Synchronization and Bias in a Simple Macroeconomic Model

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Abstract

This note illustrates how agents' beliefs about economic outcomes can dynamically synchronize and de-synchronize to produce business-cycle-like fluctuations in a simple macroeconomic model. I consider a simple macroeconomic model with multiple equilibria. The equilibria correspond to different ways that agents can use a sunspot variable to forecast future output, which are self-fulfilling. Agents are assumed to learn to use the sunspot variable through econometric learning. I show that if different agents measure output with a slight bias (though the average bias in the economy vanishes), this leads to a complex nonlinear dynamic of synchronization of beliefs about the equilibrium being played. The economy fluctuates between long eras where the agents' beliefs are almost homogenous and therefore they are coordinated on the use of the sunspot, and eras of dispersed beliefs where the coordination mechanism fails. I show that the equation describing the evolution of the economy is similar to the Kuramoto model, a prototypical model of synchronization phenomena, and make some first attempts at mapping the connections.

1 Introduction

The goal of this note is to illustrate how agents' beliefs about economic outcomes can dynamically synchronize and de-synchronize to produce business-cycle-like fluctuations in a simple macroeconomic model.

In models with strategic uncertainty, such as models with multiple equilibria, agents must form beliefs about the actions of other players. In sunspot models,

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for example, this is achieved by the agents using an exogenous stochastic process, the *sunspot* variable, to coordinate on the equilibrium that is played. An interesting question is whether agents in such an economy can learn to coordinate by measuring observable quantities in that economy. This question has been studied for a number of models, and conditions have been identified to determine if an equilibrium is stable under different learning schemes (see e.g. Woodford, 1990; Evans et al., 1994).

The literature so far has treated the question as dichotomous, referring to equilibria as either learnable or not-learnable. In this note I study a third option: that the agents' beliefs about the equilibrium being played fluctuate between eras of synchronized beliefs, where the beliefs are nearly identical across agents and self-fulfilling, and eras of asynchronous beliefs. Specifically, the agents in the model are trying to learn about the relationship between two publicly observed stochastic processes ($z_t^i, i = 1, 2$) and total output (y_t). There are specific linear combinations, $y_t = \phi + \xi \cdot z_t$, such that if the entire economy believed that output fluctuates according to this formula, it would be self fulfilling. However, I introduce two assumptions into the model that cause the system to never converge to any of these rational-expectations equilibria: first, I assume that agents update their beliefs about the relationship between the sunspot variables and output using an econometric learning algorithm; and second, I assume that each agent measures output with a slight persistent bias, i.e. some agents are inherently slightly pessimistic while others are slightly optimistic.[1] Even though the agents are on average (across agents) correct in their measurements, this imperfection is sufficient to generate a complicated nonlinear dynamic in the belief space. While agents never end up agreeing on an equilibrium, the system does not diverge either: instead, it slowly moves between eras of lower and higher dispersion in the belief space.

The dynamics of the model are closely related to the Kuramoto model (Kuramoto, 1975), which has been used to describe synchronization phenomena across different disciplines and subject areas including synchronization of flashing fireflies, phase lock in metronomes, and synchronized applause at the end of a concert (Strogatz, 2000). The Kuramoto model describes a set of oscillators whose phases, are nonlinearly coupled, not unlike how the learning process in my model links the agents' beliefs about the equilibrium being played. This is, to my knowledge, the first time that this link has been made, and potentially opens the door to incorporating into macroeconomics the rich phenomena that the Kuramoto model can describe.

The organization of this note is as follows: in the next section I review some of

[1] In a recent paper Patton and Timmermann (2010) provide empirical evidence that individual professional forecasters do seem to be persistently biassed in their predictions compared to the average forecast. This provides some motivation for my assumptions.

the relevant economic literature and give some background about the Kuramoto model for readers who are not familiar with it. Section 3 describes the model, which is a slightly modified version of the model of Benhabib, Wang, and Wen (2012). Section 4 includes the main analysis: describing the rational-expectations equilibria of the model and their stability properties, the results of numerical simulations for the full model, and proving the relationship to the Kuramoto model. Finally, some concluding remarks are left to section 5.

2 Related Literature and Background

2.1 Macroeconomics and Learning

Learning has a long history in macroeconomics, but the stochastic recursive description in this paper originates with Marcet and Sargent (1989). For a comprehensive account of the state of this field see Evans and Honkapohja (2012). Some papers that study learning in the presence of multiple equilibria and sunspots are Woodford (1990); Guesnerie and Woodford (1990); Evans et al. (1994); Evans and Honkapohja (2003a,b); Honkapohja and Mitra (2004).

2.2 Sunspots and Sentiments

Traditionally, the term *sunspot* is used in macroeconomics to describe a situation where the dynamic equations of a system lead to indeterminacy, and therefore a new stochastic process, the sunspot, can be introduced for the agents to coordinate their actions on (for example Benhabib and Farmer, 1994; Christiano and Harrison, 1996). In these models the realization of the stochastic process determines the equilibrium being played. In a more recent paper Angeletos and La'O (2013) describe a different situation where there is a unique equilibrium in which the agents use a random variable that they call the *sentiment* to choose their actions. While similar in spirit, these are formally different situations. This note makes use of the model introduced by Benhabib et al. (2012), which is similar to the latter in that the role of the stochastic process is not to choose between equilibria.

