The Digital Economy, GDP and Consumer Welfare

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Abstract

There are significant recent concerns regarding whether or not the benefits of the Digital Economy are being appropriately measured. A feature of the Digital Economy is the proliferation of new and "free" goods and services. We derive explicit terms for the contributions of a new and free goods (or services) to welfare change. We also derive a lower bound on the currently unmeasured marginal additions to real GDP growth that arise from new and free goods, giving a means by which the underestimation of GDP growth can be quantified.

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1. Introduction

There is a significant on-going debate regarding the potential of the Digital Economy to generate productivity, economic growth and welfare gains; see e.g. Gordon (2016) and Cowen (2011) versus Sichel (2016), Mokyr, Vickers and Ziebarth (2015) and Brynjolfsson and McAfee (2011). There is an accompanying debate around whether or not current economic measurement by national statistical offices is appropriately capturing the benefits of the Digital Economy; see, for example, Groshen *et al*. (2017), Hulten and Nakamura (2017), Ahmad and Schreyer (2016), Byrne, Fernald and Reinsdorf (2016), Syverson (2016), Brynjolfsson and Oh (2012) and Greenstein and McDevitt (2009). If measurement is lacking, through methodological challenges, statistical agency budgets or data availability, then we are severely hampered in our ability to understand the impact of new technologies, goods and services on the economy, and consequently the prospects for future productivity, economic growth and welfare.

In this paper we develop a new framework for measuring welfare change and real GDP growth in the presence of new and "free" goods (and services). The increased proliferation of such goods is often used to characterize the nature of the Digital Economy; new, sometimes very specialized, goods appear with increasing rapidity, and "free" goods (such as information and entertainment services) are becoming part of daily consumption for many. Free goods often have an implicit price.[1] This price is not usually observed so a price of zero is applied. Thus the positive quantities of these goods that are consumed have a zero measured value and hence are not measured in standard statistical agency collections.

Our framework provides a means by which to understand the potential mismeasurement that arises from not fully accounting for these goods. We use this framework to derive an explicit term that is the marginal value of a new good on welfare change, providing a means for estimating welfare change mismeasurement if the good is omitted from statistical agency collections.

[1] See Nakamura, Samuels and Soloveichik (2016) for examples of how to think about the valuation of "free" media.

A problem in assessing the full impact of the introduction of a new good on real GDP growth is that we would really need national statistical offices to recalculate their estimates of real GDP including the new goods with, for example, estimated Hicksian shadow prices (Hicks 1940; Diewert 1980; Hausman 1981, 1996; Feenstra, 1994), for the period before they are sold in positive quantities. However, we are able to use our framework to derive a lower bound on the addition to real GDP growth from the introduction of a new good, without having to recalculate GDP numbers published by national statistical offices.

Free goods are addressed through generalizing the standard microeconomic model of household cost minimization. It is then possible to re-work our welfare change and real GDP growth adjustments terms to allow for there to be free goods.

The rest of the paper is organised as follows. The next section sets out some preliminary definitions that will be used in the subsequent sections. Section 3 looks at the problem of new goods, and shows how the impact of new goods on welfare change and real GDP growth can be estimated to a high degree of approximation. Section 4 extends this framework to the case of free goods (and services), and section 5 concludes.

2. Preliminaries

We assume that a consumer has a utility function, f(q), which is continuous, quasiconcave and increasing in the components of the nonnegative quantity vector $q \geq 0_N$. For each strictly positive price vector $p >> 0_N$ and each utility level u in the range of f, we can define the dual cost function C as follows:

(1) $C(u,p) \equiv \min_q \{p\cdot q \; ; f(q) \geq u\}$.

We are given the price and quantity data, (p^t, q^t) for periods t = 0,1. We assume that the consumer minimizes the cost of achieving the utility level $u^t \equiv f(q^t)$ for t = 0,1 so observed expenditure in each period is equal to the minimum cost of achieving the given utility level in each period; i.e., we have

(2) $p^t \cdot q^t = C(f(q^t),p^t)$ for t = 0,1.

