Relational Incentive Contracts with Persistent Private Information

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This version: August 26, 2011

*I would like to thank participants in the Nuffield Theory Workshop at Oxford for valuable comments and Leverhulme Trust Major Research Fellowship F08519B for financial support of this research.

Abstract

This paper investigates relational incentive contracts with private information about agent types drawn from an interval and persistent over time. For a sufficiently productive relationship, a pooling contract exists in which all agent types continuing the relationship choose the same action. Necessary and sufficient conditions are given for some separation to be feasible; the parties can then do better than with full pooling. When future actions are optimal, however, separation of *all* types is never possible; the finest separation achievable is into partitions containing a non-degenerate interval of types. Separation always involves lower output initially than after separation has occurred.

Keywords: Relational incentive contracts, private information, efficiency wages
JEL classification: C73, D82, D86

1 Introduction

In a variety of economic contexts, agents of different types are pooled together in groups, with those within each group persistent over time and all treated the same despite their differences. Employees, for example, are grouped in grades, with those in a grade all paid the same. Toyota, as described by Asanuma (1989), places its suppliers into a small number of categories that receive differential treatment. In these examples, pooling of types is only partial because the different grades or categories are treated differently. Moreover, the set of agents in a particular pool is persistent over time.

Relational contract models are increasingly seen as a fruitful way to analyse on-going employment and supply relationships, see Malcomson (1999) on employment, Malcomson (2012) on supply relationships. A relational contract is an agreement, at least part of which is not legally enforceable, in which the on-going relationship between the parties plays an essential role in determining what happens. This paper shows that, with a continuum of privately-observed, persistent agent types, pooling of types is an inherent characteristic of relational contracts that provide incentives for actions when the parties cannot commit themselves to behave sub-optimally in the future. Moreover, when there is sufficient difference between types for some separation to occur, there are multiple pools each containing a non-degenerate interval of types and with the set of agents in each persistent over time, as in the examples of employment and Toyota suppliers.

Partial pooling of agent types arises in the "hidden information" relational contract model of Levin (2003) in which the privately-observed agent types are *iid* random draws each period. In that model, unlike in the model studied here, pooling does not necessarily occur. Moreover, when it does, it takes a particular form. Specifically, there is a single pool consisting of an interval of types that always includes the most productive. If that single pool does not include all types, there is an interval of less productive types each of which is separated. In particular, there are never separate pools each containing a non-degenerate interval of types. Moreover, because types are *iid* random draws each period, persistence of a particular agent in a particular pool is not systematic. Yang (2009) considers persistent types but allows for just two, so there is no possibility of multiple pools containing more than one type. Athey and Bagwell (2008) analyse a model of collusion between firms in an oligopoly in which cost shocks are both private information and persistent. But collusion between firms has very different characteristics from employment or supply relationships. In particular, only one side of the market participates in the relational contract and monetary payments are not used because they make breach of antitrust rules more apparent.

The paper closest in spirit to the present paper is MacLeod and Malcomson (1988), which also analyses relational contracts with a continuum of persistent, privately-observed agent types that are partitioned into separate pools. There, however, the assumptions of the model limit the potential rewards and punishments to such an extent that full separation of types could not be sustained even if agents could be induced to fully reveal their types. The model in the present paper does away with those limitations on rewards and punishments, thus establishing that partial pooling of types is fundamental to relational contract models with a continuum of persistent agent types that are private information.

Partial pooling of types can also occur in models of procurement with private information about agent types, see Laffont and Tirole (1993). In those models, such pooling arises because the principal (the regulator) accepts less efficient procurement in order to extract rent from the agent (the provider). If rent extraction were of no concern (for example, if the principal valued rents to the provider equally with rents to others) or could be achieved by payments between the parties *before* the agent learns her type, the principal would wish to commit to a fixed price contract, which would induce the agent to act efficiently. There would then be full, not partial, pooling of *all* types that actually provide. In the model used here, partial pooling is more fundamental in the sense that it arises even when rents can be redistributed by upfront payments before types are learnt.

In the model used here, the agent's type affects the cost of supplying effort to the principal and is persistent over time. It is specific to the relationship with that particular principal and information private to the agent. This framework corresponds to an extension of the classic model in Shapiro and Stiglitz (1984) to private information about the agent worker's disutility of effort, though it also allows for a continuous, not just binary, effort choice, as in MacLeod and Malcomson (1989). In addition, the types of both principal and agent may each change to one that makes a mutually beneficial continued relationship infeasible. Deviation by one party is taken by the other as a signal of a change to such a type, so there is then no need for explicit punishments in order to sustain equilibria, unlike in Shapiro and Stiglitz (1984) where employees are induced to work hard because they will be punished by being fired if they do not. Thus the criticism by Bewley (1995) that such punishment does not correspond to actual behaviour does not apply. Bewley (1995) summarizes the views expressed to him by managers on this issue as follows: "Good management practice uses punishment largely as a way to weed out bad characters and incompetents and to protect the group from malefactors. Many managers stress that punishment should rarely be used as a way to obtain cooperation." In the present model, each party responds optimally to what it believes is the other's type and the match comes to an end only when at least one of them believes that types are such that a mutually beneficial relationship is no longer even potentially feasible. That captures behaviour of employers in dismissing employees only because they believe they are "bad characters".

In the present model, provided the relationship is sufficiently productive, there always exists an equilibrium relational contract with full pooling of all agent types for which the relationship continues. With such a contract, the agent ends the relationship if the cost of effort is above a critical value but otherwise provides the same effort independent of type and the principal pays the same remuneration to all agent types who continue the relationship. So it is always an equilibrium for employers to expect the same amount of work from employees with different characteristics and to pay them the same. If further separation of agent types is feasible, such a contract is, however, always dominated by one with some separation among those types who continue the relationship. But, with a relational contract in which future actions are optimal, *full* separation of continuing agent types is never feasible and it may not be possible to achieve any separation of continuing types at all — necessary and sufficient conditions for that are given. The essential reason is that an agent obtains an informational rent, both now and in the future, by pretending to be a less productive type, whereas in a relational contract with optimal future actions an agent revealing actual type loses the future rent. That loss of future rent is greater for more productive types so, to induce separation, current effort must increase faster with type productivity than if there were no loss of future rent. It turns out that separation requires a discrete jump in effort between separated types and, as a result, only partial separation is possible. A workplace will, therefore, always have some employees with different characteristics who provide the same effort and receive the same remuneration. As with relational contracts without private information, however, it is not in general possible even for separated types to sustain the efficient level of effort that would be possible if the relational contract could be replaced by a legally enforceable one. Moreover, during the initial period of a relational contract with some separation of types, effort for all but the least productive type is below the level that could be sustained without private information. So there is an additional cost to information being private except "at the bottom", that is, for the least productive feasible matches. As in Watson (1999) and Watson (2002), the relationship starts out "small" but here it is for the purpose of building up trust, in the sense of convincing the other party about type. The increase in effort after the initial period also implies that remuneration increases except in the least productive of feasible relationships. As in MacLeod and Malcomson (1989) and Levin (2003), remuneration consists of two components, a component that does not depend on performance and a bonus that does. After the initial period of a separating relational contract, it is optimal for the former to be set just to compensate the agent each period for the opportunity cost of working for the principal and the rest to take the form of a bonus.

The structure of the paper is as follows. Section 2 sets out the model. Section 3 derives incentive compatibility conditions for the actions of the agent and the principal in a relational contract. Section 4 derives equilibrium conditions for relational contracts and results on pooling contracts. Section 5 analyses continuation contracts that are optimal following revelation of information about the agent's type. Section 6 studies relational contracts with some separation of types and shows that these exhibit partial pooling. Section 7 contains concluding remarks. Proofs of propositions are in an appendix.

2 Model

A principal uses an agent to carry out a specific task each period. The relationship between the two can, in principle, continue indefinitely. At each date t, the principal is one of two persistent types, $p_t \in \{0, 1\}$. Persistence takes the form that, if $p_t = 1$, then $p_{t+1} = 1$ with probability $\rho_p \in (0, 1)$ and $p_{t+1} = 0$ with probability $1 - \rho_p$. If $p_t = 0$, the principal's type remains that for ever after. The probability that p_1 starts equal to 1 is $\tilde{\rho}_p \in (0, 1)$. The principal's payoff in period t if matched with the agent is $p_t e_t - w_t$, where $e_t \in [0, \bar{e}]$ is the agent's effort in period t and w_t is payment to the agent in period t. Effort e_t cannot be verified by third parties, so a legally enforceable agreement for the performance of the task is not available. It can be thought of as anything unverifiable the agent may do that affects the payoff to the principal. The principal's payoff from not being matched with the agent for period t is $\underline{v} \geq 0$.

The agent's payoff in period t if matched with the principal is $w_t - c(e_t, a_t)$, where $c(e_t, a_t)$ is the cost of effort e_t to agent type $a_t \in [\underline{a}, \overline{a}]$, with a_t observed only by the agent. The agent's type in period 1 is distributed $F(a)$. It has persistence of the following form. If $a_t \in (\underline{a}, \overline{a}]$, then $a_{t+1} = a_t$ with probability $\rho_a \in (0, 1)$ and $a_{t+1} = \underline{a}$ with probability $1 - \rho_a$. If $a_t = \underline{a}$, the agent's type remains that for ever after. The agent's

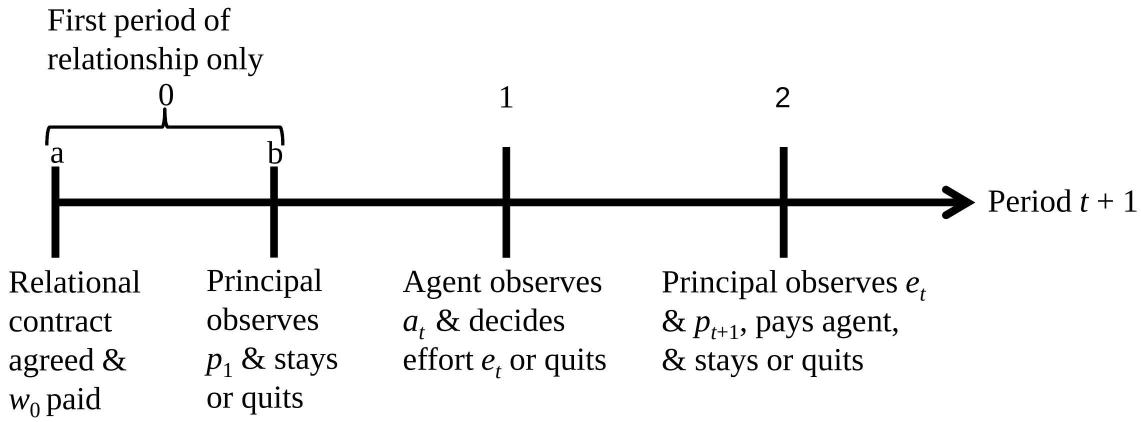

Figure 1: Timing of events in period t

payoff from choosing not to be matched with the principal for period t is $\underline{u} > 0$, which ensures that it is always more efficient for the parties to end the relationship than to continue it with the agent not putting in effort at any future date. Principal and agent share the common discount factor δ. The function c is assumed to have the following conventional properties.

Assumption 1 *For all $a \in [\underline{a}, \overline{a}]$: (1) $c(0, a) = 0$ and $c(\overline{e}, a)$ is bounded above; (2) for all $\tilde{e} \in [0, \overline{e}]$, $c(\tilde{e}, a)$ is twice continuously differentiable, with $c_1(\tilde{e}, a) > 0$, $c_2(\tilde{e}, a) \leq 0$ with strict inequality for $\tilde{e} \in (0, \overline{e}]$, $c_{11}(\tilde{e}, a) > 0$, and $c_{12}(\tilde{e}, a) < 0$; (3) $c_1(0, a) < 1$ and $c_1(\overline{e}, a) > 1$. Moreover, $c(\tilde{e}, \underline{a}) > \tilde{e} - (\underline{u} + \underline{v})$ for all $\tilde{e} \in (0, \overline{e}]$.*

Part 3 of Assumption 1 ensures that efficient effort is strictly interior to $[0, \overline{e}]$ for all a when $p_t = 1$. The final part ensures that continuation of the relationship cannot be mutually beneficial for $a = \underline{a}$.

The timing of events within period t is given in Figure 1. If t is the first period of the relationship ($t = 1$), the principal and the agent first decide (at stage 0a) whether to agree a relational contract (to be formally defined shortly) and, if they do, make the initial payment w_0 that is part of that contract. Then the principal (at stage 0b) observes p_1 and decides whether to continue the relationship. The other stages are the same for all periods. At stage 1, the agent observes a_t and either incurs effort e_t or ends the relationship. At stage 2, the principal observes e_t and p_{t+1}, pays the agent and decides whether to continue the relationship.

MacLeod and Malcomson (1989) and Levin (2003) allow for payment to have two components, a fixed component that is paid conditional on the relationship continuing but not conditional on the agent's output and a bonus that can be conditioned on output. As Levin (2003) shows, having these two components is important for the analysis. They are incorporated here by allowing the parties to commit to payment of fixed compensation $\underline{w}_t$ conditional only on the relationship being continued by both parties for period t. The difference $w_t - \underline{w}_t$ is then the bonus component paid by the principal that can be conditioned on the agent's output in period t. There is no restriction on the magnitude or sign of $\underline{w}_t$ (a negative amount would involve the agent paying the principal) but, to avoid the complication of adding a decision by the agent of whether to accept the bonus, $w_t - \underline{w}_t$ is restricted to being non-negative. (This restriction does not restrict the set of payoffs attainable with equilibrium relational contracts.)

Let $h_t = h_{t-1} \cup (e_{t-1}, w_{t-1})$, for $t \geq 2$, with $h_1 = \{w_0\}$, denote the commonly observed history at stage 1 of period t conditional on the relationship not having ended before then. At that stage, the agent can condition actions on (a_t, h_t). (The agent could also condition action on a_τ for $\tau < t$ but, since the only alternative to $a_\tau = a_t$ is $a_\tau = \underline{a}$ and the game would then have ended at τ, no purpose is served by extending the notation to allow for that.) A strategy σ^a for the agent consists of a decision rule for whether to accept w_0, a decision rule $\gamma_t^a(a_t, h_t) \in \{0, 1\}$ for each t for whether to continue the relationship at stage 1, and an effort choice $e_t(a_t, h_t)$ for each t conditional on continuation. At stage 2 of period t, the principal can condition actions on p_{t+1} and (h_t, e_t). A strategy σ^p for the principal consists of a decision rule for whether to pay w_0 (before observing p_1), a decision rule $\tilde{\gamma}_0^p(w_0, p_1) \in \{0, 1\}$ for whether to continue the relationship at stage 0b of period 1 after observing p_1, a decision rule $\tilde{\gamma}_t^p(h_t, e_t, p_{t+1}) \in \{0, 1\}$ for each t for whether to continue the relationship at stage 2, and a payment choice $w_t(h_t, e_t)$ for each t conditional on continuation. (If ending the relationship, it is always a best response not to pay a bonus.) Formally, a *relational contract* is a w_0, a $\underline{w}_t$ for each t, and a strategy pair (σ^p, σ^a).

The joint payoff to the principal and the agent from being matched in period t conditional on (p_t, a_t) is given by $p_t e_t - c(e_t, a_t)$. *Efficient* effort maximises this joint payoff. Conditional on $p_t = 0$, efficient effort is zero for all a. Conditional on $p_t = 1$, efficient effort $e^*(a_t)$ is, under Assumption 1, uniquely determined by

$$c_1(e^*(a_t), a_t) = 1 \tag{1}$$

and satisfies $e^*(a_t) \in (0, \bar{e})$ for all a_t. The maximal *joint gain* from being matched in period t conditional on a_t and $p_t = 1$ is

$$s^*(a_t) = e^*(a_t) - c(e^*(a_t), a_t) - (\underline{u} + \underline{v}). \tag{2}$$

By Assumption 1, this is strictly negative for $a_t = \underline{a}$. So, if $\underline{a}$ occurs (just as if $p_t = 0$ occurs), no mutually beneficial future relationship is feasible.

The natural equilibrium concept for this game is perfect Bayesian equilibrium in the strategies of the parties to the relational contract. To avoid the measurability details that can arise with mixed strategies when action spaces are continuous (see Mailath and Samuelson (2006, Remark 2.1.1)), attention is restricted to pure strategies. In a Bayesian equilibrium, beliefs about the other party's type when an event occurs that is on the equilibrium path for some type are defined by Bayes' rule. For an action that is not on the equilibrium path for *any* type, it is assumed that a party who takes such an action is believed to be of the lowest type, $\underline{a}$ in the case of the agent, 0 in the case of the principal. For brevity, a relational contract that is a perfect Bayesian equilibrium with these beliefs is referred to as an *equilibrium relational contract*. Such contracts are also referred to in the literature as *self-enforcing*.

