth!

Or put in words...

- Objects: Add 1 computer $\Rightarrow$ make 1 worker more productive; for a million workers, need 1 million computers

  Output per worker $\sim$ # of computers per worker

- Ideas: Add 1 new idea $\Rightarrow$ make unlimited # more productive or better off.
  - E.g. cure for lung cancer, drought-resistant seeds, spreadsheet

  Income per person $\sim$ the aggregate stock of knowledge, not on the number of ideas per person.

But it is easy to make aggregates grow: population growth!

IRS $\Rightarrow$ bigger is better.

Where does growth ultimately come from?

More people $\Rightarrow$ more ideas $\Rightarrow$ higher income / person

That's IRS associated with the nonrivalry of ideas

Evidence for Semi-Endogenous Growth (Bloom et al 2020)

- Document a new stylized fact:

  Exponential growth is getting harder to achieve.

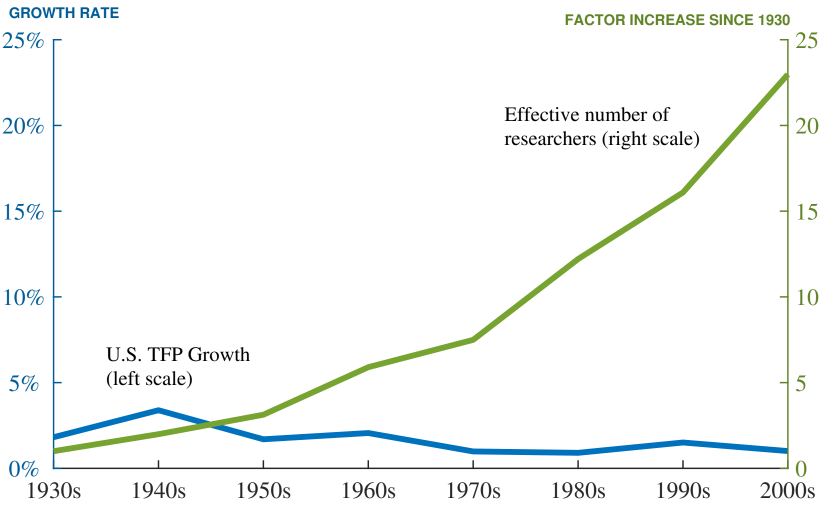
Economic growth	=	Research productivity	$\times$	Number of researchers
e.g. 2% or 5%		$\downarrow$ (falling)		$\uparrow$ (rising)

- Consistent with the SEG model:

$$\frac{\dot{A}_t}{A_t} = R_t A_t^{-\beta}$$

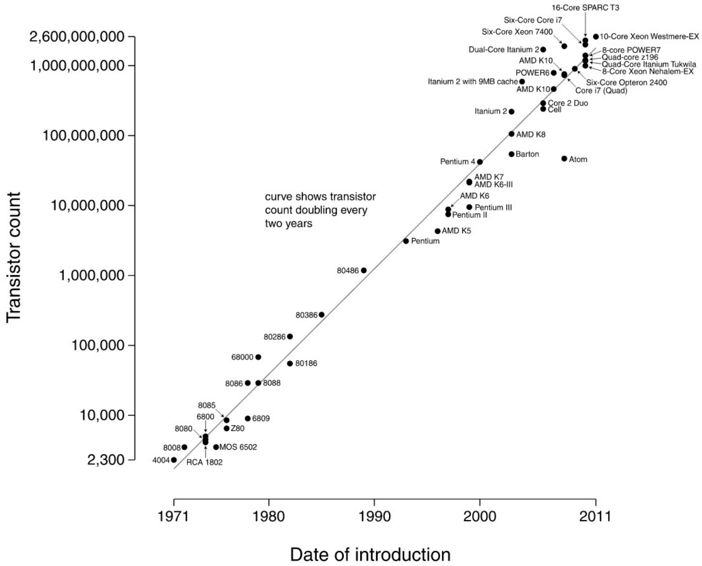
$\beta > 0 \Rightarrow$ *ideas are getting harder to find*