Valid measures of utility change over the two periods under consideration are the following Hicksian *equivalent and compensating variations* (Hicks, 1942):

(3) $Q_E(q^0,q^1,p^0) \equiv C(f(q^1),p^0) - C(f(q^0),p^0)$;
(4) $Q_C(q^0,q^1,p^1) \equiv C(f(q^1),p^1) - C(f(q^0),p^1)$.

The above variations are special cases of the following Samuelson (1974) family of quantity variations: for $p >> 0_N$, define:[2]

(5) $Q_S(q^0,q^1,p) \equiv C(f(q^1),p) - C(f(q^0),p)$.

Hence there is an entire family of cardinal measures of utility change defined by (5), with one measure for each reference price vector p.

The variations defined by (3) and (4) are not observable (since $C(f(q^1),p^0)$ and $C(f(q^0),p^1)$ are not observable) but the following Laspeyres and Paasche variations, V_L and V_P, are observable:

(6) $V_L(p^0,p^1,q^0,q^1) \equiv p^0 \cdot (q^1 - q^0)$;
(7) $V_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot (q^1 - q^0)$.

Note that V_L and V_P are difference counterparts to the Laspeyres and Paasche quantity indexes, $Q_L = p^0 \cdot q^1 / p^0 \cdot q^0$ and $Q_P = p^1 \cdot q^1 / p^1 \cdot q^0$, respectively. Hicks (1942) showed that V_L

[2] These measures of overall quantity change are difference counterparts to the family of Allen quantity indexes in normal ratio index number theory. The Allen quantity index for reference price vector p is defined as the ratio $C(f(q^1),p)/C(f(q^0),p)$.

approximates Q_E and V_P approximates Q_C to the accuracy of a first order Taylor series approximation; see also Diewert and Mizobuchi (2009; 345-346). The observable Bennet (1920) variation or indicator of quantity change V_B is defined as the arithmetic average of the Laspeyres and Paasche variations in (6) and (7):

$$(8)\ V_B(p^0,p^1,q^0,q^1) \equiv ½(p^0 + p^1)\cdot(q^1 - q^0)$$
$$= p^0\cdot(q^1 - q^0) + ½(p^1 - p^0)\cdot(q^1 - q^0)$$
$$= V_L(p^0,p^1,q^0,q^1) + ½\ \Sigma_{n=1}^N\ (p_n^1 - p_n^0)(q_n^1 - q_n^0).$$

Thus the Bennet variation is equal to the Laspeyres variation $V_L(p^0,p^1,q^0,q^1)$ plus a sum of N Harberger (1971) consumer surplus triangles of the form $(1/2)(p_n^1 - p_n^0)(q_n^1 - q_n^0)$.

An alternative decomposition of the Bennet variation is the following one:

$$(9)\ V_B(p^0,p^1,q^0,q^1) \equiv ½(p^0 + p^1)\cdot(q^1 - q^0)$$
$$= p^1\cdot(q^1 - q^0) - ½(p^1 - p^0)\cdot(q^1 - q^0)$$
$$= V_P(p^0,p^1,q^0,q^1) - ½\ \Sigma_{n=1}^N\ (p_n^1 - p_n^0)(q_n^1 - q_n^0).$$

Thus the Bennet variation is also equal to the Paasche variation $V_P(p^0,p^1,q^0,q^1)$ minus a sum of N Harberger consumer surplus triangles of the form $(1/2)(p_n^1 - p_n^0)(q_n^1 - q_n^0)$.

