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The geometry of mortality change: Convex hulls for demographic analysis

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Abstract

We introduce convex hulls as a data visualization and analytic tool for demography. Convex hulls are widely used in computer science, and have been applied in fields such as ecology, but are heretofore underutilized in population studies. We briefly discuss convex hulls, then we show how they may be applied profitably to demography. We do this through three examples, drawn from the relationship between child mortality and adult survivorship (${}_5q_0$ and ${}_{45}p_{15}$ in life table notation). The three examples are: (i) using convex hulls for outlier identification; (ii) for studying sex differences in mortality; and (iii) for studying period and cohort differences. We find, respectively, that convex hulls can be useful in robust compilation of demographic databases, and that the gap/lag framework for sex differences or period/cohort differences is more complex when mortality data are arrayed by two components as opposed to a unidimensional measure such as life expectancy. The potential applicability of these methods goes beyond mortality.

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Introduction

We propose convex hulls as a technique of demographic analysis, illustrated by three examples. The convex hull of a set of points is the polygon defined by a perimeter in which the line segment connecting any two points lies on or inside the perimeter.[1] An informal heuristic is that if a set of points are pegs in a board, the convex hull is the shape of a rubber band stretched around the outermost pegs, such that all the pegs are enclosed by the band. Figure 1 is an example: the data are seven random points in a plane (1A). There are a number of ways to draw a perimeter, of which one is shown (1B). The convex hull, which is unique, is illustrated as a white polygon (1C). The dashed line segments (1D) demonstrate why the region in 1B is not convex. Line segments connecting any two points in the data may be an edge of the convex hull, or interior, but cannot pass outside of it. Convex hulls exist in all dimensions: as a range (line segment) for unidimensional data, as polygons in $\mathbb{R}^2$ ("2D"), as polyhedra in $\mathbb{R}^3$ ("3D"), and as polytopes in higher dimensional spaces. In this paper, we only consider applications in two dimensions.

Calculating a convex hull of multidimensional data is analogous to sorting unidimensional data, in the sense that it determines the boundaries of the data, which in the univariate case is the minimum and maximum (Barnett 1976). For cross-classified data, the x range is the orthogonal projection of the convex hull onto the x-axis, similarly with the y data, and so on if there are more dimensions. Using convex hulls in data analysis is not a new idea. "Tukey peeling", also called convex peeling (Hodge and Austin, 2004), has some similarities to our first application. It entails obtaining robust estimates in multivariate analysis by removing one or more convex hulls from the data, pre-analysis. It dates back to the early 1970s (Huber, 1972), and is further elaborated in Tukey (1975) and Bebbington (1978). The properties of convex hulls of data are fairly well understood, provided the data are reasonably well behaved. There is a large literature here; see, f.e., Efron (1965), Eddy (1980), Aldous et al. (1991), Blackwell (1992), Snyder and Steele (1993), Hueter (1994), Massé (2000), Suri et al. (2013). To

[1] For a concise and more formal definition of convexity, cf. Kemeny and Snell (1962, p. 123); also Kemeny et al. (1966), pp. 312–3: "A convex set C is a set such that whenever u and v are points of C, the entire line segment between u and v also belongs to C." A convex hull consists of vertices, edges that connect these vertices, and the (interior) convex polygon defined by the edges; see figure 1 (p. 3).

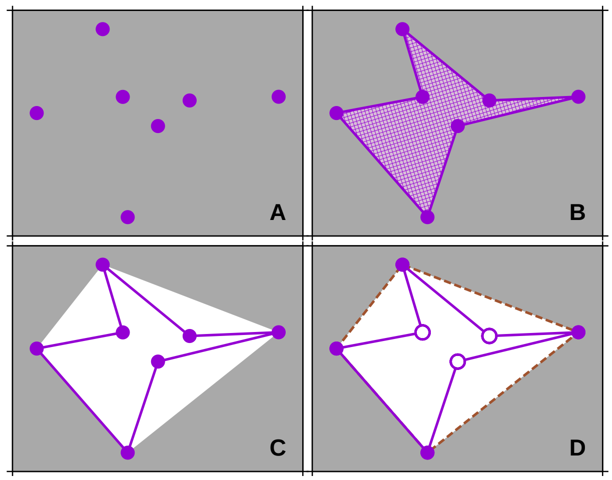


Figure 1: **A**: seven randomly-distributed points in a plane. **B**: one possible perimeter and its (non-convex) polygon (shaded). **C**: the convex hull, in white. **D**: dashed line segments illustrate non-convexity of the original perimeter. Vertices of the convex hull are shown as filled disks, while original points that are members of the convex set (i.e., the white region), but are not hull vertices, are shown as open circles.

the best of our knowledge, these techniques have not been applied in depth in demography. Appearances of "convex hull" in the demographic literature are sparse and arise in conjunction with linear programming solutions (f.e., Georgakis and Tziafetas, 1982), rather than as an analytic tool on its own terms. Wrigley and Schofield (1981) discuss "demographic terrain" (p. 247), similar in spirit to convex hull analysis as we conceptualize it (see also Goldstone 1986, Galloway 1994). Neighboring academic fields, such as ecology, have used convex hulls more (f.e., Getz and Wilmers 2004, Cornwell et al. 2006). Our three applications illustrate the usefulness of convex hulls to population studies.

Materials and Methods

Using data on all countries in the Human Mortality Database (2017, Barbieri et al. 2015), we analyze life table probabilities of child mortality (${}_5q_0$)

Table 1: **Country list: Start and end years.**

	Period		Cohort	
Country	start	end	start	end
Australia	1921	2014		
Austria	1947	2014		
Belarus	1959	2014		
Belgium†	1841	2015		
Bulgaria	1947	2010		
Canada	1921	2011		
Chile	1992	2005		
Czech Republic	1950	2014		
Denmark	1835	2014	1835	1923
Estonia	1959	2013		
Finland	1878	2015	1878	1924
France*	1816	2014	1816	1923
E Germany	1956	2013		
W Germany	1956	2013		
Greece	1981	2013		
Hungary	1950	2014		
Ireland	1950	2014		
Iceland	1838	2013	1838	1922
Israel	1983	2014		
Italy	1872	2012	1872	1921
Japan	1947	2014		
Latvia	1959	2013		
Lithuania	1959	2013		
Luxemburg	1960	2014		
Netherlands	1850	2012	1850	1921
New Zealand (Maori)	1948	2008		
New Zealand (Non-Maori)	1901	2008		
Norway	1846	2014	1846	1923
Poland	1958	2014		
Portugal	1940	2015		
Russia	1959	2014		
Slovakia	1950	2014		
Slovenia	1983	2014		
Spain	1908	2014		
Sweden	1751	2014	1751	1923
Switzerland	1876	2014	1876	1923
Taiwan	1970	2014		
U.K./England & Wales*	1841	2013	1841	1922
U.K./Scotland	1855	2013	1855	1922
U.K./Northern Ireland	1922	2013		
Ukraine	1959	2013		
United States	1933	2015		

Notes: †: missing 1914–1918.
*: total population (i.e., not only civilian)

and adult survivorship (${}_{45}p_{15}$). Table 1 lists the included countries. Throughout, we refer to cross-classification of child mortality and adult survivorship as the *mortality relationship*, and, as applicable, the *mortality hull*. We perform three analyses, the first of which is to examine outlier countries, in which we systematically delete one country at a time, and quantify how the convex hull changes. The second analysis compares male and female convex hulls, on a country-by-country basis. The third compares period and cohort data, on a per-country and per-sex basis. Convex hull calculation is well studied (Preparata and Shamos 1985, de Berg et al. 2008), and is available in many software packages. We used IDL ver. 8.6 (Exelis Visual Information Solutions, Inc., Boulder, Colorado, USA).

Figure 2 (p. 6) is an example of the convex hull approach to the mortality relationship, showing the mortality hull, separately by sex, for the entire data set. Individual countries are color-coded, although most of the data are densely clustered and therefore overlapping. Superposed on figure 2 is a more conventional approach, namely a regression fit of ${}_{45}p_{15}$ as a quadratic function of ${}_{5}q_{0}$, along with its 95% prediction interval.[2] Near the center of mass, the regression line does a good job of representing the tendency of the mortality relationship. However, we think figure 2 is a good illustration of the shortcomings of the parametric curve fitting approach.[3] Particularly away from the center, and even with the prediction interval, the regression line does not represent the variation of the data as well as the convex hull.

The geometric (i.e., convex hull) approach is not meant to be a replacement for curve fitting, but a complement to it. Nonetheless, in many areas of population studies, convex hulls may better capture the inherent variation of the data, especially in situations where the quantities of interest do not have a homoskedastic relationship.[4] The logic of our approach is that when comparing the mortality relationship (or any other multivariate classification) of two or more populations, convex hulls are a natural way to see how the data interleave. This approach is superior to comparing the bi-

[2]This is wider than the 95% confidence interval of the regression curve. See, f.e., Snedecor and Cochran (1989), p. 168; DeGroot and Schervish (2002), p. 614; etc.

[3]An excellent example of the regression approach to two life table quantities working well is Woods and Hinde (1987), p 45.

[4]For quantities analyzed on log scale (f.e., as is often the case with death rates, ${}_{n}M_{x}$), log the data first, then calculate the hull. Although the logarithmic transformation is monotone, it is not affine, so it need not preserve convexity.

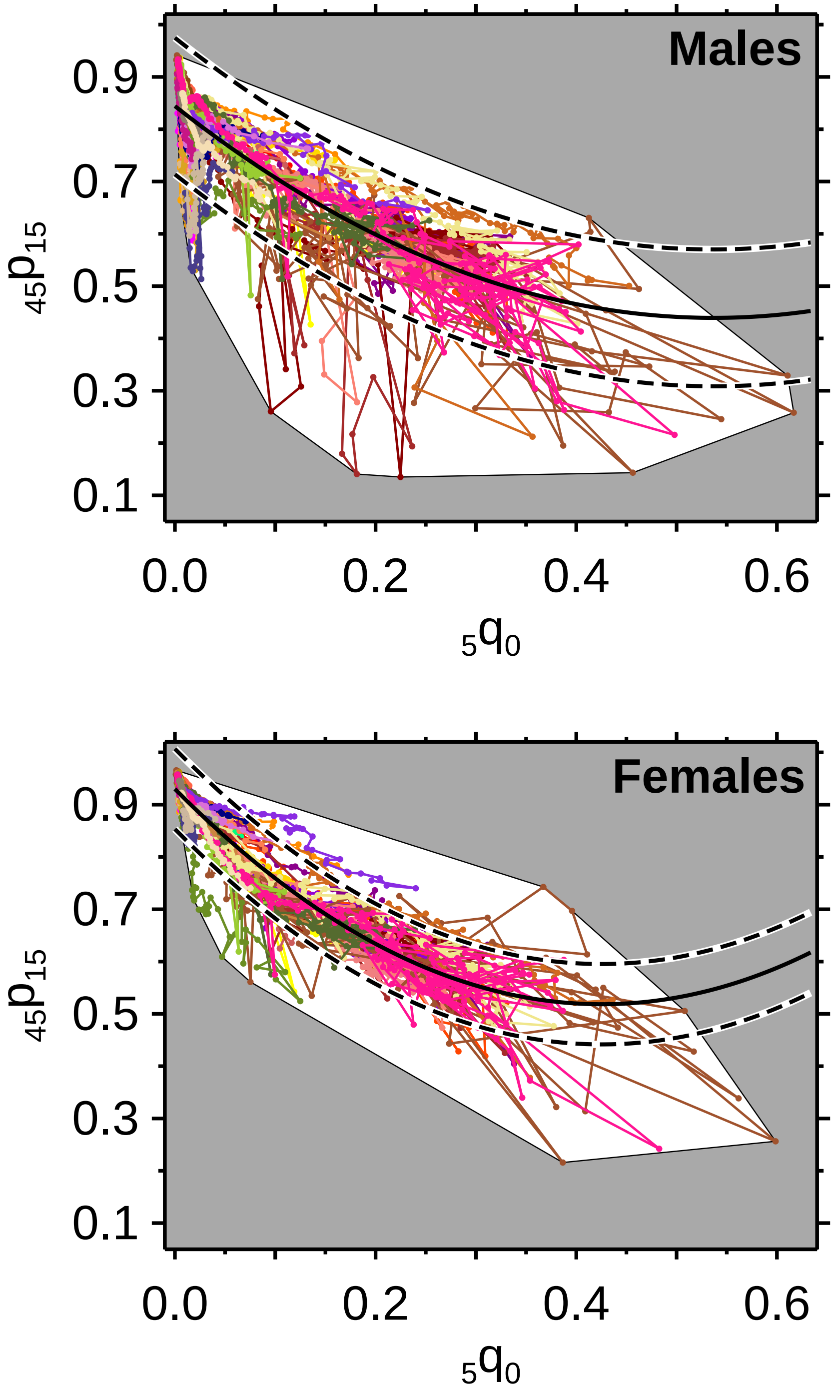


Figure 2: Adult survivorship vs child mortality, by sex. With convex hull, and quadratic regression line (solid) and its associated 95% prediction interval (dashed). Inside the hull, line segments connect chronologically-consecutive points on a per-country basis.

variate ranges, which would replace the hulls with rectangles of potentially much greater area. It is an alternative to fitting curves for each population, and then comparing the regressions.

