Financial Frictions in Production Networks*

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July 13, 2013

Abstract

We show that the input-output structure of an economy has significant qualitative and quantitative implications for the impact of financial frictions on aggregate economic activity. We first study a simple example of two different production networks: a horizontal and a vertical economy. We construct these economies so that in the absense of frictions they are allocationally equivalent. However, when firms face collateral constraints, the two economies exhibit very different equilibrium properties. In particular, we find that the vertical economy features a higher sensitivity of aggregate output, the aggregate labor wedge, and TFP to tightened collateral constraints, relative to the horizontal economy. We call the ratio of the drop in output in a network economy vs. a representative agent economy the "network liquidity multiplier". We further show that in order to obtain any implementable allocation, the vertical economy requires greater amounts of aggregate liquidity than the horizontal. Next, we solve a more general model for arbitrary input-output structures, and show that the centrality of sectors matter for how their collateral constraints affect aggregate output. We calibrate this model in order to match the input-output matrix of the U.S. economy and use this to explore the extent to which these interrelationships among sectors can help explain the drop in output during the latest recession. We find a network liquidity muliplier of around 3.8 for the U.S. economy. Furthermore, in order to generate the observed drop in output at the trough of the recession, our calibrated model would require a reduction in liquidity of less than 1/6th the drop in liquidity required in a representive firm model.

*Email: sbigio@columbia.edu, jenlao@chicagobooth.edu. We thank Bob Hall and Vasco Carvalho for valuable feedback and suggestions. We also thank our discussant Marco Ortiz for his thorough and useful discussion of our paper. In addition, we have benefitted from the comments of seminar participants at the Chicago Booth Junior Finance Symposium, the Central Reserve Bank of Peru 30th Research Meeting of Economists, and the 7th Annual Conference on Macroeconomics Across Time and Space hosted by the Federal Reserve Bank of Philadelphia and NBER.

1 Introduction

The Great Recession has highlighted the potential importance of financial factors in business cycle fluctuations. The failures of financial institutions, the increases in funding costs and intermediation spreads, the collapse of the commercial paper and syndicated loans markets–all of these events seem to have led to a significant and prolonged effect on firm output and unemployment. However, the mechanisms by which these events have translated into real economic declines remain in dispute.

In particular, financial macro models that associate output losses to events in financial markets face empirical challenges. First, when constraints are placed on the funding of investment, these models cannot deliver significant output fluctuations.[1] This point is furthermore emphasized by evidence from Chari, Christiano, and Kehoe (2010) who show that in the aggregate, non-financial corporations can finance their capital expenditures entirely from their retained earnings and dividends alone. This suggests that perhaps only a fraction of firms may be constrained in accessing credit; work by Kahle and Stulz (2012) suggests that this may in fact be the case. Viewing this through the lens of a representative firm model leads to the potential conclusion that financial constraints on firms can play only a minor role in business cycle fluctuations.

This paper argues that financial frictions can have much more power in explaining aggregate declines when one takes into account that firms engage in a substantial amount of inter-firm trade. Consider Figure 1 which plots the sectoral input-output matrix for the 65 NAICS sectors in the United States. This figure gives, for each (row) sector, its usage in production (in shares) of the goods produced by all other (column) sectors. Warmer colors indicate a more usage, whereas darker colors indicate less usage. There are two key features of this figure. First, there is a substantial amount of inter-firm trade *within* sectors–this is indicated by the red diagonal running from the northwest corner to the southeast corner. Second, there are key industries that supply to almost all other industries. This latter fact is indicated by the red and orange columns that appear for some sectors. These industries are in general manufacturing industries which supply inputs to many other sectors. All in all, there is significant inter-firm and inter-sectoral trade both within and across sectors in the modern US economy, as well as other economies (see e.g. Jones, 2011).

To understand why interfirm or intersectoral trade may be important determinants for the aggregate impact of financial frictions, consider an economy in which firms produce in isolation, that is, engage in no inter-firm trade. Furthermore, suppose that these firms need to obtain funding for their factors of production, i.e. labor and capital. In this case, if

[1] See for example the work of Chari, Kehoe and McGrattan (2007) and Bigio (2009).

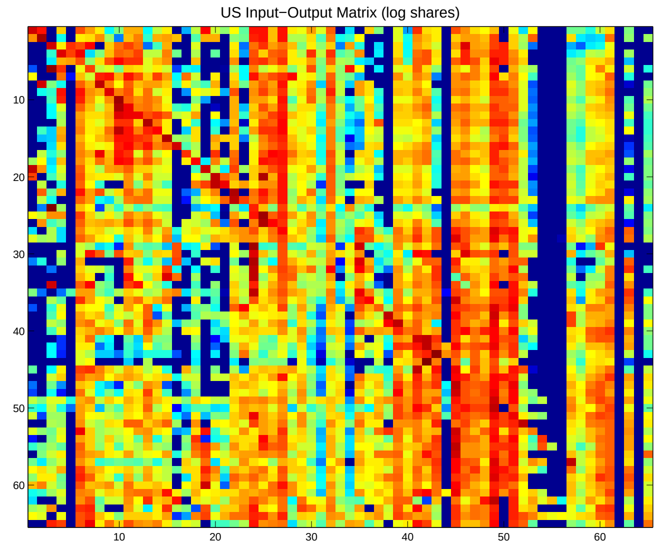


Figure 1: The US Input-Output Table (2006)

a fraction of firms produce less due to some financial shock, this has no direct effect on the other firms in the economy.[2] On the other hand, consider another economy in which firms are interconnected: firms not only purchase labor and rent capital, but also sell and purchase intermediate inputs from one another. All of these transactions are potentially subject to financing constraints. In this interconnected economy, a financial shock to only a subset of constrained firms affects not only these firms but also their trading partners. More specifically, this shock affects the production of the firms that supply intermediate goods to the affected sector, as well as the production of the firms that purchase the affected sector's goods. This in turn affects the firms to which these firms are connected to, and so on and so forth, resulting in a multiplier effect of the financial shock. Thus, what can be interpreted as small aggregate financial shocks in one sector may affect not only this sector, but propagate to other sectors. The total effect is thus amplified as the financial shock moves down the supply chain.

Hence, the key idea in our paper is that when firms are interconnected, working capital must finance not only the primary factors of production such as labor and capital, but all transactions including purchases of intermediate goods. This feature has several general implications for the aggregate effects of financial shocks. First, it implies that the funding of intermediate inputs may require much higher amounts of liquidity than the financing of

[2] other than through perhaps aggregate effects

just labor or capital expenditures.[3] Thus, in stark contrast to a representative firm model, when firms engage in substantial interfirm trade, the aggregate expenditure on the original factors of production (capital and labor) is not sufficient for indicating the bite of financing constraints. Second, a financial shock that may affect only a small portion of interconnected firms can lead to substantial propagation. Thus, aggregate losses due to financial shocks are much larger in models of production networks than those implied by representative firm models. Finally, in a production network, exactly which firms are constrained and how these firms interact with other firms matters. A firm's location within the network will determine how much its own financial state affects the rest of the economy.

Framework and Results. In this paper we formalize these ideas by analyzing the distorting effects of financial shocks in economies in which firms are organized into different production network structures. Our goal is to show how production networks matter when firms are subject to financial frictions. We first do this within a very simple example. We consider two basic economies: a horizontal one and a vertical one. In the horizontal economy, firms produce and operate in isolation, their individual products are then aggregated into a final good. In the vertical economy, firms are arranged in a supply chain–each firm buys the good produced by the firm above it and uses it as an input into their production. Finally, in either economy there is a household that supplies labor and consumes the final good. We introduce these two economies in a way such that without any frictions, they are allocationally equivalent: that is, without frictions the equilibrium in either economy yields the exact same allocation, and moreover, this allocation is efficient.

We then ask how allocations change when firms are subject to liquidity constraints. We impose a pledgeability constraint as in Bernanke Gertler (1989), Kiyotaki Moore (1997), or Bernanke, Gertler, Gilchrist (1999): firms each period must pledge a fraction of their revenue in order to finance their inputs.[4] This introduces a financial friction such that firms are forced to produce less than their optimal amount. In particular it creates a wedge between the marginal cost and marginal product of any firm. We then compare the equilibrium allocations across the two economies when firms face these constraints.

A summary of our results from this simple exercise are as follows.

Result 1: Aggregate Output, and the Network Liquidity Multiplier. A uniform tightening of constraints leads to a larger drop in aggregate output in the vertical economy than in the horizontal economy. This is solely due to these network effects. We call the ratio between

[3] This point has been informally emphasized by Vincenzo Quadrini in a discussion of Shourideh and Zetlin-Jones (2012).

[4] Underlying this ad-hoc formulation is a more micro-founded explanation: firms require liquid funds to finance its inputs due to a limited enforcement problem. We describe this firm problem in Appendix A.

the aggregate output drop in the vertical economy and that in the horizontal economy the "network liquidity multiplier."

Result 2: The Aggregate Labor Wedge. An aggregate labor wedge arises in either economy between the aggregate marginal rate of transformation of aggregate labor and the marginal rate of substitution. This labor wedge depends on the tightness of the individual firm credit constraints. We find that depending on the network structure, the aggregate labor wedge may overstate the actual financial wedges faced by individual firms.

Result 3: Productivity Measures and Misallocation. A uniform tightening in the collateral constraint implies no change in TFP in the horizontal economy, but leads to a fall in TFP in the vertical economy due to misallocation.

Result 4: Aggregate Liquidity. More vertical transactions imply that more aggregate liquidity in the economy is needed. Specifically, consider any feasible allocation. The amount of liquidity needed in the vertical economy to implement this allocation as an equilibrium is strictly greater than the amount of liquidity needed in the horizontal economy. The intuition for this is quite simple. In the horizontal economy, firms only need to have enough funding to finance their value added. However, in the vertical economy, firms must finance not only their own value added, but also must finance the added value of their intemediate inputs. Thus, there is a double counting of input financing that must occur in the vertical economy.

Result 5: Sectoral Analysis. We look more closely at sectoral effects and document the two following results.

First, the impact of individual financial constraints on aggregate output differs by network location. In vertical economies, the most downstream firm has the greatest impact on aggregate output as well as aggregate distortions. That is, a tightening of constraints on the downstream firms leads to a greater decline in output than a tightening of constraints on upstream firms.

Second, there are no direct spill-over effects in the horizontal economy. However, there are direct spill-over effects in the vertical economy. A tightening of firm i's constraint acts like an adverse demand shock on upstream firms (suppliers), while it works as an adverse supply shock on firms downstream (customers). The output of upstream firms are more affected than the output of downstream firms.

The results above arise simply from comparing horizontal and vertical economies. However, the world is not organized into simple horizontal and vertical economies. Modern economies feature very intricate and complex networks of production. Thus in order to apply our results to actual economies, we analyze our problem within a general production network structure as in Acemoglu et al. (2012). In this general structure, there are a finite number of sectors and each of these sectors may demand or supply intermediate goods from one another. We solve for the equilibrium in this general economy when firms face financial constraints and then use this to explore the effect of financial shocks in different production networks.

Result 6: Sectoral Influence related to Centrality. The influence of any sector on aggregate output depends on a measure of its "centrality".

This is consistent with earlier work in frictionless economies. In particular, we present results for star, circle, triangle networks, and a random network to illustrate this point.

Finally, one of the main reasons for working with the general network structure is to calibrate the model to data. We use the input-output matrix on the U.S. economy to calibrate the production network. We use yearly data from 1998 until 2011 on 65 NAICS sectors at the three-digit level, and data from COMPUSTAT on industry costs and sales. We examine the model's implications for output, employment, and the use of inputs. In the calibrated version of our model, we obtain the following results.

Result 7: The U.S. Network Liquidity Multiplier. If liquidity is drawn down by in 1% in every sector, we find that aggregate GDP falls by 2.7%. In contrast, the representative agent economy, output would fall only by .7%. The Network Liquidity Multiplier, i.e. the ratio of the drop in output in the network economy to that in representative agent economy, is therefore approximately 3.8%. This summarizes the extra quantitative kick one gets out of taking into account the U.S. input-output matrix.

Result 8: Implied Drop in Liquidity. Given the observed drop in output during the financial crisis, what was the implied fall in liquidity in constrained firms? In our calibrated model using the network structure of the U.S. economy, we find that at the trough of the recession, the required reduction in liquidity would have been around 1.3%. A model with a representative firm would require at least a 7.6% reduction in liquidity.

Finally, we look at the sectoral data to understand interactions that lead to these aggregate effects. We obtain the following two results.

Result 9: Sectors most affected by liquidity shocks. When there is an aggregate liquidity shock, which sectors are the most affected by this shock? We find that the sectors most affected from an aggregate liquidity shock are the *manufacturing sectors.* These include metal products, chemical products, fabricated metal products, hydrocarbons and other industries related to the extraction and transformation of raw materials. These are mostly upstream firms.

Result 10: Sectors which affect Aggregate Output the most. If we shock the liquidity within each sector individually, which sector affects aggregate output the most? We find that the sectors which lead to the largest drop in aggregate output are final good sectors, such as *retail.* Other important sectors are hospitals, brokerage firms, food & beverage, bars and restaurants, services, and motors. These all mostly seem like final good industries, i.e. downstream firms.

In conclusion, we believe our paper makes a step towards understanding both the qualitative and quantitative implications of production networks in models of financial frictions.

Layout. The remainder of this section discusses the related literature. The rest of the paper is divided into three major sections. First, Section 2 introduces the simple model and characterizes the general equilibrium. We explore the implications of networks and financial frictions for business cycles. Here we present the theoretical results outlined above. Section 3 examines the general case of an arbitrary N-by-N production network then analyze the effects of financial shocks under different network structures. Finally, in Section 4 we calibrate this model to the input-output structure of the U.S. economy and compute the production network multiplier during the recent crisis.

Section 5 concludes.

Related Literature. Our paper analyzes the impact of financial frictions within different production networks. Thus, the two main strands of literature to which our paper fits is (i) the literature on financial frictions, and (ii) the literature on production networks. First, there is a large and growing literature on the real effects of firm financial constraints, starting with the seminal work of Bernanke Gertler (1989), Kiyotaki Moore (1997), and Bernanke, Gertler, Gilchrist (1999). This earlier literature has focused on financial accelerators in representative firm environments. A more recent literature has looked at different forms of firm heterogeneity: entrepreneurial ability and net worth (Buera), adverse selection (Bigio 2012, Kurlat 2012), dispersed information (La'O, 2010), among others. Buera and Moll (2012) in fact study three variants: heterogeneity in final good productivity, heterogeneity in investment costs, and heterogeneity in recruiting costs of labor. In contrast to all of this previous work, ours is the first paper, to our knowledge, to highlight the role of the organization of production structures or networks in amplifying financial shocks. Thus, even without firm heterogeneity, the underlying organizational structure of the economy becomes a key determinant for how important financial frictions are for aggregate fluctuations.

The second strand of literature our paper falls under is that of production networks. In this sense, our paper shares the spirit of the early literature on real business cycles and the role of sectoral shocks. We build on Long and Plosser's (1983) multi-sectoral model of real business cycles. Following Long and Plosser, a debate ensued between Horvath (1998, 2000) and Dupor (1999) over whether sectoral shocks could lead to strong observable aggregate TFP shocks. More recently, this work has been extended and generalized by Acemoglu et al. (2011), for arbitrary production networks, thereby providing a general mathematical framework to answer these questions. The results of the Acemoglu et. al. paper are related to that of Gabaix (2011), who showed that firm level shocks may translate into aggregate fluctuations when the firm size distribution is power law distributed, i.e. sufficiently heavy-tailed. We use the Acemoglu et al. (2011) paper as a basis for our model in Section 3.[5][6]

Our paper is most related to that of Jones (2011a and 2011b). In these papers, Jones shows that in economies with intermediate goods there arises a multiplier effect on productivity. In Jones (2011a), he considers a particular network in which all firms purchase a uniform intermediate good (which is itself composed of all produced goods); in this, all firms are equally important in terms of the network. In Jones (2011b) on the other hand, he considers a more general network, similar to that in Section 3 of our paper, and applies this framework by considering different input-output structures of various economies. He then computes implied productivity multipliers, and uses this to explain cross-country differences in long-run growth.

We think our paper very much shares the same spirit as Jones (2011a and 2011b)–we too want to highlight how trade in intermediate goods leads to amplification. However, although our framework is similar, our paper differs from Jones (2011a and 2011b) in three important respects. First, we focus more on the idea that firm-level distortions are caused by financing frictions. Under this interpretation, we can use our model to understand the role of drops of aggregate liquidity and its multiplier effect on output (as opposed to the multiplier on TFP). We can also use our model to answer the question of how the network structure itself affects the amount of liquidity needed in the economy. Second, our framework is then suitable for understanding short run phenomena such as the recent financial crisis, which we analyze in Section 4. We think that this gives us an advantage in terms of identifying distortions in the data. In particular, Jones (2011) writes, "there is a fundamental identification problem: we see data on observed intermediate goods shares, and we do not know how to decompose that data into distortions and differences in technologies." Our approach gives us some guidance in dealing with this identification issue. Under the assumption that the recent crisis was caused by tightening of financial constraints rather than movements in technology shares,

[5] Relatedly, Cooper and Haltiwanger (1990), Jovanovic (1987), and Durlauf (1993), have explored setups in which strong complementarities across firms can generate aggregate fluctuations from micro shocks.

[6] Finally, most papers including ours takes the production network of the economy as exogenous, while a recent paper by Oberfield (2011) provides a theory of the formation and evolution of input-output structures.

we can do a simple exercise in which we assume that pre-crisis, differences in intermediate good shares were due primarily to technology differences. However, any changes in shares during the crisis arose primarily from financial friction distortions, rather than movements in sector-specific shares. This allows us to then calibrate technology shares and financial distortions. Although ours is not a perfect strategy, we can then make some progress in obtaining a number for the liquidity multiplier.

Third, while Jones (2011a and 2011b) were more focused on the aggregate effects of input-output structures, we in addition focus our attention on individual sectors, and the differential impact of these sectors on aggregate variables due to their network *location*. In the language of Acemoglu et. al (2011), we focus on properties of the influence vector which summarizes how much each firm or sector matters. In the simple model this allows us to study the effect that firm financial constraints have on aggregate output and aggregate distortions–we find that in vertical economies, the downstream firms have the most impact. Furthermore, when we move to the calibrated version, our general network then helps us to identify particular sectors of the U.S. economy which are the most important in terms of aggregate output declines.

Basu (1995) (need to add)

Our paper also provides implications for the literature on wedges and misallocation, see e.g. Chari, Kehoe, McGrattan (2007), Hsieh and Klenow (2009), Restuccia and Rogerson (2008), Banerjee and Duflo (2005), Midrigan and Xu (2012). This literature has emphasized that distortions at the micro or sectoral level can lead to aggregate distortions or wedges that we observe in aggregate data.

Our paper also highlights the importance of vertical supply chains. There has been other work in this area. Bak et al. (1993) stresses the importance of supply chains in aggregate fluctuations. Kiyotaki and Moore (1997) study what they call credit chains. In their setting firms borrow from each other, and a temporary shock to the liquidity of some firms causes a chain reaction in which other firms also get in financial difficulty. Their model differs from ours because of the timing of payments is an important component, whereas our model is static. More recently, Kim and Shin (2011) study an environment in which firms face recursive moral hazard in a supply chain resulting in interlocking claims and obligations. These financial linkages serve as the "glue" that ties firms together in the supply chain. Levine (2010) builds a theory of production chains in which chains are fragile because they are subject to the weakest link, as in Kremer's (1993) O-ring story. Optimal chain length in Levine's model is determined by the trade-off between the gains to specialization and the higher failure rate associated with longer chain length.

Furthermore, our paper makes a step in trying to quantify the empirical effects of networks. A recent paper by Di Giovanni, Levchenko, Mejean (2012) provide empirical evidence on how firm-specific shocks generate to aggregate fluctuations. They test whether this can be accounted for by the granularity effect as in Gabaix (2010) or the linkages effect as in Acemoglu et. al (2012). They find that firm linkages are about twice as important as granularity. Furthermore, Raddatz (2010) provides some evidence for the credit chains mechanism. Chodorow-Reich (2013) provides evidence on constraints affecting the use of variable inputs.

Finally, at a very basic level our paper is related to Friedman's original quantity theory of money, which states that $MV = PT$, where M is money, V is velocity, P is the aggregate price level and T is the real value of aggregate transactions. First, hold velocity constant and normalize the aggregate price level to 1. Money in our economy can be thought of as anything that can be used to purchase goods–in our model we call this liquidity. Thus, we show that more vertical networks implies a greater number or value of transactions. More transactions implies more liquidity is needed, similarly as in the quantity theory. This is the main idea behind our second result: more vertical economies imply more transactions, which implies more liquidity is needed to implement any particular allocation.