It is possible to relate the observable Bennet variation to a theoretically valid Samuelson variation of the form defined by (5). However, in order to do this, we need to assume a specific functional form for the consumer's cost function, C(u,p). If the cost function has a flexible,[3] translation-homothetic normalized quadratic functional form, then Proposition 1 in Diewert and Mizobuchi (2009; 353) relates the observable Bennet variation, $V_B(p^0,p^1,q^0,q^1)$ defined by (8) or (9) to the unobservable equivalent and compensating variations defined by (3) and (4); i.e., we have the following exact equality:

[3] Diewert (1974) defined a flexible functional form as one that provides a second order approximation to a twice continuously differentiable function at a point.

$$(10)\ V_B(p^0,p^1,q^0,q^1) = ½Q_E(q^0,q^1,p^0) + ½Q_C(q^0,q^1,p^1).$$

That is, with certain assumptions on the functional form for the consumer's cost function (and using normalized price vectors), the observable Bennet variation can be shown to be *exactly equal* to the arithmetic average of the unobservable equivalent and compensating variations.[4] Hence, there is a strong justification from an economic perspective for using the Bennet quantity variation. Also, it has a strong justification from an axiomatic perspective (Diewert, 2005).

Finally, we can note that value change can be decomposed into Bennet quantity and price variations, as follows:

$$(11)\ p^1\cdot q^1 - p^0\cdot q^0 = V_B(p^0,p^1,q^0,q^1) + I_B(p^0,p^1,q^0,q^1),$$

where $V_B(p^0,p^1,q^0,q^1) \equiv ½(p^0 + p^1)\cdot(q^1 - q^0)$ and $I_B(p^0,p^1,q^0,q^1) \equiv ½(q^0 + q^1)\cdot(p^1 - p^0)$. Equation (11) can thus provide a decomposition into quantity and price components for any value change, including a change in nominal GDP.

3. The New Goods Problem

We can now apply the above results to measure the benefits of the introduction of a new good (or service) to a consumer who cannot purchase the good in period 0 but can purchase it in period 1. First, we have to make an additional assumption. We assume that there is a shadow or reference price for the new good in period 0 that will cause the

[4] Normalised prices are needed for this result to be true: "If there is a great deal of general inflation between periods 0 and 1, then the compensating variation will be much larger than the equivalent variation simply due to this general inflation, and an average of these two variations will be difficult to interpret due to the change in the scale of prices. To eliminate the effects of general inflation between the two periods being compared, it will be useful to scale the prices in each period by a fixed basket price index of the form $\alpha \cdot P$, where $\alpha \equiv [\alpha_1, \ldots, \alpha_N] > 0_N$ is a nonnegative, nonzero vector of price weights." Diewert and Mizobuchi (2009, 352-353). They recommend choosing α so that a fixed-base Laspeyres price index is used to deflate nominal prices (footnote 34, page 368).

consumer to consume 0 units of the new good in period 0. This type of assumption dates back to Hicks (1940; 114).[5]

Let the new good be indexed by the subscript 0 and let the N dimensional vectors of period t prices and quantities for the continuing commodities be denoted by p^t and q^t for t = 0,1. The period 1 quantity of commodity 0 purchased during period 1 is also observed and is denoted by q_0^1. The period 0 shadow price for commodity 0 is not observed but we make some sort of estimate for it, denoted as $p_0^{0*} > 0$. The period 0 quantity is observed and is equal to 0; i.e., $q_0^0 = 0$. Thus the price and quantity data (for the N+1 commodities) for period 0 is represented by the 1+N dimensional vectors (p_0^{0*}, p^0) and $(0, q^0)$ and the price and quantity data for period 1 is represented by the 1+N dimensional vectors (p_0^1,p^1) and (q_0^1, q^1). We adapt our first expression for the Bennet variation, (8), to accommodate the extra new commodity. We find that our new Bennet variation is equal to the following expression:

$$
\begin{aligned}
(12)\ V_B([p_0^{0*},p^0], [p_0^1,p^1], [0,q^0], [q_0^1,q^1]) \\
&= ½(p^0 + p^1)\cdot(q^1 - q^0) + ½(p_0^{0*} + p_0^1)(q_0^1 - 0) \\
&= p^0\cdot(q^1 - q^0) + ½(p^1 - p^0)\cdot(q^1 - q^0) + p_0^1(q_0^1 - 0) - ½(p_0^1 - p_0^{0*})\,(q_0^1 - 0) \\
&= p^0\cdot(q^1 - q^0) + ½(p^1 - p^0)\cdot(q^1 - q^0) + p_0^1q_0^1 - ½(p_0^1 - p_0^{0*})q_0^1.
\end{aligned}
$$