Evidence: Aggregate U.S. Economy

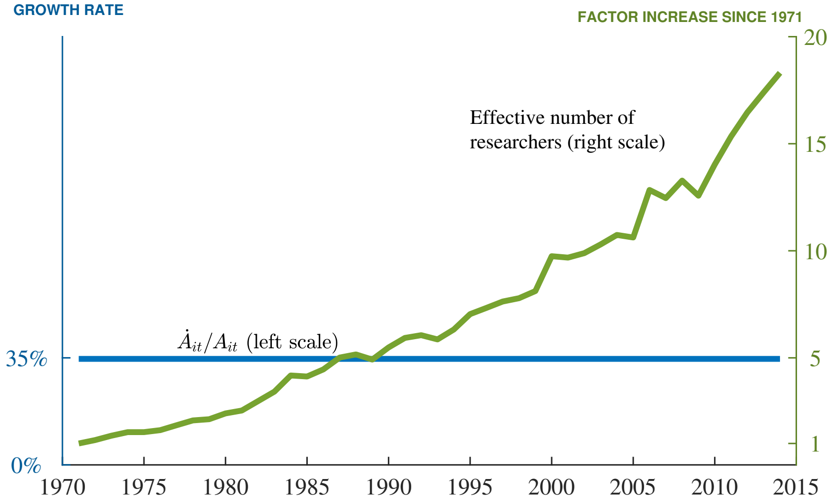


Bloom, Jones, Van Reenen, and Webb (2020)

The Steady Exponential Growth of Moore's Law



Evidence: Moore's Law



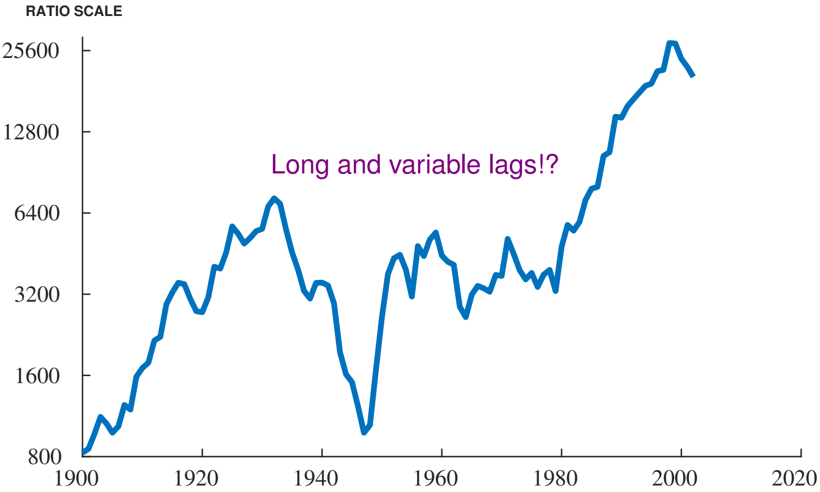
Bloom, Jones, Van Reenen, and Webb (2020)

Summary of Evidence

- Moore's Law
  - 18x harder today to generate the doubling of chip density
  - Have to double research input every decade!
- Qualitatively similar findings in rest of the economy
  - Agricultural innovation (yield per acre of corn and soybeans)
  - Medical innovations (new drugs or mortality from cancer/heart disease)
  - Publicly-traded firms
  - Aggregate economy

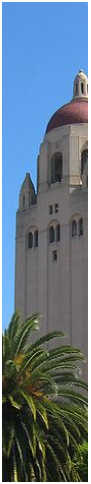
New ideas are getting harder to find!