2 Two Simple Economies

In this section we consider a very simple example which illustrates the main idea of how the production network structure interacts with the financial frictions. We consider the implications of collateral constraints in two economies which differ in their organizational structure of production. The vertical economy is an economy in which firms are arranged in a vertical supply chain. The horizontal economy is constructed to be allocationally equivalent. We then characterize the general equilibrium in both economies

2.1 The Model

Vertical Economy. There are three firms that use labor and intermediate inputs to produce output. We assume that these firms are perfectly competitive, in that they take prices as given.[7] The firms are organized in a vertical supply chain. Specifically, their production

[7] One can think of this simply as three sectors, each composed of a continuum of perfectly competitive firms.

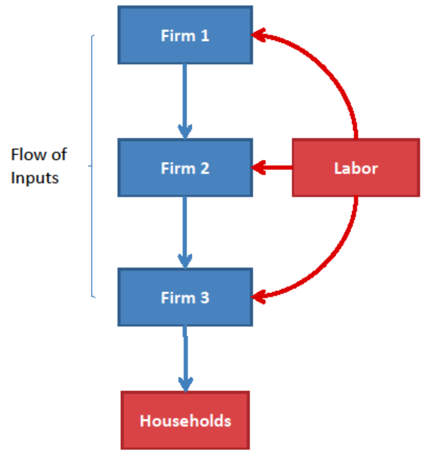


Figure 2: Vertical Economy

functions are given by

$$
\begin{aligned}
x_1 &= z_1 \ell_1^{\alpha_1} \\
x_2 &= z_2 \ell_2^{\alpha_2} x_1^{\beta_2} \\
x_3 &= z_3 \ell_3^{\alpha_3} x_2^{\beta_3}
\end{aligned}
$$

where y_{vi} is the amount produced by firm i and n_{vi} is the amount of labor employed by firm i. Thus, firm 1 uses labor as its sole input, however for $i = 2, 3$, firm i also uses as input the output of firm $i - 1$. Finally, the final consumption good is the output of firm three, that is $Y_v = y_{v3}$. We can therefore write the aggregate production function of the economy in terms of labor as follows

$$Y = x_3 = z_3 \ell_3^{\alpha_3} \left(z_2 \ell_2^{\alpha_2}\right)^{\beta_3} \left(z_1 \ell_1^{\alpha_1}\right)^{\beta_2 \beta_3} \tag{1}$$

For simplicity, assume CRS: $\alpha_3 + \alpha_2 \beta_3 + \alpha_1 \beta_2 \beta_3 = 1$. Figure 2 illustrates the flow of inputs and output in the vertical economy.

Horizontal Economy. Now consider an equivalent, but completely horizontal economy. There are three representative firms that use labor to produce a good. The production

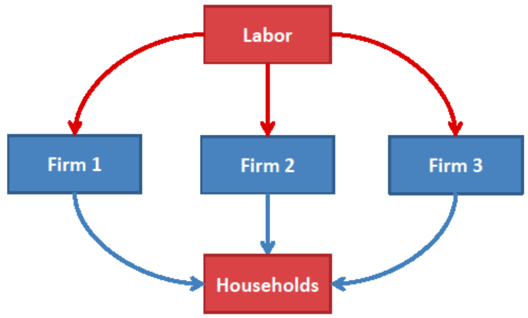


Figure 3: Horizontal Economy

functions of these representative firms are as follows

$$
\begin{aligned}
x_1 &= z_1 \ell_1^{\alpha_1} \\
x_2 &= z_2 \ell_2^{\alpha_2} \\
x_3 &= z_3 \ell_3^{\alpha_3}
\end{aligned}
$$

where y_{hi} is the amount produced by firm i and n_{hi} is the amount of labor employed by firm i. These three goods aggregated into a final consumption good, Y_h. We normalize this consumption basket so as to make it equivalent to the final good in the vertical economy: $Y_h = Y_v$

$$
Y = x_1^{\beta_2 \beta_3} x_2^{\beta_3} x_3
$$

Therefore, both economies have the same aggregate production function. However, unlike the vertical economy, the firms in the horizontal economy operate in isolation from one another, only combining at the end in terms of consumption, as in Dixit-Stiglitz. Figure 3 illustrates the flow of inputs and output in the horizontal economy.

Households and Market Clearing. To close the economy we introduce households. In either economy, there is a representative household with preferences given by

$$
U(C) - V(L)
$$

where $U : \mathbb{R} \to \mathbb{R}$ is increasing and concave, $V : \mathbb{R} \to \mathbb{R}$ is increasing and convex, C

is the final good consumption, and L is labor supplied competitively to the market. The household's budget constraint is given by $C = hL + \sum_{i=1}^{3} \pi_i$ where w is the competitive real wage rate, π_i are the profits if any of firm i, and where we have normalized the price of the final good to 1. Finally, for markets to clear, we have that consumption is equal to aggregate output $C = Y$, and labor supply equals labor demand $L = \ell_1 + \ell_2 + \ell_3$.

Remarks and Notation. We will use $\varepsilon \in \{v, h\}$ to denote the economy of interest, where $\varepsilon = v$ denotes the vertical economy and $\varepsilon = h$ the horizontal.

Our first remark is that in this paper we abstract from investment. The model is static, so that firms only have static inputs. As one will see later, the financial constraint will be on working capital.

Second, note that we are taking the network structure of these economies as exogenous. As will be seen later, there will be incentives for firms to merge or vertically integrate when there are frictions. This then begs the question of why firms are structured in the economy as they are. The theory of the firm and its boundaries is an interesting one, however, here we abstract from these considerations and take the firm boundaries as given. See Antras (2011) for a sequential production chain regarding the optimal allocation of ownership rights along a supply chain.

Equilibrium Definition. We define the competitive equilibrium in either economy as follows

Definition 1. *A competitive equilibrium in economy $\varepsilon \in \{v, h\}$ is a collection of quantities $\{\ell_1, \ell_2, \ell_3, x_1, x_2, x_3, L, Y\}$ and prices $\{p_1, p_2, p_3, h\}$ such that*

(i) each representative firm maximizes profits,

(ii) the representative household maximizes utility,

(iii) markets clear.

This is a standard definition for a Walrasian equilibrium in a production economy.

Frictionless Benchmark As a benchmark, we first consider the equilibrium in either economy in the absense of frictions. By construction, the vertical and horizontal economies are allocationally equivalent.

Proposition 1. *In either economy without frictions, there exists a unique equilibrium allocation. In either economy, the unique equilibrium allocation is given by*

$$\tilde{\alpha}_3 \frac{Y}{\ell_3} = \tilde{\alpha}_2 \frac{Y}{\ell_2} = \tilde{\alpha}_1 \frac{Y}{\ell_1} = \frac{V'(N)}{U'(Y)} \tag{2}$$

along with resource constraints $L = \ell_1 + \ell_2 + \ell_3$ *and* $Y = z_3 \ell_3^{\alpha_3} \left(z_2 \ell_2^{\alpha_2}\right)^{\beta_3} \left(z_1 \ell_1^{\alpha_1}\right)^{\beta_2 \beta_3}$*, where*

$$\tilde{\alpha}_3 = \alpha_3, \qquad \tilde{\alpha}_2 = \alpha_2 \beta_3, \qquad \tilde{\alpha}_1 = \alpha_1 \beta_2 \beta_3$$

denote each firm's labor share of the aggregate production function.

The aggregate production function simply transforms each type of labor into aggregate output. In equilibrium, the marginal rate of transformation of each type of labor is equal to the marginal rate of substitution. Thus, absent any frictions, the way production of an economy is broken down into different firms or network structures is irrelevant. This is a similar to the Modigliani-Miller () result. Here, instead of considering how an individual firm is sliced up in terms of financing, we consider how a macroeconomic production function is sliced up into different units of production. In the absense of any frictions, for a given aggregate production function, the way it is organized into different production units does not matter for allocations.

Proposition 1 further implies that we may write

$$\bar{\alpha} \frac{Y}{L} = \frac{V'(L)}{U'(Y)}$$

where $\bar{\alpha} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$ is the total labor share of output. Thus, in the absense of any frictions, these economies admit a representative firm, with production function $Y = \bar{Z} N^{\bar{\alpha}}$, where $\bar{Z} \equiv z_1^{\beta_2 \beta_3} z_2^{\beta_3} z_3$ is aggregate productivity.

2.2 Equilibrium With Frictions

After establishing the allocational equivalence of the two economies in the absense of frictions, we now consider the implications of adding financial frictions. Financial frictions introduce distortions into the economy, relative to the frictionless benchmark; however, depending on the network structure, these frictions distort the two economies in different ways.

Consider the entire literature on financial frictions. Starting with the seminal work of Bernanke Gertler (1989), Kiyotaki Moore (1997), and Bernanke, Gertler, Gilchrist (1999), there has been a large literature which has looked at the effects of frictions on the economy. These all have in common that these frictions end up looking like wedges between the marginal rate of substitution and the marginal rate of transformation either for capital and labor.

In this paper we will abstract from capital, and simply introduce financial frictions by adding collateral constraints on input purchases. We assume that each firm faces a constraint in which their expenditure on inputs is constrained to be less than or equal to a fraction χ of their revenue. One can think of this as follows: firms can credibly commit to pay only a fraction χ of their revenue to laborers or suppliers, and can abscond with the rest after production and sales are realized. Hence, expenditure on inputs cannot exceed the pledgeable portion of their revenue. In the horizontal economy this pledgeability constraint is given by the following.

$$h\ell_i \leq \chi_i p_i x_i \tag{3}$$

On the other hand, in the vertical economy, only firm 1 uses labor as a sole input so this firm faces a constraint as in (3), while firms 2 and 3 face the following constraint

$$h\ell_i + p_{i-1}y_{i-1} \leq \chi_i p_i x_i \tag{4}$$

Thus, firms face working capital constraints.[8]

The financial frictions introduce distortions into the two economies. However, we note that these constraints are not directly comparable across the two economies, as the vertical economy firms must finance both labor and intermediate goods whereas firms in the horizontal economy need only finance their wage bill. We take this into consideration in our analysis. That is, our results will not depend on how χ_{hi} directly compares to χ_{vi}, but instead will rest on how these constraints manifest themselves differently in terms of distorting the two economies.

Equilibrium Characterization. We first examine these distortions at the individual firm level. The pledgeability constraint faced by any firm introduces a wedge between the firm's marginal benefit and marginal cost of production. In the horizontal economy, each firm's production is pinned down by

$$h = \phi_{hi} p_{hi} \alpha_i \frac{x_{hi}}{n_{hi}}$$

where

$$\phi_{h1} = \min\left\{1, \frac{\chi_{h1}}{\alpha_1}\right\}, \qquad \phi_{h2} = \min\left\{1, \frac{\chi_{h2}}{\alpha_2}\right\}, \qquad \phi_{h3} = \min\left\{1, \frac{\chi_{h3}}{\alpha_3}\right\} \tag{5}$$

This simply states that the marginal cost is equal to the marginal benefit, times some wedge. This individual wedge represents the distortion for that firm away from its optimal labor usage due to the collateral constraint. That is, for any firm i, the wedge $\phi_{hi} \in [0, 1]$. When $\chi_{hi} < \alpha_i$, the firms pledgeability constraint is binding, and the wedge is given by $\phi_{hi} = \chi_{hi}/\alpha_i$.

[8]This working-capital constraint is similar to the static-input financing example in Chari, Kehoe, and McGrattan ().

The constraint is binding whenever χ_{hi} is less than the labor share of output of firm i. On the other hand, if $\chi_{hi} \geq \alpha_i$, then the firms pledgeability constraint is not binding–firms operate at their optimal level and have enough funds to cover their expenses. In this case, there is no wedge between the firm's marginal benefit and marginal cost of production, i.e. $\phi_{hi} = 1$.

In the vertical economy, firms solve a cost minimization problem in terms of it's expenditure on each of its inputs: labor and the intermediate good. This cost minimization implies that their expenditure on each good is equal to the ratio of the relative shares of each input in production.

$$\frac{hn_i}{p_{v,i-1}x_{v,i-1}} = \phi_{vi}\alpha_i$$

Given this condition, each firm's production is then pinned down by the following condition

$$w_v = \phi_{vi}p_{vi}\alpha_i\frac{x_{vi}}{n_{vi}}$$

where

$$\phi_{v1} = \min\left\{1, \frac{\chi_{h1}}{\alpha_1}\right\}, \qquad \phi_{v2} = \min\left\{1, \frac{\chi_{v2}}{\alpha_2 + \beta_2}\right\}, \qquad \phi_{v3} = \min\left\{1, \frac{\chi_{v3}}{\alpha_3 + \beta_3}\right\} \tag{6}$$

Again, for any firm i, the wedge $\phi_{hi} \in [0, 1]$ represents the distortion in optimal production level due to the collateral constraint. This the same condition as in the horizontal economy; however, the only difference here is that for firms 2 and 3, the constraint is binding whenever $\chi_{vi} < \alpha_i + \beta_i$, that is, when the pledgeability ratio χ is less than the total output share of inputs (the labor share plus the share of intermediate goods).

Combining the individual firm conditions with market clearing and household optimality conditions, we reach the following proposition which fully characterizes the equilibrium allocation.

Proposition 2. *Suppose firms face pledgeability constraints*

(i) In the horizontal economy, the unique equilibrium allocation is given by

$$(\phi_{h1})\,\tilde{\alpha}_1\frac{Y}{\ell_1} = V'(L)/U'(Y) \tag{7}$$

$$(\phi_{h2})\,\tilde{\alpha}_2\frac{Y}{\ell_2} = V'(L)/U'(Y) \tag{8}$$

$$(\phi_{h3})\,\tilde{\alpha}_3\frac{Y}{\ell_3} = V'(L)/U'(Y) \tag{9}$$

and resource constraints $L = \ell_1 + \ell_2 + \ell_3$ *and* $Y = z_3\ell_3^{\alpha_3}\left(z_2\ell_2^{\alpha_2}\right)^{\beta_3}\left(z_1\ell_1^{\alpha_1}\right)^{\beta_2\beta_3}$

(ii) In the vertical economy, the unique equilibrium allocation is given by

$$\left(\phi_{v1}\phi_{v2}\phi_{v3}\right)\tilde{\alpha}_1\frac{Y}{\ell_1} = V'(L)/U'(Y) \tag{10}$$
$$\left(\phi_{v2}\phi_{v3}\right)\tilde{\alpha}_2\frac{Y}{\ell_2} = V'(L)/U'(Y) \tag{11}$$
$$\left(\phi_{v3}\right)\tilde{\alpha}_3\frac{Y}{\ell_3} = V'(L)/U'(Y) \tag{12}$$

and resource constraints $L = \ell_1 + \ell_2 + \ell_3$ *and* $Y = z_3\ell_3^{\alpha_3}\left(z_2\ell_2^{\alpha_2}\right)^{\beta_3}\left(z_1\ell_1^{\alpha_1}\right)^{\beta_2\beta_3}$

In either economy, for each type of labor there is now a wedge between its marginal rate of transformation into aggregate output and its marginal rate of substitution. In the horizontal economy this wedge between the MRT and the real wage is simply the same wedge that arises in the individual firm decisions. This is due to the fact that each firm in the horizontal economy operates in isolation. Whatever distortion shows up at the firm level only affects the marginal rate of transformation for that type of labor, but not for others. On the other hand, in the vertical economy, this is not the case. The individual wedge of firm 2 affects the wedge for both firm 2 and firm 1. Similarly, the wedges of firm 3 affect the wedges found in firms 1 and 2. Thus, the downstream financial frictions distort upstream input use. As in Jones (2011), it has a multiplier effect, which we will evaluate in the next subsection.

Finally, note that these economies with financial frictions are isomorphic to an economy without frictions, but with taxes given by $(1-\tau_i) = \phi_i$ for $\tau \in [0,1]$. That is, it is isomorphic to an economy where firms face taxes (but not subsidies). This relates to the literature on taxation and supply chains, for example, the input-output model of Jones (2011).

Lemma 1. *The economies are isomorphic to an economy without frictions, but with individual taxes given by* $(1-\tau_i) = \phi_i$

The solution to this problem produces a wedge.. Thus, the firm's problem is equivalent to a the problem of a firm facing a sales tax of $(1-\tau_i) = \phi_i$. Thus, the corresponding tax for firm i is $\tau_i \equiv 1-\phi_i$. This tax has an immediate interpretation. This is true in all models of financial frictions–frictions show up as a wedge. Thus, the environment is isomorphic to the input-output model with distortions of Jones (2011).

2.3 Aggregate Effects of Collateral Constraints

We now look at how these distortions at the individual firm level affect the economy at the aggregate level. We consider how the tightening of these collateral constraints affects the following aggregate variables: (i) aggregate output, (ii) the aggregate labor wedge, and (iii) measures of aggregate productivity.

For simplicity, we specify a particular utility function in order to solve for aggregate output in closed form. We suppose that utility over consumption and labor is given by

$$U(C) - V(L) = \log C - L \tag{13}$$

The assumption of log-linear utility over consumption and linear disutility of labor is not crucial for any of our results, but simplifies the expressions considerably.

2.3.1 Aggregate Output

Given this specification of utility, we now solve for aggregate output in closed form. This is given in the following Lemma.

Lemma 2. *(i) Aggregate output in the frictionless economy is given by* $\bar{Y} = \bar{z}(\tilde{\alpha}_3)^{\tilde{\alpha}_3}(\tilde{\alpha}_2)^{\tilde{\alpha}_2}(\tilde{\alpha}_1)^{\tilde{\alpha}_1}$*.*

(ii) Suppose firms face pledgeability constraints which imply vectors $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$ *and* $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$*. Under these constraints, aggregate output in the horizontal economy is given by*

$$Y_h = \bar{Y}(\phi_{h3})^{\tilde{\alpha}_3}(\phi_{h2})^{\tilde{\alpha}_2}(\phi_{h1})^{\tilde{\alpha}_1} \tag{14}$$

and aggregate output in the vertical economy is given by

$$Y_v = \bar{Y}(\phi_{v3})^{\tilde{\alpha}_3}(\phi_{v2}\phi_{v3})^{\tilde{\alpha}_2}(\phi_{v1}\phi_{v2}\phi_{v3})^{\tilde{\alpha}_1} \tag{15}$$

Lemma 2, thus provides closed-form expressions for aggregate output in the frictionless economy, as well as in the vertical and horizontal economies with financial frictions. Note that because ϕ is strictly less than 1 whenever the collateral constraint is binding, this implies that output in either the horizontal or vertical economy is lower when constraints are binding than when they are not, as expected.

Using these expressions for aggregate output, we now consider the aggregate effect of tightening individual collateral constraints. We obtain the following result.

Proposition 3. *Tightening individual firm collateral constraints in the horizontal economy lead to the following drops in aggregate output*

$$\frac{d\log Y_h}{d\log \phi_{h1}} = \tilde{\alpha}_1 > 0, \qquad \frac{d\log Y_h}{d\log \phi_{h2}} = \tilde{\alpha}_2 > 0, \qquad \frac{d\log Y_h}{d\log \phi_{h3}} = \tilde{\alpha}_3 > 0$$

Tightening individual firm collateral constraints in the vertical economy lead to the following *drops in aggregate output*

$$\frac{d\log Y_v}{d\log\phi_{v1}} = \tilde{\alpha}_1, \qquad \frac{d\log Y_v}{d\log\phi_{v2}} = \tilde{\alpha}_2 + \tilde{\alpha}_1, \qquad \frac{d\log Y_v}{d\log\phi_{v3}} = \tilde{\alpha}_3 + \tilde{\alpha}_2 + \tilde{\alpha}_1$$

Proposition 3 gives the effect on any individual firm's constraint on the aggregate output of the economy. One may think of these effects as the *influence* of each constraint. Note that the influence of each constraint depends not only on the aggregate technology, i.e. the firm's labor share of aggregate production $\tilde{\alpha}_i$, but also on the network structure of the economy. Hence, influence is not equivalent across the two economies.

Consider first the horizontal economy. The effect on aggregate output of tightening any of the collateral constraints is simply equal to its labor share of total output $\tilde{\alpha}_i$. Tightening a collateral constraint leads to a 1-for-1 decrease in the labor employed at firm i. Why is this? With log-utility over consumption and linear disutility of labor, income and substitution effects cancel so that the real wage remains constant, and thus labor simply falls 1-for-1 with the collateral constraint. Furthermore, there are no effects on the labor chosen by other firms. Thus, the percentage effect of a fall in labor of any firm on aggregate output is simply that firm's labor share.