Looking at the last equation on the right hand side of (12), we see that the first term, $p^0\cdot(q^1 - q^0)$ is simply the change in consumption valued at the real prices of period 0, a Laspeyres variation as in (6); the second term, $½(p^1 - p^0)\cdot(q^1 - q^0)$, is the sum of the consumer surplus terms associated with the continuing commodities; the next term, $p_0^1q_0^1$, is the value of consumption of the new commodity in period 1, valued at the normalized price for commodity 0 in period 1 (this is the usual price times quantity contribution term to the value of real consumption of the new commodity in period 1 which would be

[5] There is quite a bit of literature on this topic and for alternative approximate welfare gain estimates; see Hausman (1981) (1996) and Feenstra (1994) and the references in these publications. Diewert has been applying the above Hicksian reservation analysis in the ratio context (i.e., in the context of the true cost of living index) for a long time; see Diewert (1980; 498-505), (1987; 378) (1998; 51-54). The weakness in these theories is the difficulty in determining the appropriate reservation prices.

recorded as a contribution to period 1 GDP); and the last term, $-\frac{1}{2}(p_0^1 - p_0^{0*})q_0^1 = \frac{1}{2}(p_0^{0*} - p_0^1)q_0^1$ is the additional consumer surplus contribution of commodity 0 to overall welfare change, which would not be recorded as a contribution to GDP. Note that the first two terms are a measure of welfare change we would get by just ignoring the new commodity in both periods. Thus the last two terms give the overall contribution to welfare change due to the introduction of the new commodity.

If we assume that the reservation price for the new commodity in period 0, p_0^{0*}, is equal to the observable price for the new commodity in period 1, p_0^1, then the last term in (12), the consumer surplus term for the new commodity, vanishes. However, it is likely that the reservation price for period 0, p_0^{0*}, is much higher than the corresponding actual price for commodity 0 in period 1, p_0^1.[6] Thus if we assume that $p_0^{0*} = p_0^1$ and evaluate (12), then the downward bias in the resulting Bennet measure of welfare change will be equal to a Harberger-type triangle, $-\frac{1}{2}(p_0^1 - p_0^{0*})(q_0^1 - 0) = \frac{1}{2}(p_0^{0*} - p_0^1)q_0^1$.

It is of interest to gauge the extent to which GDP growth is underestimated by not fully capturing the introduction of the new good. Diewert (2005; 335) showed that value change can be expressed as follows:

$$(13)\ p^1 \cdot q^1 - p^0 \cdot q^0 = p^0 \cdot q^0\ [\tfrac{1}{2}(1+Q)(P-1) + \tfrac{1}{2}(1+P)(Q-1)],$$

where P and Q are price and quantity indexes, respectively, that satisfy $P \times Q = p^1 \cdot q^1 / p^0 \cdot q^0$.[7] We can see that (13) can be decomposed into two components, a price change indicator, I_E, and a quantity change indicator, V_E:[8]

$$(14)\ I_E = \tfrac{1}{2}\ p^0 \cdot q^0\ (1+Q)(P-1);$$

[6] Hausman (1996) found that for cereals, the reservation price was about twice the price at the introduction of the new commodity.

[7] That is, the formulae for the indexes P and Q are such that the product test from the axiomatic approach to index numbers is satisfied.

[8] Diewert (2005; 333-337) derived these indicators in introducing the economic approach to indicators of price and quantity change, and called them “economic” indicators. Hence, the subscript “E” in I_E and V_E stands for “economic”.

