What Can Rational Investors Do About Excessive Volatility?*

Bernard Dumas† Alexander Kurshev‡ Raman Uppal§

27 June 2005

*We are grateful to Wei Xiong for extensive discussions. We are also grateful for comments from Andrew Abel, Suleyman Basak, Fernando Broner, Joao Cocco, John Cotter, Xavier Gabaix, Francisco Gomes, Joao Gomes, Tim Johnson, Leonid Kogan, Karen Lewis, Deborah Lucas, Albert Marcet, Anna Pavlova, Jose Scheinkman, Andrew Scott, Alex Stomper, Skander van den Heuvel, Pierre-Olivier Weill, Hongjun Yan, Moto Yogo, Joseph Zechner and participants in workshops held at London Business School, INSEAD, University of Vienna, Wharton, Massachusetts Institute of Technology University College Dublin, and Universita Pompeu Fabra.

†INSEAD, University of Pennsylvania (The Wharton School), CEPR and NBER. Mailing address: INSEAD, boulevard de Constance, 77305 Fontainebleau Cedex, France. Email: bernard.dumas@insead.fr.

‡London Business School. Mailing address: IFA, 6 Sussex Place Regent's Park, London, United Kingdom NW1 4SA. Email: akurshev.phd2003@london.edu

§London Business School and CEPR. Mailing address: IFA, 6 Sussex Place Regent's Park, London, United Kingdom NW1 4SA. Email: ruppal@london.edu.

What Can Rational Investors Do About Excessive Volatility?

Abstract

We determine and analyze the trading strategy that would allow an investor to take advantage of the excessive stock price volatility that has been documented in the empirical literature on asset pricing. We construct a general equilibrium model where stock prices are excessively volatile because there are two classes of agents and one class places is overconfident about a public signal. As a result, these agents change their expectations too often, sometimes being excessively optimistic, sometimes being excessively pessimistic. We analyze the trading strategy of the rational investors who are not overconfident about the signal. While rational risk-arbitrageurs benefit from trading on their belief that the market is being foolish, when doing so they must hedge future fluctuations in the market's foolishness. We find that fixed-income instruments can be used for the purpose of hedging. Thus, our analysis illustrates that risk arbitrage cannot be based on just a current price divergence; it must include also a protection against trading risk. We also show that the presence of a few rational traders is not sufficient to eliminate the excessive volatility effect of overconfident investors generated by the presence of overconfident investors. Furthermore, overconfident investors of this kind tend to survive for a long time before being driven out of the market by rational investors.

1 Introduction

As Shiller (1981) and LeRoy and Porter (1981) have pointed out, it may very well be the case that "stock prices move too much to be justified by changes in subsequent dividends."[1] The volatility of stock prices, if it is excessive relative to the volatility of fundamentals, may be an indication that the financial market is not information efficient. If so, there must exist a trading strategy that allows a rational, intertemporally optimizing investor (a "risk arbitrageur") to take advantage of this inefficiency. The main goal of the present paper is to calculate and understand that strategy.[2]

In the model we construct, some investors are non-Bayesian in the sense that that they give too much credence to public signals just as in Scheinkman and Xiong (2003), who call these investors "overconfident".[3] We refer to "excessive volatility" as a situation in which, for the given utility functions of agents, the level of volatility is larger than it would be under rational Bayesian learning.[4] Because some investors in our model are overconfident about the public signal, they change their minds too often about economic prospects, and this is the source of excessive volatility. Of course, it is well-known that complete irrationality in the manner of positive "feedback traders" à la De Long, Shleifer, Summers, and Waldmann (1990b) can amplify the volatility of stock prices. The added volatility creates "noise-trader risk" for rational arbitrageurs, thereby creating a limit to arbitrage. However, feedback traders may not be the best representation of irrational behavior as they constitute excessively easy game for rational investors. Furthermore, models of feedback trading do not discuss the budget constraint of the feedback traders, and therefore, leave unclear the origin of the gains that the rational arbitrageurs would make at their expense.[5] For these reasons, we prefer to model our irrational traders as being intertemporal optimizers, even if they are non-Bayesian in their learning.

We aim to derive and then analyze the optimal dynamic trading strategy of the rational investors in this model. There are two aspects to the portfolio strategy adopted by rational investors. First, these investors may not agree today with the market about its current estimate of the growth rate of dividends; when the rational investors are more optimistic than the market, they increase their investment in equity while decreasing their investment in bonds, because equity and bonds are positively correlated. Second,

[1] A controversy about excess volatility has been going on since the publication of Shiller (1981) and the matter is not fully settled today. The empirical method of Shiller has been criticized. Flavin (1983) and Kleidon (1986) have pointed out that stock prices and dividends could not be detrended by a deterministic trend based on realized returns, as Shiller had done. Furthermore, if the process for prices and/or dividends is not stationary, the ergodic theorem does not apply and volatility, defined originally across the possible sample paths, cannot be measured over time. Even in the case of stationarity, a near-unit root may exist in the behavior of these two variables, causing the statistic to reject Shiller's variance inequality in finite samples when it should not be rejected. Good methodological evaluations are provided by West (1988a,b) and Cochrane (1991). Generally speaking, as had been pointed out by LeRoy and Porter, variance-bound tests should not be implemented on the basis of the historical sequence of dividends taken at face value. The sequence must first be used to estimate the stochastic process of dividends. Stock prices must then be calculated from an extrapolation of the process to infinity before a variance bound can be placed on them. Even after Mankiw, Romer, and Shapiro (1985, 1991) improve upon the methodology used in Shiller's tests, they reach the same conclusion as he had. Their method is applied to non U.S. markets by De Long and Becht (1998) and De Long and Grossman (1998) and they also reach similar conclusions.

[2] The question being answered in our paper is the same as the one raised by Williams (1977) and Ziegler (2000) in a simpler setting in which the expected growth rate of dividends is constant (although unobserved) and in which there are fewer securities. In these two papers, the investor whose strategy one is studying is assumed to be of negligible weight in the market, in contrast to our model. For a comprehensive review of the literature on models with incomplete information, see Feldman (2005).

[3] In contrast to the model in Scheinkman and Xiong (2003), our model is of a general equilibrium economy, and rather than modeling agents as being risk neutral and then introducing short-sale constraints to limit the size of positions that agents take, we allow for short sales and agents who are risk averse, so that in our model it is risk aversion that induces agents to limit the size of their short positions.

[4] This definition of excessive volatility is not identical to that of Shiller (1981). The difference arises mostly from Shiller's use of constant discount factors and unconditional expected values to price equity. Here, we use the equilibrium discount factors that derive from intertemporal optimization with learning.

[5] Other limitations of these models are discussed in Loewenstein and Willard (2005).

even when the two groups of investors happen to agree today, rational investors are aware that irrational investors will revise their estimate differently from the way their own estimate is revised. This second effect makes the rational investor hold fewer shares of equity than would be optimal in a market without excess volatility and take a negative position in bonds (which would be zero in the absence of excess volatility). Overall, a rational risk-arbitrageur finds it beneficial to trade on his/her belief that the market is being foolish but when doing so, he/she must hedge future fluctuations in the market's foolishness. Thus, our analysis illustrates that "risk arbitrage" cannot be based on just a current price divergence; it must also be based on a model of irrational behavior and a prediction concerning the dynamics of this divergence. Further, it illustrates that the risk arbitrage must include a protection in case there is a deviation from that prediction. Our model should be of use to hedge funds who play the price convergence game. They often have at their disposal perfect-market pricing models which allow them to spot pricing anomalies. But, that is not sufficient information to be able to put in place a "risk arbitrage" strategy, including the optimal timing of trades into the strategy, of trades out of the strategy, plus the accompanying hedges. For that purpose, hedge funds also need a model of the equilibrium stochastic process of price spreads. We provide one such model.

The profitability of the rational "risk arbitrage" strategy and the survival time of irrational investors are two sides of the same coin. We also derive the speed of impoverishment of the irrational traders, or the speed of enrichment of the rational ones. Previous work (Kogan, Ross, Wang, and Westerfield (2003); Yan (2004)) has examined the survival of traders who are permanently overoptimistic or overpessimistic. Here, we study the survival of traders who are sometimes overoptimistic and sometimes overpessimistic, depending on the sequence of signals they have received. We find that, in contrast to what is typically assumed in standard models of asset pricing in frictionless markets, in our model the presence of a few rational traders is not sufficient to eliminate the effect of overconfident investors on excess volatility, and that even a moderate-sized group of overconfident investors can do a lot of damage and may survive for a long time before being driven out of the market by rational investors.[6]

We now relate our model to models of excessive volatility in the existing literature. Some headway into the design of a portfolio strategy has already been made in past research which dealt with the logical link that exists between the phenomenon of excessive volatility and the predictability of stock returns.[7] Campbell and Shiller (1988a,b) and Cochrane (2001, page 394 *ff*), have pointed out that the dividend-price ratio would be constant over time if dividends were unpredictable (specifically, if they followed a geometric Brownian walk) and expected returns were constant. Since the dividend-price ratio is changing, its changes must be predicting either future changes in dividends or future changes in expected returns. This statement is true in any economic model, unless there are violations of the transversality conditions.[8] Empirically, the dividend-price ratio hardly predicts subsequent dividends.

[6] Alchian (1950) and Friedman (1953) are given credit for articulating the doctrine according to which agents who do not predict as accurately as others are driven out of the market. De Long, Shleifer, Summers, and Waldmann (1990a, 1991) have indicated that the doctrine may not be correct, but their approach has recently been criticized by Loewenstein and Willard (2005). Sandroni (2000) shows that agents whose beliefs are most accurate (in the entropy sense), with probability one are the only ones in the long run who survive (when all utility discount rates are equal to each other) in a complete financial market. In contrast to much of this literature, our definition of "survival" will be based on consumption shares not wealth shares.

[7] That relation is analogous to the relation established by Froot and Frankel (1989) between variance-bounds tests à la Shiller and regression tests of predictability.

[8] There may be some violations of the terminal condition for stock prices, causing stock prices to deviate from the present discounted value of future dividends. A violation of the terminal condition means that investors pay a price today that reflects an expectation of a price at some distant date in the future that will not reflect the present discounted value of dividends. Such a deviation is a called a "bubble". Recently, the theory of bubbles has endeavored to explain the process by which bubbles burst. Fluctuations in the probability of a bubble bursting can potentially generate excessive volatility. But, it is not clear whether a theory of bubbles can ever be developed without assuming the presence of some irrational investors.

It must, therefore, predict returns. But, if it predicts returns, it can serve as valuable information for a rational trader, or arbitrageur, entering the market. That aspect is present in our model below.

In the literature on excess volatility, there are at least two kinds of models not based on differences in beliefs that have been considered.[9] One class of models shows that Bayesian, rational learning alone can serve to develop theoretical models with volatility that matches the data, by assuming that investors do not know the true stochastic process of dividends. For instance, Barsky and De Long (1993) write that: "Major long-run swings in the U.S. stock market over the past century are broadly consistent with a model driven by changes in current and expected future dividends in which investors must estimate the time-varying *long-run* dividend growth rate" [our emphasis]. As investors do not know the expected growth rate of dividends, prices are revised when they receive information about it. These price revisions go beyond the change in the current dividend because the current dividend also contains information about future dividends. A similar argument has been made by Timmermann (1993, 1996) and Bullard and Duffy (1998). Brennan and Xia (2001) calibrate a model in which a single type of investors populate the financial market and learn about the expected growth rate of dividends and, separately, about the expected growth rate of output. In that model, as in ours, the expected growth rate of dividends is unobservable and needs to be filtered out, which then contributes positively to the volatility of the stock price. They find that they can match all moments of stock returns. However, their model is not really "closed" since aggregate consumption is not set equal to aggregate dividends plus endowments.[10]

A second class of models studying excess volatility focuses on the discount rate. Recall that in deriving their bounds, Shiller (1981) and LeRoy and Porter (1981) had made the assumption that discount rates, by which future dividends are discounted to obtain the current price, were constant into the future. The literature on the equity-premium puzzle has developed a number of models, such as habit formation models (see Constantinides (1990); Abel (1990); Campbell and Cochrane (1999)), in which the effective discount rate is strongly time varying even though the consumption stream remains very smooth. Using models of that kind, Menzly, Santos, and Veronesi (2004) have recently calibrated a model of the U.S. stock market in which the volatility of stock returns was *larger* than the one observed in the data.[11]

The balance of this paper covers the following material. In Section 2, we discuss our modeling choices against the background of the literature that we have just surveyed. In Section 3, we determine the equilibrium. In Section 4, we discuss the impact of irrational traders on asset prices, return volatilities and risk premia and we analyze how many rational investors are needed to reduce the excessive volatility. In Section 5, we identify the main factors driving the portfolio strategy of the rational trader. In Section 6, we discuss the survival of irrational traders over time and the profits made at their expense by the rational ones. Section 7 contains the conclusion. All the mathematical derivations are collected in appendixes.

[9] In a third line of investigation, Bansal and Yaron (2004) and Hansen, Heaton, and Li (2005) find that allowing for a small long-run predictable component in dividend growth rates can generate several observed asset-pricing phenomenon, including volatility of the market return.

[10] Needless to say, in the real world, consumption is not equal to just dividends. There is also labor income and, in any case, the real world is not a pure-exchange economy, since physical investment takes place. Another way to close the model would be to allow for labor income and physical investment.

[11] Besides time-varying discount factors and learning about the dividend process, there exists at least one other theoretical reason for which stock prices may exhibit high volatility. Financial markets are vastly incomplete. The private valuation of nontraded risks may "rock the boat" of prices of traded securities, as in Citanna and Schmedders (2002) or Bhamra and Uppal (2005). However, it is not clear that market incompleteness could generate the magnitude of excessive volatility that one observes in the stock market.

2 Modeling choices and information structure

In our model below, we allow for the presence of irrational traders.[12] It is well known that rationality entails two dimensions, which may not be completely independent of each other: rationality in information processing or learning (that is, application of Bayes' law) and rationality of decision making (that is, intertemporal optimality). While our irrational traders suffer from some learning disability, we want them to remain full-fledged intertemporal optimizers, so that welfare analysis and the analysis of gains and losses of the two categories of traders remain meaningful.

One way to achieve that goal has recently been proposed by Scheinkman and Xiong (2003). In their model of a "tree" economy, a stream of dividends is paid. Some aspect of the stochastic process of dividends is not observable by anyone. Risk neutral investors receive information in the form of the current dividend and some public signals. Rational agents are people who either know the true correlation between innovations in the signal and innovations in the unobserved variables or rationally learn about it from the information they receive. Irrational (they call them "overconfident") agents are people who steadfastly refuse to learn the value of this correlation. For instance, they insist on this correlation being a positive number when, in fact, it is zero. This causes them to give too much weight to the signals. Thus, when they receive a signal, they overreact to it, which then generates excessive stock price movements.[13]

Here, we consider a setting similar to that in Scheinkman and Xiong (2003) except that investors are risk averse (and are allowed to sell short) and only one group of agents is overconfident. More specifically, there are two groups of investors: Group A, who are overconfident and Group B, who are rational. Both groups are risk averse and have the same level of constant relative risk aversion. The risk aversion does not prevent investors from short selling but it induces them not to sell infinite amounts.

We now describe the key features of our model. We adopt notation that is similar to the one used in the paper by Scheinkman and Xiong (2003).

2.1 Process for aggregate output

The dividend (output) paid by the aggregate economy at time t is equal to $\delta_t dt$.[14] The stochastic process for δ is:

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dZ_t^\delta, \tag{1}$$

where Z^δ is a Wiener under the effective probability measure, which governs empirical realizations of the process, and σ_δ is the volatility of the growth rate of dividends. The conditional expected growth

[12] A recent paper by David (2004), developed concurrently with ours, uses differences of opinion in the financial market and non-Bayesian learning. It contains a full-fledged calibration of securities price moments. However, like in Brennan and Xia (2001), his model is not closed. Also, in order to properly fit the rate of interest, David assumes that risk aversion is smaller than 1.

[13] There exist basically three ways to sustain heterogeneity of beliefs between agents. Differences in the basic model agents believe in, or in some fixed model parameter as proposed in Harris and Raviv (1993), Kandel and Pearson (1995), and Cecchetti, Lam, and Mark (2000) and used more recently by David (2004); under this approach, agents are *non-Bayesian*. Another modeling possibility is *differences in priors*, while agents remain Bayesian, as in Biais and Bossaerts (1998), Detemple and Murthy (1994), Basak (2004), Duffie, Garleanu, and Pedersen (2002), and Buraschi and Jiltsov (2002). A third, more sophisticated possibility that, however, includes noise traders, is to let agents receive *private signals* as in the vast "Noisy-Rational Expectations" literature originating from the work of Grossman and Stiglitz (1980), Hellwig (1980), and Wang (1993). In the case of private signals, agents also learn from price, a channel that is not present here.

[14] In Scheinkman and Xiong (2003), the dividend is a stochastic flow dD. That, however, cannot be an equilibrium formulation in a pure-exchange economy, in which total dividend must be equal to total consumption $c\,dt$.

rate of dividends, f_t, is also stochastic:

$$df_t = -\zeta\left(f_t - \overline{f}\right)dt + \sigma_f dZ_t^f; \quad \zeta > 0, \tag{2}$$

where Z^f is also a Wiener under the effective probability measure, σ_f is the volatility of the change in f_t, $\overline{f}$ is the long-run mean, and ζ is the parameter driving the reversion of f_t to the long run mean.

2.2 Information structure and filtering

The conditional expected growth rate of dividends, f, is not observed by any agent. Both classes of investors must estimate, or filter out, the current value of f and its future behavior. They do that from the observation of the current dividend and the observation of a public signal, s, which has the following process:

$$ds_t = f_t dt + \sigma_s dZ_t^s, \tag{3}$$

where Z^s is a Wiener under the effective probability measure as well, σ_s is the volatility of changes in s_t. All three Wieners, $\{Z^\delta, Z^f, Z^s\}$, are uncorrelated with each other (under the effective probability measure and any measure equivalent to it) so that, instantaneously, innovations in the signal, dZ^s, convey no information about innovations dZ^f in the unobserved variable. Everyone, however, knows that the drift of s at time t is equal to f_t, the drift of the dividend process. So, the signal provides some long-run information about the drift of the dividend process, as does the dividend itself. That is the only true reason for which the signal is informative.