The cross-classification of child mortality and adult survivorship — what we are calling the mortality relationship — has been studied without hulls, often as ${}_{45}q_{15}$ vs ${}_{5}q_{0}$ (the same thing, for all intents and purposes). In populations with incomplete data, it is common to have only estimates of ${}_{45}p_{15}$ and ${}_{5}q_{0}$ (or similar), from which the rest of the life table is imputed (Timæus and Moultrie, 2013). Even without the use of convex hulls, data quality can be assessed by comparing the mortality relationship of a single country to model predictions (Woods 1993, 2000, p. 375; Rao et al. 2005), or to a battery of countries with good data quality (Reniers et al. 2011, Gerland 2014). Examining the mortality relationship (or other, similar, cross classifications of life table quantities) is a staple of methodological work on model life tables (f.e., Coale and Demeny 1983, Murray et al. 2003, Wilmoth et al. 2012). Convex hull analysis permits quantification of these comparisons.

Results

i) *Country peeling*: Outlier quantification in the HMD

In using convex hulls to identify outliers, we take a country-centered approach, removing one country at a time and seeing how the resulting hull differs from the *master hull* (i.e., the hull of the entire data set, shown in figure 2). This *country peeling* differs from Tukey peeling. In figure 2, the female hull is defined by 13 points from three countries and the male hull by 12 points from six countries. Under Tukey peeling, we would remove these 13 or 12 points, respectively, and examine the modified data set, or peel the next hull. With country peeling, we remove entire countries, one at a time, instead of all the vertices of the master convex hull. *A priori*, the country peeling will have no effect except when the country being peeled is one of those three countries for females (six for males).

When a country is peeled, a new, smaller, hull is calculated to reflect the country-peeled data set. Table 2 lists the component countries of the master hull and the number of points each of those countries contributes to the master hull (column A). The next three columns of table 2 give a

Table 2: **Country peeling**

Peeled country	Number of points contrib. to master CH	Number of points outside peeled hull	Ratio of peeled to master hull: area	Ratio of peeled to master hull: diameter	Number of countries in peeled hull	Number of sides in peeled hull
	(A)	**(B)**	**(C)**	**(D)**	**(E)**	**(F)**
Males						
Belarus	1	5	0.9994	1.0	8	12
Estonia	1	2	0.9997	1.0	7	11
Finland	2	2	0.9736	1.0	7	11
France	1	2	0.9919	1.0	7	11
Iceland	6	13	0.8092	0.9504	11	15
Russia	1	4	0.9977	1.0	6	12
Females						
Belarus	1	3	0.9998	1.0	3	12
Iceland	8	16	0.6527	0.9333	10	16
N.Z. (Maori)	4	27	0.9829	1.0	3	10

counterfactual *as-if-adding* scenario. That is to say, if a country were never in the data set to begin with, and was then added, how much of an outlier would it be relative to the convex hull of the prior mortality relationship. This can be measured by how many points of the country lie outside the prior (i.e., peeled) hull (column B), or as the how large the peeled hull is, relative to the as-if-added hull (viz., the master hull) (column C).

The diameter of a convex hull is the greatest distance between any two vertices. Column D gives the diameter length of the peeled hull relative to that of the master hull. This column reveals an idiosyncrasy of the HMD data, namely that Iceland populates the hull at both ends, high child mortality/low adult survivorship (in the 19th century), and low child mortality/high adult survivorship (in the 21st century). The endpoints of the diameter of the master hull (known as the antipodes) need not be drawn from the same country, but in the HMD data set, they are. Thus, the diameter changes if and only if Iceland is the peeled country, as column D shows. Column E gives the number of component countries of the peeled hull. For both sexes, Iceland stands out as having a peeled hull with nearly the same number of component countries as the master hull, in contrast to the other peeled countries where this number declines more. Column F

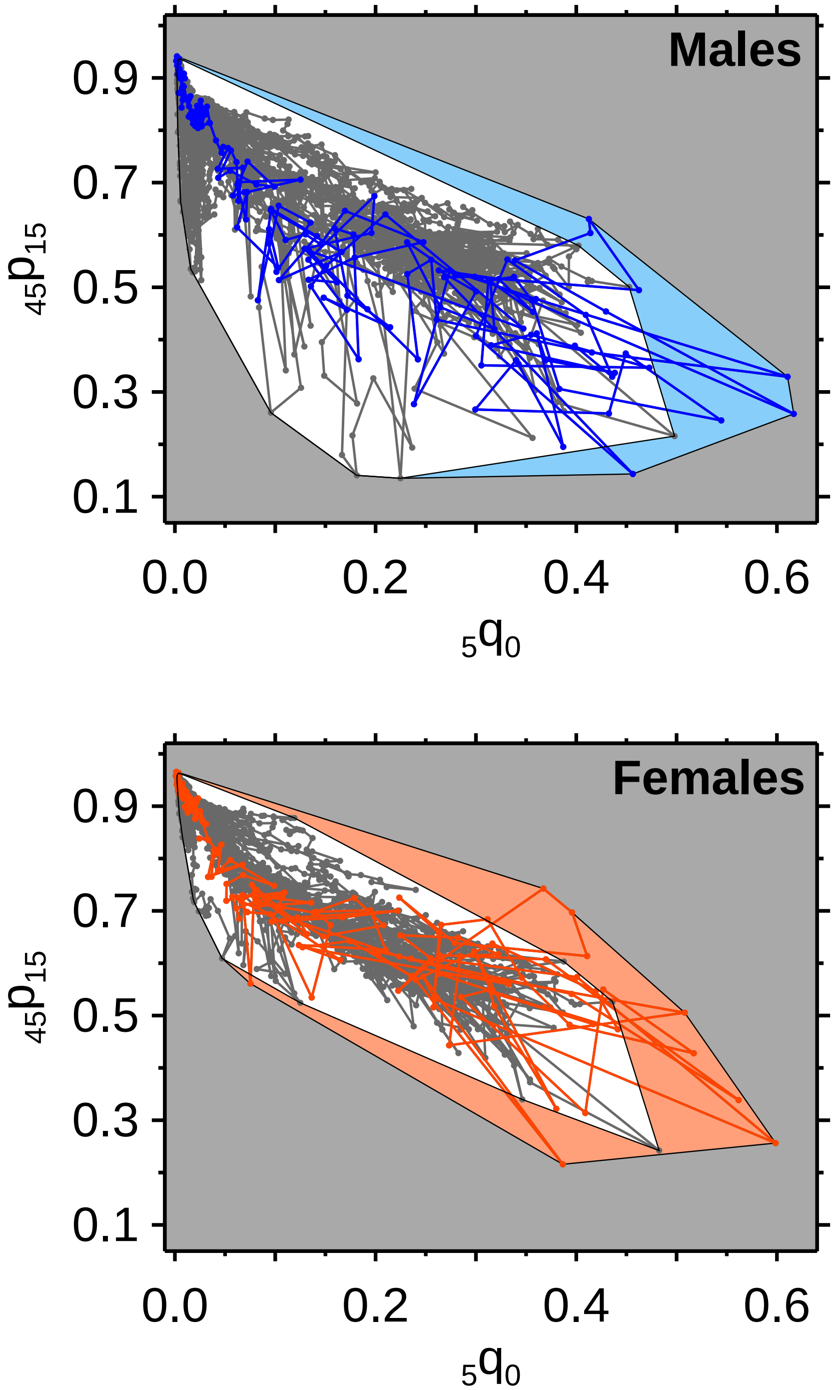


Figure 3: Adult survivorship vs child mortality, by sex. For each sex, two hulls are shown: the outer, colored, hull is the master convex hull (the same as shown in figure 2). The inner, white, hull is the result of peeling Iceland, the data of which is shown in color.

gives the number of sides of the peeled hull.[5] This is a measure of the topological complexity of the hull. In all cases except Iceland, the peeled hull has the same number of sides, or fewer, compared to the master hull. The Iceland-peeled hull for both sexes has more sides than the master hull. As with the prior columns, this descriptive statistic identifies Iceland as being qualitatively different.

Country peeling is a technique for outlier detection: it answers the question, "if country X were being added to the HMD for the first time, how different would it be from the existing countries?" Table 2 shows that most countries are similar in the mortality relationship. Indeed, for females, only three countries would be flagged as outliers if being added (one at a time) to the HMD for the first time. The most severe outlier for both sexes, as quantified by table 2, is Iceland. Country peeling for Iceland is shown graphically in figure 3. The inner, peeled, hull (in white, consisting of points from 10 countries for females and 11 for males) is smaller in area and has a smaller diameter than the master hull.

Iceland is an outlier in large part because it has a long data series (the fourth longest in the HMD, cf. table 1), and it goes from the being worst ${}_5q_0$ performer among the small set of HMD polities in the 1830s and 1840s (when it was a colony of Denmark), to, often, the best in the twenty-first century. Iceland is now a highly developed country with excellent health statistics. Historically, Iceland experienced mortality crises (Schleisner 1851, Tomasson 1977), some of which were associated with the tail end of Europe's "Little Ice Age" (Vasey, 2001). Most countries are not outliers, reflecting commonalities in the mortality relationship in the HMD data. The take home message of this section is that convex hulls are an effective tool for qualitatively identifying outliers, as well as quantifying their degree of outlierness. The following section looks at intra-country analysis of the mortality relationship by sex, and shows that convex hulls are not just for outlier detection.

[5] This is the same as the number of vertices. Some algorithms will potentially return three colinear points — i.e., what is really one side of a hull would be two sides, if counting vertices. This can arise with gridded data, but is unlikely with empirical data. We check for this, but there are no such instances.

ii) Sex differences

Much of population studies concerns time series of demographic phenomena (life expectancy, total fertility rate, etc.). *Gaps and lags* is one way to conceptualize the movement of two time series where each one is measuring the same quantity for different (but related) populations. Figure 4 illustrates this for American males and females; each panel shows a different mortality measure: figure 4A, child mortality (${}_5q_0$); figure 4B, adult survivorship (${}_{45}p_{15}$). The separations between the male and female series can be regarded as a gap (along the vertical axis, shown in white), or as a lag (along the horizontal axis, shown in black). In 1945, the male-female gap in child mortality was 11 per thousand. Or, one could say that the males would take 4.5 more years to achieve the equivalent ${}_5q_0$ as females in 1945 (a lag). Goldstein and Wachter (2006) formalized the gaps and lags framework, using periods and cohorts as the population dichotomy. As we show here without the formalism, this framework also applies to sex differences.

In the univariate time series approach, gaps can be recast as lags. In terms of mortality decline, males and females plough the same ground, but the female mortality advantage (or gap) means that males do so later. Figure 5 shows the convex hull approach to this problem. Consider first the "ISL" panel, for Iceland. We see exactly the phenomenon of males following in the path of females: the hulls are substantially overlapping. Given that the time period is the same for each sex, we should not expect total overlap. Males begin the series with mortality levels higher than seen in females, and females end the series with lower mortality than seen in males for the same time interval. Thus, we expect two regions, at opposite ends of the space, where the hulls do not overlap. This is precisely what the convex hull plot for Iceland shows.

For the United States (figure 5, "USA"), the convex hull analysis reveals a different pattern. Unlike the univariate time series in figure 4, the males' mortality relationship does not follow in the footsteps of the females'. The convex hulls are disjoint, indicating that males and females are not playing follow the leader, but are taking different paths. The disparate lags in figure 4 (i.e., 35 years for adult survivorship but only 4.5 years for child mortality) drive the disjointness of the male and female hulls. The long lag for the adult survivorship data is thought to be due principally to behavioral influences, especially tobacco use (Pampel 2002). While a careful read of figure 4 would allow one to predict the divergent paths over time,

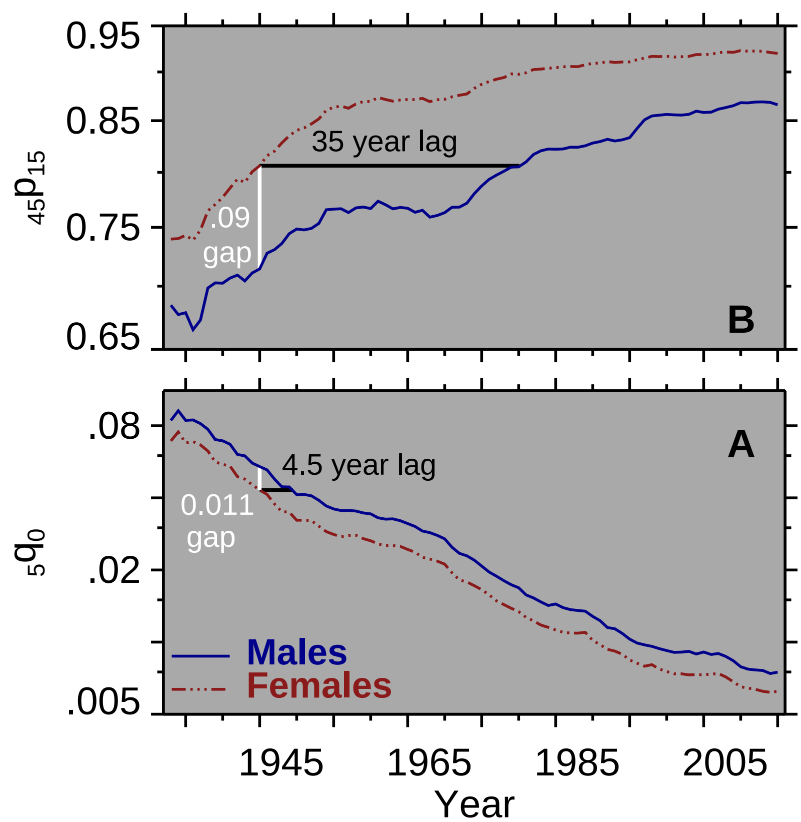


Figure 4: Sex differences in mortality, *gaps and lags* perspective. Female advantage, which is the typical, can be viewed as period gap, or as a lag of males, taking them a certain number of years to catch up. A: $_{5}q_{0}$, child mortality; B: $_{45}p_{15}$, adult survivorship. All data from HMD, for the United States.