(15) $V_E = ½\, p^0 \cdot q^0 (1+P)(Q - 1)$

If P and Q in (14) and (15) are replaced by superlative indexes,[9] such as the Fisher or Törnqvist, then the resulting indicators can also be called superlative. A corollary of Proposition 9 of Diewert (2005; 338) is that the Bennet indicator of quantity change, V_B, approximates any superlative indicator to the second order at any point where the two quantity vectors are equal (i.e., $q^0 = q^1$) and where the two price vectors are equal (i.e., $p^0 = p^1$).

The U.S. uses the superlative Fisher quantity index (the geometric mean of the Laspeyres and Paasche indexes given in section 2) for constructing real GDP, so we consider the following expression for the Fisher superlative quantity change indicator, V_E^F:

(16) $V_E^F \equiv ½\, p^0 \cdot q^0 (1+P^F)(Q^F - 1) \approx ½(p^0 + p^1)\cdot(q^1 - q^0) = V_B$,

where $P^F \equiv [(p^1 \cdot q^0/p^0 \cdot q^0)(p^1 \cdot q^1/p^0 \cdot q^1)]^{1/2}$ is the Fisher price index, or GDP deflator in our context, and $Q^F \equiv [(p^0 \cdot q^1/p^0 \cdot q^0)(p^1 \cdot q^1/p^1 \cdot q^0)]^{1/2}$ is the Fisher quantity index, or real GDP growth in our context.[10] Recall that the Bennet indicator of quantity change, V_B, is the symmetric arithmetic average of first-order approximations to the Hicksian equivalent and compensating variations of equations (3) and (4). Alternatively, under the Diewert-Mizobuchi (2009) assumptions on the functional form for the consumer's cost function, V_B is exactly equal to the arithmetic average of the equivalent and compensating variations. Hence, the Fisher superlative quantity change indicator, V_E^F in (16), can be interpreted as an approximation to a welfare change indicator, V_B.

Re-arranging (16), we get an expression for an approximation to the Fisher quantity index:

[9] See Diewert (1976) on superlative index numbers.

[10] If real GDP growth is not constructed using a superlative index such as the Fisher, but rather e.g. using a Laspeyres index as is standard in many countries, there will still be an approximation as in (16), but it may not be as accurate.

(17) $Q^F \approx [(p^0 + p^1)\cdot(q^1 - q^0)]/[\, p^0\cdot q^0\,(1+P^F)] +1$

Note that the numerator is two times the Bennet variation, V_B. From (12) we have the following:

$$(18)\ 2V_B = (p^0 + p^1)\cdot(q^1 - q^0)$$
$$= 2\, p^0\cdot(q^1 - q^0) + (p^1 - p^0)\cdot(q^1 - q^0) + 2p_0{}^1q_0{}^1 - (p_0{}^1 - p_0{}^{0*})q_0{}^1$$

Then substitute the second line of (18) into (17). If Q^F omits the new good in period 0, and we assume that P^F (the aggregate GDP deflator between adjacent periods) is unaffected by the introduction of the new good, then the (approximate) amount missing from Q^F is $(p_0{}^{0*} - p_0{}^1)q_0{}^1/[p^0\cdot q^0\,(1+P^F)]$, which can simply be added to Q^F if $p_0{}^{0*}$ is known or can be estimated. Other things constant, P^F will typically fall (very slightly) with the inclusion of the new good, so this is a lower bound on the amount to add. If this fall is negligible, then real GDP growth can be adjusted, to a second-order approximation, for not fully capturing the introduction of a new good as follows:

$$(19)\ Q^A = Q^F + (p_0{}^{0*} - p_0{}^1)q_0{}^1/[p^0\cdot q^0\,(1+P^F)]$$

Hence, Q^A provides a lower bound on real GDP growth adjusted for the introduction of a new good.

4. The “Free” Goods Problem

Consider a household whose preferences over N market goods and services and M commodities that are available to the household with no visible charge can be represented by the utility function f(x,z) where $x \geq 0_N$ and $z \geq 0_M$ are vectors which represent the consumption of market commodities and of free commodities respectively. We assume that f(x,z) is defined over the nonnegative orthant in R^{N+M} and has the following properties: (i) continuity, (ii) quasiconcave in x and y and (iii) f(x,z) is increasing if all components of x increase and increasing if all components of z increase.