3 Incentive compatibility

This section analyses the relational contracts that are incentive compatible for principal and agent. Start with the agent. For a given relational contract, let $\gamma_t^p(h_t, e_t)$ denote the probability that the principal will, given history h_t and effort e_t, continue the relationship after observing p_{t+1} (that is, the probability $\tilde{\gamma}_t^p(h_t, e_t, p_{t+1}) = 1$). Also, let

$A_t(h_t)$ denote the set of agent types a with history h_t at t. For a best response effort, the payoff gain $U_t(a, h_t)$ to agent type $a \in A_t(h_t)$ of continuing the relationship at stage 1 of period t given history h_t is

$$\begin{aligned} U_t(a, h_t) = \max_{\tilde{e} \in [0, \bar{e}]} \Big\{ -c(\tilde{e}, a) - \underline{u} + \underline{w}_t + \gamma_t^p(h_t, \tilde{e}) \Big[w_t(h_t, \tilde{e}) \\ - \underline{w}_t + \delta\rho_a \max\left\{0, U_{t+1}(a, (h_t, \tilde{e}, w_t(h_t, \tilde{e})))\right\} \\ + \delta(1 - \rho_a) \max\left\{0, U_{t+1}(\underline{a}, (h_t, \tilde{e}, w_t(h_t, \tilde{e})))\right\} \Big] \Big\}. \end{aligned} \tag{3}$$

(Explicit dependence of payoff gains on the contract is suppressed in the notation to avoid making it unnecessarily cumbersome.) The interpretation is as follows. A type a agent continuing the relationship for period t and choosing effort $\tilde{e}$ incurs cost of effort $c(\tilde{e}, a)$, forgoes utility $\underline{u}$ available if not matched with the principal, and receives payment $\underline{w}_t$. With probability $\gamma_t^p(h_t, \tilde{e})$ the principal continues the relationship, in which case the principal pays the bonus $w_t(h_t, \tilde{e}) - \underline{w}_t$. In that case, the agent receives expected gain from the future of $U_{t+1}(a, (h_t, \tilde{e}, w_t(h_t, \tilde{e})))$ if (1) type does not change (which happens with probability ρ_a) and (2) this expected future gain is non-negative, so it is worth continuing the relationship. With probability $(1 - \rho_a)$, the agent's type changes to $\underline{a}$ at $t + 1$ with future gain of $\max[0, U_{t+1}(\underline{a}, (h_t, \tilde{e}, w_t(h_t, \tilde{e})))]$.

It is individually rational for agent type a to continue the relationship for period t conditional on h_t only if

$$U_t(a, h_t) \geq 0 \tag{4}$$

and end it only if

$$U_t(a, h_t) \leq 0. \tag{5}$$

(4) serves to anchor the lowest payoff gain the agent can achieve from a relational contract. But there is another lower bound implied by (3), specifically $-\underline{u} + \underline{w}_t$, because the agent could always continue the relationship and guarantee that payoff by setting $e_t = 0$. It is convenient to combine this with the individual rationality conditions (4) and (5). Because $U_t(a, h_t)$ is necessarily non-decreasing in a, incentive compatibility requires continuation of the relationship to be determined by a cutoff level $\alpha_t(h_t)$ for each t such that all types above the cutoff continue the relationship and none below do. Together these conditions imply

$$\begin{aligned} U_t(\alpha_t(h_t), h_t) &\geq \max[0, \underline{w}_t - \underline{u}], \text{ all } h_t, t; \\ U_t(a, h_t) &\leq 0, \text{ for } a < \alpha_t(h_t), \text{ all } a \in A_t(h_t), \text{ all } h_t, t; \\ \alpha_t(h_t) &= \underline{a} \text{ for all } h_t \text{ if } \underline{w}_t > \underline{u}. \end{aligned} \tag{6}$$

Consistent with (6), the agent's decision rule for continuation of the relationship is

$$\gamma_t^a(a, h_t) = \begin{cases} 1, & \text{if } a \geq \alpha_t(h_t); \\ 0, & \text{otherwise;} \end{cases} \quad \text{for all } a \in A_t(h_t), \text{ all } h_t, t. \tag{7}$$

For stage 0a of the first period of the relationship, neither party has information about p and a beyond their initial distributions. With w_0 the payment at this stage, the agent's initial payoff gain from agreeing a relational contract is

$$w_0 + \gamma_0^p(w_0) \int_{\alpha_1(h_1)}^{\bar{a}} U_1(\tilde{a}, h_1)\, dF(\tilde{a}), \tag{8}$$

where $\gamma_0^p(w_0)$ is the probability the principal continues the relationship at stage 0b of the first period of the relationship (after observing p_1 and given payment w_0), that is, the probability $\tilde{\gamma}_0^p(w_0, p_1) = 1$. The agent will enter into a relational contract only if the payoff gain in (8) is non-negative.

For notational convenience in what follows define, for a given relational contract,

$$\begin{aligned}\tilde{U}_t(a', a, h_t) = &-c(e_t(a', h_t), a) \\ &+ \delta\rho_a \gamma_t^p(h_t, e_t(a', h_t)) \max\{0, U_{t+1}(a, (h_t, e_t(a', h_t), w_t(h_t, e_t(a', h_t))))\}, \\ &\text{for all } a, a' \in A_t(h_t), \text{ all } h_t. \end{aligned} \tag{9}$$

This consists of the components of the maximand in (3) that depend on the agent's actual type a evaluated at the effort for type a' specified by $e_t(a', h_t)$.

Proposition 1 *Necessary conditions for $e_t(a, h_t)$ in a relational contract to satisfy the agent's incentive condition (3) at t for all $a \in A_t(h_t)$ are that*

$$\begin{aligned}\tilde{U}_t(a', a', h_t) - \tilde{U}_t(a', a, h_t) &\geq U_t(a', h_t) - U_t(a, h_t) \\ &\geq \tilde{U}_t(a, a', h_t) - \tilde{U}_t(a, a, h_t), \text{ for all } a, a' \in A_t(h_t). \end{aligned} \tag{10}$$

For a relational contract for which the principal ends the relationship if the agent is believed to be type $\underline{a}$, these conditions, together with $e_t(\underline{a}, h_t) = 0$, are sufficient.

The results in Proposition 1 can be directly related to the standard results familiar from mechanism design problems for one-period models. A one-period model is equivalent to having $\delta = 0$ so $\tilde{U}_t(a', a, h_t) = -c(e_t(a', h_t), a)$. For that case, the standard procedure is to divide all terms in (10) by $a' - a$ and take the limit as $a' \to a$ to get a condition on the derivative $c_2(e_t(a, h_t), a)$ that is used to construct the difference between the payoffs of different types and also, given $c_{12} < 0$, to establish the requirement that $e_t(a, h_t)$ is non-decreasing in a. Here the additional terms in $\tilde{U}_t(a', a, h_t)$ take account of the future consequences for the continuation contract from $t + 1$ on of an agent of type a choosing the effort corresponding to type a' at t. The derivative formulation is less useful here because it turns out that, for some continuation contracts, the additional terms in $\tilde{U}_t(a', a, h_t)$ are not differentiable in a at $a' = a$. The assumption that $c_{12} < 0$ does, however, ensure that the following result holds.

Proposition 2 *Suppose a relational contract specifies $e_\tau(a', h'_\tau) \geq e_\tau(a'', h''_\tau)$ for all $\tau \geq t$, where $a', a'' \in A_t(h_t)$ and h'_τ, h''_τ are the histories at $\tau \geq t$ from choosing the effort paths for a' and a'' respectively from t on when the parties adhere to the relational contract. If choosing $e_\tau(a', h'_\tau)$ for all $\tau \geq t$ yields as high a payoff gain to agent type a' at t as choosing $e_\tau(a'', h''_\tau)$, then it does so for all agent types $a \in A_t(h_t)$ with $a \geq a'$.*

Incentive compatibility for the principal is more straightforward. Suppose at stage 2 of period t, the principal were to deviate from the relational contract by paying $w_t \neq w_t(h_t, e_t)$ but not to end the relationship. Then at $t + 1$, the agent would believe the principal's type to be $p_{t+1} = 0$ and thus that no output would ever be produced in the relationship in the future no matter what effort the agent chose. The parties jointly lose $\underline{u} + \underline{v} > 0$ for each period the relationship continues with no output produced (continuation is a negative sum game), so the agent would believe the principal's best response to be to end the relationship at any time in the future that the agent would gain from its continuation. Thus it would be a best response for the agent to end the relationship at stage 1 of $t+1$ unless $\underline{w}_{t+1} > \underline{u}$, in which case the best response would be to continue it but set $e_{t+1} = 0$. Moreover, these are best responses at any subsequent date for which the relationship is continued. But then the principal cannot do better by continuing the relationship at stage 2 of period t than by ending it. Thus, it is a best response for the principal to pay $w_t(h_t, e_t)$ when observing e_t specified by the relational contract for some agent type at stage 2 of period t as long as continuing the relationship yields a non-negative payoff gain. Let $P_t(a)$ denote the payoff gain to the principal from the relational contract when $p_{t+1} = 1$ conditional on continuing the relationship at stage 2 of period t and the agent being type a. Then, the incentive requirement for the principal to pay $w_t(h_t, e_t)$ when $p_{t+1} = 1$ as specified in the relational contract can be written

$$E_{a|h_t,e_t}\left[P_t(a)\right] \geq 0, \quad \text{for all } h_t, t. \tag{11}$$

It is individually rational for the principal to continue the relationship at t if $p_{t+1} = 1$ and (11) is satisfied. It may also be for $p_{t+1} = 0$ — for example, if $w_t(h_t, e_t) - \underline{w}_t + \underline{w}_{t+1} < -\underline{v}$, the principal can guarantee a strictly positive payoff gain from continuing the relationship at t even if $p_{t+1} = 0$ by setting $w_{t+1} = \underline{w}_{t+1}$ and ending the relationship at stage 2 of $t+1$. Let $\underline{P}_t(a)$ denote the principal's payoff gain from continuing the relationship when $p_{t+1} = 0$ conditional on the agent being type a. Then individual rationality for the principal can be specified as

$$\tilde{\gamma}_t^p(h_t, e_t, p_{t+1}) = \begin{cases} 1, & \text{if } p_{t+1} = 1 \text{ and (11) is satisfied,} \\ & \quad \text{or if } p_{t+1} = 0 \text{ and } E_{a|h_t,e_t}\left[\underline{P}_t(a)\right] > 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{for all } h_t, e_t, p_{t+1}, t. \tag{12}$$

For stage 0a of the first period of the relationship, the principal has no information about p and a beyond their initial distributions. Let P_0 and $\underline{P}_0$ denote the principal's expected payoff gain from continuing the relationship at stage 0b of its first period conditional on $p_{t+1} = 1$ and $p_{t+1} = 0$, respectively, given the initial distribution of a. Then, corresponding to (12),

$$\tilde{\gamma}_0^p(w_0, p_1) = \begin{cases} 1, & \text{if } p_1 = 1 \text{ and } P_0 \geq 0, \text{ or if } p_1 = 0 \text{ and } \underline{P}_0 \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

The principal's initial payoff gain from starting a relational contract is

$$-w_0 + \tilde{\rho}_p \max\left\{0, P_0\right\} + \left(1 - \tilde{\rho}_p\right) \max\left\{0, \underline{P}_0\right\}. \tag{14}$$

Proposition 3 *It is a best response for the principal to end a relational contract at stage 2 of period t if $p_{t+1} = 0$ and $\underline{w}_\tau + \underline{v} \geq 0$ for all $\tau \geq t+1$ or if the agent is believed to be type $a_t = \underline{a}$. In both cases, this is the unique best response if $\underline{w}_\tau + \underline{v} > 0$ for all $\tau \geq t+1$.*

4 Equilibrium relational contracts

The previous section derived incentive compatibility conditions for the actions by the agent and the principal. This section is concerned with the effort functions $e_t(a, h_t)$

and the functions $\alpha_t(h_t)$, which specify the lowest type of agent who will continue the relationship, that can form part of an equilibrium relational contract. Denote a sequence of these, one for each t, by $\langle e, \alpha \rangle$. For pure strategy equilibria, the relational contract and the agent's type fully determine the history h_t at each t of a continuing relationship so, to simplify notation in describing equilibria, the history argument is omitted. Proposition 1 specifies conditions for the payoff gain $U_t(a)$ from a contract to be incentive compatible for the agent. For an equilibrium relational contract, the payoff gains resulting from $\langle e, \alpha \rangle$ must be non-negative for both parties at the start of the relational contract. They must also satisfy the principal's incentive compatibility condition (11) and the individual rationality conditions (6), (12) and (13). But these conditions do not, by themselves, guarantee that those payoffs are feasible given $\langle e, \alpha \rangle$ because the payoffs to the agent and the principal have to be consistent with the total output produced.

To determine what is feasible, it is helpful to specify the joint gain or surplus to the principal and the agent from continuing the relationship. This joint gain can be measured at two points in each period, stage 1 and stage 2. Let $S_t^i(a)$ denote the joint gain from continuing the relationship at stage i of period t for given a conditional on $\langle e, \alpha \rangle$, on $p_t = 1$ for $i = 1$ and on $p_{t+1} = 1$ for $i = 2$. For contracts for which the relationship comes to an end at t for $a_t = \underline{a}$ and for $p_{t+1} = 0$ (which is shown below to be optimal), the two measures can be defined recursively as

$$S_t^1(a) = e_t(a) - c(e_t(a), a) - \underline{u} - \underline{v} + \gamma_t^p(e_t(a))\, S_t^2(a), \quad \text{all } a, t; \tag{15}$$

$$S_t^2(a) = \delta \rho_a \gamma_{t+1}^a(a)\, S_{t+1}^1(a), \quad \text{all } a, t. \tag{16}$$

These joint gains depend only on agent type and effort, not on the individual gains of the parties. The joint gain to starting the relational contract with $\langle e, \alpha \rangle$ and unknown (p_1, a_1) is

$$S_0 = \tilde{\rho}_p \int_{\alpha_1}^{\bar{a}} S_1^1(a)\, dF(a). \tag{17}$$

A necessary condition for a relational contract to start is that this initial joint gain is non-negative. Provided it is, it is always possible to find a w_0 such that the agent's and the principal's initial gains, given by (8) and (14) respectively, are both non-negative.

Feasibility requires that the agent receives whatever part of the joint gain is not received by the principal. It follows from (3) that, for a relational contract that ends at $t+1$ if $a_{t+1} = \underline{a}$,

$$U_t(a) = -c(e_t(a), a) - \underline{u} + \underline{w}_t + \gamma_t^p(e_t(a)) \left[S_t^2(a) - P_t(a) \right], \quad \text{all } a, t. \tag{18}$$

This condition is an accounting identity that must be satisfied by the payoffs. It corresponds to what Levin (2003) calls the *dynamic enforcement constraint*.

Relational contracts with two particular characteristics play an important role in what follows. The first characteristic is stationary effort that, for each a, is the same for all $t \geq \tau$ conditional on the relationship continuing. For stationary effort $e_t(a) = e(a)$ for $t \geq \tau$ if agent type a continues the relationship at t, it follows from (7) that $\gamma_t^a(a) = 1$. It follows from (12) that, if the principal also continues the relationship at stage 2 of period t when $p_{t+1} = 1$, $\tilde{\gamma}_t^p(e(a), p) = 1$ and so $\gamma_t^p(e(a)) = \rho_p$. Then, from (15) and (16),

$$S_t^2(a) = \frac{\delta \rho_a}{1 - \delta \rho_p \rho_a} \left[e(a) - c(e(a), a) - \underline{u} - \underline{v} \right], \quad \text{for all } t \geq \tau - 1. \tag{19}$$

Combined with the dynamic enforcement constraint (18), this gives

$$\begin{aligned}\delta\rho_p\rho_a e(a) - c(e(a), a) - \delta\rho_p\rho_a(\underline{u}+\underline{v}) \\ = \left(1-\delta\rho_p\rho_a\right)\left[U_t(a)+\underline{u}-\underline{w}_t+\rho_p P_t(a)\right], \quad \text{for all } t \geq \tau - 1. \quad (20)\end{aligned}$$

Continuation of the relationship at t for $a_t = \underline{a}$ or for $p_{t+1} = 0$ yields a strictly negative surplus, so any relational contract that allows for either of these has a lower surplus at every previous date than that given by (19). This reduces the highest effort that satisfies the dynamic enforcement constraint (18) at the same time as the individual rationality conditions in (6) and (12) for continuation of the relationship.