On the other hand, in the vertical economy the constraints downstream have a greater impact on aggregate output than those upstream. For example, a percentage change in the collateral constraint of firm 1 leads to the a fall in aggregate output equal to its labor share–the same as in the horizontal case. In contrast, a percentage change in the collateral constraint on firm 3 has a greater effect than its own labor share–instead it is the sum of the labor shares of all firms $1, 2$, and 3. The reason for this is that not only is there a direct effect on aggregate output from firm 3 employment, but it also directly affects the labor chosen by firms 1 and 2, due to reduced demand for their products. We study these spill-over effects more closely in Subsection (). For now, we see that in terms of aggregate output, downstream constraints have an amplified effect.

Finally, suppose that all constraints were to tighten at the same time–how much would aggregate output fall in response to this aggregate tightening?

Proposition 4. *Suppose firms face pledgeability constraints which imply vectors* $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$ *and* $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$*, and suppose we scaled down all collateral constraints by* ρ *percent so that each firm faces a collateral constraint given by* $\phi_i(1-\rho)$*. Then aggregate output falls more in the vertical economy than in the horizontal economy*

$$-\frac{d\log Y_v}{d\log\theta} < -\frac{d\log Y_h}{d\log\theta} < 0$$

That is, the liquidity multiplier is greater in the vertical economy than in the horizontal.

Formally, suppose we scaled down all constraints by ρ percent. The drop in aggregate output due to this fall in aggregate liquidity is given by $d\log Y/d\log\rho < 0$. We call this object $|d\log Y/d\log\rho|$ the *liquidity multiplier*–it tells us how much aggregate output falls due to a 1 percent decrease in collateral constraints across the board. In the horizontal economy, this multiplier is equal to the aggregate labor share $\bar{\alpha} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$; if the aggregate labor share is equal to 1, as in CRS production function, then the liquidity multiplier must be equal to 1. On the other hand, the multiplier in the vertical economy must necessarily be greater than the aggregate labor share. In fact, in our calibration results in Section 4, we find a liquidity multiplier of around 3.5 in the U.S. economy. In conclusion, we have so far shown that the liquidity multiplier is greater in the vertical economy than in the horizontal economy.

2.3.2 The Labor Wedge

One variable which captures distortions at the aggregate level is the aggregate labor wedge. A large literature has documented large labor wedges in the data as well as the countercyclicality of this wedge (e.g., Hall, 1997; Rotemberg and Woodford, 1999; Chari, Kehoe, and McGrattan, 2007; Shimer, 2009).[9] Following this literature, we define the aggregate labor wedge $(1-T)$ implicitly by

$$(1-T)\,\bar{\alpha}\frac{Y}{L} = \frac{V'(L)}{U'(C)}$$

That is, the aggregate labor wedge is simply the wedge between the aggregate marginal rate of transformation of aggregate labor, and the marginal rate of substitution. In the frictionless economy, $\tau = 0$ so that there is no wedge or aggregate distortion.

Using our equilibrium characterization in Proposition 2, we may also back out the aggregate labor wedge in the horizontal and vertical economies when there are financial frictions. We present the result of this exercise in the following proposition.

Proposition 5. *Suppose firms face pledgeability constraints which imply vectors $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$ and $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$. Under these constraints, the aggregate labor wedge in the horizontal economy is given by*

$$1 - T_h = \frac{\tilde{\alpha}_1\phi_{h1} + \tilde{\alpha}_2\phi_{h2} + \tilde{\alpha}_3\phi_{h3}}{\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3} \tag{16}$$

[9] A recent paper by Karabarbounis (2013) shows that most of the time variation in the labor wedge is on the household rather than on the firm side. However, this is not true during the current recession.

and in the vertical economy is given by

$$1 - T_v = \frac{\tilde{\alpha}_3\left(\phi_{v3}\right) + \tilde{\alpha}_2\left(\phi_{v2}\phi_{v3}\right) + \tilde{\alpha}_1\left(\phi_{v1}\phi_{v2}\phi_{v3}\right)}{\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3} \tag{17}$$

Proposition 5 gives the implied labor wedges in the two economies when firms face collateral constraints. Our results here do not rely on the particular form of utility over consumption and labor specified in (13); they apply to a general class of utility functions. See proof in the Appendix.

Thus, the pledgeability constraints introduce aggregate labor wedges between the aggregate marginal product of labor and the real wage. Recall from Proposition 2, that the pledgeability constraints introduce individual labor wedges between a particular firm's marginal rate of transformation and the real wage. Proposition 5 makes clear that the aggregate labor wedge is simply a *weighted sum* of these individual labor wedges, and that these weights are given by each firm's aggregate labor share, $\tilde{\alpha}_i$. However, depending on how the constraints $\phi's$ affect the individual labor wedges, the aggregate labor wedge differs across the two economies. In the horizontal economy all constraints are weighted according to their respective aggregate labor share. On the other hand, in the vertical economy, the downstream constraint, ϕ_3, has the greatest impact on the aggregate wedge. This is because the downstream constraint distorts upstream labor choices, and shows up in each individual labor wedge.

We think this result is important when one thinks about how financial friction models map to the data. One often needs to makes sense of how small distortions on average at the micro level can translate into large quantitative distortions at the macro level. Network structures can help account for this discrepancy.

In particular, depending on the network structure, the aggregate labor wedge given in the data can be greater than the average individual labor wedge of the firm. This is illustrated in Figure 4. In this figure, we set the constraints equal across all firms $\phi_i = \phi$, for all i. We then represent these individual labor wedges instead in terms of the individual tax on each firm $\tau_i = 1 - \phi_i = 1 - \phi$, and plot the aggregate labor tax T as a function of the individual taxes.[10]

The red dashed line plots the aggregate labor wedge T in the horizontal economy as a function of the individual taxes τ. Given that we have set τ equally across all firms, this is also equal to the aggregate labor wedge for the representative firm who faces collateral

[10] For this plot we use $\alpha_i = 1/3$ and $\beta = 2/3$ for all firms i. The qualitative results would not change for different technology parameters.

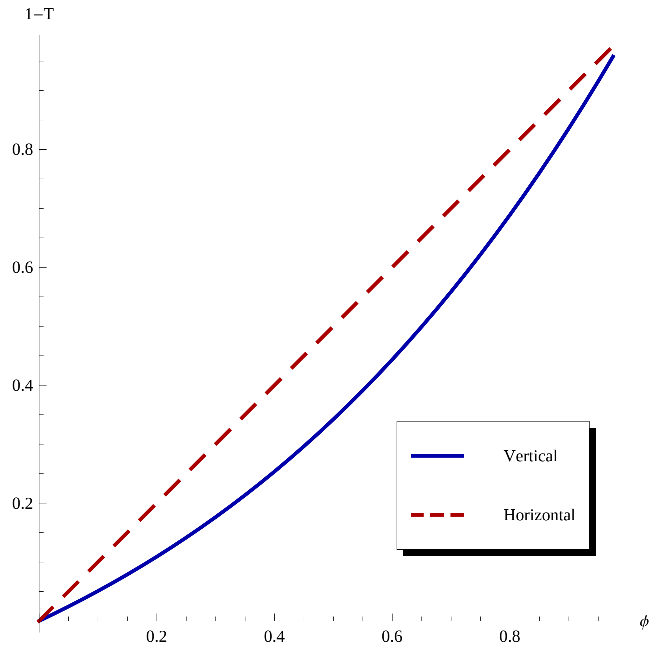


Figure 4: The Aggregate Labor Wedge

constraint $\phi = 1 - \tau$. As expected, this is a 45 degree line, as the average individual tax translates 1-for-1 into the aggregate tax. The blue solid line, on the other hand, gives the aggregate labor wedge in the vertical economy as a function of the individual taxes τ. When $\tau = 0$ and when $\tau = 1$, the aggregate tax T is equal to τ, and hence coincides with the aggregate tax in the horizontal economy. We have already shown that without frictions, the economies are equivalent, so that they should coincide when $\tau = 0$; when there is a 100% tax, they should also coincide since no firm in the economy can produce.

In contrast, for $\tau \in (0, 1)$, the aggregate tax in the vertical economy is strictly greater than the aggregate tax in the horizontal economy, and, in particular the difference is greater for intermediate values of τ. This follows from Proposition 5: the aggregate distortion in the vertical economy is an amplified version of the distortions of individual firms; in particular, it amplifies the distortions of downstream firms. Depending on the network structure, it is thus possible that small distortions at the micro level can translate into larger distortions at the macro level.

Finally, another way to interpret the labor wedge is to think of it as the implied shadow interest rate that firms face on their working capital. Again, through the lens of a representative model, a high shadow interest rate at the aggregate level would seem incompatible with the interest rates firms pay at the micro level. However, depending on the network structure, small changes in the shadow interest rate at the individual firm level can be amplified at the aggregate level.[11]

In conclusion, viewing the data through the lens of either a representative firm or a multiple-firm horizontal economy, may overstate the distortions needed at the micro level.

2.3.3 Productivity Measures and Misallocation

Finally, we study the implications of our simple model of collateral constraints and networks on productivity. Productivity is often measured in two ways, either as labor productivity, or as TFP. Here, we study both.

We define these measures in the standard way. First, in the frictionless economy, we have that $Y = \bar{A}L^{\bar{\alpha}}$. Thus, abstracting from capital, we define implied TFP as simply $Y/L^{\bar{\alpha}}$. This object is also known as the efficiency wedge or the Solow residual. Labor productivity, on the other hand is simply defined as Y/L.

The following proposition characterizes how these productivity measures vary with the collateral constraint. Specifically, we perform the same our exercise as with aggregate output: suppose all constraints were to tighten by the same percentage–how would TFP and labor productivity change in response to this aggregate tightening?

Proposition 6. *Suppose firms face pledgeability constraints which imply vectors $\{\phi_{v1}, \phi_{v2}, \phi_{v3}\}$ and $\{\phi_{h1}, \phi_{h2}, \phi_{h3}\}$, and suppose we scaled down all collateral constraints by ρ percent so that each firm faces a collateral constraint given by $\phi_i(1-\rho)$. (i) In the horizontal economy, TFP remains constant while labor productivity goes up*

$$-\frac{d\log\left(Y/L^{\bar{\alpha}}\right)}{d\log\theta} = 0 \quad \textit{and} \quad -\frac{d\log\left(Y/L\right)}{d\log\theta} > 0$$

(ii) In the vertical economy, implied TFP decreases while labor productivity is non-monotonic

$$-\frac{d\log\left(Y/L^{\bar{\alpha}}\right)}{d\log\theta} < 0 \quad \textit{and} \quad -\frac{d\log\left(Y/L\right)}{d\log\theta} \gtrless 0$$

To make Proposition 6 more easily interpretable, we illustrate these implications in Figure 5. Here, we plot for both economies log TFP and log Labor Productivity as a function of the individual taxes τ.

[11] A number of other papers have shown that financial frictions lead to static labor wedges. Chari, Kehoe, McGrattan (2007) consider a static-input financing friction and show that this leads to an efficiency wedge. Buera and Moll (2012) also find a labor wedge due to financial frictions in a search model of labor. Here, we show that if individual firms face constraints, the network structure is important in determining how each constraint contributes to the aggregate labor wedge.

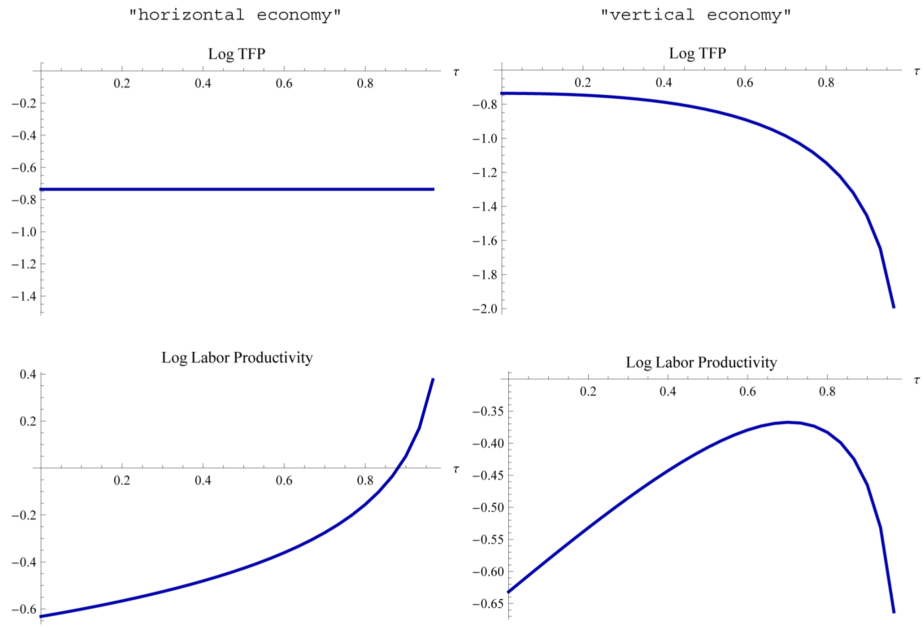


Figure 5: Productivity Measures

First, for the horizontal economy, we see that aggregate TFP is constant in τ. In the vertical economy, on the other hand, a tightening of constraints across firms implies a reduction in TFP. This is because there is misallocation of resources. As one tightens the credit constraints across all firms, some firms affect aggregate output more than other firms. In particular, the downstream firms. Hence, even if the drop in employment in these sectors is proportional to the sectors output, the drop in aggregate output is much more. Hence, although there are no aggregate productivity shocks, a uniform drop in constraints across all firms leads to a decrease in efficiency. We discuss how this translates into misallocation below.

On the other hand, for the vertical economy, labor productivity is increasing in τ. This implies that tighter credit constraints leads to an increase in labor productivity. This is easy to think about in a representative firm environment. As one tightens a constraint on a firm, that firm is forced to higher less workers. However, due to diminishing marginal returns to labor, this implies that labor productivity Y/L should rise. This is precisely what is happening in the horizontal economy. At the same time, tightening the constraint should have no effect on total factor productivity, which is correctly measured productivity of labor.

Finally, in the vertical economy labor productivity is non-monotonic. This is due to the fact that there are the two opposing effects. First, there is the fact that TFP of labor is decreasing. On the other hand, there is the effect that firms are moving down their production function which has decreasing marginal returns to scale. For low values of ρ, the second effect dominates, whereas for high values of ρ, where firms are incredibly constrained the second effect dominates.

Misallocation. We discuss briefly why TFP falls due to misallocation. There is a large and growing literature on misallocation in growth and development. See, for example, Restuccia and Rogerson (2008), Banerjee and Duflo (2005), and Hsieh and Klenow (2009). Jones (2010). We use the insights from this literature and apply it to our simple network economy to understand why TFP drops in the vertical economy but not in the horizontal economy. There is a form of misallocation that occurs.

There are a number of ways to look at misallocation. Here we build a measure similar to Hsieh and Klenow (2009). Hsieh and Klenow (2009) consider a particular type of horizontal economy–firms are heterogenous and monopolistically competitive, their output is aggregated via a CES production technology. They show that TFP in their economy is inversely related to the dispersion in marginal revenue products across industries, where marginal revenue product of labor is given by $MRP_i \equiv \alpha_i p_i y_i / n_i$.

One can build a similar measure in our economies. MRP only looks at the effect on individual revenue, without taking into account how individual output affects aggregate output. In horizontal economies this works, as all firms affect the aggregate in an equal way. However, the same is not true in more complicated structures. Due to the difference in network structure, looking at dispersion in marginal revenue products does not take into account the influence each firm has on aggregate variables due to the network structure. Thus, one instead needs to look at the dispersion of the following measure which we call the "Marginal Aggregate Revenue Product"

$$MARP \equiv \tilde{\alpha}_i \frac{PY}{n_i} \tag{18}$$

where we normalize the aggregate price level to 1. We now look at the dispersion in MARP as a proxy for measuring misallocation in our economy.

Consider first the frictionless economy where we know there is zero misallocation. From Proposition () we have that $MARP_i = \tilde{\alpha}_i Y / n_i = w, \forall i$, where w is the real wage. Thus, the MARP is equated across all firms, and therefore exhibits no dispersion.

Second, consider the horizontal economy. In this economy we have a similar condition:

$$MARP_i = \frac{w}{\phi_i}, \quad \forall i$$

Suppose the constraints are equal so that $\phi_i = \phi$ for all firms. In this case, there is again no misallocation–marginal revenue products are equalized across all sectors. Although all firms are constrained, their the marginal product of labor on aggregate output is equal across all firms. There is therefore no misallocation.

Finally, in the vertical economy, we have that

$$MARP_3 = \frac{w}{\phi_{v3}}, \qquad MARP_2 = \frac{w}{\phi_{v2}\phi_{v3}}, \qquad MARP_1 = \frac{w}{\phi_{v1}\phi_{v2}\phi_{v3}}$$

Here, setting $\phi_i = \phi$ for all firms, there is dispersion in these marginal aggregate revenue product, and hence, misallocation in the vertical economy. Furthermore, if one decreases the ϕs, the variance in these MARPs increase. Therefore, misallocation is increases and hence TFP decreases as one tightens the collateral constraint.

Although $MARP$ does not translate into aggregate TFP as nicely as one would have wanted, differences in it give some sense of misallocation. In a frictionless world, this would be equated across all firms and equal to the real wage. Finally, note that one would not have picked this up simply by looking at MRP_i. As we said before, MRP only catches distortions at the firm level. Our model would imply that $MRP_i = w/\phi_i$ for all firms. Thus, MRP is constant across all firms in both economies by construction. However, we show that this hides the distortions that are in fact present at the aggregate level. We need a better measure of misallocation in network economies with vertical supply chains.

This implies that because they only considered a horizontal economy, the misallocation measures found in Hsieh and Klenow (2009) may in fact be *lower* than the actual underlying misallocation. That is, the horizontal economy provides a *lower bound* for the amount of misallocation generated in a network economy with frictions.

2.4 Aggregate Liquidity

We now examine the aggregate amount of "liquidity" needed to implement allocations in either economy. We will show that for any equilibrium allocation, more liquidity is needed in the vertical economy than in the horizontal economy. Let us first define the aggregate amount of pledgeable funds, i.e. liquidity, as follows.

Definition 2. *Let M denote the aggregate amount of liquidity as defined by*

$$M \equiv \chi_1 p_1 x_1 + \chi_2 p_2 x_2 + \chi_3 p_3 x_3$$

That is, we define liquidity M_ε to be the aggregate amount of pledgeable funds. As we've mentioned previously, we cannot directly compare the constraints across the two economies.

Hence, fixing the vectors $\{\chi_{h1}, \chi_{h2}, \chi_{h3}\}$ and $\{\chi_{v1}, \chi_{v2}, \chi_{v3}\}$ and then comparing the liquidity across the two economies would be uninformative. However, we can instead ask what is the liquidity needed in either economy to implement a given allocation. First, we define a feasible allocation in the economy as follows.

Definition 3. *An allocation $\{\ell_1, \ell_2, \ell_3, L, Y\}$ is feasible if it satisfies $\ell_1 + \ell_2 + \ell_3 = L$ and$Y = z_3 \ell_3^{\alpha_3} \left(z_2 \ell_2^{\alpha_2}\right)^{\beta_3} \left(z_1 \ell_1^{\alpha_1}\right)^{\beta_2 \beta_3}$.*

Thus, an allocation is feasible if it satisfies the economy's resource constraints. Now, suppose we fix a feasible allocation. In order to implement this allocation as an equilibrium outcome, what is the minimum amount of liquidity needed in order to do so? We answer this in the following proposition.

Proposition 7. *Fix some feasible allocation $\{\ell_1, \ell_2, \ell_3, L, Y\}$. Then,*

(i) the aggregate liquidity needed in the horizontal economy to acheive this allocation is given by

$$M_h = \frac{V'(L)}{U'(Y)} L \tag{19}$$

(ii) the aggregate liquidity needed in the vertical economy to acheive this allocation is given by

$$M_v = \frac{V'(L)}{U'(C)} \left(L + \frac{\beta_2}{\alpha_2} \ell_2 + \frac{\beta_3}{\alpha_3} \ell_3 \right) \tag{20}$$

Therefore

$$M_v > M_h$$

Thus, we find that the amount of liquidity needed to implement any feasible allocation is strictly greater in the vertical economy than in the horizontal economy. Note that this proposition is stated in terms of the allocation alone, not in terms of the constraints ϕ, thereby making the two measures directly comparable.