Breakthrough Patents from Kelly, Papanikolaou, Seru, Taddy (2021)



Literature Review

- Early Semi-Endogenous Growth Models
  - Arrow (1962), Phelps (1966), Nordhaus (1969), Judd (1985)
  - Jones (1995), Kortum (1997), Segerstrom (1998)
- Broader Literature: Models with IRS are SEG models!
  - Trade models: Krugman (1979), Eaton-Kortum (2002), Ramondo et al (2016)
  - Firm dynamics: Melitz (2003), Atkeson-Burstein (2019), Peters-Walsh (2021)
  - Sectoral heterogeneity: Ngai-Samaniego ('11), Bloom etc ('20), Sampson ('20)
  - Technology diffusion: Klenow-Rodriguez (2005), Buera-Oberfield (2020)
  - Economic geography: Redding-RossiHansberg (2017)



Historical Growth Accounting

In LR, all growth from population growth. But historically...?

Extended Model

- Include physical capital K, human capital per person h, and misallocation M

$$Y_t = K_t^{\alpha}(Z_t h_t L_{Yt})^{1-\alpha}$$

$$Z_t \equiv A_t M_t$$

$$A_t^* = R_t^{\gamma} = (s_t L_t)^{\gamma}$$

- Write in terms of output per person and rearrange:

$$y_t = \left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} A_t M_t h_t \ell_t (1 - s_t)$$

- In LR, all growth from population growth. But historically...?

Growth Accounting Equations

$$\underbrace{d\log y_t}_{\text{GDP per person}} = \underbrace{\frac{\alpha}{1-\alpha} d\log\frac{K_t}{Y_t}}_{\text{Capital-Output ratio}} + \underbrace{d\log h_t}_{\text{Educational att.}} + \underbrace{d\log \ell_t}_{\text{Emp-Pop ratio}} + \underbrace{d\log(1-s_t)}_{\text{Goods intensity}}$$

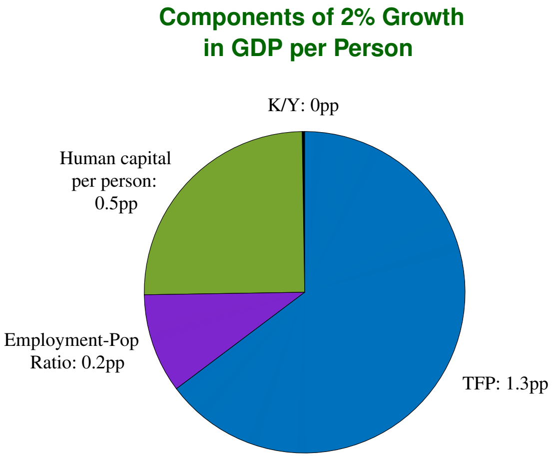
$$+ \underbrace{d\log M_t + d\log A_t}_{\text{TFP growth}}$$

where

$$\text{TFP growth} \equiv \underbrace{d\log M_t}_{\text{Misallocation}} + \underbrace{d\log A_t}_{\text{Ideas}} = \underbrace{d\log M_t}_{\text{Misallocation}} + \underbrace{\gamma\, d\log s_t}_{\text{Research intensity}} + \underbrace{\gamma\, d\log L_t}_{\text{LF growth}}$$

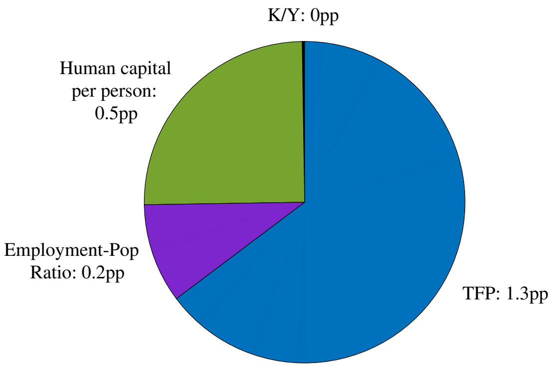
All terms are zero in the long run, other than γn. Assume $\gamma = 1/3$

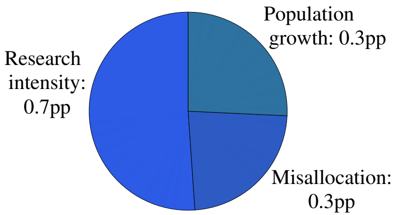
Historical Growth Accounting in the U.S., 1950s to Today