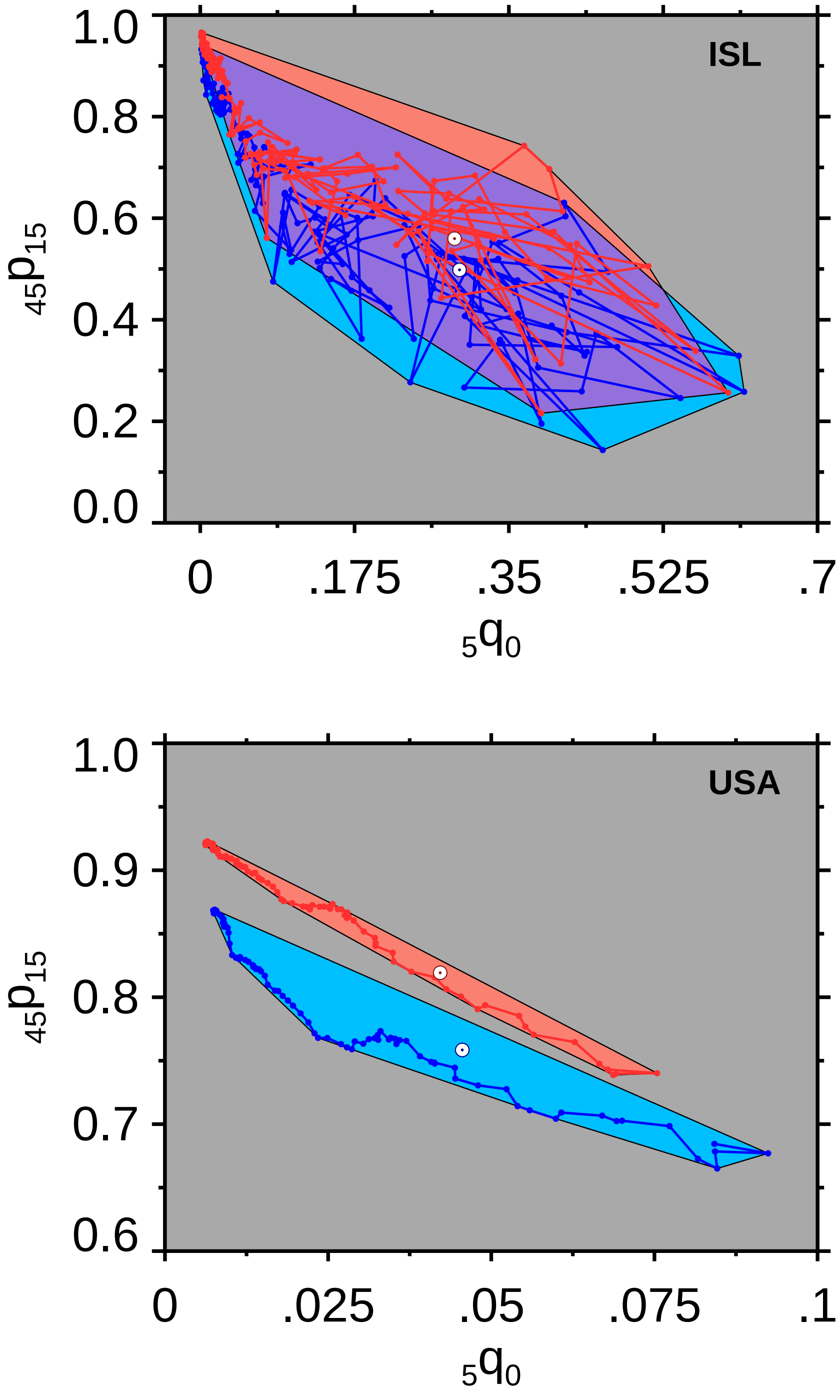


Figure 5: Male (blue) and female (red) sex-specific hulls for the mortality relationship, Iceland (ISL) and United States (USA). Overlap shown in purple. Bullseyes mark the centroids of the hulls.

Table 3: **Male and female hulls: Descriptive and comparative statistics.**

Country	Female to male Area Ratio	Female to male Diameter Ratio	Intersection area as % of M	Intersection area as % of F	% of points in intersection M	% of points in intersection F	Euclidean dist. btwn. centroids	Centroid in opposite hull? M	Centroid in opposite hull? F
	(A)	**(B)**	**(C)**	**(D)**	**(E)**	**(F)**	**(G)**	**(H)**	**(I)**
Australia	0.518	0.869	18.9	36.4	0.0	58.3	0.041	No	No
Austria	0.346	0.777		*disjoint*			0.072	—	
Belarus	0.231	0.295		*disjoint*			0.162	—	
Belgium	0.666	0.985	58.4	87.6	0.0	35.7	0.022	Yes	Yes
Bulgaria	0.292	1.049	0.3	1.1	11.1	0.0	0.074	No	No
Canada	0.439	1.044	34.0	77.4	0.0	31.2	0.016	Yes	Yes
Chile	0.274	0.585		*disjoint*			0.074	—	
Czech Republic	0.337	0.831		*disjoint*			0.086	—	
Denmark	0.936	0.941	78.9	84.3	18.2	40.0	0.039	Yes	Yes
Estonia	0.291	0.368		*disjoint*			0.163	—	
Finland	0.285	0.574	20.9	73.4	0.0	35.7	0.210	No	Yes
France	0.412	0.793	36.4	88.3	0.0	30.0	0.161	No	Yes
E Germany	0.385	0.913		*disjoint*			0.077	—	
W Germany	0.346	0.635		*disjoint*			0.067	—	
Greece	0.308	0.819		*disjoint*			0.059	—	
Hungary	0.286	0.854		*disjoint*			0.109	—	
Ireland	0.484	0.976	12.1	25.0	8.3	27.3	0.039	No	No
Iceland	0.914	1.009	80.8	88.4	20.0	50.0	0.062	Yes	Yes
Israel	0.568	0.613		*disjoint*			0.043	—	
Italy	0.428	0.851	39.6	92.5	0.0	10.0	0.083	No	Yes
Japan	0.809	0.881	19.0	23.5	0.0	37.5	0.044	No	No
Latvia	0.233	0.338		*disjoint*			0.187	—	
Lithuania	0.194	0.336		*disjoint*			0.163	—	
Luxemburg	0.366	0.573	3.9	10.7	0.0	20.0	0.083	No	No
Netherlands	0.645	0.940	61.2	94.9	0.0	72.7	0.032	Yes	Yes
N.Z. (Maori)	1.022	1.472	73.2	71.6	33.3	40.0	0.009	Yes	Yes
N.Z. (Non-Maori)	0.580	0.754	52.5	90.5	0.0	57.1	0.062	Yes	Yes
Norway	0.804	0.905	68.8	85.5	0.0	42.9	0.042	Yes	Yes
Poland	0.220	0.828		*disjoint*			0.118	—	
Portugal	0.622	0.912		*disjoint*			0.083	—	
Russia	0.274	0.307		*disjoint*			0.219	—	
Slovakia	0.368	1.093		*disjoint*			0.097	—	
Slovenia	0.322	0.404		*disjoint*			0.108	—	
Spain	0.555	0.987	42.7	77.0	0.0	9.1	0.049	Yes	Yes
Sweden	0.883	0.988	83.3	94.3	0.0	50.0	0.035	Yes	Yes
Switzerland	0.632	0.890	56.8	89.8	0.0	33.3	0.066	Yes	Yes
Taiwan	0.907	0.969		*disjoint*			0.079	—	
England and Wales	0.296	0.815	25.6	86.4	0.0	57.1	0.123	No	Yes
Scotland	0.748	0.959	44.1	59.0	8.3	50.0	0.042	Yes	Yes
Northern Ireland	0.661	1.131	52.0	78.7	0.0	40.0	0.007	Yes	Yes
Ukraine	0.225	0.230		*disjoint*			0.166	—	
United States	0.332	0.896		*disjoint*			0.061	—	

the hull approach reveals this much more clearly. The Iceland data goes back to 1838, and high variance contributes to the overlap of the hulls. The American data begin in 1933 and show less variance, with thinner hulls. Of the twenty countries with disjoint hulls, (table 3), all have data beginning after the Second World War, except Portugal (1940) and the United States (1933) (table 1). With improvements in nutrition, the advent of antibiotic drugs, and so on, the postwar mortality regime is lower variance (at least in the HMD member countries), which favors these disjoint hulls. However, Austria (1947), Bulgaria (1947), Ireland (1950), Japan (1947), Luxembourg (1960), and New Zealand/Maori (1948) are all exclusively postwar data, yet have overlapping hulls, so the overlap is not exclusively driven by noisy prewar data.

Table 3 gives comparative descriptive statistics of the sex-specific hulls, on a per-country basis. Columns A and B give the area and diameter ratio, respectively, of the female to male male hull. Male mortality has a higher variance than that of females, and as a result, all the male hulls have larger area than the country-corresponding female hull, except New Zealand/Maori. The topology of the hulls is complex, and despite the area statistics, six male hulls have shorter diagonals than the corresponding female hull. Columns C and D give the intersection area as a percent of male and female fulls. The most interesting aspect of these columns is that almost half the hulls (20/42) are disjoint across the sexes, indicating that male mortality decline, as measured by the mortality relationship, does not follow in the footsteps of the female mortality decline.

Columns columns E and F give the percentage of points of male and female hulls contained within their intersection. Even for the non-disjoint hulls, this statistic may be zero, since the hulls trace out area in regions where they are not populated by data. As noted, the male hulls are larger, but of the 22 hulls that overlap, for 16 of them there are no male data points populating the overlapping region. Conversely, none of the overlapping regions are devoid of female data points.[6] This is an indication that that male hulls are larger in a meaningful sense. The distance between the centroids of each hull is given in column G. The male and female hull centroids that are furthest apart are Russia (0.22) and Finland (0.21); these hulls are dis-

[6]An overlapping region must contain at least one hull vertex, thus it is impossible for both columns E and F to be zero in the same country, unless the hulls are disjoint. (Edges that touch but which do not share vertex are not considered to be an overlapping region.)

joint and overlapping, respectively. This shows that variance as well as location drives the overlap/non-overlap of the hulls. Indicators for whether the hull centroids lie inside the opposite-sex hull are given in columns H and I. In 13 of the 22 overlapping hulls, the overlap region contains the centroids of both convex hulls.