The intuition for this result is quite simple. In the horizontal economy, firms need only to finance their own cost of labor, that is, their own added value. Thus, the aggregate amount of liquidity needed to implement a feasible allocation is simply just the sum of the equilibrium wage bills

$$M_h = h\ell_1 + h\ell_2 + h\ell_3$$

where the real wage w is equal to the marginal rate of substitution $V'(L) / U'(C)$ in equilibrium. In the vertical economy, on the other hand, it is as if there is double counting. In addition to the value of their own labor, firms in the vertical economy must also finance their expenditure on intermediate goods, thereby also pledging collateral for the labor purchased upstream.

$$M_v = h\ell_1 + \left(h\ell_2 + \frac{1}{\chi_1}h\ell_2\right) + \left(h\ell_3 + \frac{1}{\chi_2}h\ell_3 + \frac{1}{\chi_2\chi_1}h\ell_3\right)$$

Furthermore, from comparing (19) to (20), we see that the greater the output shares β_2 and β_3 of the intermediate goods, the greater the amount of liquidity needed in the vertical economy relative to the horizontal in order to implement the same allocation.

Moreover, note that in the vertical economy the aggregate amount of liquidity is greater than aggregate expenditure on labor; yet, despite this difference in liquidity and labor expenditure, collateral constraints are still binding. Chari, Christiano, Kehoe (2009) find that in the aggregate, among public companies, retained earnings plus dividends are greater than capital expenditures. One potential conclusion that they draw from this is that financial frictions do not matter, as firms could clearly finance their own capital expenditures with their own liquidity. There are a number of caveats to this finding–first, that these are only public companies; second, that this is only looking at the aggregate rather than individual firms and hence not taking into account distributional effects. Our findings here challenge this conclusion in another way–which firms are constrained and where they are in the production network matter. Here, in the vertical supply chain, the aggregate amount of funds in equilibrium is greater than the aggregate expenditure on labor (not including intermediate inputs), yet firms are still constrained by their collateral. If instead this were a representative firm economy or a horizontal economy, this would not be the case. Thus, the conclusion we obtain from this simple exercise is that the aggregate amount of available funds may not indicate the bite of financial frictions.

Finally, we see a relationship here to the *Quantity Theory of Money*. Effectively, one can think of the pledgeability constraints on the firms as analogous to cash-in-advance constraints, but on the firm's side. Money can be thought of as anything used to make transactions. Hence, the aggregate amount of pledgeable funds M is similar to the amount of money in the economy. With this interpretation in mind, our results are similar to the following representation of the original quantity theory of money,

$$MV = PT$$

where M is money, V is the velocity of money, P is the aggregate price level, and T is the aggregate number of transactions. Suppose that the aggregate price level P is normalized to 1 and that the velocity V is a constant. This implies that the level of transactions in the economy is proportional to amount of money. This general concept holds true in our model: more money is needed in the vertical economy than in the horizontal economy because there is a greater level of transactions occurring between firms. Therefore, given any equilibrium allocation, the more transactions made in the economy, that is, the more times goods change hands between firms, the more money is necessary to complete these transactions and implement the allocation.

2.5 Individual Spill-over Effects in the Vertical Economy

We have already explored the aggregate effects of tightening credit constraints. However, underlying these results is the behavior of individual firms and prices in response to any liquidity shock. To understand these interactions, we now look at the spill-over effects from tightening the collateral constraints of an individual firm. In particular, we examine how the tightening of a constraint of one firm affects the production of other firms in the economy.

We can easily answer this question in the horizontal economy environment.

Proposition 8. *Consider the horizontal economy. A tightening in the pledgeability constraint of firm i leads to a fall in firm i's employment and output and an increase in its relative price. However, there are no direct spill-over effects on the output, employment, and prices of firms $j \neq i$.*

In the horizontal economy, there are no direct spill-over effects because there are no linkages among firms. The only effects that could arise would be from the centralized labor and consumption market–income effects on the real wage. However, these are aggregate effects which would affect all firms equally. For simplicity, we abstract from these aggregate effects in this exercise. Our specification for utility () effectively kills these indirect aggregate effects. With log-utility over consumption and linear disutility of labor, income and substitution effects cancel so that the real wage remains constant. Thus, the decision problem remains the same for all firms $j \neq i$, and hence they employ and produce the same quantities as before.

On the other hand, in the vertical economy there are direct spill-over effects to other firms from a tightening of the constraint of one firm. To understand this more generally, we extend the vertical economy from 3 firms to K firms and tighten the constraint of sector i. The effects of this individual firm tightening on the output, employment, and relative prices of the other $K-1$ firms are described in the following proposition.

Proposition 9. *Consider the vertical economy with K firms. A tightening in the pledgeability constraint of firm i leads to a fall in firm i's employment, output, intermediate input use, and an increase in its relative price. Furthermore, this has direct spill-over effects on firms $j \neq i$ described by the following:*

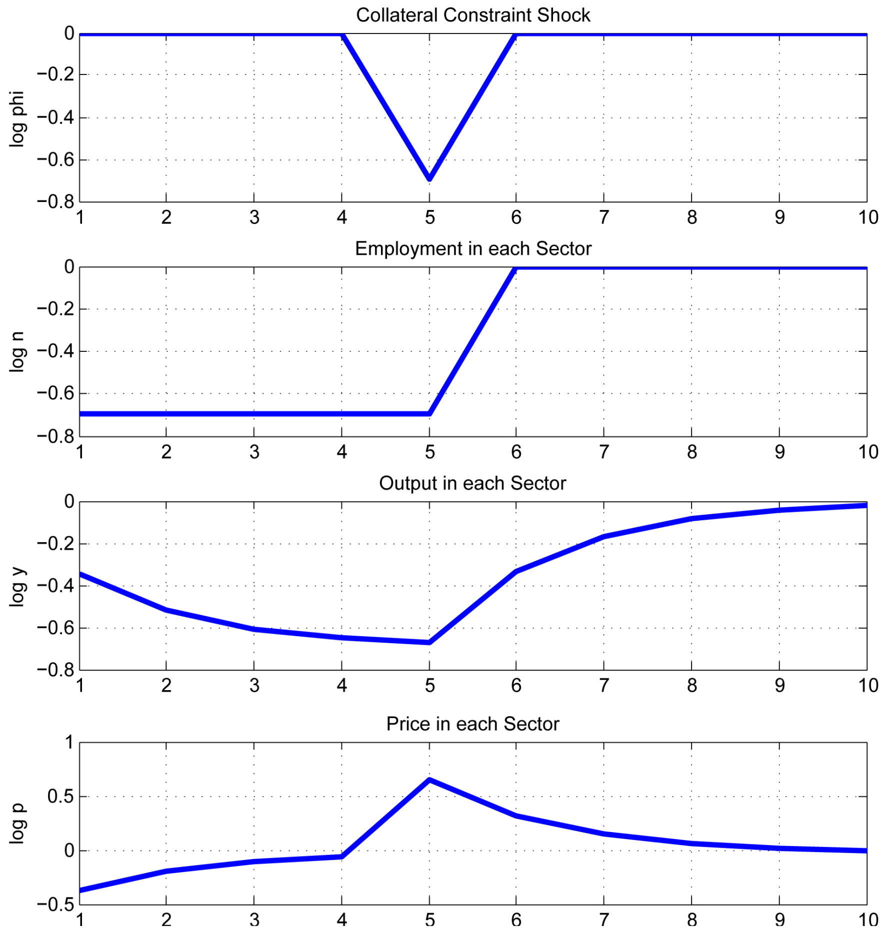


Figure 6: Spill-over Effects

(i) For firms $j < i$, employment,intermediate input use, and output fall. Its equilibrium relative price falls.

(ii) For firms $j > i$, employment remains unchanged but intermediate input use and production fall. Its equilibrium relative price rises.

We can further summarize these effects in Figure 6. Here we solve for the equilibrium in a vertical supply chain economy with $K = 10$ firms. We shock the collateral constraint of firm 5 and study what happens in equilibrium to all firms.

Consider the effects of tightening the collateral constraint of firm 5. This implies that this firm purchases both less labor and less intermediate inputs. Thus, its labor and inputs decrease and hence its output falls. Given that its supply decreases, its price thereby increases.

For firm 4, there is now less demand for its good from firm 5. This implies that in equilibrium its output and price falls. In order to produce less output, it both hires less labor and buys less intermediate inputs. Furthermore, this implies that the demand for the good of firm 3 falls. Firm 3 therefore undergoes the same qualitative effects as firm 4: its output, employment, and intermediate input use all falls. But this implies that the demand for the good of firm 2 falls, and so on.

For firm 6, the price of its input (the output of firm 5) is now higher. Thus, it demands less of its intermediate inputs, however, its employment remains unchanged. The reason employment in this firm remains unchanged is the due to the fact that Cobb-douglas production technology implies that the share of expenditure spent on each input does not change. Thus, the share spent on labor remains the same, and as long as the real wage hasn't moved (which we have ruled out with specification for utility), then labor demand remains unchanged. Therefore, firm 6 produces less and because its supply decreases, its price thereby increases. But this implies that the price of the input for firm 7 has now increased, and so on.

Thus, there are numerous spill-over effects coming from the tightening of any firm's collateral constraint in the vertical economy. In summary, for firms $j < i$ there is less demand for their good, so this acts like a demand effect. For firms $j > i$, there is an increase in input prices, so this acts like a supply effect. For all firms, these are adverse effects, so that production decreases across the board. One would not see this in a horizontal model nor a representative firm model.

Again, if we had allowed for a different specification of utility, we would also have had indirect aggregate effects coming from changes in the real wage. However, these are aggregate effects, thereby affecting all firms equally. Thus, for the above exercise, in which we only care about individual firm direct effects, we simply abstract from these considerations.

This leads us to the our fifth result:

Result 5: Sectoral Analysis. We look more closely at sectoral effects and document the two following results.

First, the impact of individual financial constraints on aggregate output differs by network location. In vertical economies, the most downstream firm has the greatest impact on aggregate output as well as on aggregate distortions. That is, a tightening of constraints on the downstream firms leads to a greater decline in output than a tightening of constraints on upstream firms.

Second, there are no direct spill-over effects in the horizontal economy. However, there are direct spill-over effects in the vertical economy. A tightening of firm i's constraint acts like an adverse demand shock on upstream firms (suppliers), while it works as an adverse supply shock on firms downstream (customers). The output of upstream firms are more affected than the output of downstream firms.

2.6 Summary

In this section we studied two economies, a horizontal economy in which firms did not transact with each other and a vertical economy in which firms were arranged in a supply chain. These economies were allocationally equivalent under no frictions. We found that financial frictions then drove wedges between each firm's marginal benefit and marginal cost of production. However, due to the different economy structures this has important effects for aggregate variables. We summarize our main results from this simple model as follows.

Result 1. In response to tightened collateral constraints, aggregate output falls more in the vertical economy than in the horizontal economy. We call the network liquidity multiplier the difference between the aggregate output drop in the vertical economy and that in the horizontal economy.

Result 2. Aggregate labor wedge or tax is greater in the vertical economy than in the horizontal economy. The aggregate labor wedge may overstate the distortions faced at the individual level.

Result 3. A uniform tightening in the collateral constraint implies no change in TFP in the horizontal economy, but leads to a fall in TFP in the vertical economy due to misallocation.

Result 4. Take any implementable allocation. More liquidity is needed in the vertical economy to implement this allocation as an equilibrium than in the horizontal economy.

Result 5. In the vertical economy, liquidity constraints on downstream firms have the greatest impact on aggregate output. On the other hand liquidity shocks affect output of upstream firms more than they affect that of downstream firms.

Thus results 1-3 focus more on the aggregate effects of different network structures, while result 4 suggests that for any allocation, more liquidity is needed in the vertical economy. Finally result 5 goes more into the sectoral analysis. All of these results suggest that looking at network structures is important when thinking about the aggregate and individual impact of financial frictions. In the following section we allow for a more general model. And finally we calibrate the general model to the data.

3 The General Production Network

In this section we consider a more general input-output structure. This section primarily illustrates how one can embed financial frictions into a more general network structure, and allows us to think about the influence of any particular constraint within the general network.

In the following section we then calibrate this model using the input-output matrix of the US economy.

3.1 The Model

Our model follows that of Acemoglu et al (2012), which is essentially a static variant of the multi-sector model of Long and Plosser (1983). On top of this model, we embed our simple representation of financial frictions, making our model similar in spirit to that of Jones (2011). Furthermore, in order to use this model later for calibration, we allow for more generality in preferences and technology.

The economy is populated by a representative household and n production sectors.

Households. The representative household has the following preferences over consumption and labor

$$U(C) - V(L)$$

with

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \qquad \text{and} \qquad V(L) = \frac{L^{1+\epsilon}}{1+\epsilon}$$

Here, $\gamma \geq 0$ parametrizes the income elasticity of labor supply,[12] and $\epsilon \geq 0$ parameterizes the Frisch elasticity of labor supply. The household consumes a basket C of the n differentiated goods produced by the n sectors, given by

$$C \equiv \prod_{i=1}^{n} c_i^{\beta_i}.$$

where c_i is the consumption of good i, and β_i is the household's Cobb-Douglas expenditure share over good i. We let $\sum_{i=1}^{n} \beta_i = 1$. Finally, the budget constraint of the household is given by

$$\sum_{i=1}^{n} p_i c_i = \sum_{i=1}^{n} \pi_i + hL \tag{21}$$

where p_i is the price of good i, h is the competitive wage, and π_i are the profits from sector i.

Production. Each good in the economy is produced by a sector, and each production sector consists of a continuum of firms. Goods are differentiated across sectors, but not

[12] Note that risk aversion and intertemporal substitution play no role in our setting because all idiosyncratic risk is insurable and the model is static. Therefore, γ only controls the sensitivity of labor supply to income for given wage.

across firms within a sector. The production of any given sector can either be consumed or used by other sectors as an input for production.

Each sector produces output using Cobb-Douglas technologies given by the following

$$x_i = \left[z_i^{\eta_i} \ell_i^{\eta_i} \left(\prod_{j=1}^{n} x_{ij}^{w_{ij}} \right)^{1-\eta_i} \right]^{\alpha_i}. \tag{22}$$

where x_i is the output of sector i, ℓ_i is the amount of labor hired by that sector, $\eta_i \in (0,1)$ is the share of labor of that sector, x_{ij} is the amount of commodity j used in the production of good i, and z_i is sector-specific productivity. Finally, α_i parameterizes the decreasing returns to scale of any sector. For now, we will set $\alpha_i = 1$.

The exponent w_{ij} denotes the share of good j in the total intermediate input use of firms in sector i.[13] Here, we assume that $\sum_{j \in N_i} w_{ij} = 1, \forall i \in \{1, ..., n\}$. This, along with the assumption that $\alpha_i = 1$, implies that the sectoral production functions all exhibit constant returns to scale. We will later relax this assumption, among others, in our calibration.

Sector i maximizes profits given by

$$\Pi_i = \max_{\ell_i, x_i} p_i x_i - h\ell_i - \sum_{j=1}^{n} p_j x_{ij}$$

subject to its financial constraint given by[14]

$$h\ell_i + \sum_{j=1}^{n} p_j x_{ij} \leq \phi_i p_i x_i$$

This constraint is the analog of the collateral constraints introduced in Section 2: the expenditure of firm i is constrained to be less than ϕ_i of its earnings.

Market Clearing. The resource constraints of the economy are given by

$$x_i = c_i + \sum_{j=1}^{n} x_{ji}, \forall i \in \{1, \ldots N\} \tag{23}$$

for each good, and

$$\sum_{i=1}^{n} \ell_i = L$$

for labor.

[13] In general, w_{ii} need not be equal to 0; sectors may use their own product as an input.

[14] As before, the limited enforcement problem is described in Appendix A.

Notation and the Input-Output Matrix. We let $\mathbf{z}$ denote the vector of log productivities and $\boldsymbol{\phi}$ denote the vector of the log $\phi's$.

$$\mathbf{z} = \begin{bmatrix} \log z_1 \\ \log z_2 \\ \vdots \\ \log z_n \end{bmatrix} \text{ and } \boldsymbol{\phi} = \begin{bmatrix} \log \phi_1 \\ \log \phi_2 \\ \vdots \\ \log \phi_n \end{bmatrix}$$

Furthermore, let $\boldsymbol{\beta}$ denote the vector of expenditure shares of the representative household for each good: $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots \beta_n)$, and let $\boldsymbol{\alpha}$ denote the vector of decreasing returns to scale of each sector: $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_n)$. Finally, we let W denote the input-output matrix of the economy with entries w_{ij}:

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & & \\ \vdots & & \ddots & \\ w_{n1} & & & w_{nn} \end{bmatrix}$$

We adopt the convention that $w_{ij} = 0$ if sector j is not an input supplier to sector i. Note that the rows of W sum to one due to the constant returns-to-scale assumption. However the columns need not sum to one, and in fact their sum is what is known as the weighted outdegree. This corresponds to the share of sector i's output in the input supply of the entire economy.[15]

We define an equilibrium for this economy as follows.

Definition 4. *A competitive equilibrium consists of a vector of prices $(p_1, p_2, \ldots, p_n)$, a wage h, a consumption bundle $(c_1, c_2, \ldots, c_n)$, input, output and labor allocations $\left(x_i, \ell_i, \{x_{ij}\}_{j=1}^n\right)$ such that (a) the representative household maximizes utility (b) the representative firms in each sector maximizes profits, and (c) prices clear commodity markets and the wage clears the labor market.*

3.2 Exact Equilibrium and Sample Networks

We first solve the equilibrium of this economy for a special case in which the representative household is endowed with one unit of labor that is supplied inelastically, firms have constant returns to scale, and firms are equivalent in all ways except through the input-output structure W. This special case makes our model exactly equal to that of Acemoglu et al

[15] Finally, input-output relationships between different sectors can equivalently represented by a directed weighted graph on n vertices, each corresponding to a particular sector of the economy.

(2012), but with the addition of collateral constraints. This case admits a neat and simple closed-form representation for equilibrium real value added, or equivalently, aggregate output, which we characterize in the following proposition.

Proposition 10. *Suppose that*

(i) the representative household is endowed with one unit of labor $L = 1$, which is supplied inelastically.

(ii) all sectors exhibit constant returns to scale: $\alpha_i = 1 \ \forall i$

(iii) the labor shares of all firms are equal $\eta_i = \eta \ \forall i$ and the expenditure shares of the household over all goods are equal $\beta_i = 1/n \ \forall i$.

Then, in the competitive equilibrium of this economy, the logarithm of real value added (GDP) is given by

$$\log Y = \log (GDP) = v' (\mathbf{z} + \boldsymbol{\phi}) + \mu$$

where μ is a constant and v is an n-dimensional vector given by

$$v \equiv \frac{1}{n} \mathbf{1}' (I - (1 - \eta) W)^{-1} \tag{24}$$

Thus, in this case, the equilibrium value of real value added, or equivalently aggregate output, is simply a log-linear function of the underlying productivities and collateral constraints in the economy. The coefficients on these shocks are given by the elements of the vector v, which is called the *influence vector*. The influence vector, which itself depends on the input-output matrix W, captures how sectoral changes in collateral constraints propagate to other sectors of the economy and ultimately affect aggregate output.

Proposition 10 gives us a direct analog to the results presented for our simple model in Proposition 3: the effect on aggregate output in response to an individual sector's tightening of collateral constraints is given by the element v_i. Here, we see that in a more general model, aggregate output depends on the intersectoral network of the economy through the Leontief inverse $(I - (1 - \eta) W)^{-1}$. As in Acemoglu et al (2012), the influence vector is closely related to the Bonacich centrality vector corresponding to the intersectoral network. That is, sectors that take more "central" positions in the network representation of the economy play a more important role in determining aggregate output. Finally, note that the Liquidity Multiplier in this economy is simply given by $\mathbf{1}'v$.

Sample Networks. Next, we use our results in Proposition 10 to plot the influence vector for some sample network economies. Figure 7 plots the influence vector for various input-output structures, W. Each column corresponds to a different input-output structure for an economy with with $n = 7$ sectors.

The top panel in each column plots the graphical representations of the input-output matrix in each economy. As in the input-output matrix, each entry corresponds to w_{ij}: the share of factor j in the intermediate use of firm i. Darker colors represents higher usage and lighter colors represent less usage. We do this for four selected network structures: a circle network, a star producer network, a triangle network, and a random network.

The bottom panel plots a bar graph of that economy's influence vector–the height of each bar represents v_i, the influence of sector i, for sectors 1 through 7.

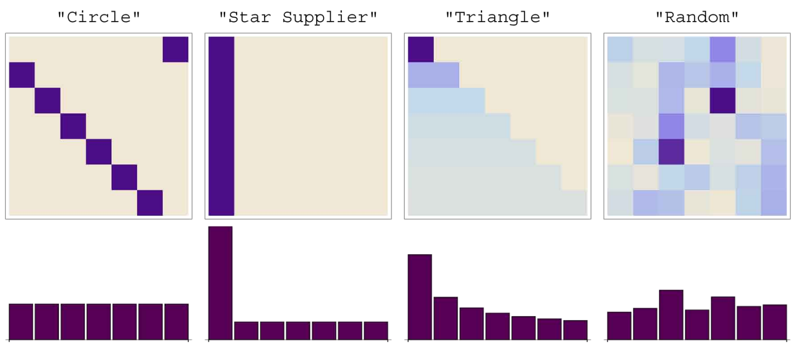


Figure 7: Networks and Influence Vectors

Circle Network. The first input-output structure we present is a circle network economy. Here, each sector i uses as inputs the good produced by sector $i-1$. This implies that sectors are arranged in a circle, one sector supplying to another.[16] This makes all sectors equivalent in terms of production. Hence, the circle network has a flat influence vector–all sectors have equal weight in terms of aggregate production.