The second particular characteristic is effort that is the same for all agents (apart from $\underline{a}$) with the same history. Let

$$A_t^+(h_t) = \{a \mid a \in A_t(h_t), a \neq \underline{a}\}, \quad \text{for all } t,$$

denote the set of a with history h_t at t, excluding $\underline{a}$ (which, because the agent's type may change to $\underline{a}$ at any date, is consistent with every history). Also, let $a_t^-(a) = \min \tilde{a} \in A_t^+(h_t)$ for $a \in A_t^+(h_t)$ denote the lowest agent type, apart from $\underline{a}$, in the set of types with the same history at t as a. For a contract that specifies the same effort $e_t(a)$ for all $a \in A_t^+(h_t)$, the only incentive compatibility condition for the agent as long as $\underline{w} \geq -\underline{v}$ is that choosing that effort provides a non-negative gain to all $a \in A_t^+(h_t)$ from having the relationship continue for $t+1$ because for any other effort it is, by Proposition 3 and given the principal's beliefs, a best response for the principal to end the relationship at t without paying a bonus. That implies

$$U_t(a) \geq \max\left[0, \underline{w}_t - \underline{u}\right], \quad \text{for all } a \in A_t^+(h_t). \quad (21)$$

The following result concerns the parts of contracts applying to date τ on for agents with history h_τ, called *continuation contracts for* h_τ.

Proposition 4 *Consider continuation contracts for h_τ with the following properties:*

1. *all agent types $a \in A_\tau^+(h_\tau)$ continue the relationship and choose the same effort $e_t = e(a_\tau^-(a))$ at $t \geq \tau$ if they believe $p_t = 1$, but otherwise end the relationship;*
2. *the principal continues the relationship at $t \geq \tau$ if $e_t = e(a_\tau^-(a))$ and $p_{t+1} = 1$, but otherwise ends the relationship.*

The following results apply to continuation contracts for h_τ that satisfy these properties:

a. *There exist such continuation contracts that are continuation equilibria at τ if and only if*

$$\delta\rho_p\rho_a e\left(a_\tau^-(a)\right) - c\left(e\left(a_\tau^-(a)\right), a\right) - \delta\rho_p\rho_a(\underline{u}+\underline{v}) \geq 0, \quad \textit{for all } a \in A_\tau^+(h_\tau), \quad (22)$$

or, equivalently,

$$S_t^2\left(a_\tau^-(a)\right) \geq \frac{c\left(e\left(a_\tau^-(a)\right), a_\tau^-(a)\right)}{\rho_p}, \quad \textit{for all } t \geq \tau. \quad (23)$$

b. *If (22) holds, there exist such continuation contracts with $w_t(e(a_\tau^-(a)))$ and $\underline{w}_t$ independent of $t \geq \tau$ for which any $P_t(a_\tau^-(a)) \geq 0$ and $U_t(a_\tau^-(a)) \geq 0$ consistent with (20) that are also independent of $t \geq \tau$ are continuation equilibrium payoff gains.*

3. *If (22) holds with equality, to be a continuation equilibrium any such continuation contract must have $U_t(a_\tau^-(a)) = P_t(a_\tau^-(a)) = 0$, $\underline{w}_t = \underline{u}$ and*

$$w_t\left(e\left(a_\tau^-(a)\right)\right) - \underline{w}_t = \frac{c\left(e(a_\tau^-(a)), a_\tau^-(a)\right)}{\rho_p} = S_t^2\left(a_\tau^-(a)\right), \qquad \text{for all } t \geq \tau. \quad (24)$$

Proposition 4 establishes not only that (22), or equivalently (23), is necessary for a continuation contract for h_τ with stationary effort from τ on to be a continuation equilibrium but also that, for any stationary effort that satisfies (22), there exists a relational contract that distributes the payoff gains to the parties in any way consistent with individual rationality and (20). Clearly, there exists an effort that satisfies (22), or equivalently (23), for given a if and only if

$$\max_{\tilde{e} \in [0, \bar{e}]} \left[\delta \rho_p \rho_a \tilde{e} - c(\tilde{e}, a)\right] \geq \delta \rho_p \rho_a \left(\underline{u} + \underline{v}\right). \quad (25)$$

The next result establishes conditions for there to exist an equilibrium relational contract that is pooling in the sense that those agent types who continue the relationship all choose the same effort and are paid the same by the principal.

Proposition 5 *Consider continuation contracts for h_τ with the properties in Proposition 4 except that agent types $a < \hat{a}$ for some $\hat{a} \in A_\tau(h_\tau)$ end the relationship at τ.*

1. *There exists a continuation contract for h_τ with these properties that is a continuation equilibrium at τ if and only if (22), or equivalently (23), is satisfied when $\hat{a}$ is substituted for $a_\tau^-(a)$.*

2. *There exists an equilibrium relational contract for the whole game for which a continuation contract for h_1 with these properties that satisfies the conditions in 1 is a continuation equilibrium if and only if $S_0 \geq 0$.*

3. *There exists an $\hat{a}$ for which some $(e(\hat{a}), \hat{a})$ satisfies (22) and (23) with $\hat{a}$ substituted for $a_\tau^-(a)$ if and only if (25) is satisfied for $a = \bar{a}$.*

Proposition 5 gives necessary and sufficient conditions for pooling of all agent types $a \geq \hat{a}$, and of all agent types $a < \hat{a}$ (in their case by ending the relationship), with the same history at τ to form part of a continuation contract for h_τ that is a continuation equilibrium from τ on. The right-hand side of (23) with $\hat{a}$ substituted for $a_\tau^-(a)$ is strictly positive for $e(\hat{a}) > 0$, so it certainly requires $S_t^2(\hat{a}) > 0$ to satisfy that condition. This corresponds to the result in the relational contract literature without private information that there must be a strictly positive joint gain or surplus from continuing the relationship for it to be sustainable. The lowest a for which a stationary contract with strictly positive effort can be sustained is that for which (25) holds with equality. In that case, (22) must also hold with equality and then, by Proposition 4, the fixed part of the compensation $\underline{w}_t$ is just sufficient to compensate the agent each period for the opportunity cost $\underline{u}$ of working for the principal. The rest of the compensation comes in the form of a bonus.

Proposition 5 establishes that, provided the relationship is sufficiently productive that (25) is satisfied for $a = \overline{a}$, there exists an equilibrium pooling relational contract with the same strictly positive effort by all agent types that continue the relationship. But the pooling contracts it describes are inefficient in not tailoring $e_t(a)$ to each type a that continues the relationship, which is what full efficiency would do. The following sections explore what can be achieved with contracts involving some separation.

5 Optimal continuation contracts

To analyse relational contracts with separation, one needs to know the payoffs from the continuation contracts that follow separation. It is helpful to consider first the case in which a type is the *only* type with its history apart from $\underline{a}$ (which is consistent with every history) and so is fully separated from other types in $(\underline{a}, \overline{a}]$. That corresponds to a the only type in $A_\tau^+(h_\tau)$ and thus $a_\tau^-(a) = a$. In that case, both parties know the joint gain $S_\tau^1(a)$ from a given continuation contract for h_τ and, since this joint gain can be redistributed before the continuation contract starts, it is optimal for them to select a continuation contract that maximises $S_\tau^1(a)$ subject to the incentive constraints, called here an *optimal continuation contract at* τ. From (7), having a continue the relationship at τ corresponds to $\gamma_\tau^a(a) = 1$. So, from (16), a higher value of $S_\tau^1(a)$ corresponds to a higher value of $S_{\tau-1}^2(a)$. Moreover, it follows from Levin (2003, Theorem 2) that, if an optimal contract exists, there are stationary contracts that are optimal. Thus, the payoffs to the parties in an optimal continuation contract can be determined by studying stationary continuation contracts that are optimal from among those that Proposition 4 establishes are continuation equilibria.

Proposition 6 *Suppose, for agent type a the only type in $A_\tau^+(h_\tau)$ so $a_\tau^-(a) = a$,*

$$\delta\rho_p\rho_a e^*(a) - c(e^*(a), a) - \delta\rho_p\rho_a(\underline{u} + \underline{v}) < 0 \tag{26}$$

but (25) is satisfied. Then, in an optimal continuation contract at τ, the agent ends the relationship at t if $a_t = \underline{a}$, the principal ends the relationship at t if $p_{t+1} = 0$, $e_t(a) = e(a)$ that satisfies (22) with equality, $U_t(a) = P_t(a) = 0$, $\underline{w}_t = \underline{u}$ and

$$w_t(e_t(a)) - \underline{w}_t = \frac{c(e_t(a), a)}{\rho_p} = S_t^2(a), \qquad \textit{for all } t \geq \tau. \tag{27}$$

Proposition 6 establishes that, when efficient effort is unattainable with a relational contract, effort in an optimal continuation contract for a satisfying (25) that is the only type in $A_t^+(h_t)$ satisfies (22) with equality. Denote by $\hat{\alpha}$ the lowest a for which (25) is satisfied. It follows that, for any $a \geq \hat{\alpha}$ that is the only type in $A_t^+(h_t)$, effort in an optimal continuation contract is given by

$$\hat{e}(a) = \begin{cases} e^*(a), \text{ if } e^*(a) \text{ satisfies (22);} \\ e(a) \text{ that satisfies (22) with equality, otherwise;} \end{cases} \quad \text{for } a \geq \hat{\alpha}. \tag{28}$$

With respect to payments to the agent, Proposition 6 has the implication that the fixed part, $\underline{w}_t$, is set at $\underline{u}$ in an optimal continuation contract. This is the highest level consistent with the agent ever initiating a separation. From (27), the bonus equals the surplus $S_t^2(a)$. The bonus and effort just pass the surplus back and forth so that

$U_t(a) = P_t(a) = 0$. The reason is that a higher bonus makes it possible to induce higher effort so, when efficient effort is unattainable, it is optimal to have the bonus at the highest level consistent with the principal continuing the relationship, with the result that the principal's future payoff gain from continuing the relationship by paying the bonus is zero. Symmetrically, the highest effort the agent is willing to exert is the level that makes the agent's payoff gain from continuing the relationship before effort is exerted zero. Thus, if one party were to terminate the relationship, the other would lose out by not getting the surplus back. The implication is that, if an employer or a purchaser were to end a relationship, the other party, be it employee or supplier, would suffer a discrete loss of utility as a result. With an optimal continuation contract, as in the efficiency wage model of Shapiro and Stiglitz (1984), having the contract terminated by the other party imposes a utility loss.

It is important not to misinterpret the result in Proposition 6 that $U_t(a) = 0$ and $P_t(a) = 0$ in an optimal continuation contract. In particular, it does not contradict the result in MacLeod and Malcomson (1989) that at least one party needs to receive a strictly positive gain or surplus for the relationship to continue. The reason is that these payoffs are measured at different stages within the period. $U_t(a)$ is measured at stage 1 of each period, $P_t(a)$ at stage 2. Because $P_t(a) = 0$, the agent receives all the strictly positive expected surplus $S_t^2(a)$ at stage 2. Indeed, as shown in Proposition 6, that surplus just equals the bonus component of compensation. Similarly, because $U_t(a) = 0$, the principal receives all the surplus $S_t^1(a)$ at stage 1 and that too is strictly positive. Thus there is a surplus to the agent at stage 2 and to the principal at stage 1. But measured after the payment is received, the agent's future payoff gain is zero and, measured before the payment is made, the principal's future payoff gain is also zero.

The analysis so far in this section applies to continuation contracts for an agent whose type is fully separated from other types in $(\underline{a}, \overline{a}]$. Separation of agent types may, however, be only partial. The following result is useful for analysing optimal continuation contracts with partial pooling.

Proposition 7 *Suppose $\langle e, \alpha \rangle$ is part of an equilibrium relational contract with*

$$e_t\left(a_{t+1}^-(a)\right) < e^*\left(a_{t+1}^-(a)\right), \quad \text{for } a \in A_t^+(h_t)\text{, for some } t. \tag{29}$$

Then $e_t(a)$ maximises $S_{t-1}^2(a)$ for $a \in A_{t+1}^+(h_t, (e_t(a), w_t(a)))$, subject to incentive compatibility and individual rationality at t for all $a \in A_{t+1}^+(h_t, (e_t(a), w_t(a)))$, only if it does so for $a_{t+1}^-(a)$. If all types $a \in A_t^+(h_t)$ are to remain pooled from t on, an optimal continuation contract at t for all $a \in A_t^+(h_t)$ corresponds to that in Proposition 6 that is optimal for $a_t^-(a)$.

Proposition 7 establishes that, if two agent types $a' > a''$ that have the same history h_t at t are going to continue to be pooled at t and efficient effort for a'' is not achieved at t, then what is optimal at t for type a' must be optimal for type a''. That applies, in particular, to $a'' = a_{t+1}^-(a')$, the lowest type with that history that continues to be pooled with a'. This result is *conditional on* a' and a'' continuing to be pooled at t. It does not rule out that a' might prefer a lower effort by a'' if it would enable a' to separate from a'' more easily at t. Proposition 7 is, nevertheless, useful for extending Proposition 6 to optimal continuation contracts with partial pooling.

6 Contracts with separation of continuing types

The previous section studied optimal continuation contracts. This section studies contracts with some separation of types when the continuation contract following separation is optimal. A natural question is whether there exist equilibrium contracts with all types fully separated. If there were to, an optimal continuation contract would have to satisfy Proposition 6. The next result determines conditions under which separation is possible with such a continuation contract.

Proposition 8 *Consider $[\hat{a}, a''] \subseteq A_t(h_t)$ with $\hat{a}$ satisfying (25) in a relational contract that is to be continued for all $a_\tau \in [\hat{a}, a'']$ if $p_{\tau+1} = 1$, and ended for $a_\tau = \underline{a}$ and for $p_{\tau+1} = 0$, for all $\tau > t$. Suppose that, conditional on $a \in (\hat{a}, a'']$ being separated from all $a' \in [\hat{a}, a)$ at period t of the relational contract, the efforts of a and a' from $t+1$ on are $\hat{e}(a) < e^*(a)$ and $\hat{e}(\hat{a}) < e^*(\hat{a})$ respectively, where $\hat{e}(a)$ is defined in (28). Necessary conditions for it to be feasible for the relational contract to be a continuation equilibrium for $a \in [\hat{a}, a'']$ for all $\tau \geq t$ and separate agent type a'' from agent type $\hat{a}$ in period t for given $e_t(\hat{a}, h_t)$, $U_t(\hat{a})$ and $\underline{w}_t$ are that there exists an $a \in (\hat{a}, a'']$ and $\tilde{e} \leq \hat{e}(a)$ such that*

$$c_2(\tilde{e}, a) \leq c_2(e_t(\hat{a}, h_t), a) + \frac{\delta \rho_p \rho_a}{1 - \delta \rho_p \rho_a} c_2(\hat{e}(\hat{a}), a) \tag{30}$$

and

$$c(\tilde{e}, a) + \left\{ c(e_t(\hat{a}, h_t), \hat{a}) - c(e_t(\hat{a}, h_t), a) + \frac{\delta \rho_p \rho_a}{1 - \delta \rho_p \rho_a} \left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a) \right] \right\} \leq c(\hat{e}(a), a) - \rho_p P_t(a) - U_t(\hat{a}) - \underline{u} + \underline{w}_t. \tag{31}$$

For $e_t(\hat{a}, h_t) \leq \hat{e}(\hat{a})$, (31) and

$$c_2(\tilde{e}, \tilde{a}) \leq c_2(e_t(\hat{a}, h_t), \tilde{a}) + \frac{\delta \rho_p \rho_a}{1 - \delta \rho_p \rho_a} c_2(\hat{e}(\hat{a}), \tilde{a}), \quad \text{all } \tilde{a} \in [\hat{a}, a], \tag{32}$$

satisfied for some $a \in (\hat{a}, a'']$ with $\hat{e}(a) \geq \tilde{e}$ and $P_t(a) \geq 0$ are sufficient for it to be feasible for the relational contract to do so when the agent believes $p_t = 1$.

Although somewhat technical, Proposition 8 is a fundamental building block for studying relational contracts with separation. Condition (30) corresponds to the standard condition for separation in a one-period model that, with $c_{12} < 0$, effort must be non-decreasing in a. A one-period model corresponds to $\delta = 0$, in which case (30) reduces to exactly that standard condition. The term involving δ takes account of how the future rent that is lost by revelation of type changes with type. For an optimal continuation contract, with effort $\hat{e}(a)$ for agent type a who is separated from $\hat{a}$, $U_{t+1}(a) = 0$ as in Proposition 6, so agent type a receives no rent from $t+1$ on. But by not separating from $\hat{a}$, agent type a can guarantee an information rent in the future, for which there must be compensation at t for revelation to occur. The term involving δ on the right-hand side of (30) captures the change in that future information rent as type changes. In a relational contract, a further condition is required for revelation, specifically that there is sufficient joint gain from continuation of the relationship to make incentive compatible the effort $\tilde{e}$ for revelation to occur. Condition (31) formalises that. It would be consistent with the individual rationality conditions (6) for the agent and (12) for the principal to have $U_t(\hat{a}) + \underline{u} - \underline{w}_t = P_t(a) = 0$, the lowest values these could take and still have the relationship continue. That may not, however, be consistent with incentive compatibility for other agent types in $A_t(h_t)$ — for example, if $\hat{a}$ is not the lowest such type. But, because the expression in braces in (31) is strictly positive for $a > \hat{a}$, it follows from those individual rationality conditions that $\tilde{e}$ is strictly less than $\hat{e}(a)$, the highest level sustainable by type a once that type has been revealed.

