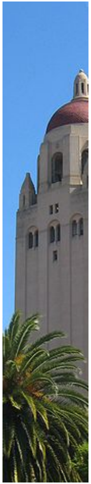

Recipes and Economic Growth: A Combinatorial March Down an Exponential Tail

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Combinatorics and Pareto

- Weitzman (1998) and Romer (1993) suggest combinatorics important for growth.
  - Ideas are combinations of ingredients
  - The number of possible combinations from a child's chemistry set exceeds the number of atoms in the universe
  - But absent from state-of-the-art growth models?
- Kortum (1997) and Gabaix (1999) on Pareto distributions
  - Kortum: Draw productivities from a distribution $\Rightarrow$ Pareto tail is essential
  - Gabaix: Pareto distribution (cities, firms, income) *results from* exponential growth

Do we really need the fundamental idea distribution to be Pareto?

Two Contributions

- A simple but useful theorem about extreme values
  - The increase of the max extreme value depends on
    (1) the way the number of draws rises, and
    (2) the shape of the upper tail
  - Applies to any continuous distribution

- Combinatorics and growth theory
  - Combinatorial growth: Cookbook of 2^N recipes from N ingredients, with N growing exponentially (population growth)

    Combinatorial growth with draws from thin-tailed distributions (e.g. the normal distribution) yields exponential growth

  - Pareto distributions are not required — draw faster from a thinner tail

Theorem (A Simple Extreme Value Result)

Let Z_K denote the maximum value from K i.i.d. draws from a continuous distribution $F(x)$, with $\bar{F}(x) \equiv 1 - F(x)$ strictly decreasing on its support. Then for $m \geq 0$

$$\boxed{\lim_{K\to\infty} \text{Pr}\left[\, K\bar{F}(Z_K) \geq m \,\right] = e^{-m}}$$

As K increases, the max Z_K rises so as to stabilize $K\bar{F}(Z_K)$.

The shape of the tail of $\bar{F}(\cdot)$ and the way K increases determines the rise in Z_K

Intuition

$$K\bar{F}(Z_K) = \varepsilon + o_p(1)$$

$$\Rightarrow \quad \bar{F}(Z_K) = \text{Pr}\left[\text{ Next draw } > Z_K\right] \sim \frac{1}{K}$$

- Theory of records: Suppose K i.i.d. draws for temperatures.
  - Unconditional probability that tomorrow is a new record high = $1/K$
  - This result is similar, but conditional instead of unconditional
- Apart from randomness from conditioning, $\bar{F}(Z_K)$ falls like $1/K$ for any distribution!

Proof of Theorem 1

- Given that Z_K is the max over K i.i.d. draws, we have

$$\begin{aligned}\text{Pr}\left[Z_K \leq x\right] &= \text{Pr}\left[z_1 \leq x, z_2 \leq x, \ldots, z_K \leq x\right] \\ &= (1 - \bar{F}(x))^K\end{aligned}$$

- Let $M_K \equiv K\bar{F}(Z_K)$ denote a new random variable. Then for $0 < m < K$

$$\begin{aligned}\text{Pr}\left[M_K \geq m\right] &= \text{Pr}\left[K\bar{F}(Z_K) \geq m\right] \\ &= \text{Pr}\left[\bar{F}(Z_K) \geq \frac{m}{K}\right] \\ &= \text{Pr}\left[Z_K \leq \bar{F}^{-1}\left(\frac{m}{K}\right)\right] \\ &= \left(1 - \frac{m}{K}\right)^K \rightarrow e^{-m} \quad \text{QED.}\end{aligned}$$

Example: Kortum (1997)

- Pareto: $\bar{F}(x) = x^{-\beta}$
- Apply Theorem 1:

$$\begin{aligned} K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\ KZ_K^{-\beta} &= \varepsilon + o_p(1) \\ \frac{K}{Z_K^{\beta}} &= \varepsilon + o_p(1) \\ \frac{Z_K}{K^{1/\beta}} &= (\varepsilon + o_p(1))^{-1/\beta} \end{aligned}$$