Star Supplier Network. The second network we examine is a star supplier network. In this network structure, every sector requires as input the good of one key sector. Here, we arbitrarily make that key sector 1; that is, sector 1 supplies its good to all sectors, including itself. One might think of this sector as utilities or transportation.[17] We find that this sector's constraint has the largest impact on aggregate output than the other sectors, while all other sectors have only a minor impact as they don't affect the supply of intermediate inputs to other sectors.

Triangular Network. The third network is a triangular network. In this network, each firm i supplies their output to firms $j \geq i$. Thus, firm 1 supplies to all sectors, whereas sector 7 only supplies its output to its own sector. Here, we see that when expenditure shares are

[16] It is much like our vertical economy, except that sector 1 only uses the input of sector n (instead of all sectors).

[17] In the data this seeems plausible–there are a few key sectors such as utilities or transportation which supplies to all other industries.

flat, sector 1 has the most influence, and that sectoral influence decreases as we move from sector 1 to sector 7.

Random Network. Finally, we study a random network. Here, the entries w_{ij} are uniformly drawn from the interval $[0, 1]$ so that the entries of each row sum to 1, but not necessarily the column. This implies an influence vector which depends greatly on which column is the most full. This coincides with the result that the centrality of any sector matters.

In conclusion, we see that the structure of the network matters for the influence of each sector. In particular, we see that the centrality of any sector is closely related to its influence. In the circle network, all firms were equivalent, and hence had equal influence on aggregate output, while on the other hand, in the star network, sector 1 was central in that its output is used in the production of all other sectors. In this case a tightened collateral constraint of sector 1 would have a much larger effect on aggregate output than all other sectors.

Result 5: Sectoral Influence related to Centrality. With equal expenditure shares, the influence of any sector on aggregate output depends on a measure of its "centrality".

This was purely a qualitative exercise to illustrate the different influence vectors which can arise in response to different network structures. In Section 4 we calibrate the model to the input-output structure of the US economy, and use this to obtain quantitative results.

3.3 Correlation between Influence Vector and Sales

A question that potentially arises in relation to network economies is whether the influence of any firm is correlated with that firm's equilibrium share of sales. If it is perfectly correlated, then one might wonder why we need to think about the network structure at all, and instead one could just look at equilibrium sales to get a sense of sectoral influence. We discuss this issue here, first within the context of no financial frictions, and then in our economy with financial frictions.

Network Economies without frictions. Due to the way we've set up our economy, note that if we turn off all financial frictions, then our economy reduces to the frictionless economy of Acemoglu et al (2012). We can do this simply by setting $\phi_i = 1$ for all i, so that $\boldsymbol{\phi} = (0, 0, \ldots, 0)$. By Proposition , this implies that aggregate GDP

$$y = \log(GDP) = v'\mathbf{z}+\mu$$

which is the same result presented Acemoglu et al (2012). Thus, aggregate GDP is simply a log-linear function of underlying productivity shocks, where the influence vector v is given in (24).

An interesting result in Acemoglu et al (2012) is that the influence vector is also equivalent to the sales vector of the economy. That is, each element of the influence vector is equal to that sector's proportion of equilibrium share of sales in the economy.

$$v_i = \frac{p_i x_i}{\sum_{j=1}^{n} p_j x_j}$$

Thus, the impact of a shock to any firm is simply scaled by the size of that firm's sales, making this consistent with the model of Gabaix (2010).

One may view this as a somewhat discouraging result for network economies. In particular, in order to compute the influence vector, one need not know anything about the underlying input-output structure W. Instead, the equilibrium share of sales for each sector serves as a sufficient statistic for that sector's influence.

This, however, is not the case when one introduces frictions into the economy, as we show next.

Network Economies with frictions. Going back to our model with frictions, we show that with binding collateral constraints, or more generally, wedges between the marginal cost and marginal product of inputs, the previous result that the influence vector is equivalent to the sales vector does not obtain.

Lemma 3. *With financial frictions, the influence vector is not equal to the vector of equilibrium shares of sales*

$$v_i \neq \frac{p_i x_i}{\sum_{j=1}^{n} p_j x_j}$$

In particular, the equilibrium sales vector is given by

$$s' = \frac{h}{n}\mathbf{1}'\left(I - (1-\eta)\tilde{W}\right)^{-1}$$

where

$$\tilde{W} \equiv \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & 0 & \\ \vdots & 0 & \ddots & \\ 0 & & & \phi_n \end{bmatrix} W$$

Thus, equilibrium sales depend not only on the input-output structure, but also on the financial frictions–the ϕ's. Furthermore, the equilibrium sales vector is not equal to the influence vector v, as the latter only depends on W. Note that, if we were to set $\phi_i = 1$ for all i, so that there were no frictions, then $\tilde{W} = IW = W$, and as expected $v = s$, the influence vector and the sales vector would be equivalent. However, this is no more the case when collateral constraints are binding.

We are perhaps not the first to point this out. Our results here seem consistent with Hulten (1974). Hulten finds certain conditions under which the influence vector is equal to equilibrium sales. One of these conditions is that there are no market imperfections, which our model clearly violates.

Hence, with frictions, equilibrium shares of sales are no longer a sufficient statistic for computing elements of the influence vector. In order to compute the influence vector in any economy with frictions, one must use information on the underlying input-output structure W of the economy in question.

4 Calibration

In this section, we calibrate the general version of the production network described in Section 3 to the input-output matrix of the U.S. economy at three-digit level sectors from the North American Industry Classification System (NAICS).[18]

Our main strategy for the calibration is the following–we back out wedges for each sector. The spirit behind these exercises is to follow the strategy that Chari, Kehoe, McGrattan (2007) apply for business cycle accounting but to apply the strategy to obtain wedges for the industry sectoral levels. The ultimate goal is to use the Input-Output (IO) structure of the US economy to obtain a measure of the liquidity multiplier in the US economy.

The theoretical analysis presented earlier shows that when production is organized horizontally and labor is supplied inelastically the liquidity multiplier is 0. If instead labor is supplied vertically or labor is supplied elastically, the response depends on the labor supply elasticity and the response of output to hours worked. Since production in the US is organized via a highly interconnected network, as we show here, these interconnections may yield a much larger response of aggregate output to liquidity. By calibrating the IO matrix to that of the U.S. economy, this section shows the liquidity multiplier turns out to be 3.8 times higher than the corresponding multiplier in the horizontal economy. First, we begin describing the calibration strategy.

Preference Parameters. First, in terms of preferences, we set $\gamma = 2$ and $\epsilon = 1$.[19] The value for γ is standard in the macro literature. Instead, ϵ, the Frisch elasticity of labor

[18] For this general model the solution is quite complicated and the equilibrium cannot be expressed in closed form. We give the analytical solution of this model in Appendix B and results are found numerically.

[19] Because labor and consumption are separable, there is the possibility of a contracting supply of labor if the income effect outweighs the substitution effect. Later, we try to the calibration with GHH preferences to kill the income effect.

supply, is higher than the average value of micro studies, but low for macro models without labor market frictions. For this, see Hall (2009). Finally, Hall (2009) states that the Labor Wedge methodology doesn't explicitly take into account unemployment. Hence, one should interpret it as a reduced form for unemployment, and therefore finds $\epsilon = 1$ an appropriate value.

To calibrate the household expenditure shares, we use data reported by the Bureau of Economic Analysis (BEA). The BEA organizes the summary level of input-output matrixes at the three-digit level according to the NAICS system. There are 65 NAICS sectors at the three-digit level in total, which include the federal government, state-level governments and government-state agencies. We treat these as sectors as in the rest of the economy. We use data for all years from 1998 until 2011. The Use of Commodities by Industries after Redefinitions (Uses) table is our source of information to calibrate the β_i shares of consumption. This table reports the expenditures of all industries and households for every commodity produced in every industry in billions of US dollars. We use the following property of our model to calibrate these expenditure shares

$$\beta_i = \frac{p_i c_i}{\sum_{j=1:65} p_j c_j}.$$

Unfortunately, prices and quantities are not reported independently, but expenditure is reported. Hence, let $u^c_{i,t} \equiv p_{i,t} c_{i,t}$ be the consumption expenditures of the household on sector i's commodities in year t. The Uses table reports these personal consumption expenditures $u^c_{i,t}$[20], as well as investment uses and exports. For now, we ignore these other uses.[21] Thus, we may compute $\beta_{i,t}$ by

$$\beta_{i,t} = \frac{u^c_{i,t}}{\sum_{j=1:65} u^c_{j,t}}.$$

and hence we compute the household's consumption share of sector j's commodity at time t.

Figure 8 reports the evolution of the imputed log-expenditure shares, $\log \beta_{i,t}$, for each industry from 2006-2010. On the x-axis we have each industry, ordered from lowest log expenditure share to highest. As is evident from this figure, the expenditure shares for each industry are quite stable over 1998-2011 period. The exceptions are mining and petroleum industries, which have the lowest household expenditure shares to begin with.

Technology Parameters and Collateral Constraints. We now explain how we calibrate the technology parameters and collateral constraints in our model. Here we go

[20] Column F010 of that table

[21] We intend to correct this deficiency in a forthcoming version.

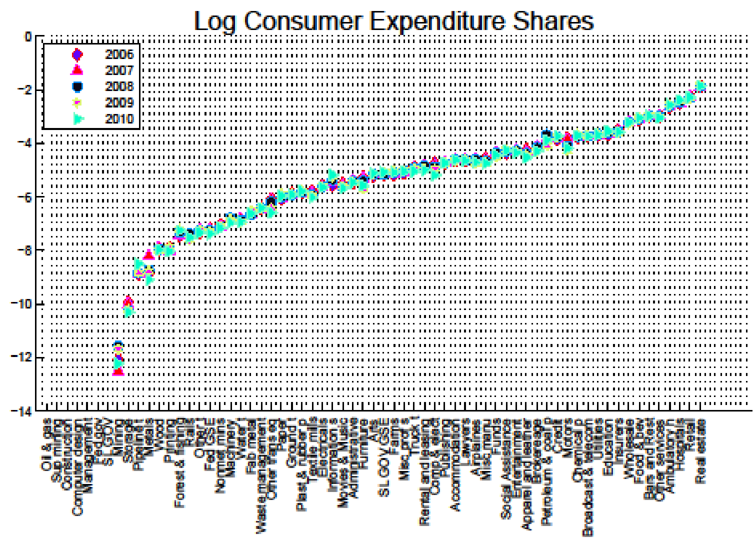


Figure 8: Calibrated Expenditure Shares

back to the more general model which allows for decreasing returns to scale in production, α_i, and different labor shares η_i across firms. We allow the firms technology be given by

$$x_i = z_i \left[\ell_i^{\eta_i} \left(\prod_{j=1}^{K} x_{ij}^{w_{ij}} \right)^{1-\eta_i} \right]^{\alpha_i}$$

with, without loss of generality, $\sum_j w_{ij} = 1$. Therefore, for each sector i and for each period t, the set of technology parameters we must calibrate are $\alpha_i, \eta_i, \{w_{ij}\}_{j=1:25}$.

To calibrate these technology parameters, we use the Industry-by-Industry Direct Requirements after Redefinitions table (Requirements), reported by the BEA. This table reports the production required, both directly and indirectly, from each industry per dollar of delivery to final use of the industry. The Requirements table includes information on labor use as well.[22] Again, let $u_{ij,t} = p_{j,t}x_{ij,t}$ be the expenditure by sector i on the output produced by sector j in year t. The Requirements table reports these sectoral expenditures $u_{ij,t}$, as well the sectoral expenditure on labor, which we denote by $u_{i0,t}$ (so that labor is considered the zero-th sector). Figure 1 in our introduction shows the contour plot for the log entries of this matrix in 2006. Finally, let $v_{i,t} = p_{it}x_{it}$ denote the total revenue of sector i in time t. This is industrial output of sector i in the Requirements table. Simple accounting should imply that $v_{i,t} = u_{it}^c + \sum_{j=1:65} u_{ji,t}$, that is, revenue of sector i should be equal to

[22] We leave out of the analysis three of the entries in these tables: (1) scrap, used and secondhand goods, (2) non-comparable imports and rest-of-the-world adjustment and (3) taxes on production and imports, less subsidies without including taxes.

total expenditure on good i by all other sectors and the households.

Calibrating the parameters η_i and $\{w_{ij}\}_{j=1:65}$ is straight-forward. However, the main challenge for the calibration is to obtain values for the coefficients of decreasing returns to scale parameter α_i separately from ϕ_i. We cannot obtain these parameters directly because the input shares cannot be independently identified from the wedges ϕ_i. This same dilemma is faced in Jones (2011). He writes, "there is a fundamental identification problem: we see data on observed intermediate goods shares, and we do not know how to decompose that data into distortions and differences in technologies. This identification problem is not solved in anything I have done." We will first explain this identification issue in our context, and then explain how we deal with it.

In terms of the model, from the firm i's optimality conditions over its inputs we have that

$$h\ell_i = \phi_i \alpha_i \eta_i p_i x_i \tag{25}$$
$$p_j x_{ij} = \phi_i \alpha_i (1-\eta_i) w_{ij} p_i x_i, \forall j \tag{26}$$

Adding all of these together, we get that

$$h\ell_i + \sum p_j x_{ij} = \phi_i \alpha_i p_i x_i \tag{27}$$

We can thus define μ_i as the ratio of total cost of inputs to total revenues as follows

$$\mu_i \equiv \frac{h\ell_i + \sum p_j x_{ij}}{p_i x_i}$$

This is the cost share of output. From equation (27), we see that our model implies that

$$\mu_i = \phi_i \alpha_i$$

In terms of the data, we have that $u_{i0} = h\ell_i$, $u_{ij} = p_j x_{ij}$, and $v_i = p_i x_i$. Thus, we can build μ_i from the data directly as follows

$$\mu_i^{data} = \frac{u_{i0} + \sum_{j=1:65} u_{ij}}{v_{i,t}}$$

so that the numerator is total expenditure of sector i on inputs (including labor), and the denominator is total revenue of sector i (including expenditure by households). These sectoral cost ratios are in fact very unstable compared to the expenditure shares of consumption in the US.

Using this, one can back out the labor share η_i from (25) and the intermediate shares w_i from (26) as follows

$$\eta_i = \frac{h\ell_i}{\mu_i p_i x_i} = \frac{u_{i0}}{\mu_i^{data} v_{i,t}} \quad \text{and} \quad (1-\eta_i)\, w_{ij} = \frac{p_j x_{ij}}{\mu_i p_i x_i} = \frac{u_{ij}}{\mu_i^{data} v_{i,t}}, \forall j$$

This identifies separately the parameters η_i and $\{w_{ij}\}_{j=1:65}$. What remains then to calibrate is ϕ_i and α_i. As one can see, given the data, these parameters are not separately identified, as $\mu_i^{data} = \phi_i \alpha_i$.

We choose to calibrate α_i and ϕ_i as follows. Through the lens of our model, we attribute the instability of the costs-to-sales ratio of each sector as the outcome of movements in the wedges $\phi_{i,t}$. Hence, we calibrate α_i to be the maximal value in this series. We thus take our estimate for $\alpha_{i,t}$ as,

$$\alpha_{i,t} = \alpha_i = \max \left\{\mu_{i,s}^{Data}\right\}_{s \in \{1998,1999,..,2011\}}$$

so that it is constant across the sample. With this value, we then proxy the sectoral wedges via the following relationship

$$\phi_{i,t} = \frac{\mu_{i,t}^{Data}}{\alpha_i}.$$

Note that our identification strategy implies that maximum value for our calibrated ϕ_{it} must be 1. The model tell us that ϕ_{it} cannot be greater than 1, as when $\phi_{it} = 1$, then the firm is at its optimum.

One reason we believe that this is a reasonable strategy is the belief that it is rather unlikely that technology shares move a lot at high frequency (yearly data). Under this assumption, we would thus expect yearly changes in $\mu_{i,t}$ to be attributed to changes in the wedges ϕ_{it} rather than from low-frequency movements in the technology parameters α. By setting α_i to the maximal value of μ_{it} in the series, we attribute all movement in the cost-to-sales ratios during the crisis to movements in wedges.

Figure 9 presents the imputed values for $\phi_{i,t}$ for every sector from 2006 until 2010. On the x-axis, we have each industry, ordered by their 2009 proxy for ϕ_{it} from lowest to highest. The 2009 proxy for ϕ_{it} is given by the red line. As should be clear from this figure, 2009 is a year with particularly low values of imputed $\phi_i's$ for most sectors. In other words, 2009, the year of the peak of the Great Recession is a year showing particularly large sectoral wedges. Furthermore, as noted above, these measures vary much more from year-to-year than the household's expenditure shares. This gives us more confidence that yearly movements in cost-to-sales ratios are driven more by wedges as opposed to technology shares.

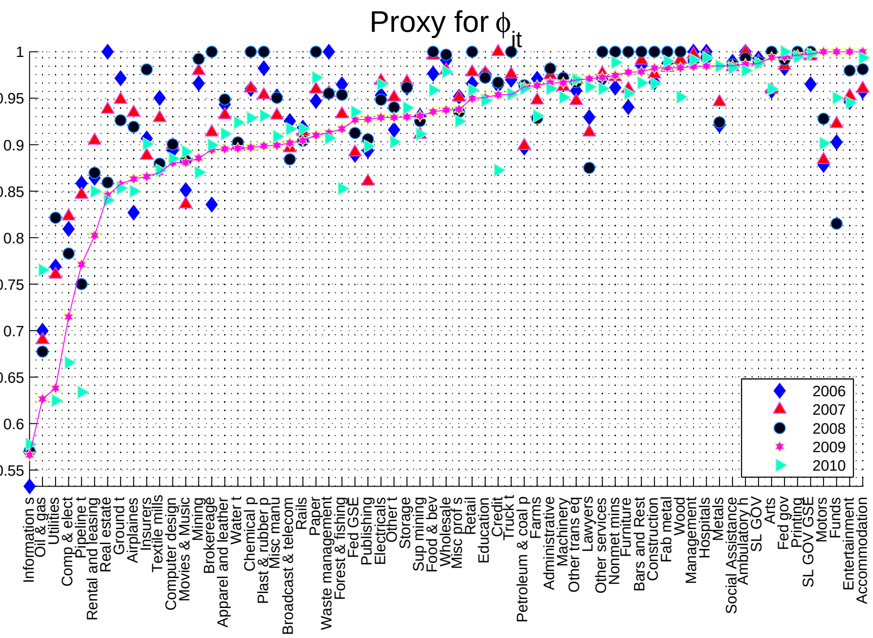


Figure 9: Calibrated ϕs

This completes the description of how we calibrate our model.

4.1 Analysis of our Sectoral Wedge Proxies

Before moving to our main results, we look more closely at our proxies for $\phi_{i,t}$. Here we ask whether our proxies of $\phi_{i,t}$ are capturing comovements with industry output. The following table presents results for three regressions where the outcome variable is the deviation of sectoral output between a given year and 2006: $y_{i,t} - y_{i,2006}$. The first column regresses $y_{i,t} - y_{i,2006}$ on $\phi_{i,t} - \phi_{i,2006}$, that is, the change in the sectoral wedge between year t and 2006. Figure 10 presents the correlation between these two variables. The figure shows a positive correlation among these variables which is validated by the high statistical significance reported in the first column of table.

Next, if one takes into consideration changes in household expenditure shares in each industry, this explanatory power of $\phi_{i,t}$ becomes even stronger. This corresponds to the second column of this table, where the R-squared has increased.

