Robust Mechanisms Under Common Valuation

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Abstract

We study robust mechanisms to sell a common-value good. We assume that the mechanism designer knows the prior distribution of the buyers' common value but is unsure of the buyers' information structure about the common value. We use linear programming duality to derive mechanisms that guarantee a good revenue among all information structures and all equilibria. Our mechanism maximizes the revenue guarantee when there is one buyer. As the number of buyers tends to infinity, the revenue guarantee of our mechanism converges to the full surplus.

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1 Introduction

In this paper we study robust mechanism design for selling a common-value good. A robust mechanism is one that works well under a variety of circumstances, in particular under weak assumptions about participants' information structure. The goal of robust mechanism design is to reduce the "base of common knowledge required to conduct useful analyses of practical problems," as envisioned by Wilson (1987).

The literature on robust mechanism design has so far largely focused on private value settings.[1] Common value is of course important in many real-life markets (particularly financial markets) and has a long tradition in auction theory. Robustness with respect to information structure is especially relevant in a common-value setting, since it is hard in practice to pinpoint exactly what is a signal (or a set of signals) for a participant and to quantify the correlation between the signal and the common value, not to mention specifying the joint distribution of signals for all participants that correctly captures their beliefs and higher order beliefs about the common value.

We are inspired by a recent paper of Bergemann, Brooks, and Morris (2016) which, among other results, shows that in a first price auction and for a fixed prior distribution of values, there is a strictly positive lower bound such that for any information structure that is consistent with the prior the equilibrium expected revenue is always above that lower bound. For example, if there are two buyers who have quasi-linear utility and a uniform $[0, 1]$ prior distribution for the common value, then the first price auction without a reserve price guarantees an expected revenue of $1/6$ for any information structure consistent with the prior and any equilibrium from the information structure. Clearly, the maximum revenue that any mechanism can possibly achieve in equilibrium in expectation is $1/2$ which is the expected common value (the full surplus), so the revenue guarantee of $1/6$ is already 33% of the best case scenario. This is quite an attractive prospect for a ambiguity averse seller (in the sense of Gilboa and Schmeidler (1989)) who has large uncertainty about buyers' information structure and equilibrium play.

The natural followup question is whether we can achieve an even better revenue guarantee with an alternative mechanism. To answer this question we work with a max-min setup, where we first minimize the expected revenue over the set of information structures and equilibria for a given mechanism, and then maximize the minimized revenue over the

[1] See Chung and Ely (2007), Brooks (2013), Frankel (2014), Carroll (2015, 2016), Yamashita (2015, 2016), Carrasco, Farinha Luz, Monteiro, and Moreira (2015), Chen and Li (2016), Hartline and Roughgarden (2016), among others; we follow this literature by adopting a max-min approach for robust mechanisms.

set of mechanisms. Bergemann and Morris (2016) give the powerful insight that we can combine information structure and equilibrium into a single entity called *Bayes Correlated Equilibrium*, which is a joint distribution over actions and value subject to obedience and consistency constraints. Minimizing revenue over Bayes correlated equilibria for any fixed mechanism is a linear programming problem, and we can equivalently solve the dual problem which is a maximization problem over the dual variables of constraints associated with Bayes correlated equilibrium. These dual variables have the interpretation as transition rates for a continuous-time Markov process over discrete states, similar to the transition probabilities in Myerson (1997) as dual variables for constraints of a complete-information correlated equilibrium. Moreover, we can combine the maximization over the dual variables with the maximization over the mechanism design variables, so we have a single maximization problem which is equivalent to but more tractable than the original max-min problem.

This duality approach yields new mechanisms with better revenue guarantee than the first price auction. These mechanisms are simple to implement in practice and are described as follows. Suppose there is one common-value good to sell, and $I \geq 1$ buyers with quasi-linear utility. Let the message space for each buyer i be the interval $[0, 1]$. We think of a message $z_i \in [0, 1]$ as the demand of buyer i. Buyer i gets the good with probability $q_i(z_i, z_{-i})$ (q_i could also be buyer i's quantity of allocation if the good is divisible) and pays $P_i(z_i, z_{-i})$. If $z_1 \geq z_2 \geq \cdots \geq z_I$, then

$$q_i(z_i, z_{-i}) = \sum_{j=i}^{I-1} \frac{z_j - z_{j+1}}{j} + \frac{z_I}{I}, \qquad P_i(z_i, z_{-i}) = X(\exp(z_i/A) - 1), \tag{1}$$

and analogously for any other ordering of $(z_1, z_2, \ldots, z_I)$. That is, the lowest buyer gets $1/I$ of his demand, the second lowest buyers gets that plus $1/(I-1)$ of the difference between his and the lowest demand, and so on. Thus, the total probability/quantity of allocation is equal to the highest demand. Moreover, each buyer's payment depends only on his demand and is independent of his final allocation, like an all-pay auction. Finally, $A > 0$ and $X > 0$ in Equation (1) are constants that are optimized for the prior distribution of value. We call the mechanism in Equation (1) the *exponential price mechanism*.

We prove that the exponential price mechanism gives the optimal revenue guarantee when there is one buyer ($I = 1$). In this case we know sharp upper bound on the revenue guarantee. For example, if the prior is the uniform distribution on $[0, 1]$, then the designer can guarantee (among all information structures and all equilibria) a revenue of at most $1/4$: fix any mechanism, there is the private-value information structure, and its equilibrium revenue must be less than $1/4$ which is obtained by the private-value optimal mechanism (a posted price of $1/2$). Roesler and Szentes (2016) study the optimal information structure for a buyer when the seller is best responding to this information structure. Roesler and Szentes's optimal information structure gives a subtle upper bound on the seller's revenue guarantee. For example, when the prior is the uniform distribution on $[0, 1]$, the seller can guarantee a revenue of at most 0.2036. We prove that the exponential price mechanism exactly guarantees the Roesler-Szentes upper bound for any prior distribution when $I = 1$.[2]

As the number I of buyers increases, the exponential price mechanism guarantees a better revenue, as we numerically demonstrate in Figure 1 and Figure 2. We prove that as the number of buyers tends to infinity, the revenue guarantee of the exponential price mechanism (over all information structures and all equilibria) becomes arbitrarily close to the full surplus (the expectation of the common value). Since the full surplus is an upper bound on the equilibrium revenue of every mechanism, the exponential price mechanism achieves the optimal revenue guarantee as $I \to \infty$. This guarantee of full surplus extraction in the limit is not obtained by the first price auction (as shown by Engelbrecht-Wiggans, Milgrom, and Weber (1983) and Bergemann, Brooks, and Morris (2016)), second price auction[3], or all-pay auction[4]. And unlike the mechanisms of Crémer and McLean (1985, 1988), our mechanism is detail free and depends only on the support of the prior distribution, and extracts the full surplus for all information structures in the limit.

[2]In other words, Roesler and Szentes (2016) characterize for one buyer:

$$\min_{\text{info. structure}} \quad \max_{\text{mechanism, equilibrium}} \quad \text{Revenue},$$

while we characterize:

$$\max_{\text{mechanism}} \quad \min_{\text{info. structure, equilibrium}} \quad \text{Revenue},$$

and show it is equal to their min-max value. Equilibrium here is a mapping from signals of the information structure to messages in the mechanism, such that there is no incentive to deviate.

[3]For a second price auction with a reserve price (potentially zero), suppose there is one informed buyer who knows the common value v, and $I - 1$ uninformed buyers who only knows the prior. The following is an equilibrium: the informed buyer truthfully bids v, and all uninformed buyers bid 0. Clearly, this equilibrium does not obtain the full surplus in revenue as $I \to \infty$.

[4]The minimum-revenue information structure in Bergemann, Brooks, and Morris (2016) for the first price auction also fails to extract the full surplus in revenue for an all-pay auction as $I \to \infty$.

