Ascending Auctions with Multidimensional Signals *

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Abstract

We study an ascending auction in which agents bid for an indivisible good and observe multidimensional Gaussian signals. We provide novel predictions of ascending auctions that arise only when agents observe multidimensional signals. The first novel prediction is that an ascending auction can have multiple symmetric equilibria. Each equilibria induces a different allocative efficiency and different profits for the seller. The second novel prediction is that, with multidimensional signals, public signals can be detrimental for profits (even in symmetric environments). In fact, a precise enough public signal can induce profits arbitrarily close to 0. Both of these novel predictions arise in a model with two-dimensional signals that combines a classic model of private values and a classic model of common values. Hence, the only difference between the model we study and the classic models of ascending auctions is the multidimensionality of the information structure.

The equilibrium is solved using a two-step procedure. The first step is to project the signals into a one-dimensional *equilibrium statistic*. The second step is to solve for the equilibrium as if agents observed *only* the equilibrium statistic (and hence, as if agents observe one-dimensional signals). The *equilibrium statistic* can also be used to solve other trading mechanisms (e.g. supply function equilibria, generalized VCG mechanisms and other multi-unit auctions).

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1 Introduction

1.1 Motivation and Model

Consider N agents bidding for a house that is sold via an ascending auction. An agent's valuation for the house depends on his taste for the house, and on the future resale value of the house. An agent knows his taste for the house, but agents have dispersed information about the common shock that determines the resale value of the house (e.g. changes in the interest rates). In this environment agents observe two signals; one signal about their taste shock and a second signal about the common shock. Although this is a natural environment to study, the properties of an ascending auction in this environment are not known.

There is a fundamental difference between ascending auctions with one-dimensional signals and multidimensional signals. In environments with multidimensional signals, the drop-out time of an agent only reflects a summary statistic of all the signals this agent observed. Hence, solving for an equilibrium requires understanding what agent i learns from the drop-out time of agent j, which in turn depends on what agent j learns from the drop-out time of agent i. Even the possibility of constructing a symmetric equilibrium in such environment is an open question. The objective of this paper is to characterize the equilibria of ascending auctions when agents observe multidimensional Gaussian signals, and analyze its properties.

Our model consists of N symmetric players bidding for an object in an ascending auction. The utility of an agent if he gets the object depends on a payoff shock that has a *common* and an *idiosyncratic* component. Each agent observes J signals about the realization of his own payoff shock. The joint distribution of signals and fundamentals is Gaussian. We study a class of Nash equilibria in undominated strategies that preserve the normality of beliefs.

We first characterize the equilibria for arbitrary symmetric one-dimensional signals and a class of symmetric two-dimensional signals. We study the difference between the predictions with multidimensional signals and one-dimensional signals. We then extend our analysis to allow for general asymmetric Gaussian signals and to allow for other trading mechanisms.

1.2 Results

We begin by studying the set of Nash equilibria when agents observe one-dimensional signals. The characterization of the equilibrium strategies follows Milgrom and Weber (1982). The Gaussian information structure allows to provide a novel description of the equilibrium outcome in terms of the information structure. The surplus generated, the seller's profits and the buyers' rents are completely determined by two statistics of the information structure. The first statistic is the informativeness of the signals observed by all agents about the differences in valuation between agents. This statistic determines the total surplus generated by the ascending auction. The second statistic is the level of *payoff interdependence*. This statistic determines the division of the surplus between the buyers and the seller, but does not affect the total surplus generated. A high level of payoff of interdependence implies that an agent faces a larger "winner's curse". An agent corrects for the larger "winner's curse" in equilibrium by lowering his bid. A higher level of payoff interdependence implies that the buyers keep a larger share of the total surplus generated. We show that in fact, any division of surplus is possible as an equilibrium outcome for some information structure. The characterization of the equilibrium outcome in terms of two statistics will remain valid for multidimensional signals, and hence it is key for the interpretation and intuition of our results.

We then study two-dimensional signals. The payoff shock is decomposed as the sum of two shocks; a common shock and a taste shock. The taste shock may be correlated across agents. The first signal agent i observes is perfectly informative about agent i's taste shock. The second signal agent i observes is a noisy signal about the common shock.[1] We show that the set of equilibria can be characterized using a two-step procedure. In the first step, we project the two signals of each agent into a one-dimensional *equilibrium statistic*. In the second step, we characterize the equilibria "as if" agents observed only the one-dimensional equilibrium statistic, and hence the characterization for one-dimensional signals is applied. To the best of our knowledge, we are the first paper that characterizes the equilibrium of an ascending auction, combining the classic models of common values and private values.

In equilibrium, agents behave "as if" they observed one-dimensional signals. The difference between one-dimensional information structures and multidimensional information structures is that with multidimensional information structures the linear combination of signals that determine an agent's bidding behavior is endogenously determined. This implies that with multidimensional information structures the surplus generated and the payoff interdependence are endogenously determined, and not exogenously determined by the information structure. We study how the predictions in environments with two-dimensional information structures differ from the predictions in environments with one-dimensional information structures.

[1]If agents observed only their taste shock, this would be a classic private value environment. If agents observed only the signal on the common shock, this would be a classic common value environment.

