Risks For the Long Run: Estimation and Inference*

Ravi Bansal†
Dana Kiku‡
Amir Yaron§

March 2007

Abstract

Recent work by Bansal and Yaron (2004) on Long Run Risks suggests that these fundamental economic risks can account for key features of asset market data. In this paper we develop methods to estimate their equilibrium model by exploiting the asset pricing Euler Equations. Using an empirical estimate for the long run risk component we demonstrate that the Long Run Risk model can indeed capture a rich array of asset returns. The model, at plausible preference estimates, can account for the market as well as the 'value' and 'size' premium. We show that time averaging effects—that is, a mismatch in the sampling and the agent's decision interval leads to significant biases in the estimates for risk aversion and the elasticity of intertemporal substitution. Our evidence suggests that accounting for these biases is important for interpreting the magnitudes of the preference parameters and the economic implications of the model for asset prices.

Preliminary and Incomplete

*We thank Andy Abel, Lars Hansen, John Heaton, Tom Sargent, Martin Schneider, seminar participants at University of California-Berkeley, Copenhagen School of Business, Norwegian School of Management, University of Chicago, and Wharton for useful comments.

†Fuqua School of Business, Duke University, ravi.bansal@duke.edu.

‡The Wharton School, University of Pennsylvania, kiku@wharton.upenn.edu.

§The Wharton School, University of Pennsylvania and NBER, yaron@wharton.upenn.edu.

1 Introduction

Asset market data ought to contain valuable information about investors' behavior (see Cochrane and Hansen (1992)). It should also contain information about the sources of risks that concern investors and hence drive asset markets. Asset market features, such as the low real returns on bills, the equity premium, large returns on value (high book-to-market) stocks relative to growth (low book-to-market) stocks, among others, provide an important market laboratory to simultaneously learn about sources of risks and preferences of investors. A challenging task is to account for these asset market facts with well identified risk sources and plausible investor behavior. Bansal and Yaron (2004) develop a Long Run Risk (LRR) asset pricing model and show that it can account for the risk free rate, equity premium and volatility puzzles. Further, they suggest that the same long run risks in consumption should empirically account for a rich cross section of asset returns with reasonable risk preferences. In this paper we develop methods and empirically evaluate the ability of the LRR model to account for asset market data using Euler Equation based estimation methods.

An elegant approach to evaluate the empirical plausibility of an asset pricing model, developed in Hansen and Singleton (1982), is to exploit its asset pricing Euler Equations using the Generalized Method of Moment (GMM) estimation technique. This approach provides a convenient way to impose the model restrictions on asset payoffs and learn about investor behavior. *A priori* it is not entirely clear how to proceed with this estimation as the intertemporal marginal rate of substitution in the LRR model, based on the Epstein and Zin (1989) and Weil (1989) preferences, incorporates the return on the consumption asset which is not directly observed by the econometrician. In this paper we present methods for estimating models with these recursive preferences using Euler Equations and a GMM estimator.

To make estimation feasible in the LRR model we exploit the dynamics of aggregate consumption growth and the model's Euler restrictions to solve for the unobserved return on the claim over the future consumption steam. The LRR model proposed in Bansal and Yaron (2004) has three risk sources (state variables) in the aggregate consumption dynamics: (i) high frequency or short run risks in consumption, (ii) low frequency or long run movements in consumption, and (iii) fluctuations in consumption uncertainty, i.e., consumption volatility risk. We derive expressions for the Intertemporal Marginal Rates of Substitution (IMRS) in terms of these risk sources for a wide range of risk aversion and intertemporal substitution parameters. We document that our methods for characterizing the model's pricing kernel are very accurate. Earlier work by Epstein and Zin (1991) also pursues the strategy of exploiting the Euler Equation-GMM method for estimation; however, they assume that the return on the consumption asset coincides with the observed value weighted market return. This premise, we show, can distort the estimated preferences and lead to false rejections of the model.

More recently several other papers explore the ability of long run risks to account for asset market data. Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2005) show that long run risks in cash flows are an important risk source in accounting for asset returns. Bekaert, Engstrom, and Xing (2005), Bansal, Gallant, and Tauchen (2005), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2004), and Lettau and Ludvigson (2005), also explore implication of LRR for asset returns. However, these papers, unlike the focus of this paper, do not evaluate the empirical plausibility of the LRR model from the perspective of the Euler Equation-GMM based estimation approach for a rich cross-section of assets.

Exploiting the estimation methods we develop, we find considerable empirical support for the LRR model at plausible preference configurations. Our evidence suggests that the investor concerns about long run risks are empirically important for understanding asset returns. More specifically, our Euler Equation-GMM based estimation of the LRR model shows that: (i) the long run risk component is highly persistent, displays fluctuations that are correlated with business cycles, and is economically and statistically significantly predictable by theoretically motivated variables, (ii) in the cross section, assets with large mean returns (e.g., value and small assets) are more sensitive to innovations in the long run risk variable; that is, these returns have larger betas with respect to the long run risk component while having negligible dependence on the betas constructed for short horizon consumption innovations; (iii) the model is not rejected by the overidentifying restrictions. The pricing errors for the various assets we consider seem small and plausible. In the annual data, the estimated risk aversion is in excess of twenty while estimates for the IES are less than one. We show, however, that after accounting for time averaging effects, the most likely value for the population risk aversion and IES are closer to 10 and 2 respectively.

Time averaging plays an important role in interpreting our estimates and evidence. The decision interval of the agent and the frequency with which an econometrician observes consumption data need not coincide. In the context of the LRR model, if consumption data is observed only on a coarser interval (e.g., annual), while the decision interval of the agent is on a finer interval (e.g., monthly), then the estimates of the risk aversion will be much larger than their true value, while the estimates of the IES will be lower than their true values, and typically less than one. This effect is important as much of the earlier evidence, indeed, finds estimates that are in the region of high risk aversion and low IES (see Campbell (1999), Hall (1988)). Our evidence indicates that much of the earlier evidence and the associated views regarding low values of IES and high risk aversion could simply be an artifact of time averaging effects in estimation.

In sum, the evidence in this paper shows that the long run risk model is quite capable in quantitatively pricing the time series and cross section of returns, and doing so with plausible parameter estimates. These parameter estimates can be quite difficult to precisely estimate using annual data. They tend to produce a somewhat misspecified model that leads to preference parameter estimates that are biased towards what is often found in the literature.

The paper continues as follows: Section 2 presents the model and its testable restrictions. Section 3 presents the data, while Section 4 provides the results of our empirical analysis. Section 5 presents Monte-Carlo evidence regarding time averaging. Section 6 provides concluding remarks.

2 Model

In this section we specify a model based on Bansal and Yaron (2004). The underlying environment is one with complete markets and the representative agent has Epstein and Zin (1989) type preferences which allow for the separation of risk aversion and the elasticity of intertemporal substitution. Specifically, the agent maximizes her life-time utility, which is defined recursively as,

$$V_t = \left[(1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \Big(E_t \big[V_{t+1}^{1-\gamma} \big] \Big)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} , \tag{1}$$

where C_t is consumption at time t, $0 < \delta < 1$ reflects the agent's time preferences, γ is the coefficient of risk aversion, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, and ψ is the elasticity of intertemporal substitution

(IES). Utility maximization is subject to the budget constraint,

$$W_{t+1} = (W_t - C_t) R_{c,t+1} \ , \tag{2}$$

where W_t is the wealth of the agent, and $R_{c,t}$ is the return on all invested wealth.

Consumption and dividends have the following joint dynamics:

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\ \sigma^2_{t+1} &= \bar{\sigma}^2 + \nu(\sigma^2_t - \bar{\sigma}^2) + \sigma_w w_{t+1}, \end{aligned} \tag{3}$$

where Δc_{t+1} is the growth rate of log consumption. As in the long run risks model of Bansal and Yaron (2004) (BY), $\mu_c + x_t$ is the conditional expectation of consumption growth, and x_t is a small but persistent component that captures long run risks in consumption growth. The parameter ρ determines the persistence in the conditional mean of consumption growth, $\mu_c + x_t$. For parsimony, as in Bansal and Yaron (2004), we have a common time-varying volatility in consumption, which, as shown in their paper, leads to time-varying risk premia. The unconditional variance of consumption is $\bar{\sigma}^2$ and ν governs the persistence of the volatility process.

As in Epstein and Zin (1989), it is easily shown that, for any asset j, the first order condition yields the following asset pricing Euler condition,

$$E_t \left[\exp\left(m_{t+1} + r_{j,t+1} \right) \right] = 1, \tag{4}$$

where m_{t+1} is the log of the intertemporal marginal rate of substitution and $r_{j,t+1}$ is the log of the gross return on asset j.

2.1 Estimation Feasibility

It can be shown that with the Epstein and Zin (1989) preferences, the log of the Intertemporal Marginal Rate of Substitution (IMRS), m_{t+1}, is

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}, \tag{5}$$

where $r_{c,t+1}$ is the continuous return on the consumption asset. As this return is unobservable and endogenous to the model we need to solve for it using the consumption dynamics. Epstein and Zin (1991) circumvent the unobservability of $r_{c,t+1}$ by equating it with the observed market return, $r_{m,t+1}$. Bansal and Yaron (2004) explicitly solve for this return given the consumption dynamics.

To solve for the return on wealth we use the log-linear approximation for the continuous return on the wealth portfolio, namely,

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} + \Delta c_{t+1} - z_t \ , \tag{6}$$

where $z_t = \log(P_t/C_t)$ is log price to consumption ratio (the valuation ratio corresponding to a claim that pays consumption) and the κ's are log linearization constants with κ_0 and κ_1 being,

$$\kappa_1 = \frac{\exp(\bar{z})}{1+\exp(\bar{z})} \ , \tag{7}$$

$$\kappa_0 = \log(1+\exp(\bar{z})) - \kappa_1 \bar{z} \ , \tag{8}$$

and $\bar{z}$ is the mean of the log price-consumption ratio.

To derive the time series for $r_{c,t+1}$ we require a solution for log price-consumption ratio, which we conjecture follows, $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$. The solution for the A's depends on all the preference parameters and the parameters that govern the state variables. Given this solution and the solution for the κ's, which are also endogenous to the model, one can create the return to the consumption asset.