Historical Growth Accounting in the U.S., 1950s to Today

Components of 2% Growth in GDP per Person

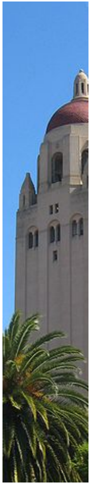




Components of 1.3% TFP Growth

Summary of Growth Accounting

- Even in a semi-endogenous growth framework where all LR growth is γn,
  - Other factors explain more than 80% of historical growth
- Transitory factors have been very important, but all must end:
  - rising educational attainment
  - rising LF participation
  - declining misallocation
  - increasing research intensity
- Implication: Unless something changes, growth must slow down!
  - The long-run growth rate is $\approx$ 0.3%, not 2%

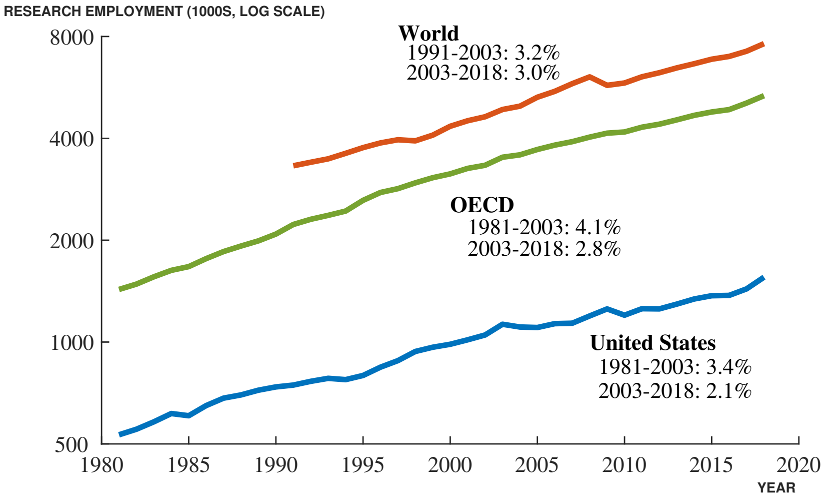


Why Future Growth might be Slower

Why Future Growth might be Slower

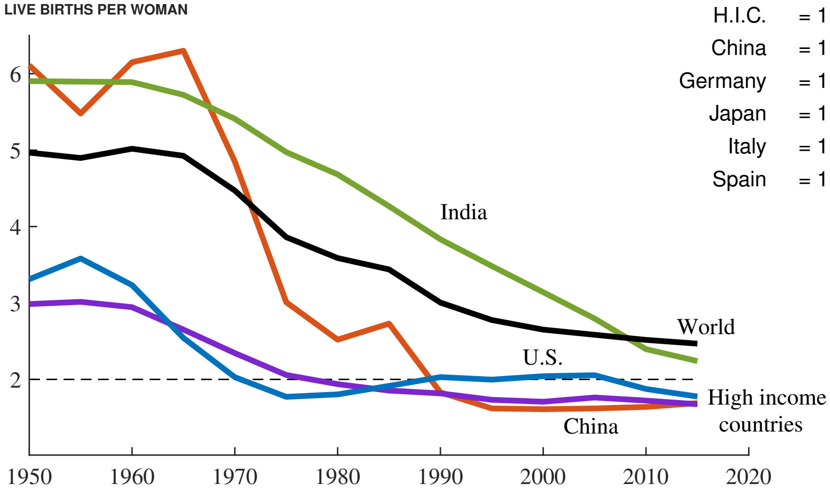
- Growth accounting exercise just presented: $\gamma n \approx 0.3\%$
- Slowdown in the growth rate of research
- Slowing population growth

Research Employment in the U.S., OECD, and World



The Total Fertility Rate (Live Births per Woman)

U.S.	= 1.8
H.I.C.	= 1.7
China	= 1.7
Germany	= 1.6
Japan	= 1.4
Italy	= 1.3
Spain	= 1.3



What happens if future population growth is negative?