The first prediction that is different with multidimensional information structures is that ascending auctions may have multiple symmetric equilibria. If there number of agents is large enough, the size of the common shock is large enough and the private signal agent i observes about the common shock is noisy enough, then there are three equilibria. One equilibrium resembles an equilibrium with private values; the surplus created and the profits earned by the seller are large. Another equilibrium resembles an equilibrium with common values; the surplus created and the profits earned by the seller are low. The third equilibrium is in the middle. The multiplicity of equilibria shows that the degree of payoff interdependence and the total surplus generated in the auction is endogenously determined, and not determined by the primitives of the information structure.

The multiplicity of equilibria is driven by a complementarity in the weights agents place on their private signals to determine their drop-out time in the auction. To illustrate the source of complementarity, consider the case in which the signal about the common shock is very noisy. If all agents different than agent i place a large weight on the signal about the common shock, then agent i will face a larger winner's curse. Hence, agent i will also place more weight on his private signal about the common shock. This will result in an equilibrium in which the total surplus generated is low and in which the level of payoff interdependence is large. There exists another equilibrium with very different properties. If all agents different than agent i place a small weight on the signal about the common shock, agent i will find it optimal to also place a small weight on the signal about the common shock. This results in an equilibrium that resemble a model with private values. That is, the total surplus generated will be high and payoff interdependence will be low.

The second prediction that is different between one-dimensional signals and multidimensional signals is that in multidimensional environments the linkage principle fails.[2] In environments with multidimensional signals, public signals can be detrimental for profits and an ascending auction can yield lower payoff than a first price auction.[3] This is despite the fact that a more precise public signal increases the total surplus.[4] The failure of the linkage principle may happen for any distribution of payoff shocks, and even when payoff shocks are arbitrarily close to common knowledge. Importantly, in our analysis each signal an agent observes satisfy the properties assumed in Milgrom and Weber (1982). Hence, even if all signals satisfy the standard

[2] The linkage principle states that public signals increase profits and ascending auctions yield higher profits than first price auctions (see Krishna (2009) for a textbook discussion).

[3] We do not solve for the equilibria in first price auctions. Nevertheless, Bergemann, Brooks, and Morris (2015) show that the profits in a first price auction are bounded away from 0.

[4] In a model with one-dimensional signals the equilibrium is efficient, and hence public signals do not change the total surplus.

properties assumed in the auction literature, the equilibrium when agents observe these signals simultaneously is radically different.

The linkage principle is based on the fact that a public signal decreases the payoff interdependence from a common shock (and hence, the winner's curse). Nevertheless, we show that a public signal may increase the winner's curse from a different common shock. The public signal increases the information an agent has about the rank of his valuation with respect to others. As the ranking of payoff shocks between agents becomes more precise, the winner's curse from common shocks increases. Note that in equilibrium agents bid "as if" they observed one-dimensional signals. The reason that the results in Milgrom and Weber (1982) do not apply, is that the equilibrium statistic does not satisfy the affiliation property. That is, in equilibrium an agent's bid is determined by a one-dimensional statistic that may be *negatively* correlated across agents.

We extend our analysis to allow for arbitrary Gaussian information structures and allow for other trading mechanisms. We show that every game that has an ex-post equilibrium when the agents observe one-dimensional signals also has an equilibrium when the agents observe multidimensional signals.[5] Moreover, the Nash equilibrium can be computed using a two step procedure. First, for each agent we find a one-dimensional equilibrium statistic of the signals he observes. Second, we compute the equilibrium as if agents observed *only* the one-dimensional equilibrium statistic. It is worth highlighting that the one-dimensional equilibrium statistic does not depend on the game, but only on the information structure.

1.3 Literature

To the best of our knowledge, the only paper in the literature studying ascending auctions with multidimensional signals that are not independently distributed across agents is Jackson (2009). He shows the non-existence of equilibria for a class of examples. The model studied therein is similar to our model — with a private and a common signal — except the distribution of signals and fundamentals is non-Gaussian (moreover, his signals have a finite support). Our paper provides reassurance that it is possible to construct equilibria under more complex information structures. In Section 7.1 we discuss the example in Jackson (2009) in more detail, and argue therein that the finite support of the information structure is important in the argument to prove non-existence of equilibrium. The extent to which it is possible to construct equilibria

[5] There is a large class of mechanisms that have an ex-post equilibria when agents observe one-dimensional signals. These mechanisms include classic trading mechanisms (e.g. supply function competition or generalized VCG mechanisms), as well as mechanisms recently proposed by the literature (a detailed discussion is provided after Proposition 4).

with multidimensional non-Gaussian information structures is still an open question.

Wilson (1998) studies an ascending auction with log-normal random variables with a two-dimensional information structure. He assumes some of the random variables are drawn from a diffuse prior. The diffuse prior implies that the random variables do not have a well defined cumulative distribution function. Hence, these are not technically random variables, and the updating is not technically done by Bayes' rule. Relaxing the assumption of diffuse priors is not only a technical contribution, but it is also fundamental to derive the novel predictions in the ascending auction. In Section 7.2 we discuss how to analyze the equilibrium under diffuse priors. This allow us to provide some insights on how the analysis can be extended to non-Gaussian information structures.

The closest connection in the literature to the equilibrium statistic we propose can be found in Dasgupta and Maskin (2000). They show that if agents' signals are independently distributed, then there is a way to project the multidimensional signals into a one-dimensional statistic. Our equilibrium statistic coincides with the one in Dasgupta and Maskin (2000) in the case of independent signals. In this case, the one-dimensional statistic corresponds to an agent's expectation of their own payoff shock conditional *only* on his private information. Hence, when signals are independently distributed there is no "equilibrium component" to the equilibrium statistic (see Section 5.6). Levin, Peck, and Ye (2007) and Goeree and Offerman (2003) study ascending auctions in which agents observe independent signals, and hence the bidding strategy can be analyzed using the same one-dimensional statistic as in Dasgupta and Maskin (2000). The novel predictions we find hinge critically on how the equilibrium statistic of one agent depends on the information the agent learns from the drop-out time of other agents.