2 Model

Information

The mechanism designer has a single good to sell. Let $\mathcal{I} = \{1, 2, \ldots, I\}$ be a finite set of buyers, $I \geq 1$. The buyers have a common value $v \in V = \{0, \nu, 2\nu, \ldots, 1\}$ for the good and have quasi-linear utility, where $\nu > 0$ is a constant. Let $p \in \Delta(V)$ be the prior distribution of common value; the prior p is known by the designer as well as by the buyers. (The designer only knows the prior p about the value.)

Each buyer i *may* possess some additional information $s_i \in S_i$ about the common value beyond the prior, where S_i is a finite set of signals. We have $\tilde{p} \in \Delta(V \times \prod_{i \in \mathcal{I}} S_i)$ such that $\text{marg}_V \, \tilde{p} = p$,[5] so buyer i's information about the common value is informed by $\tilde{p}(\,\cdot \mid s_i)$. As discussed in the introduction, the information structure $(S_i, \tilde{p})_{i \in \mathcal{I}}$ is *not* known by the designer.

Mechanism

A mechanism is a set of allocation rules $q_i : M \to [0, 1]$ and payment rules $P_i : M \to \mathbb{R}$ satisfying $\sum_{i \in \mathcal{I}} q_i(m) \leq 1$, where M_i is the message space of buyer i and is a finite set, and $M = \prod_{i \in \mathcal{I}} M_i$ the space of message profiles. A mechanism defines a game in which the buyers simultaneously submit messages and have utility

$$U_i(v, m) = v \cdot q_i(m) - P_i(m). \tag{2}$$

The allocation $q_i(m)$ can be interpreted as the probability of getting the good in the case of an indivisible good, and as the share of the good in the case of a divisible good.

We assume that a mechanism always has an opt-out option for each buyer i: there exists a message $m_i \equiv 0 \in M_i$ such that $q_i(0, m_{-i}) = P_i(0, m_{-i}) = 0$ for every $m_{-i} \in M_{-i}$.

In this paper we focus on *symmetric* mechanism, which satisfies

$$\begin{aligned} q_i(m'_i, m'_{-i}) &= q_1(m_1 = m'_i, m_{-1} = m'_{-i}) \equiv q(m'_i, m'_{-i}) \\ P_i(m'_i, m'_{-i}) &= P_1(m_1 = m'_i, m_{-1} = m'_{-i}) \equiv P(m'_i, m'_{-i}) \end{aligned} \tag{3}$$

for every $i \in \mathcal{I}$ and $m' \in M$. By $m_{-1} = m'_{-i}$ we mean that m_{-1} and m'_{-i} have the same elements but not necessarily the same ordering of elements; for example we may have

[5] Let $\text{marg}_V \, \tilde{p}$ be the marginal distribution of $\tilde{p}$ over V.

$m_{-1} = (a, b, c)$ and $m'_{-i} = (c, b, a)$. Intuitively, in a symmetric mechanism every buyer is treated in the same way. For a symmetric mechanism we abbreviate $q_1(m)$ to $q(m)$ and $P_1(m)$ to $P(m)$.

Equilibrium

Given a mechanism $(q_i, P_i)_{i\in\mathcal{I}}$ and an information structure $(S_i, \tilde{p})_{i\in\mathcal{I}}$, we have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is defined by strategy $\sigma_i : S_i \to \Delta(M_i)$ for each buyer i such that for every $s_i \in S_i$, the support of $\sigma_i(s_i)$ is among the best responses to others' strategies:

$$\operatorname{supp} \sigma_i(s_i) \subseteq \operatorname*{argmax}_{m_i\in M_i} \sum_{(v,s_{-i})\in V\times S_{-i}} U_i(v, (m_i, \sigma_{-i}(s_{-i})))\tilde{p}(v, s_{-i} \mid s_i), \tag{4}$$

where $U_i(v, (m_i, \sigma_{-i}(s_{-i})))$ is linearly extended from Equation (2).

The ex ante distribution $\mu \in \Delta(V \times M)$ generated by any BNE $(\sigma_i)_{i\in\mathcal{I}}$ of any information structure $(S_i, \tilde{p})_{i\in\mathcal{I}}$ satisfies the following two conditions:

$$\sum_{m\in M} \mu(v, m) = p(v), \quad v \in V, \tag{Consistency}$$

$$\sum_{(v,m_{-i})\in V\times M_{-i}} \mu(v, m) \left(U_i(v, (m_i, m_{-i})) - U_i(v, (m'_i, m_{-i}))\right) \geq 0, \quad i \in \mathcal{I}, (m_i, m'_i) \in M_i \times M_i. \tag{Obedience}$$

A distribution $\mu \in \Delta(V \times M)$ that satisfies the above two conditions is called a *Bayes Correlated Equilibrium* (BCE) of the mechanism $(q_i, P_i)_{i\in\mathcal{I}}$. For any BCE μ, there exists an information structure and a BNE of that information structure that generates μ. See Bergemann and Morris (2016) for more details.

For notational brevity, we sometimes omit the set to which a summation variable belongs when it is obvious; for example, summing over m means summing over $m \in M$.

Designer's problem

The mechanism designer wants to solve:

$$\max_{(q_i,P_i)_{i\in\mathcal{I}}} \min_{\mu\in\Delta(V\times M)} \sum_{(v,m)} \sum_{i} \mu(v,m)P_i(m) \tag{5}$$

$$\text{such that } \mu \text{ is a BCE of } (q_i,P_i)_{i\in\mathcal{I}}.$$

Definition 1. A mechanism *guarantees* a revenue R if every BCE of this mechanism has an expected revenue larger than or equal to R.

3 Main Results

Our main results are a class of mechanisms that give good revenue guarantee. Consider a symmetric mechanism with k messages besides the opt-out message: $M_i = \{0, 1, \ldots, k\}$, for every buyer $i \in \mathcal{I}$. The allocation $q(m_1, m_{-1})$ is given by:

$$\begin{aligned} &q(0, m_{-1}) = 0, && m_{-1} \in M_{-1}, \\ &q(m_1+1, m_{-1}) - q(m_1, m_{-1}) = \left(\frac{1}{|\operatorname{rank}(m_1, m_{-1})|}\sum_{j\in\operatorname{rank}(m_1,m_{-1})}\frac{1}{j}\right)\cdot\frac{1}{k}, && 0 \le m_1 \le k-1, \end{aligned} \tag{6}$$

where $\operatorname{rank}(m_1, m_{-1}) \subseteq \{1, 2, \ldots, I\}$ is the set of ranks (from the top) of m_1 in $(m_1, m_2, \ldots, m_I)$; for example, $\operatorname{rank}(20, 10, 20, 40, 30) = \{3, 4\}$, because $m_1 = 20$ and $m_3 = 20$ are tied for the third and the fourth place in this list; and $\operatorname{rank}(20, 10, 30, 40, 30) = \{4\}$ because in this list $m_1 = 20$ is unambiguously ranked fourth, even though there is a tie for the second and the third rank. We think of a message m_i as the demand of a fraction m_i/k of the good; the allocation in Equation (6) is increasing with the demand at a rate equal to the reciprocal of the demand's rank: a rate of 1 for the highest demand, of 1/2 for the second highest demand, of 1/3 for the third highest demand, and so on. Moreover, we break tie in a symmetric way and randomize over all feasible ranks, in the case when $|\operatorname{rank}(m_1, m_{-1})| > 1$. It is easy to check that Equation (6) uniquely defines an allocation function (i.e., the feasibility condition is always satisfied); the total amount of allocation is at most $\max(m_1, m_2, \ldots, m_I)/k$.

The payment of our mechanism is:

$$P(m_1, m_{-1}) = X\left(\left(1+\frac{1}{a}\right)^{m_1} - 1\right), \tag{7}$$

where $X > 0$ and $a > 0$ are constants that are optimized for a given prior distribution p. That is, the payment of every buyer depends only on his message and is independent of his final allocation.

As $k \to \infty$ and $a = A \cdot k$, the mechanism from Equations (6) and (7) converges to (1), where we reparametrize $m_i \in \{0, 1, \ldots, k\}$ to $z_i \equiv m_i/k \in [0, 1]$, where z_i is buyer i's demand. Thus, we abuse the terminology and refer to the mechanism from Equations (6) and (7) as the exponential price mechanism as well.