- Suppose population *declines* exponentially at rate η: $R_t = R_0 e^{-\eta t}$
- Production of ideas

$$\frac{\dot{A}_t}{A_t} = R_t A_t^{-\beta} = R_0 A_t^{-\beta} e^{-\eta t}$$

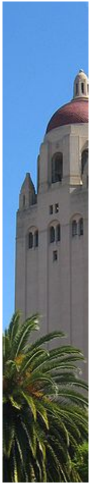
- Integrating reveals that A_t asymptotes to a constant!

$$A^* = \begin{cases} A_0 \left(1 + \frac{\beta g_{A0}}{\eta}\right)^{1/\beta} & \text{if } \beta > 0 \\ A_0 \exp\left(\frac{g_{A0}}{\eta}\right) & \text{if } \beta = 0 \end{cases}$$

Source: Jones (2022) “The End of Economic Growth...”

The Empty Planet Result

- Fertility has trended down: 5, 4, 3, 2, and less in rich countries
  - For a family, nothing special about “above 2” vs “below 2”
  - But macroeconomics makes this distinction critical!
- Standard result shown earlier: $n > 0 \Rightarrow$ **Expanding Cosmos**
  - Exponential growth in income and population
- Negative population growth $\Rightarrow$ much more pessimistic **Empty Planet**
  - Stagnating living standards for a population that vanishes
  - Could this be our future?



Why Future Growth might be Faster?

(Or at least not as slow as the preceding section implies!)

1. Finding Lost Einsteins
2. Automation and artificial intelligence

Finding Lost Einsteins

- How many Edisons and Doudnas have we missed out on historically?
  - The rise of China, India, and other emerging countries
    - China and India each have as many people as U.S.+Europe+Japan
  - Brouillette (2022): Only 3% of inventors were women in 1976; only 12% in 2016
  - Bell et al (2019): Poor people missing opportunities

- Increase global research by a factor of 3 or 7?
  - For $\gamma = 1/3$: Increase incomes by $3^\gamma - 1 = 40\%$ and $7^\gamma - 1 = 90\%$
  - Could easily raise growth by 0.2pp to 0.4pp for a century

Automation and A.I.

- Suppose research involves many tasks X_i that can be done by people or by machines

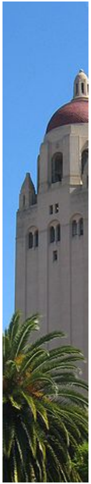
$$\dot{A}_t = A_t^{1-\beta} X_1^{\alpha_1} X_2^{\alpha_2} \cdot \ldots \cdot X_n^{\alpha_n}, \quad \sum \alpha_i = 1$$
$$= A_t^{1-\beta} K_t^{\alpha} R_t^{1-\alpha}$$

α is the fraction of research tasks that have been automated

- Long-run growth rate:

$$g_A = \frac{n}{\beta - \alpha}$$

- Rising automation could raise economic growth
  - Singularity if $\alpha = \beta$ (or at least all possible ideas get discovered quickly)
  - Labs, computers, WWW: recent automation has not offset slowing growth



Conclusion: Key Outstanding Questions

Important Questions for Future Research

- How large is the degree of IRS associated with ideas, γ?
- What is the social rate of return to research?
  - Are we underinvesting in basic research?
- Better growth accounting: contributions from DARPA, NIH, migration of European scientists during WWII, migration more generally
- Automation ongoing for 150 years, but growth slowing not rising: why?

Combinatorics and Pareto

- Weitzman (1998) and Romer (1993) suggest combinatorics important for growth.
  - Ideas are combinations of ingredients
  - The number of possible combinations from a child's chemistry set exceeds the number of atoms in the universe
  - But absent from state-of-the-art growth models?
- Kortum (1997) and Gabaix (1999) on Pareto distributions
  - Kortum: Draw productivities from a distribution $\Rightarrow$ Pareto tail is essential
  - Gabaix: Pareto distribution (cities, firms, income) *results from* exponential growth

Do we really need the fundamental idea distribution to be Pareto?