Perry and Reny (1999) show that the linkage principle may fail in multi-unit auctions.[6] The linkage principle has also been shown to fail in environments in which the payoff structure is asymmetric (see Krishna (2009)) and in environments with independent and private values (see Thierry and Stefano (2003)). In contrast to the previous literature we show that the linkage principle may fail in natural symmetric environments, which is due only to the multdimensionality of the information structure. The key insight is that the total surplus and the profits are determined by how the signals are endogenously aggregated to determine an agent's drop-out time. Hence, our paper provides a new channel by which the linkage principle may fail.

Klemperer (1998) shows that in an ascending auction with two players, an ex-ante difference

[6] As in our model, a public signal may change the allocation of the goods across the agents. Nevertheless, in their paper agents observe one-dimensional signals, and hence the change in allocation is due to the non-flat demand. In fact, with constant marginal valuation Ausubel (2004) shows that in multi-unit auctions the linkage principle holds.

in the payoff environment may lead to explosive behaviors.[7] Levin and Kagel (2005) shows that the explosive behavior is attenuated if there are more "regular" bidders. In contrast to the literature studying ascending auctions with asymmetric agents, our results are driven only by the information structure and not by assumptions on the payoff structure. In fact, all agents use the same bidding strategy. Importantly, our results hold in environments in which payoff shocks are arbitrarily close to independently distributed. Another interesting difference between our model and the models with asymmetric agents, is that in our model public signals have an effect on the allocation of the good, and not only on profits.

The connection of our paper to the mechanism design literature and other trading mechanisms is mentioned throughout the paper, as we provide our results.

The paper is organized as follows. In Section 2 we provide the model. In Section 3 we study one-dimensional signals. In Section 4 we study two-dimensional signals. In Section 5 we generalize the methodology to allow for multidimensional asymmetric signals. In Section 6 we generalize the methodology to other trading mechanisms. In Section 8 we conclude. All proofs and some additional results that are mentioned in the main text are collected in the appendix.

2 Model

2.1 Payoff Structure

We study N agents bidding for an object in an ascending auction. The utility of agent i if he wins the object at price p is given by:

$$u_i(\theta_i, p) = \exp(\theta_i) - p, \tag{1}$$

where $\exp(\cdot)$ denotes the exponential function and θ_i is a payoff shock. If an agent does not win the object he gets a utility equal to 0.

[7] Bulow and Klemperer (2002) study an ascending auction with asymmetric agents, and study circumstances under which increasing the supply of asset always decreases the equilibrium price. As in Klemperer (1998), a small difference in the utility function of agents can lead to large difference in outcomes, as long as one player has the highest valuation almost always. As we explain in Section 6, in our environment an increase in the number of goods being auctioned always decreases the price. Hence, our model does not exhibit this unusual behavior that appears in asymmetric auctions.

2.2 Information Structure

The only source of uncertainty is the realization of the payoff shocks $(\theta_1, ..., \theta_N)$. Each player observes J signals:

$$\mathbf{s}_i \triangleq (s_{i1}, ..., s_{iJ}),$$

where vectors are denoted in bold font. We occasionally use bold fonts to also denote vectors of length N, nevertheless there will be no ambiguity in the use of bold fonts. We assume that the random variables $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{(J+1)N}$ are jointly normally distributed, and all random variables have a mean equal to 0. The assumption that the means are equal to 0 allows reduce the amount of notation, but the analysis follows through in a straightforward way if the means of the random variables are not 0.

We assume that the information structure is symmetric. That is, for all $i, \ell, k \in N$ the joint distribution of variable $(\theta_i, \theta_k, \mathbf{s}_i, \mathbf{s}_k)$ is the same as the joint distribution of variables $(\theta_\ell, \theta_k, \mathbf{s}_\ell, \mathbf{s}_k)$. We later generalize the analysis by studying asymmetric environments.

We make the following definitions:

$$\bar{\theta} \triangleq \frac{1}{N} \sum_{i \in N} \theta_i \;\; ; \;\; \Delta\theta_i \triangleq \theta_i - \bar{\theta} \;\; ; \;\; \bar{\mathbf{s}} \triangleq \frac{1}{N} \sum_{i \in N} \mathbf{s}_i \;\; ; \;\; \boldsymbol{\Delta}\mathbf{s}_i \triangleq \mathbf{s}_i - \bar{\mathbf{s}}. \tag{2}$$

That is, variables with an over-bar correspond to the average of the variable over all agents, while variables preceded by a Δ correspond to the difference between a variables and the average variable. We refer to variables that have an over-bar as the common component of random variables and a variables preceded by a Δ as the idiosyncratic component of a random variable. For example, $\bar{\theta}$ is the common component of θ_i while $\Delta\theta_i$ is the idiosyncratic component of θ_i.[8]

2.3 Ascending Auction

We study an ascending auction.[9] In an ascending auction an auctioneer rises the price continuously. At each moment in time, an agent can decide to drop-out of the auction, in which case the agent does not pay anything and does not get the object. The last agent to drop-out of the auction wins the object and pays the price at which the second to last agent dropped out of the auction. We restrict attention to ascending auctions in which agents are symmetric and agents use symmetric strategies. This allow us to simplify the notation and provide the main insights

[8]Some statistical properties of the orthogonal decomposition in (2) can be found in the online appendix (see Section 10).