We define two cost or expenditure functions that are dual to f. The first cost function is the consumer's *regular cost function*, C(u,p,w), that is the solution to the following cost minimization problem which assumes (hypothetically) that the household faces positive prices for market and free goods and services so that $p >> 0_N$ and $w >> 0_M$ in (1):[11]

(20) $C(u,p,w) \equiv \min_{x,z} \{p\cdot x + w\cdot z: f(x,z) \geq u, x \geq 0_N, z \geq 0_M\}$.

We also define the household's *conditional cost function*, c(u,p,z), which is the solution to the cost minimization problem defined by (21) below where the household minimizes the cost of market goods and services needed to achieve utility level u, conditional on having the vector $z \geq 0_M$ of free goods and services at its disposal:

(21) $c(u,p,z) \equiv \min_{x} \{p\cdot x: f(x,z) \geq u, x \geq 0_N\}$.

It can be shown (using feasibility arguments) that c(u,p,z) has the following properties where $u \in$ Range f, $p >> 0_N$, and $z \geq 0_M$: (i) for fixed u and z, c(u,p,z) is nonnegative and linearly homogeneous, concave and nondecreasing in p and (ii) for fixed u and p, c(u,p,z) is nonincreasing and convex in z. If in addition, f(x,z) is linearly homogeneous in x and z (the homothetic preferences case), then c(u,p,z) is linearly homogeneous in u,z for fixed p.

If the household faced positive prices $w >> 0_M$ for its "free" goods and services, then the regular cost function minimization problem defined by (20) could be decomposed into a two stage minimization problem using the conditional cost function c; i.e., we have, using definition (20):

(22) $C(u,p,w) \equiv \min_{x,z} \{p\cdot x + w\cdot z: f(x,z) \geq u; x \geq 0_N, z \geq 0_M\}$

$= \min_{z} \{c(u,p,z) + w\cdot z: z \geq 0_M\}$.

[11] We assume u is in the range of f(x,z).

Suppose $z^* \geq 0_M$ solves the cost minimization problem that is defined in the second line of (22) and suppose further that $c(u,p,z^*)$ is differentiable with respect to the components of z at $z = z^*$. Then the first order necessary conditions for z^* to solve the cost minimization problem imply that the following first order conditions hold:

$$(23)\ \nabla_z c(u,p,z^*) = -w\ .$$

With $z = z^*$, we can go to the cost minimization problem defined by (21) and find an x solution which we denote by x^*; i.e., x^* is a solution to:

$$(24)\ \min_x \{p \cdot x: f(x,z^*) \geq u,\ x \geq 0_N\}.$$

It can be seen that (x^*,z^*) is a solution to the regular cost minimization problem defined by (20) so that:

$$(25)\ C(u,p,w) \equiv \min_{x,z} \{p \cdot x + w \cdot z: f(x,z) \geq u,\ x \geq 0_N,\ z \geq 0_M\}$$
$$= p \cdot x^* + w \cdot z^*.$$

Thus the imputed marginal valuation prices $w \equiv -\nabla_z c(u,p,z^*) \geq 0_M$ are appropriate prices to use when valuing the services of free goods in order to construct cost of living indexes or measures of money metric utility change.

Note that due to the fact that c(u,p,z) is decreasing and convex in the components of z, the marginal price for an additional unit of $z_{m,}$, $w_m(u,p,z) \equiv -\partial c(u,p,z)/\partial z_m$, will be nonincreasing in z_m; i.e., it will usually decrease as we add extra units of z_m to the household's holdings of free goods and services.