For notational ease let the state variables for the model be $Y_t' = [1 \quad x_t \quad \sigma_t^2]$, and $\mathbf{A}' = [A_0 \quad A_1 \quad A_2]$, then the solution for $z_t = \mathbf{A}'Y_t$, where[1]

$$\mathbf{A}' = \left[A_0 \qquad \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \qquad -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\ (1-\kappa_1\nu)} \left[1 + \left(\frac{\kappa_1 \varphi_e}{1-\kappa_1\rho} \right)^2 \right] \right] . \tag{9}$$

As the A's depend on κ's and hence on the average price-consumption ratio, $\bar{z}$, a solution to the system is fixed on $\bar{z}$. That is, to solve for the κ_1 and κ_0 one needs the numerical solution

[1] The expression for A_0, as well as for Γ_0 in equation (13), is given in the Appendix.

for $\bar{z}$, which satisfies

$$\bar{z} = \mathbf{A}(\bar{z})\bar{Y}. \tag{10}$$

This is quite easy to implement in practice.

Given the solution for z_t, the IMRS, in terms of the state variables and innovations can be stated as

$$m_{t+1} = \mathbf{\Gamma}' Y_t - \mathbf{\Lambda}' \zeta_{t+1} \tag{11}$$

where the three sources of risks are

$$\zeta'_{t+1} = \begin{bmatrix} \sigma_t \eta_{t+1} & \sigma_t e_{t+1} & \sigma_w w_{t+1} \end{bmatrix} \tag{12}$$

and the three dimensional vectors $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$, follow

$$\mathbf{\Gamma}' = \begin{bmatrix} \Gamma_0 & -\frac{1}{\psi} & -(\gamma-1)(\gamma-\frac{1}{\psi})\frac{1}{2}\big[1+(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho})^2\big] \end{bmatrix} \tag{13}$$

and

$$\mathbf{\Lambda}' = \begin{bmatrix} \gamma & (\gamma-\frac{1}{\psi})\frac{\kappa_1\varphi_e}{1-\kappa_1\rho} & -(\gamma-1)(\gamma-\frac{1}{\psi})\frac{\kappa_1}{2\ (1-\kappa_1\rho)}\big[1+(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho})^2\big] \end{bmatrix}. \tag{14}$$

Note that equation (11) for the pricing kernel has an approximation error emanating from the linear approximation around the theoretical value of average price to consumption ratio. We show that this approximation error is quite small and does not materially affect the results.

Given the expression for (11), it immediately follows that the risk premium on any asset j is

$$E_t[r_{j,t+1} - r_{f,t} + 0.5\sigma^2_{t,r}] = \sum_{i=1}^{3} \lambda_i \sigma^2_{i,t} \beta_{i,j} \tag{15}$$

where $\beta_{i,j}$ is the beta with respect to the i^{th} risk source of ζ_{t+1} for asset j, and λ_i is the i^{th} entry of Γ.

2.2 Special Case: IES=1

The IES is a critical parameter in the Long Run Risk Model. Many papers specialize the Epstein and Zin (1989) preferences to the case in which IES is set to one (e.g., Giovannini and Weil (1989), Tallarini (2000), Hansen, Heaton, and Li (2005), Hansen and Sargent (2006)). This has great analytical convenience in certain situations. It is important to note that our *estimation methodology* nests the case of IES=1 in a continuous fashion. Namely, the IMRS components as given in (11) adjust in continuous way as one takes the limit of the IES parameter to one. Specifically, as is well known, in the IES=1 case the price-consumption ratio, z, is constant and is determined by δ. A virtue of our approach of approximating the return, $r_{c,t+1}$, and accounting for the dependence of the approximating constants (i.e., κ_0, κ_1) on all the model parameters, is that the pricing kernel is continuous in IES. That is,

$$\lim_{\psi\to 1}\kappa_1 = \delta \quad \lim_{\psi\to 1}\Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi\to 1}\Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta) \tag{16}$$

Evaluating the pricing kernel (11) under the above restrictions gives exactly the same solution as in Giovannini and Weil (1989), Tallarini (2000) and Hansen, Heaton, and Li (2005). This approach does not confine the econometrician to prespecified value of the IES. That is, in estimation the IES is a free parameter.

2.3 Pricing Kernel Approximation Error

In our empirical work, we rely on the approximate analytical solutions of the model presented above and discussed in more details in the Appendix. In this section, we evaluate the accuracy of the log-linear approximation by comparing the approximate analytical solution for the price to consumption ratio to its numerical counterpart. The magnitude of the approximation error in the price-consumption ratio allows us to assess the reliability of the log-linear analytical solution for the stochastic discount factor, and consequently, model implications based on log-linear approximation.

Notice first that the value function in the Epstein-Zin preferences is given by,

$$V_t = (1-\delta)^{\frac{\psi}{\psi-1}} W_t (W_t/C_t)^{\frac{1}{\psi-1}}, \tag{17}$$

i.e., the life-time utility of the agent, normalized by the level of either consumption or wealth, is proportional to the wealth to consumption ratio. Hence, the solution to the wealth-consumption ratio (or, alternatively, price to consumption) based on the log-linearization of the wealth return in equation (6) will determine the dynamics of the value function. Recall also that the evolution of the IMRS (see equation (5)), through the return on wealth, depends on the valuation of the consumption claim. Thus, the log-linear solution for the IMRS, as well, hinges on the accuracy of the log-linear approximation of the price-consumption ratio.

To solve the model numerically, we use the approach proposed by Tauchen and Hussey (1991). This method is based on a discrete representation of the conditional density of the state variables, x and σ^2. In particular, we solve the pricing equation by approximating the integral in (4) by a finite sum using the Gauss-Hermite quadrature. Note that the resulting solutions, in their turn, are subject to an approximation error. In order to minimize this error and ensure the high quality of the benchmark numerical solutions, we use sufficiently large number of grid points in the quadrature rule.[2] In addition, for simplicity in this exercise we shut-off the channel of time-varying consumption volatility. Aside from this restriction, we evaluate the numerical and log-linear analytical solutions using the same parametrization of consumption growth dynamics that we subsequently employ in our simulation experiment (see Section 5 and Panel A of Table VI). Our benchmark calibration of preferences is $\delta = 0.9989$, $\gamma = 10$ and $\psi = 2$; however, we also consider alternative combinations of risk aversion and IES parameters. The approximation errors results are given in Table I.

Overall, we find that log-linear analytical solutions are remarkably close to the numerical results. In particular, for risk aversion of 10 and IES of 2, the numerical and analytical mean (volatility) of the log price to consumption ratio are 4.724 (0.0318) and 4.716 (0.0321) respectively.[3] Thus, the approximation error, expressed as a percentage of the corresponding numerical value, is about 0.17% for the mean and 0.86% for the standard deviation of the log price-consumption ratio. As the elasticity of intertemporal substitution decreases to 0.5, the percentage error falls to about 0.02% for $\bar{z}$ and 0.42% for σ_z. Although the approximation somewhat deteriorates as the magnitude of risk aversion increases, the deviation between analytical and numerical solutions remains quite small. For example, holding IES at 2 and varying risk aversion between 5 and 15 results in 0.03%–0.51% error band for the mean and

[2] Specifically, we discretize the dynamics of the expected growth component, x_t, using 100-point rule. We find that increasing the number of grid points leads to virtually identical numerical results.

[3] All the numbers reported in this section are in monthly terms.

0.17%–2.17% for the standard deviation of the log price-consumption ratio.

As discussed above, the dynamics of the price to consumption ratio has a direct bearing on the time-series properties of the IMRS. The fairly small approximation error in the price-consumption ratio, that we document, guarantees the accuracy of the pricing implications of the log-linearized solutions. Indeed, we find that approximate analytical and numerical solutions deliver very similar quantitative implications along all dimensions of the model, including levels and variances of the risk-free rate, price-dividend ratios, returns on consumption and dividend claims, and the pricing kernel.[4] This evidence confirms that empirical findings that we present below are robust to the log-linear approximation of the model.

3 Data

In this paper we use data on consumption and asset prices for the time period from 1930 till 2002 — the longest available sample. We take the view that this sample better represents the overall variation in asset and macro economic data. Importantly, this long sample also helps in achieving more reliable statistical inference. In addition, annual data is less prone to measurement errors that arise from seasonalities and other measurement problems highlighted in Wilcox (1992). The decision interval of the agent is assumed to be monthly which is motivated by standard payment cycles and is a common assumption in the literature. It is fairly straightforward to assume a different decision frequency (e.g., weekly or quarterly); however, this change will not alter the empirical findings in a significant manner.

In our empirical tests, we employ portfolios with opposite size and book-to-market characteristics that are known to provide investors with different premia over the years. In addition, our asset menu comprises the aggregate stock market portfolio and a proxy of a risk-less asset. The construction of portfolios is standard (see Fama and French (1993)). In particular, for the size sort, we allocate individual firms across 10 portfolios according to their market capitalization at the end of June of each year. Book-to-market deciles are likewise re-sorted at the end of June by ranking all the firms into 10 portfolios using their book-to-market values as of the end of the previous calendar year. NYSE breakpoints are used in both sorts. For each portfolio, including the aggregate market, we construct value-

[4]For brevity, the detailed evidence is not reported here (it is yet available upon request).

weighted monthly returns as well as per-share price and dividend series as in Campbell and Shiller (1988), Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2005). Monthly data are then time-aggregated to an annual frequency and converted to real using the personal consumption deflator. Panel A of Table II provides descriptive statics for returns, dividend growth rates and price-dividend ratios for the five portfolios of interest — small and large (i.e., firms in the top and bottom market capitalization deciles), growth and value (firms with the lowest and highest book-to-market ratios, respectively), and the aggregate stock market. The first column illustrates the well-known size and value premia. Over the sample period, small stocks have outperformed large firms by about 9%; the spread in returns on value and growth firms has averaged about 6.4%. Both high book-to-market and small firms have experienced higher growth rate of dividends and have been much more volatile than their corresponding counterparts. The bottom line of Panel A reports the mean and the standard deviation of the risk-free rate. The real interest rate is constructed by subtracting the 12-month expected inflation from the annualized yield on the 3-month Treasury bill taken from the CRSP treasury files.

Finally, we take seasonally adjusted per-capita data on real consumption and gross domestic product (GDP) from the NIPA tables available on the Bureau of Economic Analysis website. Aggregate consumption is defined as consumer expenditures on non-durables and services. Summary statistics of consumption and GDP growth rates are presented in Panel B of Table II. Growth rates are constructed by taking the first difference of the corresponding log series. In addition, Panel B displays the mean and the standard deviation of the default premium measured as the difference in yields on Baa and Aaa corporate bonds published by the Board of Governors of the Federal Reserve System.

4 Empirical Findings

Estimating and testing equation (4) involves computing the pricing kernel in equation (11). To achieve this we require to specify the dynamics of consumption growth rate and x_t. An econometrician who relies on the long risk model but relies on an annual data and decision interval will focus on annual consumption growth rates, $\Delta c^a_{t+1} \equiv \log(C^a_t/C^a_{t-1})$, and the long run risk component, x^a_t.