When mortality is summarized in more than one dimension, a male-female gap cannot be construed to be the same as a lag. Table 3 shows a diversity of relationships. The male and female mortality hulls are, in many cases, disjoint within the same national population. For some countries, the hulls are quite different from disjoint, but contain each other's center, at least as measured by the centroid. Predicting the future of male mortality as catching-up to female mortality makes sense on a unidimensional basis, but should be done with caution when dealing with ${}_5q_0$ and ${}_{45}p_{15}$ considered together.

iii) Period and cohort

As an example of convex hulls applied to period and cohort data, the Finnish mortality relationship is shown in figure 6; graphs for the ten other countries with cohort data (see table 1) are in Appendix III (p. 34). The female (red) and male (blue) convex hulls represent the period data in the same years as the cohort data series (yellow). Where the period and cohort hulls overlap is shaded orange for females and green for males. The colored regions in these graphs are like-for-like comparisons of years (1878–1924 in the Finland example). The union of the red or blue period hull and the white (non-convex) region forms the convex hull of the entire period data set (1878–2015 in the Finland example). Consider the data projected onto the horizontal axis (child mortality). Cohort ${}_5q_0$ is a combination of ${}_1q_x$ for x=0,...,4, in five consecutive years of time. The most important component, ${}_1q_0$, is from the same calendar year for both period and cohort data. Thus, the period and cohort spreads along the horizontal axis are very similar. The forty-five year time/age span of ${}_{45}p_{15}$ brings out more profound differences. For Finnish females, the ranges are similar in length, but overlap little, and hulls are disjoint. Whereas among the males, the range of cohort ${}_{45}p_{15}$ data is much smaller than that of the period data, and the cohort hull is semi-embedded in the period hull. Reflecting the influenza pandemic (Ansart et al. 2009) and the Finnish Civil War (Turpeinen 1979),

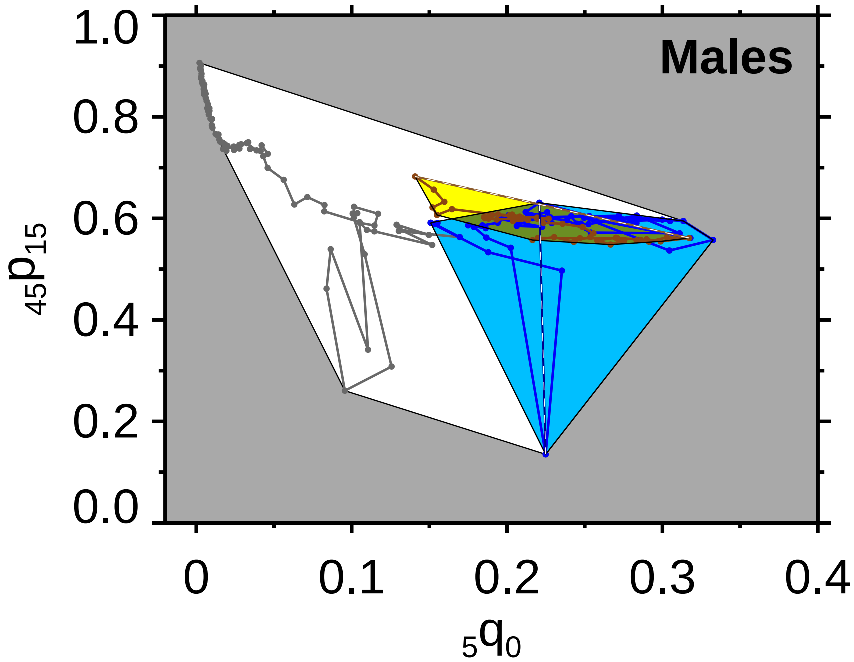
Table 4: **Cohort and period hulls: Descriptive and comparative statistics.**

		period/cohort		perimeter/area		% of cohort	diagonal	canonical corr.	
Country	sex	area	area*	cohort	period	outside per.	correlation	1st	2nd
		(A)	**(B)**	**(C)**	**(D)**	**(E)**	**(F)**	**(G)**	
Denmark	M	1.525	1.157	43.09	34.41	*disjoint*	0.978	0.956	0.295
	F	1.697	1.339	54.21	33.17	95.2	0.996	0.958	0.389
Finland	M	5.118	4.912	52.27	24.15	32.2	0.573	0.854	0.097
	F	1.090	1.167	51.05	40.43	*disjoint*	0.905	0.856	0.417
France	M	3.220	2.829	29.77	15.09	36.6	0.952	0.936	0.211
	F	1.431	1.866	46.13	33.82	90.6	0.996	0.943	0.238
Iceland	M	2.235	1.407	18.14	9.33	35.3	0.999	0.887	0.110
	F	2.524	1.498	19.14	10.01	44.6	0.987	0.874	0.076
Italy	M	3.635	2.288	30.32	16.34	68.7	0.654	0.949	0.331
	F	2.272	1.544	48.38	29.87	*disjoint*	0.950	0.968	0.414
Netherlands	M	1.525	1.394	31.93	28.19	99.4	0.960	0.978	0.543
	F	1.433	0.950	36.91	28.99	98.6	0.999	0.975	0.418
Norway	M	2.077	1.980	54.54	29.12	82.9	0.997	0.937	0.259
	F	1.385	1.384	52.61	33.89	88.9	0.946	0.927	0.256
Sweden	M	3.318	2.874	35.26	15.00	35.4	1.000	0.934	0.205
	F	3.853	3.359	45.39	16.77	40.8	0.994	0.927	0.199
Switzerland	M	3.258	2.353	70.82	27.39	*disjoint*	0.997	0.987	0.128
	F	2.319	1.689	79.52	37.50	*disjoint*	1.000	0.986	0.131
England and Wales	M	2.550	2.188	40.73	22.52	86.6	0.890	0.970	0.035
	F	0.776	0.614	41.63	49.33	95.3	1.000	0.970	0.344
Scotland	M	1.045	1.364	54.33	51.08	90.2	0.994	0.919	0.269
	F	0.844	0.891	55.33	51.47	92.0	0.961	0.927	0.279

* comparison of full-extent period data, normalized (i.e., per number of years)

the minimum ${}_{45}p_{15}$ data point in the period hull corresponds to 1918. The 1939–40 Finno-Soviet war did not occur during the time span of the cohort data; its effects can be seen in the white region.

As seen in Finnish females and the graphs in Appendix III, the cohort hull “floats” above the period hull. This is the analogue of the gap phenomenon in the time series approach (Goldstein and Wachter 2006). Nonetheless, with the mortality relationship, the story is more complicated than gaps and lags. The overlapping regions are generally small, and five sets of hulls are disjoint, including Finnish females. Considered one dimension at a time, the time series experience the gaps and lags. In general, however, the period hulls do not lag the cohort hulls (viz., cover the same ground) — in fact, some of the period and cohort hulls are disjoint.



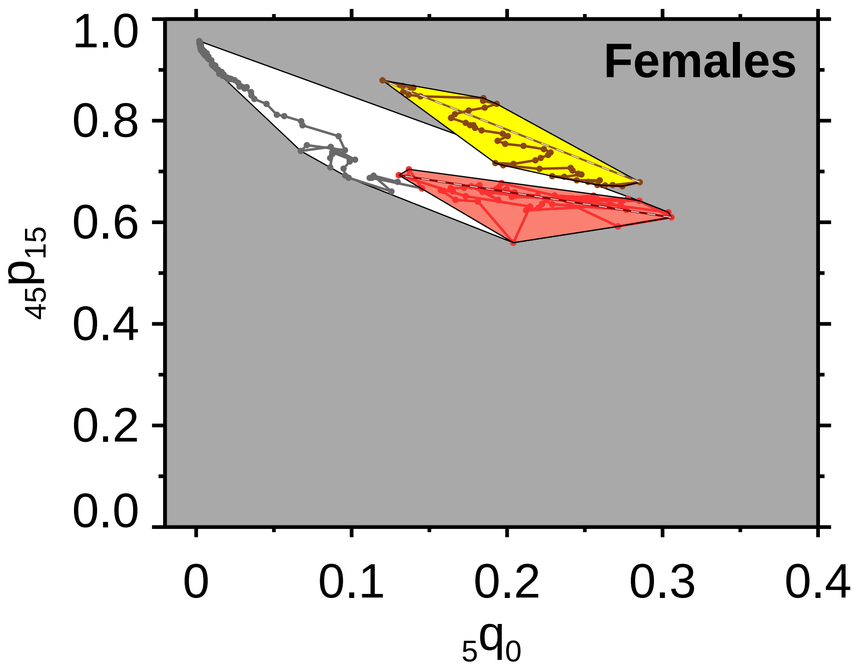


Figure 6: Period and cohort mortality relationship for Finland; data are harmonized so that the period and cohort years coincide. For the period hulls, males are blue and females are red. The cohort mortality relationship is shown in yellow (or green, where it overlaps with male period data). Dashed lines denote the diagonals of the hulls. Underplotted in white is the convex hull for the entire extent of the period data; the red or blue hull partially overlaps this white hull, by definition. The *visible* white region is therefore not convex, but the union of the white and red/blue regions (ignoring the overlapping yellow hull) is convex.

Table 4 (p. 17) gives descriptive statistics for the cohort and period hulls. As in the previous section, we report ratios of one hull to another. Apart from column B, all the statistics refer to the period data that are coincident with the cohort data. Column A gives the period to cohort area ratio, which, in the HMD data, is always > 1, except females in Great Britain (analyzed below). This regularity occurs because the range of the ${}_{45}p_{15}$ is greater for the period data. Column B provides the same statistic, but includes all of the available period data (i.e., including the white region in figure 6), and therefore the period hull encloses many more data points than that of the cohort. To achieve a meaningful statistic, we normalized the area by the number of years; this is not necessary in any other column. As with column A, most of the ratios are > 1, with females in Great Britain and the Netherlands being exceptions. As a measure of spread, columns C and D give the perimeter to area ratios for the cohort and period hulls, respectively.[7] This is notable because only one population (England and Wales, females) has period perimeter-to-area ratio exceeding that for cohort.

Column E gives the percentage of the cohort hull area that lies outside the period hull. Most have some overlap, but five hulls (out of 22) are disjoint, including both sexes for the Swiss data. Finnish males are the most overlapping, with all but 32% of the cohort hull enveloped by the period hull. We also introduce the *diagonal correlation* (column F), or the cosine of the angle between the diagonals.[8] The diagonal correlation provides a rough, but useful, dimensionless measure of how parallel, so to say, the two hulls are. While diagonals may be perfectly parallel — and three of the hulls indeed have perfect diagonal correlation (i.e., rounded up to 1.0) — the hulls themselves are polygons which do not have correlations in the traditional sense. Nonetheless, the Finnish example demonstrates the utility of the diagonal correlation measure. For males, the diagonal correlation is 0.57 while for females it is 0.91. Compare these to the hulls illustrated in

[7] As measured by this statistic, a circle has the minimum spread of any convex shape. For a circle, the distance to the furthest point from the center is the radius, r, and the perimeter to area ratio is $2/r$. A square of the same area has a perimeter of $4\sqrt{\pi}r$, and a perimeter to area ratio of $(2/r)(2/\sqrt{\pi}) > 2/r$. The distance to the furthest point from the center of the square is $r\sqrt{\pi/2} > r$. Thus, the square of the same area has more spread and a greater perimeter to area ratio, and so on.

[8] In the literature, we have not found any references to the term diagonal correlation as it relates to polygons, nor the use of this quantity as we define it.

figure 6, where the alignment (using a non-rigorous, intuitive meaning) of the two hulls seems much better for females.

The diagonal correlation is more informative than at least one conventional approach: column G gives the first and second canonical correlations (Hotelling, 1936) of the period and cohort data (not just the hull vertices).[9] While Finnish males stand out in the hull diagonal correlations (column F), there is nothing unusual about their canonical correlations (column G). This is not to say that standard techniques cannot detect something different about Finnish males relative to other HMD countries. In country×sex-specific OLS regressions of year-matched ${}_{45}p_{15}$ and ${}_{5}q_{0}$ data (period and cohort, pooled) including a dummy variable for period or cohort, the dummy is *not* statistically significant only for Finnish males. Thus, equipped only with OLS regression, no visualization, and no convex hulls, one could discern that something is different about Finland. Nonetheless, if customary techniques were unable to confirm putative outlier status identified by the hull approach, the idea of convex hulls for demographic data analysis would be moot.

Table 4 also points to England and Wales, females, as being different. Figure 7 focuses on the cohort data for England and Wales, females. Consider two subhulls, one from the start of the data until the temporal midpoint, and another from the midpoint+1 until the end of the data. These subhulls are superposed on the hull of the cohort mortality relationship in figure 7. The subhull for the earlier half of the birth cohorts has horizontal hatching, and that for the later half has vertical hatching. The union of these two subhulls forms a non-convex region which envelops all the cohort data for England and Wales females. The ratio of the area of the hatched region to the area of the convex hull is called the convexity index (Tanimoto, 1987, p. 427); in this example, it is 0.497. Using the data midpoint as the pivot point of the subhulls, this the lowest convexity index in the HMD data set among females.

The bottom panel of figure 7 gives a time series plot of ${}_{45}p_{15}$ and ${}_{5}q_{0}$; the midpoint (1881) is shown as a vertical gray rule, and a 19-year period from 1880 is shaded light gray. This part of the figure helps explain why England and Wales females have the lowest convexity index, and are outliers, in general, in table 4. During the period 1880–99, cohort child mortality stagnates

[9]Canonical correlations calculated with Stata v.13.1, StatCorp LLC, College Station, Texas.