Group A investors perform their filtering under the delusion that the signal s has correlation $\phi \in]0,1[$ with f when, in fact, it is has zero correlation.[15] The "model" Group A investors have in mind is:

$$ds_t = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s\sqrt{1-\phi^2}dZ_t^s, \tag{4}$$

while Group B is rational (and so knows or learns that $\phi = 0$). Now, there are *two informative roles* played by the signal s: from the point of view of all people, the signal provides some information about the drift of the dividend process. But because of the assumed non-zero correlation in the eyes of the irrational investors, it also provides them with short-run, incorrect information about the current shock to the dividend growth rate. We can amplify or turn down the second role relative to the first one by varying the parameter σ_s.

From filtering theory (see Lipster and Shiryaev (2001, Theorem 12.7, page 36)), the conditional expected values, $\widehat{f}^A$ and $\widehat{f}^B$, of f according to individuals of Group A (deluded; $\phi \neq 0$) and Group B (rational; $\phi = 0$) are respectively:[16,17]

$$\begin{aligned} d\widehat{f}_t^A &= -\zeta\left(\widehat{f}_t^A - \overline{f}\right)dt + \frac{\gamma^A}{\sigma_\delta^2}\left(\frac{d\delta}{\delta} - \widehat{f}_t^A dt\right) + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\left(ds - \widehat{f}_t^A dt\right), && (5)\\ d\widehat{f}_t^B &= -\zeta\left(\widehat{f}_t^B - \overline{f}\right)dt + \frac{\gamma^B}{\sigma_\delta^2}\left(\frac{d\delta}{\delta} - \widehat{f}_t^B dt\right) + \frac{\gamma^B}{\sigma_s^2}\left(ds - \widehat{f}_t^B dt\right), && (6)\end{aligned}$$

[15] This is not just a prior, or it is an infinitely precise prior. They refuse to learn the true correlation.

[16] Observe once again that output δ serves as a signal, which causes an update of the rate of growth of output, just as the signal s does.

[17] Under the effective probability measure, under which Z^s and Z^δ are Brownian motions, the stochastic differential equations for δ and s are given by Equations (1) and (4). These could be substituted into (5) and (6) to get a complete Markovian description of the process for $\left\{\delta, f, s, \widehat{f}^A, \widehat{f}^B\right\}$.

The number γ^A (γ^B) is the steady-state variance of f as estimated by Group A (B).[18] These variances would normally be deterministic functions of time. But for simplicity we assume, as did Scheinkman and Xiong (2003), that there has been a sufficiently long period of learning for people of both groups to converge to their level of variance, irrespective of their prior, while, at the same time, A types have refused to use the same information to infer the correlation number, which is the exact degree to which they are being irrational. For the purpose of obtaining a martingale, "static" formulation (as done in Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), we now rewrite these stochastic differential equations in terms of processes that are Brownian motions under subjective probability measures. Consider a two-dimensional process $W^B = \left(W_\delta^B, W_s^B\right)$ that is Brownian under the probability measure that reflects the expectations of Group B. By the definition of $\widehat{f}^B$, we can then write:

$$\frac{d\delta_t}{\delta_t} = \widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^B, \tag{7}$$

$$ds_t = \widehat{f}_t^B dt + \sigma_s dW_{s,t}^B. \tag{8}$$

A similar two-dimensional process $W^A = \left(W_\delta^A, W_s^A\right)$ that is Brownian under A's probability measure could be defined and a similar substitution could be made to represent A's expectations. The relation between them is:

$$dW_{\delta,t}^B = dW_{\delta,t}^A - \frac{\widehat{f}_t^B - \widehat{f}_t^A}{\sigma_\delta} dt, \tag{9}$$

$$dW_{s,t}^B = dW_{s,t}^A - \frac{\widehat{f}_t^B - \widehat{f}_t^A}{\sigma_s} dt. \tag{10}$$

Because the effective measure is not defined on either agent's σ-algebra, we can ignore it for the purpose of calculating the equilibrium. Instead, we use B's probability measure as the reference measure. From Equations (9) and (10), we can determine that the change from B's measure to A's measure is given by:

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \|\nu\|^2 dt - \int_0^t \nu_t^\intercal dW_t^B\right), \tag{11}$$

or

$$\frac{d\eta_t}{\eta_t} = -\nu_t^\intercal dW_t^B, \tag{12}$$

where

[18] The steady-state variances of f as estimated by Group A and Group B are, respectively:

$$\gamma^A \triangleq \frac{\sqrt{\left(\zeta + \frac{\phi\sigma_f}{\sigma_s}\right)^2 + (1-\phi^2)\,\sigma_f^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \left(\zeta + \frac{\phi\sigma_f}{\sigma_s}\right)}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}},$$

$$\gamma^B \triangleq \frac{\sqrt{\zeta^2 + \sigma_f^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \zeta}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}}.$$

As has been pointed out by Scheinkman and Xiong (2003), γ^A decreases as ϕ rises, which is the reason that Group B is called overconfident. γ^A starts at the value γ^B when $\phi = 0$ and would reach $\gamma^A = 0$ when $\phi \to 1$. The signal can lead Group A ultimately to complete (and foolish) unconditional certainty. The numerator of the diffusion of $\widehat{f}^A$ with respect to s, $\phi\sigma_s\sigma_f + \gamma^A$, however, also starts from γ^B at $\phi = 0$ but then would rise to the value $\sigma_s\sigma_f > \gamma^B$ as $\phi \to 1$. Thus, the signal *increases* the conditional uncertainty that Group A faces because of their own learning, compared to that faced by B.

$$\nu_t = \left(\widehat{f}_t^B - \widehat{f}_t^A\right)\begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \tag{13}$$

This is a simple application of Girsanov's theorem. It tells us how current disagreement $\widehat{f}^B - \widehat{f}^A$ gets reflected into probability beliefs about future events.

Substituting (9) and (10) into (5) and (6) gives:[19]

$$\begin{aligned} d\widehat{f}_t^A &= \left[-\zeta\left(\widehat{f}^A - \overline{f}\right) + \left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\left(\widehat{f}_t^B - \widehat{f}^A\right)\right]dt & (14) \\ &\quad + \frac{\gamma^A}{\sigma_\delta^2}\sigma_\delta dW_{\delta,t}^B + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\sigma_s dW_{s,t}^B \\ d\widehat{f}_t^B &= -\zeta\left(\widehat{f}^B - \overline{f}\right)dt + \frac{\gamma^B}{\sigma_\delta}dW_{\delta,t}^B + \frac{\gamma^B}{\sigma_s}dW_{s,t}^B. & (15) \end{aligned}$$

The Markovian system made of (7), (8), (14) and (15) completely characterizes the evolution of the economy in the eyes of population B. The viewpoint of population A will be handled by means of the change of measure η. For later reference, we also write the process for the difference of opinion $\widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A$:

$$d\widehat{g}_t = -\left(\zeta + \frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\widehat{g}_t dt + \frac{\gamma^B - \gamma^A}{\sigma_\delta}dW_{\delta,t}^B + \frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s}dW_{s,t}^B. \tag{16}$$

When $\widehat{g} > 0$, Group B of investors is comparatively optimistic or Group A comparatively pessimistic. Also, $\widehat{g}$ (or its absolute value) can be viewed as a measure of the dispersion of beliefs or opinions. Because $\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right) < 0$, a positive realization of the signal increment $dW_{s,t}^B$ causes Population A to become more optimistic relative to where it was before.

The joint dynamics of the four state variables $\left\{\delta, \eta, \widehat{f}^B, \widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A\right\}$ are provided by Equations (7), (12), (15) and (16). They are driven by only two Brownians, W_δ^B and W_s^B. This is because variable f is unobserved by anyone and is only a latent variable. It is important to keep in mind that the four variables are not independent of each other. Since there are only two Brownians, the diffusion matrix of $\left\{\delta, \eta, \widehat{f}^B, \widehat{g}\right\}$ is a 4×2 matrix:

$$\begin{bmatrix} \delta\sigma_\delta > 0 & 0 \\ -\eta\frac{\widehat{g}}{\sigma_\delta} & -\eta\frac{\widehat{g}}{\sigma_s} \\ \frac{\gamma^B}{\sigma_\delta} > 0 & \frac{\gamma^B}{\sigma_s} > 0 \\ \frac{\gamma^B - \gamma^A}{\sigma_\delta} > 0 & \frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s} < 0 \end{bmatrix}. \tag{17}$$

From the first and third row of the diffusion matrix, we see that δ and $\widehat{f}^B$ are always positively correlated with each other.[20] The second row of the diffusion matrix shows that the diffusion vector of η has the

[19] The conditional variance of $\widehat{f}^A$ is equal to:

$$\left[\phi\sigma_s\sigma_f + \frac{\gamma^A}{\sigma_s}\right]^2 + \left[\frac{\gamma^A}{\sigma_\delta}\right]^2 = -2\zeta\gamma^A + \sigma_f^2,$$

which is a monotonically increasing function of ϕ, rising from : $-2\zeta\gamma^B + \sigma_f^2$ at $\phi = 0$ to σ_f^2 at $\phi = 1$. The irrational Group A changes its mind in a more volatile way than does Group B.

[20] $\widehat{g}$ is positively correlated with δ and $\widehat{f}^B$ if $\left|\frac{\gamma^B - \gamma^A}{\sigma_\delta}\right| > \left|\frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s}\right|$.

sign of $\widehat{g}$. The covariance of variables δ and η will play a central role in what follows. From the first and the second rows of the diffusion matrix, we see that, instantaneously, this covariance is equal to $-\delta\eta\widehat{g}$. These two variables covary positively when $\widehat{g} < 0$ and negatively in the opposite case.[21]

When Group B is currently comparatively pessimistic ($\widehat{f}^B - \widehat{f}^A < 0$), Group A views positive innovations in δ as more probable than Group B does, which is coded as positive innovations in the change of measure η for those states of nature in which δ has positive innovations.

In the special case of pure Bayesian learning, in which everyone is rational ($\phi = 0$) and differences in beliefs can arise only from differences in priors, $\widehat{g}$ has zero diffusion and reverts to zero deterministically (see Equation (16)). Even in that case, as long as $\widehat{g}$ has not reached the value 0, η fluctuates randomly as public signals (δ, s) are realized.

Of the four state variables, two will have a *direct effect* on the economy. They are δ and η. We propose to call δ "the fundamental" and η "sentiment". The fundamental moves on its own but sentiment is correlated with the fundamental because realizations of the dividend provide information. It will soon become apparent that changes in η are equivalent to changes in the relative weights, or consumption shares, of the two subpopulations. The other two variables, $\widehat{f}^B$ and $\widehat{g}$, have an *indirect effect* in that they act only on the first two and serve to keep the system in the Markovian form: $\widehat{f}^B$ is the current estimate of the drift of δ and $\widehat{g}$ determines the diffusion of η. $\widehat{g}$ will be called "disagreement" and later "dispersion of beliefs".

To summarize:

Proposition 1 *There are two distinct effects of imperfect learning:*

- *Effect #1: $\widehat{g}$ has nonzero diffusion. Even if the two groups of investors happened to agree today ($\widehat{g} = 0$), all investors still know that they will revise their future estimates of the growth rate. In particular, the rational investors are conscious of the fact that irrational investors will revise their estimate in a manner that differs from theirs, so that they know that they will not agree tomorrow. This effect is instantaneous. It acts indirectly on the economy.*
- *Effect #2 is cumulative and direct: $\widehat{g}$ conditions the diffusion of η. The rational group of investors may not agree today with the irrational ones about its estimate of the current rate of growth of dividend: $\widehat{g} \neq 0$. This effect is cumulative: $\widehat{g}$ is stochastic and conditions the diffusion of η, which implies that η has a diffusion that can take large positive or negative values. We can say that* disagreement drives sentiment*. This effects acts directly on the economy.*

3 Individual optimization and equilibrium

In this section, we first describe the optimization problem faced by each investor and then, assuming complete financial markets, the equilibrium in this economy, which includes a characterization of the instantaneously riskless interest rate and the market price of risk. We conclude this section by explaining how the complete-markets equilibrium can be implemented via dynamic trading in long-lived securities.

[21] We shall have occasion to verify that the same remains true at every maturity.

3.1 Preferences of agents and their optimization problems

In this paper, we are interested in the interaction between two groups one of which is rational and the other one not. Differences in risk aversion and differences in the rate of impatience are not our main focus. So, we restrict our analysis to a situation in which both groups have power utility with the same risk aversion, $1-\alpha$, and rate of impatience, ρ.

Assuming a complete financial market,[22] the problem of population B is to maximize the expected utility from lifetime consumption:

$$\sup_{c} \mathbb{E}^B \int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left(c_t^B\right)^\alpha dt; \alpha < 1 \tag{18}$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^B dt = \overline{\theta}^B \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt, \tag{19}$$

where ξ^B is the change of measure from agents B's probability measure to the risk neutralized measure and $\overline{\theta}^B$ is the share of equity with which B is initially endowed. The first-order condition for consumption equates marginal utility to $\lambda^B \xi_t^B$, where λ^B is the Lagrange multiplier of the budget constraint (19):

$$e^{-\rho t} \left(c_t^B\right)^{\alpha-1} = \lambda^B \xi_t^B. \tag{20}$$

Group A is assumed to have the same utility function (with risk aversion $1-\alpha$ and rate of time preference ρ) and an initial share $\overline{\theta}^A = 1 - \overline{\theta}^B$ of the equity, and thus, an analogous optimization problem. The only difference is that Population A uses a probability measure that is different from that of Population B. The problem of Group A is to maximize the expected utility from lifetime consumption:

$$\sup_{c} \mathbb{E}^A \int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left(c_t^A\right)^\alpha dt, \tag{21}$$

subject to the static budget constraint:

$$\mathbb{E}^A \int_0^\infty \xi_t^A c_t^A dt = \overline{\theta}^A \mathbb{E}^A \int_0^\infty \xi_t^A \delta_t dt, \tag{22}$$

where ξ^A is the change of measure from agents A's probability measure to the risk neutralized measure.[23]

Using B's probability measure as the reference measure, the problem of A can be restated as:

$$\sup_{c} \mathbb{E}^B \int_0^\infty \eta_t \times e^{-\rho t} \frac{1}{\alpha} \left(c_t^A\right)^\alpha dt, \tag{23}$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^A dt = \overline{\theta}^A \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt. \tag{24}$$

[22] Details on the menu of securities are given in Section 3.4 below. David (2004) says that the fluctuating difference of measure η between the two groups makes the market "effectively incomplete". That is a matter of semantics. Analytically, the equilibrium can be obtained by complete-market methods. It would probably be more descriptive of the analytical structure that is reflected in Equation (23) below, to say that the fluctuating η causes the utility function of agents A to become "effectively state dependent" (i.e. non von Neuman-Morgentsern) relative to the probability measure of Group B.

[23] ξ^A is the density that makes prices martingales under A's probability measure. ξ^B is the density that makes prices martingales under B's probability measure. For any event E : $\mathbb{E}^A \left[\xi^A \mathbf{1}_E\right] = \mathbb{E}^B \left[\eta \xi^A \mathbf{1}_E\right] = \mathbb{E}^B \left[\xi^B \mathbf{1}_E\right]$ which implies: $\xi^B = \eta \xi^A$. The martingale pricing density is defined relative to each agent's probability measure. But the risk neutral measure is the same in the end.

The first-order condition for consumption in this case is

$$\eta_t \times e^{-\rho t} \left(c_t^A\right)^{\alpha-1} = \lambda^A \xi_t^B, \tag{25}$$

where λ^A is the Lagrange multiplier of the budget constraint (24).

3.2 Complete-market equilibrium

An equilibrium is a price system and a pair of consumption-portfolio processes such that: (i) investors choose their optimal consumption-portfolio strategies, given their perceived price processes; (ii) the perceived security price processes are consistent across investors; and (iii) commodity and securities markets clear.

The aggregate resource constraint (clearing of the commodity market), from Equations (20) and (25), is:

$$\left(\frac{\lambda^A \xi_t^B e^{\rho t}}{\eta_t}\right)^{\frac{1}{\alpha-1}} + \left(\lambda^B \xi_t^B e^{\rho t}\right)^{\frac{1}{\alpha-1}} = \delta_t. \tag{26}$$

Solving this equation:

$$\xi_t^B e^{\rho t} = \left[\frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta_t}\right]^{1-\alpha}, \tag{27}$$

and, therefore:

$$c_t^A = \delta_t \times \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}, \tag{28}$$

$$c_t^B = \delta_t \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}. \tag{29}$$

The consumption-sharing rule is linear in δ because both groups have the same risk aversion. But the slope of the linear relation (the share of consumption allocated to each group) is stochastic and driven by η because of the improper use of the signal by Group A.

The equilibrium value of ξ^B – the martingale pricing density under B's probability – depends on η, the probability density of A relative to B. In addition to reflecting the abundance or scarcity of goods, as is usual in the absence of state preference or heterogeneous beliefs, the state prices also incorporates an harmonic average of the probability beliefs of the two populations. As η fluctuates, average probability belief or "sentiment" fluctuates with it. In writing his/her budget constraint based on ξ^B, B anticipates A's beliefs. This reflects "higher-order expectations."

We highlight the fact that:

Proposition 2 *The second derivative of the function $\xi^B(\delta,\eta)$ given by Equation (27) with respect to η has the same sign as α.*[24] *When $\alpha < 0$ (risk aversion greater than 1), fluctuations in the expectations of Group A reduce the average values of all the stochastic discount factors written with respect to B's measure. The cross derivative of the function $\xi^B(\delta,\eta)$ is unambiguously negative, which will depress state prices in those states of nature in which the two variables δ and η are positively correlated (which is when $\widehat{g} < 0$ and Group B is relatively pessimistic).*

[24]The first logarithmic derivative $\frac{\eta}{\xi^B} \times \frac{\partial \xi^B}{\partial \eta}(\delta,\eta)$ is equal to the consumption share of Group A, $\frac{c^A}{\delta}$.

Had we, instead, used A's measure as reference measure, we would also have found that fluctuations in the expectations of Group B reduces the values of all the stochastic discount factors. The risk created by the fluctuations in the expectations of others (and, in fact, one's own as well) depresses financial prices. The effect is reciprocal.

