

Deterrence and the Optimal Use of Prison, Parole, and Probation

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Abstract: In this article we derive the sentence—choosing from among the sanctions of prison, parole, and probation—that achieves a target level of deterrence at least cost. Potential offenders discount the future disutility of sanctions and the state discounts the future costs of sanctions. Prison has higher disutility and higher cost per unit time than parole and probation, but the cost of prison *per unit of disutility* can be lower or higher than the cost of parole and probation per unit of disutility. The optimal order of sanctions depends on the relative discount rates of potential offenders and the state, and the optimal duration of sanctions depends on the relative costs per unit of disutility among the sanctions and on the target level of deterrence. We focus on the case in which potential offenders discount the disutility of sanctions at a higher rate than the state discounts the costs of sanctions. In this case, if prison is more cost-effective than parole and probation—that is, has a lower cost per unit of disutility—prison should be used exclusively. If prison is less cost-effective than parole and probation, probation should be used if the deterrence target is low enough, and prison followed by parole should be used if the deterrence target is relatively high. Notably, it may be optimal to employ a prison term even if prison is less cost-effective than parole and probation and even if prison is not needed to achieve the target level of deterrence, because of what we refer to as the front-loading advantage of imprisonment.

Key words: crime; imprisonment; parole; probation; prison costs; deterrence; sanctions

JEL codes: H23; K14; K42

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1. Introduction

Although imprisonment is widely employed in the United States, the use of parole and probation—in which offenders are supervised outside of prisons—is far greater. For example, at the end of 2014, there were 1.56 million individuals incarcerated in state and federal correctional facilities, and 4.71 million individuals subject to parole or probationary supervision.¹ Parole and probation are also common sanctions in many other countries.² It seems surprising, therefore, that while there is a voluminous literature in which the deterrent effect of imprisonment is considered,³ examination of the deterrent effect of parole or probation has been virtually absent.⁴ The contribution of our article is to provide a comprehensive theoretical analysis of the merits of parole and probation as means of achieving desirable deterrence, together with, or as an alternative to, imprisonment. We undertake this task by deriving the sentence—chosen from among the sanctions of prison, parole, and probation—that achieves a target level of deterrence at least cost.⁵

¹ See Carson (2015, p. 1) and Kaeble et al. (2015, p. 1). The vast majority—82 percent—of those supervised outside of prison are on probation (generally without having served any time in prison).

² For example, France had 60,896 individuals incarcerated in 2015 and 143,143 on probation or parole at the end of 2014 (the latest date for which such data are available). At the same time, England and Wales had 85,843 individuals incarcerated in 2015 and 78,521 on probation or parole. See Walmsley (2016, pp. 10-11) and Aebi and Chopin (2016, pp. 18-19, Table 1.1, columns 1.2.1-1.2.3 and 1.2.9). Australia had 39,005 individuals incarcerated and 66,793 in “community-based corrections” in June of 2016. See Australian Bureau of Statistics (2016) (with the qualification that “For the community-based corrections population, offenders may be counted more than once if they have two or more different types of community-based corrections orders operating simultaneously”).

³ See, for example, the survey articles by Levitt and Miles (2007) and Polinsky and Shavell (2007).

⁴ For instance, in the two surveys of the economics of enforcement and criminal punishment mentioned in the preceding footnote, a total of 278 articles are cited, only one of which concerns parole or probation. There are two relevant omissions from these surveys—Miceli (1994) and Garoupa (1997)—that we discuss below in comment (b) following Proposition 1.

⁵ There are other functions that parole can serve that we do not consider here, notably, reducing the cost of incapacitating offenders after it has been determined that they pose a low risk of recidivism, or providing a reward to prisoners who behave well. See, for example, Bernhardt et al. (2012) and Polinsky (2015).

Parole and probation are both forms of out-of-prison supervision of offenders, where parole is added to a prison term and probation is a substitute for a prison term.⁶ While the degree of state supervision need not be the same under parole and probation, nor their costs, we will treat parole and probation as equivalent sanctions per unit time. Thus, their difference in our analysis will be purely semantic—when a period of out-of-prison supervision is combined with a prison term, we will refer to this period as parole, but when such a period is used by itself, we will refer to it as probation.

In our model, potential offenders discount the future disutility of sanctions and the state discounts the future costs of sanctions. Prison imposes higher disutility on offenders and generates higher costs for the state per unit time than do parole and probation, but the cost of prison *per unit of disutility* can be lower or higher than the cost of parole and probation per unit of disutility. We demonstrate that when prison and parole are used together, their optimal order depends on the relative discount rates of potential offenders and the state, and that the optimal mix and duration of sanctions depend on the relative costs per unit of disutility among the sanctions and the desired level of deterrence.

We begin our analysis with the benchmark case in which neither offenders discount disutility nor the state discounts costs. In this case, the optimal sentence is either a prison term or a probation term, depending on whether imprisonment or out-of-prison supervision has the lower cost per unit of disutility.⁷

⁶ For a discussion of the distinction between probation and parole by the Bureau of Justice Statistics, Office of Justice Programs, U.S. Department of Justice, see <<http://www.bjs.gov/index.cfm?ty=qa&iid=324>>. Although parole is not usually referred to as a *sanction*, as prison and probation are, we will, for economy of language, refer to it as such here.

⁷ In the formal statements of results we also consider the case in which imprisonment and out-of-prison supervision have the same cost per unit of disutility, but we will not discuss this case in the introduction.

We then consider the case in which the offender's disutility discount rate and the state's cost discount rate are positive and equal. In this case, the result is similar in spirit to that in the benchmark case: the most cost-effective sanction—the one whose undiscounted cost per unit of disutility is lowest—should be used exclusively or to the greatest extent feasible consistent with attaining the target level of deterrence. Moreover, it does not matter when the sentence begins provided that the deterrence target is achieved. And if a prison term is employed together with a parole term, it is immaterial which sanction is used first. These latter points follow, in essence, from the observation that, if the offender's and the state's discount rates are equal, the discounted cost per unit of disutility of a sanction does not depend on when the sanction is imposed.

The primary case we examine—the one that we believe is most realistic⁸—is that in which potential offenders discount the disutility of sanctions at a higher rate than the state discounts the costs of sanctions. In this case, we demonstrate that, in contrast to the results in the first two cases, even if prison is less cost-effective than probation—has a higher undiscounted cost per unit of disutility—and even if prison is not needed to achieve the deterrence target, it may be optimal to employ a prison term. This result is due to what we will refer to as the *front-loading advantage* of prison sanctions. Specifically, because prison imposes disutility at a higher level per unit time than does probation—prison front-loads disutility—a shorter term can be used to achieve any given level of deterrence. Everything else equal, a shorter term will reduce the deterrence-diluting effect of offender discounting of disutility, which is beneficial. But a shorter term also will reduce the cost-diluting effect of the state's discounting of costs, which is detrimental. When the offender's discount rate exceeds the state's discount rate, the first effect dominates the second, resulting in a net advantage from using imprisonment. It is this

⁸ See note 18 below.

advantage that may make it worthwhile to use imprisonment even when imprisonment has a higher undiscounted cost per unit of disutility than probation.

The preceding point can be illustrated by an example. Suppose that potential offenders have a discount rate of fifteen percent and that the state has a discount rate of two percent. Let prison impose the equivalent of \$30,000 of disutility per year, while probation imposes \$2,500 of disutility per year. Given the offender discount rate, a 0.33 year—four-month—prison term would have the same deterrent effect as a 5.87 year probation term.⁹ Suppose that prison costs the state \$30,000 per person-year and probation costs the state \$2,000 per person-year. Thus, the undiscounted cost of prison per unit of disutility is \$1.00 ($= \$30,000/30,000$), while the undiscounted cost of probation per unit of disutility is \$0.80 ($= \$2,000/2,500$). Despite the fact that prison is twenty-five percent more expensive than probation per unit of disutility, the four-month prison term is preferable to the deterrent-equivalent 5.87 year probation term. The prison term has a present-value cost of \$9,967, while the probation term has a present-value cost of \$11,072. This result is due to the front-loading advantage of imprisonment.

We also demonstrate in the primary case that sanctions should never be delayed (the timing of sanctions would not matter in the first two cases). This is because, if offenders discount disutility at a higher rate than the state discounts costs, any delay in the imposition of a sanction dilutes deterrence faster than it reduces costs.

Additionally, we show in the primary case that if both prison and parole are employed, it is optimal to use prison first (the *order* of sanctions would not matter in the first two cases). The intuition underlying this result is as follows. Suppose for simplicity that the state's discount rate is zero. Consider a parole term followed by a prison term that together achieve a certain level of

deterrence. By reversing the order of the terms, with prison now used first, deterrence will rise because the sanction that produces the higher level of disutility per unit time now occurs earlier—it is diluted less by offender discounting. Given the assumption that the state’s discount rate is zero, the state’s costs are unaffected. Since deterrence is higher, one or both of the terms can be shortened to restore the original level of deterrence. This will result in lower costs, implying that it is preferable to employ prison before parole.

Section 2 presents the basic model employed in the analysis. Section 3 derives the optimal sentence in the benchmark case in which neither offenders discount disutility nor the state discounts costs. Section 4 performs a parallel analysis when the offender’s and the state’s discount rates are positive and equal. Section 5 determines the optimal order of prison and parole and the optimal sentence when the rate at which offenders discount disutility exceeds the rate at which the state discounts costs. Section 6 concludes with several comments, including a discussion of the case when the offender’s discount rate is less than the state’s discount rate. Proofs are presented in the Appendix.

2. The State’s Problem

The offender’s sentence begins at time $t = 0$. At any time $t \geq 0$, the state can impose a prison sanction or an out-of-prison supervision sanction or no sanction at all. We will refer to the last choice as imposing the *null sanction*. The prison sanction inflicts greater disutility on an offender and causes the state to incur higher cost per unit time than does the out-of-prison supervision sanction. The null sanction generates zero disutility and zero cost. The three

⁹ The prison term is assumed to be 1/3 of a year and the calculations are done in continuous time. The numbers reported in the text are rounded to the nearest hundredth of a year and to the nearest dollar.

possible sanctions will be numbered in decreasing order of severity—denoting prison with 1, out-of-prison supervision with 2, and the null sanction with 3. Thus, let

θ_i = disutility of sanction i per unit time, where $\theta_1 > \theta_2 > \theta_3 = 0$; and

c_i = cost of sanction i per unit time, where $c_1 > c_2 > c_3 = 0$.