Intuitively, the exponential price mechanism tries to be egalitarian and allocate some quantity of the good to every buyer. Since the exponential payment is a convex function of quantity, it makes sense to split the good among all buyers. Of course, a buyer with a higher demand gets more quantity because such buyer is paying more. If $z_1 > z_2 > \cdots > z_I$, then buyer i gets exactly $(z_i - z_{i+1})/i$ more than the allocation of buyer $i+1$; we have the factor $1/i$ because the quantity $(z_i - z_{i+1})/i$ is also acquired by all buyer $j > i+1$, and by definition there are i of them. The intuition for the exponential functional form of the payment rule is best illustrated when there is a single buyer and is presented in Section 4.2.1.

When there is a single buyer ($I = 1$), the exponential price mechanism becomes:

$$q(m_1) = m_1/k, \qquad P(m_1) = X\left(\left(1 + \frac{1}{a}\right)^{m_1} - 1\right), \qquad m_1 \in \{0, 1, \ldots, k\}, \tag{8}$$

since $\text{rank}(m_1) = \{1\}$ by definition. In fact, this mechanism achieves the optimal revenue guarantee:

Theorem 1. *Suppose there is one buyer, and as $\nu \to 0$ the prior p converges to a distribution with a positive density. There exist constants $A > 0$ and $X > 0$ such that the exponential price mechanism with $a = A \cdot k$ and the given X achieves the optimal revenue guarantee (i.e., solution to Problem (5)) as $k \to \infty$ and $\nu \to 0$.*

That is, for any $\epsilon > 0$, there exists $\bar{\nu}$ and $\bar{k}$ such that for any $\nu \leq \bar{\nu}$ and $k \geq \bar{k}$, the exponential price mechanism with $a = A \cdot k$ and the given X guarantees a revenue within ϵ of the best possible from Problem (5).

We compute the optimal revenue guarantee of Theorem 1 for various prior distributions in Figure 2 (page 21).

Our second result states the exponential price mechanism guarantees in expected revenue the full surplus (the expected common value) as the number of buyers tends to infinity. In this sense the mechanism is asymptotically optimal.

Theorem 2. *Let* $a = \frac{k}{\log(I)}$ *and* $X = \frac{1}{2I\log(I)}$*. The exponential price mechanism guarantees a revenue of* $\sum_v v \cdot p(v)$ *as* $k \to \infty$ *and* $I \to \infty$*.*

That is, for any $\epsilon > 0$, there exists $\bar{I}$ and $\bar{k}$ such that for any $I \geq \bar{I}$ and $k \geq \bar{k}$, the exponential price mechanism with $a = \frac{k}{\log(I)}$ and $X = \frac{1}{2I\log(I)}$ guarantees a revenue within ϵ of $\sum_v v \cdot p(v)$.

The values of a and X in Theorem 2 depend only on the support of the prior (which is in $[0, 1]$) and is independent of the other details of the prior. Thus, the convergence of the mechanism's revenue guarantee to the full surplus holds for every prior supported on $[0, 1]$.

We illustrate the revenue guarantee of the exponential price mechanism as a function of the number of buyers in Figure 1; we also compare with the first price auction with the reserve price chosen to maximize the revenue guarantee (Bergemann, Brooks, and Morris, 2016). For this figure we take the prior to be the uniform distribution on $[0, 1]$ and $\nu \to 0$. We see that the revenue guarantee of the exponential price mechanism is fairly close to the full surplus of 0.5 when there are 20 buyers.

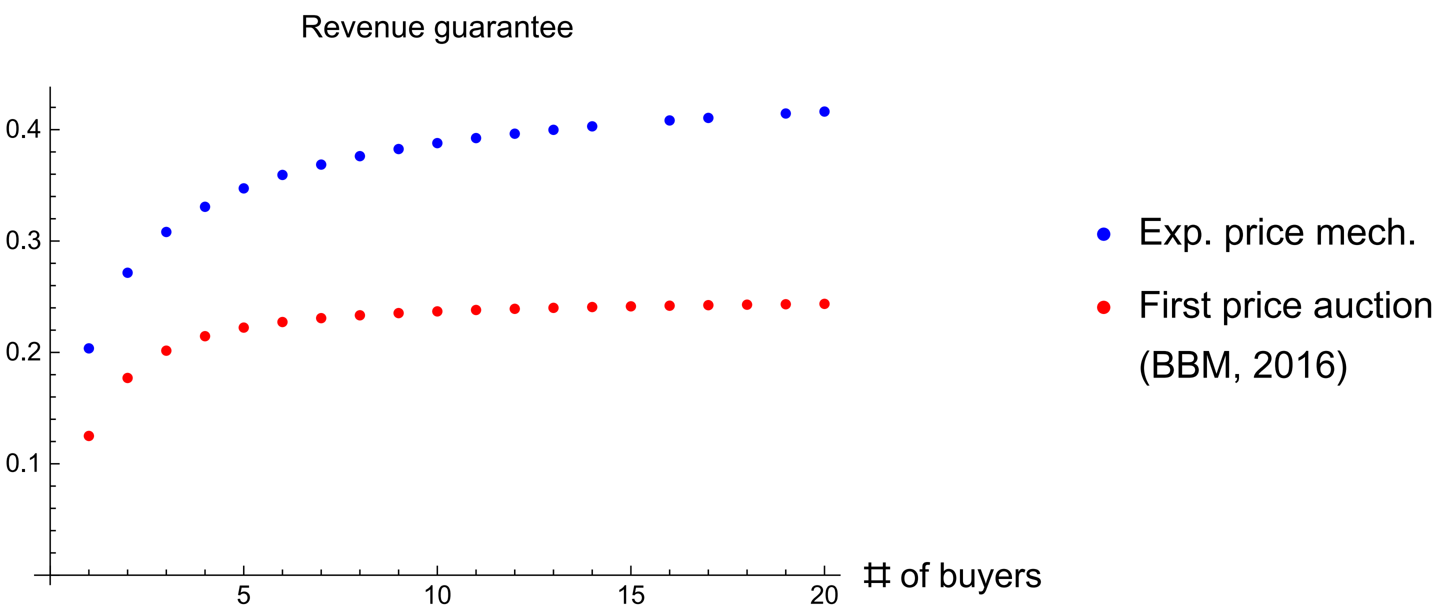


Figure 1: Revenue guarantees of the exponential price mechanism and of the first price auction with the optimal reserve price.

4 Duality Approach to Robust Mechanism

To prove Theorem 1 and Theorem 2, we introduce a duality approach. For a given mechanism $(q_i, P_i)_{i\in\mathcal{I}}$, the BCE that minimizes the expected revenue is found by the following problem:

$$\min_{\mu} \sum_{(v,m)} \sum_{i} P_i(m)\mu(v,m) \tag{9}$$

subject to:

$$\sum_{(v,m_{-i})} (U_i(v,m) - U_i(v,(m_i', m_{-i})))\mu(v,m) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i,$$

$$\sum_{m} \mu(v,m) = p(v), \quad v \in V,$$

$$\mu(v,m) \geq 0, \quad v \in V, m \in M,$$

where U_i is the utility function defined by Equation (2).

The dual problem to Problem (9) is:

$$\max_{(\alpha_i,\gamma)_{i\in\mathcal{I}}} \sum_{v} p(v)\gamma(v) \tag{10}$$

subject to:

$$\gamma(v) + \sum_{i} \sum_{m_i'} [U_i(v,m) - U_i(v,(m_i', m_{-i}))]\alpha_i(m_i' \mid m_i) \leq \sum_{i} P_i(m), \quad v \in V, m \in M,$$

$$\alpha_i(m_i' \mid m_i) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i,$$

where $\alpha_i(m_i' \mid m_i)$ is the dual variable for the obedience constraint of not playing m_i' when "recommended" to play m_i in (9), and $\gamma(v)$ is the dual variable for the consistency constraint of $\sum_m \mu(v,m) = p(v)$. By the linear programming duality theorem, Problems (9) and (10) have the same optimal value; their solutions are characterized by the complementary slackness conditions.