Two Contributions

- A simple but useful theorem about extreme values
  - The increase of the max extreme value depends on
    (1) the way the number of draws rises, and
    (2) the shape of the upper tail
  - Applies to any continuous distribution

- Combinatorics and growth theory
  - Combinatorial growth: Cookbook of 2^N recipes from N ingredients, with N growing exponentially (population growth)

    Combinatorial growth with draws from thin-tailed distributions (e.g. the normal distribution) yields exponential growth

  - Pareto distributions are not required — draw faster from a thinner tail

Theorem (A Simple Extreme Value Result)

Let Z_K denote the maximum value from K i.i.d. draws from a continuous distribution $F(x)$, with $\bar{F}(x) \equiv 1 - F(x)$ strictly decreasing on its support. Then for $m \geq 0$

$$\lim_{K \to \infty} \Pr \left[\; K\bar{F}(Z_K) \geq m \; \right] = e^{-m}$$

As K increases, the max Z_K rises so as to stabilize $K\bar{F}(Z_K)$.

The shape of the tail of $\bar{F}(\cdot)$ and the way K increases determines the rise in Z_K

Intuition

$$K\bar{F}(Z_K) = \varepsilon + o_p(1)$$

$$\Rightarrow \quad \bar{F}(Z_K) = \text{Pr}\left[\text{ Next draw } > Z_K\right] \sim \frac{1}{K}$$

- Theory of records: Suppose K i.i.d. draws for temperatures.
  - Unconditional probability that tomorrow is a new record high = $1/K$
  - This result is similar, but conditional instead of unconditional
- Apart from randomness from conditioning, $\bar{F}(Z_K)$ falls like $1/K$ for any distribution!

Proof of Theorem 1

- Given that Z_K is the max over K i.i.d. draws, we have

$$\begin{aligned}\text{Pr}\left[Z_K \leq x\right] &= \text{Pr}\left[z_1 \leq x, z_2 \leq x, \ldots, z_K \leq x\right] \\ &= (1 - \bar{F}(x))^K\end{aligned}$$

- Let $M_K \equiv K\bar{F}(Z_K)$ denote a new random variable. Then for $0 < m < K$

$$\begin{aligned}\text{Pr}\left[M_K \geq m\right] &= \text{Pr}\left[K\bar{F}(Z_K) \geq m\right] \\ &= \text{Pr}\left[\bar{F}(Z_K) \geq \frac{m}{K}\right] \\ &= \text{Pr}\left[Z_K \leq \bar{F}^{-1}\left(\frac{m}{K}\right)\right] \\ &= \left(1 - \frac{m}{K}\right)^K \to e^{-m} \quad \text{QED.}\end{aligned}$$

Example: Kortum (1997)

- Pareto: $\bar{F}(x) = x^{-\beta}$
- Apply Theorem 1:

$$\begin{aligned} K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\ KZ_K^{-\beta} &= \varepsilon + o_p(1) \\ \frac{K}{Z_K^{\beta}} &= \varepsilon + o_p(1) \\ \frac{Z_K}{K^{1/\beta}} &= (\varepsilon + o_p(1))^{-1/\beta} \end{aligned}$$

- Exponential growth in K leads to exponential growth in Z_K

$$g_Z = g_K/\beta$$

β = how thin is the tail = rate at which ideas become harder to find

Example: Drawing from a Weibull Distribution

- Weibull: $\bar{F}(x) = e^{-x^\beta}$ (notice $\beta = 1$ is just exponential)

$$
\begin{aligned}
K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\
Ke^{-Z_K^\beta} &= \varepsilon + o_p(1) \\
\Rightarrow \quad \log K - Z_K^\beta &= \log(\varepsilon + o_p(1)) \\
\Rightarrow \quad Z_K &= \left(\log K - \log(\varepsilon + o_p(1))\right)^{1/\beta} \\
\Rightarrow \quad \frac{Z_K}{(\log K)^{1/\beta}} &= \left(1 - \frac{\log(\varepsilon + o_p(1))}{\log K}\right)^{1/\beta}
\end{aligned}
$$