- Exponential growth in K leads to exponential growth in Z_K

$$g_Z = g_K/\beta$$

β = how thin is the tail = rate at which ideas become harder to find

Example: Drawing from a Weibull Distribution

- Weibull: $\bar{F}(x) = e^{-x^{\beta}}$ (notice $\beta = 1$ is just exponential)

$$
\begin{aligned}
K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\
Ke^{-Z_K^{\beta}} &= \varepsilon + o_p(1) \\
\Rightarrow \quad \log K - Z_K^{\beta} &= \log(\varepsilon + o_p(1)) \\
\Rightarrow \quad Z_K &= \left(\log K - \log(\varepsilon + o_p(1))\right)^{1/\beta} \\
\Rightarrow \quad \frac{Z_K}{(\log K)^{1/\beta}} &= \left(1 - \frac{\log(\varepsilon + o_p(1))}{\log K}\right)^{1/\beta}
\end{aligned}
$$

$$
\frac{Z_K}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}
$$

Drawing from a Weibull (continued)

$$\frac{Z_K}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}$$

- Z_K grows with $(\log K)^{1/\beta}$
  - If K grows exponentially and $\beta = 1$, then Z_K grows linearly
  - More generally, growth rate falls to zero for any β
- Definition of **combinatorial growth:** $K_t = 2^{N_t}$ with $N_t = N_0 e^{g_N t}$

$$g_Z = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta}$$

Combinatorial growth with draws from a thin-tailed distribution delivers exponential growth!

Theorem (A general condition for combinatorial growth)

Consider the full growth model (skipped in these slides) but with $z_i \sim F(z)$ as a general continuous and unbounded distribution, where $F(\cdot)$ is monotone and differentiable. Let $\eta(x)$ denote the elasticity of the tail cdf $\bar{F}(x)$; that is, $\eta(x) \equiv -\frac{d \log \bar{F}(x)}{d \log x}$. Then

$$\lim_{t \to \infty} \frac{\dot{Z}_{Kt}}{Z_{Kt}} = \frac{g_N}{\alpha}$$

if and only if

$$\lim_{x \to \infty} \frac{\eta(x)}{x^{\alpha}} = \text{Constant} > 0$$

for some $\alpha > 0$.

Remarks

$$\frac{\dot{Z}_{Kt}}{Z_{Kt}} \to \frac{g_N}{\alpha} \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\eta(x)}{x^\alpha} = \text{Constant} > 0$$

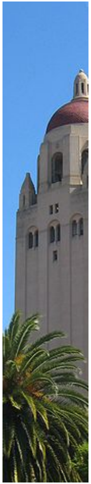
- Thinner tails require faster draws but still require power functions:
  - It's just that the elasticity itself is now a power function!
- Examples
  - Weibull: $\bar{F}(x) = e^{-x^\beta} \Rightarrow \eta(x) = x^\beta$
  - Normal: $\bar{F}(x) = 1 - \int_{-\infty}^{x} e^{-u^2/2} du \Rightarrow \eta(x) \sim x^2$ – like Weibull with $\beta = 2$
- Intuition
  - Kortum (1997): $\bar{F}(x) = x^{-\beta} \Rightarrow \eta(x) = \beta$ so $K_t = e^{nt}$ is enough
  - Here: $\bar{F}(x) = e^{-x^\beta}$ so must march down tail exponentially faster, $K_t = 2^{e^{nt}}$

For what distributions do combinatorial draws $\Rightarrow$ exponential growth?

- Combinatorial draws lead to exponential growth for many familiar distributions:
  - Normal, Exponential, Weibull, Gumbel
  - Gamma, Logistic, Benktander Type I and Type II
  - Generalized Weibull: $\bar{F}(x) = x^{\alpha} e^{-x^{\beta}}$ or $\bar{F}(x) = e^{-(x^{\beta}+x^{\alpha})}$
  - Tail is dominated by "exponential of a power function"
- When does it not work?
  - lognormal: If it works for normal, then $\log x \sim$ Normal means percentage increments are normal, so tail will be too thick!
  - logexponential = Pareto
  - Surprise: Does *not* work for all distributions in the Gumbel domain of attraction (not parallel to Kortum/Frechet).