To explore this result further, consider the separable specification $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$ with, to satisfy Assumption 1, $\phi(a) > 0$ and $\phi'(a) < 0$ for all a. For that specification, Proposition 8 can be extended in the following way.

Corollary 1 *For $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$, with $\phi(a) > 0$ and $\phi'(a) < 0$ for all a, condition (30) is equivalent to*

$$\hat{c}(\tilde{e}) \geq \hat{c}(e_t(\hat{a}, h_t)) + \frac{\delta\rho_p\rho_a}{1 - \delta\rho_p\rho_a}\hat{c}(\hat{e}(\hat{a}))\,. \tag{33}$$

Moreover, the single condition

$$\left[\hat{c}(e_t(\hat{a}, h_t)) + \frac{\delta\rho_p\rho_a}{1 - \delta\rho_p\rho_a}\hat{c}(\hat{e}(\hat{a}))\right]\phi(\hat{a}) \leq \hat{c}(\hat{e}(a))\,\phi(a) - \rho_p P_t(a) - U_t(\hat{a}) - \underline{u} + \underline{w}_t \tag{34}$$

is both necessary and sufficient for there to exist $\tilde{e} \leq \hat{e}(a)$ such that (30), (31) and (32)are satisfied. Under the conditions of Proposition 8, no $a \in (\hat{a}, \overline{a}]$ can be separated from $\hat{a}$ if

$$\hat{c}(\hat{e}(a))\,\phi(a) < \frac{\delta\rho_p\rho_a}{1 - \delta\rho_p\rho_a}\hat{c}(\hat{e}(\hat{a}))\,\phi(\hat{a})\,, \quad \textit{for all } a \in (\hat{a}, \overline{a}]\,. \tag{35}$$

Some separation of types from $\hat{a}$ is possible if there exists an $a \in (\hat{a}, \overline{a}]$ such that

$$\hat{c}(\hat{e}(a))\,\phi(a) \geq \frac{1}{1 - \delta\rho_p\rho_a}\hat{c}(\hat{e}(\hat{a}))\,\phi(\hat{a})\,. \tag{36}$$

Condition (33) is equivalent to (30) for $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$ with $\phi'(a) < 0$. Moreover, (33) is independent of a so, if it holds for any a, it holds for all a. Thus, with this specification, (33) implies (32) so, by Proposition 8, (31) and (33) together are both necessary and sufficient for a to be separated from $\hat{a}$ provided $\tilde{e} \leq \hat{e}(a)$. The left-hand side of (31) is increasing in $\tilde{e}$ for given a, so separation of a from $\hat{a}$ is easiest to obtain with $\tilde{e}$ as low as possible. Moreover, the left-hand side of (33) is continuous and strictly increasing in $\tilde{e}$ so, given that (33) is necessary, it is also necessary that there exists an $\tilde{e}$ that satisfies (33) with equality. Equality in (33) combined with (31) for $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$ yields (34), which is, therefore, necessary for a to be separated from $\hat{a}$. It is also sufficient because, by specifying $\tilde{e}$ such that (33) holds with equality, (33) and (34) imply that (31) and (32) hold for $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$, and that $\tilde{e} \leq \hat{e}(a)$. Condition (35) follows directly from (34) and the individual rationality conditions (6) for the agent and (12) for the principal to continue the relationship because $\hat{c}(e_t(\hat{a}, h_t)) \geq 0$ necessarily. Condition (36) follows from (34) because the highest feasible effort for $\hat{a}$ at t is $e_t(\hat{a}, h_t) = \hat{e}(\hat{a})$ and, by having all types $a \in A_t(h_t)$ strictly lower than $\hat{a}$ not continue the relationship, payments can be set such that the conditions (6) and (11) for type $\hat{a}$ hold with equality.

It follows from (34) that separation is more easily achieved with lower $e_t(\hat{a}, h_t)$ and with lower $\hat{e}(\hat{a})$. The former illustrates the benefits of starting a relationship "small". The latter illustrates the limitations that arise from the parties being unable to commit themselves to inefficient actions in the future. If the parties could commit to an inefficiently low effort for type $\hat{a}$ in period $t+1$, it would be easier to achieve separation of types at t.

To show that both (35) and (36) are possibilities, consider the following example.

Example 1 *Let $\hat{c}(\tilde{e}) = \tilde{e}^2$. Then efficient effort for $p_t = 1$ defined by (1) is $e^*(a) = 2/\phi(a)$. By (28), for $\hat{e}(a) < e^*(a)$, $\hat{e}(a)$ is given by (22) holding with equality so*

$$\hat{e}(a) = \frac{\delta\rho_p\rho_a + \sqrt{\left(\delta\rho_p\rho_a\right)^2 - 4\phi(a)\,\delta\rho_p\rho_a\,(\underline{u} + \underline{v})}}{2\phi(a)} < \frac{\delta\rho_p\rho_a}{\phi(a)} < e^*(a)\,,$$

the inequalities following from $(\underline{u} + \underline{v}) > 0$ and $\delta\rho_p\rho_a < 1$. Thus, in this example, efficient effort $e^*(a)$ is not attainable for any a. Moreover,

$$\lim_{\underline{u}+\underline{v}\to 0} \hat{e}(a) = \frac{\delta\rho_p\rho_a}{\phi(a)}$$

and, for $(\underline{u} + \underline{v})$ sufficiently close to zero, $\hat{e}(a)$ can be arbitrarily close to this limit. Then, in the limit as $(\underline{u} + \underline{v})$ goes to zero, (36) is satisfied if there exists $a \in (\hat{a}, \overline{a}]$ such that

$$\phi(a) < \left(1 - \delta\rho_p\rho_a\right)\phi(\hat{a})\,,$$

so some separation from $\hat{a}$ is certainly possible if there exists an $a \leq \overline{a}$ sufficiently high that $\phi(a)$ satisfies this condition. On the other hand, for this example, (35) can be written

$$\phi(a) > \frac{1 - \delta\rho_p\rho_a}{\delta\rho_p\rho_a}\phi(\hat{a})\,, \quad \textit{for all } a \in [\hat{a}, \overline{a}]\,,$$

which, since $\phi(a)$ is strictly decreasing, holds for all $a \in [\hat{a}, \overline{a}]$ if it holds for $a = \overline{a}$. It will, in particular, hold if $\delta\rho_p\rho_a$ is sufficiently close to 1.

Proposition 8 can be used to derive characteristics of *partition contracts*, that is, contracts for which $[\underline{a}, \overline{a}]$ is divided into intervals with all types in the same interval choosing the same effort and types in different intervals choosing different efforts. Partition contracts are not the only possible equilibrium contracts exhibiting partial pooling of types. Laffont and Tirole (1993, p. 383) describe, in the context of a two-period procurement model, continuation equilibria that exhibit infinite reswitching in which two types choose actions that generate the same outcome but some type intermediate between them chooses an action that generates a different outcome. But partition contracts seem very natural, while allowing for a type to be fully separated if it is in a partition with upper support the same as its lower support.

The following proposition gives some implications of Proposition 8 for equilibrium partition contracts. (For the statement of the proposition, recall that $a_t^-(a) = \min \tilde{a} \in A_t^+(h_t)$ for $a \in A_t^+(h_t)$, that is, the lowest agent type apart from $\underline{a}$ that has the same history at t as a.)

Proposition 9 *Consider period t of an equilibrium relational contract in which all $a \in [\underline{a}_t, \overline{a}_t]$ have the same history h_t and $\underline{a}_t \geq \alpha_t(h_t)$. Suppose that effort for type a at τ is $e_\tau(a, h_\tau) = \hat{e}(a_t^-(a)) < e^*(a_t^-(a))$ for all $\tau > t$. Then*

1. *$e_t(a, h_t) < \hat{e}(a)$ for $a \in (\underline{a}_t, \overline{a}_t]$ and $e_t(\underline{a}_t, h_t) \leq \hat{e}(\underline{a}_t)$;*
2. *for $a, \hat{a} \in [\underline{a}_t, \overline{a}_t]$ with $a > \hat{a}$ and separated from it, $e_t(a, h_t) - e_t(\hat{a}, h_t)$ is bounded below by some $\varepsilon > 0$;*
3. *it is not feasible to separate all $a \in [\underline{a}_t, \overline{a}_t]$.*

Part 1 of Proposition 9 shows that effort of any type a at t is no greater than $\hat{e}(a)$, effort in an optimal continuation contract if the agent's type had been revealed, and is strictly less for any a that is not the lowest type in a partition. It follows essentially from (31) and the individual rationality conditions (6) and (12) for the parties to continue the relationship because, under the conditions specified, the term in braces in (31) can be shown to be non-negative, and strictly positive for a not the lowest type in a partition. This is because of the limitations on the spread of rewards to the agent that is incentive compatible for the principal given the future joint gain generated by a relational contract. That limits the difference in efforts between the highest and the lowest agent types with the same history. Part 2 shows that the effort of any type a separated from $\hat{a}$ at t must be discretely greater than that of $\hat{a}$. This follows essentially from (30) given $c_{12} < 0$. The result arises because type $a > \hat{a}$ can continue to obtain a future informational rent that increases with type by choosing the effort for type $\hat{a}$ but receives no informational rent in the future if it reveals its type. So, to induce a to separate from $a - da$ (with $da > 0$), the difference between the current payoff of $a - da$ and of a choosing $e_t(a, h_t)$ must compensate for the difference in their loss of future informational rents from choosing $e_t(a - da, h_t)$. That requires $e_t(a, h_t)$ discretely greater than $e_t(a - da, h_t)$. Part 3 then follows because it is not possible to have a discrete jump in $e_t(a, h_t)$ for every $a \in [\underline{a}_t(h_t), \overline{a}_t(h_t)]$ given that $e_t(\overline{a}_t(h_t), h_t)$ is bounded above by $\hat{e}(\overline{a}_t(h_t)) < e^*(\overline{a}_t(h_t))$.

If it were possible to fully separate at t all types $a \in [\underline{a}_t, \overline{a}_t]$ with the same history h_t, by Proposition 6 optimal continuation equilibria would result in $e_\tau(a, h_\tau) = \hat{e}(a)$ for all $\tau > t$, so the assumptions of Proposition 9 would be satisfied. But then Part 3 of Proposition 9 would imply that full separation of $a \in [\underline{a}_t, \overline{a}_t]$ is not feasible, a contradiction that implies the following corollary.

Corollary 2 *When $\hat{e}(a) < e^*(a)$ for some $a \in (\underline{a}, \overline{a}]$, there exists no equilibrium relational contract with optimal continuation that separates all agent types $a \in [\underline{a}, \overline{a}]$.*

The essential reason for the non-existence of fully separating contracts is the discrete jump in effort between types that is required to separate them and, with a continuum of types and effort constrained to a finite interval, it is not possible to have that occur for all types. An implication is that, as noted in Laffont and Tirole (1993, p. 382), the usual revelation principle does not apply to repeated relationships in the absence of commitment because truthful revelation of types in one period would result in full separation in subsequent periods.

There may, however, exist partition contracts with less than full revelation of types. Information about types is valuable because it enables better tailoring of continuation action to type. An obvious question is how fine the partitions can be. That motivates the following definition.

Definition 1 *A* finest partition contract *is a relational contract with agent types partitioned by $\underline{a} < a^1 \leq \ldots \leq a^n < \overline{a}$ and the characteristics that, for given $a^1, e_1(a^1, h_1)$ and $p_1 = 1$:*

1. *all agent types $a \in [\underline{a}, a^1)$ end the relationship in period 1;*
2. *all agent types $a \in [a^i, a^{i+1})$ for $i = 1, \ldots n$, with a^{n+1} defined as $a^{n+1} = \overline{a}$, choose the same effort in period 1 and effort $\hat{e}(a^i)$ defined by (28) in all subsequent periods for which the relationship continues;*
3. *a^i, for $i = 2, \ldots n$, is the lowest type that can be separated from a^{i-1}, with n given by the highest integer such that $a^n < \overline{a}$ when a^i is defined in this way.*

A finest partition contract satisfies the assumptions of Proposition 9, so the results in that proposition apply. In particular, Part 1 implies that effort in period 1 for any types a who are separated in that period is below that from period 2 on for all but a^1. Thus there is a cost to information being private except "at the bottom", that is, for the least productive feasible matches. As in Watson (1999) and Watson (2002), the relationship starts out "small" but here it is for the purpose of building up trust, in the sense of convincing the other party about type. The increase in effort after the initial period also implies that remuneration increases except for the least productive of feasible relationships. Further characteristics of finest partition contracts are derived for the separable case $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$.

Proposition 10 *Consider $c(\tilde{e}, a) = \hat{c}(\tilde{e})\,\phi(a)$, with $\phi(a) > 0$, $\phi'(a) < 0$ and $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$. For given $a^1 = \alpha_1(h_1)$ that satisfies (25), $e_1(a^1, h_1) \leq \hat{e}(a^1)$ and $p_1 = 1$, there exists a unique equilibrium finest partition contract. It has a finite number of partitions, with a^i for $i = 2, \ldots, n$ specified iteratively by*

$$\hat{c}\Big(\hat{e}\Big(a^2\Big)\Big)\,\phi\Big(a^2\Big) = \left[\hat{c}\Big(e_1\Big(a^1, h_1\Big)\Big) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\Big(\hat{e}\Big(a^1\Big)\Big)\right]\phi\Big(a^1\Big) \tag{37}$$

$$\hat{c}\Big(\hat{e}\Big(a^i\Big)\Big)\,\phi\Big(a^i\Big) = \frac{1}{1-\delta\rho_p\rho_a}\hat{c}\Big(\hat{e}\Big(a^{i-1}\Big)\Big)\,\phi\Big(a^{i-1}\Big), \quad i = 3, \ldots, n, \tag{38}$$

and $e_1(a^i, h_1)$ by

$$\hat{c}\Big(e_1\Big(a^i, h_1\Big)\Big) = \hat{c}\Big(e_1\Big(a^{i-1}, h_1\Big)\Big) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\Big(\hat{e}\Big(a^{i-1}\Big)\Big), \quad i = 2, \ldots, n. \tag{39}$$

With such a contract, no further separation is possible after period 1.

Proposition 10 establishes that, even though full separation of agent types is not feasible when the parties cannot commit to inefficient actions in the future, there exists an equilibrium relational contract with some separation if there is some a^1 satisfying (25) and $e_1(a^1, h_1)$ for which a^2 defined by (37) is less than $\overline{a}$. Moreover, when separation in the first period is the finest achievable in that period, no further separation is possible in subsequent periods. That is perhaps surprising because two things change once some separation has taken place. The first is that, to sustain effort $\hat{e}(a^i)$, the lowest type a^i in partition i has payoff gain $U_t(a^i) = 0$ for $t \geq 2$, whereas incentive compatibility for separation requires $U_1(a^i) > 0$ for $i \geq 2$. Because within each partition

a^i plays the role of $\hat{a}$ in Corollary 1, that makes the condition (34) for separation less stringent because it allows that condition to be satisfied by a lower value of a on the right-hand side for a given value of the left-hand side. The second is that the continuation contract has $e_t(a^i, h_t) = \hat{e}(a^i)$ for $t \geq 2$. With a^i taking the place of $\hat{a}$ in (34), that increases the left-hand side because in the finest partition contract $e_1(a^i, h_1) < \hat{e}(a^i)$ for $i \geq 2$, which makes the requirement (34) for separation more stringent. These two effects exactly offset each other — not by coincidence but because of the nature of the incentive compatibility conditions for separation in period 1. As a result, no further separation is possible in subsequent periods. Moreover, by Proposition 6, $\hat{e}(a^i)$ is optimal for a^i for $t \geq 2$. Thus, Proposition 7 implies that $\hat{e}(a^i)$ is also optimal from period 2 on for all $a \in [a^i, a^{i+1})$.

Proposition 10 also establishes that, with a finite partition contract, types are separated into only a finite number of partitions. Employees are placed in a finite number of grades that include different abilities, a characteristic of many employment situations, and suppliers are grouped into a finite number of bands, as Asanuma (1989) explains for Toyota, despite the underlying types being continuous.