$$
\frac{Z_K}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}
$$

Drawing from a Weibull (continued)

$$\frac{Z_K}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}$$

- Z_K grows with $(\log K)^{1/\beta}$
  - If K grows exponentially and $\beta = 1$, then Z_K grows linearly
  - More generally, growth rate falls to zero for any β
- Definition of **combinatorial growth:** $K_t = 2^{N_t}$ with $N_t = N_0 e^{g_N t}$

$$g_Z = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta}$$

Combinatorial growth with draws from a thin-tailed distribution delivers exponential growth!

Theorem (A general condition for combinatorial growth)

Consider the full growth model (skipped in these slides) but with $z_i \sim F(z)$ as a general continuous and unbounded distribution, where $F(\cdot)$ is monotone and differentiable. Let $\eta(x)$ denote the elasticity of the tail cdf $\bar{F}(x)$; that is, $\eta(x) \equiv -\frac{d \log \bar{F}(x)}{d \log x}$. Then

$$\lim_{t \to \infty} \frac{\dot{Z}_{Kt}}{Z_{Kt}} = \frac{g_N}{\alpha}$$

if and only if

$$\lim_{x \to \infty} \frac{\eta(x)}{x^{\alpha}} = \text{Constant} > 0$$

for some $\alpha > 0$.

Remarks

$$\frac{\dot{Z}_{Kt}}{Z_{Kt}} \to \frac{g_N}{\alpha} \iff \lim_{x\to\infty} \frac{\eta(x)}{x^\alpha} = \text{Constant} > 0$$

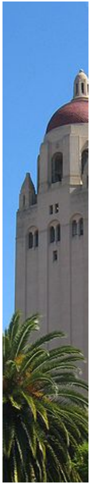
- Thinner tails require faster draws but still require power functions:
  - It's just that the elasticity itself is now a power function!
- Examples
  - Weibull: $\bar{F}(x) = e^{-x^\beta} \Rightarrow \eta(x) = x^\beta$
  - Normal: $\bar{F}(x) = 1 - \int_{-\infty}^{x} e^{-u^2/2} du \Rightarrow \eta(x) \sim x^2$ – like Weibull with $\beta = 2$
- Intuition
  - Kortum (1997): $\bar{F}(x) = x^{-\beta} \Rightarrow \eta(x) = \beta$ so $K_t = e^{nt}$ is enough
  - Here: $\bar{F}(x) = e^{-x^\beta}$ so must march down tail exponentially faster, $K_t = 2^{e^{nt}}$

For what distributions do combinatorial draws $\Rightarrow$ exponential growth?

- Combinatorial draws lead to exponential growth for many familiar distributions:
  - Normal, Exponential, Weibull, Gumbel
  - Gamma, Logistic, Benktander Type I and Type II
  - Generalized Weibull: $\bar{F}(x) = x^{\alpha} e^{-x^{\beta}}$ or $\bar{F}(x) = e^{-(x^{\beta}+x^{\alpha})}$
  - Tail is dominated by "exponential of a power function"
- When does it not work?
  - lognormal: If it works for normal, then $\log x \sim$ Normal means percentage increments are normal, so tail will be too thick!
  - logexponential = Pareto
  - Surprise: Does *not* work for all distributions in the Gumbel domain of attraction (not parallel to Kortum/Frechet).

Scaling of Z_K for Various Distributions

Distribution	cdf	Z_K behaves like	Growth rate of Z_K for $K = 2^N$
Exponential	$1 - e^{-\theta x}$	$\log K$	g_N
Gumbel	$e^{-e^{-x}}$	$\log K$	g_N
Weibull	$1 - e^{-x^\beta}$	$(\log K)^{1/\beta}$	$\frac{g_N}{\beta}$
Normal	$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} dx$	$(\log K)^{1/2}$	$\frac{g_N}{2}$
Lognormal	$\frac{1}{\sqrt{2\pi}} \int e^{-(\log x)^2/2} dx$	$\exp(\sqrt{\log K})$	$\frac{g_N}{2} \cdot \sqrt{N}$
Gompertz	$1 - \exp(-(e^{\beta x} - 1))$	$\frac{1}{\beta} \log(\log K)$	Arithmetic
Log-Pareto	$1 - \frac{1}{(\log x)^\alpha}$	$\exp(K^{1/\alpha})$	Romer!