[9]We follow Krishna (2009) in the formal description of the ascending auction.

of the paper. We generalize the analysis in Section 5.

The strategy of player i is a set of functions $\{p^i_\ell\}_{\ell\in\{2,...,N\}}$, with $p^i_\ell : \mathbb{R}\times\mathbb{R}^{N-\ell} \to \mathbb{R}_+$. The function $p^i_\ell(\mathbf{s}_i, p_{\ell+1}, ..., p_N)$ is the drop-out time of agent i, when ℓ agents are left in the action and the drop-out times so far are $p_N < ... < p_{\ell+1}$. Obviously, the function $p^i_\ell(\mathbf{s}_i, p_{\ell+1}, ..., p_N)$ must satisfy

$$p^i_\ell(\mathbf{s}_i, p_{\ell+1}, ..., p_N) \geq p_{\ell+1}.$$

That is, agent i cannot drop-out of the auction at a price lower than the price at which another agent has already dropped out. Note that we are restricting attention to symmetric environments, and hence it is not necessary to specify the identity of the agent that drops out of the auction, but only the price at which this agent dropped out.

The outcome of the ascending auction is described by the order at which each agent drops out and the price at which each agent drops out. We describe the order at which each agent drops out of the action by a permutation π.[10] $\pi(i)$ is the number of agents left in the auction when agent i dropped out of the auction. The identity of the last agent to drop-out of the auction is given by $\pi^{-1}(1)$. The price at which agents drop-out of the auction is denoted by $p_1 > .... > p_N$. Hence, for any strategy profile the expected utility of agent i is:

$$\mathbb{E}[\mathbb{1}\left\{\pi^{-1}(1) = i\right\}(e^{\theta_i} - p_2)],$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. We study the Nash equilibria of the ascending auction.

3 One-Dimensional Signals

We begin by studying one-dimensional signals. We use the Gaussian structure of the signals to provide a novel characterization of how the surplus is split between the buyers and the seller. This allow us to explain how the information structure impacts the outcome in an ascending auction. This is a fundamental part of the analysis of the model with multidimensional signals.

3.1 Description of Signals and Single Crossing Condition

We first study the case in which agents observe symmetric one-dimensional signals (s_i). We provide the expectation of θ_i conditional on the realization of all signals $(s_1, ..., s_N)$.

[10] A permutation is a bijective function $\pi : N \to N$.

Lemma 1 (Conditional Expectation)**.**
The expected value of θ_i conditional on all signals is given by:

$$\mathbb{E}[\theta_i|s_1,...,s_N]=\mathbb{E}[\Delta\theta_i|\Delta s_i]+\mathbb{E}[\bar{\theta}|\bar{s}]=\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}\Delta s_i+\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}\bar{s}. \tag{3}$$

Lemma 1 provides a description of agent i's expectation of his own payoff shock conditional on all signals. A key parameter in the analysis will be:

$$m\triangleq\frac{\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}}{\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}}. \tag{4}$$

Note that m is equal to the ratio between both regression coefficients in (3). m provides a measure on how informative is the signals of agents about $\bar{\theta}$ relative to $\Delta\theta_i$. We make one assumption on the information structure that is analogous to the classic single crossing condition. We assume that:

$$m\geq-(N-1). \tag{5}$$

Assumption (5) is similar to the single crossing condition. If $m\geq 0$ then:[11]

$$\forall i,\ell\in N,\quad corr(s_i,\theta_i)^2\geq corr(s_i,\theta_\ell)^2, \tag{6}$$

If (6) is satisfied, then the signal agent i observes is more informative about his own payoff shock than the payoff shock of agent ℓ. In Section 3.6 we provide an interpretation of $m=-(N-1)$.

We now provide a description of the set of distributions of signals and fundamentals.

Lemma 2 (Distribution of Equilibrium Statistic and Fundamentals)**.**
The variance covariance matrix of payoff shocks and signals $(\Delta\theta_i,\bar{\theta},\Delta s_i,\bar{s})$ is equal to:

$$\begin{pmatrix} \sigma^2_{\Delta\theta_i} & 0 & corr(\Delta s_i,\Delta\theta_i)\sigma_{\Delta\theta_i}\sigma_{\Delta s_i} & 0\\ 0 & \sigma^2_{\bar{\theta}} & 0 & corr(\bar{s},\bar{\theta})\sigma_{\bar{\theta}}\sigma_{\bar{s}}\\ corr(\Delta s_i,\Delta\theta_i)\sigma_{\Delta\theta_i}\sigma_{\Delta s_i} & 0 & \sigma^2_{\Delta s_i} & 0\\ 0 & corr(\bar{s},\bar{\theta})\sigma_{\bar{\theta}}\sigma_{\bar{s}} & 0 & \sigma^2_{\bar{s}} \end{pmatrix}. \tag{7}$$

As all random variables are Gaussian with mean 0, Lemma 2 completely characterizes the joint distribution of signals and payoff shocks. It is easy to check that this is determine by 6

[11]See the appendix for a proof that $m\geq 0$ is sufficient for (6).

parameters. Nevertheless, note that $(var(\theta_i), var(\bar{\theta}))$ are primitives of the payoff structure, while $var(s_i)$ is just a normalization as the variance of signals does not affect the equilibrium outcome. Hence, the three coefficients

$$(corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta}), m), \tag{8}$$

completely determine the information structure. The coefficient $corr(\Delta s_i, \Delta\theta_i)$ is the informativeness of the idiosyncratic component of the signals about the idiosyncratic component of the payoff shock. The coefficient $corr(\bar{s}, \bar{\theta})$ is the informativeness of the common component of the signals about the common component of the payoff shock. The parameter m is a measure of the amount of payoff interdependence.