Given the constant multipliers λ^A and λ^B, and given the exogenous process for δ, and η, we have now characterized the complete-market equilibrium. It would only remain to relate the Lagrange multipliers λ^A and λ^B to the initial endowments. This requires the calculation of the wealth of each group, as will be done in Equation (36).

Since Equations (27), (28), and (29) give the pricing measure and each group's consumption as a function of the current value of the dividend, δ_t and the current value of the change of measure between the two groups, η_t, we need to carry along four state variables in the Markovian recursive formulation: $\left\{\delta, \eta, \widehat{f}^B, \widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A\right\}$.[25]

3.3 Rate of interest and prices of risk

The rate of interest and the price of risk in this equilibrium are implied in the value (27) of the pricing measure. Let r denote the rate of interest on an instantaneous maturity deposit and the vector κ^i as the market prices of risk in the eyes of Group $i = \{A, B\}$. Then, as shown by Cox and Huang (1989), r and κ^i are given by the drift and the diffusion, respectively, of the risk-neutralized measure for Population i, ξ_t^i:

$$\xi_t^i = \delta_0^{\alpha-1} \exp\left(-\int_0^t r dt - \frac{1}{2}\int_0^t \left\|\kappa^i\right\|^2 dt - \int_0^t \left(\kappa^i\right)^{\intercal} dW^i\right). \tag{30}$$

The interest rate and the market prices of risk can be obtained by applying Itô's lemma to (27) and are given in the next proposition.

Proposition 3 *In equilibrium, the instantaneous interest rate is*

$$\begin{aligned} r\left(\eta, \widehat{f}^B, \widehat{g}\right) &= \rho + (1-\alpha)\widehat{f}^B - \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 - (1-\alpha)\widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \\ &\quad -\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right)\widehat{g}^2 \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}, \end{aligned} \tag{31}$$

and the market prices of risk in the eyes of Population B and A are:[26]

$$\kappa^A(\eta, \widehat{g}) = \begin{bmatrix} (1-\alpha)\sigma_\delta \\ 0 \end{bmatrix} - \widehat{g} \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}\begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}, \tag{32}$$

$$\kappa^B(\eta, \widehat{g}) = \begin{bmatrix} (1-\alpha)\sigma_\delta \\ 0 \end{bmatrix} + \widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}\begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \tag{33}$$

[25] Notice that the current signals s^A and s^B are not state variables. This is because the instantaneous information they provide about the growth rate of dividends is negligible next to the cumulative information already coded into $\widehat{f}^A$ and $\widehat{f}^B$.

[26] The risk-neutral measures for Populations A and B differ only in the market prices of risk. That is, the instantaneously riskless interest rate perceived by all agents is the same, and so the difference in the risk neutral measures is purely a difference in the market prices of risk perceived by the two populations.

The rate of interest, from Equation (31) is an increasing function of Population B's expected rate of growth of the dividend, $\widehat{f}^B$. In fact, expectation of future growth is impounded only in the rate of interest and not in the prices of risk.

From Equation (31), we also see that the rate of interest is influenced in a nonmonotonic and asymmetric way by the difference in beliefs $\widehat{g}$, as it is in David (2004). There are two ways to understand this phenomenon. One is to say, at the intuitive level, that this happens because $\widehat{g}$ contributes both to the averages of $\widehat{f}^A$ and $\widehat{f}^B$, and also to the difference between them. The second form of explanation is more analytical. It is based on the observation that, in Equation (31), the fourth term arises from the cross derivative of ξ^B with respect to η and δ, which is always of the same sign, multiplied by the covariance between these two variables, which changes sign with $\widehat{g}$. When $\widehat{g} > 0$ (Group A is comparatively pessimistic), the rate of interest is lower. Group A views positive innovations in δ as less probable than Group B does, which is coded as negative innovations in the change of measure η for those states of nature in which δ has positive innovations. Group A views this configuration as a reason to invest in the riskless asset because, for them, that is equivalent subjectively to positive innovations in marginal utility in those states of nature in which δ has positive innovations.

From Equation (31) for the interest rate, we also see that the fifth term containing $\widehat{g}^2$ arises from the second derivative of ξ^B with respect to η, which, as we saw, is negative whenever risk aversion is greater than 1 ($\alpha < 0$). When that is true, disagreement increases the equilibrium rate of interest because it depresses all the stochastic discount factors.

We now study the expressions for the market price of risk in Equations (32) and (33). Under agreement ($\widehat{g} = 0$), the prices of risk κ^i include a reward for output risk W_δ, and this reward is $(1-\alpha)\sigma_\delta$, but zero reward for signal risk W_s. As soon as there is disagreement ($\widehat{g} \neq 0$), both populations of investors realize that "sentiment", i.e., the probability measure of the other population, will fluctuate randomly. Hence, they start charging a premium for the risk arising from the vagaries of others.

It is noteworthy that neither the rate of interest nor the prices of risk depend directly on the parameter ϕ measuring irrationality. They depend on it indirectly via the current value of the probability difference, η, and the current value of the difference of opinion, $\widehat{g}$.

3.4 Securities-market implementation of the complete-market equilibrium

There are three Brownians in the economy, that is, the Brownians driving δ, f, and s. However, since the growth rate f is not observed, only two of the three variables that they drive, $\{\delta, f, s\}$, are observable and can be used to define "states of nature" or as a basis for writing the terms of a security's contract. Correspondingly, there are only two Wieners of consequence: W_δ^B and W_s^B. Therefore, three linearly independent securities are required to implement the equilibrium.

The choice of menu is largely arbitrary. Let there be a riskless, instantaneous bank deposit with a rate of interest r. "Equity" or total wealth pays the aggregate dividend δ. We introduce also a consol bond with infinite maturity paying a coupon $1 \times dt$ in each time period of length dt. That makes three securities, two of which are instantaneously risky.

Consider then the price F of equity whose flow payoff at time t is δ_t. The price of this security is also the total financial wealth of the economy. Its equilibrium price, using as reference measure the measure of Group B, can be obtained directly from the pricing measure (27):

$$\begin{aligned} F\left(\delta,\eta,\widehat{f}^B,\widehat{g},t\right) &\triangleq \frac{1}{\xi_t^B}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty \xi_u^B\delta_u du \\ &= \delta^{1-\alpha}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty e^{-\rho(u-t)}\left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}\right]^{1-\alpha}\delta_u^\alpha du. \end{aligned} \tag{34}$$

Similarly, the price of the consol bond is:

$$\begin{aligned} P\left(\eta,\widehat{f}^B,\widehat{g},t\right) &\triangleq \frac{1}{\xi_t^B}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty \xi_u^B du \\ &= \delta^{1-\alpha}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty e^{-\rho(u-t)}\left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}\right]^{1-\alpha}\delta_u^{\alpha-1} du. \end{aligned} \tag{35}$$

Using the same approach, we can compute the wealth of Group B investors as being the price of a "security" whose flow payoff at time t is their consumption. From Equation (29), B's wealth is:

$$\begin{aligned} F^B\left(\delta,\eta,\widehat{f}^B,\widehat{g},t\right) &\triangleq \frac{1}{\xi_t^B}\mathbb{E}^B_{\delta,\eta,\hat{f}^B,\widehat{g}}\int_t^\infty \xi_u^B c_u^B du \\ &= \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left[\frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta}\right]^{1-\alpha}}\mathbb{E}^B_{\delta,\eta,\hat{f}^B,\widehat{g}}\int_t^\infty e^{-\rho(u-t)}\left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta_u}\right]^{-\alpha} du. \end{aligned} \tag{36}$$

To compute the expected values in (34), (35) and (36), we need the joint conditional distribution of η_u and δ_u, given $\delta_t,\eta_t,\widehat{f}_t^B,\widehat{g}_t$ at t. That joint distribution is not easy to obtain but its characteristic function, $\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\left[(\delta_u)^\varepsilon(\eta_u)^\chi\right]$, where $\varepsilon,\chi\in\mathbb{C}$ can be obtained in closed form. Guided by the functional form of the coefficients of the associated partial differential equation and following Heston (1993) and Kim and Omberg (1996) and undoubtedly others, we show in Appendix B that:

$$H\left(\delta,\eta,\widehat{f}^B,\widehat{g},t;u,\varepsilon,\chi\right)\triangleq\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\left[(\delta_u)^\varepsilon(\eta_u)^\chi\right]=\delta^\varepsilon\eta^\chi\times H_f\left(\widehat{f}^B,t,u;\varepsilon\right)\times H_g\left(\widehat{g},t,u;\varepsilon,\chi\right), \tag{37}$$

where functions $H_f\left(\widehat{f}^B,t,u;\varepsilon\right)\triangleq\mathbb{E}^B_{\widehat{f}^B}\left[(\delta_u)^\varepsilon\right]$ and H_g are defined explicitly by (B2) and (B5).[27] Then, Appendix C provides two alternative methods to write the functions F, P and F^B explicitly as well as "growth conditions" sufficient to guarantee that the time integrals in (34) and (35) converge.[28]

4 Effect of irrationality on asset prices and return volatilities

In this section, we study the effect of disagreement and irrationality on asset prices and their volatilities. These are easily obtained by straightforward applications of Itô's lemma to the explicit expressions for

[27] Appendix B also contains technical conditions for the function H_g to be well defined.

[28] As far as pricing is concerned, the limiting case in which $\theta^A\to1$ and $\theta^B\to0$ or vice versa (so that one population dominates the market), exhibits some similarities with the model of a homogeneous-agent economy in Brennan and Xia (2001). The closeness of the Brennan and Xia model to these limiting cases means that their model serves as a useful pricing benchmark.

the value of the stock market F and the value of bonds P and their derivatives. Generally, the diffusion vector of a security price is equal to the gradient of the price function premultiplying the diffusion matrix of state variables (Equation (17)).

In order to illustrate the effect of irrationality on securities prices, we specify numerical values for the parameters of the model. Even though our objective is not to match the magnitude of particular moments in the data, we would like to work with parameter values that are reasonable. The parameter values that we specify are based on the estimation of models similar to ours undertaken in Brennan and Xia (2001). In addition, we have set the volatility of the signal σ_s equal to 0.1. The particular values chosen for the parameters are listed in Table 1.[29]

The dynamics of equilibrium prices are evidently driven by the state variables $\delta, \eta, \widehat{f}^B, \widehat{g}, t$. Among them, δ is only a scale variable multiplying the total value of the stock market and the wealth of each population and not affecting at all the price of a bond. The state variable η always appears in the ratio $\frac{\lambda^B \eta}{\lambda^A}$, incorporating Lagrange multipliers and the current probability measure difference between the two populations. It captures the relative Negishi weights of the two populations, with $\frac{\lambda^B}{\lambda^A}$ representing the initial (time-0) weights and the initial distribution of wealth and η representing the changes in the weights that have occurred as a result of the gains and losses accumulated by the irrational group because of its learning mistakes. When we do not vary that ratio, we set it equal to 1 to represent the situation in which the two groups currently have roughly similar sizes.[30] But we also vary the relative weights of the two populations because we wish to know how many rational investors are needed for the market to behave almost as it would under full rationality.

We present our results from the analysis of the model in plots. Each plot in the figures has two curves on it, with the dotted line representing the case where $\phi = 0$ and all agents are rational, and the dashed line representing the case where Population A is irrational, which corresponds to $\phi = 0.95$.

4.1 Average belief vs. dispersion of beliefs

In addition to δ and η, the two other state variables are the beliefs of Population B, $\widehat{f}^B$, and the difference in beliefs, $\widehat{g}$. However, as we have seen, $\widehat{g}$ contributes both to the average of $\widehat{f}^A$ and $\widehat{f}^B$ and also to the difference between them. For purposes of interpretation and exposition, it is clearer to define $\widehat{f}^M$, the population average belief about the expected rate of growth (where the weights are each population's share of consumption):

$$\widehat{f}^M \triangleq \frac{\widehat{f}^B \times \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \widehat{f}^A \times \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} = \widehat{f}^B - \widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}. \tag{38}$$

[29] Observe that the range of parameter values that can be considered is restricted by the need to satisfy the growth conditions for the prices of equity and bonds to be well defined (Equations (C15) and (C17)). This limits, in particular, the range of values for the discount rate or for risk aversion that can be considered. Because of this constraint, the risk aversion we consider is somewhat too low by itself to account for the equity premium. The presence of irrational traders, however, will suffice to bring the equity premium up to realistic levels.

[30] If we vary the parameter ϕ, we adjust this ratio in such a way that the time-0 lifetime budget constraints of the two populations still hold, with unchanged time-0 endowments of securities.

The rate of interest can then be written:

$$
\begin{aligned}
r\left(\eta, \widehat{f}^B, \widehat{g}\right) &= \rho + (1-\alpha)\widehat{f}^M - \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 \\
&\quad -\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\widehat{g}^2 \times \frac{\left(\frac{1}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2},
\end{aligned} \tag{39}
$$

and the effect of $\widehat{g}$, which appears in the last term is now purely quadratic and symmetric. In this way, $\widehat{g}$ represents the effect of pure *dispersion* of beliefs in the population. For reasons we have already explained à propos Equation (31), if risk aversion is greater than 1, $\alpha < 0$, dispersion of beliefs increases the rate of interest above what it would be under homogeneous beliefs.[31] From the analytical point of view, once $\widehat{f}^M$ has been introduced as a variable, the second derivative of ξ^B with respect to η is the sole cause of the influence of $\widehat{g}$, which drives the variance of η. The effect of the cross derivative of ξ^B with respect to η and δ and of the covariance between these two variables has now been absorbed into $\widehat{f}^M$.

It is also conceivable to recognize average beliefs, $\widehat{f}^M$, and dispersion of beliefs, $\widehat{g}$, as two drivers for the prices of other securities. It is unfortunately not possible to define a concept of "average belief" in a manner that would be valid for all assets, specifically for assets of all maturities. The way in which beliefs compound over time and get discounted into prices via marginal utility, when η is stochastic (and generally correlated with δ), would imply a different concept of average beliefs for different maturities.[32] The average belief $\widehat{f}^M$ that we have defined in (38) applies only to the rate of interest, which is an instantaneous-maturity asset. Nonetheless, we use that concept below as a convenient, albeit only an approximate, interpretation device.[33]

For these reasons, we wish to introduce a change of state variables from $\left\{\delta, \eta, \widehat{f}^B, \widehat{g}\right\}$ to $\left\{\delta, \eta, \widehat{f}^M, \widehat{g}\right\}$ and new pricing functions for the price of equity, the price of the consol bond, and the wealth of B:

$$\widetilde{F}\left(\delta, \eta, \widehat{f}^M, \widehat{g}, t\right) \triangleq F\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right), \tag{40}$$

$$\widetilde{P}\left(\delta, \eta, \widehat{f}^M, \widehat{g}, t\right) \triangleq P\left(\eta, \widehat{f}^B, \widehat{g}, t\right), \tag{41}$$

$$\widetilde{F}^B\left(\delta, \eta, \widehat{f}^M, \widehat{g}, t\right) \triangleq F^B\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right). \tag{42}$$

In the simpler case in which α is an integer (which can obtain only when $\alpha \leq 0$; i.e., risk aversion is greater than 1), the solutions (C4, C5, C6) given in the appendix but rewritten in terms of the new

[31] David (2004) assumed a risk aversion lower than 1, precisely in order to bring down the rate of interest. See our numerical illustrations below. Had we assumed a lifetime utility of the recursive, Epstein and Zin (1989) type, we could have distinguished risk aversion from elasticity of intertemporal substitution. It is likely that the condition for the rate of interest to be reduced (increased) by dispersion of beliefs would have hinged on the elasticity of substitution being higher (lower) than 1, not on the level of risk aversion.

[32] This is related to the observation made by Allen, Morris, and Shin (2004) that, under risk aversion, market-average beliefs do not satisfy the law of iterated expectations. Note that $\widehat{f}^M$ is the drift of δ under an average probability belief/measure M, defined by the change of measure (from B's measure to M):

$$\frac{\left[\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}{\left[\left(\frac{1}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}.$$

Notice that this process does not have zero drift. It generates probability densities that do not sum to 1 and do not satisfy the law of iterated expectations.

[33] Keeping $\widehat{f}^M$ fixed is not sufficient to keep the average measure M fixed.

variables can be interpreted as follows:

$$\widetilde{F}\left(\delta,\eta,\widehat{f}^M,\widehat{g},t\right)=\delta\frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}\int_t^\infty e^{-\rho(u-t)}\times\left\{\sum_{j=0}^{1-\alpha}C_{1-\alpha}^j\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}}\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\delta_u}{\delta}\right)^\alpha\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right]\right\}du, \tag{43}$$

$$\widetilde{P}\left(\eta,\widehat{f}^M,\widehat{g},t\right)\triangleq\frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}\int_t^\infty e^{-\rho(u-t)}\times\left\{\sum_{j=0}^{1-\alpha}C_{1-\alpha}^j\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}}\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\delta_u}{\delta}\right)^{\alpha-1}\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right]\right\}du, \tag{44}$$

$$\widetilde{F}^B\left(\delta,\eta,\widehat{f}^M,\widehat{g},t\right)=\delta\frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}\int_t^\infty e^{-\rho(u-t)}\times\left\{\sum_{j=0}^{-\alpha}C_{-\alpha}^j\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}}\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\delta_u}{\delta}\right)^\alpha\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right]\right\}du. \tag{45}$$

In the next two sections, we analyze the prices of securities, the volatility of their returns and the correlations of these returns.

4.2 Prices

We start by analyzing the effect of irrationality on the level of asset prices. Figure 1 plots the interest rate, r, the price of equity, F, and the consol bond price, P, against three of the four state variables.[34] Comparing the *levels* of dashed curves, which are for the case where Group A investors exhibit irrationality ($\phi = 0.95$), to those of the dotted curves, which are for the case where both groups of investors are rational ($\phi - 0$), illustrates that:

Proposition 4 *In all cases, price levels are reduced by the presence of irrational traders (compare levels of dashed curves to those of dotted curves) provided* $\alpha < 0$ *(risk aversion is greater than 1).*

This is because irrational traders add "noise" for which all traders require a risk premium.