2.3 The Kuramoto Model

The Kuramoto model (Kuramoto, 1975) describes a system of N oscillators whose phases $\{\psi_t^i\}_{i=1}^N$ ($\psi_t^j \in [-\pi, \pi]$), are coupled in the manner described by the equation

$$\frac{d}{dt}\psi_t^i = \omega^i - \frac{K}{N}\sum_{j=1}^{N}\sin(\psi_t^i - \psi_t^j), \quad i = 1, \cdots, N,$$

where $t \in \mathbb{R}$ is a continuous time parameter, ω^i is the natural frequency of the oscillator i, and K is the coupling coefficient. This might describe, for example, a system of metronomes set to different frequencies whose phases are coupled by sending vibrations through a platform that they are placed on. Thorough introductions to the model and reviews of the current state of the literature include Strogatz (2000) and Acebrn et al. (2005).

By defining

$$R_t e^{i\psi_t} = \frac{1}{N}\sum_{i=1}^{N} e^{i\psi_t^i},$$

the equations take the more convenient form

$$\frac{d}{dt}\psi_t^i = \omega^i - R_t K \sin(\psi_t^i - \psi_t).$$

I refer the reader to the above references for a rigorous treatment of the model, and just briefly mention that in the case of large N, there are three interesting solutions to the model. First, if the phases are distributed uniformly along the circle, then $R_t = 0$ and we have each phase moving with its natural frequency $\psi_t^i = \omega^i t$. This is the *incoherent solution.*

A second solution can occur if K is sufficiently large and the ω^i's are not too dispersed. This is the synchronized solution, which is given by all the phases moving with the same frequency with a constant phase difference between them: $\psi_t^i = \bar{\omega} t + \phi^i$, and $\psi_t = \bar{\omega} t$. The parameter $\bar{\omega}$ is the average of the frequencies, and the phase differences ϕ^i can be found by plugging this solution into the equations. One finds

$$\bar{\omega} = \frac{1}{N}\sum_{i=1}^{N}\omega^i, \qquad \sin\phi^i = \frac{\omega^i - \bar{\omega}}{\bar{R}K}, \qquad \bar{R} = \frac{1}{N}\left|\sum_{i=1}^{N} e^{i\phi^i}\right|.$$

In addition, there may be a mixed solution, where the oscillators whose natural frequency ω^i is close enough to $\bar{\omega}$ are moving in phase-lock while the others are drifting around the circle.

The stability of the solutions depends on the strength of the coupling coefficient K. In a finite sample, the system may spend a large amount of time (compared to $\bar{\omega}^{-1}$) in a state approximating the synchronized phase, and then transition to the incoherent phase, and continuing back and forth. Thus, the system has two characteristic time scales: a short one defined by the natural frequency $\bar{\omega}^{-1}$ and a long one defined by the frequency of transitions.

Finally, by choosing different formulas for the coupling between the oscillators, one can get even more complex ordered phenomena in the Kuramoto model. For example, by choosing the coupling between every two oscillators $K_{i,j}$ to be some specific function of $|i-j|$, one gets solutions were parts of the system is synchronized in different times, and those areas move in wave-like patterns. These solutions have proved instructive in various applications, and may well be useful for economists as well.

3 Model Setup

The model setup is based on Benhabib, Wang, and Wen (2012).

3.1 Households and Firms

3.1.1 Households

A representative household values streams of consumption $C_t \geq 0$ and labor $N_t \geq 0$ according to

$$U = \sum_{t=0}^{\infty} \beta^t [\log C_t - \psi N_t], \quad \beta \in (0,1), \quad \psi > 0,$$

and is subject to the budget constraint

$$P_t C_t \leq W_t N_t + \Pi_t,$$

where P_t, W_t and Π_t are the prices of the consumption good, the nominal wage, and the profits from ownership of firms, respectively.

The household's first-order conditions are

$$C_t = \frac{1}{\psi} \cdot \frac{W_t}{P_t}, \tag{1}$$

$$N_t = \frac{1}{\psi} - \frac{\Pi_t}{W_t}. \tag{2}$$

3.1.2 Final Good Producers

The consumption good is produced by competitive final good producers using a continuum of intermediate goods indexed by $j \in [0,1]$, with the stochastic technology

$$Y_t = \left[\int_0^1 \epsilon_{jt}^{\theta} Y_{jt}^{1-\theta} dj \right]^{\frac{1}{1-\theta}}, \quad \theta > 0 \tag{3}$$

where ϵ_{jt} are iid random variables, and can be interpreted as preference shocks.

Denoting the price of good j at time t by P_{jt}, the demand for intermediate good j is given by

$$\left(\frac{Y_{jt}}{Y_t}\right)^\theta = \frac{P_t}{P_{jt}}\varepsilon_{jt}^\theta.$$

From which we also get the relationship:

$$P_t^{1-1/\theta} = \int_0^1 \epsilon_{jt} P_{jt}^{1-1/\theta} dj.$$

3.1.3 Intermediate Goods Producers

Each variety of intermediate good j is manufactured by a monopolist using labor as the only input and with the production function: $Y_{jt} = AN_{jt}$. The intermediate good manufacturers must decide on their level of production simultaneously at the beginning of the period without observing the shocks ϵ_{jt}. After these decisions have been made, prices are set so that markets clear, similarly to a Cornout competition.