We define "global" willingness to pay measures for free goods using the conditional cost function. Consider a household that holds no free goods, has utility $u^* = f(x^*,0_M)$ where x^* is the observed market goods consumption vector and the household faces the vector of market goods prices p. We assume that the household minimizes the market cost of achieving its utility level so that $Y^* \equiv p \cdot x^* = c(u^*,p,0_M)$. Now suppose that the household acquires the vector of free goods $z^* > 0_M$. Since $c(u^*,p,z)$ is decreasing in z, the amount of income Y^{**} that the household would require to attain the same level of utility u^* is reduced to $Y^{**} \equiv c(u^*,p,z^*) < c(u^*,p,0_M) = Y^*$. Thus in theory, the consumer should be willing to pay $Y^* - Y^{**}$ to acquire the bundle of free goods z. Thus define the "global" *willingness to pay function* for the acquisition of z^* as follows:

$$(26)\ W_P(u^*,p,z^*) \equiv c(u^*,p,0_M) - c(u^*,p,z^*).$$

If the household holds the amount $z^{**} > 0_M$ of free goods and services, then we can develop an analogous willingness to sell measure as follows. Let x^{**} denote the household's observed market goods consumption vector and we again assume that the household faces the vector of market goods prices p. Let $u^{**} \equiv f(x^{**},z^{**})$. We assume that the household minimizes the market cost of achieving its utility level u^{**} so that $p \cdot x^{**} = c(u^{**},p,z^{**})$. Now suppose that the household disposes of its vector of free goods z^{**}. The amount of income that the household would require to attain the same level of utility u^{**} is increased to $c(u^{**},p,0_M) > c(u^{**},z^{**})$. Thus in theory, the consumer should be willing to sell its free goods for the amount $c(u^{**},p,0_M) - c(u^{**},z^{**})$. Thus define the *"global" willingness to sell function* for the disposal of z^{**} as follows:

$$(27)\ W_S(u^{**},p,z^{**}) \equiv c(u^{**},p,0_M) - c(u^{**},p,z^{**}).$$

For welfare measurement purposes, it is useful to define *marginal* willingness to sell functions. Thus let e_m be a unit vector of dimension M with a 1 in component m and zeros elsewhere for m = 1,...,M. Assume that the household holds $z \geq 1_M$ units of the free goods and services, faces market prices p, has $x > 0_N$ units of market goods and services and $p \cdot x = c(u,p,z)$ where $u = f(x,z)$. Define the *m^{th} marginal willingness to sell function*, $W_m(u,p,z)$ as follows:

(28) $W_m(u,p,z) \equiv c(u,p,z-e_m) - c(u,p,z)$; $m = 1,...,M$.

Presumably, survey, experimental or indirect methods could be used in order to obtain approximate measures for these marginal willingness to sell functions. Let $W(u,p,z)$ denote the vector $[W_1(u,p,z),...,W_M(u,p,z)]$. It can be seen that $W(u,p,z)$ is a discrete approximation to the marginal valuation price vector $w \equiv -\nabla_z c(u,p,z)$ that was defined earlier by (23).[12]

Assuming that we have valuations for the free goods, we can extend the Bennet welfare change variation of (12) to include these goods. Following the set up for regular goods in the previous section, let a new "free" good be indexed by the subscript 0 and let the N dimensional vectors of period t prices and quantities for the continuing commodities be denoted by w^t and z^t for $t = 0,1$. The period 1 quantity of commodity 0 purchased during period 1 is also observed and is denoted by z_0^1. The period 0 shadow price for commodity 0 is not observed but we make some sort of estimate for it, denoted as $w_0^{0*} > 0$. The period 0 quantity is observed and is equal to 0; i.e., $z_0^0 = 0$. Thus the price and quantity data (for the N+1 commodities) for period 0 is represented by the 1+N dimensional vectors (w_0^{0*}, w^0) and $(0, z^0)$ and the price and quantity data for period 1 is represented by the 1+N dimensional vectors (w_0^1,w^1) and (z_0^1, z^1).