Consider now the variations of the price functions. We first vary average belief $\widehat{f}^M$, holding all other variables at their benchmark values (including $\widehat{g} = -1\%$) (see the second column of Figure 1). In this economy, the rate of interest increases linearly and the price of bonds decreases with an increase in the expectation of future growth of either population (see the graph in the third row and second column of Figure 1). In other words, the yields of bonds increase with the increase in the short rate, which arises from higher expected growth. This is because higher growth of dividends implies lower marginal utility of future consumption. In the case of equity (see the graph in the second row and the second column of Figure 1), the same effect is present and, when risk aversion is greater than 1, that effect dominates the effect of increased future dividends. The ratio of the price of equity to current dividends drops, as did the price of the bond, with an increase in average belief of future growth. That is one way in which our model differs from that of Brennan and Xia (2001). When Brennan and Xia increase their investors' estimate of expected dividend growth, they keep constant their investors' estimate of expected aggregate consumption growth. The ratio of the price of equity to current dividend rises. When increasing their investors' estimate of expected aggregate consumption while keeping expected dividend growth constant, the price of equity drops. In our model, consumption equals dividend. When investors become more

[34] No plots are shown against the variable δ, which is only a scale variable for equity and has no effect on bonds.

optimistic about future dividend growth, the price-dividend ratio drops – the second effect dominates – if risk aversion is greater than 1.

Varying now the dispersion of beliefs $\widehat{g}$ (third column of Figure 1) keeping other variables at their benchmark values, we verify that the rate of interest, by design, is a quadratic function of $\widehat{g}$, symmetric around $\widehat{g} = 0$. Almost as a mirror image of the rate of interest, the prices of the bond and of equity decline as the square of $\widehat{g}$ increases. All three variations are due to the fact disagreement drives sentiment. Disagreement translates into a larger diffusion of the sentiment variable η, which is a genuine source of risk in the marketplace, as was indicated in Proposition 1. It appears that the price functions for equity and the bond are symmetric under rational, that is, pure Bayesian, learning (dotted line), where disagreement can arise only from differences in priors. However, they are no longer symmetric when Group A is irrational. As mentioned earlier, this is because our concepts of average belief and dispersion of beliefs $\widehat{g}$ is not uniformly applicable to all maturities.

Next, we vary the relative weight of the two populations, keeping $\widehat{g}$ and $\widehat{f}^M$ at their benchmark values ($\widehat{g} = -1\%$; $\widehat{f}^M = \overline{f}$). The result is shown in the first column of Figure 1, where we have not placed on the x-axis the variable $\frac{\eta\lambda^A}{\lambda^B}$ itself but $\frac{c^A}{\delta}$, the consumption share of Population A.[35] The rate of interest exhibits symmetric, concave variations around the point $\frac{c^A}{\delta} = .5$, as is apparent from Equation (39). This feature obtains irrespective of the sign of $\widehat{g}$ as long as $\widehat{g} \neq 0$.

When the share of the irrational Population A is very low, the prices of all securities are at their highest level, which is, of course, equal to the level reached when both populations are rational. For instance, for the bond ($1/\lambda^A \to 0$):

$$P\left(\eta, \widehat{f}^B, \widehat{g}, t\right) = \delta^{1-\alpha}\mathbb{E}^B_{\delta,\widehat{f}^B}\int_t^\infty e^{-\rho(u-t)}\delta_u^{\alpha-1}du.$$

When, in contrast, the share of the irrational population is very high, prices of all securities are lower than in the rational case:

$$P\left(\eta, \widehat{f}^B, \widehat{g}, t\right) = \delta^{1-\alpha}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty e^{-\rho(u-t)}\frac{\eta_u}{\eta}\delta_u^{\alpha-1}du = \delta^{1-\alpha}\mathbb{E}^A_{\delta,\widehat{f}^A}\int_t^\infty e^{-\rho(u-t)}\delta_u^{\alpha-1}du.$$

The replacement of B's measure with A's measure is the only difference between these two security price levels. It amounts to replacing $\widehat{f}^B$ with $\widehat{f}^A$, as the two populations have different beliefs about future growth, and simultaneously to replacing γ^A with $\gamma^B > \gamma^A$, as the two populations do not treat the signal the same way and Group A is overconfident about their conditional estimate of growth (see Footnote 18). The first effect is neutralized in our figure by the fact that we are keeping the average belief $\widehat{f}^M$ at a given level.[36] So, the second effect, the difference between γ^B and γ^A is the reason for the price gap between the rational solution (dotted line) and the irrational solution (dashed line) in the neighborhood of $c^A/\delta = 1$. By being overconfident conditionally, Group A creates noise, in the form of additional unconditional or long-run $\widehat{g}$ risk, which in turn increases η risk, which itself is priced in the market.

Between the two extremes, for intermediate value of the share of consumption of the two groups, prices are even lower than they are at the two extremes.[37] To understand this phenomenon, we can break

[35] Note that $\frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}} = \frac{c^A}{\delta}$. This is just a change of scale to ease the display of the graph. The relation between η and c^A/δ is monotonically increasing.

[36] When Group B is alone, $\widehat{f}^B = \widehat{f}^M$. When Group A is alone, $\widehat{f}^A = \widehat{f}^M$, the same number again.

[37] The statement is made under the assumption that $\alpha < 0$ or risk aversion is greater than 1.

up the prices of securities given by (43) and (44) into two components corresponding to the analysis of the expected values of products such as $\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\delta_u}{\delta}\right)^{\alpha}\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right]$ or $\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\delta_u}{\delta}\right)^{\alpha-1}\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right]$ as a product of expected values plus a covariance. The first component reflects the effect of the randomness of η; the second captures the covariation between η and δ. This decomposition is represented in the graphs of the first column of Figure 1 by means of the solid line: below the solid line is the effect of the randomness of η alone resulting from the irrational behavior of Group B and above the line is the effect of the covariation. Because $0 \leq \frac{j}{1-\alpha} \leq 1$, Jensen's inequality implies that $\mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\eta_u}{\eta}\right)^{\frac{j}{1-\alpha}}\right] < \mathbb{E}^B_{\delta,\eta,\widehat{f}^M,\widehat{g}}\left[\left(\frac{\eta_u}{\eta}\right)\right] = 1$ (η being a martingale). Hence, the effect of the randomness of η alone is unambiguously negative.[38] The effect of the covariation is also negative, irrespective of the sign of $\widehat{g}$ but that is much harder to prove.

We emphasize this result in the following:

Proposition 5 *Their exist weights of the irrational investor group in the market such that above those weights the prices of equity and the bond are lower than they would have been had either population been alone in the stock exchange. The reason for prices being depressed is that investors charge a premium for the randomness of the sentiment variable. They speculate on the behavior of others and require a reward for the risk taken.*

Harrison and Kreps provide an example of a speculative-behavior equilibrium in which investors are risk neutral and are subject to a no-short-sale constraint. In such a setting, when an investor views the stock as being overvalued, they cannot sell it short. They go along with the overvaluation and hold the stock positively because there is a chance that a category of investors will grow to be very optimistic in the future and will want to buy the stock. Scheinkman and Xiong (2003) is an example of a dynamic equilibrium of that type. The "option to resell" adds value to the stock over and above the value it has in the eyes of the current owners.

In our setting, investors are not risk neutral and we do not constrain them from selling short. Instead, investors limit the portfolio positions they take naturally as a consequence of being risk averse. In fact, in the setting we consider, no investor would pay for the option to sell at a future market price because they could simply sell short without cost. Moreover, in our model, both categories of investors generally hold the stock simultaneously. So, in our model, it is possible for today's valuation to be either above or below what it would be if each category of investors had to value the anticipated dividend stream based on its own expectations process. If risk aversion is less than 1, prices can only be below the private valuation of the rational investors. They are also below the private valuation of irrational investors provided the weight of the rational group is not too large. A relatively small group of irrational investors can do a lot of damage to the level of prices in the financial markets.

A similar phenomenon is described in Cao and Ou-Yang (2005). Their setting is less restricted than ours so that the conditions they find for the phenomenon to arise are more complex. Furthermore, they mostly develop conditions applicable under near risk neutrality. But the intuition in both papers

[38] It is not difficult to show that:

$$\frac{E\left[\left(1+\eta_t^{\frac{1}{1-\alpha}}\right)^{1-\alpha}\right]}{\left(1+\eta_0^{\frac{1}{1-\alpha}}\right)^{1-\alpha}},$$

where $\frac{\eta_t}{\eta_0}$ is a positive random variable, is smaller than 1 if and only if $1-\alpha<1$.

is the same.[39] Each investor faces the prospect of the other investor becoming more or less optimistic than he/she is today and incorporates that prospect in the price he/she is willing to pay today. Our conclusion is that the prospect of the other investor becoming less optimistic dominates whenever risk aversion is less than 1.[40]

4.3 Return volatility and correlation

As we saw, the *levels* of security prices are reduced when the "irrationality parameter" ϕ is increased from $\phi = 0$ to $\phi = 0.95$. However, the main effect of irrationality of Group A on the *volatility* of asset prices arises from the greatly increased volatility of state variable $\widehat{g}$ and, because of it, also from the volatility of η. The diffusion of the securities' prices is, of course, equal to the gradient of each price function premultiplying the diffusion of state variables.

Figure 2 plots the volatility of stock returns, the volatility of bond returns, and the correlation between stock and bond returns. As before, each quantity is plotted against three variables:[41] the share of aggregate endowment consumed by Group A, c^A/δ, which indicates the relative weight of the two populations, average beliefs, $\widehat{f}^M$, and dispersion of beliefs, $\widehat{g}$.

From Figure 2, we see that irrational investors create "noise" that increases the volatility of both risky assets—the stock and the bond. The volatility increases without bounds when there is disagreement ($\widehat{g} \neq 0$). The volatility of bond returns increases also under the effect of irrationality and then also without bounds. The values produced by the model for the volatility of bond returns (and interest rates) are regrettably too high to fit real-world data.[42]

The plots in the last row of Figure 2 show that the presence of overconfident investors, as well as the disagreement between investor groups, increase the correlation between stock and bond returns because, as we pointed out, the prices of the two assets move in the same direction when expectations fluctuate. The correlation increases under irrationality but agreement ($\widehat{g} = 0$) and it approaches 1 as disagreement is introduced ($\widehat{g} \neq 0$). In fact, under rationality, including the case of rationality and agreement, the correlation between stocks and bonds is negative for our choice of parameter values. The reason for this phenomenon is the impact of the dividend received. In the absence of a signal apart from the current dividend, the correlation between stocks and bonds is negative because the ex post stock return is increased by a higher-than-expected dividend whereas the ex post bond return is reduced by the prospect of increased growth it induces. If, however, a favorable public signal arrives, the enhanced growth prospects bring down both securities prices simultaneously. For sufficiently small values of σ_s, i.e., sufficiently high signal precisions, the rational case correlation becomes positive. And, in the case of irrationality, the overconfident response of irrational traders to the public signal is large enough to cause the latter effect to dominate.

To determine how the level of volatility in the market varies with the relative weight of Groups A and B, we plot in the first column of Figure 2 the volatility of the stock market return, the bond return and their correlation as a function of Population B's consumption. This shows that the excess volatility in the market increases with an increase in the relative weight of Population A. The plot also shows that

[39] It would appear that the results of Cao and Ou-Yang (2005) are mostly based on the effect of the covariance between η and δ (in our notation).

[40] Our statement is limited to the case of isoelastic utility with equal risk aversions for both groups of investors.

[41] The level of aggregate endowment (dividend) δ, which is only a scale variable for the equity price, has no impact on the moments of the rates of return.

[42] With a risk aversion smaller than 1, David (2004) was able to match the volatility of interest rates much better. Alternatively, if one wanted to match interest-rate volatility, one could introduce habit formation.

it is not enough to have just a few rational investors to get the volatility down to the level warranted by fundamentals.

5 The optimal portfolio of a rational investor B

In this section, we first study the wealth of the investor and then analyze the portfolio strategy of the rational investor.

5.1 The wealth of Group B

Figure 3 illustrates the variations of the wealth of Group B relative to three of the four state variables. As was the case for total wealth, or equity, the effect of a rise in average anticipated growth ($\widehat{f}^M$) on the wealth of Group B is to decrease it. That is illustrated in the second plot of Figure 3. To prepare for the forthcoming subsection on portfolio choice, let us note that this means that a positive random shock in $\widehat{f}^M$ is "good news" for Group B (as well as for Group A).[43] In states of nature in which average growth expectation is high, they have arranged to have less accumulated savings to finance future consumption. Equivalently, when making up its portfolio, Group B seeks to own securities with high unexpected returns in state of nature that have *negative* shocks to $\widehat{f}^M$. Bonds will have this property.

Not surprisingly given that we have defined $\widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A$, implying that $\widehat{g} > 0$ represents a situation in which Group B is more optimistic than the average about future growth, there is a directional effect on the wealth of Group B. The wealth of Group B is a decreasing function of $\widehat{g}$. A positive random shock in $\widehat{g}$ is also "good news" for Group B (but not for Group A). In states of nature in which they are optimistic about the future, they have arranged to have less accumulated savings to finance future consumption. Consequently, when making up its portfolio, Group B will seek to protect itself against future *negative* shocks to $\widehat{g}$.

As $\widehat{g}$ is reduced to sufficiently negative values, however, the wealth of B starts dropping. As we saw 'a propos total wealth, the disagreement translates into a larger diffusion of the sentiment variable η, which is a genuine source of risk in the marketplace (Effect #2 in Proposition 1). That almost symmetric, non-directional effect is superimposed on the directional effect resulting in the nonmonotonic curve we see in Figure 3.

The first plot of Figure 3 shows that varying η, and with it the relative weight of the two populations, the wealth of Group B, not surprisingly, drops monotonically with the share of consumption of Group A, c^A/δ.

[43]Denote by $J(F^B, Y)$ the value function of the investor's lifetime utility under a Merton-like dynamic-programming formulation of the investor's optimization problem, where F^B denotes the investor's wealth and Y any state variable. Then one can show, by applying the Cox-Huang transformation, that

$$F_Y^B = -\frac{J_{F^B Y}}{J_{F^B F^B}}. \tag{46}$$

From the concavity of the utility function, we know that $J_{F^B F^B} < 0$ so that F_Y^B has the same sign as $J_{F^B Y}$. Merton (1971) defines as "good news" a positive shock to a state variable Y such that $J_{F^B Y} < 0$, because that implies that a positive shock to Y induces a decrease in marginal utility and a rise in current consumption. Lower current financial wealth is associated with higher current consumption.

5.2 The portfolio composition of Population B

To solve the portfolio-choice problem of Agent B, we apply the risk-sensitivity method of Cox and Huang (1989) to the expression for the wealth of Investor B, which is given in (36).[44] All that is needed for this purpose are the price functions $\widetilde{F}$, $\widetilde{P}$, $\widetilde{F}^B$ and the 4×2 diffusion matrix of the state variables:

$$\begin{bmatrix} \theta_F^B & \theta_P^B \end{bmatrix} \times \begin{bmatrix} \text{gradient of } \widetilde{F} \\ \text{gradient of } \widetilde{P} \end{bmatrix} \times \begin{bmatrix} \text{diffusion} \\ \text{matrix of} \\ \text{four state} \\ \text{variables} \end{bmatrix} = \begin{bmatrix} \text{gradient of } \widetilde{F}^B \end{bmatrix} \times \begin{bmatrix} \text{diffusion} \\ \text{matrix of} \\ \text{four state} \\ \text{variables} \end{bmatrix}.$$

In this way, we can obtain directly the total portfolio $\left(\Theta^B\right)^\intercal = \begin{bmatrix} \theta_F^B & \theta_P^B \end{bmatrix}$ of investor B. These are the fractions of the outstanding total supply of securities held by Investor B.[45] The signs of the terms of the gradient of Population B's wealth with respect to the four state variables have been revealed to us by Figure 3.

Figure 4 gives Population B's portfolio holding (expressed as a percentage of wealth). In all the plots of this figure, the variable on the x-axis is the dispersion in beliefs, $\widehat{g}$ (the benchmark value of which is 1%). There are two columns of plots, with the one on the left giving the position in equity and the one on the right giving the position in bonds. There are three rows of plots in each column: the first row gives the overall investment in stocks and in bonds, while the other two rows decompose this total holding into two components. The second row gives the static (mean-variance or myopic) investment in stocks and in bonds;[46] and, the third row gives the investment in stocks and bonds in order to hedge intertemporally against changes in all the state variables.[47]

Let us start with the case of rationality (dotted line) of Group A and agreement ($\widehat{g} = 0$). There is then nothing to distinguish the two populations. The top row of Figure 4 shows that both groups are 100% invested in equity and 0% in bonds.

Continuing with the case of rationality but introducing some current disagreement ($\widehat{g} \neq 0$), we see that Population B continues to be 100% invested in equity irrespective of the disagreement but it engages in speculation in the bond market. The demand curve for bonds is symmetric around the benchmark value of $\widehat{g}$ and is nonmonotonic. As we have had occasion to point out, the reason for the prevalence of this nonmonotonicity is that the variable $\widehat{g}$ has two impacts, one of which is directional, the other one non-directional.

Decomposing the rational demand for the bond into static and hedging components (Figure 4, second column, rows two and three), we discover that the zero demand for the bond at the point $\widehat{g} = 0$ is the sum of a negative static component and a positive hedging component. The static component reflects beliefs about returns. When Group B is, for instance, optimistic about future growth ($\widehat{g} > 0$), it shorts the bond and invests the proceeds in the short-term deposit. This is because it expects the short rate to rise relative to the current value of long rates. Equity would be an inferior way to speculate about future expected growth, or about future spot interest rates, because the realized one-period output flow

[44] That method is based on the "martingale representation theorem".

[45] Over the two populations, these fractions sum to 1 for equity and to 0 for the bond so that the equilibrium holdings of Population A follow immediately from those of B.

[46] The static portfolio component is obtained by rewriting the wealth function $\widetilde{F}^B$ as a function of $\left(\delta, \xi^B, \widehat{f}^M, \widehat{g}, t\right)$ instead of $\left(\delta, \eta, \widehat{f}^M, \widehat{g}, t\right)$. The gradient with respect to ξ^B gives the hedge portfolio protecting against ξ^B. But we know from Cox and Huang (1989) that that is precisely the static (Markowitz) component. That is because the diffusion vector of ξ^B is equal to the vector of expected returns premultiplied by the diffusion matrix of securities returns.

[47] The intertemporal hedge itself could be broken down into three hedges against the three state variables: η, $\widehat{f}^M$, and $\widehat{g}$.