The intermediate good producers' problem is therefore

$$\max_{Y_{jt}} \mathbb{E}_{jt}[(P_{jt} - W_t/A)Y_{jt}],$$

where $\mathbb{E}_{jt}$ represents the firms expectation operator conditioned on the information (and beliefs) available to firm j at time t, which will be described below. The first-order-condition is

$$Y_{jt} = \mathbb{E}_{jt}\left[A(1-\theta)\frac{P_t}{W_t}Y_t^\theta \epsilon_{jt}^\theta\right]^{1/\theta}.$$

Substituting (1) into the above, we get

$$Y_{jt} = \mathbb{E}_{jt}\left[\frac{A(1-\theta)}{\psi}Y_t^{\theta-1}\epsilon_{jt}^\theta\right]^{1/\theta} = \mathbb{E}_{jt}\left[Y_t^{\theta-1}\epsilon_{jt}^\theta\right]^{1/\theta}, \tag{4}$$

where in the last step, without loss of generality, I choose units of output such that $\psi = A(1-\theta)$.

Notice that in order to make the production decision, the firm needs to forecast the overall output Y_t, and its own preference shock ϵ_{jt}. I assume that the firm bases its decision on a signal s_{jt} that is obtained from consumer surveys to be described below.

3.2 Information

There is a large number of forecasters indexed by $i \in [0,1]$. The forecasters observe a two-dimensional stochastic process z_t and use that to forecast output Y_t. The variable z_t is in fact entirely unrelated to the fundamentals of the economy, but the forecasters do not know that and have to develop a belief about this relationship. In the baseline model we assume that z_t is a two-dimensional standard normal random variable, independent across time, and we limit the belief space so that the forecasters are limited to beliefs of the form $y_t = \log Y_t = \phi^i + \xi^i \cdot z_t$, with $(\phi^i, \xi^i) \in \mathbb{R}^3$.

The intermediate-good firms do not get to see the signal z_t directly, but instead rely on a survey of the forecasters to estimate its demand curve. However, the firm is limited in its ability to conduct market research, and we assume that it eventually obtains a signal that mixes the forecasters' beliefs about output with their preference for the firm's good:

$$s_{jt} = \lambda \varepsilon_{jt} + (1-\lambda)\left(\left\langle \phi^i \right\rangle_t + \left\langle \xi^i \right\rangle_t \cdot z_t\right), \quad \lambda \in (0,1), \tag{5}$$

where $\varepsilon_{jt} = \log \epsilon_{jt}$, and $\langle \phi^i \rangle_t$ and $\langle \xi^i \rangle_t$ denote the average beliefs of forecasters about ϕ and ξ respectively. A possible interpretation of this information constraint is that the firm asks the forecasters questions that aim to gauge the demand for their good, and the result is a signal that mixes the preference for the specific good ε_{jt} with the overall forecast for output $y_t = \langle \phi^i \rangle_t + \langle \xi^i \rangle_t \cdot z_t$.

The intermediate good firms also each have a belief that output follows $y_t = \phi^j + \xi^j \cdot z_t$, which they use to interpret the signal they obtain from the survey.

3.3 Learning

Throughout this paper it is assumed that at any point in time all agents have a belief about the system, which can be summarized by a point $(\phi^j, \xi^j) \in \mathbb{R}^3$, and they act as if they have no uncertainty about it. At the end of the period, the variable z_t is revealed, and firms update their beliefs. This non-Bayesian form of learning is sometimes called econometric learning, and has been used extensively in macroeconomics.[2]

The updating process can be written recursively:

$$\begin{aligned}
\begin{pmatrix} \phi^j_{t+1} \\ \xi^j_{t+1} \end{pmatrix} &= \begin{pmatrix} \phi^j_t \\ \xi^j_t \end{pmatrix} + g_t {\Upsilon^j_{t+1}}^{-1} \begin{pmatrix} 1 \\ z_t \end{pmatrix} (y_t - \phi^j_t - \xi^j_t \cdot z_t), \\
\Upsilon^j_{t+1} &= \Upsilon^j_t + g_t(z_t \cdot z_t' - \Upsilon^j_t),
\end{aligned}$$

[2]For a review of the literature, see Evans and Honkapohja (2012)

where g_t is the gain function. The choice $g_t = 1/t$ corresponds to least-square learning (RLS), and replicates the OLS estimator. In the next section we consider a few different choices for the gain function. Also notice that the matrix Υ_t^j only depends on the starting point Υ_0^j and on the realization of z_t, which is common knowledge, which implies that Υ_t^j will quickly diverge to the unit matrix I_2. Therefore, for simplicity I assume that $\Upsilon_t^j = I_2$ throughout.

I modify the above by assuming that each firm in the economy has a persistent bias in its measurement of output, denoted by $\Delta\phi^j$. Since this bias generates a biased estimate of ϕ, it can be interpreted as an assumption that some agents are inherently too optimistic or pessimistic. With this assumption, the recursive learning formula is

$$\begin{pmatrix} \phi_{t+1}^j \\ \xi_{t+1}^j \end{pmatrix} = \begin{pmatrix} \phi_t^j \\ \xi_t^j \end{pmatrix} + g_t \begin{pmatrix} 1 \\ z_t \end{pmatrix} (y_t + \Delta\phi^j - \phi_t^j - \xi_t^j \cdot z_t). \tag{6}$$

Finally, the beliefs of the forecasters at the beginning of every period are assumed to be identically distributed to those of the firms. This simplifying assumption is similar to assuming that firms do get to observe z_t but with a very large error, so that this information is not useful for making their own prediction about output, and that the surveys are conducted by polling representatives of other firms.