Mechanism designer's problem in (5) can be written as:

$$\max_{(P_i,q_i,\alpha_i,\gamma)_{i\in\mathcal{I}}} \sum_v p(v)\gamma(v) \tag{11}$$

subject to:

$$\gamma(v) \le \sum_i P_i(m) + \sum_i \sum_{m_i'} [v(q_i(m_i', m_{-i}) - q_i(m)) - P_i(m_i', m_{-i}) + P_i(m)]\alpha_i(m_i' \mid m_i), \quad v \in V, m \in M,$$

$$q_i(m) \ge 0, \quad \sum_{i'} q_{i'}(m) \le 1, \quad q_i(0, m_{-i}) = P_i(0, m_{-i}) = 0, \qquad i \in \mathcal{I}, m \in M$$

$$\alpha_i(m_i' \mid m_i) \ge 0, \qquad (m_i, m_i') \in M_i \times M_i, i \in \mathcal{I},$$

where we label the opt-out message as $0 \in M_i$.

The advantage of problem (11) over the equivalent problem (5) is that we work with a maximization problem instead of a max-min problem. Moreover, we work with $(\alpha_i)_{i\in\mathcal{I}}$, where each α_i has $|M_i \times M_i|$ dimensions, instead of μ which has $|V \times \prod_{i\in\mathcal{I}} M_i|$ dimensions; the reduction in dimensions is significant if $|V|$ is large. Lastly, if we find a tuple $(\alpha_i, q_i, P_i)_{i\in\mathcal{I}}$ that satisfies the constraints of Problem (11), then the value of (11) under such $(\alpha_i, q_i, P_i)_{i\in\mathcal{I}}$ is by definition a lower bound on the optimal revenue guarantee. On the other hand, finding a feasible tuple $(\mu, q_i, P_i)_{i\in\mathcal{I}}$ for Problem (5) (i.e., μ is a BCE of $(q_i, P_i)_{i\in\mathcal{I}}$) does not yield by itself any conclusion about the revenue guarantee, since there may exist another BCE μ' of $(q_i, P_i)_{i\in\mathcal{I}}$ with a lower revenue than μ.

Problem (11) is a bilinear programming problem: fixing the mechanism $(P_i, q_i)_{i\in\mathcal{I}}$ the maximization problem over $(\alpha_i, \gamma)_{i\in\mathcal{I}}$ is linear; and fixing the dual variables $(\alpha_i)_{i\in\mathcal{I}}$ the maximization problem over $(P_i, q_i, \gamma)_{i\in\mathcal{I}}$ is also linear.

Problem (11) can be summarized as:

$$\max_{(q_i,P_i,\alpha_i)_{i\in\mathcal{I}}} \sum_v p(v) \cdot \min_m \text{Rev}(v, m), \tag{12}$$

subject to the feasibility constraints, where

$$\text{Rev}(v, m) \equiv \sum_i \left(P_i(m) + \sum_{m_i'} (U_i(v, m_i', m_{-i}) - U_i(v, m))\alpha_i(m_i' \mid m_i) \right). \tag{13}$$

Since $U_i(v, m)$ is a linear function of v, so is $\text{Rev}(v, m)$ for any fixed m. We interpret $\alpha_i(m_i' \mid m_i)$ as buyer i's rate of deviation from message m_i to m_i', and $\text{Rev}(v, m)$ as the revenue generated by the message profile m, *plus* the incentive to deviate from m given value v and rates of deviation $(\alpha_i)_{i\in\mathcal{I}}$. By minimizing $\text{Rev}(v,m)$ over m, we are ignoring message profile m that either (1) has a large revenue, or (2) there is a large incentive to deviate from m by a buyer. Intuitively, (1) and (2) combines to give equilibrium message profile with minimum revenue.

Problem (11) is bounded above by $\sum_v v \cdot p(v)$, by the following lemma:

Lemma 1. *For every $v \in V$, we have:*

$$\min_m \text{Rev}(v,m) \le v. \tag{14}$$

Proof. Fix an arbitrary $v \in V$. Consider the problem:

$$\begin{aligned}
&\max_{\gamma,(\alpha_i)_{i\in\mathcal{I}}} \ \gamma \qquad (15)\\
&\text{subject to:}\\
&\gamma + \sum_i \sum_{m_i'} (U_i(v,m) - U_i(v,(m_i',m_{-i})))\alpha_i(m_i' \mid m_i) \le \sum_i P_i(m), \quad m \in M,\\
&\alpha_i(m_i' \mid m_i) \ge 0, \quad i \in \mathcal{I}, (m_i,m_i') \in M_i \times M_i.
\end{aligned}$$

The dual to the above problem is:

$$\begin{aligned}
&\min_\mu \ \sum_m \mu(m) \sum_i P_i(m) \qquad (16)\\
&\text{subject to:}\\
&\sum_{m_{-i}} \mu(m)(U_i(v,m) - U_i(v,(m_i',m_{-i}))) \ge 0, \quad i \in \mathcal{I}, (m_i,m_i') \in M_i \times M_i,\\
&\sum_m \mu(m) = 1,\\
&\mu(m) \ge 0, \quad m \in M,
\end{aligned}$$

which is minimizing the revenue over *complete-information* correlated equilibria μ (for the fixed v). For any μ satisfying the constraints, we have $\sum_{m_{-i}} \mu(m)U_i(v,m) = \sum_{m_{-i}} \mu(m)(vq_i(m) - P_i(m)) \ge 0$ for every $i \in \mathcal{I}$ and $m_i \in M_i$ because of the presence of the opt-out message $0 \in M_i$. Therefore, $\sum_m \mu(m)\sum_i(vq_i(m) - P_i(m)) \ge 0$, and $\sum_m \mu(m)\sum_i P_i(m) \le \sum_m \mu(m)\sum_i vq_i(m) \le v$. Thus the optimal solution of (15) is bounded above by v. □

4.1 A Lower Bound

We work with symmetric mechanism $q(m) \equiv q_1(m)$ and $P(m) \equiv P_1(m)$ (cf. Equation (3)) and symmetric $\alpha(m_i' \mid m_i) \equiv \alpha_i(m_i' \mid m_i)$.

Instead of directly solving Problem (12), we make some educated guess on (q_i, P_i, α_i) to get a lower bound on the maximum value of Problem (12). Suppose $M_i = \{0, 1, \dots, k\}$ for every buyer $i \in \mathcal{I}$, where $q(0, m_{-1}) = 0 = P(0, m_{-1})$ for every $m_{-1} \in M_{-1}$.

We focus on

$$\alpha(j' \mid j) = \begin{cases} a & j' = j+1 \\ 0 & j' \neq j+1 \end{cases}, \quad (j, j') \in \{0, 1, \dots, k\}^2. \tag{17}$$

Condition (17) says that the local obedience constraint in BCE is binding: if the above α satisfies the complementarity slackness condition with a BCE μ, then a buyer is indifferent between messages j and $j+1$ if he is "recommended" to submit j in the BCE μ. This is a discrete analogue of the first order condition at j.

Condition (17) implies that there are two kinds of messages: the "interior" message $j \in \{0, 1, \dots, k-1\}$, and the "boundary" message $j = k$. Thus there are $I + 1$ kinds of message profiles $m \in M = \{0, 1, \dots, k\}^I$, depending on the number of boundary messages in m. For $0 \leq n \leq I$, define the class of message profiles:

$$M(n) = \{m \in M : |\{i \in \mathcal{I} : m_i = k\}| = n\}. \tag{18}$$

The sets $M(n)$, $0 \leq n \leq I$, form a partition of M. Our second assumption is that

$$\mathrm{Rev}(v, m) = \mathrm{Rev}(v, m') \ \forall v \in V, \qquad \text{if } m \text{ and } m' \text{ belong to the same } M(n). \tag{19}$$

Condition (19) attemps to make $\mathrm{Rev}(v, m)$ over m as redundant as possible, to minimize the number of items inside the min operator in Equation (12).