δ would introduce noise in ex post returns that is absent with bonds. Indeed, the static demand for equity is not very different from the number 1.

The bond is a hedge, even in the absence of disagreement, for a reason which is the mirror image of the reason for which it is a speculative instrument in the presence of disagreement. Even when investors today happen to agree about expected future growth, they know that tomorrow both of their opinions will have evolved. The hedge operates as follows. If, tomorrow, the average belief $\widehat{f}^M$ rises to a higher level than today, bond prices are reduced so that a person holding a long position in the bond collects an offsetting return. The protection operates also in the case of a reduced belief of growth. Investors also use the bond as a hedge against future disagreement $\widehat{g} = \widehat{f}^B - \widehat{f}^A$. Both hedges work in the same direction because we are currently looking at the demand of Group B. Because beliefs are mean reverting, hedging demand slopes upward. Given the high volatility of interest rates in our model, the hedging demand is also quite sizable. It represents around half of the wealth of Group B in our numerical example.[48]

The strong hedging pressure brings down the expected excess return on bond, which explains why the static demand is strongly negative.

The major difference between the two cases of irrationality and rationality is found in the equity column of Figure 4 (compare dashed and dotted lines). In the case of irrationality (dashed line) and agreement, the rational investor B invests into equity (and to a smaller extent into bonds) a smaller fraction of his/her wealth than in a rational market. The reason is that rational, risk averse investors are deterred by the presence of the irrational traders which are a source of risk in their eyes. They prefer to take refuge in the riskless short-term asset unless they are extremely optimistic about future growth. The reduced equity position is a way to hedge fluctuations in the beliefs (see bottom row, first column).

If, however, the rational investors are currently more optimistic than the irrational ones ($\widehat{g} > 0$), they overcome their fear and invest in equities. The opposite is, of course, true when they are pessimistic. This demand behavior is mostly due to static demand (see first column, second row), which exploits predictability in stock returns. However, because investors may change their minds in subsequent periods, the increased static demand in case of optimism is tempered by an accompanying increased hedging demand for stocks (first column, third row).

Predictability has not been documented so far. It is present because the stock price is a decreasing function and the conditionally expected stock return an increasing function, of the average estimate of expected growth. These two relations induce a negative statistical relation between current stock price and future expected return. Predictability of stock returns has two components: predictability of future dividends and predictability of future expected returns. And predictability of future expected returns has two components; predictability of future interest rates and predictability of future risk premia on equity. We have seen that Group B investors exploit predictability of interest rates (and risk premia on bonds) by means of bonds: they short them when they are optimistic. They now exploit the other two components by means of an equity investment: they go long into the equity when they are optimistic. Their main purpose in doing that is to exploit predictability of the nearby dividends. The difference of motivation (nearby dividend rather than price movement) explains that for equity they go long at the same time that they short the bond whose only predictable return will come from a movement in price.

For bonds, there is not a clear difference between the demands for them in the cases of rationality and irrationality. The motives behind the demand for bonds remain what they were. The static demand reflects a desire to take a bet on future growth by means of term structure bets. The hedging

[48] We should also mention that equity serves as a hedge against η risk.

demand remains strong at a level equal to about half the size of the investor's wealth for our parameter configuration (second column, bottom line).

Proposition 6 *In the case of irrationality as in the case of rationality, the bond serves both to take a view on future expected growth rates and to hedge against their revisions that will be reflected in future bond prices, whereas in the case of irrationality, equity is used by rational investors mostly to exploit their disagreement with the irrational investors about the value of the nearby dividends.*

In a sense, from the point of view of rational investors, the distribution of the roles between the two risky instruments that are available is as follows. Equity is the short-term risky investment in that its holdings by rational investors are motivated by the prospect of good cash flows. The bond, on the other hand, is the long-term investment in that its holdings are motivated by future asset price movements.

We have demonstrated that a rational risk-arbitrageur finds it beneficial to trade on his/her belief that the market is being foolish. When doing so, however, he/she must hedge future fluctuations in the market's foolishness. This illustrates the general idea that risk arbitrage cannot just be based on a current price divergence. It must also be based on a model of irrational behavior and a prediction concerning the speed of convergence. The risk arbitrage must include a protection in case there is a deviation from that prediction, a form of risk that David (2004) has called "trading risk" and that we called "sentiment risk". We have found that, for the current form of heterogeneous beliefs, bonds are an essential accompaniment of equity investment, in order to hedge the trading or sentiment risk that is present in the financial market.

6 Vindication: Profits of rational investors *vs.* survival of overconfident agents

We now return to the question we asked originally concerning the potential for gains that the excessive volatility creates for the rational investors who follow the portfolio strategy that we described. By asking whether rational risk arbitrageurs can take advantage of overconfident investors, we simultaneously ask whether rational investors eliminate the overconfident investors from the economy very quickly, or whether overconfident investors can survive for some time. Survival of irrational traders is an issue that is the focus of recent papers by Berrada (2004), Kogan, Ross, Wang, and Westerfield (2003) and Yan (2004). Here, however, we consider a different kind of irrational agents, who change their mind too frequently.

One way to measure the survival of irrational agents in the economy is to study the evolution of the expected value of the share of total dividend that will be consumed by them under the objective probability measure.

The expected value of Population A's consumption-dividend ratio under the objective probability measure is:

$$\mathbb{E}^P\left[\frac{c_u^A}{\delta_u}\right] = \mathbb{E}^P\left(\frac{\lambda^B}{\lambda^A}\eta_u\right)^{\frac{1}{1-\alpha}}\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta_u\right)^{\frac{1}{1-\alpha}}\right]^{-1}. \tag{47}$$

To compute this expectation, we need conditional distribution of η_u, given $\eta_t, f_t, \widehat{f}_t^A, \widehat{f}_t^B$ at t. As in the previous section, we first obtain its characteristic function:

$$\mathbb{E}^P_{\eta,\widehat{g}^A,\widehat{g}^B}\left[\eta_u\right]^\chi = \eta^\chi \times H_P\left(\widehat{g}^A, \widehat{g}^B, t; \chi; u\right), \tag{48}$$

where $\widehat{g}^A \triangleq \widehat{f}^A - f$, $\widehat{g}^B \triangleq \widehat{f}^B - f$, and

$$\begin{aligned} H_P\left(\widehat{g}^A, \widehat{g}^B, t; \chi; u\right) &= \exp\Big\{A_P\left(\chi; u-t\right) + C^A\left(\chi; u-t\right) \times \left(\widehat{g}^A\right)^2 + C^B\left(\chi; u-t\right) \times \left(\widehat{g}^B\right)^2 \\ &\quad + 2C^{AB}\left(\chi; u-t\right) \times \widehat{g}^A \times \widehat{g}^B\Big\}, \end{aligned} \tag{49}$$

for certain functions of time A_P, C^A, C^B, and C^{AB} that are given in Appendix C.

Then, by Fourier inversion, survival denoted by $S\left(\eta, \widehat{g}^A, \widehat{g}^B, t; u\right)$, is:

$$\begin{aligned} S\left(\eta, \widehat{g}^A, \widehat{g}^B, t; u\right) &\triangleq \mathbb{E}^P_{\eta, \widehat{g}^A, \widehat{g}^B}\left[\frac{c^A_u}{\delta_u}\right] \\ &= \int_0^\infty \left(\frac{\lambda^B}{\lambda^A}\eta_u\right)^{\frac{1}{1-\alpha}} \left[1 + \left(\frac{\lambda^B}{\lambda^A}\eta_u\right)^{\frac{1}{1-\alpha}}\right]^{-1} \\ &\quad \times \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta}\right)^{-i\chi} H_P\left(\widehat{g}^A, \widehat{g}^B, t; i\chi; u\right) d\chi\right] \frac{d\eta_u}{\eta_u}. \end{aligned} \tag{50}$$

Figure 5 illustrates the case where irrational agents ($\phi = 0.95$) start out having 50% of the total wealth. We plot against future dates the expected percentage of the total dividends consumed by Group A. The first conclusion is that, ultimately, irrational agents become extinct. But, the more interesting observation is that, in contrast to what is typically assumed in models of rational asset pricing, irrational agents do not lose their wealth right away. For instance, if the initial share of consumption of the overconfident investors was 50%, then even after 200 years they consume 25% of the aggregate dividends.

7 Conclusion

Assuming that there is excess volatility in capital markets, our objective was to analyze the trading strategy that would allow an investor to take advantage of this excess volatility. To achieve our goal, we first constructed a general equilibrium model where stock prices are excessively volatile using the same device as in Scheinkman and Xiong (2003). That is, there are two classes of agents, and one class (irrational or overconfident) believes that the magnitude of the correlation between the innovations in the signal and innovations in the unobserved variable, the growth rate of dividends, is larger than it is actually. Consequently, when a signal is received, this class of agents overreacts to it, which then generates excessive stock price movements. We then analyzed the trading strategy of the rational investors who knows that the true magnitude of this correlation is zero.

Our analysis shows that the portfolio of rational investors consists of two components: a static (i.e., Markowitz) portfolio based only on *current* expected stock returns and risk (this is also a hedge against changes in the pricing measure), and a portfolio that hedges the investor against the future changes in output that are not impounded in the change of the pricing measure. The second component of the portfolio hedges the investor against *future* revisions in the market's expected dividend growth, and a portfolio that hedges against future *disagreement* in revisions of expected dividend growth. There are two aspects to the portfolio strategy of rational investors: First, these investors may not agree today with the market about its current estimate of the growth rate of dividends: when the rational investors are more optimistic than the market, they increases their investment in equity. Second, even when the two groups of investors happen to agree today, rational investors are aware that irrational investors will revise their estimate in a manner that differs from theirs. This second effect makes rational investors hold fewer shares of equity than would be optimal in a market without excess volatility and causes them to take a negative position in bonds (which would be zero in the absence of excess volatility).

In short, we find that rational risk-arbitrageurs finds it beneficial to trade on their belief that the market is being foolish but when doing so, they must hedge future fluctuations in the market's foolishness. Thus, our analysis illustrates that risk arbitrage cannot be based just on a current price divergence; it must also be based on a model of irrational behavior and a prediction concerning the speed of price convergence, and that the risk arbitrage must include a protection in case there is a deviation from that prediction.

We also find that, in contrast to what is typically assumed in standard models of asset pricing in frictionless markets, the presence of a few rational traders is not sufficient to eliminate the effect of overconfident investors on excess volatility and that overconfident investors may survive for a long time before being driven out of the market by rational investors.

A Proofs for propositions

Proof of Proposition 3

Taking into account (7) and (12) and applying Itô's lemma, we get:

$$\frac{d\delta_t^{\alpha-1}}{\delta_t^{\alpha-1}} = -(1-\alpha)\left(\widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^B\right) + \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 dt, \tag{A1}$$

$$\frac{d\left(\frac{\eta_t}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^B}\right)^{\frac{1}{1-\alpha}}} = \frac{1}{2}\frac{\alpha}{(1-\alpha)^2}\nu_t^\intercal \nu_t dt - \frac{1}{1-\alpha}\nu_t^\intercal dW_t^A, \tag{A2}$$

$$\begin{aligned}\frac{d\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} &= \frac{1}{2}\frac{\alpha}{1-\alpha}\frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}\nu_t^\intercal \nu_t dt \\ &\quad - \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}\nu_t^\intercal dW_t^A.\end{aligned} \tag{A3}$$

Then, from (27):

$$\begin{aligned}\frac{d\xi_t^B}{\xi_t^B} &= -\rho dt - (1-\alpha)\left(\widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^A\right) + \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 dt \\ &\quad + \frac{1}{2}\frac{\alpha}{1-\alpha}\frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}\nu_t^\intercal \nu_t dt - \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}\nu_t^\intercal dW_t^A \\ &\quad + (1-\alpha)\sigma_\delta \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}\nu_t^\intercal \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt.\end{aligned} \tag{A4}$$

Collecting drift and diffusion terms in $\frac{d\xi_t^A}{\xi_t^A}$ and taking into account (30), we obtain (31) and (33).

As η is a change from A's measure to B's measure, $\xi^B = \xi^A \times \eta$. Consequently, $\kappa^B = \kappa^A + \nu$, which leads to (32).

B The characteristic function

We want to compute

$$H\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi\right) \triangleq \mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\left[(\delta_u)^\varepsilon (\eta_u)^\chi\right].$$

This function satisfies the linear PDE:

$$0 \equiv \mathcal{D}H\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi\right) + \frac{\partial H}{\partial t}\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi\right), \tag{B1}$$

with the initial condition $H\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, t; \varepsilon, \chi\right) = \delta^\varepsilon \eta^\chi$, and where $\mathcal{D}$ is the differential generator of $\left(\delta_t, \eta_t, \widehat{f}_t^B, \widehat{g}_t, t\right)$ under the probability measure of Group B. In doing that, we shall use the following function:

$$H_f\left(\widehat{f}^B,t;u,\varepsilon\right) \triangleq \mathbb{E}^B_{\delta,\widehat{f}^B}\left[(\delta_u)^{\varepsilon}\right]=\exp\left\{\varepsilon\left[\overline{f}\,(u-t)+\frac{1}{\zeta}\left(\widehat{f}^B-\overline{f}\right)\left[1-e^{-\zeta(u-t)}\right]\right]+\frac{1}{2}\varepsilon\,(\varepsilon-1)\,\sigma_\delta^2\,(u-t)\right.$$
$$\left.+\frac{\varepsilon^2\gamma^B}{2\zeta^2}\left[1-e^{-\zeta(u-t)}\right]^2+\frac{\varepsilon^2\sigma_f^2}{4\zeta^3}\left[2\zeta\,(u-t)-2\left[1-e^{-\zeta(u-t)}\right]-\left[1-e^{-\zeta(u-t)}\right]^2\right]\right\}. \tag{B2}$$

Spelling out (B1) we have:

$$\begin{aligned}
0 \equiv\;& \frac{\partial H}{\partial\delta}\delta\widehat{f}^B-\frac{\partial H}{\partial\widehat{f}^B}\zeta\left(\widehat{f}^B-\overline{f}\right)-\frac{\partial H}{\partial\widehat{g}}\widehat{g}\left(\zeta+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s^2}+\frac{\gamma^A}{\sigma_\delta^2}\right)\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\delta^2}\left(\delta\sigma_\delta\right)^2+\frac{1}{2}\frac{\partial^2 H}{\partial\eta^2}\left(\eta\widehat{g}\right)^2\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\widehat{g}^2}\left[\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta}\right)^2+\left(\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s}\right)^2\right]\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\left(\widehat{f}^B\right)^2}\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\left(\gamma^B\right)^2\\
&-\frac{\partial^2 H}{\partial\delta\partial\eta}\delta\eta\widehat{g}+\frac{\partial^2 H}{\partial\delta\partial\widehat{g}}\delta\left(\gamma^B-\gamma^A\right)+\frac{\partial^2 H}{\partial\delta\partial\widehat{f}^B}\delta\gamma^B\\
&-\frac{\partial^2 H}{\partial\eta\partial\widehat{g}}\eta\widehat{g}\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta^2}+\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s^2}\right)\\
&-\frac{\partial^2 H}{\partial\eta\partial\widehat{f}^B}\eta\widehat{g}\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\gamma^B\\
&+\frac{\partial^2 H}{\partial\widehat{g}\partial\widehat{f}^B}\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta^2}+\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s^2}\right)\gamma^B\\
&+\frac{\partial H}{\partial t},
\end{aligned} \tag{B3}$$

with terminal condition:

$$H\left(\delta,\eta,\widehat{f}^B,\widehat{g},u,u;\varepsilon,\chi\right)=\delta^\varepsilon\times\eta^\chi.$$