3.2 Examples of Information Structures

We provide some examples of one-dimensional signals studied in the literature.

Example 1 (Example 2 in Dasgupta and Maskin (2000))**.**
Each agent observes a signal:

$$s_i = \theta_i + \varepsilon_i,$$

where ε_i is independently distributed across agents. It is possible to check that, $corr(\Delta s_i, \Delta\theta_i)$ and m are increasing in $var(\varepsilon_i)$.

Example 2 (Reny and Perry (2006))**.**
We provide a linear-Gaussian version of the signals studied in Reny and Perry (2006).[12] *Each agent observes signal s_i and there is a common shock ω. The signals and the shock have a correlation $corr(s_i, \omega)$. The payoff shock is given by:*

$$\theta_i = \omega + s_i.$$

It is possible to check that m is increasing in $cov(\omega, \bar{s})$, while $corr(\Delta s_i, \Delta\theta_i) = 1$.

Example 3 (Noise-Free Signals)**.**

[12] The set of signals studied in this example are neither more general or less general than the ones studied by Reny and Perry (2006). The signals studied by Reny and Perry (2006) are drawn from a compact support and they allow for utility functions that are non-linear. Beyond some technical differences, it is clear that the signals studied in this example have the same economic interpretation.

Consider the following information structure:

$$s_i = \frac{1}{\mu} \cdot \bar{\theta} + \Delta\theta_i. \tag{9}$$

Unless $\mu = 1$, the signal s_i does not allow agent i to perfectly infer θ_i. Nevertheless, agent i can infer θ_i from the realization of all signals $(s_1, ..., s_N)$. We study these signals in Section 3.6.

Example 4 (Common Values)**.**
We can accommodate for common value environments by considering the information structure:

$$s_i = \bar{\theta} + \varepsilon_i.$$

In this case, agents do not have any information about $\Delta\theta_i$, so it is effectively a common value environment. [13]

3.3 Characterization of Equilibrium for One-Dimensional Signals

We now characterize the equilibrium of the ascending auction. We relabel agents such that the realization of signals satisfy $s_1 > ... > s_N$. As signals are noisy, we might have that the order over payoff shocks is not preserved. For example, we may have $\theta_{i+1} > \theta_i$ (even though by construction $s_{i+1} \leq s_i$). The expectation of θ_i assuming that signals $(s_1, ..., s_{i-1})$ are equal to s_i (that is, assuming that all signals higher than s_i are equal to s_i) is denoted by $\mathbb{E}[\theta_i | s_i, ..., s_i, ..., s_N]$.

Lemma 3 (Equilibrium of Ascending Auction)**.**
The ascending auction has a Nash equilibrium in which agent i's drop-out time is given by:

$$p_i = \mathbb{E}[\exp(\theta_i) | s_i, ..., s_i, s_{i+1} ..., s_N]. \tag{10}$$

In equilibrium, agent $i = 1$ gets the object and pays a price of $p_2 = \mathbb{E}[\exp(\theta_2) | s_2, s_2, s_3, ..., s_N]$.

Lemma 3 shows that the player with the i^{th} highest signal drops out of the auction at his expected valuation, assuming that the $i - 1$ signals that are higher than i's signals are equal to s_i. This is the classic equilibrium characterization found in Milgrom and Weber (1982). The

[13] Note that the right way to accommodate for common values is by taking the limit $corr(\Delta s_i, \Delta\theta_i) \to 0$. All our results will have a well defined limit $corr(\Delta s_i, \Delta\theta_i) \to 0$, which corresponds to the case of common values. By taking the limit $var(\Delta\theta_i) \to 0$, we do not necessarily approach a common value environment (see Section 3.6).

equilibrium characterized in Lemma 3 is an ex-post equilibrium. That is, even if every agent knew the realization of the signals of all other agents, the strategy profile described by (10) would still be an equilibrium (see the proof of Lemma 3). It is well known in the literature that the ascending auction has many equilibria. In this paper we restrict attention to the class of equilibria in which agents use continuous drop-out times, and which implements a symmetric ex-post equilibria when agents observe one-dimensional signals.[14]

3.4 Total Surplus

We now characterize the equilibrium surplus. The expected surplus is equal to the expected valuation of the buyer that wins the object. That is, the expected surplus is given by:

$$S(s_1, ..., s_N) \triangleq \max \left\{ \mathbb{E}[\exp(\theta_1)|s_1, ..., s_N], ..., \mathbb{E}[\exp(\theta_N)|s_1, ..., s_N] \right\}. \tag{11}$$

Since we relabel agents such that signals satisfy $s_1 > ... > s_N$, it is easy to check that $S(s_1, ..., s_N) = \mathbb{E}[\exp(\theta_1)|s_1, ..., s_N]$. We provide the comparative statics of the ex-ante expected surplus.