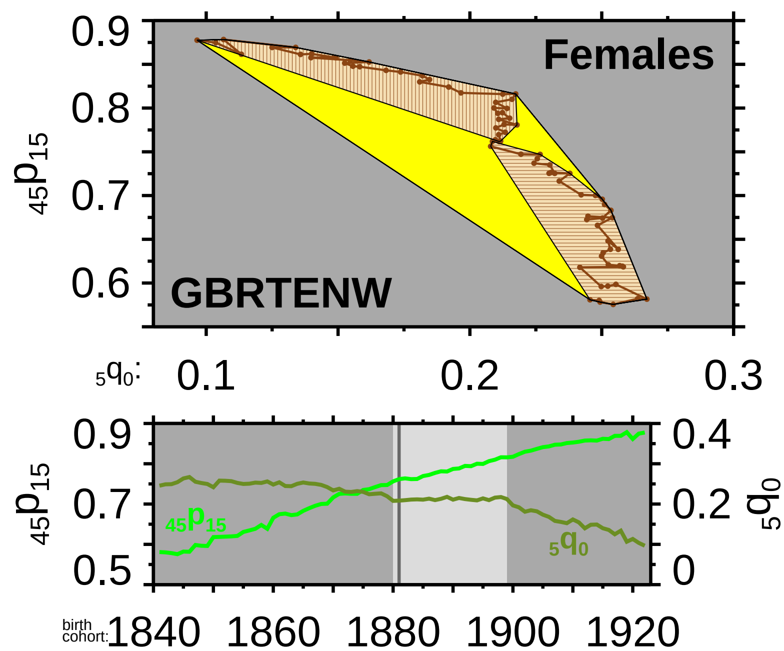


Figure 7: Top panel ("GBRTENW"): cohort mortality hull for England and Wales females. Also shown are the sub-hulls for the first half of the data (birth cohorts 1841–1881, with horizontal hatching) and the second half (1882–1922, with vertical hatching). Bottom panel: time series plot of ${}_5q_0$ (right axis) and ${}_{45}p_{15}$ (left axis); the axes have different ranges but the same extent, so slopes are comparable. A 19-year period starting in 1880 is indicated by different shading; this is a period of stagnation of cohort ${}_5q_0$.

while adult survivorship continues to rise. This unusual pattern causes the mortality relationship to rise without much horizontal displacement. This is seen clearly in the second (vertically-hatched) subhull in figure 7. As a result, the cohort convex hull is relatively larger than that of the other hulls, which is accounts for the unusual descriptive statistics for England and Wales females in table 4. The convexity index analysis helps bring this into focus.[10] The usual pattern of improvements along both axes of the mortality relationship is interrupted in this example, creating an outlier. Mortality decline in Victorian England is well-studied, and while trends in ${}_5q_0$ have been shown before (Woods et al., 1988), we are unaware of comments on the unusual stagnation of cohort ${}_{45}p_{15}$ relative to ${}_5q_0$, compared to other countries. This is another illustration of the strength of the convex hull approach.

Similar to the analysis of the male and female mortality relationship, the convex hull analysis of periods and cohorts shows that when mortality is cross-classified as the mortality relationship, period and cohort relationships are not well described in terms of gaps and lags, but have a more complex relationship. That is not especially surprising — given that the mortality relationship is not life expectancy and does not have the same dimensionality — it should behave differently. Nonetheless, convex hull analysis helps bring out some interesting aspects of the mortality history more efficinetly than looking one dimension at a time. What is more, the convex hull approach to the mortality relationship allows quantitative characterizations of the patterns and how they relate to one another.

Conclusion

The goal of this work is to introduce convex hull analysis to demography, as a tool for exploratory data analysis (in the sense of Tukey, 1977). Further refinements are possible, such as hypothesis testing (see, f.e., Rogers, 1978) — although there are substantial difficulties using convex hulls as inferential tools when the analyzed quantities don't follow standard distributions such as normal, which is often the case in demography.

[10] It helps that, coincidentally, the start of the period of ${}_5q_0$ stagnation is very close to the midpoint of cohort series, thus the temporally-later sub-hull has a much greater area than it otherwise would.

A number of the interesting findings brought out by the convex hulls — for example female mortality in Victorian England and Wales, or Iceland's move from worst to first — can be divined by more standard techniques. We suggest that convex hulls bring these patterns into sharp relief when they might otherwise hide in the data. Independent verification of what is happening using more time-tested techniques does not seem to us to be a weakness of the convex hull approach. Indeed, one of the themes of Gnanadesikan's fine textbook (1997) on multivariate methods is that more often than not, there are multiple ways to get the same substantive answer to a question involving cross-classified data. We see convex hulls as fitting neatly into that idea.

One of the strengths of our approach is that we use convex hulls as a descriptive tool, and as such there are no assumptions that can be violated. However, convex hulls are only as good as the data used to construct them, so are not without potential limitations. Convex hulls are determined by extreme values. As such, convex hulls should prove to be quite useful in the identification of defective data in that outliers become readily apparent. Where outlying observations cause especial difficulties, the data may be (appropriately enough) Tukey peeled one or more times prior to the main analysis.

This is the first work of which we are aware that makes extensive use of convex hulls as an analytic tool or framework for population data. We hope our analysis demonstrates convex hull analysis as a promising tool for demographers and we encourage population scientists to consider their use. Convex hulls are a tool that compliment standard approaches, and they are not proposed as a replacement for anything. The applicability of convex hulls in population studies is not limited to the mortality relationship. For example, application to the demographic transition (Kirk, 1996) seems especially promising. Historical demography seems like another area in which convex hulls could be profitably applied, with log(GRR) plots (Wachter, 2014, p. 133) particularly inviting.

Acknowledgments

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Works Cited

Aldous, David J., Bert Fristedt, Philip S. Griffin, and William E. Pruitt. 1991. "The number of extreme points in the convex hull of a random sample." *Journal of Applied Probability* 28(2):287–304.

Ansart, Séverine, Camille Pelat, Pierre-Yves Boelle, Fabrice Carrat, Antoine Flahault, and Alain-Jacques Valleron. 2009. "Mortality burden of the 1918–1919 influenza pandemic in Europe." *Influenza and Other Respiratory Viruses* 3(3):99–106.

Barbieri, Magali, John R. Wilmoth, Vladimir M Shkolnikov, Dana Glei, Domantas Jasilionis, Dmitri Jdanov, Carl Boe, Timothy Riffe, Pavel Grigoriev, and Celeste Winant. 2015. "Data resource profile: The Human Mortality Database (HMD)." *International Journal of Epidemiology* 44(5):1549–1556.

Barnett, V. 1976. "The ordering of multivariate data." *Journal of the Royal Statistical Society, Series A* 139(3):318–355.

Bebbington, A. C. 1978. "A method of bivariate trimming for robust estimation of the correlation coefficient." *Applied Statistics* 27(3):221–226.

Blackwell, Paul. 1992. "Convex hulls of selected subsets of a Poisson process." *Journal of Applied Probability* 29(4):814–824.

Coale, Ansley J. and Paul Demeny. 1983. *Regional model life tables and stable populations*. Academic Press, San Diego, second ed.

Cornwell, William K., Dylan W. Schwilk, and David D. Ackerly. 2006. "A trait-based test for habitat filtering: Convex hull volume." *Ecology* 87(6):1465–1471.

de Berg, Mark, Otfried Cheong, Marc van Kreveld, and Mark Overmars. 2008. *Computational geometry: Algorithms and applications*. Springer, Berlin, third ed.

DeGroot, Morris H. and Mark J. Schervish. 2002. *Probability and statistics*. Addison-Wesley, Reading, MA, third ed.

Eddy, William F. 1980. "The distribution of the convex hull of a Gaussian sample." *Journal of Applied Probability* 17(3):686–695.

Efron, Bradley. 1965. "The convex hull of a random set of points." *Biometrika* 52(3/4):331–343.

Galloway, Patrick R. 1994. "A reconstruction of the population of North Italy from 1650 to 1881 using annual inverse projection with comparisons to England, France, and Sweden." *European Journal of Population* 10(3):223–274.

Georgakis, Giannis and George Tziafetas. 1982. "Stochastic projection of the population distribution in Greece." *European Demographic Information Bulletin* 13(3):120–134.

Gerland, Patrick. 2014. "UN Population Division's methodology in preparing base population for projections: Case study for India." *Asian Population Studies* 10(3):274–303.

Getz, Wayne M. and Christopher C. Wilmers. 2004. "A local nearest-neighbor convex-hull construction of home ranges and utilization distributions." *Ecography* 27(4):489–505.

Gnanadesikan, Ramanathan. 1997. *Methods for statistical data analysis of multivariate observations*. Wiley, New York, second ed.

Goldstein, Joshua R. and Kenneth W. Wachter. 2006. "Relationships between period and cohort life expectancy: Gaps and lags." *Population Studies* 60(3):257–269.

Goldstone, J. A. 1986. "The demographic revolution in England: A re-examination." *Population Studies* 40(1):5–33.

Hodge, Victoria and Jim Austin. 2004. "A survey of outlier detection methodologies." *Artificial Intelligence Review* 22(2):85–126.

Hotelling, Harold. 1936. "Relations between two sets of variates." *Biometrika* 28(3/4):321–377.

Huber, Peter J. 1972. "Robust statistics: A review." *Annals of Mathematical Statistics* 43(4):1041–1067.

Hueter, Irene. 1994. "The convex hull of a normal sample." *Advances in Applied Probability* 26(4):855–875.

Human Mortality Database. 2017. `http://www.mortality.org/`. Accessed 16 March 2017.

Kemeny, John G. and J. Laurie Snell. 1962. *Mathematical models in the social sciences*. MIT Press, Cambridge, Massachusetts.

Kemeny, John G., J. Laurie Snell, and Gerald L. Thompson. 1966. *Introduction to finite mathematics*. Prentice-Hall, Englewood Cliffs, New Jersey, second ed.

Kirk, Dudley. 1996. "Demographic transition theory." *Population Studies* 50(3):361–387.

Massé, Bruno. 2000. "On the LLN for the number of vertices of a random convex hull." *Advances in Applied Probability* 32(3):675–681.

Murray, C. J. L., B. D. Ferguson, A. D. Lopez, M. Guillot, J. A. Salomon, and O. Ahmad. 2003. "Modified logit life table system: Principles, empirical validation, and application." *Population Studies* 57(2):165–182.

Pampel, Fred C. 2002. "Cigarette use and the narrowing sex differential in mortality." *Population and Development Review* 28(1):77–104.

Preparata, Franco P. and Michael Ian Shamos. 1985. *Computational geometry: An introduction*. Springer, New York.

Rao, Chalapati, Alan D. Lopez, Gonghuan Yang, Stephen Begg, and Jiemin Ma. 2005. "Evaluating national cause-of-death statistics: Principles and application to the case of China." *Bulletin of the World Health Organization* 83(8):618–625.

Reniers, Georges, Bruno Masquelier, and Patrick Gerland. 2011. "Adult mortality in Africa." In Richard G. Rogers and Eileen M. Crimmins (eds.), *International handbook of adult mortality*, chap. 7, pp. 151–170. Springer, Dordrecht.

Rogers, L. C. G. 1978. "The probability that two samples in the plane will have disjoint convex hulls." *Journal of Applied Probability* 15(4):790–802.

Schleisner, P. A. 1851. "Vital statistics of Iceland." *Journal of the Statistical Society of London* 14(1):1–10.

Snedecor, George W. and William G. Cochran. 1989. *Statistical methods*. Iowa State University Press, Ames, eighth ed.

Snyder, Timothy Law and J. Michael Steele. 1993. "Convex hulls of random walks." *Proceedings of the American Mathematical Society* 117(4):1165–1173.

Suri, Subhash, Kevin Verbeek, and Hakan Yıldız. 2013. "On the most likely convex hull of uncertain points." In Hans L. Bodlaender and Giuseppe F. Italiano (eds.), *Algorithms – ESA 2013: 21st annual European symposium, Sophia Antipolis, France, September 2–4, 2013. Proceedings*, pp. 791–802. No. 8125 in Lecture Notes in Computer Science, Springer, Berlin.

Tanimoto, Steven L. 1987. *The elements of artifical intelligence: An introduction using LISP*. Computer Science Press, Rockville, Maryland.

Timæus, Ian M and Tom A Moultrie. 2013. "Fitting model life tables to a pair of estimates of childhood and adult mortality." In Tom Moultrie, Rob Dorrington, Allan Hill, Kenneth Hill, Ian Timæus, and Basia Zaba (eds.), *Tools for demographic estimation*, chap. 32, pp. 348–356. International Union for the Scientific Study of Population, Paris.

Tomasson, Richard F. 1977. "A millennium of misery: The demography of the Icelanders." *Population Studies* 31(3):405–427.

Tukey, John W. 1975. "Mathematics and the picturing of data." In Ralph D. James (ed.), *Proceedings of the International Congress of Mathematicians, Vancouver 1974*, vol. 2, pp. 523–531.

———. 1977. *Exploratory data analysis*. Addison-Wesley, Reading, Massachusetts.

Turpeinen, Oiva. 1979. "Fertility and mortality in Finland since 1750." *Population Studies* 33(1):101–114.

Vasey, Daniel E. 2001. "A quantitative assessment of buffers among temperature variations, livestock, and the human population of Iceland, 1784 to 1900." *Climatic Change* 48(1):243–263.

Wachter, Kenneth W. 2014. *Essential demographic methods*. Harvard University Press, Cambridge.

Wilmoth, John, Sarah Zureick, Vladimir Canudas-Romo, Mie Inoue, and Cheryl Sawyer. 2012. "A flexible two-dimensional mortality model for use in indirect estimation." *Population Studies* 66(1):1–28.

Woods, R. I., P. A. Watterson, and J. H. Woodward. 1988. "The causes of rapid infant mortality decline in England and Wales, 1861–1921. Part I." *Population Studies* 42(3):343–366.

Woods, Robert. 1993. "On the historical relationship between infant and adult mortality." *Population Studies* 47(2):195–219.