4 Analysis

First, consider a firm whose beliefs are given by (ϕ^j, ξ^j). Defining $x_{jt} = (\theta-1)(y_t - \phi^j) + \theta\varepsilon_{jt}$, we have from (4)

$$y_{jt} = \theta^{-1} \log \mathbb{E}_{jt}\left[e^{(\theta-1)y_t + \theta\epsilon_{jt}} | s_{jt}\right] = (1-\theta^{-1})\phi^j + \theta^{-1} \log \mathbb{E}_{jt}\left[e^{x_{jt}} | s_{jt}\right]. \tag{7}$$

Let us assume that $\log \epsilon_{j,t} \sim \mathrm{N}(0, \sigma_\varepsilon^2)$, so that $\mathbb{E}_{jt}[x_{jt}] = 0$, $\mathbb{E}_{jt}[s_{jt}] = (1-\lambda)\phi^j$, and the subjective variance-covariance matrix of (x_{jt}, s_{jt}) is

$$\Sigma = \begin{pmatrix} \theta^2\sigma_\varepsilon^2 + (1-\theta)^2\|\xi^j\|^2 & \theta\lambda\sigma_\varepsilon^2 - (1-\lambda)(1-\theta)\|\xi^j\|^2 \\ \text{sym.} & \lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\|\xi^j\|^2 \end{pmatrix}.$$

Therefore, $x_{jt}|s_{jt} \sim \mathrm{N}(m(\|\xi^j\|^2)(s_{jt} - (1-\lambda)\phi^j), \hat{\Sigma}(\|\xi^j\|^2))$, where

$$m(\|\xi^j\|^2) = \frac{\theta\lambda\sigma_\varepsilon^2 - (1-\lambda)(1-\theta)\|\xi^j\|^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\|\xi^j\|^2},$$
$$\hat{\Sigma}(\|\xi^j\|^2) = \frac{(\theta+\lambda-2\theta\lambda)^2\|\xi^j\|^2\sigma_\varepsilon^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\|\xi^j\|^2}.$$

Therefore, from (7)

$$y_{jt} = (1-\theta^{-1})\phi^j + \theta^{-1}\left[m(\|\xi^j\|^2)(s_{jt} - (1-\lambda)\phi^j) + \frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)\right].$$

Using this in (3),

$$\begin{aligned}
(1-\theta)y_t &= \log \int_0^1 e^{\theta\varepsilon_{jt}+(1-\theta)y_{jt}} dj = \\
&= \log \int_0^1 e^{\theta\varepsilon_{jt}+(1-\theta)\{(1-\theta^{-1})\phi^j+\theta^{-1}[m(\|\xi^j\|^2)(s_{jt}-(1-\lambda)\phi^j)+\frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)]\}} dj = \\
&= \log \int_0^1 e^{\theta\varepsilon_{jt}+(1-\theta)\{(1-\theta^{-1})\phi^j+\theta^{-1}[m(\|\xi^j\|^2)(\lambda\varepsilon_{jt}+(1-\lambda)(\langle\phi^i\rangle-\phi^j+\langle\xi^i\rangle\cdot z_t))+\frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)]\}} dj,
\end{aligned}$$

where (5) is used in the last step. Since ε_{jt} is independent of beliefs, we can integrate

$$\begin{aligned}
(1-\theta)y_t = \log \int_0^1 & e^{\frac{\sigma_\varepsilon^2}{2}[\theta+(\theta^{-1}-1)\lambda m(\|\xi^j\|^2)]^2} \times \\
& \times e^{+(1-\theta)\{(1-\theta^{-1})\phi^j+\theta^{-1}[(1-\lambda)m(\|\xi^j\|^2)(\langle\phi^i\rangle-\phi^j+\langle\xi^i\rangle\cdot z_t)+\frac{1}{2}\hat{\Sigma}(\|\xi^j\|^2)]\}} dj. \qquad (8)
\end{aligned}$$

Equation (8) describes the mapping from the full belief space to actual output.

4.1 Rational Expectations Equilibria

Equation (8) simplifies tremendously in the case that all agents have common beliefs $\phi^j = \phi$, $\xi^j = \xi$. We have

$$\begin{aligned}
(1-\theta)y_t = -\frac{(1-\theta)^2}{\theta}\phi + \frac{1}{2}[\theta + (\theta^{-1}-1)m(\|\xi\|^2)\lambda]^2\sigma_\varepsilon^2 \\
+ (\theta^{-1}-1)\left[m(\|\xi\|^2)(1-\lambda)\xi\cdot z_t + \frac{1}{2}\hat{\Sigma}(\|\xi\|^2)\right].
\end{aligned}$$

The actual output is a linear function of z_t, therefore the above defines a mapping from the commonly perceived law of motion $y_t = \phi + \xi \cdot z_t$ to the actual law of motion above:

$$\phi \to -\frac{(1-\theta)}{\theta}\phi + \frac{1}{2\theta}\hat{\Sigma}(\|\xi\|^2) + \frac{[\theta + (\theta^{-1}-1)m(\|\xi\|^2)\lambda]^2\sigma_\varepsilon^2}{2(1-\theta)}, \qquad (9a)$$

$$\xi \to \frac{1}{\theta}m(\|\xi\|^2)(1-\lambda)\xi. \qquad (9b)$$

A rational expectations equilibrium is a fixed point of the above mapping. There always exist a fixed point of the form:

$$\xi^C = 0, \qquad\qquad \phi^C = \frac{\theta\sigma_\varepsilon^2}{2(1-\theta)}.$$

The superscript C stands for 'certainty', since output is constant in this fixed point. Note that in this case, since output is known, the signal reveals ε_{jt} so firms choose output optimally.