Proposition 1 (Comparative Statics: Surplus)**.**
The ex-ante expected surplus $\mathbb{E}[S(s_1, ..., s_N)]$ *is strictly increasing in* $corr(\Delta s_i, \Delta\theta_i)$ *and constant in* $corr(\bar{s}, \bar{\theta})$ *and* m*.*

Proposition 1 shows that the expected surplus depends only on how informative is the join of the information structure $(s_1, ..., s_N)$ about the idiosyncratic component of the payoff shocks $(\Delta\theta_1, ..., \Delta\theta_N)$. This is natural as the surplus comes from assigning the object efficiently across agents. Since the degree of payoff interdependence m does not affect the efficiency of the auction, the surplus depends only on the degree to which the object is assigned efficiently.

3.5 Split of Surplus: Profits and Rents

We now characterize the seller's profits and the buyers' rents. The buyers' rents are given by:

$$V(s_1, ..., s_N) \triangleq \mathbb{E}[exp(\theta_1) - p_2|s_1, ..., s_N].$$

The buyers' rents plus the seller's profits is equal to the total surplus. We characterize the seller's profits in terms of the total surplus.

[14] See Krishna (2009) for a textbook discussion.

Theorem 1 (Profits)**.**

The seller's profits are equal to:

$$p_2 \;=\; exp\Bigg((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1,...,s_N] - \mathbb{E}[\theta_2|s_1,...,s_N]) \Bigg) \times S(s_1,...,s_N) \qquad (12)$$

Theorem 1 characterizes how the total surplus is split between the buyer and the seller (the buyers' rents are the complement of the surplus).[15] Remember that m does not change the total surplus, and hence m is the key parameter that determines how the surplus is split between the buyers and the seller. We provide the comparative static on how the profits change with the information structure.

Corollary 1 (Comparative Statics: Profits and Rents)**.**

The expected profits ($\mathbb{E}[p_2]$) are decreasing in m and constant in $corr(\bar{s}, \bar{\theta})$. The buyers' rents ($\mathbb{E}[V(s_1,...,s_N)]$) are increasing in m and constant in $corr(\bar{s}, \bar{\theta})$.

Corollary 1 shows that the profits are decreasing in the degree of payoff interdependence m. Note that Corollary 1 does not provide the comparative static of the seller's profits with respect to $corr(\Delta s_i, \Delta\theta_i)$. This is because the seller's profits can be decreasing or increasing with respect to $corr(\Delta s_i, \Delta\theta_i)$.[16] If m is big enough, then the seller's profits are decreasing in $corr(\Delta s_i, \Delta\theta_i)$. The intuition is that more information about the idiosyncratic shock may have an excess effect on the "winner's curse". This is one of the fundamental insights on the paper as this is the intuition behind the failure of the linkage principle when we study multidimensional information structures in Section 4.

If $m = 1$, then the expected value of an agent's payoff shock conditional on all signals is the same as the expectation conditional only on his private information (that is, $\mathbb{E}[\theta_i|s_1,...,s_N] = \mathbb{E}[\theta_i|s_i] \propto s_i$). This corresponds to the case in which agents have private values. In this case, the price paid is equal to the expected second highest valuation:

$$p_2 = \mathbb{E}[exp(\theta_2)|s_1,...,s_N].$$

If $m > 1$, the price paid is higher than the expected second highest valuation. If $m < 1$, the price paid is lower than the expected second highest valuation. If $m \to \infty$, then the seller's profits

[15] Note that: $(\mathbb{E}[\theta_1|s_1,...,s_N] - \mathbb{E}[\theta_2|s_1,...,s_N]) \geq 0$, and hence the term multiplying $S(s_1,...,s_N)$ is between 0 and 1 (as long as $m \in [-(N-1), \infty)$).

[16] We believe the buyers' rents are increasing in $corr(\Delta s_i, \Delta\theta_i)$. Nevertheless, we have no been able to prove this.

converge to 0. If $m \to -(N-1)$ then the seller's profits will converge to the total surplus.[17]

It is easy to check that, if $m < -(N-1)$ then $p_2 > S(s_1, ..., s_N)$. That is, the buyers must be getting negative rents. This implies that, if $m < -(N-1)$, then the strategy profile (10) is not a Nash equilibrium. Hence, The Gaussian information structures allows to provide necessary conditions on the information structure such that this strategy profile is a Nash equilibrium.

3.6 Noise-Free Signals

We now provide a corollary that studies noise-free signals (see (9)). We can write the information structure in terms of $(corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta}), m)$ as follows:

$$corr(\Delta s_i, \Delta\theta_i) = corr(\bar{s}, \bar{\theta}) = 1 \text{ and } m = \mu. \tag{13}$$

Hence, the parameter μ only changes m. This allow us to show that the total surplus can be split in any proportion between buyers and the seller, depending on the information structure.

Corollary 2 (Any Division of Surplus is Possible)**.**
If agents observe signal of the form (9)*, then:*

$$\lim_{\mu\to\infty} \mathbb{E}[p_2] = 0 \quad \textit{and} \quad \lim_{\mu\to-(N-1)} \mathbb{E}[p_2] = \mathbb{E}[S(s_1, ..., s_N))]. \tag{14}$$

Corollary 2 has two components. On one hand, as $\mu \to \infty$ the seller's profits converge to 0.[18] This implies that the price converges in distribution to 0. On the other hand, as $\mu \to -(N-1)$ the seller's profits converge to the total surplus. This implies that the buyers' rents converge to 0.