The solution of this PDE is:

$$H\left(\delta,\eta,\widehat{f}^B,\widehat{g},t,u;\varepsilon,\chi\right)\triangleq\delta^\varepsilon\eta^\chi\widetilde{H}\left(\widehat{f}^B,\widehat{g},t,u;\varepsilon,\chi\right),$$

where:

$$\widetilde{H}\left(\widehat{f}^B,\widehat{g},t,u;\varepsilon,\chi\right)=H_f\left(\widehat{f}^B,t,u;\varepsilon\right)\times H_g\left(\widehat{g},t,u;\varepsilon,\chi\right), \tag{B4}$$

$$H_g\left(\widehat{g},t,u;\varepsilon,\chi\right)=\exp\left\{A_0\,(u-t)+B_0\,(u-t)\times\widehat{g}+C\,(u-t)\times\widehat{g}^2\right\}, \tag{B5}$$

and where:

$$C\,(\chi;u-t)=\frac{2a\left(1-e^{-q(u-t)}\right)}{q-b+(q+b)\,e^{-q(u-t)}}, \tag{B6}$$

$$a = \frac{1}{2}\chi(\chi-1)\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right) \tag{B7}$$

$$b = -2\left[\zeta+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s^2}+\frac{\gamma^A}{\sigma_\delta^2}+\chi\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta^2}+\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s^2}\right)\right] \tag{B8}$$

$$c = 2\times\left[\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta}\right)^2+\left(\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s}\right)^2\right] \tag{B9}$$

$$k = -\chi\left[1+\frac{\gamma^B}{\zeta}\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\right] \tag{B10}$$

$$l = \chi\frac{\gamma^B}{\zeta}\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right) \tag{B11}$$

$$m = \left[\left(\gamma^B-\gamma^A\right)+\frac{\gamma^B}{\zeta}\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta^2}+\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s^2}\right)\right] \tag{B12}$$

$$n = -\frac{\gamma^B}{\zeta}\left(\frac{\gamma^B-\gamma^A}{\sigma_\delta^2}+\frac{\gamma^B-\left(\phi\sigma_s\sigma_f+\gamma^A\right)}{\sigma_s^2}\right) \tag{B13}$$

$$q = \sqrt{b^2-4ac}. \tag{B14}$$

$$B_0(\varepsilon,\chi;u-t) = \varepsilon B(\chi;u-t) \tag{B15}$$

where[49]

$$B(\chi;u-t) = \frac{2\left[\vartheta_1+\vartheta_2 e^{-\frac{1}{2}q(u-t)}+\vartheta_3 e^{-q(u-t)}+\vartheta_4 e^{-\zeta(u-t)}+\vartheta_5 e^{(-q-\zeta)(u-t)}\right]}{q-b+(q+b)e^{-q(u-t)}}, \tag{B16}$$

and

$$\vartheta_1 = \frac{4am+k(q-b)}{q}, \tag{B17}$$

$$\vartheta_2 = -2\left(\frac{4am-bk}{q}+q\frac{4an-bl+2l\zeta}{q^2-4\zeta^2}\right), \tag{B18}$$

$$\vartheta_3 = \frac{4am-k(q+b)}{q}, \tag{B19}$$

$$\vartheta_4 = \frac{4an+l(q-b)}{q-2\zeta}, \tag{B20}$$

$$\vartheta_5 = \frac{4an-l(q+b)}{q+2\zeta}. \tag{B21}$$

whereas:

$$A_0(\varepsilon,\chi;u-t) = A_1(\chi;u-t)+\varepsilon^2 A_2(\chi;u-t),$$

[49]

$$\begin{aligned}\widehat{C}(u-t) &= \exp\left\{\frac{b}{2}(u-t)+c\int_t^u C(\tau-t)\,d\tau\right\}\\ &= \exp\left\{\frac{b}{2}(u-t)+\ln(2q)-\ln\left(q-b+(q+b)e^{-q(u-t)}\right)-\frac{b+q}{2}(u-t)\right\}\\ &= \frac{2qe^{-\frac{1}{2}q(u-t)}}{q-b+(q+b)e^{-q(u-t)}}\end{aligned}$$

where

$$A_1(\chi;u-t) = \frac{1}{4}\left[2\ln(2q) - 2\ln\left(q-b+(q+b)e^{-q(u-t)}\right) - (b+q)(u-t)\right], \tag{B22}$$

$$\begin{aligned}
A_2(\chi;u-t) = {} & 2m\left[\vartheta_1 D_1(0;u-t) + \vartheta_2 D_1\left(\frac{q}{2};u-t\right) + \vartheta_3 D_1(q;u-t)\right. \\
& \left. + \vartheta_3 D_1(\zeta;u-t) + \vartheta_5 D_1(q+\zeta;u-t)\right] \\
& + 2n\left[\vartheta_1 D_1(\zeta;u-t) + \vartheta_2 D_1\left(\frac{q}{2}+\zeta;u-t\right) + \vartheta_3 D_1(q+\zeta;u-t)\right. \\
& \left. + \vartheta_3 D_1(2\zeta;u-t) + \vartheta_5 D_1(q+2\zeta;u-t)\right] \\
& + c\left[\vartheta_1^2 D_2(0;u-t) + \vartheta_2^2 D_2(q;u-t) + \vartheta_3^2 D_2(2q;u-t) + \vartheta_4^2 D_2(2\zeta;u-t)\right. \\
& + \vartheta_5^2 D_2(2q+2\zeta;u-t) + 2\vartheta_1\vartheta_2 D_2\left(\frac{q}{2};u-t\right) + 2\vartheta_1\vartheta_3 D_2(q;u-t) \\
& + 2\vartheta_1\vartheta_4 D_2(\zeta;u-t) + 2\vartheta_1\vartheta_5 D_2(2\zeta;u-t) + 2\vartheta_2\vartheta_3 D_2\left(\frac{3q}{2};u-t\right) \\
& + 2\vartheta_2\vartheta_4 D_2\left(\frac{q}{2}+\zeta;u-t\right) + 2\vartheta_2\vartheta_5 D_2\left(\frac{3q}{2}+\zeta;u-t\right) + 2\vartheta_3\vartheta_4 D_2(q+\zeta;u-t) \\
& \left. + 2\vartheta_3\vartheta_5 D_2(2q+\zeta;u-t) + 2\vartheta_4\vartheta_5 D_2(q+2\zeta;u-t)\right],
\end{aligned} \tag{B23}$$

and denoting by $\mathcal{H}$ the Hypergeometric function:

$$\begin{aligned}
D_1(p;u-t) &= \int_t^u \frac{e^{-p(\tau-t)}}{q-b+(q+b)e^{-q(\tau-t)}}d\tau \qquad \text{(B24)} \\
&= \begin{cases} \frac{q(u-t)-\ln(2q)+\ln\left(q-b+(q+b)e^{-q(u-t)}\right)}{q(q-b)}, & p=0, \\ \frac{1}{p(q-b)}\left[\mathcal{H}\left(1,\frac{p}{q},1+\frac{p}{q},-\frac{q+b}{q-b}\right) - e^{-p(u-t)}\mathcal{H}\left(1,\frac{p}{q},1+\frac{p}{q},-\frac{q+b}{q-b}e^{-q(u-t)}\right)\right], & p>0, \end{cases} \\
D_2(p;u-t) &= \frac{1}{q(q-b)}\left[\frac{1}{2q} - \frac{e^{-p(u-t)}}{q-b+(q+b)e^{-q(u-t)}} + (q-p)D_1(p;u-t)\right].
\end{aligned}$$

Proposition B1 *The function $H_g(\widehat{g},t,u;\varepsilon,\chi)$ is well-defined for $\chi\in[0,1]$ and $u\geq t$.*

Proof. The radicand in Equation (B14) for q can be written as a quadratic trinomial of χ:

$$b^2-4ac = q_2\chi^2+q_1\chi+q_0, \tag{B25}$$

where

$$q_2 = -\frac{4\sigma_f^2\phi^2}{\sigma_\delta^2}, \tag{B26}$$

$$q_1 = 8\phi\sigma_f\left(\frac{\phi\sigma_f}{\sigma_\delta^2}-\frac{\zeta}{\sigma_s}\right), \tag{B27}$$

$$q_0 = 4\left[\left(\zeta+\frac{\phi\sigma_f}{\sigma_s}\right)^2+\left(1-\phi^2\right)\left(\frac{\sigma_f^2}{\sigma_s^2}+\frac{\sigma_f^2}{\sigma_\delta^2}\right)\right]. \tag{B28}$$

As $q_2\leq 0$, $q_0>0$, and

$$q_2+q_1+q_0 = 4\left(\zeta^2+\frac{\sigma_f^2}{\sigma_\delta^2}+\frac{\sigma_f^2}{\sigma_s^2}\right)>0, \tag{B29}$$

then, when $\chi \in [0,1]$, $b^2 - 4ac > 0$, and $q = \sqrt{b^2 - 4ac}$ is real and strictly positive.

Taking into account that $c \geq 0$, and for $\chi \in [0,1]$, $a \leq 0$ and

$$b = -2\left[\chi\sqrt{\zeta^2 + \frac{\sigma_f^2}{\sigma_\delta^2} + \frac{\sigma_f^2}{\sigma_s^2}} + (1-\chi)\sqrt{\left(\zeta + \frac{\phi\sigma_f}{\sigma_s}\right)^2 + (1-\phi^2)\left(\frac{\sigma_f^2}{\sigma_s^2} + \frac{\sigma_f^2}{\sigma_\delta^2}\right)}\;\right] < 0, \tag{B30}$$

we obtain that $q + b \geq 0$ and

$$q - b + (q+b)\, e^{-q(u-t)} \geq q - b > 0. \tag{B31}$$

Consequently, when $\chi \in [0,1]$ and $u \geq t$, functions $C(\chi; u-t)$, and $B(\chi; u-t)$ are well-defined, integrals $A_1(\chi; u-t)$ and $A_2(\chi; u-t)$ are convergent and their closed-form expressions (B22) and (B23) are obtained correctly. ■

Note that we consider only $\chi \in [0,1]$, because in Appendix C, we consider only the values: $\chi = \frac{j}{1-\alpha}$, $j = 0, ..., 1-\alpha$, when $\alpha \in \mathbb{Z}$.

C The wealth and price functions

Knowing the characteristic function (37) from Appendix B, the securities market prices (34), (35) and (36) can be obtained by one of two methods. One is general. It is the inverse Fourier transform, for which the formulae are given below and that can be computed by means of the Fast Fourier Transform, but for which we have no proof of convergence.[50] By Fourier inversion, the stock price is:

$$\begin{aligned} F\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right) &= \delta \frac{1}{\left[\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^A, t, u; \alpha\right) \\ &\times \left\{\int_0^\infty \left[\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta}\right)^{-i\chi} H_g\left(\widehat{g}, t, u; \alpha, i\chi\right) d\chi\right] \frac{d\eta_u}{\eta_u}\right\} du. \end{aligned} \tag{C1}$$

Similarly, the price of the consol bond is:

$$\begin{aligned} P\left(\eta, \widehat{f}^A, \widehat{g}, t\right) &= \frac{1}{\left[\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^A, t, u; \alpha - 1\right) \\ &\times \left\{\int_0^\infty \left[\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta}\right)^{-i\chi} H_g\left(\widehat{g}, t, u; \alpha - 1, i\chi\right) d\chi\right] \frac{d\eta_u}{\eta_u}\right\}. \end{aligned} \tag{C2}$$

The wealth of Group B investors is:

$$\begin{aligned} F^B\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right) &= \delta \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^A, t, u; \alpha\right) \\ &\times \left\{\int_0^{+\infty} \left[\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{-\alpha} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta}\right)^{-i\chi} H_g\left(\widehat{g}, t, u; \alpha, i\chi\right) d\chi\right] \frac{d\eta_u}{\eta_u}\right\} du. \end{aligned} \tag{C3}$$

[50] In fact, we do not know whether H_g is well defined for $\chi \in \left]-i\infty, +i\infty\right[$.

The second method is applicable in the special case in which $1-\alpha \in \mathbb{N}$ and $\alpha < 0$. Then the bracket $\left[\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}$ can be expanded into an exact finite series by virtue of the binomial formula. The overall calculation is then greatly simplified. The equity price is equal to:[51]

$$\begin{aligned} F\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right) &= \delta \frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^B, t, u; \alpha\right) \\ &\quad \times \left\{ \sum_{j=0}^{1-\alpha} C_{1-\alpha}^j \left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}} H_g\left(\widehat{g}, t, u; \alpha, \frac{j}{1-\alpha}\right)\right\} du. \end{aligned} \tag{C4}$$

Similarly, the price of a consol bond is:

$$\begin{aligned} P\left(\eta, \widehat{f}^B, \widehat{g}, t\right) &\triangleq \frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^B, t, u; \alpha-1\right) \\ &\quad \times \left\{ \sum_{j=0}^{1-\alpha} C_{1-\alpha}^j \left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}} H_g\left(\widehat{g}, t, u; \alpha-1, \frac{j}{1-\alpha}\right)\right\}, \end{aligned} \tag{C5}$$

and the wealth of Group B is:

$$\begin{aligned} F^B\left(\delta, \eta, \widehat{f}^B, \widehat{g}, t\right) &= \delta \frac{1}{\left[1+\left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f\left(\widehat{f}^B, t, u; \alpha\right) \\ &\quad \times \left\{ \sum_{j=0}^{-\alpha} C_{-\alpha}^j \left(\frac{\lambda^B}{\lambda^A}\eta\right)^{\frac{j}{1-\alpha}} H_g\left(\widehat{g}, t, u; \alpha, \frac{j}{1-\alpha}\right)\right\} du. \end{aligned} \tag{C6}$$

Analogously, we can derive all other functions: gradients, second moments, portfolio holdings.

Proposition C1 *When α is an integer, the growth condition for the function F to be well-defined is:*

$$\alpha \overline{f} + \frac{1}{2}\alpha(\alpha-1)\sigma_\delta^2 + \frac{\alpha^2 \sigma_f^2}{2\zeta^2} < \rho. \tag{C7}$$

Proof. For $u \geq t$, and $\chi \in [0,1]$, $b(\chi) < 0$, $q - b > 0$ and

$$0 \leq \frac{q+b}{q-b} e^{-q(u-t)} < 1, \tag{C8}$$

we can obtain the Taylor series below

$$\frac{1}{q-b+(q+b)\,e^{-q(u-t)}} = \frac{1}{(q-b)\left(1-\frac{q+b}{b-q}e^{-q(u-t)}\right)} = \frac{1}{q-b}\sum_{j=0}^{\infty}\left[\frac{q+b}{b-q}e^{-q(u-t)}\right]^j, \tag{C9}$$

[51]The χ argument belongs to $[0,1]$ allowing us to apply Proposition B1 to conclude that H_g is well defined.

and the series is uniformly convergent. So, we can interchange summation and integral operator in the expression for $A_1(\chi; u-t)$ and obtain:

$$\begin{aligned} A_1(\chi; u-t) &= \frac{c}{2}\int_t^u C(\tau-t)\,d\tau \\ &= \frac{c}{2}\int_t^u \frac{2a\left(1-e^{-q(\tau-t)}\right)}{q-b}\sum_{j=0}^{\infty}\left[\frac{q+b}{b-q}e^{-q(\tau-t)}\right]^j d\tau \\ &= \varrho_1 \times (u-t) + \sum_{j=0}^{\infty}\varrho_{1j}\times e^{-jq(u-t)}, \end{aligned} \tag{C10}$$

where

$$\varrho_1 = \frac{ac}{q-b} \leq 0. \tag{C11}$$

Similarly,

$$\begin{aligned} A_2(\chi; u-t) &= \int_t^u B(\tau-t)\left[B(\tau-t)\times\frac{c}{4}+m+ne^{-\zeta(\tau-t)}\right]d\tau \\ &= \varrho_2\times(u-t)+\sum_{j=0}^{\infty}\left[\varrho_{2j}\times e^{-\frac{1}{2}jq(u-t)}+\widetilde{\varrho}_{2j}\times e^{-\left(\frac{1}{2}jq+\zeta\right)(u-t)}\right], \end{aligned} \tag{C12}$$

where

$$\begin{aligned} \varrho_2 &= \frac{c\vartheta_1^2}{(q-b)^2}+\frac{2m\vartheta_1}{q-b}=\frac{4am^2-2bmk+ck^2}{q^2} \\ &= -\frac{2\phi^2\sigma_f^2\chi(1-\chi)}{q^2}\left(1+\frac{\sigma_f^2}{\zeta^2\sigma_\delta^2}\right)\leq 0. \end{aligned} \tag{C13}$$

So, the function $H_g(\widehat{g}, t, u; \varepsilon, \chi)$ can be represented as:

$$\begin{aligned} H_g(\widehat{g}, t, u; \varepsilon, \chi) &= \exp\Big\{\left[\varrho_1(\chi)+\varepsilon^2\varrho_2(\chi)\right]\times(u-t) \\ &\quad + \sum_{j=0}^{\infty}\left[h_{1j}(\widehat{g};\varepsilon,\chi)\,e^{-\frac{1}{2}jq(\chi)(u-t)}+h_{2j}(\widehat{g};\varepsilon,\chi)\,e^{-\left(\frac{1}{2}jq(\chi)+\zeta\right)(u-t)}\right]\Big\}. \end{aligned} \tag{C14}$$

Similarly to the approach in Brennan and Xia (2001, Theorem 6),[52] we can prove that, when $\alpha \in \mathbb{Z}$, the growth condition for the general economy is:

$$\alpha\overline{f}+\frac{1}{2}\alpha(\alpha-1)\sigma_\delta^2+\frac{\alpha^2\sigma_f^2}{2\zeta^2}+\max_{\chi\in\left\{\frac{j}{1-\alpha}\right\}_{j=0}^{1-\alpha}}\left[\varrho_1(\chi)+\alpha^2\varrho_2(\chi)\right]<\rho. \tag{C15}$$

Finally, from (C11) and (C13)

$$\max_{\chi\in[0,1]}\left[\varrho_1(\chi)+\alpha^2\varrho_2(\chi)\right]=\varrho_1(0)+\alpha^2\varrho_2(0)=0, \tag{C16}$$

and the growth condition (C15) turns into (C7). ■

Proposition C2 *When α is integer, the growth conditions for the price of the consol bond, P, to be well defined is:*

$$(\alpha-1)\overline{f}+\frac{1}{2}(\alpha-1)(\alpha-2)\sigma_\delta^2+\frac{(\alpha-1)^2\sigma_f^2}{2\zeta^2}<\rho. \tag{C17}$$

[52] In addition, using standard "epsilon-delta" reasoning, we should consider only the finite sum in (C14).

D The expected value of A's consumption share

In this appendix, we wish to compute the following expected value:[53]

$$H\left(\eta,\widehat{g}^A,\widehat{g}^B,t;u,\chi\right)\triangleq\mathbb{E}^P_{\eta,\widehat{g}^A,\widehat{g}^B}\left[\eta_u\right]^{\chi}.$$

Note that the expectation above is computed with respect to the true probability measure rather than the measure of B, as was done in the previous appendix, where we computed $\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\left[\left(\delta_u\right)^{\varepsilon}\left(\eta_u\right)^{\chi}\right]$.