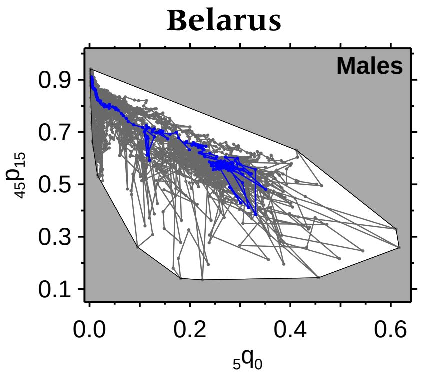
———. 2000. *The demography of Victorian England and Wales*. Cambridge University Press, Cambridge.

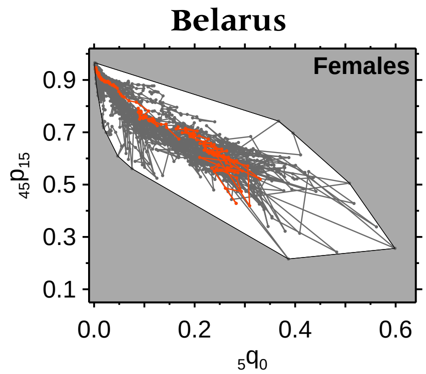
Woods, Robert and P. R. Andrew Hinde. 1987. "Mortality in Victorian England: Models and patterns." *Journal of Interdisciplinary History* 18(1):27–54.

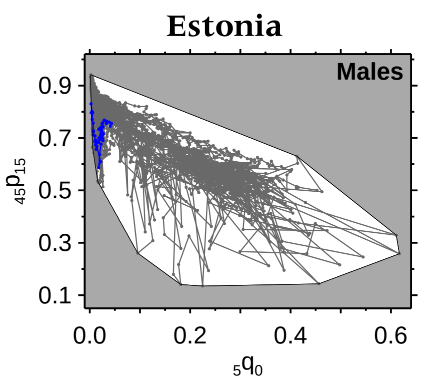
Wrigley, E. A. and R. S. Schofield. 1981. *The population history of England, 1541–1871: A reconstruction*. Edward Arnold, London.

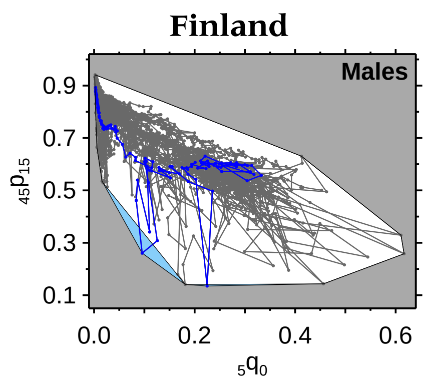
Appendix I: Peeled hulls for all countries

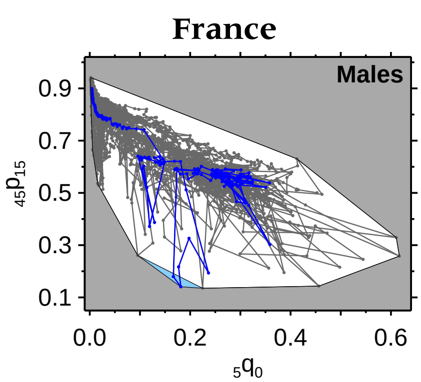
This appendix shows the country-peeled hulls for all countries in which peeling has an effect, apart from Iceland (which is shown in figure 3 in the main text).

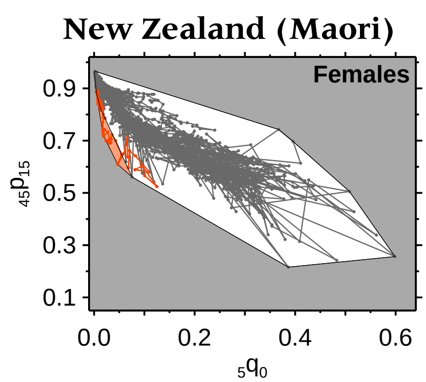


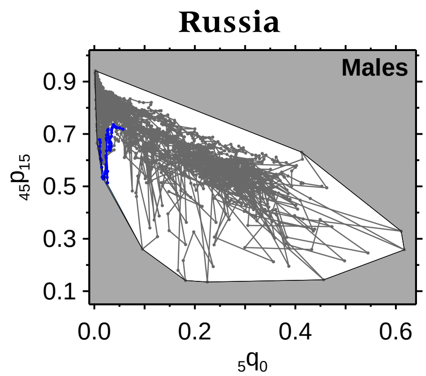






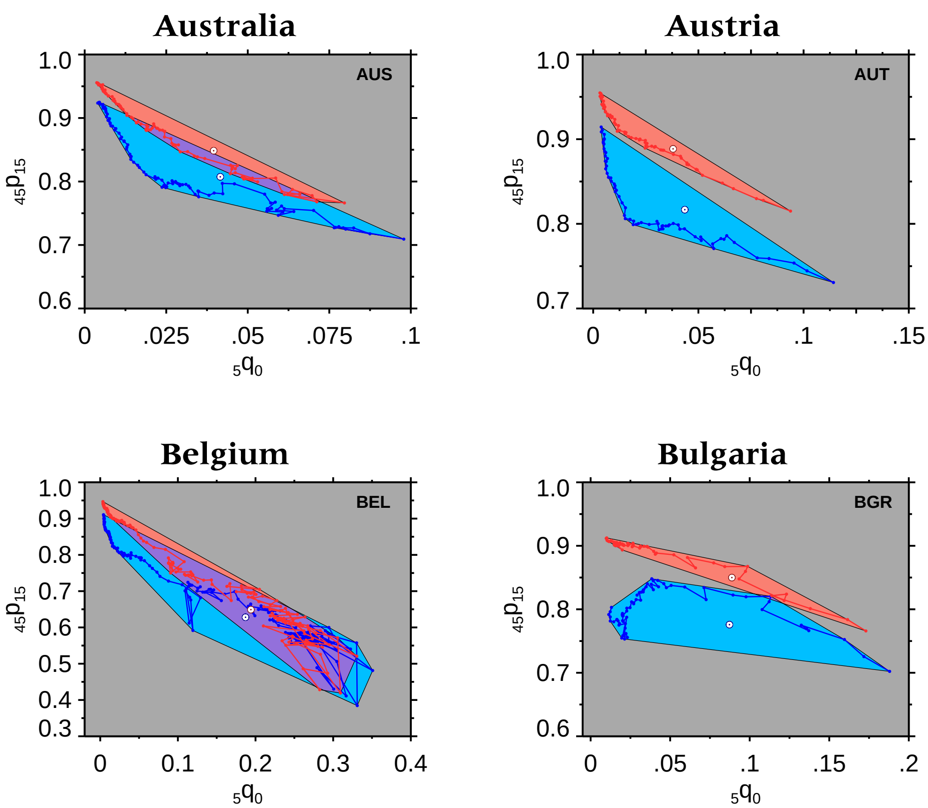


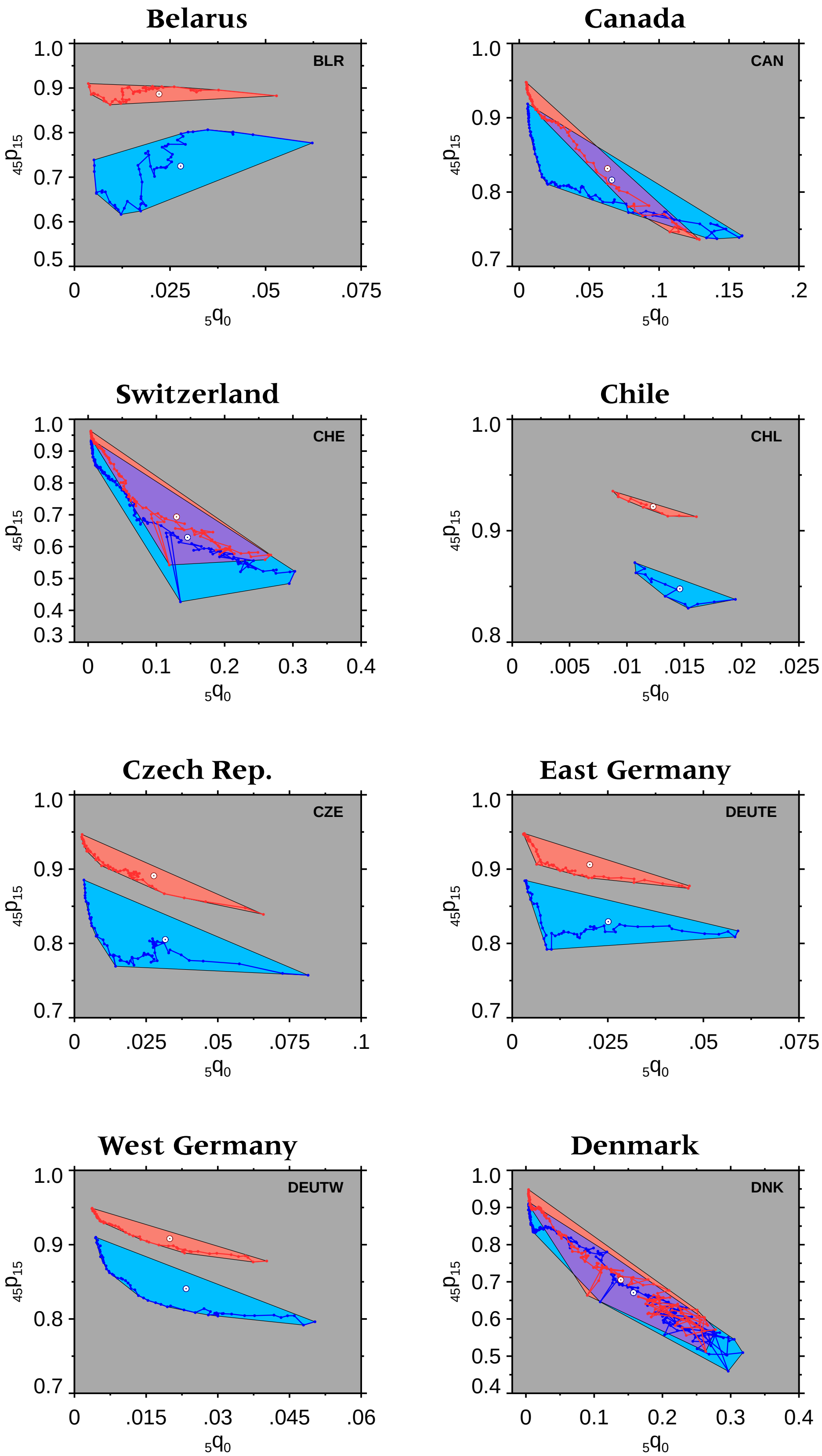




Appendix II: Sex-specific hulls for all countries

This appendix shows the male and female sex-specific hulls (as in figure 5), except for Iceland and the United States, which are shown in the main text.