We begin by providing the intuition of $\mu \to \infty$. Consider the case of $N = 2$ and suppose that agents observe signals of the form (9). In the limit $\mu \to \infty$, the signals of agents are almost perfectly *negatively* correlated (remember that $\Delta\theta_1 + \Delta\theta_2 = 0$). If agent i observes a negative realization of his signal, he expects the other agent to have a positive realization of his signal. Hence, he expects to loose the auction. The only circumstance under which both agents can have a negative realization of their signals is that the realization of $\bar{s}$ is negative and "far" from 0. Yet, as $\mu \to \infty$ this implies that $\bar{\theta}$ is "very" negative. Hence, when an agent observes a negative realization of his signal, he knows that he observed the highest signal only if $\bar{\theta}$ is very negative.

[17]Note that in the limit to common values ($corr(\Delta\theta_i, \Delta s_i) \to 0$) two things happen; $m \to \infty$ and $\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N]) \to 0$. It is easy to check that in the limit to common values the profits do not converge to 0.

[18]As $p_2 \geq 0$ for all realization of signals, $\mathbb{E}[p_2] \to 0$ implies that p_2 converges in distribution to 0

Hence, he will win the object only if his payoff shock ($\theta_i = \Delta\theta_i + \bar{\theta}$) is very low. Hence, the bid of an agent that observes a negative signal will converge to 0. Hence, the agent that wins the object will pay almost 0.

On the other hand, in the limit $m \to -(N-1)$ we can re-write (3) as follows:

$$\lim_{\mu\to-(N-1)} \mathbb{E}[\theta_i|s_1,...,s_N] = (\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})} - \frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)})\frac{1}{N}\sum_{j\neq i} s_j.$$

In the limit the expected value of θ_i conditional on $(s_1,...,s_N)$ does not depend on s_i. Hence, a buyer has no information about his own payoff shock. This implies that buyers' get not rents.

Role of Affiliation Property. Milgrom and Weber (1982) show that that in symmetric environments with one-dimensional signals ascending auctions yields weakly greater profits than first price auctions. Corollary 2 indirectly shows that this is not satisfied in the limit $\mu \to \infty$. This is because in the limit $\mu \to \infty$ the signals of agents are negatively correlated ($corr(s_i,s_j) < 0$). Hence, in the limit the affiliation property assumed is Milgrom and Weber (1982) is not satisfied.

3.7 Public Signals

Finally, we study the impact of a public signal on the seller's profits and the total surplus generated. We assume that agents observe a public signal:

$$s^p = \bar{\theta} + \bar{\varepsilon}^p,$$

where $\bar{\varepsilon}^p$ is a common value noise term independent of all other random variables in the model.

Lemma 4 (Impact of Public Signal: One-Dimensional Information Structure)**.**
The total surplus generated is constant in $var(\bar{\varepsilon}^p)$ and the seller's profits is increasing in $var(\bar{\varepsilon}^p)$.

Lemma 4 shows that the surplus generated does not depend on the precision of the public signal. This is natural, as the ascending auction is efficient when agents observe one-dimensional signals. The seller's profits are increasing in the precision of the public signal because the public signal reduces the winner's curse. Both comparative statics are standard in the literature, yet neither will hold in environments with multidimensional signals.

4 Two-Dimensional Signals

We now study a class of symmetric two-dimensional signals. The objective is two-fold. The first objective is to provide novel predictions in ascending auctions that arise only when agents observe multidimensional signals. The second objective is to illustrate how to solve for the equilibrium when agents observe multidimensional Gaussian signals. Although the information structure we study in this section is very stylized, the same methodology can be extended to general Gaussian information structures and to other trading mechanisms.

4.1 Information Structure

We assume that an agent's valuation of the object is the sum of a taste shock and a common shock. For example, an agent's valuation of a house is determined by his taste for the house and the resale value of the house. We assume that agent i knows his taste shock and each agent observes a noisy signal of the common shock that affects the valuation of all agents.

The formal description of the information structure is as follows. The payoff shock of an agent is equal to the sum of two independent payoff shocks:

$$\theta_i = \eta_i + \bar{\varphi}. \tag{15}$$

where η_i has some correlation $corr(\eta_i, \eta_j)$ across agents, while $\bar{\varphi}$ is common to all agents. It is useful to note that:[19]

$$\bar{\theta} = \bar{\eta} + \bar{\varphi} \text{ and } \Delta\theta_i = \Delta\eta_i.$$

That is, the realization of the idiosyncratic component of the payoff shock is equal to the realization of the idiosyncratic component of the shock η_i. On the other hand, the common component of the payoff shock is equal to the sum of the common component of two shocks. agent i observes two signals:

$$s_{i1} = \eta_i \ ; \ s_{i2} = \bar{\varphi} + \varepsilon_i, \tag{16}$$

where ε_i is a noise term independent across agents and independent of all other random variables in the model.

Before we study the equilibrium outcome it is convenient to study some limiting cases. If $var(\varepsilon_i) = 0$ or $var(\varepsilon_i) = \infty$ the model corresponds to a private value environment. If $var(\varepsilon_i) = 0$

[19] Remember that $\bar{\eta} \triangleq \frac{1}{N}\sum_{i\in N} \eta_i$ and $\Delta\eta_i \triangleq \eta_i - \bar{\eta}$.

then agent i can compute θ_i perfectly using just his private information. On the other hand, if $var(\varepsilon_i) = \infty$ then it is the same as if agent i observed only s_{i1}. This also corresponds to a private value environment. Note that for any value of $var(\varepsilon_i)$ the ascending auction would implement the efficient outcome if agents "ignored" signal s_{i2}. Of course, this does not happen in equilibrium

4.2 Equilibrium with Two-Dimensional Signals

We first define a one-dimensional equilibrium statistic, and then explain the intuition and how it is used in the characterization of the equilibria.