A *sentence* is a function $\delta(t)$ mapping each non-negative time to a sanction, such that the value of $\delta(t)$ changes only a finite number of times and sanctions are imposed for strictly positive lengths of time. Let \mathcal{A} be the set of sentences. A *term* of sentence $\delta(t)$ is an interval of time such that $\delta(t)$ is constant throughout the interior of the interval but changes at the upper bound of the interval if the interval is finite. Terms are thus the longest periods of time over which a sentence imposes a constant punishment, and this definition corresponds with the usual meaning of, for example, “a term of prison.” By definition, every sentence $\delta(t)$ partitions the set of non-negative times into a finite number of terms.

To conform to common usage, if an out-of-prison supervision term is employed in combination with a prison term it will be referred to as a parole term, whereas if it is used alone, it will be referred to as a probation term.

Potential offenders discount the future disutility of sanctions and the state discounts the future costs of sanctions. Let

r = rate at which potential offenders discount the disutility of sanctions; $r \geq 0$; and

ρ = rate at which the state discounts the cost of sanctions; $\rho \geq 0$.

We assume for analytical convenience that individuals live forever.¹⁰

¹⁰ Since the optimal terms of prison and out-of-prison supervision will be seen to be finite in the case that we focus on in Section 5 (in which offenders discount disutility at a higher rate than the state discounts costs), this assumption does not affect our main results.

The state seeks to achieve a target level of deterrence, which is to say to impose some specified level of discounted disutility on offenders through its choice of sanctions. Let

$$k = \text{target level of deterrence; } k > 0.$$

If the discount rate r of offenders is positive, we assume that k is strictly less than the present value of the disutility that would result from a perpetual prison term, that is, that $k < \theta_1/r$.

Otherwise, if $k = \theta_1/r$, the only sentence that could achieve the target level of deterrence would be a perpetual prison term, or if $k > \theta_1/r$, no sentence could achieve the target level of deterrence.

The state's problem is to choose a sentence $\delta(t)$ from the set of sentences Δ that achieves the target level of deterrence k at the lowest discounted cost.¹¹ Thus, using the notation $c(\theta_i) = c_i$, the state's problem is:

$$\min_{\delta(\cdot) \in \Delta} \int_0^{\infty} c(\delta(t))e^{-\rho t} dt \quad (1)$$

subject to

$$\int_0^{\infty} \delta(t)e^{-rt} dt = k. \quad (2)$$

The state's problem as just described does not impose any restrictions on the number of terms that each sanction can be used (aside from ruling out an infinite number of terms). In the case that we focus on, in which the offender's discount rate exceeds the state's discount rate, we prove in the Appendix that the optimal sentence employs a prison term and an out-of-prison supervision term at most once each. In the other cases, in which the discount rates are both zero or positive and equal, the optimal sentence is not unique, but the set of optimal sentences includes a sentence that uses at most one term of each sanction. Accordingly, in describing our

¹¹ The state's problem could be formulated more generally as maximizing social welfare, defined as the utility of offenders from committing harmful acts, less the harm done, less the cost of catching offenders, and less the social cost of sanctions. Clearly, a necessary condition for a social-welfare maximizing enforcement policy is that the disutility imposed on offenders through sanctions be achieved at the lowest possible cost to the state.

results in the following sections, we will refer to a single prison term and a single out-of-prison supervision term.¹² Let

s_1 = length of prison term; $s_1 \geq 0$; and

s_2 = length of out-of-prison supervision term (parole or probation); $s_2 \geq 0$.

3. Optimal Sentences When Offenders Do Not Discount Disutility and the State Does Not Discount Costs

To provide a benchmark against which to assess the effects of discounting, we begin with the case in which neither offenders discount disutility nor the state discounts costs. In this case the optimal sentence is given by the following proposition.

Proposition 1: If offenders do not discount disutility and the state does not discount costs, $r = \rho = 0$, then for any target level of deterrence $k \in (0, \infty)$, the optimal sentence depends solely on the relative cost per unit of disutility of the sanctions. Specifically,

(a) if $c_1/\theta_1 < c_2/\theta_2$, a prison term $s_1 = k/\theta_1$ starting at any time is optimal;

(b) if $c_1/\theta_1 = c_2/\theta_2$, any mix of sanctions that satisfies the deterrence constraint, $s_1\theta_1 + s_2\theta_2 = k$, starting at any time is optimal; and

(c) if $c_1/\theta_1 > c_2/\theta_2$, a probation term $s_2 = k/\theta_2$ starting at any time is optimal.

Comments: (a) Since there is no discounting of disutility or costs, it is obvious that the sanction that has the lowest undiscounted cost per unit of disutility will be preferred and that any mix of sanctions will be optimal if there is no difference between the sanctions in this regard. Moreover, if there is no discounting, the starting time of the sanctions is immaterial (as is whether the sanction is imposed in one term).

¹² For brevity, when we list the terms of a specific sentence, we generally omit terms of the null sanction.

(b) Miceli (1994) analyzed prison, parole, and probation when, implicitly, the offender's and the state's discount rates are both assumed to be zero, and came to conclusions that are consistent with those stated in Proposition 1.¹³ For a similar analysis, see Garoupa (1997). Neither, however, considered whether the order of sanctions matters or how discounting affects the results.

4. Optimal Sentences When Offenders Discount Disutility at the Same Rate at Which the State Discounts Costs

To better understand the pure effect of discounting on the optimal choice of sanctions, we derive here the optimal sentence when the offender's disutility discount rate and the state's cost discount rate are positive and equal, $r = \rho > 0$. In this case the sanctions used in an optimal sentence depend on their relative cost per unit of disutility¹⁴ and on the target level of deterrence.

Proposition 2: If offenders discount disutility at the same positive rate that the state discounts costs, $r = \rho > 0$, and

(a) if $c_1/\theta_1 < c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, either a finite prison term

$$s_I = (1/r)\ln\{[1 - (rk/\theta_1)e^{rt}]^{-1}\} > 0 \quad (3)$$

starting at any $t \in [0, t_M)$ is optimal, where

$$t_M = (1/r)\ln(\theta_1/rk) > 0, \quad (4)$$

¹³ Central to Miceli's analysis is the use of parole as a reward for good behavior in prison, to lower the state's cost of operating prisons (as previously noted, we do not consider this effect in order to focus on the pure deterrence effects of sanctions). This rationale for parole is required in his analysis for the possible optimality of combining a parole term with a prison term. Otherwise, as in our Proposition 1, either prison alone or probation is optimal when the cost per unit of disutility differs among the sanctions.

¹⁴ We will occasionally, including here, omit the word "undiscounted" when referring to the cost per unit of disutility of a sanction.

or an infinite prison term $s_1 = \infty$ starting at $t = t_M$, is optimal;

(b) if $c_1/\theta_1 = c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, any mix of sanctions starting at any time that satisfies the deterrence constraint is optimal; and

(c) if $c_1/\theta_1 > c_2/\theta_2$, then

(i) for a relatively low target level of deterrence $k \in (0, \theta_2/r)$, either a finite probation term

$$s_2 = (1/r)\ln\{[1 - (rk/\theta_2)e^{rt}]^{-1}\} > 0 \quad (5)$$

starting at any $t \in [0, t_M)$ is optimal, where

$$t_M = (1/r)\ln(\theta_2/rk) > 0, \quad (6)$$

or an infinite probation term $s_2 = \infty$ starting at $t = t_M$, is optimal;

(ii) for an intermediate target level of deterrence $k = \theta_2/r$, an infinite probation term $s_2 = \infty$ starting at $t = 0$ is optimal; and

(iii) for a relatively high target level of deterrence $k \in (\theta_2/r, \theta_1/r)$, either a finite prison term

$$s_1 = (1/r)\ln\{(\theta_1 - \theta_2)/(\theta_1 - rk)\} > 0 \quad (7)$$

starting at $t = 0$ followed immediately by an infinite parole term $s_2 = \infty$, or a finite parole term

$$s_2 = (1/r)\ln\{(\theta_1 - \theta_2)/(rk - \theta_2)\} > 0 \quad (8)$$

starting at $t = 0$ followed immediately by an infinite prison term $s_1 = \infty$, is optimal.

Comments: (a) Although the statement of Proposition 2 is more complicated than that of Proposition 1, the main lesson of Proposition 1—that the sanction with the lower cost per unit of disutility should be relied upon to the greatest extent possible—carries over to the present proposition. This point is a consequence of the offender's disutility discount rate and the state's cost discount rate being the same. Hence, if the sanction with the lowest *undiscounted* cost per

unit of *undiscounted* disutility (c_i/θ_i) is used at each instant in time, the *present value* of the cost divided by the *present value* of the disutility also will be lowest.

(b) When imprisonment has the lower cost per unit of disutility, a prison term can be relied upon exclusively regardless of the target level of deterrence (up to the maximum of θ_1/r), due to prison being the more potent sanction. Moreover, because disutility and costs are discounted at the same rate, it is immaterial when a prison term begins, provided that it can achieve the target level of deterrence (which imposes an upper bound, t_M given by (4), on the commencement of the term). This explains part (a) of Proposition 2.

(c) Now consider the case in which out-of-prison supervision—parole or probation—has a lower cost per unit of disutility than prison. Ideally, a probation term would then be used, but because probation has lower potency than prison, a probation term can achieve the target level of deterrence only if the target is not too high (not exceeding θ_2/r). If the target level of deterrence exceeds the level of deterrence achievable by probation, then some prison time will be required. But when parole has a lower cost per unit of disutility than prison, it will be optimal to minimize the contribution of the prison term to the present value of disutility, subject to achieving the target level of deterrence. This can be accomplished equally well by employing prison first and then switching to parole as soon as possible, while still satisfying the deterrence constraint, or by using parole first and switching to prison only when necessary in order to meet the deterrence constraint. This explains the two alternative, equally desirable, options in part (c) of Proposition 2.

(d) It is obvious from the preceding discussion that when prison and out-of-prison supervision have the same cost per unit of disutility, the choice of sanctions is immaterial

provided that the deterrence target is achieved. Additionally, because disutility and costs are discounted at the same rate, it does not matter when the sanctions commence.

(e) If we relax the assumption that a sentence uses at most one term of each sanction, then when the offender's and the state's discount rates are equal, optimal sentences may include multiple terms of a sanction. In essence, this is because the discounted cost of achieving a specified level of deterrence does not depend in the present case on when the sanctions are imposed.