	(1)	(2)	(3)
$\phi_{i,t} - \phi_{i,2006}$	1105.1606 $(7.6328e-07)$	55.5495 $(4.5982e-12)$	44.0621 $(1.2858e-08)$
$\beta_{i,t} - \beta_{i,2006}$		841.156 $(4.373e-05)$	777.0462 $(8.8635e-05)$
Neighbor 1			24.7274 $(8.1866e-06)$
Neighbor 2			14.448 (0.010027)
Neighbor 3			19.7989 (0.00088182)
Neighbor 4			12.355 (0.029164)
Neighbor 5			7.148 (0.22418)
R^2	0.087051	0.23947	0.32592

Finally, we study how $\phi_{i,t}$ affects the output of other sectors $j \neq i$. First, for every pair of sectors i and j, we compute within our model the impact that a liquidity shock ϕ_{it} of sector i has on the industry output of sector j. Figure 11 presents this information graphically. The arrows appear whenever a sector's ϕ_{it} has a particularly strong effect on the output of sector j according to our calibrated model. Furthermore, for each industry i, we rank the industries j who's shocks to ϕ_{jt} most affect the output of sector i. Figure 12 lists for every sector i, the first five sectors in this ranking–i.e. the sectors that most affect the output in sector i. The outcomes seem natural. For example, the Farming industry is most affected by shocks to Forest & fishing, wood, petroleum, oil & gas, and mining. The Motors sector is most affected by shocks to metals, Fabricated metals, Mining, Electricals, and Plastic and Rubber parts.

The third column of the table reports the same regression, but also including the changes in ϕ_{it} for the first 5 sectors (Neighbors) in this ranking. This improves the fit. We perform a placebo test by also randomly running regressions outside those 5 sectors. The conclusion from these tests is that outside the first 5 ranked sectors, these network effects disappear.

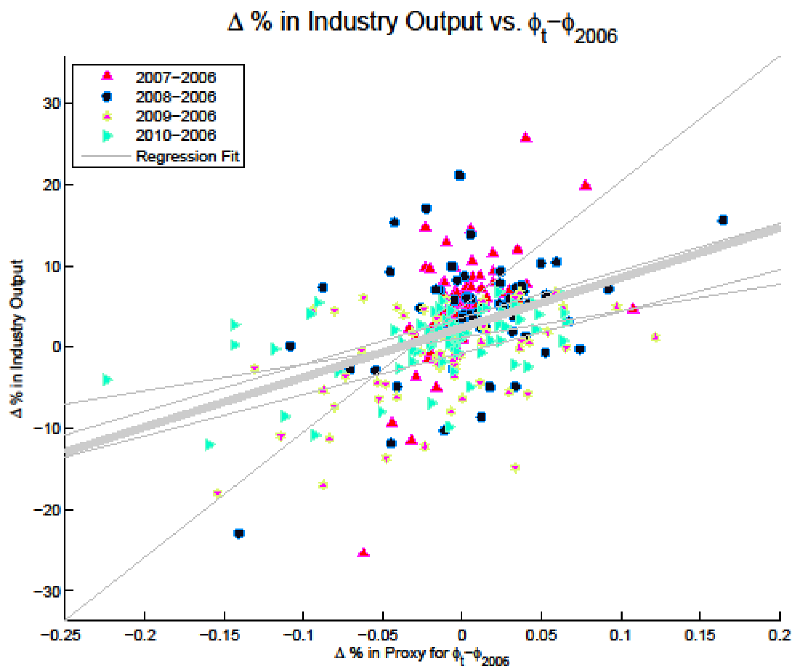


Figure 10: correlations

4.2 Aggregate Analysis

We now move on to the main results from our calibrated model.

Result 7: The U.S. Network Liquidity Multiplier. Suppose liquidity is drawn down by in 1% in every sector. What is the fall in aggregate output given our calibration? We find that aggregate GDP falls by 2.7%. Of course, the answer to this question depends on the parameters that we impute into the model as well as a given year's input-output matrix and the values of the vector $\phi_{i,t}$ in a given year. To compute this number, we look at the fall in log aggregate output to a 1% reduction of liquidity in every sector. Since the derivative depends on the calibration for a given year, we average across all years of our sample. Furthermore, higher elasticities in labor induce higher values for this estimate.

Given this result, we can then quickly calculate the Network Liquidity Multiplier. The liquidity multiplier is the ratio of the drop in output in the network economy to that in representative agent economy. In representative agent economy, output would fall only by .7%. in response to a 1% fall in ϕ. This implies that the Network Liquidity Multiplier for the U.S. is approximately 2.7/.7=3.8%.

Result 8: Implied Drop in Liquidity during the Crisis. The top panel of Figure 13 reports

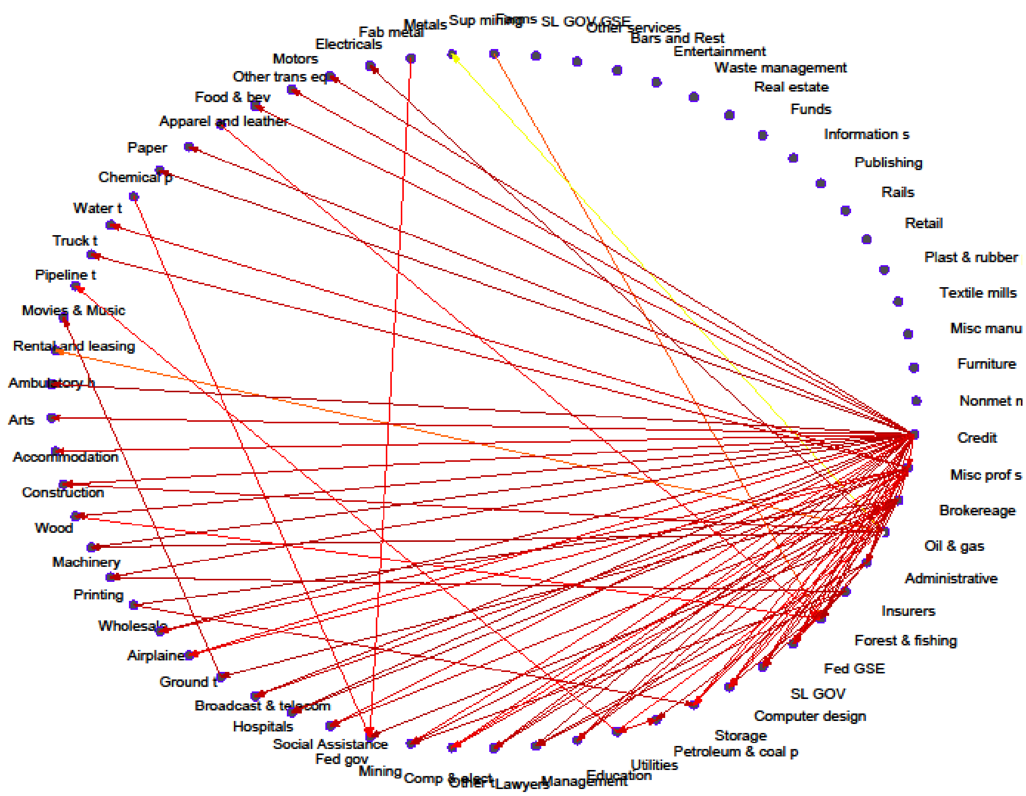


Figure 11: Network

Figure 12: Network Neighbors

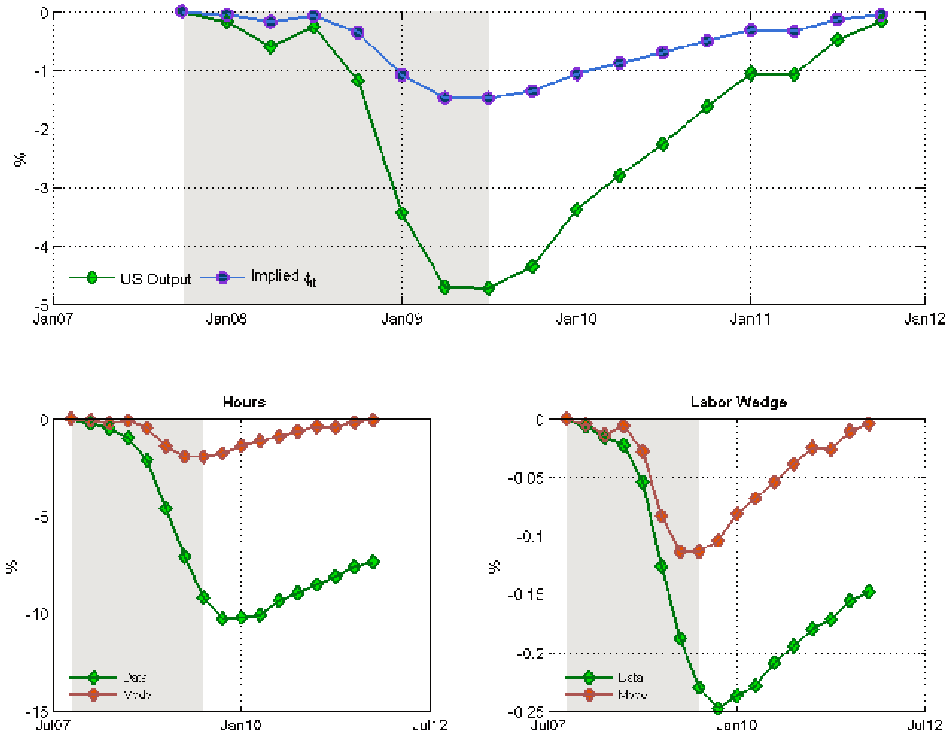


Figure 13: Top Panel: Implied Drop in Liquidity. Bottom Panel: Hours and the Labor Wedge

the response of GDP to a particular sequence of shocks to ϕ. The sequence corresponds to the quarters ranging from the third quarter of 2007 until the second quarter of 2011. The sequence of shocks is chosen so that GDP in the model follows the same path as in the data. This is given in green line–the green line plots the log deviations of U.S. GDP from its average growth path over that last 30 years. The blue line plots the aggregate liquidity shock that hits every sector symmetrically that would generate this path of GDP.

At the trough of the cycle, the required reduction in liquidity in the calibrated network would be about 1.3% for every sector. In contrast, a model with a representative firm would require at least a 7.6% reduction in ϕ. Thus, our calibrated model requires a reduction in liquidity of less than 1/6th the drop in liquidity required in a representive firm model. (need to add this calculation)

As you can see from this Figure 13, the model cannot match the observed drop in hours–hours fall much more in the data than in the model. While the model is not designed to fit the responsiveness of labor, it is important to point out what features the model may be missing. The model lacks any form of price or wage rigidities. In particular, wage rigidities could potentially lead to a much sharper reduction in hours than that suggested by Figure 13.

Result 9: Which sectors are the most affected by the aggregate implied liquidity shock? When there is an aggregate liquidity shock, which sectors are the most affected by this shock? We find that the sectors most affected from an aggregate liquidity shock are the *manufacturing sectors*. These include metal products, chemical products, fabricated metal products, hydrocarbons and other industries related to the extraction and transformation of raw materials. Figure 16 in our appendix presents the full results. According to our model–this is due to network effects: these sectors provide many intermediate inputs that are used in the production of final good industries. In addition, we find that some of the service industries such as miscellaneous professional services, administrative, brokerage and management firms are also near the top of this list.

Result 10: Which sectors have the most influence on aggregate output? If we shock the liquidity within each sector individually, which sector affects aggregate output the most? We find that the sectors which lead to the largest drop in aggregate output are final good sectors, such as *retail*. Figure 8 reports the full results–it gives the estimates of the response of aggregate GDP to the shock in ϕ_i in every sector i. Other important sectors are hospitals, brokerage firms, food & beverage, bars and restaurants, services, and motors. These all mostly seem like final good industries. This result seems consistent with our analytical results–that the most downstream goods have the greatest influence.

To understand this, Figure 8 presents a scatter plot of sectoral observations: consumption expenditures shares in each sector against the sector's aggregate impact. This Figure suggests that sectors with larger expenditures shares lead to larger aggregate effects.

4.3 Are Sectoral Wedges capturing Financial Frictions?

So far, our analysis has been concerned with the network effects of wedges during the financial crisis. We have attributed the short-run fluctuations in these wedges to financial factors. One question, however, is whether these ϕ wedges are actually picking up financial frictions.

Here, we try to partially answer this question. In particular, we analyze whether wedges during the Great Recession responded more in sectors commonly viewed as more dependent on external finance. For this purpose, we reconstruct the measure of external finance in Rajan and Zingales (2001) for the sample of firms in COMPUSTAT for the period 1990-2006. That is, we don't include the period of the Great Recession. Following Rajan and

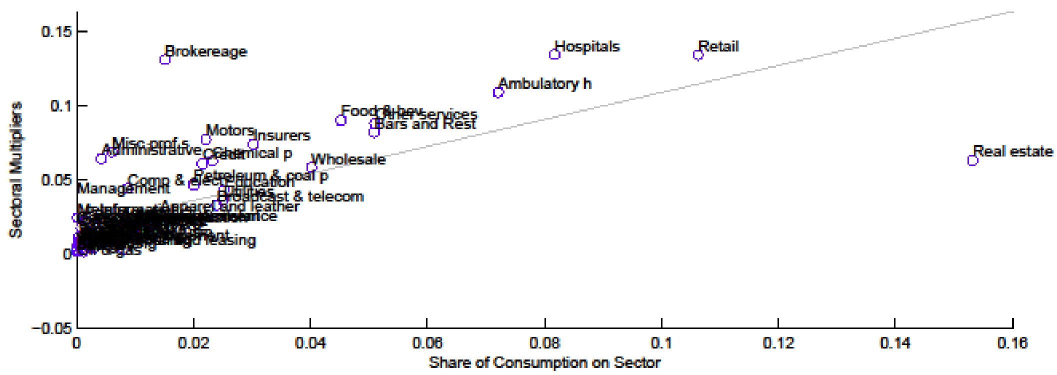


Figure 14: Correlation of Sectoral Influence with Consumption Expenditure Shares

Zingales (2001), a firm's dependence on external finance is defined as capital expenditures minus cash flow operations over total capital expenditures. Cash flow from operations is sales minus decreases in inventories, decreases in receivables, and increases in payables. We use the notation $RZ_{i,t}$ to refer to the Rajan Zingales measure of an industry (4 digits) i in quarter period t.

We run the following panel regression:

$$\eta_{j,t} = \beta_{age} \times age_{j,t} + \beta_{RZ} RZ_i + \beta_{NBER,t} \times I_{NBER,t} \times RZ_i + Controls + \varepsilon_{j,t}$$

where $age_{j,t}$ is the age of the firm j, RZ_i is the Rajan Zingales measure for the firm's industry, $I_{NBER,t}$ is a time dummy for the dates corresponding to the great recession, and $Controls$ is a set of controls that include year and quarter fixed effects. The results of this regressions come out as expected. Age is negatively correlated with markups and the Rajan Zingales measure is positively correlated. Most interestingly for us, the coefficient of the Rajan Zingales measure against the dates of the Great Recession are significant and positive for the quarters of the Great Recession. They pickup in the quarter post the Lehman Brothers crisis and smoothly vanish after. Figure 15 presents the estimate of $\beta_{NBER,t}$ together with its 95% confidence interval.

This regression provides some indirect evidence that industries with highest dependence showed particularly high markups. Rajan and Zingales find that industries with high external financial dependence are smaller in countries with worst indicators of financial development. The same indicators for the time series in the US show that sectors with higher dependence experienced increases in their markups, our measure of tighter financial constraints.

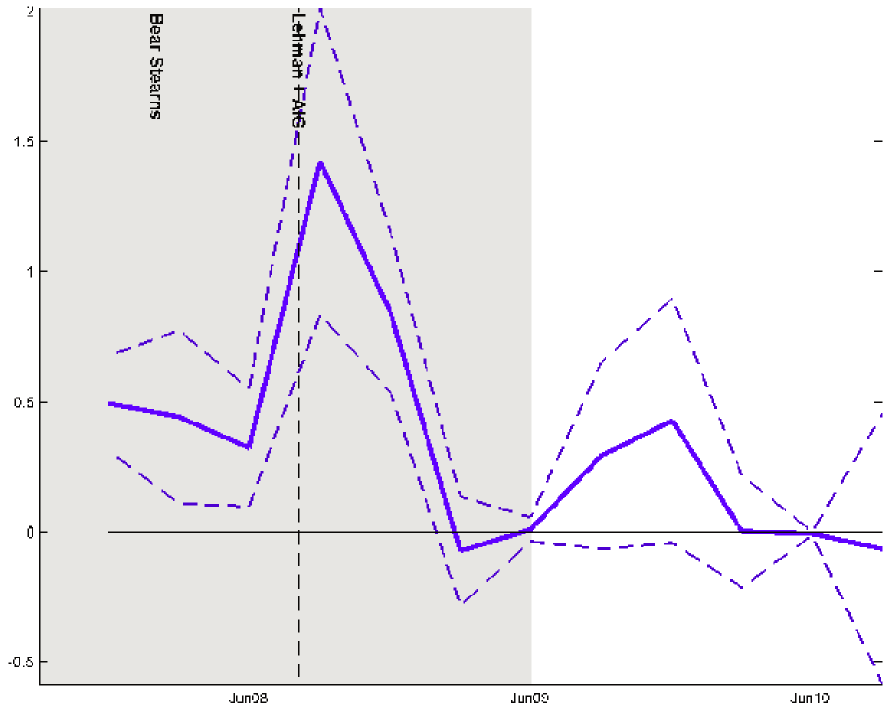


Figure 15: Time Series of β_{NBER}

5 Conclusion

This paper argues that the network of production links in an economy can matter substantially for the transmission of financial shocks. To make this point, we formulated an economy in the most simple way possible. We provided several analytic examples of liquidity shocks to analyze their propagation in particular network structures. We then took the structure of the U.S. I-O and calibrated a more general model. We asked what is the liquidity multiplier in the U.S. Our experiment showed a multiplier of 3.8. compared to that of a horizontal economy whose multiplier is 1.

There are clearly many caveats with what we have so far done.

Obviously on big issue is that we are ignoring any dynamics. The model abstracts from capital, inventory, as well as any form of durability in goods. Consumer durables would potentially amplify the output of industries in the durables sectors by causing changes in relative demand.

Another concern is that we are allowing a high degree of production elasticity substitution. In the model, sectors whose suppliers are affected by reductions in liquidity easily substitute production inputs with other inputs or labor. The model may be missing a larger amount of rigidities in the production process. [4] A final concern is that we are treating all sectors in a symmetric way. This is a problem because sectors such as retail, wholesale and warehousing and transportation are sectors that don't transform products in a the same way manufacturing industries. Often these are simple intermediaries charging a constant markup so the decreasing returns to scale assumption may not be appropriate.

We believe the exercise illustrates the that the effects of liquidity shocks can be quite dramatic if production is organized with industrial linkages. We speculate that if one were to introduce the possibility of demand changes via durable consumption preferences, nominal rigidities, or low short-run input substitution, the effects could be even more dramatic. This extensions can be studied bringing this framework into richer environments.

There are other questions that are relevant for the theoretical study of financial frictions in networks. Early work of Kiyotaki and Moore on Credit Chains noted that disruptions in the payments chain have important welfare implications. We abstracted from any strategic/time dimension form of trade credit because in our model, trade credit moves jointly with liquidity. However, we should study a model which can explain disruptions in the supply chain. In our model, the network structure is exogenous. There is a growing literature on endogenous network formation; see e.g. Oberfeld (2012).

Finally, one question which might be interesting for future research is the question of the optimal allocation of liquidity. If the government could move credit from one part of the economy to another, where would the planner choose to move credit to? As our model is based on inefficient wedges, the answer to this question would seem similar to that found in the optimal taxation literature. That is, one would not want double marginalization of goods, and this would call for zero taxation of intermediate goods. But the planner's problem is a bit different in our context, as wedges are due to firm liquidity rather than taxes. Hence, one would need to set up a planner problem which respects this type of constraint and implementability issues.

In conclusion, we hope that this paper has brought to light the interesting effects production networks may have when combined with models of financial frictions.

6 Appendix A: Proofs

Proof of Proposition 1 *Vertical Economy.* Let us first consider the equilibrium in the vertical economy. Firm 1 maximizes the following objective function.

$$\max p_1 z_1 \ell_1^{\alpha_1} - h\ell_1$$

This yields the following first-order condition

$$p_1 \alpha_1 \frac{x_1}{\ell_1} = h \tag{28}$$

Firm 2 maximizes the following objective function.

$$\max p_2 z_2 \ell_2^{\alpha_2} x_1^{\beta_2} - h\ell_2 - p_1 y_1$$

This yields FOCs

$$\alpha_2 p_2 \frac{x_2}{\ell_2} = h \qquad \text{and} \qquad \beta_2 p_2 \frac{x_2}{x_1} = p_1 \tag{29}$$

Firm 3 solves a similar problem to that of firm 2; it's FOC's are given by

$$\alpha_3 p_3 \frac{x_3}{\ell_3} = h \qquad \text{and} \qquad \beta_3 p_3 \frac{x_3}{x_2} = p_2 \tag{30}$$

Finally, we have that $Y = x_3$ and $P = p_3$. Thus, combining the focs of all firms (28) with (29) and (30), we reach the following equations.

$$\alpha_3 \frac{Y}{\ell_3} = \frac{h}{P} \tag{31}$$

$$\alpha_2 \beta_3 \frac{Y}{\ell_2} = \frac{h}{P} \tag{32}$$

$$\alpha_1 \beta_2 \beta_3 \frac{Y}{\ell_1} = \frac{h}{P} \tag{33}$$

Finally, the optimality condition for the household is given by

$$\frac{v'(L)}{u'(Y)} = \frac{h}{P} \tag{34}$$

where we use the fact that $Y = C$. Combining this with (31)-(33) yields conditions (2) in the proposition.