Belarus
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Canada
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Switzerland
CHE
Chile
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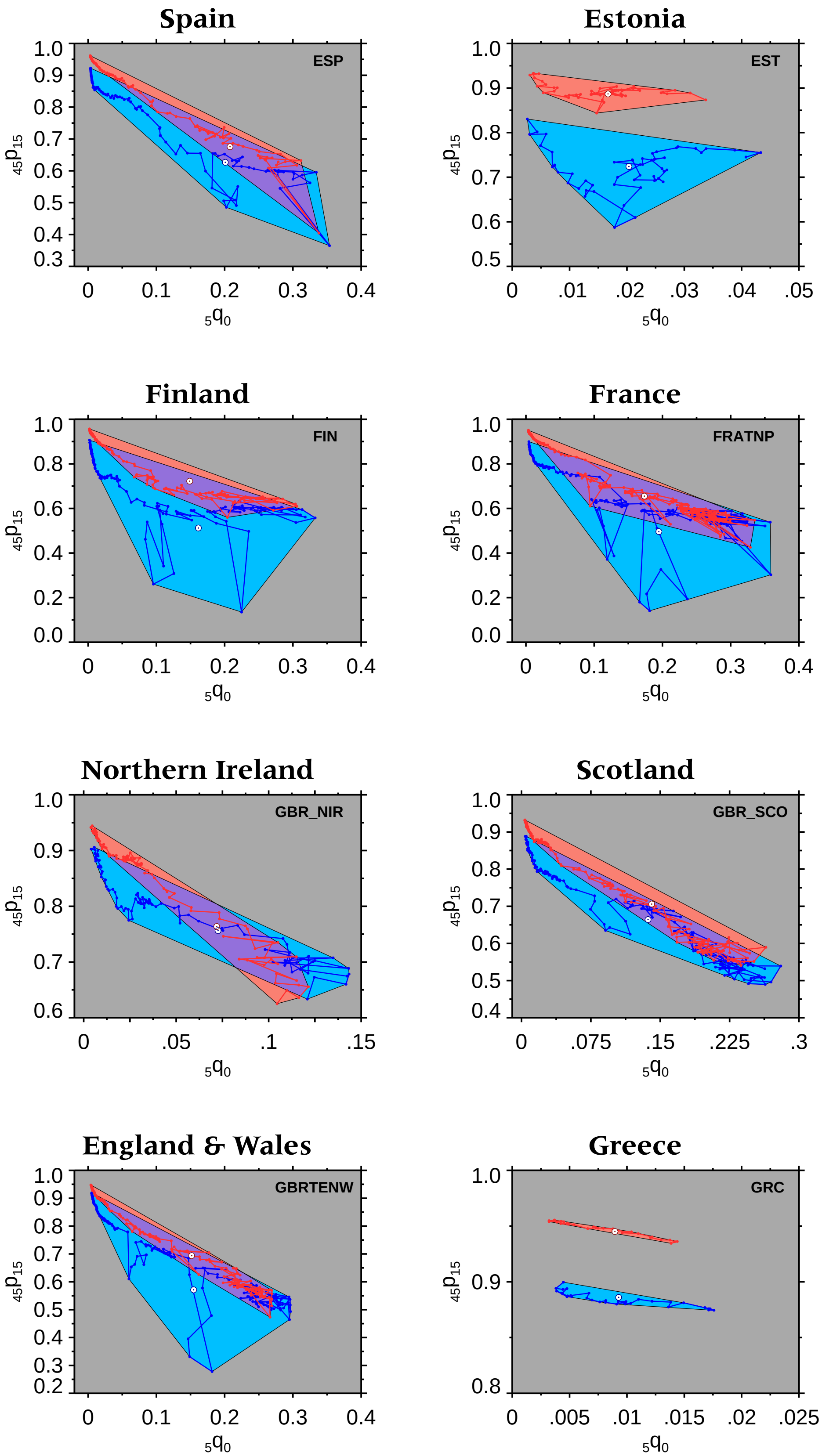
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ESP
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Finland
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France
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Northern Ireland
GBR_NIR
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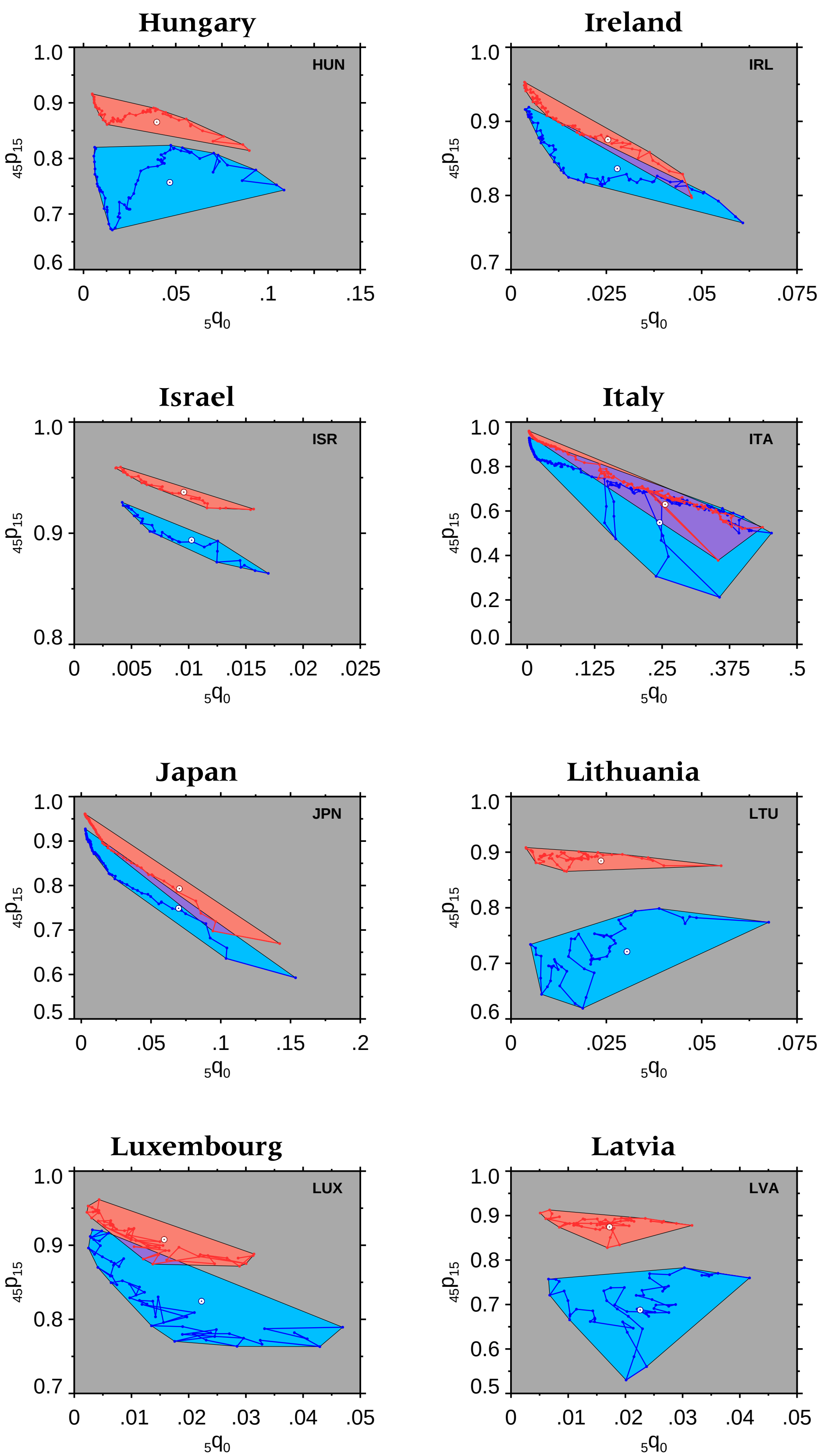
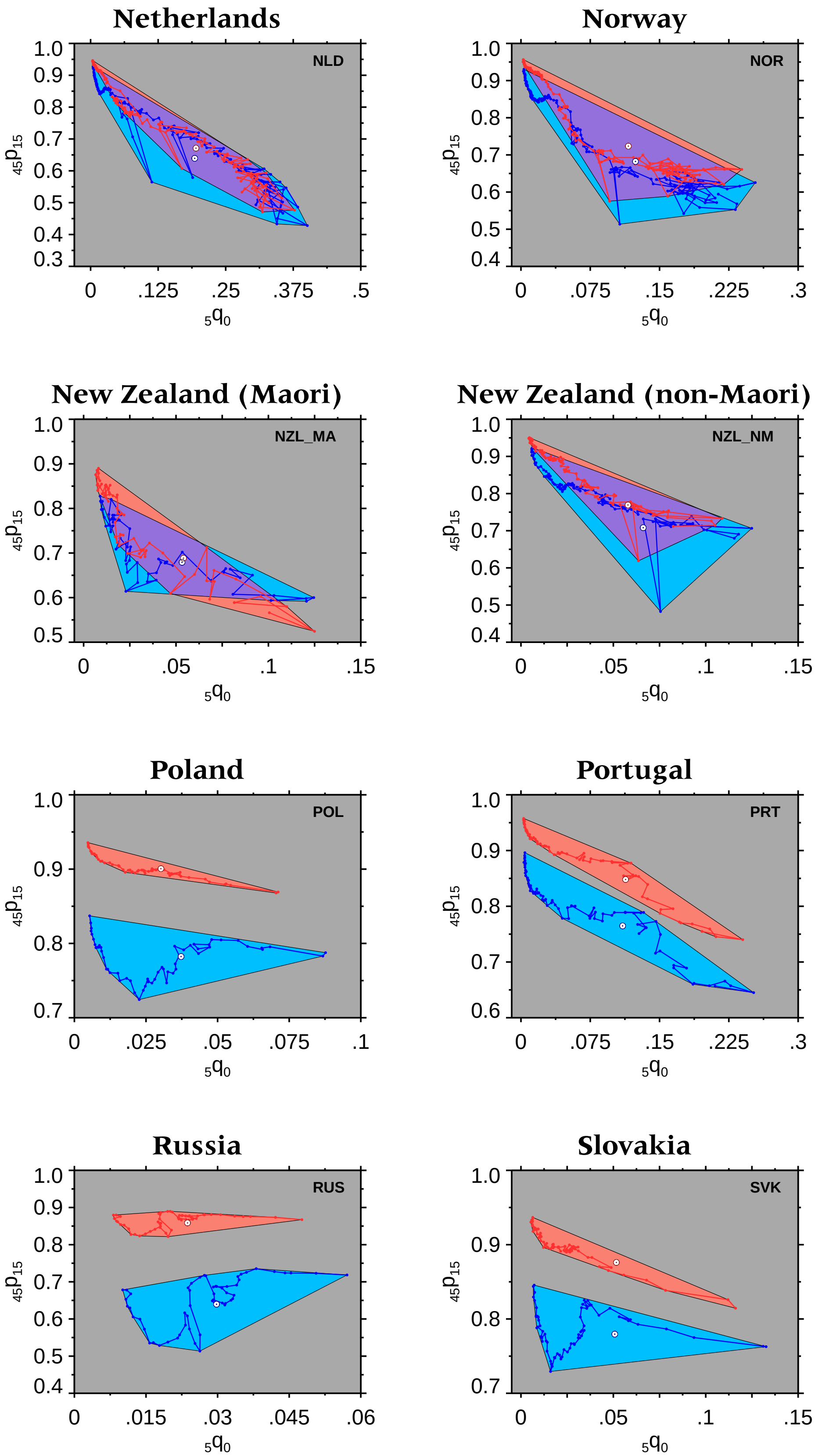
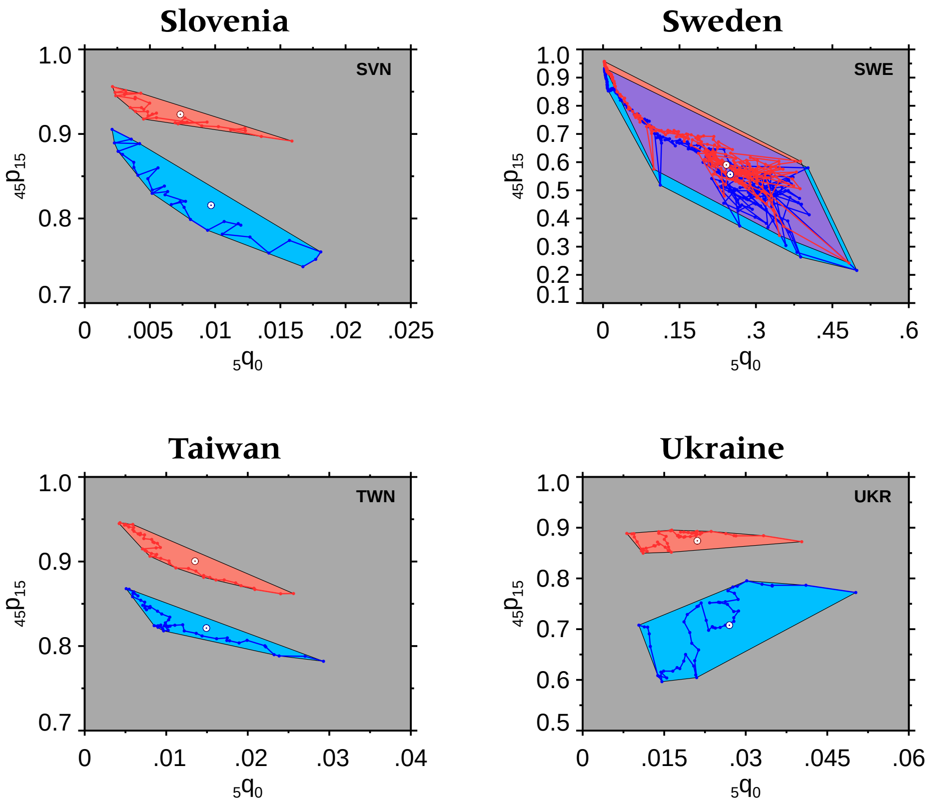
Greece
GRC
$_{45}p_{15}$
$_{5}q_{0}$


Hungary
HUN
Ireland
IRL
Israel
ISR
Italy
ITA
Japan
JPN
Lithuania
LTU
Luxembourg
LUX
Latvia
LVA
$_{45}p_{15}$
$_{5}q_{0}$

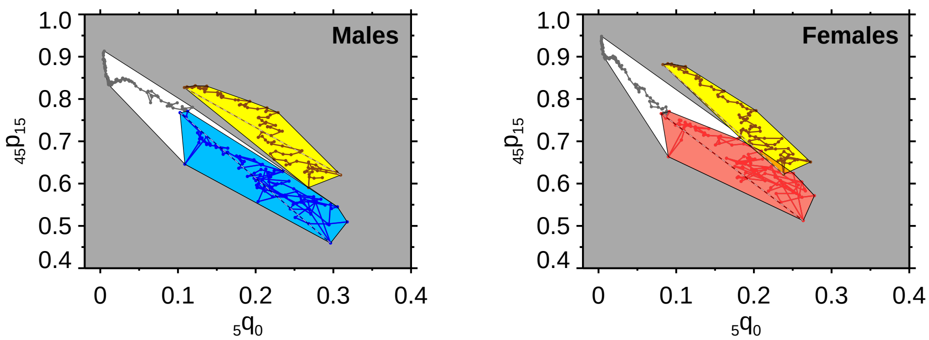

Netherlands
NLD
Norway
NOR
New Zealand (Maori)
NZL_MA
New Zealand (non-Maori)
NZL_NM
Poland
POL
Portugal
PRT
Russia
RUS
Slovakia
SVK
$_{45}p_{15}$
$_{5}q_{0}$