Numerical example: Suppose that offenders discount disutility and the state discounts costs at five percent per year. As in the example in the introduction, we assume that prison imposes the equivalent of \$30,000 of disutility per year, while parole and probation impose \$2,500 of disutility per year. Prison costs the state \$30,000 per person-year and parole and probation cost the state either \$2,000 per person-year (the value employed in the introduction) or \$3,000 per person-year. Thus, the cost of prison per unit of disutility is \$1.00 ($= 30,000/30,000$), while the cost of parole and probation per unit of disutility is either \$0.80 ($= 2,000/2,500$) or \$1.20 ($= 3,000/2,500$). We will consider target levels of deterrence of \$10,000 and \$100,000. When optimal sentences are not unique, we just discuss the optimal sentence starting at $t = 0$.

If prison is more cost-effective than parole and probation (which occurs if parole and probation cost \$3,000 per person-year), the optimal sentence is a prison term of 0.34 years if the target level of deterrence is \$10,000 and 3.65 years if the target level of deterrence is \$100,000.¹⁵

If prison is less cost-effective than parole and probation (which occurs if parole and probation cost \$2,000 per person-year), the results depend on whether the target level of deterrence is below or above the threshold corresponding to the level of deterrence achievable by

a sentence of lifetime probation. In the present example, this threshold is \$50,000 (= \$2,500/0.05). If the target level of deterrence is \$10,000, below the threshold, the optimal sentence is a 4.46 year probation term.¹⁶ But if the target level of deterrence is \$100,000, some prison time is needed to satisfy the deterrence constraint. Then the optimal sentence is either a prison term of 1.91 years followed by an infinite parole term or a parole term of 47.96 years followed by an infinite prison term.¹⁷ (Obviously, if we had assumed a finite lifetime, the second sanctions in each case would be finite.)

5. Optimal Sentences When Offenders Discount Disutility at a Higher Rate Than the State Discounts Costs

We now turn to what we believe is the most realistic case, when offenders discount disutility at a higher rate than the state discounts costs.¹⁸ We begin by showing that in this case if both prison and parole are employed, it is optimal to use prison first, and we then derive the optimal sentence given the sanctions' relative cost per unit of disutility and the target level of deterrence.

¹⁵ Since a prison term s_1 starting at $t = 0$ generates disutility with a present value of $(\theta_1/r)(1 - e^{-rs_1})$, s_1 can be solved from the deterrence constraint, $(\theta_1/r)(1 - e^{-rs_1}) = k$.

¹⁶ A probation term s_2 starting at $t = 0$ can be solved from $(\theta_2/r)(1 - e^{-rs_2}) = k$.

¹⁷ A prison term s_1 followed by an infinite parole term can be solved from $(\theta_1/r)(1 - e^{-rs_1}) + e^{-rs_1}(\theta_2/r) = k$. Similarly, a parole term s_2 followed by an infinite prison term can be solved from $(\theta_2/r)(1 - e^{-rs_2}) + e^{-rs_2}(\theta_1/r) = k$.

¹⁸ Mastrobuoni and Rivers (2016) estimated criminal discount rates at 30 percent (in their terminology, an annual discount factor of 0.74, which we have converted to a continuous discount rate). Åkerlund et al. (2016) found that individuals with high discount rates—58 percent or higher—were significantly more likely to participate in criminal acts; they concluded that “[o]ur findings are therefore consistent with the idea that criminals have extremely high discount rates.” In the same vein, Lee and McCrary (2017) concluded that their model and data supported the view that “offenders [have] short time horizons, leading them to perceive little difference between nominally long and short incarceration periods.” Studies of public discount rates, in contrast, have found rates generally to be between 3.5 percent and 7 percent. See Burgess and Zerbe (2011) and Moore et al. (2013).

Proposition 3: If offenders discount disutility at a higher rate than the state discounts costs, $r > \rho \geq 0$, then a prison term should be used before a parole term if both sanctions are employed in an optimal sentence.

Comments: (a) This result was explained in the introduction under the assumption that the offender's discount rate is positive and the state's discount rate is zero. Maintaining that assumption, we provide a similar explanation here, but one that tracks more closely the structure of the proof. Consider a prison term of length s_1 and a parole term of length s_2 , and suppose that parole is employed first. If the order of the sanctions is then switched to prison first, the undiscounted cost of the sanctions will be the same. But the present value of the disutility of the sanctions will have increased, because the more potent sanction now occurs earlier and is diluted less by discounting. It would then be possible to shorten the prison term, replacing prison time with parole time, until the level of deterrence is restored to that generated by the original sequence in which parole occurred first.¹⁹ Thus, a sentence in which prison precedes parole can always duplicate the deterrence created by a sentence in which parole precedes prison, and do so at lower cost (since parole is less expensive than prison). This intuition applies whenever the offender's disutility discount rate exceeds the state's cost discount rate, including when the state's discount rate is positive.

(b) Observe that the present case is the first to provide an explanation grounded in deterrence theory of why a prison term should precede a parole term. If the offender's and the state's discount rates were the same (whether zero or positive), the order would not matter.

¹⁹ As noted, before this substitution of parole time for prison time, the present value of disutility in the new sentence is higher than in the original sentence. If the substitution were complete, so that the new sentence consisted solely of a parole term of length $s_1 + s_2$, the present value of disutility in the new sentence would be lower than in the original sentence. Thus, there must be a limited substitution of parole time for prison time that leads to the same present value of disutility.

Moreover, as we will note in section 6, if the offender's discount rate were *less* than the state's discount rate, it would be optimal to use the parole term before the prison term.

(c) In practice, parole terms do follow prison terms when both sanctions are employed. This is true in the United States as well as in every other country's criminal justice system with which we are familiar. While there are other reasons why this should be so,²⁰ the result of Proposition 3 provides at least a partial basis for this practice.

Numerical example: We can illustrate the intuition behind Proposition 3 with the numerical example used in the introduction, in which the offender's disutility discount rate is fifteen percent and the state's cost discount rate is two percent. Let sentence A consist of one year of parole followed by one year of prison. Given the assumption that parole and prison impose \$2,500 and \$30,000 of disutility per year, respectively, the present value of the disutility generated by this sentence is \$26,299. Given the assumption that parole and prison cost \$2,000 and \$30,000 per person-year, respectively, the present value of the cost of this sentence is \$31,094.

Now reverse the order of the terms in sentence A, so that the one-year prison term occurs before the one-year parole term. This change results in a sentence in which the present value of disutility is higher than in sentence A because the more severe sanction, prison, now occurs first. Let sentence B consist of a prison term of *less than one year* followed by a parole term of greater than one year such that the sum of the two terms remains at two years but the present value of disutility is restored to the level created by sentence A. In this example, the length of the new prison term that equates the disutility generated by sentences A and B in this way is 0.85 years.

²⁰ Both of the rationales for parole mentioned in note 5 above require that parole follow prison.

The cost of sentence B tends to be cheaper than that of sentence A due to sentence B substituting some parole time for some prison time. But there is a countervailing effect under sentence B because it uses the more costly sanction first. When the offender's disutility discount rate is higher than the state's cost discount rate, the first effect dominates the second, resulting in a lower present value of cost. Specifically, the present value of the cost of sentence B is \$27,558, while that of sentence A was seen to be \$31,094. Hence, sentence A, in which parole was used before prison, cannot be optimal.

In the present case the following proposition describes the optimal sentence.

Proposition 4: If offenders discount disutility at a higher rate than the state discounts costs, $r > \rho \geq 0$, and

(a) if $c_1/\theta_1 \leq c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, a finite prison term

$$s_1 = (1/r)\ln[\theta_1/(\theta_1 - rk)] > 0 \quad (9)$$

starting at $t = 0$ is optimal; and

(b) if $c_1/\theta_1 > c_2/\theta_2$, then

(i) for a relatively low target level of deterrence $k \in (0, \kappa]$, where

$$\kappa = (\theta_2/r) - (\theta_2/r)\{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)} < \theta_2/r, \quad (10)$$

a finite probation term

$$s_2 = (1/r)\ln[\theta_2/(\theta_2 - rk)] > 0 \quad (11)$$

starting at $t = 0$ is optimal; and

(ii) for a relatively high target level of deterrence $k \in (\kappa, \theta_1/r)$, a finite prison term

$$s_1 = (1/r)\ln[(\theta_1 - r\kappa)/(\theta_1 - rk)] > 0 \quad (12)$$

starting at $t = 0$ followed immediately by a finite parole term

$$s_2 = (1/r)\ln[\theta_2/(\theta_2 - r\kappa)] > 0 \quad (13)$$

is optimal.

Comments: (a) As discussed in the introduction, when the offender's disutility discount rate exceeds the state's cost discount rate, there is a front-loading advantage of prison over parole or probation. Specifically, because prison imposes higher disutility per unit time than out-of-prison supervision, it can achieve any given level of deterrence with a shorter term. Everything else equal, this is desirable because the deterrent effect of the sanction is not diluted as much by offender discounting. However, prison also imposes higher costs per unit time than out-of-prison supervision. Everything else equal, a shorter term with higher costs per unit time is undesirable because the costs are not diluted as much by the state's discounting. But when the offender's discount rate exceeds the state's discount rate, the deterrence-related benefit of imprisonment more than offsets this cost-related detriment, resulting in a net advantage of imprisonment over out-of-prison supervision.

(b) If prison is weakly cheaper per unit of disutility than out-of-prison supervision ($c_1/\theta_1 \leq c_2/\theta_2$), prison should be used without parole both because prison is at least as cost-effective at any point in time and it has a front-loading advantage. Moreover, since prison is more potent than out-of-prison supervision, it can achieve any target level of disutility (up to θ_1/r), so there is no reason to consider out-of-prison supervision in order to satisfy the deterrence constraint. This reasoning explains part (a) of Proposition 4.

(c) If prison is more expensive per unit of disutility than out-of-prison supervision ($c_1/\theta_1 > c_2/\theta_2$), there is a tradeoff in the choice of the sanctions. Although out-of-prison supervision then would be more cost-effective at any point in time, prison still has a front-loading advantage. This advantage increases with the target level of deterrence k , since a higher k will require longer terms of sanctions, which will augment the deterrence-diluting effect of

offender discounting more than the cost-diluting effect of state discounting. Hence, when the target level of deterrence is relatively low (up to the threshold level κ), the front-loading advantage of imprisonment will be dominated by the superior cost-effectiveness of out-of-prison supervision, making it optimal to rely on a probation term. But if the target level of deterrence is relatively high (exceeding κ), the front-loading advantage of imprisonment will make imprisonment worth employing despite its higher cost per unit of disutility. This reasoning explains the general thrust of part (b) of Proposition 4, though we have two more particular points about this part that we now turn to.