If $0 \leq \lambda \leq 1/2$, there is an addition circle of fixed points given by

$$\|\xi^S\|^2 = \frac{\theta\lambda(1-2\lambda)}{(1-\lambda)^2}\sigma_\varepsilon^2, \qquad\qquad \phi^S = \phi^C\left(1 - \frac{(1-\theta)(1-2\lambda)}{1-\lambda}\right).$$

The superscript S stands for 'stochastic', as output is a linear function of the stochastic process z_t in these fixes points.

These two types of fixed points correspond to the equilibria found in Benhabib et al. (2012). As they note, the average output is lower in the stochastic equilibrium (clearly $\phi^C > \phi^S$ for all $\lambda \in [0, 1/2]$ and $\theta \in (0, 1)$). It is also straightforward to show that the certainty equilibrium is Pareto superior.

4.2 Stability without Bias

Before discussing the effect of the persistent bias, it is instructive to consider the stability properties of the rational expectations equilibria found above, thus for now we set $\Delta\phi^j = 0$. Equations (9) define a mapping from perceived to actual law of motion in the (ϕ, ξ) space with homogenous beliefs, and the stability properties of this mapping are given by the eigenvalues of the first derivatives at the fixed point. For the certainty equilibrium, the eigenvalues are:

$$\lambda_0 = -\frac{1-\theta}{\theta}, \quad \lambda_{1,2} = \frac{1-\lambda}{\lambda}.$$

For the stochastic equilibria, the eigenvalues are:

$$\lambda_0 = -\frac{1-\theta}{\theta}, \quad \lambda_1 = 1 - \frac{2(1-2\lambda)}{(1-2\lambda)\theta + \lambda}, \quad \lambda_2 = 1.$$

The unit eigenvalue in the latter case is a manifestation of the rotational symmetry in the (ξ^1, ξ^2).

Under recursive-least-squares learning, the condition for convergence is that the eigenvalues as defined above are all smaller than unity.[3] Therefore, we see

[3]Cf. §2 of Evans and Honkapohja (2012), and §3.1 for the case of heterogenous expectations.

that for $\lambda > 1/2$, when only the certainty equilibrium exists, it is also stable, and therefore the system almost surely converges to it. When $\lambda < 1/2$, both types of equilibria exist, but the certainty equilibrium is not stable since $\lambda_{1,2} > 1$. As for the stochastic equilibria, the first two eigenvalues are smaller than unity ($\lambda_0, \lambda_1 < 1$), which would imply convergence with probability one, but the third eigenvalue $\lambda_2 = 1$ is exactly the borderline case when convergence cannot be determined. Since this eigenvalue comes from an exact symmetry of the system, the mapping (9) is exactly flat in the corresponding direction, thus higher-order derivatives vanish and cannot be used to resolve the question of stability.

For the rest of this note, I focus on the case $\lambda < 1/2$, where the certainty equilibrium is not relevant. Once we reintroduce the bias ($\Delta\phi^j$), we shall see that the system tends to "drift" between the circle of stochastic equilibria. Recursive-least-squares learning is defined by (6) with $g_t = t^{-1}$, thus, the weight that agents assign to new observations decreases to zero over time. This is not a very natural assumption in a system that does not converge. Instead, a popular alternative is least-squares with forgetting, that is defined by $g_t = (1-q)/(1-q^t)$ ($q \in (0,1)$), and arises from the assumption that agents assign a weight of q^t to an observation that occurred t periods ago. This gain function approaches constant gains $g_t \to (1-q)$ as $t \to +\infty$, which implies that learning does not slow down as time passes.

Convergence under constant gains is more complicated, but it is sufficient to require that the mapping (9) is Lyapunov stable, i.e. that the eigenvalues are within the unit circle, which is guaranteed for the first two eigenvalues of the stochastic equilibria when $\theta > 1/2$ and $(1-2\lambda)(1-\theta) < \lambda$. These results are summarized in figure 1. From here on we mostly concentrate on the area of the parameter space where these conditions are satisfied, so that without the bias the system would convergence at least in the two directions orthogonal to the rotational symmetry in the (ξ^1, ξ^2) plane.

4.3 Numerical Simulation

For studying the full system, it is useful to introduce polar coordinates:

$$R_t^j e^{i\psi_t^j} = \xi_t^{1j} + i\xi_t^{2j},$$
$$R_t e^{i\psi} = \int R_t^j e^{i\psi_j} dj.$$

For example, the stochastic equilibria are given by the R^j's all equal to ξ^S, and the ψ^j all equal to some constant, and therefore $R = \xi^S$ and ψ is equal to the same constant. Indeed, holding the R_t^j's fixed, when the ψ_t^j's are close to each other, we will have $R_t \approx \langle R_t^j \rangle$, but when the ψ_t^j's are dispersed $R_t \approx 0$, so R_t is a measure of the degree of coordination between the agents.