We now go to the exponential price mechanism defined by Equations (6) and (7). Clearly, if there are n boundary messages in a message profile m, then the rest (the interior messages) have ranks among $\{n+1, n+2, \dots, I\}$, and by Equation (6) we have:

$$\sum_{i: m_i < k} q(m_i + 1, m_{-i}) - q(m_i, m_{-i}) = \frac{1}{k} \sum_{j=n+1}^{I} \frac{1}{j}. \tag{20}$$

Moreover, for an interior m_i we have:

$$P(m_i, m_{-i}) - a(P(m_i + 1, m_{-i}) - P(m_i, m_{-i})) = -X \tag{21}$$

by Equation (7).[6] Therefore, under Condition (17) we have:

$$\text{Rev}(v, m) = \frac{a\,v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1 + 1/a)^k - 1) - (I - n)X, \quad \text{if } m \in M(n). \tag{22}$$

Thus, Condition (19) is satisfied, and we have the following lower bound on the maximum value of Problem (12):

$$\Pi_I^* \equiv \max_{k\geq 1,\, a\geq 0,\, X} \left(\sum_v p(v) \cdot \min_{0\leq n\leq I} \left(\frac{a\,v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1 + 1/a)^k - 1) - (I - n)X \right) \right). \tag{23}$$

Proposition 1. *The exponential price mechanism guarantees a revenue of Π_I^* defined in (23).*

Proof. The proof is given by the construction above. □

We prove Theorem 1 and Theorem 2 by studying Π_1^* and $\lim_{I\to\infty} \Pi_I^*$. In Figure 1 (page 9) we plot Π_I^* for the uniform $[0, 1]$ distribution, as $\nu \to 0$.

4.2 Proof of Theorem 1

Let $I = 1$. Suppose as $\nu \to 0$, the prior p converges to a distribution with density ρ, where $\rho : [0, 1] \to [0, \infty)$ is positive almost everywhere.

To prove Theorem 1, we need to discuss a revelant result in Roesler and Szentes (2016). Roesler and Szentes (2016) study the optimal information structure for the buyer (and the worst for the seller) when the seller best responds to the information structure. Such

[6] In fact, Equation (7) is the solution to the difference equation

$$P(m_i, m_{-i}) - a(P(m_i + 1, m_{-i}) - P(m_i, m_{-i})) = -X$$

for $m_i \in \{0, 1, \ldots, k - 1\}$, with the initial condition of $P(0, m_{-i}) = 0$.

information structure has the following cumulative distribution function for the signals:

$$G_\pi^B(s) = \begin{cases} 1 & s \geq B \\ 1 - \pi/s & s \in [\pi, B) \\ 0 & s < \pi \end{cases}, \tag{24}$$

where $s \in [0,1]$ is an unbiased signal of the buyer for his value ($\mathbb{E}[v \mid s] = s$), $0 < \pi \leq B$ are two free parameters, and there is an atom of size π/B at $s = B$. If the buyer has this distribution of unbiased signals and observes the realization of the signal, then the seller is clearly indifferent between every posted price in $[\pi, B]$ and has a revenue of π from the optimal mechanism (which is a posted price).[7] Thus, π is an upper bound on the seller's revenue guarantee.

Given the density $\rho(v)$, $G_\pi^B(s)$ is a distribution of an unbiased signal on v if and only if ρ is a mean-preserving spread of $G_\pi^B(s)$, which holds if and only if:

$$\int_0^1 v\,\rho(v)\,dv = \int_0^1 s\,dG_\pi^B(s) = \pi + \pi\log B - \pi\log\pi \tag{25}$$

$$\min_{s\in[\pi,B]} F(s,\pi) \geq 0, \text{ where} \tag{26}$$

$$F(s,\pi) \equiv \int_{s'=0}^{s}\int_{v=0}^{s'} \rho(v)\,dv\,ds' - \int_0^s G_\pi^B(s')\,ds' = \int_{s'=0}^{s}\int_{v=0}^{s'} \rho(v)\,dv\,ds' - (s - \pi - \pi\log s + \pi\log\pi),$$

i.e., G_π^B has the same mean as ρ and second-order stochastically dominates ρ. Let $B = B(\pi)$ be defined from π by Equation (25).

Roesler and Szentes (2016) prove that the best information structure for the buyer (and the worst for the seller) when the seller best responds to the information structure is $G_{\pi^*}^{B^*}$, where π^* is the smallest π such that $\min_{s\in[\pi,B(\pi)]} F(s,\pi) \geq 0$, and $B^* \equiv B(\pi^*)$; that is, π^* is the smallest π such that ρ is a mean-preserving spread of $G_\pi^{B(\pi)}(s)$. For our purpose, by making π small we tighten the upper bound on the seller's revenue guarantee.

We now show that the exponential price mechanism can obtain the upper bound π^*. In the case of one buyer, Problem (23) simplifies to:

$$\Pi_1^* \equiv \max_{k\geq 1,\, a\geq 0,\, X} \sum_v \min\left(\frac{a\,v}{k} - X,\; X\left(\left(1 + \frac{1}{a}\right)^k - 1\right)\right) p(v), \tag{27}$$

[7] Intuitively, if the seller has a strict incentive over the posted price, then we can slightly change the buyer's information structure to lower the seller's optimal revenue and to increase the buyer's surplus, while preserving the seller's best response in posted price.

Set $a = A \cdot k$. As $\nu \to 0$ and $k \to \infty$, we have

$$\sum_v \min(av/k, X(1+1/a)^k)\, p(v) - X \longrightarrow \Pi_1 \equiv \int_0^1 \min(Av, X\exp(1/A))\, \rho(v) dv - X. \quad (28)$$

We maximize Π over A and X. Suppose $\frac{X\exp(1/A)}{A} \in [0,1]$, the first order condition is:

$$\frac{\partial \Pi_1}{\partial X} = \int_{\frac{X\exp(1/A)}{A}}^1 \exp(1/A)\, \rho(v)\, dv - 1 = 0, \quad (29)$$
$$\frac{\partial \Pi_1}{\partial A} = \int_0^{\frac{X\exp(1/A)}{A}} v\, \rho(v)\, dv - \int_{\frac{X\exp(1/A)}{A}}^1 \frac{X\exp(1/A)}{A^2}\, \rho(v)\, dv = 0.$$

If the above first order condition holds and $\frac{X\exp(1/A)}{A} \in [0,1]$, then we have $\Pi_1 = X/A$.

Going back to the construction of Roesler-Szentes, let s^* be an arbitrary selection from $\operatorname{argmin}_{s\in[\pi^*,B^*]} F(s,\pi^*)$. Since $\min_{s\in[\pi,B(\pi)]} F(s,\pi)$ is a continuous function of π, we have $F(s^*,\pi^*) = 0$. Moreover, s^* must be interior[8], so we have $\frac{\partial F}{\partial s}(s^*,\pi^*) = 0$.

Therefore, we have (the first line is $\frac{\partial F}{\partial s}(s^*,\pi^*) = 0$, and the second line is $F(s^*,\pi^*) = 0$):

$$\int_{v=0}^{s^*} \rho(v)\, dv - 1 + \pi^*/s^* = 0, \quad (30)$$
$$\int_{s=0}^{s^*}\int_{v=0}^{s} \rho(v)\, dv\, ds - (s^* - \pi^* - \pi^*\log s^* + \pi^*\log\pi^*) = -\int_0^{s^*} v\,\rho(v)\, dv + \pi^*\log s^* - \pi^*\log\pi^* = 0,$$

where in the second equality of the second line we use integration by parts and substitute in the first line. Clearly, there exist unique $A > 0$ and $X > 0$ such that $s^* = X\exp(1/A)/A$ and $\pi^* = X/A$. (We have $s^* < B^* < 1$.) Then the above equations become:

$$\int_{\frac{X\exp(1/A)}{A}}^1 \rho(v)\, dv = \exp(-1/A), \qquad \int_0^{\frac{X\exp(1/A)}{A}} v\,\rho(v)\, dv - \frac{X}{A^2} = 0,$$

which is clearly equivalent to Equation (29).