(d) It is noteworthy that the threshold level of deterrence κ , at and below which probation is optimal and above which prison followed by parole is optimal, is strictly less than θ_2/r , the maximum level of deterrence that can be achieved by probation. This implies that there is a range of the target level of deterrence over which, even though prison is more costly per unit of disutility than probation, *and even though probation is capable of achieving the target level of deterrence*, it is desirable to use a prison term (followed by a parole term). The explanation, of course, is the one provided in the previous paragraph—that imprisonment has a front-loading advantage that can offset its lower cost-effectiveness.²¹

(e) One other point of interest regarding part (b) of Proposition 4 is that when prison and parole are used together, only the prison term is lengthened as the target level of deterrence k increases. This result is explained by the fact, observed above, that the front-loading advantage of imprisonment grows as the target level of deterrence increases. In contrast, the disadvantage of imprisonment in terms of its cost-effectiveness—that it has a higher undiscounted cost per unit

²¹ Once the target level of deterrence exceeds θ_2/r , some prison time is needed to satisfy the deterrence constraint, so prison would have to be part of the optimal sentence regardless of its cost-effectiveness and regardless of the magnitude of its front-loading advantage.

of disutility at each point in time (by assumption in part (b))—remains constant. Hence, there will be a critical value of the target level of deterrence, κ , at which the front-loading advantage of prison will just offset its cost-effectiveness disadvantage. For all higher levels of deterrence, it will be increasingly beneficial to use prison rather than parole.

(f) Finally, note that in the present case the optimal prison term or the optimal probation term always begins at $t = 0$. In other words, sanctions should never be delayed. This is because, if offenders discount disutility at a higher rate than the state discounts costs, any delay of a sanction dilutes deterrence faster than it reduces costs. The starting time of sanctions often would not matter if the discount rates were the same (see Propositions 1 and 2).

Numerical example: We continue to employ the example used above, but consider another target level of deterrence, \$2,000, in addition to the \$10,000 and \$100,000 levels.

First, suppose that prison has a lower cost per unit of disutility than parole and probation, as will be the case in the example if the cost of parole and probation is \$3,000 per person-year. Then the optimal sentence is a 0.07 year prison term if the target level of deterrence is \$2,000; a 0.34 year prison term if the target level of deterrence is \$10,000; and a 4.62 year prison term if the target level of deterrence is \$100,000.

If prison has a higher cost per unit of disutility than parole and probation, which will occur if the cost of parole and probation is \$2,000 per person-year, the optimal sentence depends on whether the target level of deterrence is below or above the threshold level of deterrence, $\kappa = \$4,048$, which is lower than the level of deterrence achievable by a sentence of lifetime probation, $\theta_2/r = \$16,667$.

A target level of deterrence of \$2,000 is less than κ , in which case the optimal sentence is a probation term of 0.85 years.

A target level of deterrence of \$10,000 exceeds κ , but is less than θ_2/r . Thus, probation, which has a lower cost per unit of disutility than does prison, would be capable of achieving the target level of deterrence. Nonetheless, the optimal sentence is a prison term of 0.21 years followed by a parole term of 1.86 years.

A target level of deterrence of \$100,000 exceeds θ_2/r , so some imprisonment is needed in order to satisfy the deterrence constraint. In this case, the optimal sentence is a prison term of 4.48 years followed by a parole term of 1.86 years. Note that, as can be seen by comparing the optimal sentence in the present paragraph with that in the previous paragraph, once the target level of deterrence exceeds κ , only the prison term increases as the target level of deterrence increases.

6. Concluding Comments

We conclude with several comments about extensions and generalizations of the analysis.

Offender Discount Rate Less Than State Discount Rate: Although we do not believe that the offender's disutility discount rate is likely to be less than the state's cost discount rate, we discuss this case briefly here for completeness.²² It would then be optimal to use a parole term before a prison term if both sanctions are employed. Suppose, for example, that the offender's discount rate is zero and that the state's discount rate is positive.²³ Consider a one-year prison term and a one-year parole term. It is clear that the present value of costs will be lower if the

²² Because the analysis of this case so closely parallels that pertaining to Section 5, we do not include it in the Appendix.

²³ When the offender's discount rate r is zero, an optimal sentence does not exist given the assumption that individuals live forever, since the state can always lower the present value of the cost of any sentence by delaying its start, without affecting disutility. Notwithstanding this observation, we will for simplicity employ the assumption

more cost-intensive sanction, prison, follows the less cost-intensive sanction, parole, rather than the reverse. Since the offender discount rate is zero in this example, the level of deterrence is independent of the order of the sanctions. Hence, it would be optimal to use the parole term before the prison term.

A second difference from the case focused on in the main body of our article is that it may now be optimal to postpone the start of sanctions. This result is obvious if the offender discount rate is zero since deterrence is unaffected by the start date of the sanctions, but the present value of costs declines as the start date is delayed.

As in the primary case we analyzed in section 5, if the cost of prison per unit of disutility is less than or equal to the cost of out-of-prison supervision per unit of disutility ($c_1/\theta_1 \leq c_2/\theta_2$), prison should be used alone. If the opposite relationship holds ($c_1/\theta_1 > c_2/\theta_2$), parole followed by prison should be used regardless of the target level of deterrence. This latter result is somewhat counterintuitive since it implies that it is optimal to use a prison term even if prison is less cost-effective than probation and even if probation is capable of satisfying the deterrence constraint. The explanation is similar to that provided in the primary case we analyzed, where a parallel result occurred. There the use of imprisonment at the beginning of a sentence allowed disutility to be imposed faster, reducing the deterrence-diluting effect of the offender's discount rate. Here, the use of imprisonment late in the sentence allows the start of the parole term to be postponed, enhancing the cost-diluting effect of the state's discount rate.

Heterogeneity among potential offenders: We have been assuming for simplicity that potential offenders are identical with respect to the disutility they bear from sanctions and with respect to the rate at which they discount this disutility. We will consider here how the analysis

that $r = 0$ in explaining the intuition behind the results when $\rho > r > 0$. (Obviously, this existence problem would disappear if individuals had finite lifetimes.)

would be affected if there were heterogeneity among potential offenders with respect to these factors.

Suppose that an individual would bear disutility of $\lambda\theta_1$ per unit time in prison and $\lambda\theta_2$ per unit time on parole or probation, where there is a distribution of λ among individuals. Since λ multiplies both θ_1 and θ_2 , the relationship between the cost per unit of disutility for prison, $c_1/\lambda\theta_1$, and for out-of-prison supervision, $c_2/\lambda\theta_2$, would be the same for all offenders. Moreover, any sentence generating disutility of k for an individual with $\lambda = 1$ will generate disutility of λk for an individual with any other value of λ . It follows that a sentence that minimizes the cost of imposing disutility k on an individual with $\lambda = 1$ also minimizes the cost of imposing disutility λk on an individual with parameter λ . Thus, if it happens to be socially optimal to impose disutility of λk on an individual with parameter λ , this form of heterogeneity among offenders will not change the optimal sentence. But because any sentence will now generate different levels of deterrence among potential offenders, depending on their λ values, the optimal sentence is likely to be affected. To derive it under this generalization would require a full social welfare model of crime commission and deterrence, including consideration of the distribution of the gain from crime among potential offenders and the harm from crime.

Alternatively, suppose that the disutility per unit time is the same for all potential offenders for a given type of sanction, but that the rate at which they discount disutility, r , varies among them. In this case as well, the relationship between the cost per unit of disutility for prison, c_1/θ_1 , and for parole and probation, c_2/θ_2 , would be the same for all offenders. The complication this generalization introduces is that some potential offenders' discount rates could be less than the state's cost discount rate ρ , while other potential offenders' discount rates could exceed ρ . Our conjecture is that the optimal sequence and mix of sanctions will be determined to

a significant degree by whether the majority of potential offenders have r values exceeding ρ or that are less than ρ . As above, to derive the optimal sentence would require a full social welfare model of crime commission and deterrence.

Lastly, we will mention a few policy implications that might be drawn from our analysis.

Discount rates differ by age: It is widely believed that younger offenders, especially young males, tend to discount future utility at a rate that is significantly higher than the average for the population as a whole.²⁴ Consider how optimal sentences vary with a potential offender's discount rate r , everything else equal. We will assume in this discussion that the variation in r occurs over a range that is above the state's discount rate ρ , so that the analysis in section 5 is applicable.

Suppose that there are two groups of offenders, young ones and older ones, with the former group having a higher r . If prison is weakly more cost-effective than out-of-prison supervision, $c_1/\theta_1 \leq c_2/\theta_2$, then a prison sanction would be optimal for both groups of offenders, but the length of the term would have to be higher for the younger offenders due to their higher discount rate.

If prison is less cost-effective than out-of-prison supervision, $c_1/\theta_1 > c_2/\theta_2$, then the optimal sentence depends on whether the target level of deterrence is below or above a threshold (see Proposition 4(b)). This threshold will be lower the higher is the offender's discount rate r .²⁵ Hence, there are three possibilities—that the target level of deterrence is below the threshold for both groups; that it is between the two thresholds; or that it exceeds both thresholds.

²⁴ For example, Wilson and Herrnstein (1985, p. 205) argue that young people have higher criminal tendencies than older people due in part to their generally higher discount rates. Lee and McCrary (2009) find evidence consistent with high discount rates in their young and predominantly male sample of offenders.

The results in the first case are straightforward; both groups should be subject to probation sanctions, with the younger group receiving a longer term due to its higher discount rate. In the third case, both groups should be subject to a prison sanction followed by a parole sanction, with the younger group receiving a longer prison term and a shorter parole term because its higher discount rate strengthens the front-loading advantage of imprisonment. In the intermediate case, younger offenders should be subject to prison followed by parole, while older offenders should be subject to probation. Thus, there would be a greater tendency to use prison for younger offenders who have relatively high discount rates and out-of-prison supervision for older offenders who have relatively low discount rates.

Disutility of prison rises with income: One component of the disutility of prison is the wage income that is lost while confined. Obviously, this loss will be greater the higher is the offender's rate of compensation. A year in prison will cost a car mechanic who engages in fraud a lot less than an investment banker who engages in fraud.

Consider how optimal sanctions vary with the disutility of prison θ_1 , say due to income variations, assuming for simplicity that the disutility of parole or probation is not affected. We will continue to assume that the analysis in section 5 is applicable.