Evidence from Patents

Combinatorial growth matches the patent data

Rate of Innovation?

- Kortum (1997) was designed to match a key "fact": that the flow of patents was stationary
  - Never clear this fact was true (see below)
- Flow of patents in the model?
  - Theory of record-breaking: $p(K) = 1/K$ is the fraction of ideas that are improvements [cf Theorem 1: $\bar{F}(Z_K) = \frac{1}{K}(\varepsilon + o_p(1))$]
  - Since there are $\dot{K}$ recipes added to the cookbook every instant, the flow of patents is

$$p(K)\dot{K} = \frac{\dot{K}_t}{K_t}$$

  - This is constant in Kortum (1997) $\Rightarrow$ constant flow of patents

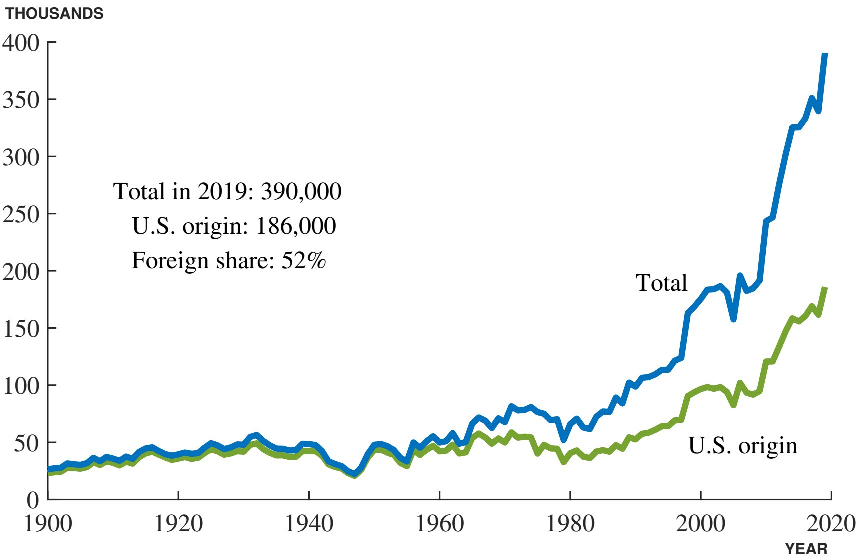
Flow of Patents in Combinatorial Growth Model?

- Simple case: $\dot{N}_t = \alpha R_t$ (i.e. $\lambda = 1$ and $\phi = 0$).
- Then

$$
\begin{aligned}
K_t &= 2^{N_t} \\
\Rightarrow \frac{\dot{K}_t}{K_t} &= \log 2 \cdot \dot{N}_t \\
&= \log 2 \cdot \alpha R_t \\
&= \log 2 \cdot \alpha \bar{s} L_0 e^{g_L t}
\end{aligned}
$$

- That is, the combinatorial growth model predicts that the number of new patents should grow exponentially over time
    - When ideas are small, it takes a growing number to generate exponential growth

Annual Patent Grants by the U.S. Patent and Trademark Office



Conclusion

- $K\bar{F}(Z_K) \sim \varepsilon$ links K and the shape of the tail cdf to how the max increases
- Weitzman meets Kortum: Combinatorial growth in recipes whose productivities are draws from a thin-tailed distribution gives rise to exponential growth
- Other applications: wherever Pareto has been assumed in the literature, perhaps we can use thin tails?
  - Many literatures: technology diffusion, trade, search, productivity
  - If ideas are "small," need enhanced theory of markups and heterogeneity