Proposition 10 is in the spirit of MacLeod and Malcomson (1988), who also derive an equilibrium relational contract consisting of a finite number of ranks that corresponds to a partitioning of continuous agent types that are persistent private information. The underlying reason for the finite partitioning is, however, different in the two cases. In MacLeod and Malcomson (1988), the finite partitioning is required to maintain incentive compatibility even if information about types were fully revealed. That is not the case in the present model. The difference arises for two reasons. First, MacLeod and Malcomson (1988) follow the efficiency wage model of Shapiro and Stiglitz (1984) in not having a bonus component to pay. Thus, the only way for the principal to induce the agent to incur effort is by the threat that failure to do so will result in the agent receiving only the payoff $\underline{u}$ that can be obtained without being matched with the principal. Second, in MacLeod and Malcomson (1988), agent types are not specific to the particular relationship, but equally valuable to competing principals, who can observe the payment made to an agent. This corresponds to the payoff $\underline{u}$ being an increasing function of a. Specifically, it is assumed there that an agent who is dismissed for not complying with the relational contract in one rank is believed by other principals to be appropriate for the rank below. Together with the absence of bonuses, this ensures that the difference in payoffs between ranks must be discrete, which restricts the number of ranks to being finite. But when payment of bonuses is permitted, or when agent types are specific to the relationship so that $\underline{u}$ is independent of the information revealed, the finite number of ranks is no longer required to maintain incentive compatibility if information about types were fully revealed. Both bonuses and relationship-specific types are characteristics of the model used here. As a result, in the present model it would be incentive compatible to have each type a in its own rank with effort at the optimal level specified in Proposition 6, and hence a continuum of ranks, if agents could be induced to fully reveal their types. Thus the discrete partitioning of types in the present model results not from the need to maintain incentive compatibility once information about types has been revealed. Instead it results from the impossibility of getting agents to fully reveal information about types in the first place, a problem that arises even with bonuses and relationship-specific types. It is in that sense more fundamental to relational contract models than the corresponding result in MacLeod and Malcomson (1988).

The partial pooling of types here also arises for different reasons from the partial pooling in models of procurement with private information about agent types discussed in Laffont and Tirole (1993). In those models, partial pooling arises because the principal (the regulator) wants to extract rent from the agent (the provider) and is willing to accept less efficient procurement to achieve that. If rent extraction were of no concern to the principal (for example, if the regulator valued rents to the provider equally with rents to others), the principal would (if feasible) commit to a fixed price contract, which would induce the agent to act efficiently (and would also be renegotiation proof in a multi-period model). There would then be *full* pooling of types who provide, not partial pooling of such types. Exactly the same applies if, as in the present paper, the initial contract between principal and agent is made before the agent learns her type because, given risk-neutral parties, the expected rent can always be transferred by an upfront payment before type is revealed. In the model used here, rent extraction is still at the heart of partial pooling — it is because agents can extract rent by pretending to be a lower type that types are not fully revealed. But the partial pooling is more fundamental in the sense that it cannot be removed by upfront payments made before the agent learns her type or by the principal not being concerned with the distribution of rent.[1]

In the relational contracts described in Proposition 10, all sorting into partitions takes place in the first period of the relationship, after which no further partitioning is feasible. There are, however, obvious extensions to the model that would result in sorting not all happening straightaway. One is if, in addition to the persistent private information about the agent's type, the agent receives privately observed *iid* shocks to type, as in Levin (2003). While that results in a model that is more complicated to analyse formally, it is intuitive that one period of observation would not then be sufficient for the principal to place the agent's persistent type in a particular band, so sorting would take place over time. A second possible extension is if experience reduces the disutility of achieving a given performance level in a way that differentially affects agent types. Then separation that is not feasible according to Corollary 1 at $t = 1$ may become feasible at some $t > 1$. Again, the basic idea is intuitive.

In the relational contracts in Proposition 10, different partitions result from different a^1 and $e_1(a^1, h_1)$ and different values of these will in general result in a different joint gain S_0 from the contract. What is optimal depends on the distribution of agent

[1]It is, moreover, not clear that procurement models with private information about agent types actually generate the discrete partitioning of types that arises here. In one-period models with a continuum of types, partial pooling can occur when the distribution of types fails to satisfy the monotone hazard rate condition of being everywhere log concave, see Laffont and Tirole (1993, Appendix A1.5). But that condition is satisfied by many standard distributions and, even if it is not satisfied everywhere, pooling occurs for only those types for which the condition does not hold. For two-period models with a continuum of persistent types, Laffont and Tirole (1993, Chapter 10) show that contracts that achieve full separation are feasible but never optimal. They do not, however, show what type of pooling is optimal — the proof that separation is not optimal uses only that pooling for an interval of the least productive types dominates separation. In the analysis in this paper, full separation is not even feasible. The same holds in procurement models when the parties can commit only to short-term contracts, even with the monotone hazard rate condition satisfied, see Laffont and Tirole (1993, Chapter 9). In that case, however, non-feasibility arises from a lack of commitment that follows from a legal constraint on long-term contracting (for example, sovereign bodies cannot commit their successors), not because of an informational constraint inherent to the environment. In contrast, in the relational contract model studied here, full separation is not feasible even though the only constraints on long-term contracting are informational.

types. Lowering $a^1 = \alpha_1(h_1)$, where that is incentive compatible, obviously increases the set of types that continue the relationship and, from (37), also lowers a^2, and so on. But it may be incentive compatible only if $e_1(\alpha_1(h_1), h_1)$ is also lowered. From (37), that lowers a^2, and so on, but it reduces effort for all $a \in [a^1, a^2)$. Moreover, it is not necessarily the case that it is optimal to use the finest possible partitions even for given $\alpha_1(h_1)$ and $e_1(\alpha_1(h_1), h_1)$. If, for example, the probability density of agents of the type a^2 defined by (37) is close to zero, it may result in a higher S_0 to have a higher a^2, which will result in higher effort from period 2 on for all types $a \in [a^2, a^3)$. Thus what is optimal depends in a number of ways on the distribution of agent types. The following result does, however, apply independently of the distribution of types.

Proposition 11 *Consider an optimal pooling relational contract with $e_t(a, h_t) = \hat{e}$ for all types $a \geq \hat{a}$, h_t and t. Then, provided there exist an $a \in [\hat{a}, \overline{a}]$ and $\tilde{e} < \hat{e}(a)$ that satisfy (31) and (32) for $e_1(\hat{a}, h_1) = \hat{e}(\hat{a}) = \hat{e}$, there exists an equilibrium partition relational contract with additional separation of types in period 1 that generates strictly higher joint gain S_0.*

Proposition 11 contrasts with a result in Athey and Bagwell (2008, Proposition 2), who show that full pooling can be optimal in their model of collusion in oligopoly when there is a continuum of fully persistent types. In the present model, some separation, if feasible, always dominates full pooling of types that continue the relationship.

7 Conclusion

This paper has derived incentive compatibility and dynamic enforcement conditions for a relational contract in which the agent's type is privately known by the agent and is persistent over time, unlike in the models of Levin (2003), MacLeod (2003) and Athey et al. (2004), where the agent's type is an *iid* random draw each period. Applied to employment, it generalizes the models of Shapiro and Stiglitz (1984) and MacLeod and Malcomson (1989) to private information about workers' disutility of effort. Provided the relationship is sufficiently productive, there always exists a pooling contract in which the agent ends the relationship if the cost of effort is above a critical value but otherwise, whatever the agent's type, the agent provides the same effort and the principal pays the same remuneration. Some separation between agent types who continue the relationship may be feasible — necessary and sufficient conditions for this have been given. If such additional separation of agent types is feasible, an optimal pooling contract is always dominated by a contract with some separation of agent types who continue the relationship.

With relational contracts for which future actions are optimal, however, it may not be possible to achieve any separation of types for which the relational contract continues and full separation of such types is never feasible. When some separation is feasible, agent types (though continuous) can be partitioned, with all types within a partition delivering the same level of performance. This is similar in spirit to a result in MacLeod and Malcomson (1988) but the mechanisms underlying the results are different. In MacLeod and Malcomson (1988), limitations on rewards (because bonuses are ruled out) and on punishments (because information about the agent's type is valuable to other potential principals) play an important role in requiring partitioning of agent types into pools to maintain incentive compatibility even if information about types were fully revealed. The present model has neither of those characteristics, so there would be no problem of maintaining incentive compatibility for each individual agent type to have a different level of effort if it were possible to induce agents to reveal their types. Instead, the partitioning arises from the difficulty of inducing agents to reveal their types in the first place. In that respect, the result here is more fundamental to relational contracts with persistent private information about agent types.

Partial pooling of types is also a characteristic of relational contracts with private information that is not persistent, as in Levin (2003) and MacLeod (2003). But there, because agent types are *iid* draws each period, there is no persistence of particular agents in particular partitions. Thus, these models cannot explain why particular employees or suppliers remain in the same group for extended periods of time.

The reason for partial pooling is also different from that in the models of procurement discussed in Laffont and Tirole (1993) when agents have private information about their types. When partial pooling arises in those models, it arises because the principal is willing to sacrifice some economic efficiency in order to extract information rent from the agent. Here, it is again more fundamental in the sense that it arises even though rent can be extracted from the agent by an upfront payment before the agent's type is revealed. Partial pooling is inherent to relational contracts with persistent private information about agent types independently of who receives rent at the initial contracting stage.

Where it is possible to separate agent types who continue the relationship, an additional cost to inducing separation is that effort in the period in which types are revealed is necessarily below the level that could be sustained without private information for all but the least productive feasible relationships. The increase in effort after the initial period also implies that remuneration increases except for the least productive of feasible relationships. As in MacLeod and Malcomson (1989) and Levin (2003), remuneration consists of two components, a component that does not depend on performance and a bonus that does. With an optimal relational contract, the former just compensates the agent each period for the opportunity cost of working for the principal and the remaining compensation comes in the form of a bonus.

In the model used here, either party's type can change in a way that makes continuation of the relationship infeasible. Setting up the model in this way means that an "off the equilibrium path" action by one party can be interpreted by the other as a change in type that renders a continued relationship infeasible and that party then ends the relationship. There is thus no need for explicit punishments in order to sustain equilibria. Each party responds optimally to what it believes is the other party's type and the relationship comes to an end only when at least one of them believes that it is not sufficiently productive to be worth continuing. In that respect, the model answers the criticism Bewley (1999) makes of efficiency wage models that employers rarely use the threat of punishment to obtain cooperation. Since, in practice, economic agents can rarely know for sure that other agents' types will not change, that seems an attractive way to model economic behaviour.

Appendix: Proofs

Proof of Proposition 1. The effort function $e_t(a, h_t)$ may not satisfy the agent's incentive condition (3) at t because agent type a deviates by choosing either (1) $\tilde{e} =$

$e_t(a',h_t) \neq e_t(a,h_t)$ for some $a' \in A_t(h_t)$ or (2) $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t(h_t)$. Given a relational contract, let $\check{U}(\hat{a},a,h_t)$ denote the maximand in (3) for agent type a choosing $\tilde{e} = e_t(\hat{a},h_t)$. Incentive compatibility to the first type of deviation corresponds to $U_t(a,h_t) = \check{U}(a,a,h_t)$. That in turn corresponds to

$$U_t(a,h_t) \geq \check{U}(a',a,h_t) = U_t(a',h_t) + \check{U}(a',a,h_t) - \check{U}(a',a',h_t)\,, \quad \forall a,a' \in A_t(h_t)\,,$$

and, with the roles of a and a' reversed,

$$U_t(a',h_t) \geq \check{U}(a,a',h_t) = U_t(a,h_t) + \check{U}(a,a',h_t) - \check{U}(a,a,h_t)\,, \quad \forall a,a' \in A_t(h_t)\,.$$

These two conditions imply

$$\begin{aligned} \check{U}(a',a',h_t) - \check{U}(a',a,h_t) &\geq U_t(a',h_t) - U_t(a,h_t) \\ &\geq \check{U}(a,a',h_t) - \check{U}(a,a,h_t)\,, \quad \forall a,a' \in A_t(h_t)\,. \end{aligned} \tag{A.1}$$

But $\check{U}(\hat{a},\hat{a},h_t) - \check{U}(\hat{a},a,h_t) = \tilde{U}_t(\hat{a},\hat{a},h_t) - \tilde{U}_t(\hat{a},a,h_t)$, so (A.1) implies that (10) is necessary. It also implies (10) is sufficient to prevent a deviation to $\tilde{e} = e_t(a',h_t) \neq e_t(a,h_t)$ for any $a' \in A_t(h_t)$.

Now consider deviation to $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t(h_t)$. By assumption, the principal observing such a deviation believes the agent to be type $\underline{a}$, which is always in $A_t(h_t)$ because there is strictly positive probability the agent's type changes to $\underline{a}$ between $t-1$ and t. For a relational contract for which the principal ends the relationship if the agent is believed to be type $\underline{a}$ and $e_t(\underline{a},h_t) = 0$, any $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t(h_t)$ must be strictly positive because $\underline{a} \in A_t(h_t)$ and $\tilde{e}$ cannot be negative. But, with the principal ending the relationship for any such $\tilde{e}$, it is always strictly preferable for a deviating agent to mimic $\underline{a}$ by setting $\tilde{e} = 0$ than to choose such an $\tilde{e}$. Thus, the maximisation problem in (3) for type a corresponds to choosing an optimal $\tilde{e} = e_t(a',h_t)$ for some $a' \in A_t(h_t)$. So $e_t(\underline{a},h_t) = 0$ and (A.1) are sufficient to deter any deviation to $\tilde{e} \neq e_t(a',h_t)$ for any $a' \in A_t(h_t)$ given the principal's response, and thus $e_t(\underline{a},h_t) = 0$ and (10) also are. ■

Proof of Proposition 2. As in the proof of Proposition 1, let $\check{U}(\hat{a},a,h_t)$ denote the maximand in (3) for agent type a choosing $\tilde{e} = e_t(\hat{a},h_t)$ given the relational contract. Then the statement in the proposition certainly holds if

$$\check{U}(a',a,h_t) - \check{U}(a'',a,h_t) \geq \check{U}(a',a',h_t) - \check{U}(a'',a',h_t)\,, \, \forall a \in A_t(h_t)\,, a \geq a',$$

or, re-arranging, if

$$\check{U}(a',a,h_t) - \check{U}(a',a',h_t) \geq \check{U}(a'',a,h_t) - \check{U}(a'',a',h_t)\,, \, \forall a \in A_t(h_t)\,, a \geq a'.$$

But $\check{U}(\hat{a},\hat{a},h_t) - \check{U}(\hat{a},a,h_t) = \tilde{U}_t(\hat{a},\hat{a},h_t) - \tilde{U}_t(\hat{a},a,h_t)$ for $\tilde{U}_t$ defined in (9), so this condition is equivalent to

$$\tilde{U}_t(a',a,h_t) - \tilde{U}_t(a',a',h_t) \geq \tilde{U}_t(a'',a,h_t) - \tilde{U}_t(a'',a',h_t)\,, \, \forall a \in A_t(h_t)\,, a \geq a'. \tag{A.2}$$

From the definition of $\tilde{U}_t$ in (9),

$$\begin{aligned} &\tilde{U}_t(\hat{a},a,h_t) - \tilde{U}_t(\hat{a},a',h_t) \\ &\quad = \sum_{\tau=t}^{\infty} \left\{ \left[c\left(e_\tau(\hat{a},h'_\tau)\,,a'\right) - c\left(e_\tau(\hat{a},h''_\tau)\,,a\right)\right] (\delta\rho_a)^{\tau-t} \prod_{j=t}^{\tau-1} \gamma_j^p\left(h_j,e_j(\hat{a},h_j)\right) \right\}, \end{aligned}$$

with the convention that $\prod_m^n = 1$ for $n < m$. By Assumption 1, $c_{12}(\tilde{e}, a) < 0$ for all $(\tilde{e}, a)$. It follows that (A.2) holds when $e_\tau(a', h'_\tau) \geq e_\tau(a'', h''_\tau)$ for all $\tau \geq t$. ■

Proof of Proposition 3. With $p_{t+1} = 0$, $p_\tau = 0$ for all $\tau \geq t+1$ so continuing the relationship for $t+1$ results in no output in the future. By continuing it at stage 2 of period t, therefore, the principal pays $w_t - \underline{w}_t \geq 0$ and at least $\underline{w}_\tau$ in each period $\tau \geq t+1$ for which the relationship continues, but receives no output. For $\underline{w}_\tau + \underline{v} \geq 0$ for all $\tau \geq t+1$, it follows that $E_{a|h_t,e_t}\left[\underline{P}_t(a)\right] \leq 0$ for all a, so it is a best response for the principal to end the relationship at stage 2 of period t.