Scaling of Z_K for Various Distributions

Distribution	cdf	Z_K behaves like	Growth rate of Z_K for $K = 2^N$
Exponential	$1 - e^{-\theta x}$	$\log K$	g_N
Gumbel	$e^{-e^{-x}}$	$\log K$	g_N
Weibull	$1 - e^{-x^\beta}$	$(\log K)^{1/\beta}$	$\frac{g_N}{\beta}$
Normal	$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} dx$	$(\log K)^{1/2}$	$\frac{g_N}{2}$
Lognormal	$\frac{1}{\sqrt{2\pi}} \int e^{-(\log x)^2/2} dx$	$\exp(\sqrt{\log K})$	$\frac{g_N}{2} \cdot \sqrt{N}$
Gompertz	$1 - \exp(-(e^{\beta x} - 1))$	$\frac{1}{\beta} \log(\log K)$	Arithmetic
Log-Pareto	$1 - \frac{1}{(\log x)^\alpha}$	$\exp(K^{1/\alpha})$	Romer!



Evidence from Patents

Combinatorial growth matches the patent data

Rate of Innovation?

- Kortum (1997) was designed to match a key "fact": that the flow of patents was stationary
  - Never clear this fact was true (see below)

- Flow of patents in the model?
  - Theory of record-breaking: $p(K) = 1/K$ is the fraction of ideas that are improvements [cf Theorem 1: $\bar{F}(Z_K) = \frac{1}{K}(\varepsilon + o_p(1))$]
  - Since there are $\dot{K}$ recipes added to the cookbook every instant, the flow of patents is

$$p(K)\dot{K} = \frac{\dot{K}_t}{K_t}$$

  - This is constant in Kortum (1997) $\Rightarrow$ constant flow of patents

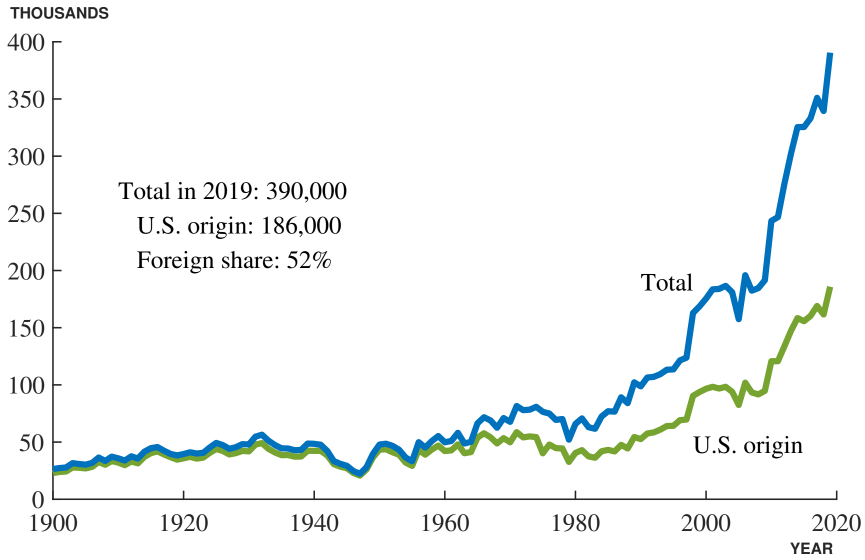
Flow of Patents in Combinatorial Growth Model?

- Simple case: $\dot{N}_t = \alpha R_t$ (i.e. $\lambda = 1$ and $\phi = 0$).
- Then

$$\begin{aligned} K_t &= 2^{N_t} \\ \Rightarrow \frac{\dot{K}_t}{K_t} &= \log 2 \cdot \dot{N}_t \\ &= \log 2 \cdot \alpha R_t \\ &= \log 2 \cdot \alpha \bar{s} L_0 e^{g_L t} \end{aligned}$$

- That is, the combinatorial growth model predicts that the number of new patents should grow exponentially over time
  - When ideas are small, it takes a growing number to generate exponential growth

Annual Patent Grants by the U.S. Patent and Trademark Office



Conclusion

- $K\bar{F}(Z_K) \sim \varepsilon$ links K and the shape of the tail cdf to how the max increases
- Weitzman meets Kortum: Combinatorial growth in recipes whose productivities are draws from a thin-tailed distribution gives rise to exponential growth
- Other applications: wherever Pareto has been assumed in the literature, perhaps we can use thin tails?
  - Many literatures: technology diffusion, trade, search, productivity
  - If ideas are "small," need enhanced theory of markups and heterogeneity