The higher is θ_1 , everything else equal, the more likely it is that prison will be more cost-effective than parole and probation. If $c_1/\theta_1 \leq c_2/\theta_2$ holds for both the car mechanic and the investment banker, prison would be used for each, but the prison term would be shorter for the investment banker because his θ_1 is higher (assuming the same target level of deterrence). If $c_1/\theta_1 \leq c_2/\theta_2$ holds for the investment banker but not for the car mechanic, then the investment

²⁵ Recall that below the threshold κ , probation is used, while above κ , prison and parole are employed. As r rises, the front-loading advantage of imprisonment rises, making it more desirable, everything else equal, to impose a sentence consisting of prison and parole rather than of probation. Equivalently, κ declines.

banker would be subject to a prison term without parole, while the car mechanic would be subject either to a probation term or a prison term followed by a parole term, depending on the target level of deterrence.

And if $c_1/\theta_1 > c_2/\theta_2$ holds for both offenders, then the optimal sentences will depend on the target level of deterrence k . If k is low enough, both offenders should receive a probation term of equal length since they are assumed to suffer the same disutility per unit time while on probation. If k is in an intermediate range (greater than the applicable κ for the investment banker but less than the applicable κ for the car mechanic), the investment banker should receive a prison term followed by a parole term, while the car mechanic should receive a probation term. And if k is high enough, both offenders should receive a prison term followed by a parole term.²⁶

In sum, because the disutility of imprisonment includes lost income, optimal sanctions for higher-earning individuals will be more likely to include prison terms, in essence because imprisonment is more cost-effective (c_1/θ_1 is lower) with respect to higher-earning individuals. But their prison terms may be shorter, due to the potency of prison being higher for such individuals.

Declining cost of parole and probation: Various technological advances have likely reduced the cost of supervising individuals on parole and probation. A prominent example would be the use of electronic leg cuffs that report the location of a parolee at all times. Another possibility would be the use of remote surveillance cameras in the parolee's home to verify, for example, that the parolee did not use drugs or engage in domestic violence. Although some

²⁶ Whether the investment banker's prison term is shorter or longer in this last case is ambiguous because, on one hand, a shorter term for the banker is needed to achieve any given level of deterrence, but on the other hand, substituting prison time for parole time may be desirable because prison is more cost-effective with respect to him.

technological advances also reduce the cost of administering prisons and jails, our conjecture is that the reduction is, in percentage terms, greater with respect to parole and probation.

Consider how a reduction in the cost of out-of-prison supervision, c_2 , would affect the optimal sentence, everything else equal, in the primary case of interest studied in section 5. One obvious effect would be to make it more likely that part (b) of Proposition 4 would apply, in which prison is less cost-effective than parole and probation, $c_1/\theta_1 > c_2/\theta_2$. Hence, rather than relying solely on imprisonment, as would be the case if part (a) of Proposition 4 were applicable, either probation or prison combined with parole would be used, depending on the target level of deterrence. Moreover, in the latter case, the optimal prison term would be shorter, and the optimal parole term would be longer. A further effect of a reduction in c_2 would be to raise the threshold level of deterrence κ , which implies that probation would be used more often. Thus, not surprisingly, a reduction in the cost of out-of-prison supervision would lead to the greater use of parole and probation.

Appendix

Proof of Proposition 1: Consider a sentence that satisfies the deterrence constraint and that can include multiple terms of each sanction. Let S_1 be the sum of the lengths of all terms of imprisonment and let S_2 be the sum of the lengths of all terms of out-of-prison supervision. The sentence generates cost $C = c_1S_1 + c_2S_2$ and disutility $\theta_1S_1 + \theta_2S_2 = k$. Substitute $S_2 = (k - \theta_1S_1)/\theta_2$ into the expression for cost to obtain

$$C = k(c_2/\theta_2) + S_1\theta_1[(c_1/\theta_1) - (c_2/\theta_2)]. \quad (\text{A1})$$

Clearly, if $c_1/\theta_1 < c_2/\theta_2$, cost is decreasing in S_1 , so optimality requires $S_1 = k/\theta_1$ and $S_2 = 0$. In particular, a sentence including a single prison term of length $s_1 = k/\theta_1$ (starting at any time) will be optimal, proving part (a). If $c_1/\theta_1 > c_2/\theta_2$, cost is increasing in S_1 , so optimality requires $S_1 = 0$ and $S_2 = k/\theta_2$. In particular, a sentence including a single probation term of length $s_2 = k/\theta_2$ (starting at any time) will be optimal, proving part (c). If $c_1/\theta_1 = c_2/\theta_2$, C is constant for all sentences satisfying the deterrence constraint, so all such sentences are optimal, proving part (b). \square

Proof of Proposition 2: Let D_i be the present value of the disutility generated by sanction i in a given sentence, where the sentence can include multiple terms of sanction i . Then the deterrence constraint requires that $D_1 + D_2 = k$. If C is the resulting present value of the costs borne by the state, then because costs and disutilities have the same discount factors when $r = \rho$, $C = (c_1/\theta_1)D_1 + (c_2/\theta_2)D_2$. Solving the deterrence constraint for $D_2 = k - D_1$ and substituting into the expression for cost, we obtain

$$C = k(c_2/\theta_2) + D_1[(c_1/\theta_1) - (c_2/\theta_2)]. \quad (\text{A2})$$

If $c_1/\theta_1 = c_2/\theta_2$, every sentence meeting the deterrence constraint generates the same cost, $k(c_2/\theta_2)$, so we obtain part (b).

If $c_1/\theta_1 < c_2/\theta_2$, cost decreases with D_1 , so any sentence satisfying the deterrence constraint with $D_1 = k$ and $D_2 = 0$ is optimal. A sentence that includes a prison term of length $s_1 = (1/r)\ln\{[1 - (rk/\theta_1)e^{rt}]^{-1}\}$ starting at $t \in [0, t_M)$, where $t_M = (1/r)\ln(\theta_1/rk)$, and otherwise uses only the null sanction, generates $D_1 = k$, and is therefore optimal. Likewise, a sentence that includes an infinite prison term starting at $t = t_M$ and otherwise uses only the null sanction generates $D_1 = k$, and is therefore optimal. Part (a) follows.

If $c_1/\theta_1 > c_2/\theta_2$, cost increases with D_1 , so if the state can satisfy the deterrence constraint without prison, that is, if $k \leq \theta_2/r$, then any sentence satisfying the deterrence constraint with $D_1 = 0$ and $D_2 = k$ is optimal. If $k < \theta_2/r$, a sentence that includes a probation term of length $s_2 = (1/r)\ln\{[1 - (rk/\theta_2)e^{rt}]^{-1}\}$ starting at $t \in [0, t_M)$, where $t_M = (1/r)\ln(\theta_2/rk)$, and otherwise uses only the null sanction, generates $D_2 = k$, and is therefore optimal. Likewise, a sentence that includes an infinite probation term starting at $t = t_M$ and otherwise uses only the null sanction generates $D_2 = k$, and is therefore optimal. Part (c)(i) follows. If $k = \theta_2/r$, an infinite probation term generates $D_2 = k$, and is therefore optimal. Part (c)(ii) follows.

If $c_1/\theta_1 > c_2/\theta_2$, so that cost increases with D_1 , and if the state cannot satisfy the deterrence constraint without some use of prison, that is, if $k > \theta_2/r$, then an optimal sentence employs the minimum D_1 necessary to satisfy the deterrence constraint. It is straightforward to show that in this case an optimal sentence does not use the null sanction. Suppose, to the contrary, that the null sanction is employed. Then, because out-of-prison supervision is cheaper per unit of disutility than prison, it would be possible to (i) replace part of a term of the null sanction with a term of out-of-prison supervision and (ii) shorten a prison term in such a way that the present value of disutility generated by the sentence does not change, but cost drops. Since, for $r > 0$,

$$\int_0^{\infty} e^{-rt} dt = 1/r \quad (\text{A3})$$

and

$$\int_0^{\infty} \delta(t) e^{-rt} dt = k, \quad (\text{A4})$$

for any sentence not using the null sanction, it must be that $(D_1/\theta_1) + (D_2/\theta_2) = 1/r$. Combining this equation with the deterrence constraint implies that for any optimal sentence,

$$D_1 = (\theta_1/r)(rk - \theta_2)/(\theta_1 - \theta_2) \quad (\text{A5})$$

and

$$D_2 = (\theta_2/r)(\theta_1 - rk)/(\theta_1 - \theta_2). \quad (\text{A6})$$

Any sentence with these values of D_1 and D_2 will generate the same present value of cost, and therefore be optimal. Both (i) a sentence with a prison term of length $s_1 = (1/r)\ln\{(\theta_1 - \theta_2)/(\theta_1 - rk)\}$ starting at $t = 0$, followed immediately by an infinite parole term, and (ii) a sentence with a parole term of length $s_2 = (1/r)\ln\{(\theta_1 - \theta_2)/(rk - \theta_2)\}$ starting at $t = 0$, followed immediately by an infinite prison term, generate the required values of D_1 and D_2 , and are therefore optimal. This establishes part (c)(iii). \square

Proof of Proposition 3: We will demonstrate here that if $r > \rho \geq 0$, then any optimal sentence that includes multiple sanctions uses them in order of decreasing severity. Proposition 3 follows immediately from this claim.

As a preliminary matter, we introduce the concept of the *duration* of a continuous flow of costs or utility. Let $\phi(t)$ be the level of a continuous flow per unit time. The duration of flow $\phi(t)$ is defined as

$$\pi(\phi, x) = \left(\int_0^{\infty} t \phi(t) e^{-xt} dt \right) / \left(\int_0^{\infty} \phi(t) e^{-xt} dt \right). \quad (\text{A7})$$

The duration can be interpreted as a weighted average of the time at which flows of costs or utility occur, where the weight at time t is proportional to the present value of the flow at t .

Additionally, let $PV(\phi, x)$ denote the present value of flow $\phi(t)$ as of time $t = 0$ using discount rate $x \geq 0$. The duration of $\phi(t)$ can be expressed in terms of this present value as

$$\tau(\phi, x) = -\partial \ln(PV(\phi, x)) / \partial x. \quad (\text{A8})$$

Thus, the duration of a flow is not only the weighted average time at which the flow occurs, but also, by (A8), a measure of the sensitivity of the flow's present value to changes in the discount rate.