Suppose the principal believes the agent's type at stage 2 of period t is $a_t = \underline{a}$. Once the agent's type has become $\underline{a}$, it remains $\underline{a}$ ever after. Because, by Assumption 1, $c(\tilde{e}, \underline{a}) > \tilde{e} - (\underline{u} + \underline{v})$ for all $\tilde{e} \in (0, \bar{e}]$, a continued mutually beneficial relationship is not possible for $a = \underline{a}$ (continuation is a negative sum game). Thus the principal believes the agent's best response is to end the relationship at any time in the future that the principal would gain from its continuation. Hence it is a best response for the principal to end the relationship at stage 2 of period t unless $\underline{w}_{t+1} + \underline{v} < 0$. But $\underline{w}_{t+1} + \underline{v} < 0$ requires $\underline{w}_{t+1} < 0$, for which it is a best response for the agent to end the relationship at stage 1 of period $t+1$. In either case, it is a best response for the principal to end the relationship at the earliest opportunity. Uniqueness of the best response if $\underline{w}_{t+1} + \underline{v} > 0$ follows directly. ■

Proof of Proposition 4. Part a. Necessity. With the strategies specified in the proposition, $\gamma_t^a(a) = 1$ for all $a \in A_\tau^+(h_\tau)$ and $\gamma_t^p(\hat{e}) = \rho_p$. Thus (20) must hold. Moreover, the principal observes the same history for all $a \in A_\tau^+(h_\tau)$ for all t and must, therefore, make the same payment. Thus, $P_t(a)$ is the same for all $a \in A_\tau^+(h_\tau)$, so (11) requires $P_t(a) \geq 0$ for $a \in A_\tau^+(h_\tau)$. Together with (21), these imply that $e(a_\tau^-(a))$ must satisfy (22). (19) for $a_\tau^-(a)$ and $e(a_\tau^-(a))$ implies that (23) corresponds to (22) for $a = a_\tau^-(a)$ and, with $c_2 < 0$, this implies that (22) is satisfied for all $a \geq a_\tau^-(a)$.

Sufficiency. Consider for $a \in A_\tau^+(h_\tau)$ the continuation contract with, for all $t \geq \tau$, $e_t(a) = e(a_\tau^-(a))$ satisfying (22), $w_t(e(a_\tau^-(a))) = w$, and $\underline{w}_t = \underline{w}$. Payoff gains are then stationary. It follows from (3) for the agent, and a corresponding calculation for the principal, that

$$\begin{aligned} U_t(a) &= \frac{-c(e(a_\tau^-(a)), a) - \underline{u} + \underline{w} + \rho_p (w - \underline{w})}{1 - \delta\rho_p\rho_a}, \quad \text{for all } a \in A_\tau^+(h_\tau),\ t \geq \tau, \\ P_t(a) &= \frac{-(w - \underline{w}) + \delta\rho_a \left[e(a_\tau^-(a)) - \underline{v} - \underline{w}\right]}{1 - \delta\rho_p\rho_a}, \quad \text{for all } a \in A_\tau^+(h_\tau),\ t \geq \tau. \end{aligned}$$

With (22) satisfied for all $a \in A_\tau^+(h_\tau)$, there certainly exist $w \geq \underline{w}$ and $\underline{w} \in [-\underline{v}, \underline{u}]$ such that $U(a) \geq \max[0, \underline{w} - u]$ and $P(a) \geq 0$, specifically when

$$\frac{c(e(a_\tau^-(a)), a_\tau^-(a)) + \underline{u} - \underline{w}}{\rho_p} \leq w - \underline{w} \leq \delta\rho_a \left[e(a_\tau^-(a)) - \underline{v} - \underline{w}\right], \qquad \text{for all } a \in A_\tau^+(h_\tau), \quad \text{(A.3)}$$

because, with $c_2 < 0$, $c(e, a)$ is decreasing in a. By the argument in the text preceding (21), with $e_t(a)$ the same for all $a \in A_\tau^+(h_\tau)$, the agent's individual rationality and incentive compatibility conditions (6) and (10) for $t \geq \tau$ for $a_t \neq \underline{a}$ can be replaced by

(21). This and the principal's incentive and individual rationality conditions in (11) and (12) for continuing the relationship as long as $p_{t+1} = 1$ are satisfied for the specified stationary continuation contract. With $\underline{w} \in [-\underline{v}, \underline{u}]$, it is a best response for the principal to end the relationship at $t \geq \tau$ if $p_{t+1} = 0$ or if $e_t \neq e(a_\tau^-(a))$ by Proposition 3. By Assumption 1, $U_t(\underline{a}) < 0$ even for the highest values of w and $\underline{w}$ satisfying (A.3), so it is a best response for the agent to end the relationship at $t \geq \tau$ if $a_t = \underline{a}$ or if $w_t \neq w$. Thus the specified continuation contract for h_τ with $e_t(a) = e(a_\tau^-(a))$ for all $a \in A_\tau^+(h_\tau)$ and $t \geq \tau$ is a continuation equilibrium at τ that satisfies all the conditions of the proposition.

Part b. The continuation contract specified in the proof of sufficiency for Part a has $w_t(e(a_\tau^-(a)))$ and $\underline{w}_t$ independent of $t \geq \tau$. Moreover, by varying w and $\underline{w}$ between the upper and lower levels in (A.3), $P_t(a_\tau^-(a))$ and $U_t(a_\tau^-(a))$ can take on any non-negative values independent of $t \geq \tau$ that are consistent with (20).

Part c. It follows from (6) that, for any agent type to end the relationship, it is necessary that $\underline{w}_t \leq \underline{u}$. For (22) to hold with equality, it follows from (20) combined with (11), (21) and $\underline{w}_t \leq \underline{u}$ that $U_t(a_\tau^-(a)) = P_t(a_\tau^-(a)) = 0$ and $\underline{w}_t = \underline{u}$ so that, consistent with (21), $U_t(a_\tau^-(a)) \geq \max[0, \underline{w}_t - \underline{u}]$. (24) then follows from (A.3) and the equivalence of (22) and (23). ■

Proof of Proposition 5. It follows from Proposition 4 with $\hat{a}$ substituted for $a_\tau^-(a)$ that, provided $a < \hat{a}$ can be induced to end the relationship at τ, there exists a relational contract with the properties specified if and only if (22) is satisfied for $a = \hat{a}$. Proposition 4 also ensures there exists such a contract for which $U_t(\hat{a}) = 0$ so, because $U_t(a)$ is necessarily strictly increasing in a for a given strictly positive effort sequence, that contract ensures the individual rationality condition in (6) for $a < \hat{a}$ to end the relationship at τ is satisfied. Thus all the conditions are satisfied for the contract specified to be a continuation equilibrium at τ, which establishes Part 1. For $S_0 \geq 0$, there then always exists a w_0 for which both parties gain from starting the relational contract and clearly $S_0 \geq 0$ is necessary for both parties to agree to a relational contract, which establishes Part 2. Finally, with $c_2 < 0$, the maximand in (25) is increasing in a. So there exists some $\hat{a} \in (\underline{a}, \overline{a}]$ that satisfies (22) and (23) with $\hat{a}$ substituted for $a_\tau^-(a)$ if and only if (25) is satisfied for $a = \overline{a}$. ■

Proof of Proposition 6. The case with a the only type in $A_\tau^+(h_\tau)$ is a special case of that studied in Proposition 4 with $a_\tau^-(a) = a$. Consider first optimality among continuation contracts that satisfy properties 1 and 2 of Proposition 4. (25) satisfied for a implies that there exists a continuation contract for h_τ with those properties and stationary payments that satisfies (22) and is a continuation equilibrium at τ. Because (22) is necessary for a continuation contract for h_τ with those properties to be a continuation equilibrium, (26) implies that there exists no such contract with efficient effort $e^*(a)$ for a. Thus, to maximize $S_{\tau-1}^2(a)$ subject to the incentive constraints, a continuation contract for h_τ that is optimal given a from among those with those properties must maximize stationary effort subject to the incentive constraints and so satisfies (22) with equality. Call this maximum effort $\hat{e}(a)$. It then follows from Proposition 4 that an optimal stationary continuation contract for h_τ has $U_t(a) = P_t(a) = 0$ and $\underline{w}_t = \underline{u}$ for all $t \geq \tau$ and has effort $\hat{e}(a)$ that, when substituted for $e(a_\tau^-(a))$, satisfies (24) for all $t \geq \tau$.

Now consider whether it is possible to do better with a contract that does not satisfy properties 1 and 2 of Proposition 4. Suppose first the agent were not to end the relationship at t for $a_t = \underline{a}$. The joint gain $S_t^1(\underline{a})$ from t on is necessarily strictly negative with $a_t = \underline{a}$, so there is then an additional negative term on the right-hand side of (16), which results in an additional positive term on the right-hand side of (20). That reduces the highest stationary effort $e(a)$ consistent with (20) and the individual rationality conditions in (6) and (12) for continuation of the relationship and so reduces $S_{t-1}^2(a)$. Similarly, if the principal were not to end the relationship at stage 2 of period t for $p_{t+1} = 0$, there would be a further strictly negative term on the right-hand side of (15) that would also result in an additional positive term on the right-hand side of (20). In this case, there would also be the further term

$$+\left[1-\gamma_t^p(e_t(\underline{a}))\right]\left[S_t^2(\underline{a})-P_t(\underline{a})\right]$$

on the right-hand side of (18) that is strictly negative and thus adds another strictly positive term to the right-hand side of (20). Thus a stationary contract that does not satisfy properties 1 and 2 of Proposition 4 is strictly dominated by one that does.

Let $\hat{S}^2(a)$ denote the joint gain $S_{\tau-1}^2(a)$ from an optimal stationary contract from τ on. From Levin (2003, Theorem 2), no non-stationary continuation equilibrium contract can achieve a higher joint gain than $\hat{S}^2(a)$ at $\tau-1$. But the same applies equally to any $t \geq \tau$ so, for any optimal continuation contract, $S_t^2(a) = \hat{S}^2(a)$. Thus any optimal continuation contract must satisfy the feasibility condition (18) with $S_t^2(a) = \hat{S}^2(a)$. Since $\hat{e}(a) < e^*(a)$, for any continuation contract with non-stationary effort to do as well as an optimal stationary continuation contract, it must have $e_t(a) > \hat{e}(a)$ for some $t \geq \tau$. Now $\hat{e}(a)$ satisfies (18) with $S_t^2(a) = \hat{S}^2(a)$ and $U_t(a) = P_t(a) = 0$ for all $t \geq \tau$. So, to satisfy (18) with $U_t(a), P_t(a) \geq 0$ as required for individual rationality, any non-stationary continuation equilibrium contract must have $\underline{w}_t > \underline{w} = \underline{u}$. But from (6), for $\underline{w}_t > \underline{u}$, individual rationality for the agent requires $U_t(a) \geq \underline{w}_t - \underline{u}$, so increasing $\underline{w}_t$ above $\underline{w} = \underline{u}$ would not permit (18) to be satisfied with $e_t(a) > \hat{e}(a)$ for any $t \geq \tau$. Thus, any optimal continuation contract must have $e_t(a) = \hat{e}(a)$ for all $t \geq \tau$. The remaining results in the proposition follow directly. ■

Proof of Proposition 7. By definition, $a_{t+1}^-(a)$ is the lowest type (apart from $\underline{a}$) that has the same history as a at $t+1$. For these to have the same history at $t+1$, it must be that

$$e_t\left(a_{t+1}^-(a)\right) = e_t(a)\,.$$

Because efficient effort $e^*(a)$ is non-decreasing,

$$\begin{aligned} &e_t\left(a_{t+1}^-(a)\right) < e^*\left(a_{t+1}^-(a)\right) \\ &\qquad\qquad \Longrightarrow e_t(a) < e^*(a) \;\text{ for all } a \in A_{t+1}^+(h_t,(e_t(a)\,,w_t(a)))\,, \end{aligned} \tag{A.4}$$

so $e_t(a)$ is below efficient effort for all such a. Thus, from (15), increasing $e_t(a)$ increases $S_t^1(a)$, and hence from (16) $S_{t-1}^2(a)$, for all $a \in A_{t+1}^+(h_t,(e_t(a)\,,w_t(a)))$.

Suppose the proposition were not true, that is, $e_t(a)$ for $a \in A_{t+1}(h_t,(e_t(a)\,,w_t(a)))$ did not maximize $S_{t-1}^2\left(a_{t+1}^-(a)\right)$ subject to incentive compatibility and individual rationality for $a_{t+1}^-(a)$. Then it would be feasible to change $e_t(a)$ while still satisfying incentive compatibility and individual rationality for $a_{t+1}^-(a)$ in such a way as to increase $S_{t-1}^2(a)$ for all $a \in A_{t+1}(h_t,(e_t(a)\,,w_t(a)))$, unless doing so violated incentive compatibility or individual rationality for some other $a \in A_{t+1}(h_t,(e_t(a)\,,w_t(a)))$. In view of (A.4), this would require an increase in $e_t(a)$. But, for given $\langle e,\alpha\rangle$ and h_t,

$U_t(a, h_t)$ is non-decreasing in a, so anything that satisfies agent individual rationality (6) for $a^-_{t+1}(a)$ also does so for all $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$ and, since the principal's payoff at t is necessarily the same for all $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$, anything that satisfies the principal's conditions (11) and (12) for $a^-_{t+1}(a)$ also does so for all $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$. Thus the only constraints that might be violated are incentive compatibility constraints for $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$ to choose $e_t(a)$. But, for an increase in $e_t(a)$ for $e_t(a) < e^*(a)$, it is always feasible to increase payment to compensate the agent for the higher effort without reducing the payoff to the principal. Consider an increase in payment just sufficient to compensate $a^-_{t+1}(a)$. Because the cost of effort is non-increasing in a, this will certainly at least compensate any other $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$ — recall that $a^-_{t+1}(A^+_{t+1}(h_t, (e_t(a), w_t(a))))$ is the lowest such a apart from $\underline{a}$ — so the payoff gain to $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$ choosing $e_t(a)$ becomes no less attractive relative to any other effort level. Then none of the incentive compatibility or individual rationality constraints for any $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$ would be violated by the increase in $e_t(a)$ and the original $e_t(a)$ could not, therefore, maximize $S^2_{t-1}(a)$, for all $a \in A^+_{t+1}(h_t, (e_t(a), w_t(a)))$.

When *all* types $a \in A^+_t(h_t)$ are to remain pooled from t on, it must be that $P_t(a) = P_t(a^-(a))$ for all $a \in A^+_t(h_t)$ because these types all choose the same effort at all future dates and the principal must, therefore, always pay them the same. Moreover, it must be that $P_t(a^-(a)) \geq 0$ because otherwise the principal would not continue the relationship even though $p_{t+1} = 1$. So for $a^-_t(a)$ all the constraints in Proposition 6 still apply and $a^-_t(a)$ could thus certainly not do better than with the continuation contract in Proposition 6. Moreover, only one effort is specified in the contract for those with history h_t, so (21) remains the only incentive compatibility condition for type $a^-_t(a)$ in a stationary continuation contract. Thus, the set of optimal stationary continuation contracts is the same as when type $a^-_t(a)$ is the only type in $A^+_t(h_t)$. By Theorem 2 in Levin (2003), if an optimal contract exists, there are stationary contracts that are optimal, so Proposition 6 applies just as when $a^-_t(a)$ is the only type in $A^+_t(h_t)$. ■

Lemma 1 *Let $A^+(h_\tau)$ denote the subset of $a \in A^+_\tau(h_\tau)$ for which a continuation contract specifies that for all $t \geq \tau$: (1) $e_t(a, h_t) = \hat{e}(a^-(a)) < e^*(a^-(a))$ for all $a \in A^+(h_\tau)$ if $p_{t+1} = 1$, where $a^-(a) = \min \tilde{a} \in A^+(h_\tau)$ for $a \in A^+(h_\tau)$; (2) the agent ends the relationship at t for $a_t = \underline{a}$; and (3) the principal ends the relationship at t for $p_{t+1} = 0$. For the continuation contract to be an equilibrium continuation contract, $U_t(a^-(a)) = P_t(a^-(a)) = 0$, $\underline{w}_t = \underline{u}$ for all $t \geq \tau$ and*

$$S^2_t(a^-(a)) = \frac{c(\hat{e}(a^-(a)), a^-(a))}{\rho_p}, \qquad \textit{for all } t \geq \tau - 1. \tag{A.5}$$