Figure 2 shows the result of a typical simulation where firms are assumed to have the bias $\Delta\phi^j = (-1)^j\, 0.08\phi^S$. One can see that the beliefs about average output fluctuates mostly between ϕ^S and ϕ^C, thus, the system does not converge but also does not diverge. More interestingly, the middle panel shows that R_t/ξ^S seems to climb towards unity, as the system tends to converge toward one of the stochastic equilibria, but falls as beliefs get more dispersed.

The precise behaviour of the system is very sensitive to the variance of the bias $\Delta\phi^j$. With high bias the system stays near $R_t = 0$ and the resulting time series for output, y_t, is right-skewed and heavy tailed. With small bias, $|\Delta\phi^j| \ll \phi^S$, the system quickly converges and stays near $R_t = 1$. Output in the latter case is symmetric and mesokurtic. This behaviour is not surprising: large persistent bias prevents agents from getting close enough to the rational-expectations-equilibria, and so beliefs remain dispersed, while with small bias the agents converge on some equilibria and only move slightly thereafter in repones to shocks. However, simulations with many different choices of parameters also suggest that the transition between these two regimes occurs abruptly at some critical value. In the next subsection I argue that this feature is a result of the similarity between this model and the Kuramoto model.

One other result that can be obtained from simulations concerns the cyclical behaviour of dispersion of beliefs. In particular, notice that the best predictor for future output $\mathbb{E}_{j,t}y_{t+1}$ is ϕ_t^j. Therefore, the statistical properties of ϕ_t^j can be thought of as analogous to the dispersion of forecasts measured in various surveys of professional forecasters. Clearly, my model is too simplistic to be compared to empirical data, but it is worth noting that in all simulations I find that high forecast dispersion (the variance of ϕ_t^j) is associated with the mean forecast being away from ϕ^S. In other words, this model predicts that forecasters will tend to disagree more when the average prediction is near the peak of trough of the cycle.

4.4 Learning about Phases Only

It is difficult to say anything more analytical about the full nonlinear system, but some insight can be obtained by considering the following simplification: suppose that all agents had fixed beliefs about ϕ and R^j. Specifically, assume that $\phi^j = \phi^S$ and $R^j = \xi^S$. This leaves ψ^j as the only parameter that the agents are trying to learn about. With these assumptions, the actual law of motion (8) simplifies to

$$y_t = \phi^S + \frac{1}{1-\theta}\log\int_0^1 e^{(1-\theta)\xi^S r_t \cos(\psi^j - \zeta_t)}dj,$$

where $r_t e^{i\zeta_t} = z_t^1 + iz_t^2$. To first order in g_t, the learning process is given by[4]

$$\psi_{t+1}^j = \psi_t^j - \frac{g_t r_t}{\xi^S} \sin(\psi_t^j - \zeta_t)\left(\xi^S r_t \Big\{ \big\langle \cos(\psi_t^k - \zeta_t)\big\rangle^* - \cos(\psi_t^j - \zeta_t)\Big\} + \Delta\phi^j\right), \tag{10}$$

$$\big\langle \cos(\psi_t^k - \zeta_t)\big\rangle^* = \frac{1}{(1-\theta)(\xi^S r_t)} \log \int_0^1 e^{(1-\theta)\xi^S r_t \cos(\psi_t^k - \zeta_t)} dk. \tag{11}$$

Figure 3 shows an example of a simulation of this system. What is remarkable about this system is that R_t, which measures the coordination between the agents, spends extended amounts of time close to ξ^S (maximum coordination), and then quickly changes to a phase of very low coordination. Not incidentally, this is precisely the dynamics of the Kuramoto model. To make the connection, first consider the term $\big\langle \cos(\psi_t^k - \zeta_t)\big\rangle^*$. This term is a logarithm of a generalized mean of $\exp(\cos(\psi^j - \zeta_t))$. In the linear approximation, which is valid when the ψ^j's are not too dispersed, the generalized mean is replaced with a simple arithmetic mean, so that (10) simplifies to

$$\psi_{t+1}^j = \psi_t^j - \frac{g_t r_t}{\xi^S} \sin(\psi_t^j - \zeta_t)\left(2\xi^S r_t \int_0^1 \left\{ \sin\left(\frac{\psi_t^j + \psi_t^k}{2} - \zeta_t\right) \sin\left(\frac{\psi_t^j - \psi_t^k}{2}\right)\right\} dk + \Delta\phi^j\right).$$

The above can be thought of as a noisy version of the Kuramoto model, which was described in section 2.3. In order to make the connection more apparent, consider the case where the ψ^j's are all close, so that to first order $(\psi_t^j + \psi_t^k)/2 \approx \psi_t^j$. The above equation is then

$$\psi_{t+1}^j = \psi_t^j - g_t r_t \sin(\psi_t^j - \zeta_t)\left[\sin(\psi_t^j - \zeta_t) \int_0^1 \sin(\psi_t^j - \psi_t^k) dk + \frac{\Delta\phi^j}{\xi^S}\right]. \tag{12}$$

Compare this to the discrete-time version of the Kuramoto model:

$$\psi_{t+1}^j = \psi_t^j - g_t \left[K \int \sin(\psi_t^j - \psi_t^k) dk + \omega^j\right]. \tag{13}$$

The similarity is clear. The only distinction is that in (12), both terms in the square brackets have a coefficient that is stochastic. The first coefficient is $r_t \sin^2(\psi_t^j - \zeta_t)$, and is always positive, therefore, it has the same effect as in the Kuramoto model, i.e. that of pulling the phases toward each other. The second term, which is