We now prove by contradiction that if $r > \rho \geq 0$, then any optimal sentence that includes multiple sanctions uses those sanctions in order of decreasing severity. Specifically, we will show that if sentence $\delta(t)$ is optimal and, during a finite period of time $[a, b]$, $\delta(t)$ uses two sanctions (chosen from among prison, out-of-prison supervision, and the null sanction) in order of *increasing* severity, then we can construct another sentence, $d(t)$, that reverses the order of the sanctions during the period $[a, b]$ and generates the same present value of disutility as $\delta(t)$ at lower cost. We may assume without loss of generality that this period is $[0, T]$ for some $T > 0$.²⁷ Call the less severe sanction l for low severity, and the more severe sanction h for high severity, where $\theta_l < \theta_h$. Let $L \in (0, T)$ be the length of time during which the less severe sanction is used. Thus, $\delta(t) = \theta_l$ if $t \in [0, L]$, and $\delta(t) = \theta_h$ if $t \in (L, T]$.²⁸

Now consider an alternative sentence, $d(t)$, that is identical to the original sentence, $\delta(t)$, except that for some $H \in (0, T)$, $d(t) = \theta_h$ if $t \in [0, H]$, and $d(t) = \theta_l$ if $t \in (H, T]$. In other words,

²⁷ In this proof, the terms of the sentence outside of the period $[a, b]$ do not affect the analysis, so we can simply ignore all terms of sentence $\delta(t)$ occurring before $t = a$.

$d(t)$ is the same as $\delta(t)$ except that in the period $[0, T]$ the order of the sanctions is reversed, and a given sanction is not necessarily used for the same length of time in the two sentences (that is, it is not necessarily true that $H = T - L$). Let H be chosen so that the present value of disutility is the same under $\delta(t)$ and $d(t)$. It is straightforward to see that such an H exists (if $H = T - L$, the present value of disutility would be higher under $d(t)$, while if $H = 0$, it would be lower).

Next observe that if both disutility and costs (not just disutility) are discounted at rate $r = \rho > 0$, then the same discount factors can be used for both disutility and cost. It follows that if $r = \rho$, the weight on θ_i in the expression for the present value of disutility for any sentence will equal the weight on c_i in the corresponding expression for the sentence's cost. Therefore, since $\delta(t)$ and $d(t)$ generate the same present value of disutility, $\delta(t)$ and $d(t)$ must also generate the same present value of costs if $r = \rho$:

$$PV(c(\delta), r) = PV(c(d), r). \quad (\text{A9})$$

We now demonstrate that when $r > \rho \geq 0$, sentence $d(t)$ is less costly than sentence $\delta(t)$, so $\delta(t)$ cannot be optimal. Since we will be comparing two sentences that differ only during the period $[0, T]$, we need only compare disutility and cost for this period, so for convenience and without loss of generality, we assume that for all $t > T$, $\delta(t) = d(t) = 0$.

The next step is to decompose each of the flows of costs $c(\delta(t))$ and $c(d(t))$ into two parts: one flow common to both sentences and a second flow that has a greater duration for sentence $\delta(t)$. We then use this decomposition to prove that $PV(c(\delta), \rho) > PV(c(d), \rho)$, implying that sentence $\delta(t)$ is not optimal, the desired contradiction.

Define $v(t)$, $w(t)$, and $z(t)$ as follows:

²⁸ It is immaterial whether $\delta(L) = \theta_l$ or $\delta(L) = \theta_{li}$. We will not comment again when similar points arise

$$v(t) = c_l \forall t \in [0, T], \text{ and } v(t) = 0 \forall t \notin [0, T]; \quad (\text{A10})$$

$$w(t) = c_h - c_l > 0 \forall t \in (L, T], \text{ and } w(t) = 0 \forall t \notin (L, T]; \quad (\text{A11})$$

and

$$z(t) = c_h - c_l > 0 \forall t \in [0, H], \text{ and } z(t) = 0 \forall t \notin [0, H]. \quad (\text{A12})$$

Observe that $v(t) \leq c(\delta(t))$ and $v(t) \leq c(d(t)) \forall t \geq 0$. The definitions of $v(t)$, $w(t)$, and $z(t)$ ensure that $\forall t \geq 0$,

$$c(\delta(t)) = v(t) + w(t) \quad (\text{A13})$$

and

$$c(d(t)) = v(t) + z(t). \quad (\text{A14})$$

The flow $z(t)$ can be converted to the flow $w(t)$ by a simple transformation: Starting with $z(t)$, increase by L the time at which positive flow begins (from $t = 0$ to $t = L$) and increase by $T - H$ the time at which positive flow ends (from $t = H$ to $t = T$). Because duration is the weighted average time at which costs are incurred, eliminating some time at the beginning of a constant flow or adding some time at its end must raise duration. Since the transformation from $z(t)$ to $w(t)$ effects both of these adjustments, it follows that:

$$\tau(w, x) > \tau(z, x) \forall x \geq 0. \quad (\text{A15})$$

We now show that (A15) implies that $PV(c(\delta), \rho) > PV(c(d), \rho)$. By (A8),

$$\tau(w, x) = -\partial \ln(PV(w, x)) / \partial x \quad (\text{A16})$$

and

$$\tau(z, x) = -\partial \ln(PV(z, x)) / \partial x. \quad (\text{A17})$$

Therefore, since $r > \rho$,

below.

$$\ln(PV(w, r)) - \ln(PV(w, \rho)) = -\int_{\rho}^r \tau(w, x) dx \quad (\text{A18})$$

and

$$\ln(PV(z, r)) - \ln(PV(z, \rho)) = -\int_{\rho}^r \tau(z, x) dx. \quad (\text{A19})$$

Substituting (A13) and (A14) into (A9) yields

$$PV(w, r) = PV(z, r). \quad (\text{A20})$$

Since by (A15), $\tau(w, x) > \tau(z, x)$ for all $x \in [\rho, r]$,

$$\int_{\rho}^r \tau(w, x) dx > \int_{\rho}^r \tau(z, x) dx. \quad (\text{A21})$$

Given the preceding derivations, we obtain

$$\begin{aligned} \ln(PV(w, \rho)) &= \ln(PV(w, r)) + \int_{\rho}^r \tau(w, x) dx > \ln(PV(w, r)) \\ &+ \int_{\rho}^r \tau(z, x) dx = \ln(PV(z, r)) + \int_{\rho}^r \tau(z, x) dx = \ln(PV(z, \rho)). \end{aligned} \quad (\text{A22})$$

The first equality in (A22) follows from (A18), the inequality from (A21), the second equality from (A20), and the last equality from (A19). By (A22),

$$PV(w, \rho) > PV(z, \rho), \quad (\text{A23})$$

so

$$PV(v, \rho) + PV(w, \rho) > PV(v, \rho) + PV(z, \rho), \quad (\text{A24})$$

and by (A13) and (A14),

$$PV(C(\delta), \rho) > PV(C(d), \rho). \quad (\text{A25})$$

Since sentence $d(t)$ satisfies the deterrence constraint, (A25) contradicts the assumption that sentence $\delta(t)$ is optimal. Thus, if $r > \rho \geq 0$, any sanctions used in an optimal sentence must appear in order of decreasing severity. \square

Before proceeding to the proof of Proposition 4, we state and prove three lemmas that will be used in the proof.

Lemma 1: If $r > \rho \geq 0$, the final term of any optimal sentence must use the null sanction.

Proof of Lemma 1: Assume $r > \rho \geq 0$. We suppose that an optimal sentence $\delta(t)$ uses imprisonment or out-of-prison supervision in its final term, and then derive a contradiction by constructing a sentence $d(t)$ that generates the same disutility as $\delta(t)$ at lower cost. It follows that an optimal sentence must conclude with an infinite term of the null sanction.

Assume that the final term of $\delta(t)$ starts at $t = S \geq 0$. Let f indicate the final sanction of sentence $\delta(t)$, so that $\theta_f \in \{\theta_1, \theta_2\}$ is the disutility per unit time produced by f and $c(\theta_f) = c_f \in \{c_1, c_2\}$ is the corresponding cost per unit time. Let $d(t)$ differ from $\delta(t)$ by shifting all sanctions in sentence $\delta(t)$ forward (into the future) by a short period of time $\tau > 0$, adding a brief term of prison from $t = 0$ to $t = \tau$, and switching to the null sanction in perpetuity at $t = T \geq S + \tau$. In other words, $d(t)$ equals θ_1 for $t \in [0, \tau]$; $\delta(t - \tau)$ for $t \in (\tau, T]$; and θ_3 for $t > T$.

First suppose that $r > \rho > 0$. Let k be the present value of the disutility generated by sentence $\delta(t)$ and C be the present value of the cost of $\delta(t)$. Define Δk as the present value of the disutility of sentence $d(t)$ less the present value of the disutility of sentence $\delta(t)$; and define ΔC to be the corresponding expression for the difference in costs. We will choose T in sentence $d(t)$ to ensure that $\Delta k = 0$ and then show that $\Delta C < 0$, implying that $\delta(t)$ cannot be optimal.

The present value of disutility generated by $d(t)$ is

$$(\theta_1/r)(1 - e^{-r\tau}) + ke^{-r\tau} - (\theta_f/r)e^{-rT}. \quad (\text{A26})$$

The first term in (A26) is the disutility due to the initial prison term of $d(t)$. The second and third terms together represent the disutility generated by the remainder of $d(t)$, from $t = \tau$ onward. The second term is the disutility that would be generated by imposing sentence $\delta(t)$ after a delay of τ ,

and the third term is the disutility lost because $d(t)$ switches from sanction f to the null sanction at time $t = T$, rather than continuing with sanction f , as does sentence $\delta(t)$.

Therefore, for Δk to equal zero, (A26) must equal k , which can be expressed as

$$(\theta_l/r)(1 - e^{-r\tau}) + k(e^{-r\tau} - 1) - (\theta_f/r)e^{-rT} = 0. \quad (\text{A27})$$

Solving (A27) for e^{-rT} and taking the natural logarithm of both sides yields

$$T = (1/r)\ln\{\theta_f/[(\theta_l - rk)(1 - e^{-r\tau})]\}. \quad (\text{A28})$$

Note that because $\theta_f/(\theta_l - rk)$ is finite and positive, T increases without bound as τ goes to zero, and we can always choose τ sufficiently low so that (A28) yields $T \geq S + \tau$, as assumed above.