Proof. It must be that $P_t(a) = P_t(a^-(a))$ for all $a \in A^+(h_\tau)$ because these types all choose the same effort at all future dates and the principal must, therefore, always pay them the same. Moreover, it must be that $P_t(a^-(a)) \geq 0$ because otherwise the principal would not continue the relationship even though $p_{t+1} = 1$ at whatever $t > \tau$ all $a \notin A^+(h_\tau)$ have a different history. By definition, $\hat{e}(a)$ satisfies (22) with equality, so the right-hand side of (20) must equal zero for $a^-(a)$ for all $t > \tau$. For this to be consistent with the condition in (6) for agent type $a^-(a)$ to continue the relationship, it must be that $P_t(a^-(a)) = 0$ for $t > \tau$. For the agent to end the relationship for any $t > \tau$ for which $a_t = \underline{a}$, it must be that $\underline{w}_t \leq \underline{u}$ for all $t > \tau$. But then, for the right-hand side of (20) to equal zero and (6) for agent type $a^-(a)$ to continue the relationship to be satisfied for all $t > \tau$, it must be that $\underline{w}_t = \underline{u}$ and $U_t(a^-(a)) = 0$ for all $t > \tau$. With $\hat{e}(a^-(a))$ satisfying (22) with equality, (19) implies (A.5). ■

Proof of Proposition 8. For it to be feasible to separate a'' from $\hat{a}$ at t, either a'' is the lowest $a \in (\hat{a}, a'']$ that it is feasible to separate or there exists some lower a that is. The relational contract in the proposition specifies that, conditional on the relationship continuing at $t+1$, efforts for a and $a' \in [\hat{a}, a'')$ from $t+1$ on are $\hat{e}(a) < e^*(a)$ and $\hat{e}(\hat{a}) \leq e^*(\hat{a})$ if separated at t, so Lemma 1 implies $U(\hat{a}) = U(a) = 0$. With $c_2 < 0$, this implies the payoff gain at $\tau + 1$ to any type $a' < a$ from choosing $\hat{e}(a)$ for all dates $t > \tau$ is strictly negative. It follows that, for all $a, a', \hat{a} \geq \alpha_{t+1}(h_{t+1})$,

$$\begin{aligned}
U_{t+1}(a, (h_{t+1}, e_{t+1}(a, h_{t+1}))) &= U(a) = 0, \\
U_{t+1}(a', (h_{t+1}, e_{t+1}(a, h_{t+1}))) &< 0, \quad \text{for } a' < a, \\
U_{t+1}(a, (h_{t+1}, e_{t+1}(\hat{a}, h_{t+1}))) &= \frac{1}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a)\right] > 0, \quad \text{for } \hat{a} < a,
\end{aligned} \tag{A.6}$$

the final condition corresponding to the future rent type a can gain from not separating from $\hat{a}$ at t and choosing $\hat{e}(\hat{a})$ for all dates $t+1$ on, as the relational contract requires for the principal not to end the relationship at $\tau > t$ even though $p_{\tau+1} = 1$. For a the lowest type feasible to separate from $\hat{a}$ at t, effort at t for $a' \in [\hat{a}, a)$ must be the same as for $\hat{a}$. Under these conditions, $\gamma_t^p(h_t, e_t(a', h_t)) = \rho_p$ so (10) for a and $a' \in [\hat{a}, a)$ corresponds to

$$\begin{aligned}
&-c(e_t(\hat{a}, h_t), a') + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a')\right] \\
&\qquad + c(e_t(\hat{a}, h_t), a) - \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a)\right] \\
&\qquad \geq U_t(a', h_t) - U_t(a, h_t) \\
&\qquad\qquad \geq -c(e_t(a, h_t), a') + c(e_t(a, h_t), a), \quad \text{all } a' \in [\hat{a}, a),
\end{aligned}$$

or, multiplying through by -1,

$$\begin{aligned}
&c(e_t(a, h_t), a') - c(e_t(a, h_t), a) \\
&\qquad \geq U_t(a, h_t) - U_t(a', h_t) \\
&\qquad\qquad \geq c(e_t(\hat{a}, h_t), a') - c(e_t(\hat{a}, h_t), a) \\
&\qquad\qquad\qquad + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}), a') - c(\hat{e}(\hat{a}), a)\right], \quad \text{all } a' \in [\hat{a}, a).
\end{aligned} \tag{A.7}$$

Necessity. By Proposition 1, (10) is necessary for incentive compatibility and, hence, (A.7) is necessary for separation of a and a' under the conditions of the proposition. By Assumption 1, $c(\tilde{e}, a)$ is differentiable, and hence continuous, in a, so (A.7) must hold with equalities in the limit as $a' \to a$. Thus, for it to hold for $a' < a$, the derivative of the left-hand side with respect to a' must be no greater than the derivative of the right-hand side at $a' = a$. This implies (30) is necessary, as is $\tilde{e} \leq \hat{e}(a)$, because no higher effort would be consistent with the dynamic enforcement constraint

(18) and the individual rationality conditions (6) and (12) for agent type a to continue the relationship and for the principal to continue it for $p_{t+1} = 1$. Use of the right-hand inequality in (A.7) for $a' = \hat{a}$ (with the argument h_t dropped) to substitute for $U_t(a)$ in (18) and of $\gamma_t^p(e_t(a)) = \rho_p$ implies

$$\begin{aligned} U_t(\hat{a}) + c(e_t(\hat{a}), \hat{a}) - c(e_t(\hat{a}), a) + \frac{\delta \rho_p \rho_a}{1 - \delta \rho_p \rho_a} \left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a) \right] \\ \leq -c(e_t(a), a) - \underline{u} + \underline{w}_t + \rho_p \left[S_t^2(a) - P_t(a) \right]. \end{aligned}$$

For $e_\tau(a) = \hat{e}(a) < e^*(a)$ for all $\tau > t$, Lemma 1 implies $S_t^2(a) = c(\hat{e}(a), a)/\rho_p$. Use of this in the above and re-arrangement yields

$$\begin{aligned} c(e_t(a), a) + c(e_t(\hat{a}), \hat{a}) - c(e_t(\hat{a}), a) + \frac{\delta \rho_p \rho_a}{1 - \delta \rho_p \rho_a} \left[c(\hat{e}(\hat{a}), \hat{a}) - c(\hat{e}(\hat{a}), a) \right] \\ \leq c(\hat{e}(a), a) - \rho_p P_t(a) - U_t(\hat{a}) - \underline{u} + \underline{w}_t. \qquad \text{(A.8)} \end{aligned}$$

It follows that (31) is necessary.

Sufficiency. Consider relational contracts with the agent ending the relationship at $\tau \geq t$ for $a_\tau = \underline{a}$, the principal doing that for $p_{\tau+1} = 0$, and effort functions conditional on the relationship continuing

$$e_t(\tilde{a}, h_t) = \begin{cases} e_t(\hat{a}, h_t), & \text{for } \tilde{a} \in [\hat{a}, a), \\ \tilde{e} \leq \hat{e}(a), & \text{for } \tilde{a} \geq a \text{ with history } h_t, \end{cases} \qquad \text{(A.9)}$$

$$e_\tau(\tilde{a}, h_\tau) = \begin{cases} \hat{e}(\hat{a}), & \text{for } \tilde{a} \in [\hat{a}, a), \\ \hat{e}(a), & \text{for } \tilde{a} \geq a \text{ with history } h_t, \end{cases} \quad \text{for all } \tau > t, \qquad \text{(A.10)}$$

with $\tilde{e}$ such as to satisfy (31) and (32).

Consider first continuation contracts from $t + 1$ with these properties. With $\hat{a}$ satisfying (25) as specified in the proposition, $\hat{e}(\hat{a})$ and $\hat{e}(a)$ both exist and, by the definition of $\hat{e}(a)$ in (28), (23) is satisfied for $\hat{a}$ with effort $\hat{e}(\hat{a})$ and for $\hat{a}$ with effort $\hat{e}(a)$. So Proposition 4 ensures that there exist continuation contracts at $t + 1$ with these properties that are continuation equilibria conditional on the efforts in (A.9). From Lemma 1, $\underline{w}_\tau = \underline{u}$ for all $\tau \geq t$ and

$$\begin{aligned} S_\tau^2(\hat{a}) &= \frac{c(\hat{e}(\hat{a}), \hat{a})}{\rho_p}, \quad \text{for all } \tau \geq t, \\ S_\tau^2(a) &= \frac{c(\hat{e}(a), a)}{\rho_p}, \quad \text{for all } \tau \geq t. \end{aligned}$$

From (19), with the efforts specified in (A.10) for $a' \in [\hat{a}, a)$,

$$\begin{aligned} S_\tau^2(a') &= \frac{\delta \rho_a}{1 - \delta \rho_p \rho_a} \left[\hat{e}(\hat{a}) - c\big(\hat{e}(\hat{a}), a'\big) - \underline{u} - \underline{v} \right] \\ &= \frac{c(\hat{e}(\hat{a}), \hat{a})}{\rho_p} + \frac{\delta \rho_a}{1 - \delta \rho_p \rho_a} \left[c(\hat{e}(\hat{a}), \hat{a}) - c\big(\hat{e}(\hat{a}), a'\big) \right], \quad \text{for } a' \in [\hat{a}, a), \tau \geq t. \end{aligned}$$

Now consider period t conditional on those continuation equilibria. The principal's payoff gain $P_t(a)$ must be the same for all $a \in A_t(h_t)$ who choose the same effort at t.

Then, from (18), the expressions just given for $S^2_\tau(a')$ and $S^2_\tau(a)$, and $\gamma^p_t(e_t(a)) = \rho_p$, the efforts in (A.9) result in

$$\begin{aligned} U_t(a') &= -c(e_t(\hat{a},h_t),a') - \underline{u} + \underline{w}_t + c(\hat{e}(\hat{a}),\hat{a}) \\ &\qquad + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}),\hat{a}) - c(\hat{e}(\hat{a}),a')\right] - \rho_p P_t(\hat{a}), \quad \text{for } a' \in [\hat{a},a), \quad \text{(A.11)} \\ U_t(a) &= -c(\tilde{e},a) - \underline{u} + \underline{w}_t + c(\hat{e}(a),a) - \rho_p P_t(a). \quad \text{(A.12)} \end{aligned}$$

Moreover, from (A.11) for $a' = \hat{a}$,

$$\rho_p P_t(\hat{a}) = c(\hat{e}(\hat{a}),\hat{a}) - c(e_t(\hat{a},h_t),\hat{a}) - U_t(\hat{a}) - \underline{u} + \underline{w}_t, \tag{A.13}$$

so

$$\begin{aligned} U_t(a') &= -c(e_t(\hat{a},h_t),a') + c(e_t(\hat{a},h_t),\hat{a}) \\ &\qquad + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}),\hat{a}) - c(\hat{e}(\hat{a}),a')\right] + U_t(\hat{a}), \quad \text{for } a' \in [\hat{a},a). \quad \text{(A.14)} \end{aligned}$$

With $\underline{w}_\tau = \underline{u} > -\underline{v}$ for all $\tau \geq t$, Proposition 3 ensures that it is a best response for the principal to end the relationship if the agent is believed to be type $\underline{a}$, which in turn implies $e_t(\underline{a}) = 0$. Thus, (10) is sufficient for incentive compatibility by Proposition 1. For the efforts specified in (A.9) and (A.10), that implies (A.7) with $e_t(a,h_t) = \tilde{e}$ is sufficient for $a' \in [\hat{a},a)$. From (A.12) and (A.14),

$$\begin{aligned} U_t(a) - U_t(a') &= -\left[c(\tilde{e},a) - c(e_t(\hat{a},h_t),a')\right] + \left[c(\hat{e}(a),a) - c(e_t(\hat{a},h_t),\hat{a})\right] \\ &\quad - \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}),\hat{a}) - c(\hat{e}(\hat{a}),a')\right] - \rho_p P_t(a) - U_t(\hat{a}) - \underline{u} + \underline{w}_t. \quad \text{(A.15)} \end{aligned}$$

From Assumption 1, $c(\tilde{e},a)$ is differentiable, and hence continuous, in its arguments, so to satisfy (A.7) requires $U_t(a) = \lim_{a'\to a} U_t(a')$. From (A.15), that requires

$$\begin{aligned} c(\tilde{e},a) + c(e_t(\hat{a},h_t),\hat{a}) - c(e_t(\hat{a},h_t),a) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\left[c(\hat{e}(\hat{a}),\hat{a}) - c(\hat{e}(\hat{a}),a)\right] \\ = c(\hat{e}(a),a) - \rho_p P_t(a) - U_t(\hat{a}) - \underline{u} + \underline{w}_t. \end{aligned}$$

With (31) satisfied, there exists $P_t(a) \geq 0$ and $U_t(\hat{a}) \geq \max[0, \underline{w}_t - \underline{u}]$ for which this is satisfied. Consider $U_t(\hat{a}) = \underline{w}_t - \underline{u} = 0$. Then from (A.13), $P_t(\hat{a}) \geq 0$ because $e_t(\hat{a},h_t) \leq \hat{e}(\hat{a})$. Because, with the efforts specified, the principal's payoff gain at t is the same for all types who choose the same effort at t, these imply that the principal's incentive condition (11) is satisfied at t for all $a' \in [\hat{a},a)$ and for all $\tilde{a} \geq a$ with history h_t. Moreover, from (A.14), $U_t(a')$ is non-decreasing in a' for $a' \in [\hat{a},a)$, $U_t(a) = \lim_{a'\to a} U_t(a')$, and $U_t(\tilde{a}) \geq U_t(a)$ for $\tilde{a} \geq a$ with history h_t. Thus the agent's individual rationality condition in (6) for continuing the relationship is satisfied for all $\tilde{a} \geq \hat{a}$ with history h_t. (Any agent types $\tilde{a} < \hat{a}$ with history h_t will then have $U_t(\tilde{a}) < 0$ and so will not continue the relationship.) Furthermore, from (A.15),

$$\frac{\partial}{\partial a'}\left[U_t(a) - U_t(a')\right] = c_2(e_t(\hat{a},h_t),a') + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a} c_2(\hat{e}(\hat{a}),a'), \tag{A.16}$$

which is the same as the derivative of the right-most term in (A.7). So, for $U_t(a) = \lim_{a' \to a} U_t(a')$, the right-hand inequality in (A.7) is satisfied with equality for all $a' \in [\hat{a}, a)$. Next note that, when (32) is satisfied, integration of both sides with respect to $\tilde{a}$ from a' to a, with $e_t(a, h_t)$ set equal to $\tilde{e} > e_t(\hat{a}, h_t)$, ensures that the left-most term in (A.7) is no less than the right-most term for $a' \in [\hat{a}, a)$. Thus (A.7) is satisfied, and hence so is (10), for a and for $a' \in [\hat{a}, a)$.

For any $\tilde{a} > a$ with the same history h_t, Proposition 2 ensures $e_t(a, h_t) = \tilde{e}$ is preferred to $e_t(\hat{a}, h_t)$ given that it is by a. Thus the relational contract specified is also a continuation equilibrium for such $\tilde{a}$ and thus separates $a'' \geq a$ from $\hat{a}$. ■

Lemma 2 *The effort function $\hat{e}(a)$, defined by equation (28), is strictly increasing for all $a \in [\hat{\alpha}, \overline{a}]$ and $c(\hat{e}(a), a)$ is strictly increasing in a for all $a \in [\hat{\alpha}, \overline{a}]$ such that $\hat{e}(a) < e^*(a)$.*

Proof. For $\hat{e}(a) = e^*(a)$, the first result follows immediately from $e^*(a)$ strictly increasing. For $\hat{e}(a) < e^*(a)$, from (28) and (22),

$$\delta\rho_p\rho_a\hat{e}(a) - c(\hat{e}(a), a) - \delta\rho_p\rho_a(\underline{u} + \underline{v}) = 0, \quad \text{for all } \in [\hat{\alpha}, \overline{a}], \tag{A.17}$$

so, differentiating with respect to a,

$$\left[\delta\rho_p\rho_a - c_1(\hat{e}(a), a)\right]\frac{\partial}{\partial a}\hat{e}(a) - c_2(\hat{e}(a), a) = 0, \quad \text{for all } \in [\hat{\alpha}, \overline{a}]. \tag{A.18}$$

Define $e^0(a)$ as the value of $\tilde{e}$ that solves (25). Because, by definition, $a = \hat{\alpha}$ satisfies (22) with equality, it follows that $\hat{e}(\hat{\alpha}) = e^0(\hat{\alpha})$. Moreover, the square bracket in (A.18) is zero for $a = \hat{\alpha}$. Because $c_1(e, a) > 0$ and $c_2(e, a) < 0$ for $e > 0$, the solution $\hat{e}(a)$ to (A.17) has $\hat{e}(a) > e^0(a)$ for all $a > \hat{\alpha}$. But then the square bracket in (A.18) is certainly negative for $a > \hat{\alpha}$. Hence, with $c_2 < 0$, it follows from (A.18) that $\partial\hat{e}(a)/\partial a$ is strictly positive for $a > \hat{\alpha}$. With $\hat{e}(a)$ strictly increasing in a, it follows immediately from (A.17) that $c(\hat{e}(a), a)$ is too for all $a \in [\hat{\alpha}, \overline{a}]$ such that $\hat{e}(a) < e^*(a)$. ■

Proof of Proposition 9. Part 1. Consider first $\underline{a}_t$. Then $e_t(\underline{a}_t, h_t) \leq \hat{e}(\underline{a}_t)$ because $e_t(\underline{a}_t, h_t) = \hat{e}(\underline{a}_t)$ is the highest sustainable effort level for $\underline{a}_t$ in a continuation contract with $\hat{e}(\underline{a}_t) < e^*(\underline{a}_t)$. Now consider $a \in (\underline{a}_t, \overline{a}_t]$. One possibility is that a is not separated from $\underline{a}_t$ at t, in which case $e_t(a, h_t) = e_t(\underline{a}_t, h_t) \leq \hat{e}(\underline{a}_t) < \hat{e}(a)$, the final inequality following because, by Lemma 2, $\hat{e}(a)$ is strictly increasing in a. The other possibility is that a is separated from $\underline{a}_t$ at t, in which case $a^-_{t+1}(a)$ must be separated from $\underline{a}_t$ at t because, by definition of $a^-_{t+1}(a)$, $e_t\left(a^-_{t+1}(a), h_t\right) = e_t(a, h_t)$ given that both $a^-_{t+1}(a)$ and a have to have the same history at $t+1$. That $e_t\left(a^-_{t+1}(a), h_t\right) < \hat{e}\left(a^-_{t+1}(a)\right)$ follows directly from (31) and the incentive conditions (6) and (11) because $c_{12} < 0$ implies that the term in braces in (31) is strictly positive for $a^-_{t+1}(a) > \underline{a}_t$. But then $e_t(a, h_t) = e_t\left(a^-_{t+1}(a), h_t\right) < \hat{e}\left(a^-_{t+1}(a)\right) < \hat{e}(a)$, again with the final inequality following because, by Lemma 2, $\hat{e}(a)$ is strictly increasing in a. That completes the proof of Part 1.