[4] I'm making the heuristic assumption that the agents continue using the RLS algorithm (6), but then simply override whatever estimator they got by replacing ϕ^j with ϕ^S and rescale ξ^j such that $\|\xi^j\| = \xi^S$. Essentially, they use the OLS estimator and then extract $\tan\psi_t^j = \xi_t^{2j}/\xi_t^{1j}$. This is not an efficient estimator of ψ, but it is consistent and allows for the analytical results that follow.

proportional to $\sin(\psi_t^j - \zeta_t)\Delta\phi^j$, like the ω^j term in (13) has the effect of pulling the phases apart. It may seem that this term should average out over time because the sine term would average out to zero, but since the bias is persistent, a correlation builds up between ψ_t^j and $\Delta\phi^j$, which means that this term affects each agent differently in a persistent manner to drive the beliefs apart. In fact, a time series analysis of long simulations shows that the correlation $\rho(\psi_t^j, \Delta\phi^j)$ is always close to ± 1 when R_t is close to ξ^S. This is demonstrated in figure 4.

The Kuramoto model has three special solutions: synchronized, incoherent, and partially-synchronized. In the synchronized, the phases ψ_t^j all move in synch, in the incoherent the phases are spread uniformly along the circle, and in the partially-synchronized the phases with $|\omega^j|$ smaller than some critical value are phase-locked while the others drift incoherently. All of these phenomena can be observed in simulations by changing the exogenous parameters. For example, the system remains in the synchronized state when $\langle|\Delta\phi^j|\rangle \ll \phi^S$, and remains in the incoherent state when $\langle|\Delta\phi^j|\rangle \gg \phi^S$. However, more research is needed to understand the relationships more deeply.

5 Conclusion

This note is intended to be an opening point to the study of synchronization and desynchronization phenomena in macroeconomics. It shows how the existence of a (small) bias in the way that agents measure economic variables can lead to complex nonlinear phenomena, such as spontaneous creation and destruction of order. Demonstrating the connection to the Kuramoto model is useful, since much is known about this model and it may be possible to compare belief-synchronization phenomena in economics to other complex systems.

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Figures

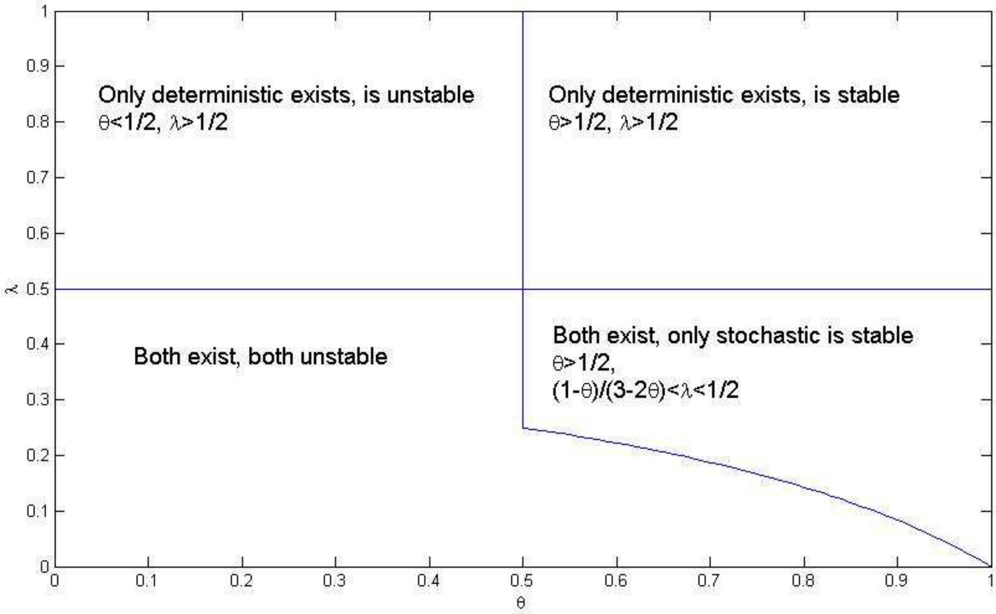

Figure 1: The phase space of the model, illustrating the regions of existence of the two types of rational-expectations equilibria, and the Lyapunov stability of the mapping (9). Lyapunov stability is a sufficient condition for convergence under learning with constant gains.