Our approach to estimating the long run component x^a_t is to use observed financial market prices and consumption data, all of which, under the null of the model, depend on the state variable x_t^a among others. The standard VAR, $Y_t^a = [\Delta c_t^a, v_t^a]$ where v_t^a is an $n-1 \times 1$ vector of predictive variables can used to measure the low frequency variable x, which corresponds to the conditional mean of consumption growth. In the appendix we show via simulation that our VAR approach, using financial market variables, provides a fairly good way to extract x from the data.

Our VAR includes consumption growth, log of GDP to consumption ratio, and the predictive variables the market's price-dividend ratio, the risk-free rate, and the default premium. Assume that,

$$Y_{t+1}^a = BY_t^a + \Upsilon_{t+1}^a, \tag{18}$$

where B is an $n \times n$ matrix and Υ_{t+1}^a is a vector of residuals, and all the variables in equation (18), without loss of generality, are demeaned. Let ι_j be a row vector of zeros with one in j^{th} column. The persistent component, x_t^a, equals the conditional expectation of consumption growth which under the specification above is

$$x_t^a = \iota_1 BY_t^a.$$

Further, the innovation to consumption growth, which corresponds to high frequency shock is

$$\eta_{t+1}^a = \iota_1 \Upsilon_{t+1}^a.$$

The long run risk shock, e_{t+1}^a, is extracted by fitting an AR(1) to x_t^a, that is

$$x_{t+1}^a = \rho^a x_t^a + e_{t+1}^a \tag{19}$$

Given the dynamics of x_t^a and shocks η_{t+1}^a and e_{t+1}^a, the pricing kernel in (11) can be computed for any configuration of preference parameters. Reducing the multivariate representation of the conditional mean of consumption growth to the single variable representation in equation (19) allows to maintain a very parsimonious and easily interpretable structure on the model and the estimation.

Table III provides the VAR estimates of equation (18) for extracting x_t^a. In predicting consumption growth, we use a two-year moving average of lagged consumption growth, the log of consumption to GDP ratio, as well as other typical asset pricing predictive variables. Specifically, we use the price-dividend ratio of the aggregate market portfolio, the short interest rate, and the default premium. The price-dividend ratio rises, as theory predicts, when x_t^a rises. In addition, when consumption is above GDP it is predicted to revert back towards GDP's level. Finally, both the default premium and risk-free rate affect x_t^a in a significant way. There are two key features of Table III. First, consumption growth is highly predictable with an adjusted R^2 of 37%. Figure 1 provides the time series of x_t^a plotted along with realized consumption growth and NBER recessions. It is clear that this variable is relatively slow moving and captures features of the business cycle. Recessions are clearly associated with a decline in x_t^a. Together, this evidence clearly shows that the data is far from an *i.i.d* view for consumption growth. Second, the persistence parameter, ρ^a, is quite large at 0.67, implying an approximate monthly persistence of $0.67^{1/12} = 0.968$.

One important issue is whether the long run response of the consumption growth in the parsimonious structure of the univariate x in equation (19) is similar to that obtained from the VAR. Figure 2 display the accumulated impulse responses to a one percent short and long run shocks under the VAR and univariate dynamics. The shock to the long run component in the VAR is defined as $\iota_1 B \Upsilon_t^a$, while the short run shock to consumption is simply η_t^a. The figure shows that quantitatively the short and long run behavior of consumption is virtually the same under the two alternative specifications. First the short run shock η_t and the long run shock e_t are virtually uncorrelated–the correlation in the data is 0.10. Consequently, the η_t shock has no impact on consumption growth beyond its immediate effect. The full VAR based consumption accumulated impulse response rises rapidly up to 7 years and there after is fairly flat at a value of about 2.8 for a 30 year horizon. This implies that a one percent shock to the conditional mean of consumption growth, x_t, leads to an upward revision of about 2.8 percent in the long run consumption level. Note that in order to make figure 2 comparable across shocks and configuration each plot refers to a one percent shock. However, a one percent shock to the long run component of consumption is a very large one as a one standard deviation shock to e_t is only 0.0084 — a shock that would lead to a long run rise in consumption of only 0.0275 percent! The main point to take away from Figure 2 is that the single variable representation of equation (19) leads to a very similar accumulated impulse response as that of the VAR.

In principle, the volatility component can be estimated by adding a standard GARCH specification for Υ_{t+1}^a, but given the annual data and for other reasons discussed further below we only use the long run component in specifying consumption growth dynamics.[5]

4.1 Returns and Betas

Before continuing on to formally estimating the model, we provide preliminary evidence linking the returns, betas, and market price of risk as described in equation (15). In order to derive the betas we use the following strategy. We first estimate the expected return for a given asset, $\bar{r}^a_{j,t+1}$, by projecting the asset return on lagged realized and expected growth of consumption, Δc_t and x_t respectively, as well as its own lagged dividend growth and price-dividend ratio. Given the innovations η^a_{t+1} and e^a_{t+1}, described above, and the innovations to the asset return, $u^a_{r,j,t+1} = r^a_{j,t+1} - \bar{r}^a_{j,t+1}$, we compute the asset's betas with respect to various consumption risks. Specifically, the long-run consumption beta is measured as $\beta^a_{e,j} \equiv \dfrac{\mathrm{Cov}(e^a_{t+1}, u^a_{r,j,t+1})}{\mathrm{Var}(e^a_{t+1})}$, while the exposure of asset returns to transient risks in consumption is constructed as $\beta^a_{\eta,j} \equiv \dfrac{\mathrm{Cov}(\eta^a_{t+1}, u^a_{r,j,t+1})}{\mathrm{Var}(\eta^a_{t+1})}$.

The cross-sectional prices of risks are estimated by regressing mean returns on the two betas, i.e.,

$$\bar{R}^a_j = \lambda^a_0 + \lambda^a_\eta \beta^a_{\eta,j} + \lambda^a_e \beta^a_{e,j} + \epsilon^a_j \tag{20}$$

To expand the degrees of freedom, in the cross-sectional regression we employ the full asset menu consisting of 10 size and 10 book-to-market sorted portfolios plus the aggregate stock market. We find that the price of short-run consumption risks, although positive, is not significantly different from zero. In particular, $\hat{\lambda}^a_\eta = 0.33$ (SE=0.252). In contrast, the market price of low-frequency fluctuations in consumption is both positive and significant: $\hat{\lambda}^a_e = 0.58$ (SE=0.092). Together, the consumption betas explain about 50% of the cross-

[5]In general, the above VAR can be augmented by the variance equation as in Bansal and Yaron (2004) model described in Section 2. In particular, the variance of consumption growth can be measured by taking an absolute value of consumption residuals, and, similarly to x^a, its dynamics can be modelled via a simple first-order autoregressive process. This would allow us to estimate the exposure of asset returns to volatility risks, $\beta^a_{w,j}$, and evaluate the market price of fluctuating economic uncertainty. Although feasible, this specification will unlikely yield a reliable estimate of the volatility component and its time-series behavior given a low sampling frequency of the data in hand. Therefore, in our empirical work, we abstract from time-varying uncertainty and focus primarily on the pricing of short- and long-run risks in consumption level. However, the volatility component is needed for matching salient features of annualized asset market data when in Section 5 we calibrate a monthly long run risk model and time aggregate it to annual frequency. We find that in the simulated annual data, as in the actual data, it is difficult to detect the presence of the volatility component.

sectional variation in risk premia. It is worthy to note, though, that the predictive ability in the cross-section is entirely attributed to the long-run beta. Table IV provides the betas for the five returns we consider. There is a clear link between average returns on these assets and their exposures to long-run risk as measured by β_e^a. In particular, note that value and growth portfolios have a β_e^a of 15.3 and 10.4 respectively reflecting the 'value premium'; in a similar fashion the long run β's for small and big firms are 16.4 and 9.1 respectively, reflecting the effect of size. Finally, the table also provides the standard CCAPM betas, β_{ccapm}^a. It is quite evident that average returns are not well captured by these traditional betas, giving rise to the well documented failure of the CCAPM.

4.2 Euler Equation Estimation Evidence

This section provides the main results of our paper. The vector of structural parameters we seek to estimate includes the preference and technology parameters governing the dynamics for consumption. We utilize an annual version of equation (4) for the five asset returns: the risk-free rate, market return, value and growth, large and small firm portfolios in estimating this parameter vector. Table V provides the estimates of the structural parameters based on the Euler equations for the Long Run Risk model for several alternative GMM weighting matrices. In the first panel, we use the identity matrix. In the second panel, we use the diagonal of the inverse of the returns' covariance matrix which gives more weight to assets with lower volatility. Each of these panels provides the structural parameters regarding preferences, namely, IES, and risk aversion (ψ and γ), where we pre-set the time discount rate, δ, to 0.9989^{12}. In addition, we provide the average pricing error for each of the returns, and the J test for overidentifying restrictions.

The results are quite illuminating. For the identity matrix, risk aversion is above 27 and the IES is 0.6. For the inverse of the covariance return matrix, the results are similar; risk aversion is 23, and IES is 0.7. In both cases the standard errors for the IES and risk aversion are quite large.[6] The main feature of the results is the fact that the model prices assets quite well. Formally the model is not rejected as the overidentifying restrictions have p-values above 0.2 for all alternative weighting matrices. Moreover, the pricing errors are quite small. The largest pricing error is only 0.054 for the return on the portfolio of small

[6]The results are quite similar when we use the continuously updated weighting matrix, as in Hansen, Heaton, and Yaron (1996), to those using the inverse of the covariance matrix of returns.

stocks when using the inverse of the covariance return matrix. For comparison, Table VI provides similar results for the more restrictive case of time separable CRRA preferences. As is well known, these preferences have more difficulty in pricing assets. The model's overidentifying restriction test is rejected when the weighting matrix used is the returns covariance matrix, and is only marginally significant for the identity weighting matrix (a p-value of 12%). Moreover, the pricing errors are substantially larger for the CRRA preferences. In particular, the maximal pricing error (for small portfolio) is now around 22%, and there is another two digit pricing error. Perhaps, the most apparent discrepancy in statistical fit between the CRRA preferences and the LRR model is the pricing errors of a portfolio of Value minus Growth, and Small minus Large. While the t-statistics on these two pricing errors are insignificant for the LRR model, they are statistically rejected for the CRRA model.