Then, in an extension of (12), welfare change including both new and free goods can be written as follows:

$$\text{(29) } V_B = p^0\cdot(q^1 - q^0) + \tfrac{1}{2}(p^1 - p^0)\cdot(q^1 - q^0) + p_0^1 q_0^1 - \tfrac{1}{2}(p_0^1 - p_0^{0*})q_0^1$$
$$+ w^0\cdot(z^1 - z^0) + \tfrac{1}{2}(w^1 - w^0)\cdot(z^1 - z^0) + w_0^1 z_0^1 - \tfrac{1}{2}(w_0^1 - w_0^{0*})z_0^1,$$

where the second line gives the contribution of the continuing and entering "free" goods.

[12] If $z_m = 0$, then we need to change the definition of $W_m(u,p,z) \equiv c(u,p,z-e_m) - c(u,p,z)$ to the corresponding marginal willingness to pay function, $W_m^*(u,p,z) \equiv c(u,p,z) - c(u,p,z+e_m)$.

If the concern is that real GDP omits the contribution from continuing free goods, then we can use the results of the previous section and re-write (19) to adjust real GDP growth, Q^F, as follows:

$$(30)\ Q_{FG}{}^{A} = Q^F + [2w^0\cdot(z^1 - z^0) + (w^1 - w^0)\cdot(z^1 - z^0) + 2w_0{}^1 z_0{}^1]/[p^0\cdot q^0\,(1+P^F)]$$

Note that this assumes that we are either able to adjust the GDP deflator, P^F, for the price changes in continuing free goods, or that they have negligible impact. As their prices are likely to fall over time, (30) can be interpreted as providing a lower bound on GDP growth adjusted for continuing free goods.

With the same qualification, including both regular and free new goods, we get the following "fully adjusted" real GDP growth, $Q_{Full}{}^{A}$:

$$\begin{aligned}(31)\ Q_{Full}{}^{A} = Q^F &+ (p_0{}^{0*} - p_0{}^1)q_0{}^1/[p^0\cdot q^0\,(1+P^F)] \\ &+ [2w^0\cdot(z^1 - z^0) + (w^1 - w^0)\cdot(z^1 - z^0) + 2w_0{}^1 z_0{}^1]\,/[p^0\cdot q^0\,(1+P^F)] \\ &+ (w_0{}^{0*} - w_0{}^1)z_0{}^1/[p^0\cdot q^0\,(1+P^F)],\end{aligned}$$

where the first line of (31) is the adjustment arising from the entry of a new good, the second line is an additional contribution from accounting for continuing free goods, and the third line is the adjustment term arising from the entry of a free good.[13]

Conclusion

This paper has developed a framework for measuring welfare change and real GDP growth when there are new and "free" goods (and services). This provides a means by which to understand the potential mismeasurement that arises from not fully accounting for these goods. This is of particular current interest, given that the frequent introduction of new goods and the presence of free goods are often used to characterize the modern "Digital Economy".

[13] Obviously, (31) can easily be generalized to the case of multiple new regular and free goods.

Perhaps appropriately, we drew on both old and new literatures to define a framework for measuring welfare change. We were able to use this framework to derive an explicit term that is the marginal value of a new good on welfare change. That is, we get a measure of the contribution to welfare of a new good, and hence the extent of welfare change mismeasurement if it is omitted from statistical agency collections.

We also showed how to work out a lower bound on the addition to real GDP growth from the introduction of a new good, without having to recalculate GDP numbers published by national statistical offices.

We then introduced free goods into a standard microeconomic model of household cost minimization and re-worked our welfare change and real GDP growth adjustments terms to allow for there to be "free" goods (with an implicit or imputable price).

Hence, we have derived explicit adjustments for both welfare change and real GDP growth that account for new and free goods, both of which are new to the literature. These expressions enable a more thorough exploration of the impacts of new and free goods, with significant potential policy implications. As an example, given that real GDP growth is a key component of national productivity growth estimates, to the extent that the adjustments add to GDP growth they may go some way to explaining the much-documented and debated productivity growth slowdown experienced by industrialized countries since 2004.

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