By (A28),

$$e^{-\rho T} = \{(\theta_l - rk)(1 - e^{-r\tau})/\theta_f\}^{\rho/r}. \quad (\text{A29})$$

Analogously to (A26), the present value of costs under $d(t)$ is

$$(c_l/\rho)(1 - e^{-\rho\tau}) + Ce^{-\rho\tau} - (c_f/\rho)e^{-\rho T}. \quad (\text{A30})$$

Therefore, for $\delta(t)$ to be optimal, it must be that $\Delta C \geq 0$, that is,

$$(c_l/\rho)(1 - e^{-\rho\tau}) + C(e^{-\rho\tau} - 1) - (c_f/\rho)e^{-\rho T} \geq 0. \quad (\text{A31})$$

Rearranging (A31) so that $e^{-\rho T}$ is on the right-hand side and then employing (A29) yields

$$[(c_l - \rho C)/c_f](1 - e^{-\rho\tau}) \geq \{(\theta_l - rk)(1 - e^{-r\tau})/\theta_f\}^{\rho/r}. \quad (\text{A32})$$

Note that $c_l - \rho C > 0$; this condition is equivalent to $C < c_l/\rho$, which must hold unless $\delta(t)$ is a perpetual prison term, which it cannot be if $k < \theta_l/r$. Thus, (A32) can be rewritten as

$$(1 - e^{-\rho\tau})/(1 - e^{-r\tau})^{\rho/r} \geq \{(\theta_l - rk)/\theta_f\}^{\rho/r} [c_f/(c_l - \rho C)]. \quad (\text{A33})$$

Since $\theta_l - rk > 0$ (implied by the assumption that $k < \theta_l/r$) and $c_l - \rho C > 0$, the right-hand side of (A33) is a positive constant. By applying L'Hopital's rule, it can be shown that the left-hand side of (A33) goes to 0 as τ goes to 0. It follows that for sufficiently low values of τ , inequality (A33) cannot hold and $\delta(t)$ cannot be optimal. We have thus demonstrated a contradiction.

Now suppose that $r > \rho = 0$. Then any sentence that does not use the null sanction in its final term generates costs with an infinite present value. Given the assumption that $k < \theta_1/r$, a finite prison term followed by the null sanction can achieve the deterrence constraint. Such a sentence will generate only finite costs, implying that a sentence using prison or out-of-prison supervision in its final term cannot be optimal when $r > \rho = 0$.

We thus conclude that when $r > \rho \geq 0$, an optimal sentence must use the null sanction in its final term. \square

Lemma 2: Assume $r > \rho \geq 0$ and let $\delta(t)$ be an optimal sentence. Then:

(a) $\delta(t)$ generates disutility of

$$D(s_1, s_2) = (\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1+s_2)}]; \quad (\text{A34})$$

(b) $\delta(t)$ generates cost of

$$C(s_1, s_2) = c_1s_1 + c_2s_2 \quad (\text{A35})$$

if $\rho = 0$ and cost of

$$C(s_1, s_2) = (c_1/\rho)(1 - e^{-\rho s_1}) + (c_2/\rho)[e^{-\rho s_1} - e^{-\rho(s_1+s_2)}] \quad (\text{A36})$$

if $\rho > 0$; and

(c) let

$$L(s_1, s_2, \lambda, \mu_1, \mu_2) = C(s_1, s_2) + \lambda(k - D(s_1, s_2)) - \mu_1s_1 - \mu_2s_2; \quad (\text{A37})$$

then there exist Kuhn-Tucker multipliers $\lambda, \mu_1 \geq 0$, and $\mu_2 \geq 0$, such that $\delta(t)$ satisfies the following Kuhn-Tucker conditions:

$$\partial L / \partial s_i = 0 \quad \forall i \in \{1, 2\}; \quad (\text{A38})$$

$$\lambda(k - D(s_1, s_2)) = 0; \quad (\text{A39})$$

and

$$\mu_i s_i = 0 \quad \forall i \in \{1, 2\}. \quad (\text{A40})$$

Proof of Lemma 2: The proof of Proposition 3 and Lemma 1 imply that when $r > \rho \geq 0$ only two numbers are needed to describe an optimal sentence completely, the ordered pair (s_1, s_2) , representing the lengths of the terms of prison and out-of-prison supervision (one of which can be zero) that precede the infinite term of the null sanction.

Thus, the disutility of sentence $\delta(t)$ is

$$D(s_1, s_2) = \int_0^{\infty} \delta(t) e^{-rt} dt = \theta_1 \int_0^{s_1} e^{-rt} dt + \theta_2 \int_{s_1}^{s_1+s_2} e^{-rt} dt + \theta_3 \int_{s_1+s_2}^{\infty} dt, \quad (\text{A41})$$

which can be written as (A34). The cost of sentence $\delta(t)$ if $\rho = 0$ clearly is (A35); its cost if $\rho > 0$ is an expression analogous to (A41) with ρ substituted for r and c_i substituted for θ_i , which can be written as (A36). This confirms parts (a) and (b) of the lemma.

Given these results, the state's problem described in section 2 by (1) and (2) can be rewritten as the following constrained optimization problem, which will be referred to as the *optimal sentencing problem*:

$$\min_{s_1, s_2} C(s_1, s_2) \quad (\text{A42})$$

$$\text{subject to} \quad k - D(s_1, s_2) = 0 \quad (\text{A43})$$

$$\text{and} \quad -s_i \leq 0 \quad \forall i \in \{1, 2\}. \quad (\text{A44})$$

The Lagrangean function corresponding to the optimal sentencing problem can be written as (A37). By the Kuhn-Tucker Theorem, if a technical condition called the constraint qualification holds, then at any solution to the optimal sentencing problem, there must be a number λ and non-negative numbers s_1, s_2, μ_1 , and μ_2 such that (A38) through (A40) also hold. We will refer to (A38) through (A40) as the Kuhn-Tucker conditions for the optimal sentencing

problem. The constraint qualification condition is satisfied for this problem, though for brevity we will not demonstrate this here. Thus, we have obtained part (c) of the lemma. \square

Lemma 3: Either (a) $(c_1 - c_2)/(\theta_1 - \theta_2) < c_1/\theta_1 < c_2/\theta_2$; or (b) $c_2/\theta_2 < c_1/\theta_1 < (c_1 - c_2)/(\theta_1 - \theta_2)$; or (c) $c_1/\theta_1 = c_2/\theta_2 = (c_1 - c_2)/(\theta_1 - \theta_2)$.

Proof of Lemma 3: (a) Suppose that $c_1/\theta_1 < c_2/\theta_2$. From this expression it can be established that $(c_1 - c_2)/(\theta_1 - \theta_2) < c_1/\theta_1$ through the following steps. Multiply both sides by $\theta_1\theta_2$; subtract $c_1\theta_1$ from both sides; factor out c_1 on the left-hand side and θ_1 on the right-hand side; multiply both sides by $-1/[(\theta_1 - \theta_2)\theta_1]$. This establishes part (a).

(b) Now suppose that $c_2/\theta_2 < c_1/\theta_1$. By a procedure analogous to that employed in part (a) it can be shown that $c_1/\theta_1 < (c_1 - c_2)/(\theta_1 - \theta_2)$, thereby establishing part (b). The only difference is that in the third step θ_1 should be factored out on the left-hand side and c_1 on the right-hand side.

(c) Finally, suppose that $c_1/\theta_1 = c_2/\theta_2$. Using the same procedure as in part (a) it can be shown that $c_1/\theta_1 = (c_1 - c_2)/(\theta_1 - \theta_2)$, thereby demonstrating part (c). \square

Proof of Proposition 4: Lemma 2 provides the Lagrangean function for the optimal sentencing problem and the Kuhn-Tucker conditions that any solution to this problem must satisfy. We will rewrite the Kuhn-Tucker conditions in terms of the parameters of the optimal sentencing problem, and then solve for the term lengths s_1 and s_2 . Last, we will argue that the Kuhn-Tucker conditions are sufficient as well as necessary for optimality.

First assume that $\rho > 0$. By Lemma 2, when $r > \rho > 0$, the Lagrangean for the optimal sentencing problem can be expressed as:

$$\begin{aligned}
 L(s_1, s_2, \lambda, \mu_1, \mu_2) &= (c_1/\rho)(1 - e^{-\rho s_1}) + (c_2/\rho)[e^{-\rho s_1} - e^{-\rho(s_1 + s_2)}] \\
 &+ \lambda(k - (\theta_1/r)(1 - e^{-rs_1}) - (\theta_2/r)[e^{-rs_1} - e^{-r(s_1 + s_2)}]) - \mu_1 s_1 - \mu_2 s_2.
 \end{aligned}
 \tag{A45}$$

Condition (A38) requires that $\partial L/\partial s_1 = 0$, which can be written as

$$\mu_1 = (c_1 - c_2)e^{-\rho s_1} + c_2e^{-\rho(s_1 + s_2)} + \lambda((\theta_2 - \theta_1)e^{-rs_1} - \theta_2e^{-r(s_1 + s_2)}). \quad (\text{A46})$$

Condition (A38) also requires that $\partial L/\partial s_2 = 0$, which can be expressed as

$$\mu_2 = c_2e^{-\rho(s_1 + s_2)} - \lambda\theta_2e^{-r(s_1 + s_2)}. \quad (\text{A47})$$

Condition (A39) requires that

$$\lambda(k - \{(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1 + s_2)}]\}) = 0. \quad (\text{A48})$$

Condition (A40) requires that

$$\mu_1 s_1 = 0 \quad (\text{A49})$$

and

$$\mu_2 s_2 = 0. \quad (\text{A50})$$

Thus, if $r > \rho > 0$, the Kuhn-Tucker conditions (A38)-(A40) become conditions (A46)-(A50).

Now assume that $\rho = 0$. The only Kuhn-Tucker condition affected by switching from $\rho > 0$ to $\rho = 0$ is (A38), requiring that $\partial L/\partial s_1 = 0$ and $\partial L/\partial s_2 = 0$. It is straightforward to show that, if one derives the conditions $\partial L/\partial s_1 = 0$ and $\partial L/\partial s_2 = 0$ when $r > \rho = 0$, the results are identical to the corresponding conditions when $r > \rho > 0$ if the latter conditions are evaluated at $\rho = 0$. Thus, the Kuhn-Tucker conditions (A46)-(A50) apply whenever $r > \rho \geq 0$, and not just when $r > \rho > 0$.

To derive the optimal term lengths for s_1 and s_2 , it will be convenient to consider four mutually exclusive and collectively exhaustive cases. In each case, we will assume that the pair of term lengths (s_1, s_2) satisfies the Kuhn-Tucker conditions, (A46)-(A50).