Table VII presents GMM results for alternative specifications for the persistence of the long run component of consumption. In each case we scale the innovation to maintain the same unconditional variance as the one from our estimates. The estimation is based on the identity weighting matrix and thus the results are directly comparable to the estimates under vertical panel A in Table V. The model is still not rejected by the overidentifying restrictions and to a large extent the pricing errors are of similar magnitude. The results illuminate an important trade-off between the persistence in x and the estimated preference parameters. As ρ increases the estimated risk aversion drops and the estimated IES coefficients rise. In fact for the case for $\rho = 0.75$ the risk aversion is now 21.6 and the IES estimate is 1.2 where as for $\rho = 0.5$ the risk aversion estimate rise to 30 and the IES drops to 0.2. This trade-off underscores an important channel in ultimately interpreting our point estimates. As we show below time averaging causes a downward bias in the estimated persistence of x, which in turn leads to an upward bias in the estimated risk aversion and a downward bias in the estimated IES.

Finally, our results are robust to alternative use of instruments. In results that are not reported here, we use more elaborate system of instruments to capture potential important variation in conditional moments. For example, adding lagged consumption growth, or risk free rate or market return as instruments, leads to similar large (above twenty) risk aversion estimates.

In sum, our model shows that once the return to wealth is appropriately accounted for, the Long Run Risk Model can account quite well for both the time series and cross sectional variation in returns. A concerning open issue is the fact that the IES seems to be below one and risk aversion is quite large. This parameter configuration in simulations of the Long Run Risk model is likely to produce data features which are counterfactual. This consequently raises an important issue in terms of interpreting the empirical evidence documented in the data in this paper and in earlier work. In the remaining sections we address this issue by highlighting the effects of time averaging. In particular, we show that even if the population risk aversion is low and the IES is larger than one, the GMM when contaminated by time averaging effects will produce estimates, as in our case above, of high risk aversion and low IES. More specifically, we write down a *monthly* Long Run Risk Model of the type specified in equation (3) and show that time averaging to *annual* data leads to downward bias in the estimated IES and an upward bias in the estimated risk aversion, while maintaining the ability to price annual asset returns.

5 Decision Interval and Time Averaging

5.1 Time Averaging and Finite Sample Effects on Estimation

In this section we examine the effects the decision interval and time averaging have on the economic plausibility of the Long Run Risk model and the interpretation of structural parameters. In particular, we wish to assess how time averaging affects the estimation procedure utilized in the previous section. To do so we calibrate a monthly version of the Long Run Risk model and then time aggregate the data to construct simulated annual variables counterpart to their observed data.

Time averaging, in our context, arises due to the fact that the sampling frequency of the data is different from the decision interval of the agent. For example, the data is sampled at an annual frequency while the decision interval may be monthly. This averaging effect, we show, has the effect of distorting the parameter estimates and consequently the interpretation of the model implications. Role of time averaging in models has been emphasized in the past; Hansen and Sargent (1983) highlight its importance in interpreting an adaptive expectations model and Heaton (1995) in the context of an asset pricing model with time-nonseparable preferences. In this paper we show the importance of this issue in the context of the long run risks model.

In our case time averaging effects emanate from two sources. First, the monthly consumption is replaced by annual consumption $C_t^a = \sum_{j=1}^{12} C_{t-j}$, where C_{t-j} corresponds to month j consumption in year t. Second, with annual data the estimates of monthly x_t are replaced by annual estimates x_t^a, which are obtained from annual consumption growth. This latter feature, as we show below, will also distort the measure of the persistence in the x_t process, and consequently the market price of risks. As Table VII already demonstrates this empirically the preference parameters are quite sensitive to the persistence in x. To gather more intuition regarding this tradeoff note that the econometrician utilize the following stochastic discount factor in estimating risk premia,

$$m_{t+1}^a = \mathbf{\Gamma^{a\prime}} Y_t^a - \mathbf{\Lambda^{a\prime}} \zeta_{t+1}^a. \tag{21}$$

Recall, the risk compensation for the long run risk on a monthly frequency is λ_e

$$\lambda_e \sigma_t^2 \;\;=\;\; \left(\gamma - \frac{1}{\psi}\right) \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \sigma_t^2. \tag{22}$$

The 'effective risk aversion', λ_e, is highly sensitive to the risk aversion parameter γ, as well as the parameter ρ governing the persistence of the long run component x_t. If the true decision interval of the agent is monthly but the econometrician uses annual data, then ρ^a is much smaller than ρ, and this tends to make the market price of risk on an annual frequency smaller for any given risk aversion and IES parameters. For example, as we show below, a monthly model in which ρ is 0.982 yields an annual (via time averaging) ρ^a of only 0.65. The fact that the annualized volatility multiplying this effective risk premia, σ_t, is about 12 times that of the monthly volatility is still not enough to compensate for the reduction in effective risk aversion due to the smaller annual persistence coefficient. Thus, to achieve the desired risk premia, and to offset the reduction in 'effective risk aversion', the econometrician estimates a very large risk aversion even though the true risk aversion is much smaller.

The message from this is simple but important. Time averaging and the appropriate decision interval can have critical affects for deducing the appropriate risk aversion parameter. Note, that these potential effects would be absent in a model that focuses on $i.i.d$ consumption growth, in which case the long run piece is absent and time averaging essentially does not effect the estimation of γ via λ_η.

In addition to the time-averaging effects, another channel that tends to push the risk aversion to a larger magnitude is the finite sample bias in estimating ρ. It is well known, see Kendell (1954), that the persistence parameter in a standard AR(1) process is biased downwards – this again tends to lower the market price of long run risks and hence estimation, to match the risk premium, tends to push the risk aversion higher. We provide, via simulations, a decomposition of the magnitude of the effects that arise solely due to time averaging and due to finite sample bias.

5.2 Consumption, Dividends, and Asset Returns

To complete the specification of the model, we need, in addition to the consumption dynamics already given in (3), to specify the dividend dynamics of each asset j,

$$\Delta d_{j,t+1} \quad = \quad \mu_d + \phi_j x_t + \pi_j \sigma_t \eta_{t+1} + \varphi_j \sigma_t u_{d,j,t+1} \tag{23}$$

where $\Delta d_{j,t+1}$ is the dividend growth rate of portfolio j. In addition, we assume that all shocks are $i.i.d$ normal and are orthogonal to each other, although we allow for cross-sectional correlations in dividend news, $u_{d,j,t+1}$. Dividends have a levered exposure to the persistent component in consumption, x_t, which is captured by the parameter ϕ_j. In addition, we allow the $i.i.d$ consumption shock η_{t+1} to influence the dividend process, and thus serve as an additional source of risk premia. The magnitude of this influence is governed by the parameter π_j.[7] The dynamics are similar to those in Bansal and Yaron (2004), Bansal, Dittmar, and Lundblad (2005), and Kiku (2006).[8] The model is assumed to have a monthly decision interval and the parameters governing the consumption and dividend dynamics are given in Panels A of Table IX and X respectively. Throughout we use a risk aversion parameter of 10 and an IES value of 2.[9]

We simulate from the monthly consumption and dividend dynamics specified in equa-

[7]Note that this type of specification is isomorphic to one in which $\pi_j = 0$ but the correlation between η_{t+1} and $u_{d,j,t+1}$ is non-zero.

[8]Note that as we are dealing with dividend per-share cashflows this configuration does not impose cointegration between consumption and dividends as analyzed in Bansal and Yaron (2006). Menzly, Santos, and Veronesi (2004) provide an alternative specification for cashflow dynamics across assets which does not rely on the LRR channels but does impose cointegration between consumption and per share dividends.

[9]Kandel and Stambaugh (1991) find a configuration with risk aversion larger than twenty and an IES that is very small as plausible. As we show below the low IES leads to risk free rate (market) volatility that is too large (small).

tions (3) and (23). We construct the appropriate time aggregated C_t^a and D_t^a level series and then construct their annual growth rates. Panel B of Table IX provides Monte-Carlo evidence regarding the annual time series of consumption growth. Specifically, we simulate the model with 876 months which results in 73 annual observations as in our data set. We replicate this over 500 simulations. The last two columns in Table IX are the median and standard deviation respectively of annual consumption growth across these simulations. The table clearly shows that the model successfully matches the mean, standard deviation, and autocorrelation of annual consumption growth.

Panel A of Table X provides the parameters governing the dividend dynamics for the five assets we consider. Panel B of this table, provides information regarding the mean, standard deviation of dividend growth and its correlation with consumption growth from the data and the model. Again, the model's statistics are based on the median and standard deviation across the 500 Monte-Carlo simulations. By and large, the model's output matches quite well with the data. The correlations between consumption and dividend growth are essentially indistinguishable from their point estimates in the data. The data and the model's mean dividend growth rates are all within standard error of each other. More importantly, the ranking across the four assets of interest is maintained. The volatility of dividend growth is matched very precisely (and within one standard error) for the market, growth and large portfolios. For the small and value portfolios the model's median volatility is quite smaller than that implied by the data. However, these two portfolios' extreme volatility are driven by few data points. Our approach is to be more conservative with respect to these volatility numbers while ensuring the model generates average returns that are comparable to what is observed in the data.

Table XI provides the data and model predictions for the mean and volatility of the return as well as the level of the price dividend ratio for each asset. Again, the model replicates quite well all of these statistics. The data is well within one standard error of the model estimates. In particular, note that the model is able to generate the 'size' and 'value' premium as in also highlighted in Kiku (2006).

5.3 GMM Estimation and Time Averaging

Equipped with plausible model-generated data for returns, we use GMM on the annual simulated data and test equation (4) in an analogous fashion to the estimation procedure we used for the observed data. Table XII provides the distribution of risk aversion and IES estimates, as well as the $J-Stat$ and p-values across the 500 simulations. This table provides several important implications. First, the mean of the risk aversion and IES estimates across simulations is quite close to those estimated directly on the annual data in Table V. This holds true for the different weighting schemes used. Noteworthy is the fact that the mean risk aversion is large (e.g., 21 for the identity weighting matrix). Figure 3 provides a histogram of the risk aversion estimates using the identity weighting matrix; the dispersion and right skewness are apparent. Moreover, the mean estimates for the IES are all less than 1. Recall, these simulations are based on a model in which risk aversion is 10 and the IES is 2. Thus, time averaging leads to severe downward bias in IES and quite significant upward bias in risk aversion. Further, note that the model is not rejected by the overidentifying restrictions. In fact the mean $J - Stat$ is quite close to that estimated by the data in Table V which is well within the 90% confidence interval generated by the simulations.