We know that

$$\begin{aligned}
d\widehat{g}^A_t &= -\widehat{g}^A_t\left(\zeta+\frac{\gamma^A}{\sigma_\delta^2}+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s^2}\right)dt+\frac{\gamma^A}{\sigma_\delta}dZ^\delta_t+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s}dZ^s_t-\sigma_f dZ^f_t\\
&= -\psi^A\widehat{g}^A_t dt+\frac{\gamma^A}{\sigma_\delta}dZ^\delta_t+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s}dZ^s_t-\sigma_f dZ^f_t \qquad \text{(D1)}\\
d\widehat{g}^B_t &= -\widehat{g}^B_t\left(\zeta+\frac{\gamma^B}{\sigma_\delta^2}+\frac{\gamma^B}{\sigma_s^2}\right)dt+\frac{\gamma^B}{\sigma_\delta}dZ^\delta_t+\frac{\gamma^B}{\sigma_s}dZ^s_t-\sigma_f dZ^f_t\\
&= -\psi^B\widehat{g}^B_t dt+\frac{\gamma^B}{\sigma_\delta}dZ^\delta_t+\frac{\gamma^B}{\sigma_s}dZ^s_t-\sigma_f dZ^f_t \qquad \text{(D2)}\\
\frac{d\eta_t}{\eta_t} &= \left(\widehat{g}^B_t-\widehat{g}^A_t\right)\widehat{g}^B_t\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)dt-\left(\widehat{g}^B_t-\widehat{g}^A_t\right)\left(\frac{1}{\sigma_\delta}dZ^\delta_t+\frac{1}{\sigma_s}dZ^s_t\right). \qquad \text{(D3)}
\end{aligned}$$

The function $H\left(\eta,\widehat{g}^A,\widehat{g}^B,t;u,\chi\right)$ satisfies the linear PDE:

$$0\equiv\mathcal{D}H\left(\eta,\widehat{g}^A,\widehat{g}^B,t;u,\chi\right)+\frac{\partial H}{\partial t}\left(\eta,\widehat{g}^A,\widehat{g}^B,t;u,\chi\right), \qquad \text{(D4)}$$

with the initial condition $H\left(\eta,\widehat{g}^A,\widehat{g}^B,t;\chi\right)=\eta^{\chi}$, where $\mathcal{D}$ is the differential generator of $\left(\eta,\widehat{g}^A,\widehat{g}^B,t\right)$ under the true probability measure. Spelling out (D4) we have:

$$\begin{aligned}
0 \equiv\ & \frac{\partial H}{\partial\eta}\eta\left(\widehat{g}^B_t-\widehat{g}^A_t\right)\widehat{g}^B_t\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)-\frac{\partial H}{\partial\widehat{g}^A}\psi^A\widehat{g}^A-\frac{\partial H}{\partial\widehat{g}^B}\psi^B\widehat{g}^B\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\eta^2}\left[\eta\left(\widehat{g}^B_t-\widehat{g}^A_t\right)\right]^2\left(\frac{1}{\sigma_\delta^2}+\frac{1}{\sigma_s^2}\right)\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\left(\widehat{g}^A\right)^2}\left(\frac{\left(\gamma^A\right)^2}{\sigma_\delta^2}+\frac{\left(\phi\sigma_s\sigma_f+\gamma^A\right)^2}{\sigma_s^2}+\sigma_f^2\right)\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial\left(\widehat{g}^B\right)^2}\left(\frac{\left(\gamma^B\right)^2}{\sigma_\delta^2}+\frac{\left(\gamma^B\right)^2}{\sigma_s^2}+\sigma_f^2\right)\\
&-\frac{\partial^2 H}{\partial\eta\partial\widehat{g}^A}\eta\left(\widehat{g}^B_t-\widehat{g}^A_t\right)\left(\frac{\gamma^A}{\sigma_\delta^2}+\frac{\phi\sigma_s\sigma_f+\gamma^A}{\sigma_s^2}\right)\\
&-\frac{\partial^2 H}{\partial\eta\partial\widehat{g}^B}\eta\left(\widehat{g}^B_t-\widehat{g}^A_t\right)\left(\frac{\gamma^B}{\sigma_\delta^2}+\frac{\gamma^B}{\sigma_s^2}\right)\\
&+\frac{\partial^2 H}{\partial\widehat{g}^A\partial\widehat{g}^B}\left(\frac{\gamma^A\gamma^B}{\sigma_\delta^2}+\frac{\left(\phi\sigma_s\sigma_f+\gamma^A\right)\gamma^B}{\sigma_s^2}+\sigma_f^2\right)+\frac{\partial H}{\partial t}. \qquad \text{(D5)}
\end{aligned}$$

[53]In this appendix, $H(\cdot)$ should not be confused with the H function used in Appendix B; that is, we use the same character, but they denote different functions.

The appropriate solution of this PDE is

$$H\left(\eta,\widehat{g}^{A},\widehat{g}^{B},t;u,\chi\right)\triangleq\eta^{\chi}H_{P}\left(\widehat{g}^{A},\widehat{g}^{B},t;u,\chi\right)$$

where:

$$\begin{aligned}&H_{P}\left(\widehat{g}^{A},\widehat{g}^{B},t;\chi;u\right)=\\&\qquad\exp\left\{A_{P}\left(u-t\right)+C^{A}\left(u-t\right)\times\left(\widehat{g}^{A}\right)^{2}+C^{B}\left(u-t\right)\times\left(\widehat{g}^{B}\right)^{2}+2C^{AB}\left(u-t\right)\times\widehat{g}^{A}\times\widehat{g}^{B}\right\},\end{aligned}\tag{D6}$$

and:

$$\begin{aligned}A_{P}\left(u-t\right)\;=\;&\int_{t}^{u}\left[C^{A}\left(\tau-t\right)\left(\frac{\left(\gamma^{A}\right)^{2}}{\sigma_{\delta}^{2}}+\frac{\left(\phi\sigma_{s}\sigma_{f}+\gamma^{A}\right)^{2}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)+C^{B}\left(\tau-t\right)\left(\frac{\left(\gamma^{B}\right)^{2}}{\sigma_{\delta}^{2}}+\frac{\left(\gamma^{B}\right)^{2}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)\right.\\&\left.+2C^{AB}\left(\tau-t\right)\left(\frac{\gamma^{A}\gamma^{B}}{\sigma_{\delta}^{2}}+\frac{\left(\phi\sigma_{s}\sigma_{f}+\gamma^{A}\right)\gamma^{B}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)\right]d\tau.\end{aligned}\tag{D7}$$

The functions of time C^{A},C^{AB} and C^{B} are defined as the elements of the matrix Z :

$$Z=\left(\begin{array}{cc}C^{A}&C^{AB}\\C^{AB}&C^{B}\end{array}\right),$$

itself defined as follows. Let matrices $X\left(u-t\right)$ and $Y\left(u-t\right)$ be the unique solution of the linear Cauchy problem

$$\left\{\begin{array}{ll}\dot{X}=Q^{11}X+Q^{12}Y, & X\left(0\right)=I,\\\dot{Y}=Q^{21}X+Q^{22}Y, & Y\left(0\right)=0,\end{array}\right.\tag{D8}$$

where I is the identity 2×2 matrix. Let:

$$Z\left(u-t\right)=Y\left(u-t\right)\left[X\left(u-t\right)\right]^{-1}.\tag{D9}$$

The coefficients are:

$$Q^{21}=\left(\begin{array}{cc}\frac{1}{2}\chi\left(\chi-1\right)\left(\frac{1}{\sigma_{\delta}^{2}}+\frac{1}{\sigma_{s}^{2}}\right)&-\frac{1}{2}\chi^{2}\left(\frac{1}{\sigma_{\delta}^{2}}+\frac{1}{\sigma_{s}^{2}}\right)\\-\frac{1}{2}\chi^{2}\left(\frac{1}{\sigma_{\delta}^{2}}+\frac{1}{\sigma_{s}^{2}}\right)&\frac{1}{2}\chi\left(\chi+1\right)\left(\frac{1}{\sigma_{\delta}^{2}}+\frac{1}{\sigma_{s}^{2}}\right)\end{array}\right),\tag{D10}$$

$$Q^{11}=-\left(Q^{22}\right)^{\top}=\left(\begin{array}{cc}\psi^{A}-\chi\left(\frac{\gamma^{A}}{\sigma_{\delta}^{2}}+\frac{\phi\sigma_{s}\sigma_{f}+\gamma^{A}}{\sigma_{s}^{2}}\right)&\chi\left(\frac{\gamma^{A}}{\sigma_{\delta}^{2}}+\frac{\phi\sigma_{s}\sigma_{f}+\gamma^{A}}{\sigma_{s}^{2}}\right)\\-\chi\left(\frac{\gamma^{B}}{\sigma_{\delta}^{2}}+\frac{\gamma^{B}}{\sigma_{s}^{2}}\right)&\psi^{B}+\chi\left(\frac{\gamma^{B}}{\sigma_{\delta}^{2}}+\frac{\gamma^{B}}{\sigma_{s}^{2}}\right)\end{array}\right),\tag{D11}$$

$$Q^{12}=\left(\begin{array}{cc}-2\left(\frac{\left(\gamma^{A}\right)^{2}}{\sigma_{\delta}^{2}}+\frac{\left(\phi\sigma_{s}\sigma_{f}+\gamma^{A}\right)^{2}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)&-2\left(\frac{\gamma^{A}\gamma^{B}}{\sigma_{\delta}^{2}}+\frac{\left(\phi\sigma_{s}\sigma_{f}+\gamma^{A}\right)\gamma^{B}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)\\-2\left(\frac{\gamma^{A}\gamma^{B}}{\sigma_{\delta}^{2}}+\frac{\left(\phi\sigma_{s}\sigma_{f}+\gamma^{A}\right)\gamma^{B}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)&-2\left(\frac{\left(\gamma^{B}\right)^{2}}{\sigma_{\delta}^{2}}+\frac{\left(\gamma^{B}\right)^{2}}{\sigma_{s}^{2}}+\sigma_{f}^{2}\right)\end{array}\right).\tag{D12}$$

Table 1: Choice of parameter values and benchmark values of the state variables

This table lists the particular choice of parameter values used for all the figures in the paper. These values are similar to the estimation results reported in Brennan and Xia (2001). The table also indicates the benchmark values of state variables, which are the reference values taken by all state variables except for the one being varied in a given graph.

Name	Symbol	Value
Parameters for aggregate endowment and the signal		
Long-term average growth rate of aggregate endowment	$\bar{f}$	0.015
Volatility of expected growth rate of endowment	σ_f	0.015
Volatility of aggregate endowment	σ_δ	0.13
Mean reversion parameter	ζ	0.2
Volatility of the signal	σ_s	0.1
Parameters for the agents		
Agent A's correlation between signal and mean growth rate	ϕ	0.95
Agent B's correlation between signal and mean growth rate	—	0
Agent A's initial share of aggregate endowment	λ^B/λ^A	1
Time-preference parameter for both agents	ρ	0.11
Relative risk aversion for both agents	$1-\alpha$	3
Benchmark values of the state variables		
The level of aggregate dividends	δ	1
The coupon rate on the consol bond	$\overline{\delta}$	0.07
The change from B's measure to A's measure	η	1
The population average belief about the expected rate of growth	$\widehat{f}^M$	$\overline{f}$
The difference in opinions: $\widehat{f}^B - \widehat{f}^A$	$\widehat{g}$	-0.01

Figure 1: Prices

This figure shows the interest rate, r, the price of equity, F, and the bond price, P, against three of the four state variables: the share of aggregate endowment consumed by Group A, $c^A/\,\delta$, average beliefs, $\widehat{f}^M$, and dispersion in beliefs, $\widehat{g}$. The dotted line in each plot represents the case where $\phi = 0$ and all agents are rational, while the dashed line represents the case where $\phi = 0.95$ implying that Population A is irrational. For parameter values and benchmark values of the variables, see Table 1.

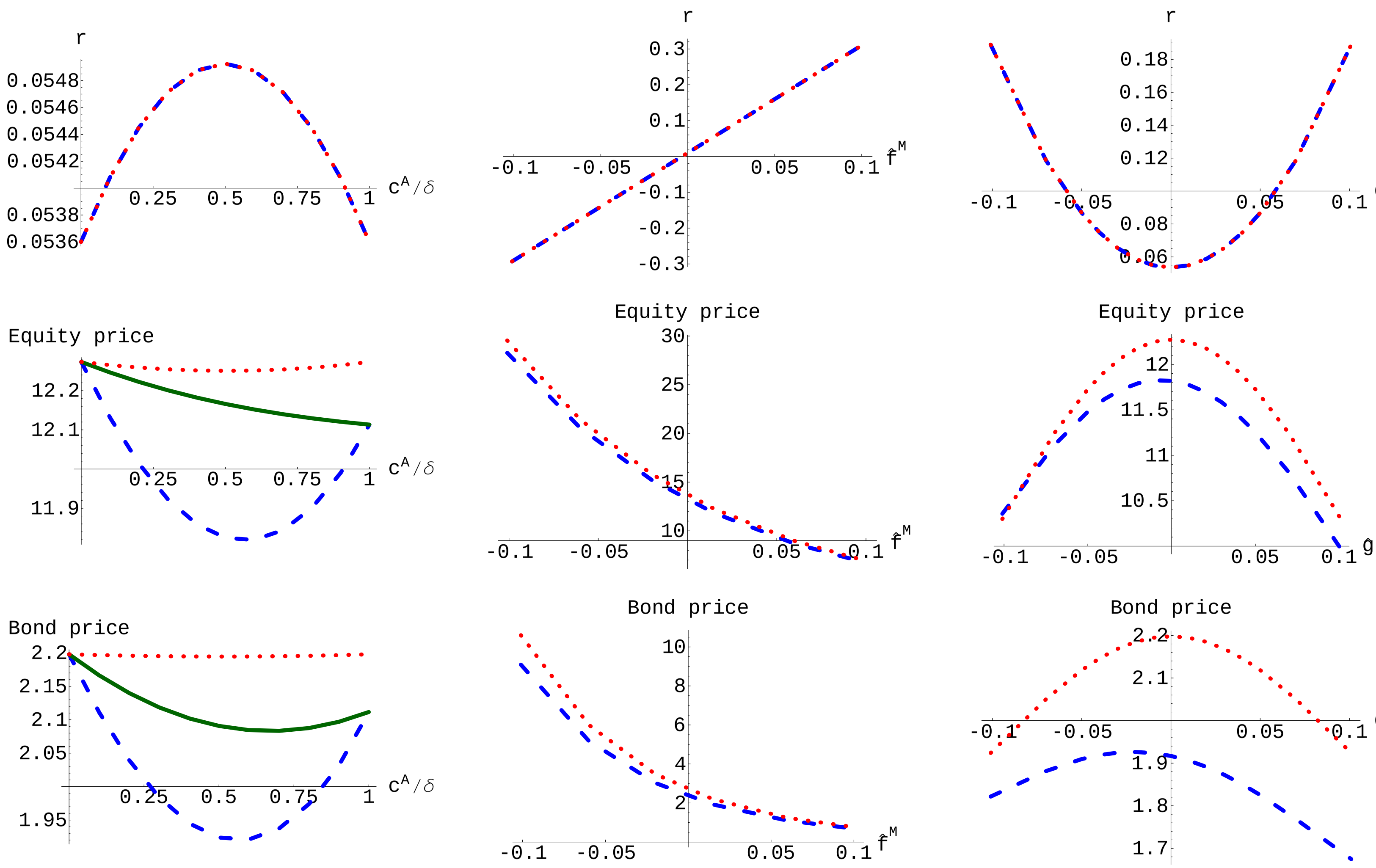


Figure 2: Volatilities and Correlations

This figure shows the volatility of stock returns, the volatility of bond returns, and the correlation between stock and bond returns. Each second moment is plotted against three of the four state variables: the share of aggregate endowment consumed by Group A, $c^A/\ \delta$, average beliefs, $\widehat{f}^M$, and dispersion in beliefs, $\widehat{g}$. The dotted line in each plot represents the case where $\phi = 0$ and all agents are rational, while the dashed line represents the case where $\phi = 0.95$ implying that Population A is irrational. For parameter values and benchmark values of the state variables, see Table 1.

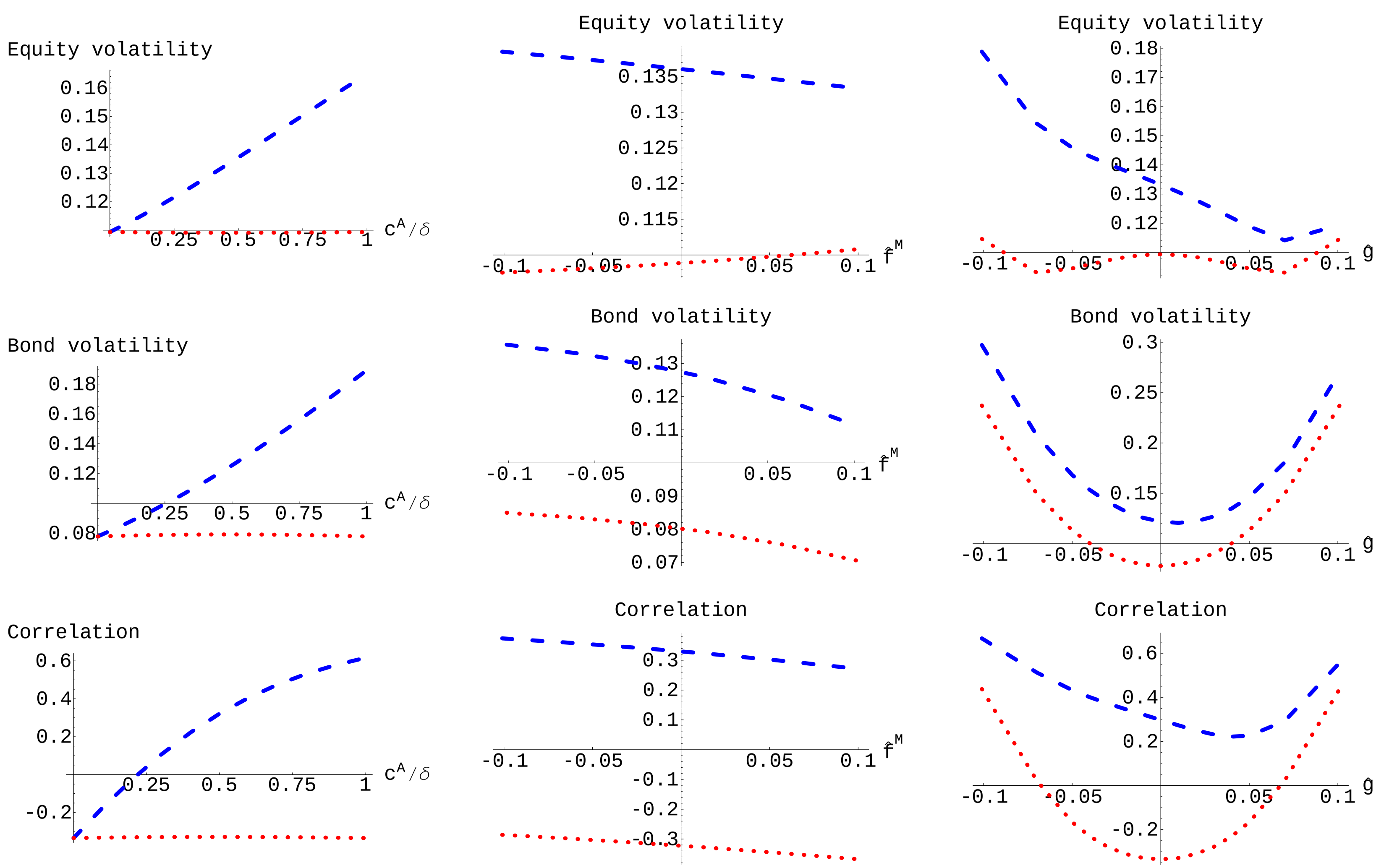


Figure 3: Wealth of Rational Investors (Population B)

This figure illustrates the variations of the wealth of Population B against three of the four state variables: the share of aggregate endowment consumed by Group A, c^A/δ, average beliefs, $\widehat{f}^M$, and dispersion in beliefs, $\widehat{g}$. The dotted line in each plot represents the case where $\phi = 0$ and all agents are rational, while the dashed line represents the case where $\phi = 0.95$ implying that Population A is irrational. For parameter values and benchmark values of the state variables, see Table 1.