Horizontal Economy. We now characterize the equilibrium in the horizontal economy. In the horizontal economy, all firms are identical and maximize the following objective.

$$\max p_i z_i \ell_i^{\alpha_i} - h\ell_i$$

the FOC's for each firm is given by

$$p_i \alpha_i \frac{x_i}{\ell_i} = h \tag{35}$$

Finally the household maximizes the following objective

$$\max P x_1^{\beta_2 \beta_3} x_2^{\beta_3} x_3 - p_1 x_1 - p_2 x_2 - p_3 x_3$$

The FOCs are given by

$$\beta_2 \beta_3 P \frac{Y}{x_1} = p_1, \quad \beta_3 P \frac{Y}{x_2} = p_2, \quad \text{and} \quad P \frac{Y}{x_3} = p_3$$

Combining these above equations with the firms FOC's (35) we get the following conditions, which are exactly the same as in the vertical economy.

$$\alpha_1 \beta_2 \beta_3 \frac{Y}{\ell_1} = \frac{h}{P} \tag{36}$$

$$\alpha_2 \beta_3 \frac{Y}{\ell_2} = \frac{h}{P} \tag{37}$$

$$\alpha_3 \frac{Y}{\ell_3} = \frac{h}{P} \tag{38}$$

The household's optimality condition over consumption and labor remains the same as in the vertical case, (34). Thus, combining this condition with the above equations (36)-(38) yields conditions (2) in the proposition.

Proof of Proposition 2 *Part (i) Vertical Economy.* Let's first consider the vertical economy with collateral constraints. First for the household's labor consumption condition over consumption and labor remains the same as before, and given by (34)

Firm 1 maximizes the following objective

$$\max p_1 z_1 \ell_1^{\alpha_1} - h\ell_1$$

subject to the collateral constraint,

$$h\ell_1 \leq \chi_1 p_1 x_1$$

If collateral constraint is binding then $h\ell_1 = \chi_1 p_1 x_1$, otherwise the firm chooses labor according to it's unconstrained FOC above in (). We may summarize this in the following way.

$$h\ell_1 = \phi_1 \alpha_1 p_1 y_1$$

where $\phi_i = \min\left\{1, \frac{\chi_i}{\alpha_i}\right\}$. That is, if the collateral constraint is not binding, $\phi_1 = 1$. Otherwise, $\phi_1 \in (0, 1)$.

Now consider the problem of firm 2. Firm 2 maximizes the following objective

$$\max p_2 z_2 \ell_2^{\alpha_2} x_1^{\beta_2} - h\ell_2 - p_1 y_1$$

subject to $h\ell_2 + p_1 x_1 \leq \chi_2 p_2 x_2$. Firm 2's cost minimization is given by

$$\min h\ell_2 + p_1 y_1$$

subject to $x_2 = z_2 \ell_2^{\alpha_2} x_1^{\beta_2}$. This implies that the firm's optimal choices for inputs must satisfy

$$h\ell_2/\alpha_2 = p_1 x_1/\beta_2 \tag{39}$$

Firm 2's expenditure on goods is given by $h\ell_2 + p_1 x_1 = (\alpha_2 + \beta_2)\, p_2 x_2$. Comparing this to the collateral constraint (), firm 2 is constrained if and only if $\alpha_2 + \beta_2 > \chi_2$. If constrained, then

$$h\ell_2 = \frac{\alpha_2}{\alpha_2 + \beta_2} \chi_2 p_2 x_2$$

Thus, we can write

$$h\ell_2 = \phi_2 \alpha_2 p_2 x_2 \tag{40}$$

where $\phi_2 = \min\left\{1, \frac{\chi_2}{\alpha_2 + \beta_2}\right\}$. Similarly for firm 3 we have that

$$h\ell_3/\alpha_3 = p_2 x_2/\beta_3 \tag{41}$$

We can similarly for firm3 write,

$$h\ell_3 = \phi_3 \alpha_3 p_3 x_3 = \phi_3 \alpha_3 PY \tag{42}$$

where $\phi_3 = \min\left\{1, \frac{\chi_3}{\alpha_3 + \beta_3}\right\}$, which combined with the household's optimality condition (34), corresponds to equation () in the proposition.

Combining (42) with firm 3's optimality condition (41), we have that

$$p_2 x_2 = \frac{\beta_3}{\alpha_3} h\ell_3 = \beta_3 \phi_3 PY$$

Finally, combining this with the optimality condition of firm 2 (40), implies that

$$h\ell_2 = \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

which, along with the household's optimality condition (34), corresponds to equation () in the proposition. And finally combining () with firm 2's optimality condition (), we have that

$$p_1 x_1 = \frac{\beta_2}{\alpha_2} h\ell_2 = \frac{\beta_2}{\alpha_2} \phi_2 \phi_3 \alpha_2 \beta_3 PY$$

Combining this with the optimality condition of firm 1 (), implies that

$$h\ell_1 = \phi_1 \phi_2 \phi_3 \alpha_1 \beta_2 \beta_3 PY$$

which, along with the household's optimality condition (34), corresponds to equation () in the proposition.

Part (ii) the Horizontal Economy. Again, each firm solves an identical problem. Firm i chooses ℓ_i to maximize

$$\max p_i z_i \ell_i^{\alpha_i} - h\ell_i$$

subject to

$$h\ell_i \leq \chi_i p_i x_i$$

thus, if it is binding then $\chi_i p_i x_i = h\ell_i$, otherwise the firm's optimality condition is $\alpha_i p_i x_i = h\ell_i$. We can therefore write that

$$\phi_i \alpha_i p_i x_i = h\ell_i, \quad \forall i \tag{43}$$

where $\phi_i = \min\left\{1, \frac{\chi_{hi}}{\alpha_i}\right\}$, $\forall i$.

On the other hand, the household's optimal consumption problem is the same as in the no frictions case. Combining conditions ()-() with the firm's conditions (43) yields

$$\begin{aligned} \phi_1 \alpha_1 \beta_2 \beta_3 PY &= h\ell_i \\ \phi_2 \alpha_2 \beta_3 PY &= h\ell_i \\ \phi_3 \alpha_3 PY &= h\ell_i \end{aligned}$$

These conditions, along with the household's optimality condition (34), correspond to equations ()-() in the proposition. QED.

Proof of Lemma 1 A firm which faces taxes on revenues has the following problem.

Proof of Lemma 2 Given our specification for utility in (13), we have that $V'(L)/U'(Y) = Y$.

Part (i) the Frictionless Economy. Thus, in the economy without frictions, we have that

$$\tilde{\alpha}_3 \frac{Y}{\ell_3} = \tilde{\alpha}_2 \frac{Y}{\ell_2} = \tilde{\alpha}_1 \frac{Y}{\ell_1} = Y$$

Thus, we have that

$$\ell_1 = \tilde{\alpha}_1, \quad \ell_2 = \tilde{\alpha}_2, \quad \text{and } \ell_3 = \tilde{\alpha}_3$$

Substituting these values for ℓ into (1) gives us our expression for aggregate output $\bar{Y} = \bar{z} (\tilde{\alpha}_3)^{\tilde{\alpha}_3} (\tilde{\alpha}_2)^{\tilde{\alpha}_2} (\tilde{\alpha}_1)^{\tilde{\alpha}_1}$.

Part (ii) the Horizontal Economy. In the horizontal economy, the unique equilibrium allocation is given by

$$\phi_{h1} \tilde{\alpha}_1 \frac{Y}{\ell_1} = \phi_{h2} \tilde{\alpha}_2 \frac{Y}{\ell_2} = \phi_{h3} \tilde{\alpha}_3 \frac{Y}{\ell_3} = Y \tag{44}$$

Thus, we have that

$$\ell_1 = \phi_{h1} \tilde{\alpha}_1, \quad \ell_2 = \phi_{h2} \tilde{\alpha}_2, \quad \text{and } \ell_3 = \phi_{h3} \tilde{\alpha}_3$$

Substituting these values for ℓ into (1) gives us our expression for aggregate output in the horizontal economy

$$Y_h = \bar{Y} (\phi_{h3})^{\tilde{\alpha}_3} (\phi_{h2})^{\tilde{\alpha}_2} (\phi_{h1})^{\tilde{\alpha}_1}$$

The Vertical Economy. In the vertical economy, the unique equilibrium allocation is given by

$$(\phi_{v1} \phi_{v2} \phi_{v3}) \tilde{\alpha}_1 \frac{Y}{\ell_1} = (\phi_{v2} \phi_{v3}) \tilde{\alpha}_2 \frac{Y}{\ell_2} = (\phi_{v3}) \tilde{\alpha}_3 \frac{Y}{\ell_3} = Y \tag{45}$$

Thus, we have that

$$\ell_1 = \phi_{v1} \phi_{v2} \phi_{v3} \tilde{\alpha}_1, \quad \ell_2 = \phi_{v2} \phi_{v3} \tilde{\alpha}_2, \quad \text{and } \ell_3 = \phi_{v3} \tilde{\alpha}_3$$

Substituting these values for ℓ into (1) gives us our expression for aggregate output in the horizontal economy

$$Y_v = \bar{Y} (\phi_{v3})^{\tilde{\alpha}_3} (\phi_{v2} \phi_{v3})^{\tilde{\alpha}_2} (\phi_{v1} \phi_{v2} \phi_{v3})^{\tilde{\alpha}_1}$$

QED.

Proof of Proposition 3 *The Horizontal Economy.* These expressions follow directly from taking the derivative of (14) with respect to ϕ_{h1}, ϕ_{h2}, and ϕ_{h3}, respectively.

The Vertical Economy. These expressions follow directly from taking the derivative of (15) with respect to ϕ_{v1}, ϕ_{v2}, and ϕ_{v3}, respectively.

Proof of Proposition 4 *The Horizontal Economy.* In the horizontal economy, aggregate output may be written as

$$Y_h = \bar{Y} \left(\theta \varepsilon_{h3}\right)^{\tilde{\alpha}_3} \left(\theta \varepsilon_{h2}\right)^{\tilde{\alpha}_2} \left(\theta \varepsilon_{h1}\right)^{\tilde{\alpha}_1}$$

Taking the derivative of this with respect to θ, we get that

$$\frac{d \log Y_h}{d \log \theta} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 > 0$$

The Vertical Economy. In the vertical economy, aggregate output may be written as

$$Y_v = \bar{Y} \left(\theta \varepsilon_{v3}\right)^{\tilde{\alpha}_3} \left(\theta^2 \varepsilon_{v2} \varepsilon_{v3}\right)^{\tilde{\alpha}_2} \left(\theta^3 \varepsilon_{v1} \varepsilon_{v2} \varepsilon_{v3}\right)^{\tilde{\alpha}_1}$$

Taking the derivative of this with respect to θ, we get that

$$\frac{d \log Y_v}{d \log \theta} = 3\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \tilde{\alpha}_3$$

It follows that $-d \log Y_v / d \log \theta < -d \log Y_h / d \log \theta < 0$. QED.

Proof of Proposition 5 *The Horizontal Economy.* In the horizontal economy, we have that

$$\begin{aligned}
\ell_3 &= \left(\phi_{h3}\right) \tilde{\alpha}_3 \frac{Y}{V'(L) / U'(C)} \\
\ell_2 &= \left(\phi_{h2}\right) \tilde{\alpha}_2 \frac{Y}{V'(L) / U'(C)} \\
\ell_1 &= \left(\phi_{h1}\right) \tilde{\alpha}_1 \frac{Y}{V'(L) / U'(C)}
\end{aligned}$$

so that aggregate labor is given by

$$L = (\tilde{\alpha}_1 \phi_{h1} + \tilde{\alpha}_2 \phi_{h2} + \tilde{\alpha}_3 \phi_{h3}) \frac{Y}{V'(L)/U'(C)}$$

This implies that

$$\bar{\alpha}\frac{Y}{L} = \frac{\bar{\alpha}}{\tilde{\alpha}_1 \phi_{h1} + \tilde{\alpha}_2 \phi_{h2} + \tilde{\alpha}_3 \phi_{h3}} \frac{V'(L)}{U'(C)}$$

From this, we can again back out the aggregate labor wedge $1 - T_h$, which is given by (16).

The Vertical Economy. In the vertical economy, we have that

$$\begin{aligned}
\ell_3 &= (\phi_{v3})\,\tilde{\alpha}_3 \frac{Y}{V'(L)/U'(C)} \\
\ell_2 &= (\phi_{v2}\phi_{v3})\,\tilde{\alpha}_2 \frac{Y}{V'(L)/U'(C)} \\
\ell_1 &= (\phi_{v1}\phi_{v2}\phi_{v3})\,\tilde{\alpha}_1 \frac{Y}{V'(L)/U'(C)}
\end{aligned}$$

so that aggregate labor is given by

$$L = (\tilde{\alpha}_1 \phi_{v1}\phi_{v2}\phi_{v3} + \tilde{\alpha}_2 \phi_{v2}\phi_{v3} + \tilde{\alpha}_3 \phi_{v3}) \frac{Y}{V'(L)/U'(C)}$$

This implies that

$$\bar{\alpha}\frac{Y}{L} = \frac{\bar{\alpha}}{\tilde{\alpha}_1 \phi_{v1}\phi_{v2}\phi_{v3} + \tilde{\alpha}_2 \phi_{v2}\phi_{v3} + \tilde{\alpha}_3 \phi_{v3}} \frac{V'(L)}{U'(C)}$$

From this, we can again back out the aggregate labor wedge $1 - T_h$, which is given by (15). QED.

Proof of Proposition 6 *The Horizontal Economy.* In the horizontal economy, we first have that labor is given by $L = \tilde{\alpha}_1 \phi_{h1} + \tilde{\alpha}_2 \phi_{h2} + \tilde{\alpha}_3 \phi_{h3}$. Hence, labor productivity and TFP are respectively given by

$$\frac{Y}{L} = \frac{\bar{Y}(\phi_{h3})^{\tilde{\alpha}_3}(\phi_{h2})^{\tilde{\alpha}_2}(\phi_{h1})^{\tilde{\alpha}_1}}{\tilde{\alpha}_1 \phi_{h1} + \tilde{\alpha}_2 \phi_{h2} + \tilde{\alpha}_3 \phi_{h3}} \quad \text{and} \quad \frac{Y}{L^{\bar{\alpha}}} = \frac{\bar{Y}(\phi_{h3})^{\tilde{\alpha}_3}(\phi_{h2})^{\tilde{\alpha}_2}(\phi_{h1})^{\tilde{\alpha}_1}}{(\tilde{\alpha}_1 \phi_{h1} + \tilde{\alpha}_2 \phi_{h2} + \tilde{\alpha}_3 \phi_{h3})^{\bar{\alpha}}}$$

Writing this in terms of idiosyncratic and aggregate components, we have that

$$\frac{Y}{L} = \frac{\bar{Y}(\varepsilon_{h3})^{\tilde{\alpha}_3}(\varepsilon_{h2})^{\tilde{\alpha}_2}(\varepsilon_{h1})^{\tilde{\alpha}_1}}{\tilde{\alpha}_1 \varepsilon_{h1} + \tilde{\alpha}_2 \varepsilon_{h2} + \tilde{\alpha}_3 \varepsilon_{h3}}\theta^{\tilde{\alpha}-1} \quad \text{and} \quad \frac{Y}{L^{\bar{\alpha}}} = \frac{\bar{Y}(\varepsilon_{h3})^{\tilde{\alpha}_3}(\varepsilon_{h2})^{\tilde{\alpha}_2}(\varepsilon_{h1})^{\tilde{\alpha}_1}}{(\tilde{\alpha}_1 \varepsilon_{h1} + \tilde{\alpha}_2 \varepsilon_{h2} + \tilde{\alpha}_3 \varepsilon_{h3})^{\bar{\alpha}}}\theta^{\tilde{\alpha}-\bar{\alpha}}$$

This implies that

$$-\frac{d\log\left(Y/L\right)}{d\log\theta}=1-\bar{\alpha}>0$$

and

$$-\frac{d\log\left(Y/L^{\bar{\alpha}}\right)}{d\log\theta}=0$$

The Vertical Economy. In the vertical economy, we first have that labor is given by $L=\tilde{\alpha}_1\phi_{v1}\phi_{v2}\phi_{v3}+\tilde{\alpha}_2\phi_{v2}\phi_{v3}+\tilde{\alpha}_3\phi_{v3}$. Hence, labor productivity and TFP are respectively given by

$$\frac{Y}{L}=\frac{\bar{Y}\left(\phi_{v3}\right)^{\tilde{\alpha}_3}\left(\phi_{v2}\phi_{v3}\right)^{\tilde{\alpha}_2}\left(\phi_{v1}\phi_{v2}\phi_{v3}\right)^{\tilde{\alpha}_1}}{\tilde{\alpha}_1\phi_{v1}\phi_{v2}\phi_{v3}+\tilde{\alpha}_2\phi_{v2}\phi_{v3}+\tilde{\alpha}_3\phi_{v3}}\quad\text{and}\quad\frac{Y}{L^{\bar{\alpha}}}=\frac{\bar{Y}\left(\phi_{v3}\right)^{\tilde{\alpha}_3}\left(\phi_{v2}\phi_{v3}\right)^{\tilde{\alpha}_2}\left(\phi_{v1}\phi_{v2}\phi_{v3}\right)^{\tilde{\alpha}_1}}{\left(\tilde{\alpha}_1\phi_{v1}\phi_{v2}\phi_{v3}+\tilde{\alpha}_2\phi_{v2}\phi_{v3}+\tilde{\alpha}_3\phi_{v3}\right)^{\bar{\alpha}}}$$

Writing this in terms of idiosyncratic and aggregate components, we have that

$$\frac{Y}{L}=\frac{\bar{Y}\left(\theta\varepsilon_{v3}\right)^{\tilde{\alpha}_3}\left(\theta^2\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_2}\left(\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_1}}{\tilde{\alpha}_1\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_2\theta^2\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\theta\varepsilon_{v3}}\quad\text{and}\quad\frac{Y}{L^{\bar{\alpha}}}=\frac{\bar{Y}\left(\theta\varepsilon_{v3}\right)^{\tilde{\alpha}_3}\left(\theta^2\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_2}\left(\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_1}}{\left(\tilde{\alpha}_1\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_2\theta^2\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\theta\varepsilon_{v3}\right)^{\bar{\alpha}}}$$