Table XIII provides analogous estimates for risk aversion, the IES, and $J-Stat$ when the moment conditions specialize the utility function to power utility as in Table VI. The risk aversion estimates are larger than those for the Epstein and Zin (1989) preferences in Table XII. Figure 3 also provides the histogram of risk aversion estimates across simulations for the power utility preferences. The distribution is large and seem to be somewhat bimodal with a mass at small and very large risk aversion levels. The model is rejected based on the mean $J - Stat$s and the pricing errors deteriorate substantially relative to the case of Epstein-Zin preferences. In this respect the results of imposing the power utility restriction are similar to those estimated in the data as seen in comparing Tables V and VI.

Together, Tables XII-XIII remarkably replicate the results in Tables V-VI, providing additional support in favor of the model proposed and estimated. A natural question to ask is why the IES is estimated with a downward bias while risk aversion is estimated with an upward bias? The answer hinges on time averaging. As explained in section 5.1 time averaging (in conjunction with stochastic volatility) reduces the persistence of the monthly x_t process and therefore the ability of the model to generate risk premia.

In Table XIV we isolate the time averaging effects from those of having a finite sample by estimating the model using a very long annual sample. The top two rows in this table provide the persistence ρ^a of x_t^a and the R^2 of consumption growth process. It is clear that even in a very long sample the autocorrelation in x_t^a is obviously lower than that of x_t but more surprisingly is also quite lower than that of ρ^{12}. That is the long sample's autocorrelation is 0.71 while the monthly's model autocorrelation (0.982) raised to the power of 12 is 0.80. Finally, the median estimate for ρ in finite samples is 0.65 (the 5% and 95% quintiles are 0.27 and 0.84 respectively), which shows that finite sample contributes to another small reduction in the persistence of the long run component. Based on this long sample, the remaining panels of this table provide the GMM estimates of risk aversion and IES for the two alternative weighting matrices. Specifically, risk aversion is around 12-15 (relative to a true population value of 10) and the IES is around 0.4-0.6 relative to a population of value 2. Note, however, that the p-values of the over-identifying restrictions of the model are significantly and uniformly rejected across the various weighting matrices. This information compared to the results in Table XII shows that finite sample properties are, to a large degree, responsible for the non rejection results of the model.

The discussion above demonstrates the important consequences of the downward bias in estimating the persistence in x for interpreting the preference parameters. One additional channel that may contribute to this downward bias is the missing stochastic volatility feature of consumption and dividend growth. When using financial market data to extract x, the presence of volatility generates a second factor in the monthly model that can bias the extracted x. Our preliminary experiments indicate that the contribution of stochastic volatility does not fully account for the bias in the estimated preference parameters. To account for this we conduct an experiment in which we eliminate stochastic volatility but increase the persistence of x and lower σ_e/σ_η to maintain the unconditional volatility of consumption as well as key risk premia moments. Based on a long sample estimates, the persistence of x is now 0.82, and the estimated risk aversion, γ is 13.5 and the IES is just about 1.

As emphasized in section 5.1 risk aversion and persistence play complementary roles in contributing to risk premia. To generate risk premia while compensating for the downward bias in persistence, as just shown above, the estimated risk aversion is increased relative to its true value. This larger risk aversion has an adverse effect in lowering the risk free rate through the precautionary effect term. To compensate for that, the IES is lowered. This is demonstrated by the fact that in the simulations the correlation between γ and μ/ψ, the non precautionary term of the risk free rate, is 0.60. Together, this evidence underscores the importance of sampling frequency and the potential consequences of extracting preferences from time aggregated data.

Epstein and Zin (1991) pursue a GMM estimation approach but in evaluating the pricing kernel in equation (5) replace the return on the consumption claim $r_{c,t+1}$ with the observed value weighted NYSE stock market return. In Table XV we use our simulated data to estimate the model with a pricing kernel based on the market return. The estimated risk aversion is quite low. This, to a large extent, is due to the large volatility induced into the pricing kernel by the volatile market return. Finally, and most importantly, the table clearly shows that the model's overidentifying restrictions are overwhelmingly rejected. Finite sample experiments also lead to vast rejection of the model. The implication of this experiment is that deriving the appropriate return on consumption is critical for appropriately assessing the LRR model.

6 Conclusions

This paper establishes a practical method for obtaining the long run risk component – an essential ingredient of the Long Run Risk model proposed in Bansal and Yaron (2004). Using this we are able to empirically construct the unobservable return on total wealth, a required input in pricing assets when using the Epstein and Zin (1989) preferences. The Long Run Risk model is quite successful at capturing the time series and cross-sectional variation in returns. The model prices assets quite well including the 'value' and 'size' premium. A calibrated version of the model generates the equity premium, volatility of the market return, and the mean and volatility of the risk free rate as well as the returns on several portfolios and their price-dividend ratios. In using the model as a data generating process we show that time averaging leads to downward biased estimates of the IES and upward bias in risk aversion — and to estimates of similar magnitude to the levels estimated using the observed data. Together, this information provides strong evidence in favor of the long run risk model, while at the same time reconciling why often the literature had found large values of risk aversion and small values of the IES.

7 Appendix

To derive asset prices we use the IMRS together with consumption and dividend dynamics given in (3) and (23). The Euler condition in equation (4) implies that any asset j in this economy should satisfy the following pricing restriction,

$$E_t\Big[\exp\big(\theta\ln\delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1} + r_{j,t+1}\big)\Big] = 1\ , \tag{24}$$

where $r_{j,t+1} \equiv \log(R_{j,t+1})$ and $r_{c,t+1}$ is the log return on wealth. Notice that the solution to (24) depends on time-series properties of the unobservable return r_c. Therefore, we first substitute $r_{j,t+1} = r_{c,t+1}$ and solve for the return on the aggregate consumption claim; after that, we present the solution for the return on a dividend-paying asset.

7.1 Consumption Claim

We start by conjecturing that the logarithm of the price to consumption ratio follows, $z_t = A_0 + A_1 x_t + A_2\sigma_t^2$. Armed with the endogenous variable z_t, we plug the approximation $r_{c,t+1} = \kappa_0 + \Delta c_{t+1} + \kappa_1 z_{t+1} - z_t$ into the Euler equation above. The solution coefficients, A's, can now be easily derived by collecting the terms on the corresponding state variables. In particular,

$$\begin{aligned}
A_0 &= \frac{1}{1-\kappa_1}\Big[\log\delta + \kappa_0 + \Big(1-\frac{1}{\psi}\Big)\mu_c + \kappa_1 A_2(1-\nu)\bar{\sigma}^2 + \frac{\theta}{2}\Big(\kappa_1 A_2\sigma_w\Big)^2\Big] \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \\
A_2 &= -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\ (1-\kappa_1\nu)}\Big[1+\Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\Big]
\end{aligned} \tag{25}$$

For more details, see the the appendix in Bansal and Yaron (2004).

Notice that the derived solutions depend on the approximating constants, κ_0 and κ_1, which, in their turn, depend on the unknown mean of the price to consumption ratio, $\bar{z}$. In order to solve for the price of the consumption asset, we first substitute expressions for κ's (equations (7) and (8)) into the expressions for A's and solve for the mean of the price to consumption ratio. Specifically, $\bar{z}$ can be found by numerically solving a fixed-point problem:

$$\bar{z} = A_0(\bar{z}) + A_2(\bar{z})\bar{\sigma}^2 \ ,$$

where the dependence of A's on $\bar{z}$ is given above.

The solution for the price-consumption ratio, z_t, allows us to write the pricing kernel as a function of the evolution of the state variables and the model parameters,

$$m_{t+1} \ = \ \Gamma_0 + \Gamma_1 x_x + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1} \ , \tag{26}$$

where

$$\begin{aligned} \Gamma_0 &= \log \delta - \frac{1}{\psi}\mu_c - (\theta - 1)\Big[A_2(1-\nu)\bar{\sigma}^2 + \frac{\theta}{2}\Big(\kappa_1 A_2 \sigma_w\Big)^2\Big] \\ \Gamma_1 &= -\frac{1}{\psi} \\ \Gamma_2 &= (\theta - 1)(\kappa_1 \nu - 1)A_2 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \lambda_\eta &= \gamma \\ \lambda_e &= (1-\theta)\kappa_1 A_1 \varphi_e = (\gamma - \frac{1}{\psi})\frac{\kappa_1 \varphi_e}{1-\kappa_1\rho} \\ \lambda_w &= (1-\theta)\kappa_1 A_2 = -(\gamma - 1)(\gamma - \frac{1}{\psi})\frac{\kappa_1}{2\ (1-\kappa_1\rho)}\big[1 + (\frac{\kappa_1\varphi_e}{1-\kappa_1\rho})^2\big] \end{aligned} \tag{28}$$

Note that λ's represent market prices of transient (η_{t+1}), long-run (e_{t+1}) and volatility (w_{t+1}) risks respectively. For more detailed discussion see Bansal and Yaron (2004).

7.2 Dividend Paying Assets

The solution coefficients for the valuation ratio of a dividend-paying asset j can be derived in a similar fashion as for the consumption asset. In particular, the price-dividend ratio for a claim to dividends, $z_{j,t} = A_{0,j} + A_{1,j}x_t + A_{2,j}\sigma_t^2$, where

$$
\begin{aligned}
A_{0,j} &= \frac{1}{1-\kappa_{1,j}}\left[\Gamma_0 + \kappa_{0,j} + \mu_{d_j} + \kappa_{1,j}A_{2,j}(1-\nu)\bar{\sigma}^2 + \frac{1}{2}\Big(\kappa_{1,j}A_{2,j} - \lambda_w\Big)^2\sigma_w^2\right] \\
A_{1,j} &= \frac{\phi_j - \frac{1}{\psi}}{1-\kappa_{1,j}\rho} \\
A_{2,j} &= \frac{1}{1-\kappa_{1,j}\nu}\left[\Gamma_2 + \frac{1}{2}\Big((\pi_j - \lambda_\eta)^2 + (\kappa_{1,j}A_{1,j}\varphi_e - \lambda_\epsilon)^2\Big)\right]
\end{aligned} \tag{29}
$$

It follows then that the innovation into the asset return is given by,

$$
r_{j,t+1} - E_t[r_{j,t+1}] = \varphi_j\sigma_t u_{j,t+1} + \beta_{\eta,j}\sigma_t\eta_{t+1} + \beta_{e,j}\sigma_t e_{t+1} + \beta_{w,j}\sigma_w w_{t+1}\ , \tag{30}
$$

where the asset's betas are defined as,

$$
\begin{aligned}
\beta_{\eta,j} &= \pi_j \\
\beta_{e,j} &= \kappa_{1,j}A_{1,j}\varphi_e \\
\beta_{w,j} &= \kappa_{1,j}A_{2,j}
\end{aligned}
$$