Case 1: $s_1 > 0, s_2 = 0$

Since $s_1 > 0$, (A49) requires that $\mu_1 = 0$. Then (A46) becomes, with $s_2 = 0$,

$$c_1e^{-\rho s_1} - \lambda\theta_1e^{-rs_1} = 0, \quad (\text{A51})$$

or, solving for λ ,

$$\lambda = (c_1/\theta_1)e^{(r-\rho)s_1} > 0. \quad (\text{A52})$$

Thus, by (A48), with $s_2 = 0$,

$$(\theta_1/r)(1 - e^{-rs_1}) = k. \quad (\text{A53})$$

Solving (A53) for s_1 yields

$$s_1 = (1/r)\ln[\theta_1/(\theta_1 - rk)] > 0. \quad (\text{A54})$$

Since we have been assuming that when $r > 0$, $k < \theta_1/r$, the argument of the natural logarithm function in (A54) exceeds unity and hence the preceding expression for s_1 is well-defined.

As noted in the proof of Lemma 2, the Kuhn-Tucker Theorem requires that the multipliers μ_1 and μ_2 be non-negative. Since in the present case $s_2 = 0$, (A50) implies that $\mu_2 \geq 0$. By substituting (A52) into (A47) and using $s_2 = 0$, the condition that $\mu_2 \geq 0$ can be expressed as

$$\mu_2 = c_2e^{-\rho s_1} - (c_1/\theta_1)e^{-\rho s_1}\theta_2 \geq 0, \quad (\text{A55})$$

which is equivalent to

$$c_1/\theta_1 \leq c_2/\theta_2. \quad (\text{A56})$$

Hence, case 1 is possible only when $c_1/\theta_1 \leq c_2/\theta_2$, and then for $k < \theta_1/r$, s_1 is given by (A54) and $s_2 = 0$.

Case 2: $s_1 = 0$, $s_2 > 0$

Since $s_2 > 0$, (A50) requires that $\mu_2 = 0$. Then (A47) becomes, with $s_1 = 0$,

$$c_2e^{-\rho s_2} - \lambda\theta_2e^{-rs_2} = 0 \quad (\text{A57})$$

or, solving for λ ,

$$\lambda = (c_2/\theta_2)e^{(r-\rho)s_2} > 0. \quad (\text{A58})$$

Thus, by (A48), with $s_1 = 0$,

$$(\theta_2/r)[1 - e^{-rs_2}] = k. \quad (\text{A59})$$

Solving (A59) for s_2 yields

$$s_2 = (1/r)\ln[\theta_2/(\theta_2 - rk)] > 0. \quad (\text{A60})$$

To guarantee that this expression for s_2 is well-defined, we assume that $k < \theta_2/r$. Below, we show that k must satisfy an even more restrictive condition for the present case to arise.

Since in the present case $s_1 = 0$, (A49) implies that $\mu_l \geq 0$. By substituting (A58) into (A46) and using $s_1 = 0$, the condition that $\mu_l \geq 0$ can be expressed as

$$c_1 - c_2 + c_2 e^{-\rho s_2} + (c_2/\theta_2) e^{(r-\rho)s_2} (\theta_2 - \theta_1 - \theta_2 e^{-rs_2}) \geq 0. \quad (\text{A61})$$

Solve (A59) for e^{-rs_2} to obtain

$$e^{-rs_2} = (\theta_2 - rk)/\theta_2. \quad (\text{A62})$$

Raising both sides of (A62) by the power of ρ/r yields

$$e^{-\rho s_2} = [(\theta_2 - rk)/\theta_2]^{\rho/r}. \quad (\text{A63})$$

Now insert (A62) and (A63) into (A61) and rearrange terms to obtain

$$c_1 - c_2 \geq (c_2/\theta_2) e^{(r-\rho)s_2} (\theta_1 - rk) - c_2 [(\theta_2 - rk)/\theta_2]^{\rho/r}. \quad (\text{A64})$$

Raise both sides of (A62) by the power of $(\rho - r)/r$ and substitute the resulting right-hand side for $e^{(r-\rho)s_2}$ in (A64); after factoring out the term $[(\theta_2 - rk)/\theta_2]^{\rho/r}$ on the right-hand side of (A64), the result is

$$c_1 - c_2 \geq [(\theta_2 - rk)/\theta_2]^{\rho/r} [\{(c_2/\theta_2)(\theta_1 - rk)/[(\theta_2 - rk)/\theta_2]\} - c_2] \quad (\text{A65})$$

After multiplying both sides of (A65) by θ_2/c_2 and then rewriting $\{(\theta_1 - rk)/[(\theta_2 - rk)/\theta_2]\} - \theta_2$ as $(\theta_1 - \theta_2)/[(\theta_2 - rk)/\theta_2]$, (A65) can be expressed as

$$[(c_1 - c_2)/(\theta_1 - \theta_2)](\theta_2/c_2) \geq [(\theta_2 - rk)/\theta_2]^{(\rho - r)/r}. \quad (\text{A66})$$

Raise both sides of (A66) by the power of $r/(r - \rho)$; multiply each side by $(\theta_2 - rk)\{[(\theta_1 - \theta_2)/(c_1 - c_2)](c_2/\theta_2)\}^{r/(r-\rho)}$; and then solve for k to obtain

$$k \leq \kappa, \quad (\text{A67})$$

where

$$\kappa = (\theta_2/r) - (\theta_2/r)\{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)}. \quad (\text{A68})$$

Since $k > 0$, in order for condition (A67) to possibly hold, it must be that $\kappa > 0$. It is straightforward to show from (A68) that κ is positive if and only if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$. By part (b) of Lemma 3, this implies that $c_2/\theta_2 < c_1/\theta_1$. Moreover, if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$, the term in braces in (A68) is less than 1, implying that $\kappa < \theta_2/r$.

In sum, Case 2 is possible only when $c_1/\theta_1 > c_2/\theta_2$ and $k \leq \kappa$, where $\kappa < \theta_2/r$ is given by (A68), in which case $s_1 = 0$ and s_2 is given by (A60).

Case 3: $s_1 > 0, s_2 > 0$

Since $s_1 > 0$ and $s_2 > 0$, (A49) and (A50) require that $\mu_1 = \mu_2 = 0$. Thus, by (A47),

$$c_2 e^{-\rho(s_1 + s_2)} - \lambda \theta_2 e^{-r(s_1 + s_2)} = 0, \quad (\text{A69})$$

which can be solved for λ :

$$\lambda = (c_2/\theta_2) e^{(r-\rho)(s_1 + s_2)} > 0. \quad (\text{A70})$$

Similarly, by (A46),

$$(c_1 - c_2) e^{-\rho s_1} + c_2 e^{-\rho(s_1 + s_2)} + \lambda((\theta_2 - \theta_1) e^{-r s_1} - \theta_2 e^{-r(s_1 + s_2)}) = 0. \quad (\text{A71})$$

After substituting λ from (A70) into (A71) and simplifying, (A71) can be written as

$$[(c_1 - c_2)/(\theta_1 - \theta_2)](\theta_2/c_2) = e^{(r-\rho)s_2}. \quad (\text{A72})$$

Observe that

$$e^{-r s_2} = (e^{(r-\rho)s_2})^{-r/(r-\rho)} = \{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)} = (\theta_2 - r\kappa)/\theta_2, \quad (\text{A73})$$

where the second equality follows from (A72) and the third equality from (A68). Taking the natural logarithms of the first and last terms in (A73) and solving for s_2 yields

$$s_2 = (1/r) \ln[\theta_2/(\theta_2 - r\kappa)]. \quad (\text{A74})$$

This expression is well-defined and positive provided that $\theta_2/(\theta_2 - r\kappa) > 1$ or, equivalently, given (A68), if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$. By Lemma 3, this implies that $c_2/\theta_2 < c_1/\theta_1$.

Since $\lambda > 0$ (see (A70)), (A48) implies that

$$(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1 + s_2)}] = k. \quad (\text{A75})$$

If one factors out e^{-rs_1} from the second term and uses (A73) to substitute for e^{-rs_2} , this expression can be rewritten as

$$(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)e^{-rs_1}[r\kappa/\theta_2] = k. \quad (\text{A76})$$

Solving (A76) for e^{-rs_1} and taking natural logarithms of both sides yields, after further manipulation,

$$s_1 = (1/r)\ln[(\theta_1 - r\kappa)/(\theta_1 - rk)]. \quad (\text{A77})$$

This expression is well-defined and positive provided that $(\theta_1 - r\kappa)/(\theta_1 - rk) > 1$ or, equivalently, $k > \kappa$.

Thus, Case 3 is possible only when $c_1/\theta_1 > c_2/\theta_2$ and $k > \kappa$, where κ is given by (A68), in which case s_1 is given by (A77) and s_2 by (A74).

Case 4: $s_1 = 0, s_2 = 0$

If $s_1 = s_2 = 0$, the sentence obviously cannot generate any disutility and therefore cannot be a solution to the optimal sentencing problem.

Together, the proof of Proposition 3 and Lemma 1 showed that when $r > \rho \geq 0$, an optimal sentence consists of a term of prison, followed by a term of out-of-prison supervision, followed by an infinite term of the null sanction (with one of the first two terms possibly of zero length). Lemma 2 showed that the ordered pair of term lengths (s_1, s_2) for an optimal sentence must satisfy the Kuhn-Tucker conditions. It can be demonstrated that if $r > \rho \geq 0$, a solution to

the optimal sentencing problem exists.²⁹ Thus, since the Kuhn-Tucker conditions have only a single solution in each of Cases 1 through 3, these conditions are necessary and sufficient for optimality; and the term lengths we derived above for each case are uniquely optimal.

If $c_1/\theta_1 \leq c_2/\theta_2$, only Case 1 ($s_1 > 0, s_2 = 0$) is possible, and the unique solution to the optimal sentencing problem is a prison term of length (A54) starting at $t = 0$. Obviously, this result holds for any target level of deterrence $k \in (0, \theta_1/r)$. We have thus demonstrated (a).

If $c_1/\theta_1 > c_2/\theta_2$ and $k \leq \kappa$, only Case 2 ($s_1 = 0, s_2 > 0$) is possible, and the unique solution to the optimal sentencing problem consists of a probation term of length (A60) starting at $t = 0$. We have thus demonstrated (b)(i).

If $c_1/\theta_1 > c_2/\theta_2$ and $k > \kappa$, only Case 3 ($s_1 > 0, s_2 > 0$) is possible, and the unique solution to the optimal sentencing problem consists of a prison term of length (A77) starting at $t = 0$, followed immediately by a parole term of length (A74). We have thus demonstrated (b)(ii). \square

²⁹ One can formally prove the existence of a solution in two steps: (a) showing that optimal term lengths (s_1, s_2) have finite upper bounds, so that the optimal sentencing problem is equivalent to a minimization over a closed and bounded set of term lengths; and (b) then applying the Weierstrass Theorem, which states that any continuous function has a minimum on a closed and bounded set.

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