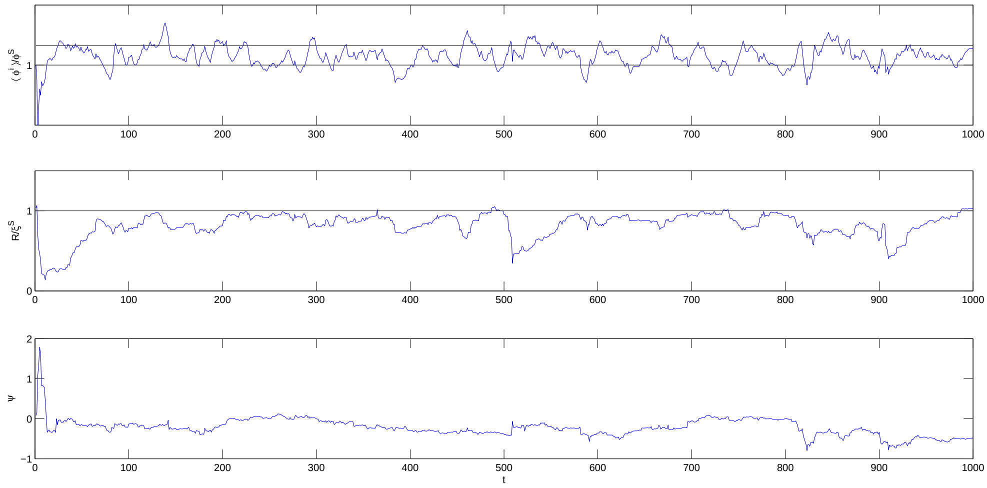

Figure 2: An example of a simulation of the full system. The top graph displays the evolution of the average belief $\left\langle \phi_t^j \right\rangle / \phi^S$, with the solid black lines denoting ϕ^S (bottom) and ϕ^C (top). The middle graph shows R_t/ξ^S, and the bottom shows the average phase ψ_t.

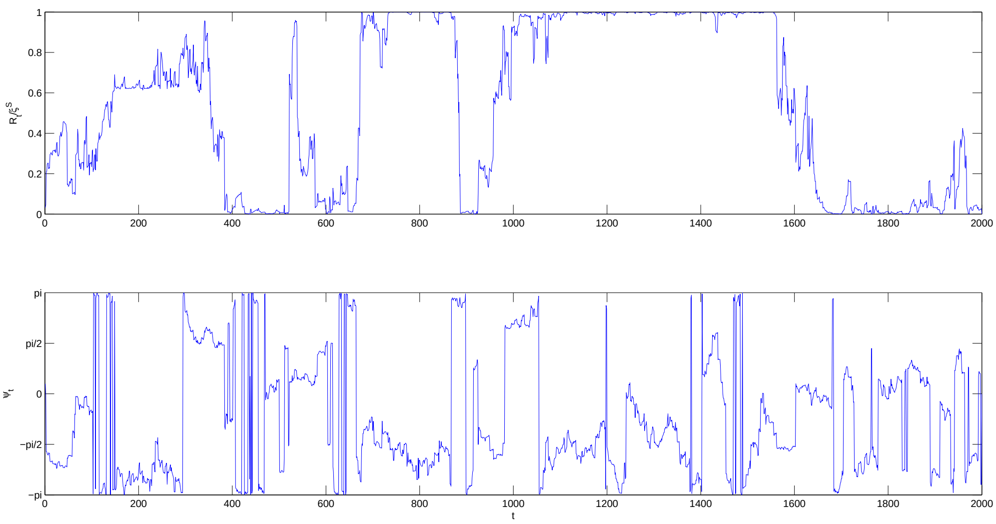

Figure 3: An example of a simulation of the system where only the phases ψ^j are updated. The top graph shows R_t/ξ^S, and the bottom shows the average phase ψ_t.

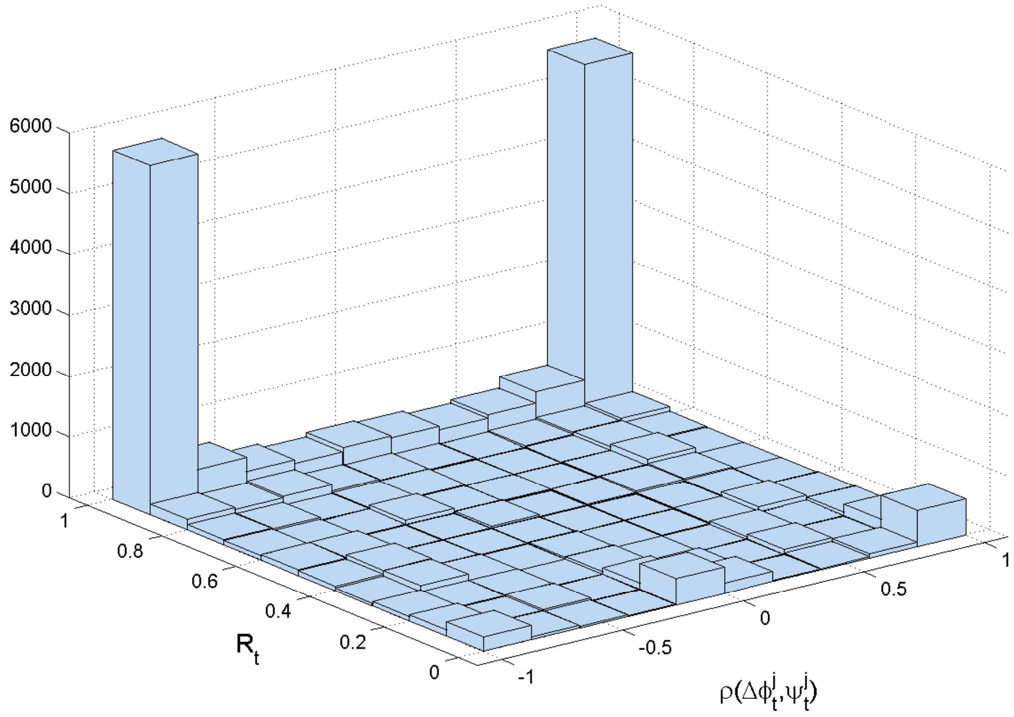

Figure 4: A bivariate histogram of R_t/ξ^S and the correlation $\rho(\psi_t^j, \Delta\phi^j)$ drawn from long simulations ($T = 10^5$).