Part 2. It follows directly from (30) and $c_2(\tilde{e}, a), c_{12}(\tilde{e}, a) < 0$ that $\tilde{e} - e_t(\hat{a}, h_t)$ is bounded below by some $\varepsilon > 0$. The result follows from $e_t(a, h_t) = \tilde{e}$.

Part 3. This follows from Part 2 because $e_t(a, h_t)$ for $a \in [\underline{a}_t, \overline{a}_t]$ is a non-decreasing function taking values in the bounded interval $[0, e^*(\overline{a}_t(h_t))]$ and there exists ε such that, where $e_t(a, h_t)$ is not constant in a, it changes by a step of at least $\varepsilon > 0$. It is, therefore, a step function. So not all types $a \in [\underline{a}_t, \overline{a}_t]$ can have different effort and hence not all types can be separated. ■

Proof of Proposition 10. For a finest partition equilibrium, it must be feasible to separate type a^{i+1} from a^i at $t=1$. With the formulation $c(\tilde{e},a)=\hat{c}(\tilde{e})\,\phi(a)$, Corollary 1 applies, so (34) must be satisfied at $t=1$ for $a=a^i$ and $\hat{a}=a^{i-1}$ for all $i\geq 2$. Moreover, (A.14) must hold for $t=1$, $a'=a^i$ and $\hat{a}=a^{i-1}$ for all $i\geq 2$ given the continuity of $U_t(a)$ established in the proof of Proposition 8. Under these conditions and with the formulation $c(\tilde{e},a)=\hat{c}(\tilde{e})\,\phi(a)$, (A.14) can be written

$$U_1\left(a^i\right)=\left[\hat{c}\left(e_1\left(a^{i-1},h_1\right)\right)+\frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\left[\phi\left(a^{i-1}\right)-\phi\left(a^i\right)\right]$$
$$+U_1\left(a^{i-1}\right),\quad i\geq 2.$$

Recursive substitution for $U_1\left(a^{i-1}\right)$ yields

$$U_1\left(a^i\right)=\sum_{j=1}^{i-1}\left[\hat{c}\left(e_1\left(a^j,h_1\right)\right)+\frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\phi\left(a^j\right)-\phi\left(a^{j+1}\right)\right]+U_1\left(a^1\right),$$
$$i\geq 2.$$

For $a<a^1$ to end the relationship at $t=1$ but a^1 to continue it, it must be that $U_1\left(a^1\right)=0$ which, from (6), implies $\underline{w}_1\leq\underline{u}$ because only $a^1=\alpha_1(h_1)>\underline{a}$ can satisfy (25), as specified in the proposition. Use of these in (34) with $a=a^{i+1}$ and $\hat{a}=a^i$ yields

$$\begin{aligned}&\left[\hat{c}\left(e_1\left(a^i,h_1\right)\right)+\frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\phi\left(a^i\right)\\&\quad+\sum_{j=1}^{i-1}\left[\hat{c}\left(e_1\left(a^j,h_1\right)\right)+\frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\phi\left(a^j\right)-\phi\left(a^{j+1}\right)\right]\\&\qquad\leq\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right)-\rho_pP_1\left(a^{i+1}\right)-\underline{u}+\underline{w}_1,\quad i=1,\ldots,n,\end{aligned}\tag{A.19}$$

with the convention that the summation is zero for $i=1$. For the formulation $c(\tilde{e},a)=\hat{c}(\tilde{e})\,\phi(a)$ in the proposition, it follows from Corollary 1 that (A.19) is necessary and sufficient for there to exist $e_1\left(a^i,h_1\right)=\tilde{e}$ that satisfies (33) for $t=1$, $a=a^i$ and $\hat{a}=a^{i-1}$ for all $i\geq 2$ such that separation of a^{i+1} from a^i at $t=1$ is feasible with the continuation efforts specified for a finest partition equilibrium.

In a finest partition equilibrium, a^i is defined as the lowest type that can be separated from a^{i-1}. With $\hat{e}(a)<e^*(a)$ for all $a\in[\underline{a},\overline{a}]$, it follows from Lemma 2 that $c(\hat{e}\,(a)\,,a)$, which corresponds to $\hat{c}(\hat{e}(a))\,\phi(a)$ in the specification in the proposition, is strictly increasing in a for all a for which the relationship will be continued. Thus, the a^{i+1} closest to a^i consistent with (A.19) has (A.19) hold with equality and $\underline{u}-\underline{w}_1$ and $P_1\left(a^{i+1}\right)$ at the lowest values consistent with the individual rationality conditions (6) with $U_1\left(a^1\right)=0$ and (11), namely 0. Moreover, the a^{i+1} that satisfies this is unique.

Thus, in a finest partition equilibrium,

$$\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right) = \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\phi\left(a^i\right)$$
$$+\sum_{j=1}^{i-1}\left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\phi\left(a^j\right) - \phi\left(a^{j+1}\right)\right],$$
$$i = 1, \ldots, n. \quad \text{(A.20)}$$

Moreover, the right-hand side of (A.20) is strictly increasing in $e_1\left(a^i, h_1\right)$ for given $\left(a^i, h_1\right)$. Thus, for a^{i+1} to be as close to a^i as possible, $e_1\left(a^i, h_1\right)$ must be as low as possible, which implies $e_1\left(a^i, h_1\right) = \tilde{e}$ such that (33) for $t = 1$ and $\hat{a} = a^{i-1}$ holds with equality. That implies (39).

For $i = 1$, (A.20) implies (37) because the summation term is zero for $i = 1$. For $i \geq 2$, (A.20) implies

$$\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right) - \hat{c}\left(\hat{e}\left(a^i\right)\right)\phi\left(a^i\right)$$
$$= \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\phi\left(a^i\right)$$
$$+\sum_{j=1}^{i-1}\left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\phi\left(a^j\right) - \phi\left(a^{j+1}\right)\right]$$
$$-\left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\phi\left(a^{i-1}\right)$$
$$-\sum_{j=1}^{i-2}\left[\hat{c}\left(e_1\left(a^j, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^j\right)\right)\right]\left[\phi\left(a^j\right) - \phi\left(a^{j+1}\right)\right], \quad i = 2, \ldots, n,$$

or, cancelling terms under the summation signs,

$$\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right) - \hat{c}\left(\hat{e}\left(a^i\right)\right)\phi\left(a^i\right)$$
$$= \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\phi\left(a^i\right)$$
$$+\left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\left[\phi\left(a^{i-1}\right) - \phi\left(a^i\right)\right]$$
$$-\left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\phi\left(a^{i-1}\right), \quad i = 2, \ldots, n,$$

or

$$\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right) - \hat{c}\left(\hat{e}\left(a^i\right)\right)\phi\left(a^i\right)$$
$$= \left[\hat{c}\left(e_1\left(a^i, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^i\right)\right)\right]\phi\left(a^i\right)$$
$$-\left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\phi\left(a^i\right), \quad i = 2, \ldots, n.$$

Use of (39) with $\hat{c}\left(e_1\left(a^i, h_1\right)\right)$ on the left-hand side to substitute for $\hat{c}\left(e_1\left(a^i, h_1\right)\right)$ in this yields

$$\begin{aligned}
&\hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right) - \hat{c}\left(\hat{e}\left(a^{i}\right)\right)\phi\left(a^{i}\right) \\
&= \left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i}\right)\right)\right]\phi\left(a^{i}\right) \\
&\quad - \left[\hat{c}\left(e_1\left(a^{i-1}, h_1\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i-1}\right)\right)\right]\phi\left(a^{i}\right) \\
&= \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i}\right)\right)\phi\left(a^{i}\right), \quad i = 2, \ldots, n. \qquad \text{(A.21)}
\end{aligned}$$

This can be written as (38). Moreover, by Lemma 2 and $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$, the right hand side of (A.21) is increasing with i because $a^i > a^{i-1}$. But $\hat{e}(\overline{a}) < e^*(\overline{a})$ and the assumptions of the model ensure $e^*(\overline{a}) < \overline{e}$ and $\hat{c}(\overline{e})$ bounded above, which is sufficient to ensure that the number of partitions is finite.

To show that no further separation is feasible after period 1 of a finest partition equilibrium, note from the definition of such an equilibrium that, for $t \geq 2$, $e_t\left(a^i, h_t\right) = \hat{e}\left(a^i\right)$ for $i = 1, \ldots, n$. Then, from Lemma 1 , $\underline{w}_t = \underline{u}$ and $U_t\left(a^i\right) = 0$ for $t \geq 2$. It follows from (34) in Corollary 1 that, to be feasible to separate $a > a^i$ from a^i at $t \geq 2$, it must be that

$$\left[\hat{c}\left(\hat{e}\left(a^{i}\right)\right) + \frac{\delta\rho_p\rho_a}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i}\right)\right)\right]\phi\left(a^{i}\right) \leq \hat{c}(\hat{e}(a))\,\phi(a) - \rho_p P_t(a) \qquad \text{(A.22)}$$

or

$$\frac{1}{1-\delta\rho_p\rho_a}\hat{c}\left(\hat{e}\left(a^{i}\right)\right)\phi\left(a^{i}\right) \leq \hat{c}(\hat{e}(a))\,\phi(a) - \rho_p P_t(a)\,. \qquad \text{(A.23)}$$

Further separation is possible only if there is an $a < a^{i+1}$ that satisfies this. By Lemma 2 and $\hat{e}(a) < e^*(a)$ for all $a \in [\underline{a}, \overline{a}]$, $\hat{c}(\hat{e}(a))\,\phi(a)$ is strictly increasing in a for all a for which the relationship will be continued, so that can be the case only if $\hat{c}(\hat{e}(a))\,\phi(a) < \hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right)$. But because the lowest value of $P_t(a)$ consistent with the incentive condition (11) is zero, (A.23) and (38) imply $\hat{c}(\hat{e}(a))\,\phi(a) \geq \hat{c}\left(\hat{e}\left(a^{i+1}\right)\right)\phi\left(a^{i+1}\right)$ for $i \geq 2$ and (A.22) and (37) imply the same for $i = 1$. Thus there can exist no $a < a^{i+1}$ that can be separated from a^i at $t \geq 2$. ∎

Proof of Proposition 11. By Propositions 6 and 7, it is optimal for the pooling relational contract to have the relationship end at t for $a_t = \underline{a}$ and for $p_{t+1} = 0$. For such a contract, it follows from Proposition 5 and the definition of $S^2(a)$ in (16) that, for a pooling equilibrium relational contract with $e_t(a, h_t) = \hat{e}$ for all types $a \geq \hat{a}$, h_t and t,

$$S^2(\hat{a}) = \frac{\delta\rho_a}{1-\delta\rho_p\rho_a}\left[\hat{e} - c(\hat{e}, \hat{a}) - \underline{u} - \underline{v}\right] \geq \frac{c(\hat{e}, \hat{a})}{\rho_p}.$$

For $(\hat{e}, \hat{a})$ optimal, it must clearly be that this holds with equality because otherwise it would be possible to have more types matched with the same effort $\hat{e}$ by lowering $\hat{a}$, thus generating a greater joint gain. With some re-arrangement, that gives

$$\delta\rho_p\rho_a\left[\hat{e} - c(\hat{e}, \hat{a}) - \underline{u} - \underline{v}\right] = \left(1 - \delta\rho_p\rho_a\right)c(\hat{e}, \hat{a})$$

or

$$\delta\rho_p\rho_a\hat{e} - c(\hat{e},\hat{a}) - \delta\rho_p\rho_a\left(\underline{u} + \underline{v}\right) = 0,$$

which, by (28), implies $\hat{e} = \hat{e}\left(\hat{a}\right)$.

Now consider an equilibrium partition relational contract that separates $a \in [\hat{a}, \tilde{a})$ from $a \in [\tilde{a}, \overline{a}]$ in period 1 but with pooling *within* each of these intervals, with continuation efforts for $t \geq 2$ of $e_t(a) = \hat{e}(\hat{a})$ for $a \in [\hat{a}, \tilde{a})$ and $e_t(a) = \hat{e}(\tilde{a})$ for $a \in [\tilde{a}, \overline{a}]$. By the definition of $\hat{e}\left(a\right)$ in (28), (23) is satisfied for $\hat{a}$ with effort $\hat{e}\left(\hat{a}\right)$ and for a with effort $\hat{e}\left(a\right)$, so Proposition 5 ensures that there exists an equilibrium continuation contract for $t \geq 2$ with these continuation efforts. Then, for all $a \in [\hat{a}, \tilde{a})$, $e_t(a)$ for $t \geq 2$ is no further from the efficient level $e^*(a)$ than is $\hat{e}$. Now consider $a \in [\tilde{a}, \overline{a}]$. By Lemma 2, the effort function $\hat{e}(a)$, defined by (28), is strictly increasing for $a \in [\hat{\alpha}, \overline{a}]$, so $\hat{e}(\tilde{a}) > \hat{e}(\hat{a}) = \hat{e}$. Thus, for all $a \in [\tilde{a}, \overline{a}]$, $e_t(a)$ for $t \geq 2$ is strictly closer to the efficient level $e^*(a)$ than is $\hat{e}$. So the joint gain from $t = 2$ on is strictly greater for $a \in [\tilde{a}, \overline{a}]$, and no less for $a \in [\hat{a}, \tilde{a})$, with this relational contract than with the pooling relational contract with $(\hat{e}, \hat{a})$ optimal.

Now consider $t = 1$. For $a = \hat{a}$, it is feasible to set $e_1(\hat{a}) = \hat{e}(\hat{a}) = \hat{e}$. Provided there exist an a and $\tilde{e}$ that satisfy (31) and (32) for $e_1(\hat{a}, h_1) = \hat{e}(\hat{a}) = \hat{e}$, Proposition 8 ensures that there exists an a that can be separated from $\hat{a}$ in period 1. Let $\tilde{a}$ be such an a and note that, from Part 2 of Proposition 9, $e_1(\tilde{a}) > \hat{e}$. Moreover, there then exists an equilibrium relational contract with pooling in period 1 of all $a \in [\tilde{a}, \overline{a}]$ with effort $e_1(\tilde{a})$, and of all types $a \in [\hat{a}, \tilde{a})$ with $e_1(a) = \hat{e}$, and the continuation equilibria specified for period 2 on. Then, for all $a \in [\hat{a}, \tilde{a})$, $e_1(a)$ is no further from the efficient level $e^*(a)$ than is $\hat{e}$ and, for all $a \in [\tilde{a}, \overline{a}]$, $e_1(a)$ is strictly closer to the efficient level $e^*(a)$ than is $\hat{e}$. So the joint gain for $t = 1$, as well as that from $t = 2$ on, is strictly greater for $a \in [\tilde{a}, \overline{a}]$, and no less for $a \in [\hat{a}, \tilde{a})$, with the equilibrium partition relational contract involving some separation than with the pooling relational contract with $(\hat{e}, \hat{a})$ optimal. ■

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