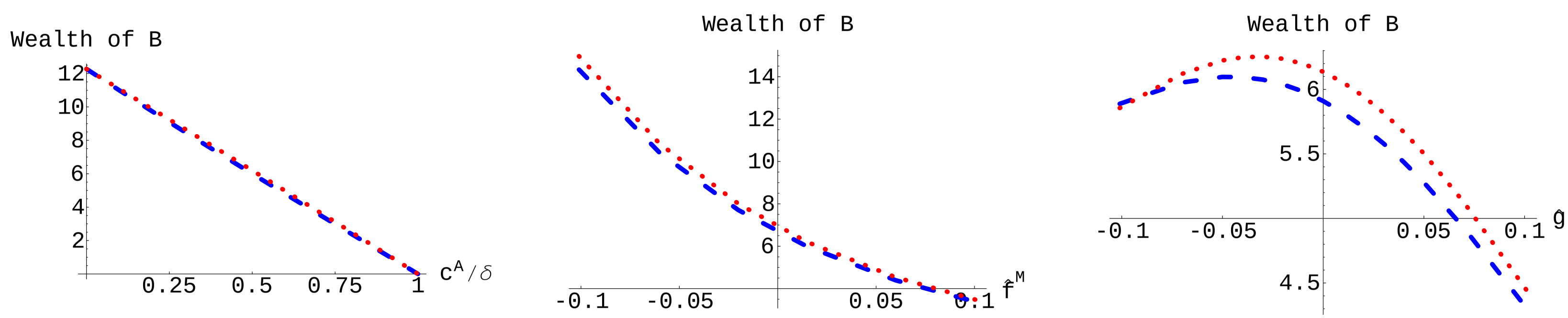


Figure 4: Portfolio Weights for Rational Investors (Population B)

This figure gives Population B's portfolio holding (expressed as a percentage of B's wealth). In all the plots, the variable on the x-axis is the dispersion in beliefs, $\widehat{g}$. There are two columns of plots, with the one on the left giving the position in equity and the one on the right giving the position in bonds. The first *row* gives the overall investment in equity and in bonds. The second row gives the static (mean-variance or myopic) investment. The third row gives the dynamic (intertemporal) hedging component of the portfolio. The dotted line in each plot represents the case where $\phi = 0$ and all agents are rational, while the dashed line represents the case where $\phi = 0.95$ implying that Population A is irrational. For parameter values and benchmark values of the state variables, see Table 1.

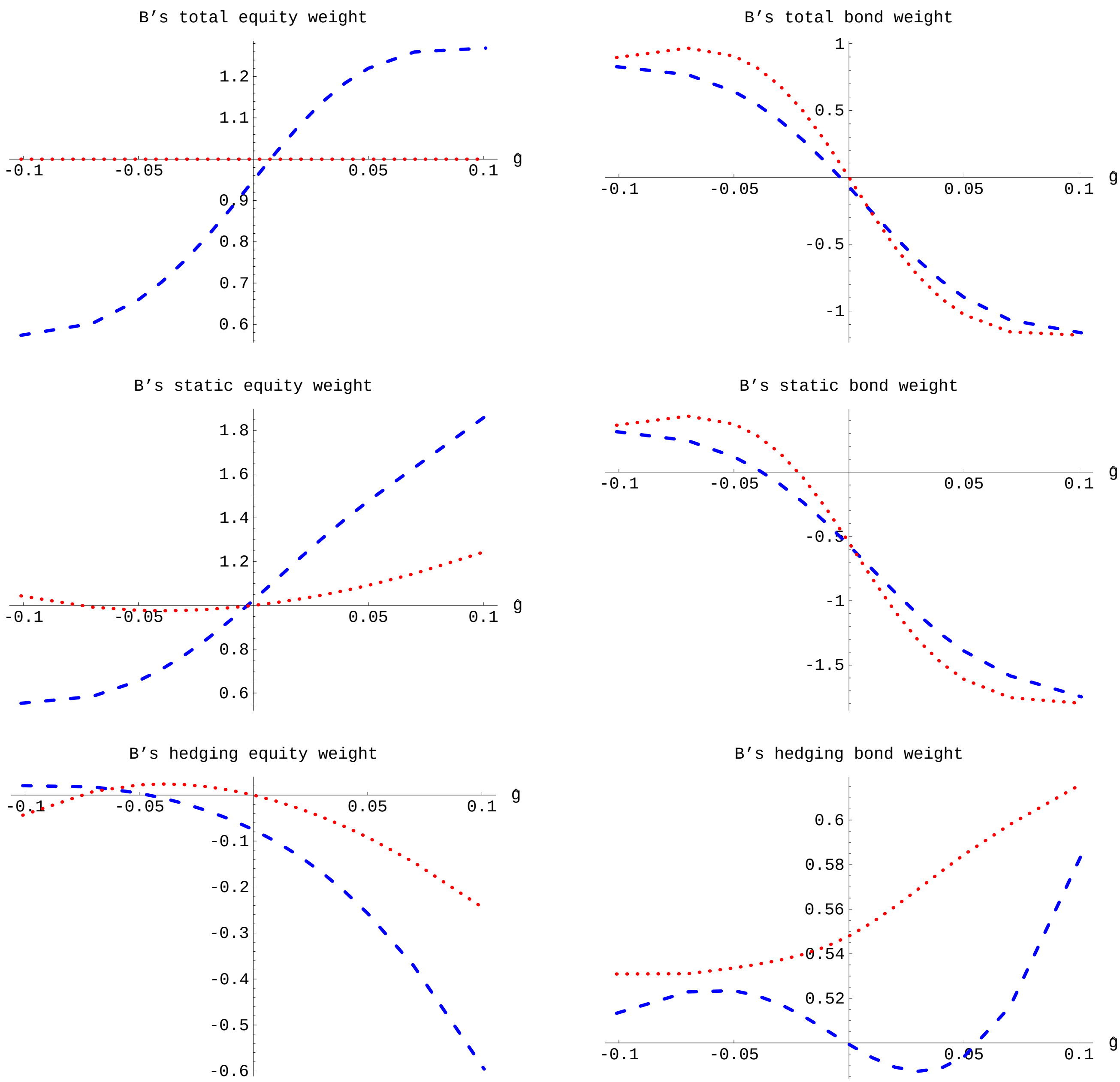
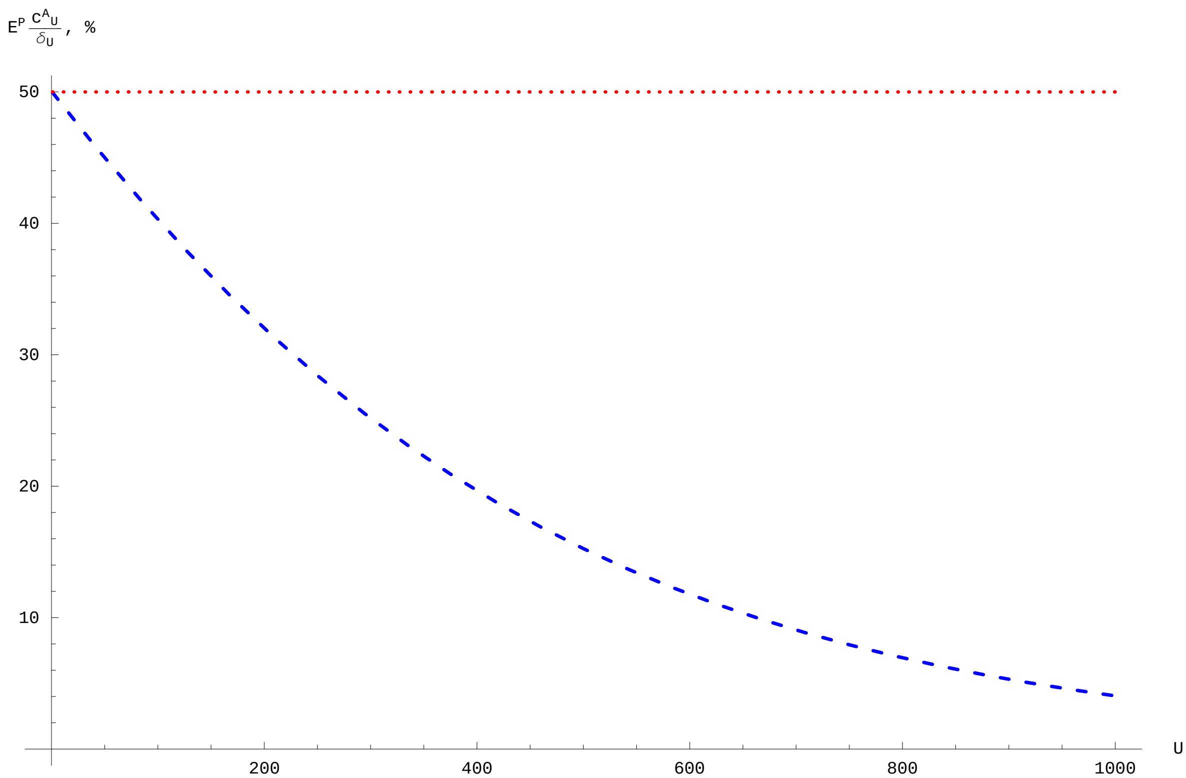


Figure 5: Survival of Population A

This figure shows the expected value of Population A's consumption share (percentage of the total dividends consumed by Group A) as a function of time measured in years, where current time is assumed to be 0 and the future time is denoted on the x-axis by u. The dotted line represents the case where $\phi = 0$ and all agents are rational, while the dashed line represents the case where $\phi = 0.95$ implying that Population A is irrational. For parameter values and benchmark values of the state variables, see Table 1.



References

Abel, A. B., 1990, "Asset Prices under Habit Formation and Catching Up with the Joneses," *American Economic Review*, 80, 38–42.

Alchian, A., 1950, "Uncertainty, Evolution and Economic Theory," *The Journal of Political Economy*, 58, 211–221.

Allen, F., S. Morris, and H. S. Shin, 2004, "Beauty Contests and Iterated Expectations in Asset Markets," Working paper, University of Pennsylvania.

Bansal, R., and A. Yaron, 2004, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59, 1481–151.

Barsky, R. B., and J. B. De Long, 1993, "Why Does the Stock Market Fluctuate?," *Quarterly Journal of Economics*, 108, 291–311.

Basak, S., 2004, "Asset Pricing with Heterogeneous Beliefs," forthcoming in *Journal of Banking and Finance*.

Berrada, T., 2004, "Bounded Rationality and Asset Pricing," Working paper, HEC Montreal.

Bhamra, H. S., and R. Uppal, 2005, "The Effect of Improved Risk Sharing on Stock Market Volatility," Working paper, London Business School.

Biais, B., and P. Bossaerts, 1998, "Asset Prices and Trading Volume in a Beauty Contest," *Review of Economic Studies*, 65, 307–40.

Brennan, M. J., and Y. Xia, 2001, "Stock Price Volatility and Equity Premium," *Journal of Monetary Economics*, 47, 249–83.

Bullard, J., and J. Duffy, 1998, "Learning and Excess Volatility," Working paper, Federal Reserve Bank of Saint-Louis.

Buraschi, A., and A. Jiltsov, 2002, "Option volume and difference in beliefs," Working paper, London Business School.

Campbell, J. Y., and J. H. Cochrane, 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107, 205–51.

Campbell, J. Y., and R. J. Shiller, 1988a, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1, 195–227.

Campbell, J. Y., and R. J. Shiller, 1988b, "Stock Prices, Earnings, and Expected Dividends," *Journal of Finance*, 43, 661–76.

Cao, H. H., and H. Ou-Yang, 2005, "Bubbles and Panics in a Frictionless Market with Heterogeneous Expectations," Working paper, Cheung Kong Graduate School of Business.

Cecchetti, S. G., P.-s. Lam, and N. C. Mark, 2000, "Asset Pricing with Distorted Beliefs: Are Equity Returns Too Good to Be True?," *American Economic Review*, 90, 787–805.

Citanna, A., and K. Schmedders, 2002, "Controlling Price Volatility through Financial Innovation," Working paper, Northwestern University.

Cochrane, J. H., 1991, "Volatility Tests and Efficient Markets: Review Essay," *Journal of Monetary Economics*, 27, 463–85.

Cochrane, J. H., 2001, *Asset Pricing*, Princeton University Press.

Constantinides, G. M., 1990, "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 519–43.

Cox, J. C., and C.-f. Huang, 1989, "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

David, A., 2004, "Heterogeneous Beliefs, Trading Risk, and the Equity Risk Premium," Working paper, Washington University.

De Long, J. B., and M. Becht, 1998, ""Excess Volatility" in the German Stock Market, 1876-1990," Working paper, Department of Economics, Harvard University.

De Long, J. B., and R. Grossman, 1998, ""Excess Volatility" in the London Stock Market, 1870-1990," Working paper, Department of Economics, Harvard University.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann, 1990a, "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98, 703–38.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann, 1990b, "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance*, 45, 379–95.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann, 1991, "The Survival of Noise Traders in Financial Markets," *Journal of Business*, 64, 1–20.

Detemple, J., and S. Murthy, 1994, "Intertemporal Asset Pricing with Heterogenous Beliefs," *Journal of Economic Theory*, 62, 294–320.

Duffie, D., N. Garleanu, and L. H. Pedersen, 2002, "Securities Lending, Shorting, and Pricing," *Journal of Financial Economics*, 66, 307–39.

Epstein, L., and S. Zin, 1989, "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

Feldman, D., 2005, "Incomplete Information Equilibria: Separation Theorems and Other Myths," Working paper, The University of New South Wales.

Flavin, M. A., 1983, "Excess Volatility in the Financial Markets: A Reassessment of the Empirical Evidence," *Journal of Political Economy*, 91, 929–56.

Friedman, M., 1953, *Essays in Positive Economics*, University of Chicago Press, Chicago, IL.

Froot, K. A., and J. Frankel, 1989, "Forward Discount Bias: Is it an Exchange Risk Premium?," *Quarterly Journal of Economics*, 104, 139–161.

Grossman, S. J., and J. E. Stiglitz, 1980, "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 393–408.

Hansen, L. P., J. C. Heaton, and N. Li, 2005, "Consumption Strikes Back? Measuring Long Run Risk," Working paper, University of Chicago.

Harris, M., and A. Raviv, 1993, "Differences of Opinion Make a Horse Race," *Review of Financial Studies*, 6, 473–506.

Hellwig, M. F., 1980, "On the Aggregation of Information in Complete Markets," *Journal of Economic Theory*, 26, 279–312.

Heston, S. L., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327–43.

Kandel, E., and N. D. Pearson, 1995, "Differential Interpretation of Public Signals and Trade in Speculative Markets," *The Journal of Political Economy*, 103, 831–872.

Karatzas, I., J. Lehoczky, and S. Shreve, 1987, "Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon," *SIAM Journal of Control and Optimization*, 25, 1157–1186.

Kim, T. S., and E. Omberg, 1996, "Dynamic Nonmyopic Portfolio Behavior," *Review of Financial Studies*, 9, 141–61.

Kleidon, A. W., 1986, "Variance Bounds Tests and Stock Price Valuation Models," *Journal of Political Economy*, 94, 953–1001.

Kogan, L., S. Ross, J. Wang, and M. Westerfield, 2003, "The Price Impact and Survival of Irrational Traders," Working paper, MIT.

LeRoy, S. F., and R. D. Porter, 1981, "The Present-Value Relation: Tests Based on Implied Variance Bounds," *Econometrica*, 49, 555–74.

Lipster, R. S., and A. N. Shiryaev, 2001, *Statistics of Random Processes II, Applications*, Springer Verlag, second edn.

Loewenstein, M., and G. A. Willard, 2005, "The Limits of Investor Behavior," Working paper, University of Maryland.

Mankiw, N. G., D. Romer, and M. D. Shapiro, 1985, "An Unbiased Reexamination of Stock Market Volatility," *Journal of Finance*, 40, 677–87.

Mankiw, N. G., D. Romer, and M. D. Shapiro, 1991, "Stock Market Forecastability and Volatility: A Statistical Appraisal," *Review of Economic Studies*, 58, 455–77.

Menzly, L., T. Santos, and P. Veronesi, 2004, "Understanding Predictability," *Journal of Political Economy*, 112, 1–47.

Merton, R. C., 1971, "Optimum consumption and portfolio rules in a continuous time model," *Journal of Economic Theory*, 3, 373–413.

Sandroni, A., 2000, "Do Markets Favor Agents Able to Make Accurate Predictions?," *Econometrica*, 68, 1303–1341.

Scheinkman, J. A., and W. Xiong, 2003, "Overconfidence and Speculative Bubbles," *Journal of Political Economy*, 111, 1183–1219.

Shiller, R. J., 1981, "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?," *American Economic Review*, 71, 421–36.

Timmermann, A., 1996, "Excess Volatility and Predictability of Stock Prices in Autoregressive Dividend Models with Learning," *Review of Economic Studies*, 63, 523–57.

Timmermann, A. G., 1993, "How Learning in Financial Markets Generates Excess Volatility and Predictability in Stock Prices," *Quarterly Journal of Economics*, 108, 1135–45.

Wang, J., 1993, "A Model of Intertemporal Asset Prices under Asymmetric Information," *Review of Economic Studies*, 60, 249–82.

West, K. D., 1988a, "Bubbles, Fads and Stock Price Volatility Tests: A Partial Evaluation," *Journal of Finance*, 43, 639–56.

West, K. D., 1988b, "Dividend Innovations and Stock Price Volatility," *Econometrica*, 56, 37–61.

Williams, J. T., 1977, "Capital Asset Prices with Heterogeneous Beliefs," *Journal of Financial Economics*, 5, 219–39.

Yan, H., 2004, "Natural Selection in Financial Markets: Does It Work?," Working paper, London Business School.

Ziegler, A., 2000, "Optimal Portfolio Choice under Heterogeneous Beliefs," *European Finance Review*, 4, 1–19.