The risk premium for any asset is determined by the covariation of the return innovation with the innovation into the pricing kernel. Thus, the risk premium for $r_{j,t+1}$ is equal to the asset's exposures to systematic risks multiplied by the corresponding risk prices,

$$
\begin{aligned}
E_t(r_{j,t+1} - r_{f,t}) + 0.5\sigma_{t,r_j}^2 &= -Cov_t\Big(m_{t+1} - E_t(m_{t+1}), r_{j,t+1} - E_t(r_{j,t+1})\Big) \\
&= \lambda_\eta\sigma_t^2\beta_{\eta,j} + \lambda_e\sigma_t^2\beta_{e,j} + \lambda_w\sigma_w^2\beta_{w,j}
\end{aligned}
$$

7.3 IES=1

When $\psi = 1$, the log of the IMRS is given in terms of the value function normalized by consumption, $vc_t = \log(V_t/C_t)$,

$$
m_{t+1} = \log\delta - \gamma\Delta c_{t+1} + (1-\gamma)vc_{t+1} - \frac{1-\gamma}{\delta}vc_t
$$

Conjecturing that $vc_t = B_0 + B_1 x_t + B_2 \sigma_t^2$ and using the evolution of vc_t:

$$vc_t = \frac{\delta}{1-\gamma} \log E_t \Big[\exp\{(1-\gamma)(vc_{t+1} + \Delta c_{t+1}\} \Big] ,$$

the solution coefficients are given by,

$$B_0 = \frac{\delta}{1-\delta} \Big[\mu + B_2(1-\nu)\bar{\sigma}^2 + \frac{1}{2}(1-\gamma)(B_2 \sigma_w)^2 \Big]$$

$$B_1 = \frac{\delta}{1-\delta\rho} \tag{31}$$

$$B_2 = -(\gamma - 1)\frac{\delta}{2\ (1-\delta\nu)} \Big[1 + \Big(\frac{\delta \varphi_e}{1-\delta\rho} \Big)^2 \Big] \tag{32}$$

As above, the pricing kernel can be expressed in terms of underlying preference parameters, state variables and systematic shocks,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_\epsilon \sigma_t \epsilon_{t+1} - \lambda_w \sigma_w w_{t+1} \tag{33}$$

where:

$$\Gamma_0 = \log\delta - \mu - (1-\gamma) \Big[\frac{1}{\delta} B_2 (1-\nu)\bar{\sigma}^2 + \frac{1}{2}(1-\gamma)(B_2\sigma_w)^2 \Big] \tag{34}$$

$$\Gamma_1 = -1$$

$$\Gamma_2 = -\frac{(\gamma-1)^2}{2} \Big[1 + \Big(\frac{\delta\varphi_e}{1-\delta\rho} \Big)^2 \Big]$$

and

$$\lambda_\eta = \gamma \tag{35}$$

$$\lambda_\epsilon = (\gamma - 1) \frac{\delta\varphi_e}{1-\delta\rho}$$

$$\lambda_w = -(\gamma-1)^2 \frac{\delta}{2\ (1-\delta\rho)} \Big[1 + \Big(\frac{\delta\varphi_e}{1-\delta\rho} \Big)^2 \Big]$$

Finally, note that in the IES=1 case, the wealth-to-consumption ratio is constant, namely, $\dfrac{W_t}{C_t} = \dfrac{1}{1-\delta}$. The price-to-consumption ratio, therefore, is equal $\dfrac{P_t}{C_t} = exp(\bar{z}) = \dfrac{\delta}{1-\delta}$. Consequently, the parameter of the log-approximation of the log-wealth return,

$$\kappa_1 \quad = \quad \frac{\exp(\bar{z})}{1+\exp(\bar{z})} = \frac{\frac{\delta}{1-\delta}}{1+\frac{\delta}{1-\delta}} = \delta.$$

Plugging $\kappa_1 = \delta$ and $\psi = 1$ into equations (26), (27) and (28), yields exactly equation (33), (34) and (35). It then follows that

$$\lim_{\psi\to 1} \kappa_1 = \delta \quad \lim_{\psi\to 1} \Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi\to 1} \Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta) \tag{36}$$

References

Bansal, Ravi, Robert F. Dittmar, and Christian Lundblad, 2005, Consumption, dividends, and the cross-section of equity returns, *Journal of Finance* 60, 1639–1672.

Bansal, Ravi, Ron Gallant, and George Tauchen, 2005, Rational Pessimism, Rational Exuberance, and Markets for Macro Risks, Working paper, Duke University.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

Bansal, Ravi, and Amir Yaron, 2006, The asset pricing-macro nexus and return-cash flow predictability, Working paper, The Wharton School, University of Pennsylvania.

Bekaert, Geert, Eric Engstrom, and Yuhang Xing, 2005, Risk, Uncertainty and Asset Prices, Working paper, .

Campbell, John, and Robert Shiller, 1988, Stock Prices, Earnings, and Expected Dividends, *Journal of Finance* 43, 661–676.

Campbell, John Y., 1999, *Asset prices, Consumption and the Business Cycle* Elsevier Science pp. 1231–1303 Handbook of Macroeconomics, Volume 1, John B. Taylor and Michael Woodford, editors.

Cochrane, John, and Lars Hansen, 1992, Asset Pricing Explorations for Macroeconomics, *NBER Macro Annual* pp. 115–165.

Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, risk aversion, and the intertemporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Epstein, Larry G., and Stanley E. Zin, 1991, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis, *Journal of Political Economy* 99, 263–286.

Fama, Eugene, and Kenneth French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.

Giovannini, Alberto, and Philippe Weil, 1989, Risk Aversion and Intertemporal Substitution in the Capital Asset Pricing Model, Working paper, NBER Working Paper No. 2824.

Hall, Robert, 1988, Intertemporal Substitution in Consumption, *Journal of Political Economy* 96, 339–357.

Hansen, Lars, John Heaton, and Nan Li, 2005, Consumption strikes back?, Working paper, University of Chicago.

Hansen, Lars, John Heaton, and Amir Yaron, 1996, Finite Sample Properties of Alternative GMM Estimators, *Journal of Business and Economic Statistics* 49, 531–560.

Hansen, Lars, and Thomas Sargent, 2006, Fragile Beliefs and the Price of Model Uncertainty, Working paper, .

Hansen, Lars Peter, and Thomas Sargent, 1983, Aggregation Over Time and the Inverse Optimal Predictor Problem for Adaptive Expectations in Continuous Time, *International Economic Review*.

Hansen, Lars Peter, and Kenneth Singleton, 1982, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* 63, 767–804.

Heaton, John, 1995, An Empirical Investigation of Asset Pricing with Temporally Dependent Preference Specifications, *Econometrica* 63, 681–717.

Kandel, Eugene, and Robert Stambaugh, 1991, Asset returns and intertemporal preferences, *Journal of Monetary Economics* 27, 39–71.

Kiku, Dana, 2006, Long Run Risks and the Value Premium Puzzle, Working paper, Duke University.

Lettau, Martin, and Sydney Ludvigson, 2005, Euler Equation Errors, Working paper, New York University.

Malloy, Christopher, Tobias Moskowitz, and Annette Vissing-Jorgensen, 2004, Long-Run stockholder consumption risk and asset returns, Working paper, University of Chicago.

Menzly, Lior, Tano Santos, and Pietro Veronesi, 2004, Understanding predictability, *Journal of Political Economy* 112(1), 1–47.

Tallarini, Thomas, 2000, Risk Sensitive Real Business Cycles, *Journal of Monetary Economics* 45(3), 507–532.

Tauchen, George, and Robert Hussey, 1991, Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models, *Econometrica* 59, 371–96.

Weil, Philippe, 1989, The Equity Premium Puzzle and the Risk-free Rate Puzzle, *Journal of Monetary Economics* 24, 401–421.

Wilcox, David, 1992, The construction of the U.S. consumption data: Some facts and their implications for empirical work, *American Economic Review* 82, 922–941.

Table I

Approximation Error

Panel A: Approximate Analytical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.592	4.754	5.058	0.059	0.021	0.032
RA	**10**	3.789	4.572	4.716	0.060	0.021	0.032
	15	4.055	4.421	4.470	0.062	0.021	0.032

Panel B: Numerical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.594	4.755	5.060	0.059	0.021	0.032
RA	**10**	3.788	4.576	4.724	0.060	0.021	0.032
	15	4.033	4.436	4.493	0.061	0.021	0.031

Panel C: Approximation Error (as a % of numerical values)

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	0.05	0.01	0.03	0.04	-0.16	-0.17
RA	**10**	-0.02	0.10	0.17	-0.42	-0.83	-0.86
	15	-0.54	0.32	0.51	-1.84	-2.16	-2.17

Table II
Summary Statistics

Panel A: Asset Data

	Returns		Div Growth		Log(P/D)	
	Mean	StdDev	Mean	StdDev	Mean	StdDev
Size Portfolios						
Small	0.166	0.40	0.066	0.27	4.07	0.62
Large	0.076	0.19	0.003	0.11	3.30	0.43
B/M Portfolios						
Growth	0.070	0.22	-0.003	0.16	3.71	0.62
Value	0.134	0.33	0.047	0.29	3.42	0.68
Market	0.083	0.20	0.007	0.11	3.33	0.46
Risk-free Rate	0.008	0.01				

Panel B: Predictive Variables

	Mean	StdDev
Consumption Growth	0.020	0.022
GDP Growth	0.022	0.051
Default Spread	0.012	0.007

Panel A of Table II presents descriptive statistics for returns, dividend growth rates and logarithms of price-dividend ratios of size and book-to-market sorted portfolios, and the aggregate stock market. Small and large portfolios represent firms in the top and bottom market capitalization deciles, growth and value correspond to the lowest and highest book-to-market decile. Returns are value-weighted, dividends and price-dividend ratios are constructed on the per-share basis, growth rates are measured by taking the first difference of the logarithm of dividend series. The bottom line of Panel A reports the mean and the standard deviation of the annualized yield on the 3-month Treasury bill. Panel B presents sample statistics for the per-capita consumption of nondurables and services, gross domestic product (GDP), and the default premium. The latter is defined as the difference in yields on Baa and Aaa corporate bonds. All asset and macro data are real, sampled on an annual frequency and cover the period from 1930 to 2002.

Table III

Predictability of Consumption Growth

Predictor	Estimate	t-stat
2-yr Moving Ave of Cons Growth	0.445	2.90
Log(Cons/GDP)	-0.070	-2.52
Default Spread	1.912	3.62
Short Interest Rate	-0.116	-2.09
Log(P/D)	0.019	2.30
$\bar{R}^2 = 0.37$		

Table III presents predictability evidence for consumption growth. The second column reports estimated regression coefficients from projecting consumption growth onto lagged predictive variables. The corresponding t-statistics are calculated using the Newey-West variance-covariance estimator with 4 lags. The data employed in the regression are annual and span the period from 1930 to 2002.