Therefore, for any $\epsilon > 0$, when k is sufficiently large and ν sufficiently small, we have $\Pi_1^* \geq \pi^* - \epsilon$. When ν is sufficiently small, the revenue guarantee must be smaller than $\pi^* + \epsilon$ by the Roesler-Szentes construction. This concludes the proof.

[8] If $B^* < 1$, we must have $F(B^*,\pi^*) > 0$, for otherwise we would have $\int_0^1 G_{\pi^*}^{B^*}(s)\, ds > \int_{s=0}^1\int_{v=0}^s \rho(v)\, dv\, ds$, which would contradict the fact that $G_{\pi^*}^{B^*}$ has the same mean as p. If $B^* = 1$, then we have $\frac{\partial F}{\partial s}(B^*,\pi^*) = \frac{\pi^*}{B^*} > 0$. In any case $s^* \neq B^*$. Since $F(\pi^*,\pi^*) > 0$, we also have $s^* \neq \pi^*$.

4.2.1 Intuition on Theorem 1.

We note that as $k \to \infty$ and $a = A \cdot k$, the exponential price mechanism in Equation (8) becomes:

$$q(z) = z, \qquad P(z) = X(\exp(z/A) - 1), \tag{31}$$

where $z \equiv m_1/k \in [0, 1]$ is the demand of the buyer.

Fix an unbiased information structure (S, G) for the buyer: $\mathbb{E}[v \mid s] = s$ for every $s \in S \subseteq [0, 1]$, and $s \in S$ has the cumulative distribution function $G(s)$.

Given a realization of signal s, the buyer solves:

$$\max_z s \cdot z - X(\exp(z/A) - 1).$$

If $s \leq \pi^* \equiv X/A$, then the buyer's optimal demand is $z = 0$, and he pays 0; if $s \geq s^* \equiv X\exp(1/A)/A$, then the buyer's optimal demand is $z = 1$, and he pays $X(\exp(1/A) - 1))$. If $s \in [\pi^*, s^*]$, then the optimal demand is given by the first order condition $s = X\exp(z/A)/A$, and the buyer pays $X(\exp(z/A) - 1)|_{s=X\exp(z/A)/A} = As - X$. Thus, the equilibrium revenue under the unbiased information structure (S, G) is:

$$\Pi_1(G) \equiv \int_{s=\pi^*}^{1} \min(As - X, X(\exp(1/A) - 1))\, dG(s), \tag{32}$$

which is similar to Π_1 in Equation (28), but with a different lower limit in the integral.

In general, we have:

$$\Pi_1(G) \geq \int_{s=0}^{1} \min(As - X, X(\exp(1/A) - 1))\, dG(s) \geq \int_{s=0}^{1} \min(As - X, X(\exp(1/A) - 1))\, \rho(s)\, ds = \Pi_1, \tag{33}$$

since ρ is a mean-preserving spread of G, and $\min(As - X, X(\exp(z/A) - 1))$ is a concave function of s; thus, Π_1 is a lower bound on the equilibrium revenue over all information structures, confirming Proposition 1 when $I = 1$.

The proof of Theorem 1 shows that $\Pi_1 = \Pi_1(G)$ when $G = G_{\pi^*}^{B^*}$ as constructed by Roesler and Szentes. This can be seen in Equation (33) as follows: the first inequality in (33) is an equality when $G = G_{\pi^*}^{B^*}$ because $G_{\pi^*}^{B^*}$ is supported on the interval $[\pi^*, B^*]$; the second inequality in (33) is an equality when $G = G_{\pi^*}^{B^*}$ because $\int_{s=\pi^*}^{s^*} s\, dG_{\pi^*}^{B^*}(s) = \int_{v=0}^{s^*} v\, \rho(v)\, dv$ and $G_{\pi^*}^{B^*}(s^*) = \int_{v=0}^{s^*} \rho(v)\, dv$ by Equation (30).

4.3 Proof of Theorem 2

For each $0 \le n \le I$, define

$$\mathrm{Rev}_n(v) = \frac{a\,v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1+1/a)^k - 1) - (I-n)X, \tag{34}$$

which is $\mathrm{Rev}(v, m)$ for any $m \in M(n)$. Let

$$v(n) = \frac{(n+1)kX}{a}(1+1/a)^k, \tag{35}$$

By construction, we have $\mathrm{Rev}_n(v(n)) = \mathrm{Rev}_{n+1}(v(n))$ for each $0 \le n \le I-1$. Set $v(-1) = 0$ and $v(I) = \infty$. Clearly, if $X > 0$, then $\mathrm{Rev}_n(v) = \min_{0 \le n' \le I} \mathrm{Rev}_{n'}(v)$ if and only if $v \in [v(n-1), v(n)]$.

We want to approximate the identity function v by $\min_{0 \le n \le I} \mathrm{Rev}_n(v)$. To do so, we set $A = 1/\log(I)$, $X = 1/(2I\log(I))$ and $a = Ak$. We have $\lim_{k\to\infty}(1+1/a)^k = I$, $\lim_{k\to\infty} v(1) = 1$. Thus,

$$\begin{aligned}\lim_{k\to\infty} \sum_{v\in V} p(v) \min_{0\le n\le I} \mathrm{Rev}_n(v) = &\sum_{v \le v(0), v\in V} p(v) \left(\frac{v}{\log(I)} \sum_{j=1}^{I} \frac{1}{j} - \frac{1}{2\log(I)} \right) \\ &+ \sum_{v > v(0), v\in V} p(v) \left(\frac{v}{\log(I)} \sum_{j=2}^{I} \frac{1}{j} - \frac{1}{\log(I)} \right)\end{aligned} \tag{36}$$

Clearly, the above equation converges to $\sum_v v \cdot p(v)$ as $I \to \infty$. This completes the proof.

5 A Generalization when $I = 2$

Given the central role that Conditions (17) and (19) play in the exponential price mechanism, in this section we study Conditions (17) and (19) in details for the case of two buyers; we arrive at a generalization of the exponential price mechanism. We compare the revenue guarantees of the two mechanisms for various prior distributions, and compare with the first price auction and with some upper bound on revenue guarantee.

Suppose $I = 2$. We consider a symmetric mechanism with $k+1$ messages: $M_1 = M_2 = \{0, 1, \ldots, k\}$. Our usual assumption on the mechanism is:

$$q(0,j) = 0 = P(0,j), \quad q(j,l) \ge 0, \quad q(j,l) + q(l,j) \le 1, \quad (j,l) \in \{0,1,\ldots,k\}^2. \tag{37}$$

Conditions (17) and (19) hold under $I = 2$ if and only if

$$\begin{aligned}
&2q(1,0) = q(j+1,l) - q(j,l) + q(l+1,j) - q(l,j), \quad (j,l) \in \{0,1,\ldots,k-1\}^2, \\
&q(1,k) = q(j+1,k) - q(j,k), \quad j \in \{0,1,\ldots,k-1\},
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
-2aP(1,0) &= P(j,l) + P(l,j) - a(P(j+1,l) - P(j,l)) - a(P(l+1,j) - P(l,j)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (j,l) \in \{0,1,\ldots,k-1\}^2, \\
P(k,0) - aP(1,k) &= P(j,k) + P(k,j) - a(P(j+1,k) - P(j,k)), \quad j \in \{0,1,\ldots,k-1\}.
\end{aligned} \tag{39}$$

It is without loss to assume that the feasibility constraint $q(j,k) + q(k,j) \leq 1$ binds for every j (if not, we can increase $q(k,j)$, which strictly increases $\text{Rev}(v,(k-1,j))$, without decreasing any other $\text{Rev}(v,m)$ or violating any feasibility constraint):

$$q(j,k) + q(k,j) = 1, \quad j \in \{0,1,\ldots,k\}. \tag{40}$$

Lemma 2. *For any allocation q that satisfies Conditions (38) and (40), we have $q(1,0) = (3k+1)/(4k^2)$ and $q(1,k) = 1/(2k)$.*

Lemma 3. *For any payment P that satisfies Condition (39), we have:*

$$P(k,k) = \left((1+1/a)^k - 1\right)^2 aP(1,0) + \left((1+1/a)^k - 1\right)(aP(1,k) - P(k,0)). \tag{41}$$

Define,

$$Y_0 \equiv 2aP(1,0), \qquad Y_1 \equiv -P(k,0) + aP(1,k), \tag{42}$$

i.e., $-Y_0$ is equal to the first line of (39), and $-Y_1$ is equal to the second line of (39).