Slovenia
SVN
Sweden
SWE
Taiwan
TWN
Ukraine
UKR
$_{45}p_{15}$
$_{5}q_{0}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
.005
.01
.015
.02
.025
.15
.3
.45
.6
.03
.04
.045
.06

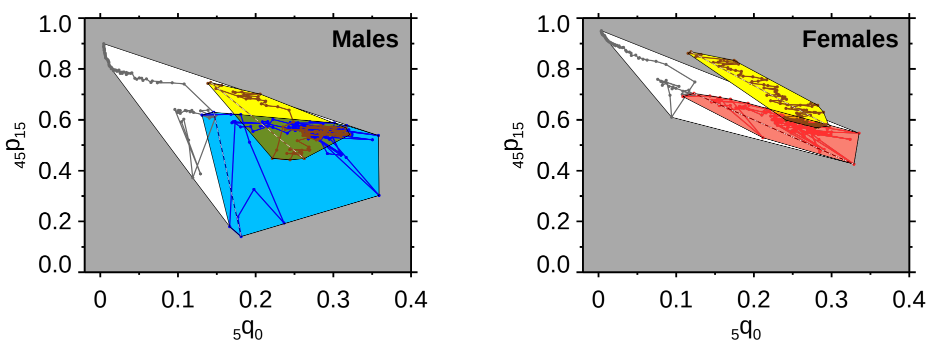
Appendix III: Period-cohort hulls for all countries

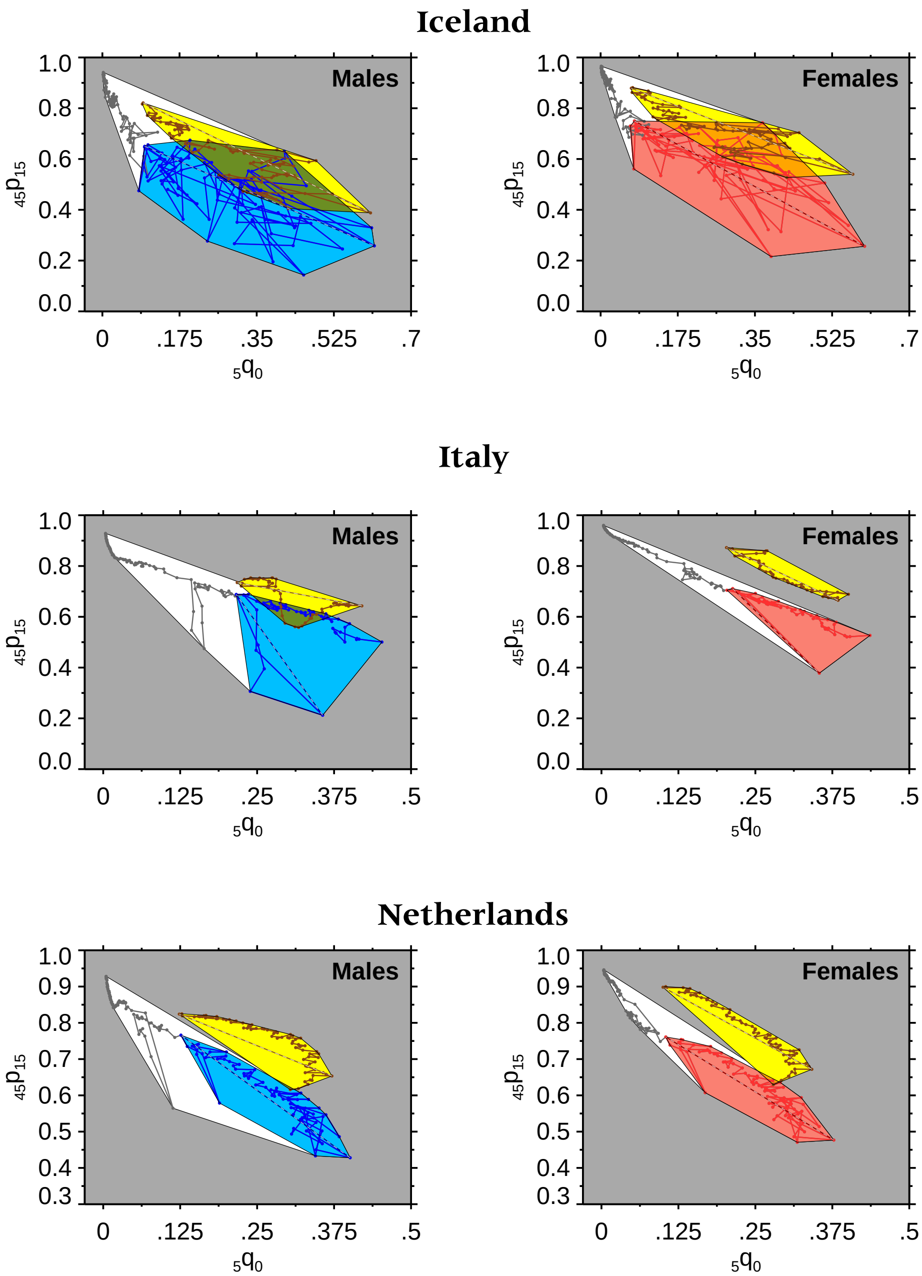
This appendix shows the period-cohort hulls for all countries apart from Finland (which is shown in figure 6 in the main text). Period hulls are colored red (females) and blue (males); cohort hulls are yellow. Where period and cohort hulls overlap, the colors are orange (females) or green (males). Period hulls are constrained to the same time range as the cohort data. Underplotted white regions are the full extent of available period data; see figure 6 caption for more details.

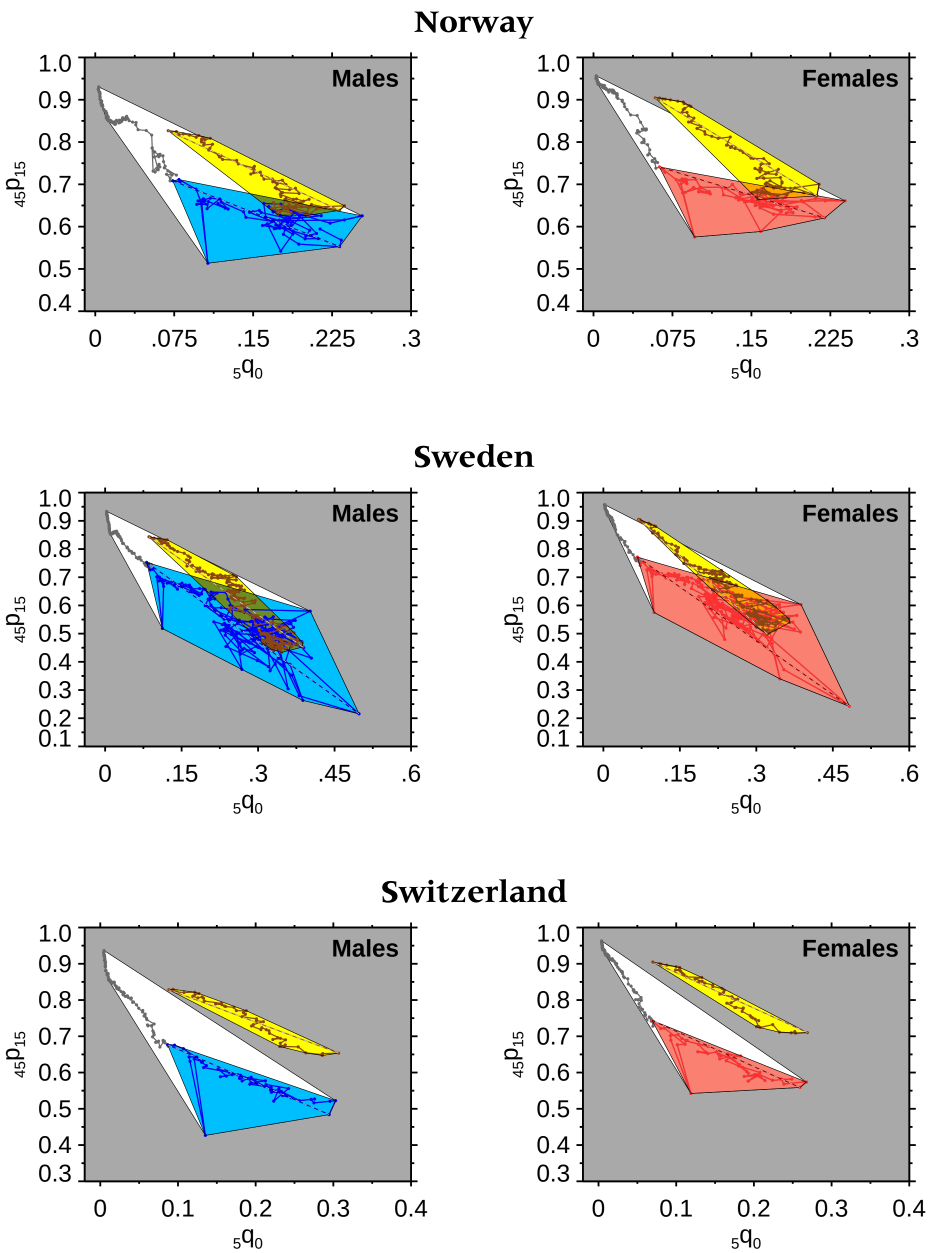
Denmark



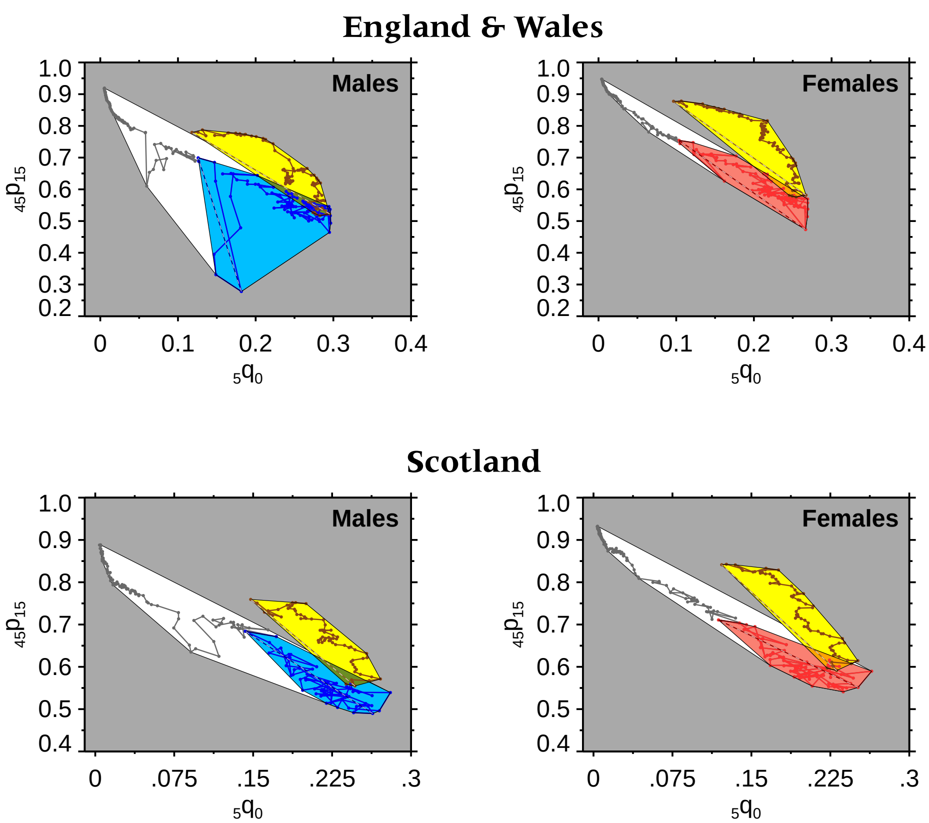
France




Iceland
Males
Females
$_{45}p_{15}$
$_5q_0$
1.0
0.8
0.6
0.4
0.2
0.0
0
.175
.35
.525
.7
Italy
Males
Females
$_{45}p_{15}$
$_5q_0$
1.0
0.8
0.6
0.4
0.2
0.0
0
.125
.25
.375
.5
Netherlands
Males
Females
$_{45}p_{15}$
$_5q_0$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
.125
.25
.375
.5



Norway
Males
Females
Sweden
Males
Females
Switzerland
Males
Females
$_{45}p_{15}$
$_{5}q_{0}$


England & Wales
Males
Females
$_{45}p_{15}$
$_{5}q_{0}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
Scotland
Males
Females
$_{45}p_{15}$
$_{5}q_{0}$
0
.075
.15
.225
.3