Table IV

Consumption Betas

	Mean Ret	β^a_η	β^a_e	β^a_{ccapm}
Small	0.166	0.51	16.40	0.71
Large	0.076	2.22	9.10	0.69
Growth	0.070	2.65	10.69	0.82
Value	0.134	0.90	15.30	0.14
Market	0.083	2.28	10.40	0.59

Table IV presents mean returns and consumption betas for firms in the lowest and highest deciles of size and book-to-market sorted portfolios — small and large, and growth and value, respectively, as well as the aggregate stock market. Consumption betas are calculated as the covariation between consumption news and innovations in asset returns scaled by the variance of the corresponding consumption shock. β^a_η represents the exposure of returns to transient shocks in consumption, β^a_e measures risks related to long-run fluctuations in consumption. Short-run consumption innovations are constructed by removing the conditional mean from the realized growth in consumption, where the former is modelled according to Table III. Long-run consumption risks are extracted by fitting an AR(1) process to the expected growth component. Innovations in returns are constructed using a log-linear approximation of returns and estimated VAR(1)-dynamics for dividend growth rates and price-dividend ratios. The frequency of the data is annual, the sample covers the period from 1930 to 2002.

Table V

Estimation Evidence: Long Run Risk Model

	Panel A: W = Identity		**Panel B**: W = diag{Var(R)}$^{-1}$	
Parameter	Estimate	SE	Estimate	SE
RA	27.70	8.67	22.89	7.37
IES	0.59	2.57	0.70	2.67
Asset	PrError	t-stat	PrError	t-stat
Small	0.038	0.30	0.054	0.53
Large	-0.020	-0.18	-0.009	-0.11
Growth	-0.032	-0.29	-0.020	-0.24
Value	0.021	0.18	0.037	0.40
Market	-0.018	-0.16	-0.006	-0.07
Risk-Free	0.012	0.09	0.000	0.00
Small-Large	0.058	1.33	0.063	1.45
Value-Growth	0.052	1.62	0.057	1.82
J-stat	5.60		5.28	
p-value	0.23		0.26	

Table V presents GMM estimates of Long Run Risk model: the risk aversion parameter (RA) and the elasticity of intertemporal substitution (IES). Three vertical panels summarize estimation results for different weighting schemes: the identity matrix (A), the inverse of the diagonal of the variance-covariance matrix of returns (B). The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio (growth and value, respectively), aggregate stock market and the risk-free rate. Average pricing errors and their t-statistics are presented for each asset. The bottom two lines report J-statistics for overidentifying restrictions and the corresponding p-values. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table VI

Estimation Evidence: CRRA Preferences

	Panel A: W = Identity		**Panel B**: W = diag{Var(R)}$^{-1}$	
Parameter	Estimate	SE	Estimate	SE
RA	4.07	1.35	42.86	7.71
Asset	PrError	t-stat	PrError	t-stat
Small	0.072	1.38	0.220	0.32
Large	-0.017	-0.70	-0.007	-0.02
Growth	-0.021	-0.83	-0.036	-0.08
Value	0.044	1.24	0.134	0.22
Market	-0.009	-0.40	0.009	0.02
Risk-Free	-0.081	-2.99	-0.003	-0.01
Small-Large	0.089	2.02	0.227	1.18
Value-Growth	0.065	2.54	0.170	1.19
J-stat	8.64		12.14	
P-value	0.12		0.03	

Table VI presents GMM estimates of the parameter of risk aversion (RA) for CRRA preferences. Three vertical panels summarize estimation results for different weighting schemes: the identity matrix (A), the inverse of the diagonal of the variance-covariance matrix of returns (B). The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio (growth and value, respectively), aggregate stock market and the risk-free rate. Average pricing errors and their t-statistics are presented for each asset. The bottom two lines report J-statistics for overidentifying restrictions and the corresponding p-values. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table VII

Estimation Evidence: The Effects of Persistence in x

	Panel A: $\rho = 0.50$		**Panel B**: $\rho = 0.75$	
Parameter	Estimate	SE	Estimate	SE
RA	30.39	14.17	21.56	5.90
IES	0.22	0.52	1.22	5.61
Asset	PrError	t-stat	PrError	t-stat
Small	0.068	0.41	0.032	0.31
Large	-0.022	-0.18	-0.017	-0.19
Growth	-0.033	-0.27	-0.030	-0.32
Value	0.042	0.29	0.017	0.18
Market	-0.016	-0.13	-0.016	-0.17
Risk-Free	-0.024	-0.17	0.012	0.10
J-stat	2.92		6.34	
p-value	0.57		0.18	

Table VII presents GMM estimates of Long Run Risk model for alternative specifications for the persistence of x: the risk aversion parameter (RA) and the elasticity of intertemporal substitution (IES). In each case the innovation to the long run innovation is scaled to maintain identical unconditional variance as in the estimated model. All estimation utilize the identity matrix as in vertical panel A in Table V. The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio (growth and value, respectively), aggregate stock market and the risk-free rate. Average pricing errors and their t-statistics are presented for each asset. The bottom two lines report J-statistics for overidentifying restrictions and the corresponding p-values. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table VIII

Model-Implied Risk Premia Decomposition

	Mean Ret	Risk Premia		
		Long-Run	Short-Run	Total
Small	0.166	0.102	0.004	0.105
Large	0.076	0.056	0.016	0.072
Growth	0.070	0.066	0.019	0.085
Value	0.134	0.095	0.006	0.101
Market	0.083	0.064	0.016	0.080

Table VIII presents the decomposition of risk premia implied by the Long Run Risk Model. Compensations for various consumption risks are determined by asset betas and the corresponding prices of risks. Specifically, the long-run risk premium is computed as $\lambda_e^a \bar{\sigma}_e^{2,a} \beta_e^a$; the compensation for short-run risks in consumption corresponds to $\lambda_\eta^a \bar{\sigma}_\eta^{2,a} \beta_\eta^a$. Risk prices are based on GMM estimates with the identity weight matrix. For comparison, the first column reports sample mean returns for the five assets. The data employed in the estimation are annual and cover the period from 1930 to 2002.

Table IX
Consumption Growth Dynamics

Panel A: Calibration of Monthly Consumption Growth

μ_c	ρ	φ_e	$\bar{\sigma}$	ν	σ_w
0.0015	0.982	0.042	0.0054	0.98	0.0000068

Panel B: Dynamics of Annual Consumption Growth

Statistic	— **Data** —		— **Model** —	
$E[\Delta c]$	1.96	(0.34)	1.83	(0.62)
$\sigma(\Delta c)$	2.21	(0.38)	2.27	(0.37)
$AC(1)$	0.44	(0.13)	0.47	(0.12)

Panel A of Table IX summarizes the calibration of parameters that govern the dynamics of monthly consumption growth:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\sigma^2_{t+1} &= \bar{\sigma}^2 + \nu(\sigma^2_t - \bar{\sigma}^2) + \sigma_w w_{t+1}
\end{aligned}$$

Panel B reports the mean, the volatility and the first-order autocorrelation of annual consumption growth. "Data" column presents summary statistics of observed per-capita consumption of non-durables and services over the period from 1930 till 2002. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table X
Dividend Growth Dynamics

Panel A: Calibration of Monthly Dividend Growth Rates

Asset	μ_d	ϕ	π	φ
Small	0.0058	5.4	1.5	7.3
Large	0.0015	2.3	3.3	5.7
Growth	0.0015	2.0	3.6	7.1
Value	0.0040	4.4	1.9	5.2
Market	0.0015	2.3	3.8	5.4

Panel B: Dynamics of Annual Dividend Growth Rates

Asset	Statistic	— **Data** —		— **Model** —	
Small	$E[\Delta d]$	6.57	(4.15)	6.63	(3.71)
	$\sigma(\Delta d)$	27.2	(4.19)	15.4	(2.15)
	$Corr(\Delta c, \Delta d)$	0.44	(0.09)	0.45	(0.14)
Large	$E[\Delta d]$	0.34	(1.09)	1.87	(2.05)
	$\sigma(\Delta d)$	10.5	(1.69)	12.4	(1.60)
	$Corr(\Delta c, \Delta d)$	0.50	(0.15)	0.54	(0.11)
Growth	$E[\Delta d]$	-0.26	(1.65)	1.77	(2.41)
	$\sigma(\Delta d)$	16.3	(1.94)	14.4	(1.80)
	$Corr(\Delta c, \Delta d)$	0.47	(0.10)	0.47	(0.11)
Value	$E[\Delta d]$	4.67	(3.40)	4.74	(2.81)
	$\sigma(\Delta d)$	28.6	(3.61)	11.8	(1.78)
	$Corr(\Delta c, \Delta d)$	0.51	(0.07)	0.57	(0.12)
Market	$E[\Delta d]$	0.74	(1.18)	1.87	(2.02)
	$\sigma(\Delta d)$	11.0	(1.92)	12.3	(1.63)
	$Corr(\Delta c, \Delta d)$	0.60	(0.14)	0.60	(0.10)

Panel A of Table X presents the calibration of monthly dividend growth rates for the cross-section of assets:

$$\Delta d_{j,t+1} \quad = \quad \mu_{d_j} + \phi_j x_t + \pi_j \sigma_t \eta_{t+1} + \varphi_j \sigma_t u_{d,j,t+1}$$

The asset menu comprises small and large market capitalization firms, growth and value portfolios that represent low and high book-to-market firms respectively, and the aggregate stock market. Panel B reports the mean and the volatility of dividend growth rates, as well as their correlation with consumption growth. "Data" column presents summary statistics of the per-share dividend series observed over 1930-2002 time period. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table XI
Asset Pricing Implications

Asset	Statistic	— **Data** —		— **Model** —	
Small	$E(R)$	16.60	(4.18)	15.40	(4.32)
	$\sigma(R)$	40.4	(3.84)	35.7	(6.48)
	$E(pd)$	4.07	(0.15)	3.59	(0.15)
Large	$E(R)$	7.58	(2.19)	7.90	(2.20)
	$\sigma(R)$	19.1	(1.79)	19.2	(2.74)
	$E(pd)$	3.30	(0.10)	3.22	(0.05)
Growth	$E(R)$	7.01	(2.40)	6.87	(2.59)
	$\sigma(R)$	21.6	(1.89)	21.1	(3.11)
	$E(pd)$	3.71	(0.15)	3.48	(0.05)
Value	$E(R)$	13.37	(3.03)	12.58	(3.05)
	$\sigma(R)$	33.1	(3.89)	26.4	(4.45)
	$E(pd)$	3.42	(0.15)	3.18	(0.11)
Market	$E(R)$	8.27	(2.10)	8.10	(2.17)
	$\sigma(R)$	20.1	(1.88)	19.6	(2.75)
	$E(pd)$	3.33	(0.11)	3.15	(0.05)
Risk-Free Rate	$E(R)$	0.76	(0.27)	1.08	(0.34)
	$\sigma(R)$	1.12	(0.22)	0.87	(0.18)

Table XI presents asset pricing moments for five equity portfolios and the risk-free rate. Small and large are portfolios of firms with low and high market capitalization, growth and value correspond to the top and the bottom book-to-market deciles. $E(R)$, $\sigma(R)$ and $E(pd)$ denote expected returns, return volatilities and means of log price-dividend ratios respectively. "Data" column presents summary statistics of the observed annual data that span the period from 1930 to 2002. Numbers in parentheses are robust standard errors calculated using the Newey-West variance-covariance estimator with 4 lags. The entries reported in "Model" column are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations. Model-implied statistics represent the median and the standard deviation (in parentheses) of the corresponding statistics across simulations.