Lemma 2 and Lemma 3 lead us unambiguously to the following problem:

$$\Pi_2^{\#} \equiv \max_{k\geq 1,\, a\geq 0,\, Y_0,\, Y_1} \sum_v \min\left(\frac{3k+1}{2k^2}av - Y_0,\, \frac{av}{2k} - Y_1,\, Y_0\left((1+1/a)^k - 1\right)^2 + 2Y_1\left((1+1/a)^k - 1\right)\right) p(v). \tag{43}$$

Comparing $\Pi_2^{\#}$ above with Π_2^* in Equation (23), the main difference is that in $\Pi_2^{\#}$ there are two variables Y_0 and Y_1, instead of a single variable X in Π_2^*; the difference between the coefficient of $\frac{3k+1}{2k^2}$ in $\Pi_2^{\#}$ and of $\frac{3}{2k}$ in Π_2^* is unimportant, since $k \to \infty$ in both maximization problems. In fact, if $k \to \infty$, Π_2^* is a special case of $\Pi_2^\#$ with $Y_0 = 2X$ and $Y_1 = X - X((1 + 1/a)^k - 1)$. Thus, we have $\Pi_2^\# \geq \Pi_2^*$ as $k \to \infty$.

Proposition 2. *Suppose there are two buyers. There exists a symmetric mechanism that guarantees a revenue of $\Pi_2^\#$ defined in (43).*

We now specify the mechanism for Proposition 2. Consider the following allocation rule:

$$q(0,l) = 0, \qquad l \in \{0, 1, \ldots, k\}, \tag{44}$$

$$q(j+1,l) - q(j,l) = \begin{cases} (2k+1)/(4k^2) & j < l \\ (3k+1)/(4k^2) & j = l \\ (4k+1)/(4k^2) & j > l \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k-1\}^2, \tag{45}$$

$$q(j+1,k) - q(j,k) = 1/(2k), \qquad j \in \{0,1,\ldots,k-1\}.$$

It is easy to check that the above allocation rule satisfies the feasibility constraint, and Conditions (38) and (40). As $k \to \infty$, the above allocation rule becomes identical to the allocation rule of exponential price mechanism in Equation (6).

Given any values of Y_0 and Y_1, we can choose the following solution to Equation (39):

$$P(j,l) - a(P(j+1,l) - P(j,l)) = \begin{cases} -Y_0/2 & 0 \leq l < k, \\ -Y_1 - P(k,j) & l = k, \end{cases}, \quad j \in \{0,1,\ldots,k-1\}, \tag{46}$$

which is equivalent to (see footnote 6):

$$P(j,l) = \begin{cases} ((1+1/a)^j - 1)\,\frac{Y_0}{2} & 0 \leq l < k \\ ((1+1/a)^j - 1)\left(Y_1 + \left((1+1/a)^k - 1\right)\frac{Y_0}{2}\right) & l = k \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k\}^2. \tag{47}$$

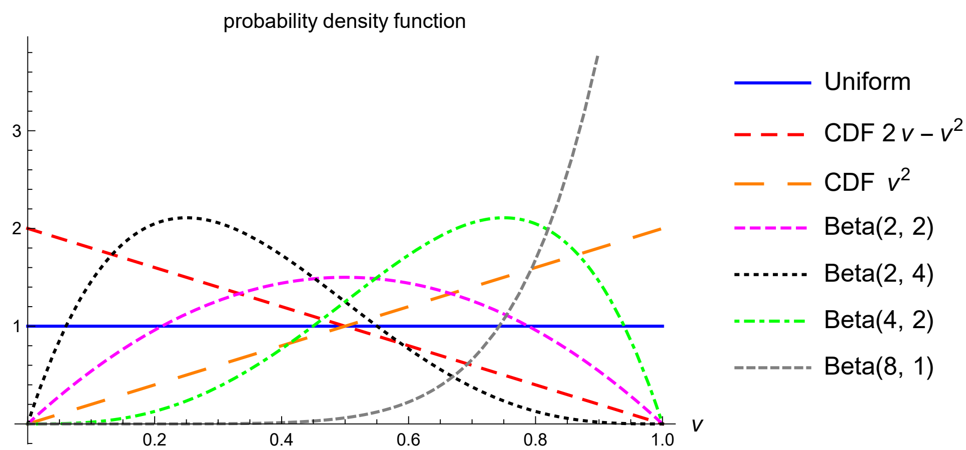
The above payment rule is identical to the payment rule of exponential price mechanism in Equation (7), except when the other player submits the boundary message k.

Figure 2 shows the revenue guarantees Π_2^* and $\Pi_2^\#$ for various prior distributions[9] as $\nu \to 0$ and compares them with the first price auction with two buyers, where the reserve

[9]Distribution Beta(b,c) has a p.d.f. of $v^{b-1}(1-v)^{c-1} \cdot \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}$ for $v \in [0,1]$. Beta$(1,1)$ is of course the uniform distribution.

price is chosen to maximize the revenue guarantee in the first price auction (Bergemann, Brooks, and Morris, 2016). We see that Π_2^* and $\Pi_2^{\#}$ are generally very close, though $\Pi_2^{\#}$ is slightly better than Π_2^* when the distribution is heavily concentrated among high values. In these examples $\Pi_2^{\#}$ is always better than the optimal revenue guarantee from first price auction with reserve price. In Figure 2 we also include the optimal revenue guarantees from the exponential price mechanism in Theorem 1 with one buyer.

Figure 2: Revenue guarantees from Proposition 1 (Π_2^*), Proposition 2 ($\Pi_2^{\#}$), first price auction with optimal reserve price (Bergemann, Brooks, and Morris, 2016), and Theorem 1 (Π_1^*).



Prior distribution	Mean	$\Pi_2^{\#}$	Π_2^*	FPA w/ optimal reserve	Π_1^*
Uniform	0.5	0.273	0.272	0.177	0.204
CDF $2v - v^2$	0.3333	0.166	0.166	0.102	0.120
CDF v^2	0.6667	0.437	0.431	0.346	0.341
Beta(2, 2)	0.5	0.302	0.301	0.230	0.229
Beta(2, 4)	0.3333	0.188	0.188	0.139	0.140
Beta(4, 2)	0.6667	0.475	0.463	0.414	0.381
Beta(8, 1)	0.8889	0.751	0.710	0.716	0.652

Finally, consider the following "wallet game" information structure: buyer i privately observes signal s_i, $i = 1, 2$, where s_i has the uniform distribution on $[0, 1]$. The common value is $v = (s_1 + s_2)/2$. Thus, the prior of the common value is the "triangle" distribution. Given this information structure, the optimal mechanism is a direct mechanism that assigns the good to buyer 1 if $s_1 \geq s_2$ and $s_1 + s_2/2 \geq 1/2$; assigns to buyer 2 if $s_2 > s_1$ and

$s_2 + s_1/2 \geq 1/2$; and does not assign the good otherwise. (The virtual value of buyer i is $s_i - 1/2 + s_j/2$, $j \neq i$.) The payment rule is given by Myerson's Lemma and makes this mechanism incentive compatible. The expected revenue of this optimal mechanism is 0.3611.[10] Thus, if the prior of the common value is the triangle distribution, 0.3611 is an upper bound on the revenue guarantee of any mechanism. We compute for the triangle distribution: $\Pi_2^* = 0.31094$ and $\Pi_2^\# = 0.31324$, both within 86% of this upper bound.

6 Conclusion

We introduce a new class of mechanisms (the exponential price mechanisms) to sell a common value good. The mechanisms are simple and practical, and can guarantee a good revenue over all information structures and equilibria. The revenue guarantee is provably optimal when there is one buyer, and converges to the full surplus as the number of buyers tends to infinity. To derive these mechanisms we introduce a linear programming duality approach, which we believe is useful for other robust mechanism design problems, e.g., for studying the revenue guarantee when buyers have both common and private values.