Looking at TFP, we have that

$$\frac{Y}{L^{\bar{\alpha}}}=\frac{\bar{Y}\left(\varepsilon_{v3}\right)^{\tilde{\alpha}_3}\left(\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_2}\left(\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_1}}{\left(\tilde{\alpha}_1\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_2\theta^2\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\theta\varepsilon_{v3}\right)^{\bar{\alpha}}}\theta^{3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3}$$

$$\frac{L^{\bar{\alpha}}\bar{Y}\left(\varepsilon_{v3}\right)^{\tilde{\alpha}_3}\left(\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_2}\left(\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}\right)^{\tilde{\alpha}_1}\left(3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3\right)\theta^{3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3-1}-Y\bar{\alpha}L^{\bar{\alpha}-1}\left(3\tilde{\alpha}_1\theta^2\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+2\tilde{\alpha}_2\theta\varepsilon_{v2}\right.}{L^{2\bar{\alpha}}}$$

$$\frac{L^{\bar{\alpha}}Y\left(3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3\right)\theta^{-1}-Y\bar{\alpha}L^{\bar{\alpha}-1}\left(3\tilde{\alpha}_1\theta^2\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+2\tilde{\alpha}_2\theta\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\varepsilon_{v3}\right)}{L^{2\bar{\alpha}}}$$

$$\left(3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3\right)\theta^{-1}-\bar{\alpha}L^{-1}\left(3\tilde{\alpha}_1\theta^2\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+2\tilde{\alpha}_2\theta\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\varepsilon_{v3}\right)>0$$

$$\begin{aligned}\frac{3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3}{\bar{\alpha}}&>\left(\frac{3\tilde{\alpha}_1\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+2\tilde{\alpha}_2\theta^2\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\theta\varepsilon_{v3}}{\tilde{\alpha}_1\theta^3\varepsilon_{v1}\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_2\theta^2\varepsilon_{v2}\varepsilon_{v3}+\tilde{\alpha}_3\theta\varepsilon_{v3}}\right)\\ \frac{3\tilde{\alpha}_1+2\tilde{\alpha}_2+\tilde{\alpha}_3}{\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3}&>\left(\frac{3\tilde{\alpha}_1\theta^3+2\tilde{\alpha}_2\theta^2+\tilde{\alpha}_3\theta}{\tilde{\alpha}_1\theta^3+\tilde{\alpha}_2\theta^2+\tilde{\alpha}_3\theta}\right)\end{aligned}$$

$$
\begin{aligned}
\left(3\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \tilde{\alpha}_3\right)\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) &> \left(3\tilde{\alpha}_1\theta^3 + 2\tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) \\
3\tilde{\alpha}_1\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) + 2\tilde{\alpha}_2\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) + \tilde{\alpha}_3\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) &> 3\tilde{\alpha}_1\theta^3\left(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3\right) + \\
3\tilde{\alpha}_1\left(\tilde{\alpha}_2\theta^2 + \tilde{\alpha}_3\theta\right) + 2\tilde{\alpha}_2\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_3\theta\right) + \tilde{\alpha}_3\left(\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_2\theta^2\right) &> 3\tilde{\alpha}_1\theta^3\left(\tilde{\alpha}_2 + \tilde{\alpha}_3\right) + 2\tilde{\alpha}_2\theta \\
3\tilde{\alpha}_1\tilde{\alpha}_2\theta^2 + 3\tilde{\alpha}_1\tilde{\alpha}_3\theta + 2\tilde{\alpha}_2\tilde{\alpha}_1\theta^3 + 2\tilde{\alpha}_2\tilde{\alpha}_3\theta + \tilde{\alpha}_3\tilde{\alpha}_1\theta^3 + \tilde{\alpha}_3\tilde{\alpha}_2\theta^2 &> 3\tilde{\alpha}_1\theta^3\tilde{\alpha}_2 + 3\tilde{\alpha}_1\theta^3\tilde{\alpha}_3 + 2 \\
\tilde{\alpha}_1\tilde{\alpha}_2\theta^2 + 2\tilde{\alpha}_1\tilde{\alpha}_3\theta + \tilde{\alpha}_2\tilde{\alpha}_3\theta &> \tilde{\alpha}_1\theta^3\tilde{\alpha}_2 + 2\tilde{\alpha}_1\theta^3\tilde{\alpha}_3 + \tilde{\alpha}_2
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\alpha}_1\tilde{\alpha}_2\theta + 2\tilde{\alpha}_1\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{\alpha}_3 &> \tilde{\alpha}_1\theta^2\tilde{\alpha}_2 + 2\tilde{\alpha}_1\theta^2\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{\alpha}_3\theta \\
\tilde{\alpha}_1\tilde{\alpha}_2\theta\left(1-\theta\right) + 2\tilde{\alpha}_1\tilde{\alpha}_3\left(1-\theta^2\right) + \tilde{\alpha}_2\tilde{\alpha}_3\left(1-\theta\right) &> 0 \\
\tilde{\alpha}_1\tilde{\alpha}_2\theta\left(1-\theta\right) + 2\tilde{\alpha}_1\tilde{\alpha}_3\left(1+\theta\right)\left(1-\theta\right) + \tilde{\alpha}_2\tilde{\alpha}_3\left(1-\theta\right) &> 0 \\
\tilde{\alpha}_1\tilde{\alpha}_2\theta + 2\tilde{\alpha}_1\tilde{\alpha}_3\left(1+\theta\right) + \tilde{\alpha}_2\tilde{\alpha}_3 &> 0
\end{aligned}
$$

so it's true. In the vertical economy, implied TFP decreases while labor productivity is non-monotonic

$$\frac{d\log\left(Y/L^{\bar{\alpha}}\right)}{d\log\theta} < 0$$

QED.

Proof of Proposition 7 *The Horizontal Economy.* In the horizontal economy, $h\ell_i = \chi_i p_i x_i$ for all i. Thus, the aggregate amount of liquidity needed is given by the sum of the wage bills, $M_h = h\ell_1 + h\ell_2 + h\ell_3$. We can translate that into real terms. To implement this as an equilbrium, it must satisfy that $h = V'(L)/U'(C)$. Thus

$$M_h = \left(\frac{V'(L)}{U'(C)}\right)\left(\ell_1 + \ell_2 + \ell_3\right)$$

The Vertical Economy. In the vertical economy, $h\ell_1 = \chi_1 p_1 x_1$ for firm 1, whereas for firms 2 and 3, $h\ell_2 + p_1 x_1 = \chi_2 p_2 x_2$ and $h\ell_3 + p_2 x_2 = \chi_3 p_3 x_3$. Thus, the aggregate amount of liquidity is given by

$$M_v = h\ell_1 + \left(h\ell_2 + p_1 x_1\right) + \left(h\ell_3 + p_2 x_2\right)$$

That is, the expenditure on both the wage bills and the intermediate goods. To implement this as an equilibrium, we know that from the firm optimality conditions, we have that $h\ell_2/\alpha_2 = p_1 x_1/\beta_2$ and $h\ell_3/\alpha_3 = p_2 x_2/\beta_3$. Furthermore, the allocation must satisfy $h = V'(L)/U'(C)$. Thus, we can express this as

$$M_v = \left(\frac{V'(L)}{U'(C)}\right)\left[\ell_1 + \left(\ell_2 + \frac{\beta_2}{\alpha_2}\ell_2\right) + \left(\ell_3 + \frac{\beta_3}{\alpha_3}\ell_3\right)\right]$$

It necessarily follows that $M_v > M_h$. QED.

Proof of Proposition 10 *Household's Problem.* The household's optimization problem is fairly straight-forward. The FOCs of the household's problem imply that household expenditure on individual goods is equal across all goods

$$p_i c_i = p_j c_j, \quad \forall i, j$$

Substituting this back into the household budget constraint (21), we have that expenditure on any good is equal to 1/nth of household income.

$$p_i c_i = h/n, \quad \forall i$$

Firm's Problem. The representative firm's profit maximization problem is given by

$$\max p_i z_i \ell_i^{\alpha} \left(\prod_{j=1}^{n} x_{ij}^{\omega_{ij}} \right)^{1-\alpha} - h\ell_i - \sum_{j=1}^{n} p_j x_{ij}$$

subject to the firm's pledgeability constraint $h\ell_i + \sum_{j=1}^{n} p_j x_{ij} \leq \phi_i p_i x_i$.

We can split the firm's problem into two auxiliary problems: (i) a cost minimization problem, then, (ii) profit maximization problem. Consider a firm which chooses to minimize cost subject to producing x_i of the composite. The optimal input use problem is given by,

$$c(x_i) = \min_{x_{ij}} h\ell_i + \sum_{j=1}^{n} p_j x_{ij} \tag{46}$$

subject to

$$x_i = z_i \ell_i^{\alpha} \left(\prod_{j=1}^{n} x_{ij}^{\omega_{ij}} \right)^{1-\alpha}$$

Let λ_i be the Lagrange Multiplier associated with the constraint. The first order conditions from this problem imply

$$\frac{h\ell_i}{\alpha} = \frac{p_j x_{ij}}{(1-\alpha)\,\omega_{ij}} \quad \text{and} \quad \frac{p_j x_{ij}}{p_k x_{ik}} = \frac{\omega_{ij}}{\omega_{ik}}, \quad \forall j, k \tag{47}$$

From this, we have that the share of firm i's expenditure on the input produced by firm j is proportional to its share in production. One can substitute this relationship into the the cost function (46) to obtain the relation between the sector i's intermediate input and the j

sector input,

$$c(x_i) = \frac{1}{\alpha} h \ell_i \quad \text{and} \quad c(x_i) = \frac{1}{(1-\alpha)\,\omega_{ij}} p_j x_{ij} \tag{48}$$

Finally, the firm's financial constraint implies that $c(x_i) = \phi_i p_i x_i$. This implies

$$\begin{aligned} \ell_i &= \phi_i \alpha \frac{1}{h} p_i x_i \\ x_{ij} &= \phi_i (1-\alpha)\,\omega_{ij} \frac{1}{p_j} p_i x_i \end{aligned}$$

Substituting these values back into the production function (22) and taking logs gives us

$$\begin{aligned} \alpha \log h &= \log z_i + \log \phi_i + \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \log p_i \\ &\quad + (1-\alpha) \sum_{j=1}^{n} \omega_{ij} \log \omega_{ij} - (1-\alpha) \sum_{j=1}^{n} \omega_{ij} \log p_j \end{aligned} \tag{49}$$

Let us define the following vectors

$$\mathbf{z} = \begin{bmatrix} \log z_1 \\ \log z_2 \\ \vdots \\ \log z_n \end{bmatrix} \text{ and } \boldsymbol{\phi} = \begin{bmatrix} \log \phi_1 \\ \log \phi_2 \\ \vdots \\ \log \phi_n \end{bmatrix}, \text{ and } \mathbf{p} = \begin{bmatrix} \log p_1 \\ \log p_2 \\ \vdots \\ \log p_n \end{bmatrix}$$

Then stacking equation (49) upon one another where each row i corresponds to equation (49) for sector i, we have that

$$\mathbf{1}\alpha \log h = (\mathbf{z} + \boldsymbol{\phi}) + (I - (1-\alpha) W)\,\mathbf{p} + \mathbf{B}$$

where $\mathbf{1}$ is a column vector of n ones and

$$\mathbf{B} = \mathbf{1}\left[\alpha \log \alpha + (1-\alpha)\log(1-\alpha)\right] + (1-\alpha) \begin{bmatrix} \sum_{j=1}^{n} \omega_{1j} \log \omega_{1j} \\ \sum_{j=1}^{n} \omega_{2j} \log \omega_{2j} \\ \vdots \\ \sum_{j=1}^{n} \omega_{nj} \log \omega_{nj} \end{bmatrix}$$

Multiplying this equation by the i-th element of the vector $\left(\frac{1}{n}(I-(1-\alpha)W')^{-1}\mathbf{1}\right)'$ gives us

$$\left(\frac{1}{n}(I-(1-\alpha)W')^{-1}\mathbf{1}\right)' \mathbf{1}\alpha \log h = (\mathbf{z} + \boldsymbol{\phi}) + \frac{1}{n}\mathbf{1}'\mathbf{p} + \mathbf{B}$$

$$\log h = v'(\mathbf{z}+\boldsymbol{\phi})+\mu$$

where μ is a constant independent of the vector of shocks given by $\mu = \frac{1}{n}\mathbf{1}'\mathbf{p}+\mathbf{v}'\mathbf{B}$. Note that

$$\frac{1}{n}\mathbf{1}'\mathbf{p} = \frac{1}{n}\sum_{i=1}^{n}\log p_i$$

corresponds to the ideal price index $n\left(p_1 p_2 \cdots p_n\right)^{1/n}$ which we may normalize to be equal to 1.

Finally, real value added in the economy is given by $\sum_{i=1}^{n} p_i c_i = h$. This implies that GDP is given by $y = \log(GDP) = \log h$. we use equation () to derive aggregate gdp in this economy.

In the competitive equilibrium of this economy, the logarithm of GDP is given by

$$\log Y = \log(GDP) = v'(\mathbf{z}+\boldsymbol{\phi})+\mu$$

old stuff

$$\begin{aligned}
\left(\frac{1}{n}\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\mathbf{1}\alpha\log h &= v'(\mathbf{z}+\boldsymbol{\phi})+\mathit{const}+\left(\frac{1}{n}\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\left(I-(1-\alpha)W\right)\mathbf{p}\\
\frac{1}{n}\left(\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\mathbf{1}\alpha\log h &= v'(\mathbf{z}+\boldsymbol{\phi})+\mathit{const}+\frac{1}{n}\mathbf{1}'\left(I-(1-\alpha)W'\right)^{-1\prime}\left(I-(1-\alpha)W\right)\mathbf{p}\\
\frac{1}{n}\left(\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\mathbf{1}\alpha\log h &= v'(\mathbf{z}+\boldsymbol{\phi})+\mathit{const}+\frac{1}{n}\mathbf{1}'\left(I-(1-\alpha)W\right)^{-1}\left(I-(1-\alpha)W\right)\mathbf{p}\\
\frac{1}{n}\left(\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\mathbf{1}\alpha\log h &= v'(\mathbf{z}+\boldsymbol{\phi})+\mathit{const}+\frac{1}{n}\mathbf{1}'\mathbf{p}\\
\left(\frac{1}{n}\left(I-(1-\alpha)W'\right)^{-1}\mathbf{1}\right)'\mathbf{1}\alpha\log h &= v'(\mathbf{z}+\boldsymbol{\phi})+\mathit{const}+\frac{1}{n}\mathbf{1}'\mathbf{p}
\end{aligned}$$

Proof of Lemma 3 Acemoglu et al. show that

$$s' = (h/n)\,\mathbf{1}'\left(I-(1-\alpha)W'\right)^{-1} = hv'$$

whereas with financial frictions we have that

$$s' = (h/n)\,\mathbf{1}'\left(I-(1-\alpha)W'\Phi'\right)^{-1}$$

We'll first show the Acemoglu result then show how this doesn't apply in our economy.

The market clearing condition for commodity i is given by

$$c_i + \sum_{j=1}^{n} x_{ji} = x_i$$

Multiplying this by p_i we have that

$$p_i c_i + p_i \sum_{j=1}^{n} x_{ji} = p_i x_i$$

Plugging in consuption levels and firms' input demands, we have that

$$\frac{h}{n} + (1-\alpha) \sum_{j=1}^{n} w_{ji} p_j x_j = p_i x_i$$

Letting $s_i = p_i x_i$ denote sales, we have that

$$s_i = \frac{h}{n} + (1-\alpha) \sum_{j=1}^{n} w_{ji} s_j$$

stacking, we have that

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1}\frac{h}{n} + (1-\alpha) \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & & & \\ w_{1n} & & & w_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

we can thus write

$$\mathbf{s} = \mathbf{1}\frac{h}{n} + (1-\alpha) W' \mathbf{s}$$

where

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

taking the transpose of this, we have that

$$\mathbf{s}' = \mathbf{1}'\frac{h}{n} + (1-\alpha) \mathbf{s}' W$$

rearranging, we have that

$$\mathbf{s}' = \frac{h}{n}\mathbf{1}'\left(I-(1-\alpha)W\right)^{-1} = hv'$$

In our paper, when we plug in firm optimality conditions () into (), we get that

$$\frac{h}{n} + (1-\alpha)\sum_{j=1}^{n}\phi_j w_{ji} p_j x_j = p_i x_i$$

Thus

$$s_i = \frac{h}{n} + (1-\alpha)\sum_{j=1}^{n}\phi_j w_{ji} s_j$$

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1}\frac{h}{n} + (1-\alpha)\begin{bmatrix} \phi_1 w_{11} & \phi_2 w_{21} & \cdots & \phi_n w_{n1} \\ \phi_1 w_{12} & \phi_2 w_{22} & \cdots & \phi_n w_{n2} \\ \vdots & & & \\ \phi_1 w_{1n} & & & \phi_n w_{nn} \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

we may rewrite this as

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \mathbf{1}\frac{h}{n} + (1-\alpha)\begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & & & \\ w_{1n} & & & w_{nn} \end{bmatrix}\begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \phi_n \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Therefore

$$\mathbf{s} = \mathbf{1}\frac{h}{n} + (1-\alpha)W'\Phi\mathbf{s}$$

where

$$\Phi = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \phi_n \end{bmatrix}$$

taking the transpose of this, we have that

$$\mathbf{s}' = \mathbf{1}'\frac{h}{n} + (1-\alpha)\mathbf{s}'\Phi'W$$

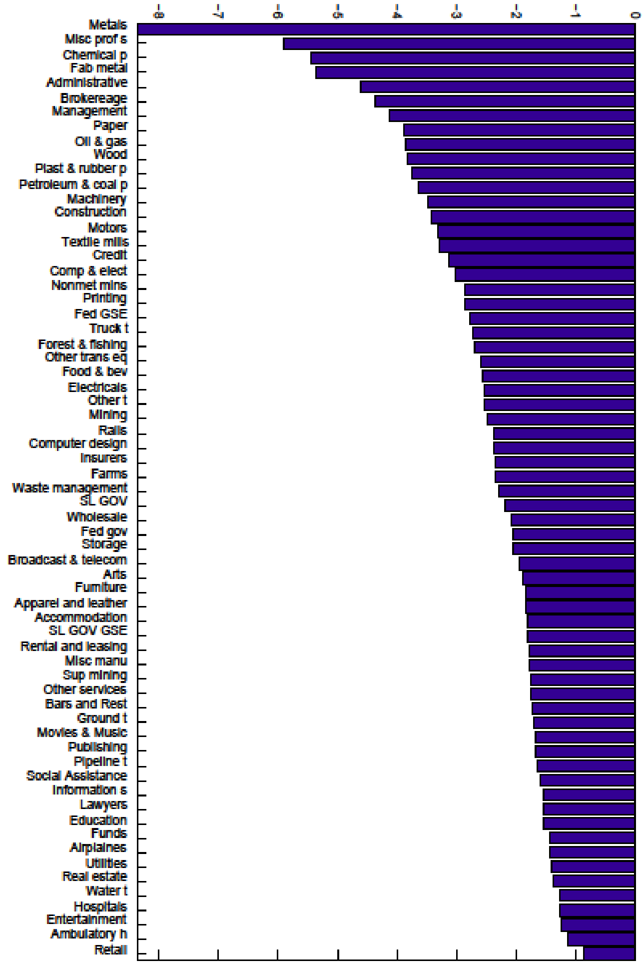


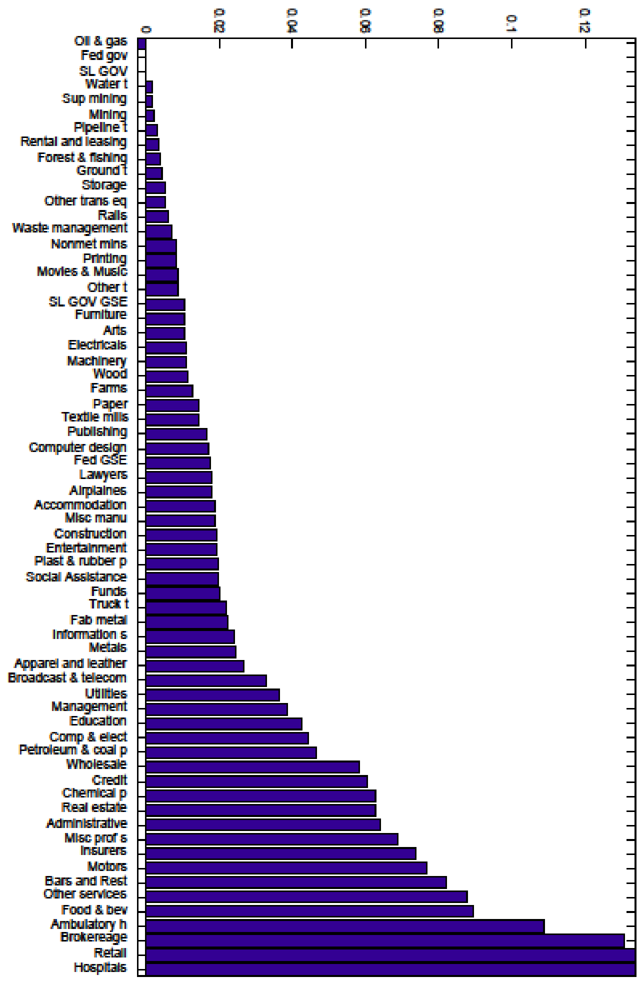
Figure 16: Sectors affected by Liquidity Shock

rearranging, we have that

$$\mathbf{s}' = \frac{h}{n}\mathbf{1}'\left(I - (1-\alpha)\,\Phi' W\right)^{-1}$$

7 Appendix B: Calibration Proofs and Analysis

8 Appendix C: Additional Calibration Figures


0
0.02
0.04
0.06
0.08
0.1
0.12
Oil & gas
Fed gov
SL GOV
Water t
Sup mining
Mining
Pipeline t
Rental and leasing
Forest & fishing
Ground t
Storage
Other trans eq
Rails
Waste management
Nonmet mins
Printing
Movies & Music
Other t
SL GOV GSE
Furniture
Arts
Electricals
Machinery
Wood
Farms
Paper
Textile mills
Publishing
Computer design
Fed GSE
Lawyers
Airplaines
Accommodation
Misc manu
Construction
Entertainment
Plast & rubber p
Social Assistance
Funds
Truck t
Fab metal
Information s
Metals
Apparel and leather
Broadcast & telecom
Utilities
Management
Education
Comp & elect
Petroleum & coal p
Wholesale
Credit
Chemical p
Real estate
Administrative
Misc prof s
Insurers
Motors
Bars and Rest
Other services
Food & bev
Ambulatory h
Brokereage
Retail
Hospitals