Table XII

Simulation Evidence: Long Run Risk Model

		Mean	5%	50%	95%
Panel A:	*RA*	21.12	8.16	19.03	41.17
W = Identity	*IES*	0.51	0.12	0.40	1.10
	J-stat	5.82	1.96	5.60	9.92
	P-value	0.29	0.04	0.23	0.74
Panel B:	*RA*	18.62	8.05	17.32	35.76
W = diag$\{\text{Var(R)}\}^{-1}$	*IES*	0.57	0.18	0.51	1.00
	J-stat	6.07	1.68	5.47	12.10
	P-value	0.28	0.02	0.24	0.79

Table XII presents the distribution of GMM estimates of the Long Run Risk Model, J-statistics for overidentifying restrictions and the corresponding p-values. RA and IES denote risk aversion and the elasticity of intertemporal substitution respectively. Three horizontal panels summarize estimation results for different weighting schemes: the identity matrix (A), the inverse of the diagonal of the variance-covariance matrix of returns (B). The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate. The entries are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations.

Table XIII

Simulation Evidence: CRRA Preferences

		Mean	5%	50%	95%
Panel A:	*RA*	28.27	3.02	24.51	74.63
W = Identity	*J-stat*	7.47	2.74	7.40	11.95
	P-value	0.26	0.04	0.19	0.74
Panel B:	*RA*	42.67	0.13	43.71	80.18
W = diag$\{\text{Var(R)}\}^{-1}$	*J-stat*	28.14	5.46	18.01	77.31
	P-value	0.07	0.00	0.00	0.36

Table XIII presents the distribution of the GMM estimate of the risk aversion parameter (RA) of CRRA preferences, J-statistics for overidentifying restrictions and the corresponding p-values. Three horizontal panels summarize estimation results for different weighting schemes: the identity matrix (A), the inverse of the diagonal of the variance-covariance matrix of returns (B). The asset menu consists of firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate. The entries are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations.

Table XIV

Simulation Evidence: Time Aggregation and Long Run Risk Model

		Long Sample
X-properties :	ρ	0.71
	R^2	0.46
Panel A:	*RA*	15.09
W = Identity	*IES*	0.39
	P-value	0.00
Panel B:	*RA*	12.30
W = diag{Var(R)}$^{-1}$	*IES*	0.48
	P-value	0.00

Table XIV presents the GMM estimates of the Long Run Risk Model, J-statistics for overidentifying restrictions and the corresponding p-values for a long simulation. RA and IES denote risk aversion and the elasticity of intertemporal substitution respectively. Three horizontal panels summarize estimation results for different weighting schemes: the identity matrix (A), the inverse of the diagonal of the variance-covariance matrix of returns (B). The asset menu comprises firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate, as well as an asset that pays aggregate consumption each period. The entries are based on a sample with 10,000 annual observations.

Table XV

Simulation Evidence: Pricing Kernel based on Market Return

	Long Sample
RA	1.78
IES	1.71
P-value	0.00

Table XV presents the GMM estimates of the Long Run Risk Model, J-statistics for overidentifying restrictions and the corresponding p-values for a pricing kernel in which the market return $r_{m,t}$ is used instead of the return on consumption, $r_{c,t}$. The model and estimation are based on monthly frequency. RA and IES denote risk aversion and the elasticity of intertemporal substitution respectively. The results are for the identity matrix. The asset menu comprises firms with small and large market capitalization, low and high book-to-market ratio, aggregate stock market and the risk-free rate, as well as an asset that pays aggregate consumption each period. The entries are based on a sample with 120,000 monthly observations.

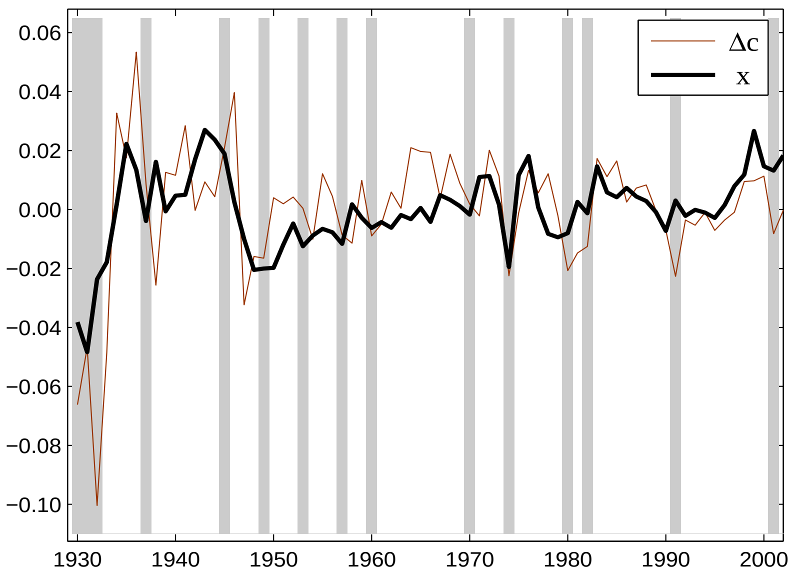


Figure 1. Realized and Expected Growth of Consumption

Figure 1 plots time series of realized (thin red line) and expected (thick black line) growth in consumption. Consumption is defined as the per-capita expenditure on non-durables and services. The expected consumption growth is constructed according to the predictability evidence presented in Table III. The data are real, sampled on an annual frequency and cover the period from 1930 to 2002. Shaded areas correspond to NBER-dated recessions.

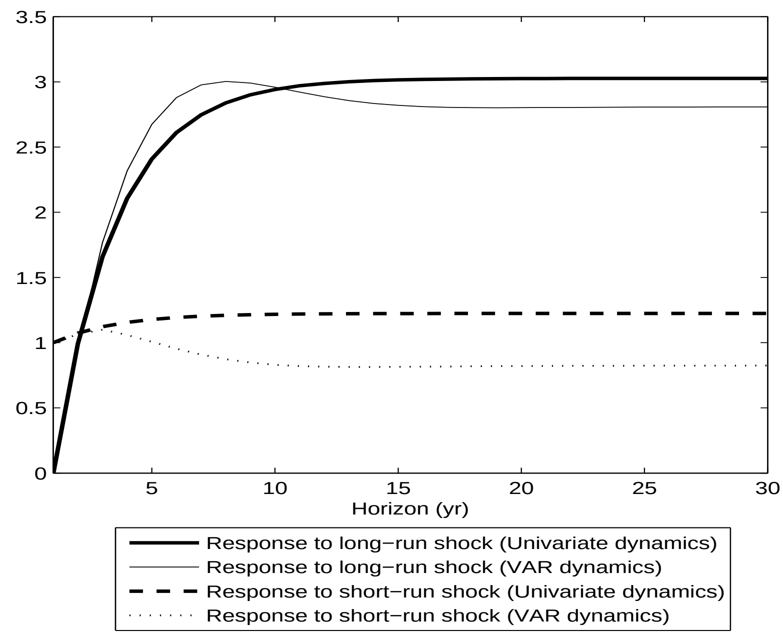
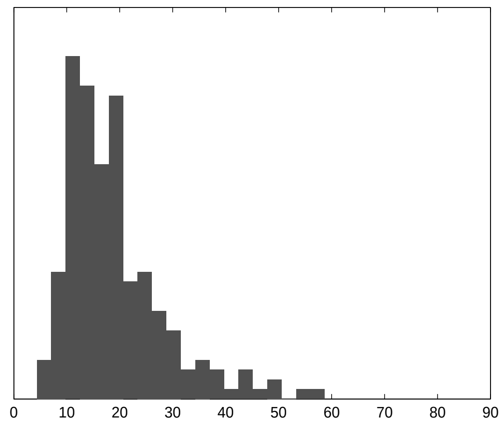
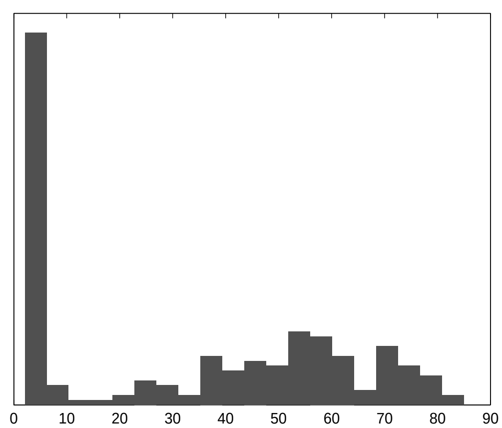


Figure 2. Accumulated Impulse Response Function to Short and Long Run Shocks

Figure 2 plots the accumulated impulse response of log consumption to a unit shock in e and η. The thick (thin) solid line represents the impulse response based on the univariate (VAR) dynamics for a one percent long run shock. The dash (dotted) line represents the impulse response based on the univariate (VAR) dynamics to a one percent short-run shock.



(a) EZ Preferences, W = Identity



(b) CRRA Preferences, W = Identity

Figure 3. Histogram of the RA Estimate

Figure 3 displays the monte carlo distribution of the GMM estimate of the parameter of risk aversion. In the top panel, (a) and (b), moment conditions are based on Epstein and Zin (1989) preferences, the bottom figures, (c) and (d) correspond to CRRA preferences. Plots are based on the identity weight matrix. The figures are based on 500 simulated samples, each with 876 months, time-aggregated to 73 annual observations.