[10] We compute: $2 \cdot \int_{s_1=0}^{1} \int_{s_2=0}^{s_1} \max(s_1 + s_2/2 - 1/2, 0)\, ds_2\, ds_1 = 13/36 \approx 0.3611$.

Appendix

A Proofs for Section 5

Proof of Lemma 2. By (40) we have $q(k,k) = 1/2$. By the second line of (38) this implies that $q(j,k) = j/(2k)$ and $q(k,j) = 1 - j/(2k)$, $j = 0, 1, \ldots, k$. Then we have

$$k - \frac{(k-1)k}{4k} = \sum_{j=0}^{k-1} q(k,j) = \sum_{j=0}^{k-1} \sum_{l=0}^{k-1} q(l+1,j) - q(l,j) = k^2 q(1,0) \tag{48}$$

where the last equality follows from the first line of (38). Thus, $q(1,0) = (3k+1)/(4k^2)$.

□

Proof of Lemma 3. Fix an arbitrary P that satisfies Condition (39).

From the second line of (39) ($\mathrm{Rev}(v,(j-1,k)) = \mathrm{Rev}(v,(j,k))$), we have

$$P(j+1,k) - P(j,k) = (1+1/a)(P(j,k) - P(j-1,k)) + (P(k,j) - P(k,j-1))/a, \tag{49}$$

for $j = 1, 2, \ldots, k-1$. Equation (49) implies that

$$P(j+1,k) - P(j,k) = (1+1/a)^j P(1,k) + \sum_{j'=1}^{j} (1+1/a)^{j-j'} (P(k,j') - P(k,j'-1))/a, \tag{50}$$

and as a consequence, for any $j = 0, 1, \ldots, k$:

$$P(j,k) = a((1+1/a)^j - 1)P(1,k) + \sum_{j'=1}^{j-1} ((1+1/a)^{j-j'} - 1)(P(k,j') - P(k,j'-1)). \tag{51}$$

We claim that

$$\begin{aligned} X(l) &\equiv \sum_{j=1}^{l-1} (1+1/a)^{l-j} (P(l,j) - P(l,j-1)) \\ &= P(l,l-1) + a((1+1/a)^l - 1)^2 P(1,0) - (1+1/a)^l P(l,0), \end{aligned} \tag{52}$$

for every $l = 1, 2, \ldots, k$. Equation (52) for $l = k$ and Equation (51) together imply Equation (41), which proves the lemma.

Clearly, (52) is true for $l = 1$. Suppose (52) is true for $l = \kappa < k$ as an induction hypothesis; we prove that this implies (52) is true for $l = \kappa + 1$.

From $\text{Rev}(v, (\kappa, j-1)) = \text{Rev}(v, (\kappa, j))$ we have:

$$\begin{aligned} &P(\kappa+1, j) - P(\kappa+1, j-1) \\ =&(1+1/a)(P(\kappa, j) - P(\kappa, j-1)) + (1+1/a)(P(j, \kappa) - P(j-1, \kappa)) \\ &- (P(j+1, \kappa) - P(j, \kappa)), \end{aligned} \tag{53}$$

summing the above equation across $j = 1, 2, \ldots, \kappa - 1$ gives:

$$\begin{aligned} &\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(\kappa+1, j) - P(\kappa+1, j-1)) \\ =&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa, j) - P(\kappa, j-1)) + \sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(j, \kappa) - P(j-1, \kappa)) \\ &- \sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(j+1, \kappa) - P(j, \kappa)) \\ =&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa, j) - P(\kappa, j-1)) + (1+1/a)^{\kappa+1}P(1, \kappa) - (1+1/a)^2(P(\kappa, \kappa) - P(\kappa-1, \kappa)). \end{aligned} \tag{54}$$

That is,

$$\begin{aligned} &X(\kappa+1) \\ =&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa, j) - P(\kappa, j-1)) + (1+1/a)^{\kappa+1}P(1, \kappa) \\ &- (1+1/a)^2(P(\kappa, \kappa) - P(\kappa-1, \kappa)) + (1+1/a)(P(\kappa+1, \kappa) - P(\kappa+1, \kappa-1)) \\ =&(1+1/a)^2[P(\kappa, \kappa-1) + a((1+1/a)^\kappa - 1)^2 P(1, 0) - (1+1/a)^\kappa P(\kappa, 0)] + (1+1/a)^{\kappa+1}P(1, \kappa) \\ &- (1+1/a)^2(P(\kappa, \kappa) - P(\kappa-1, \kappa)) + (1+1/a)(P(\kappa+1, \kappa) - P(\kappa+1, \kappa-1)), \end{aligned} \tag{55}$$

where in the last equality we have used the induction hypothesis (52) for $l = \kappa$.

From $\text{Rev}(v, (\kappa, 0)) = \text{Rev}(v, (1, 0))$ we have $(1+1/a)P(\kappa, 0) - P(1, \kappa) = P(\kappa+1, 0) -$

$2P(1,0)$. Therefore, the previous equation is equivalent to:

$$\begin{aligned} & X(\kappa+1) \\ =& (1+1/a)^2P(\kappa,\kappa-1) + a(1+1/a)^2((1+1/a)^\kappa-1)^2P(1,0) - (1+1/a)^{\kappa+1}(P(\kappa+1,0)-2P(1,0)) \\ & - (1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa)) + (1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1)) \\ =& (1+1/a)^2P(\kappa,\kappa-1) + [a(1+1/a)^2((1+1/a)^\kappa-1)^2 + 2(1+1/a)^{\kappa+1}]P(1,0) - (1+1/a)^{\kappa+1}P(\kappa+1,0) \\ & - (1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa)) + (1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1)) \end{aligned} \tag{56}$$

From $\mathrm{Rev}(v,(\kappa,\kappa)) = \mathrm{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa) - P(\kappa+1,\kappa) = -P(1,0)$. Therefore, the previous equation is equivalent to:

$$\begin{aligned} & X(\kappa+1) \\ =& (1+1/a)^2P(\kappa,\kappa-1) + [a(1+1/a)^2((1+1/a)^\kappa-1)^2 + 2(1+1/a)^{\kappa+1} + (1+1/a)]P(1,0) \\ & - (1+1/a)^{\kappa+1}P(\kappa+1,0) + (1+1/a)^2P(\kappa-1,\kappa) - (1+1/a)P(\kappa+1,\kappa-1). \end{aligned} \tag{57}$$

From $\mathrm{Rev}(v,(\kappa-1,\kappa)) = \mathrm{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa-1) + (1+1/a)P(\kappa-1,\kappa) - P(\kappa+1,\kappa-1) = P(\kappa,\kappa) - 2P(1,0)$, Therefore, the previous equation is equivalent to:

$$\begin{aligned} & X(\kappa+1) \\ =& [a(1+1/a)^2((1+1/a)^\kappa-1)^2 + 2(1+1/a)^{\kappa+1} - (1+1/a)]P(1,0) \\ & - (1+1/a)^{\kappa+1}P(\kappa+1,0) + (1+1/a)P(\kappa,\kappa). \end{aligned} \tag{58}$$

Finally, using $(1+1/a)P(\kappa,\kappa) - P(\kappa+1,\kappa) = -P(1,0)$ again we get:

$$\begin{aligned} & X(\kappa+1) \\ =& [a(1+1/a)^2((1+1/a)^\kappa-1)^2 + 2(1+1/a)^{\kappa+1} - (1+1/a) - 1]P(1,0) \\ & - (1+1/a)^{\kappa+1}P(\kappa+1,0) + P(\kappa+1,\kappa). \end{aligned} \tag{59}$$

Since $a(1+1/a)^2((1+1/a)^\kappa-1)^2 + 2(1+1/a)^{\kappa+1} - (1+1/a) - 1 = a((1+1/a)^{\kappa+1}-1)^2$, this proves (52) when $l = \kappa+1$. □

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