Definition 1 (Equilibrium Statistic)**.**
A linear combination of signals $\zeta_i \triangleq s_{i1} + \beta s_{i2}$ *is an equilibrium statistic if:*

$$\mathbb{E}[\theta_i | \mathbf{s}_i, \zeta_1, ..., \zeta_N] = \mathbb{E}[\theta_i | \zeta_1, ..., \zeta_N] \tag{17}$$

An equilibrium statistic is a linear combination of signals in which the weights on this signals satisfy statistical condition (17). The expected value of θ_i conditional on $\mathbf{s}_i$ and the equilibrium statistic of other agents $\{\zeta_j\}_{j\neq i}$ is equal to the expected value of θ_i conditional on all equilibrium statistics $\{\zeta_j\}_{i\in N}$. Although (17) is defined purely in terms of the information structure — without reference to the game or the solution concept — it is transparent to see that there is an equilibrium notion involved. If all agents use just their equilibrium statistic instead of their whole information structure, and agent i knows the equilibrium statistic of other agents, then agent i also wants to use only his equilibrium statistic.

We show that for every equilibrium statistic there exists a Nash equilibrium in which each agent i behaves as if he observed *only* his equilibrium statistic ζ_i. Similar to the analysis of one-dimensional signals, we assume that agents are ordered as follows:

$$\zeta_N < ... < \zeta_1. \tag{18}$$

If there are multiple equilibrium statistics, then there will be one Nash equilibrium for each equilibrium statistic. Different equilibrium statistics induces a different order (as in (18)), so the Nash equilibrium is described in terms of the order induced by each equilibrium statistic.

Theorem 2 (Symmetric Equilibrium with Multidimensional Signals)**.**
For every equilibrium statistic $\{\zeta_i\}_{i\in N}$ there exists a Nash equilibrium in which agent i's drop-out time is given by:

$$p_i = \mathbb{E}[\exp(\theta_i)|\zeta_i, ..., \zeta_i, ..., \zeta_N],$$

In equilibrium, agent 1 *gets the object and pays a price equal to* $p_2 = \mathbb{E}[\exp(\theta_2)|\zeta_2, \zeta_2, ..., \zeta_N]$.

Proposition 2 characterizes a natural class of equilibria in the ascending auction. There exists a class of equilibria in which agents project their signals into a one-dimensional statistic using the equilibrium statistic $\zeta_i = s_{i1} + \beta s_{i2}$. In equilibrium agents behave as if they observed *only* $\boldsymbol{\beta} \cdot \mathbf{s}_i$, which is a one-dimensional object.[20] Note that this implies that the analysis in Section 3 remains valid, with the modification that we need to replace s_i with ζ_i. We later prove that Proposition 2 would not change if we allow agents to re-enter in the ascending auction. This is because the Nash equilibrium characterized in Proposition 2 is a *posterior* equilibrium.

Finally, we provide a characterization of the set of equilibrium statistics.

Lemma 5 (Equilibrium Statistic)**.**
A linear combination of signals $\zeta_i = s_{i1} + \beta s_{i2}$ is an equilibrium statistic of information structure (16) *if and only if β is a root of the following cubic equation:*

$$\frac{-1}{var(\Delta\varepsilon_i)} + \frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})}\beta + \frac{-1}{var(\Delta\eta_i)}\beta^2 + \frac{(var(\Delta\eta_i) + var(\bar{\eta}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\eta_i)var(\bar{\eta})var(\bar{\varphi})}\beta^3 = 0 \tag{19}$$

It is transparent to see that generically (19) has 1 or 3 solutions. Moreover, it is easy to check that all roots are positive. The characterization in (19) will be basic for the analysis of the equilibria. We later generalized and explain the intuition behind (19).

Finally, we note that the information structure (16) is similar to the information structure studied by Wilson (1998), but he assumes the random variables come from a diffuse prior. He assumes that $var(\bar{\varphi})$,$var(\bar{\eta}) \to \infty$. In the limit, the unique solution to (19) is $\beta = 1$, and hence in equilibrium agents place equal weights in both signals. In this case, the equilibrium statistic of an agent is equal to his expected payoff shock conditional only on his private information ($\zeta_i = \mathbb{E}[\theta_i|s_{i1}, s_{i2}]$). We study information structures with diffuse priors in Section 7.2. Interestingly, similar information structures have also been studied in the rational expectations literature. The

[20]Note that the multiplication of vectors is using the dot product. That is, $\boldsymbol{\beta} \cdot \mathbf{s}_i = s_{i1} + \beta s_{i2}$

multiplicity of equilibria in the ascending auction is related to the multiplicity of equilibria in Ganguli and Yang (2009), Amador and Weill (2010) and Manzano and Vives (2011) .

4.3 Multiplicity of Equilibria

An ascending auction with multidimensional signals may have multiple symmetric equilibria. The multiplicity of equilibria provides a sharp illustration of how agents use their signals to determine their drop-out time, and how this is modified by what they learn from the drop-out time of other agents. The multiplicity of equilibria comes directly from the fact that (19) may have multiple roots. We formalized this in the following lemma.

Lemma 6 (Multiplicity of Equilibria)**.**
The auction has a unique (multiple) equilibrium within the class of equilibria studied in Proposition 2 if $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 < 0\ (> 0)$, *where,*[21]

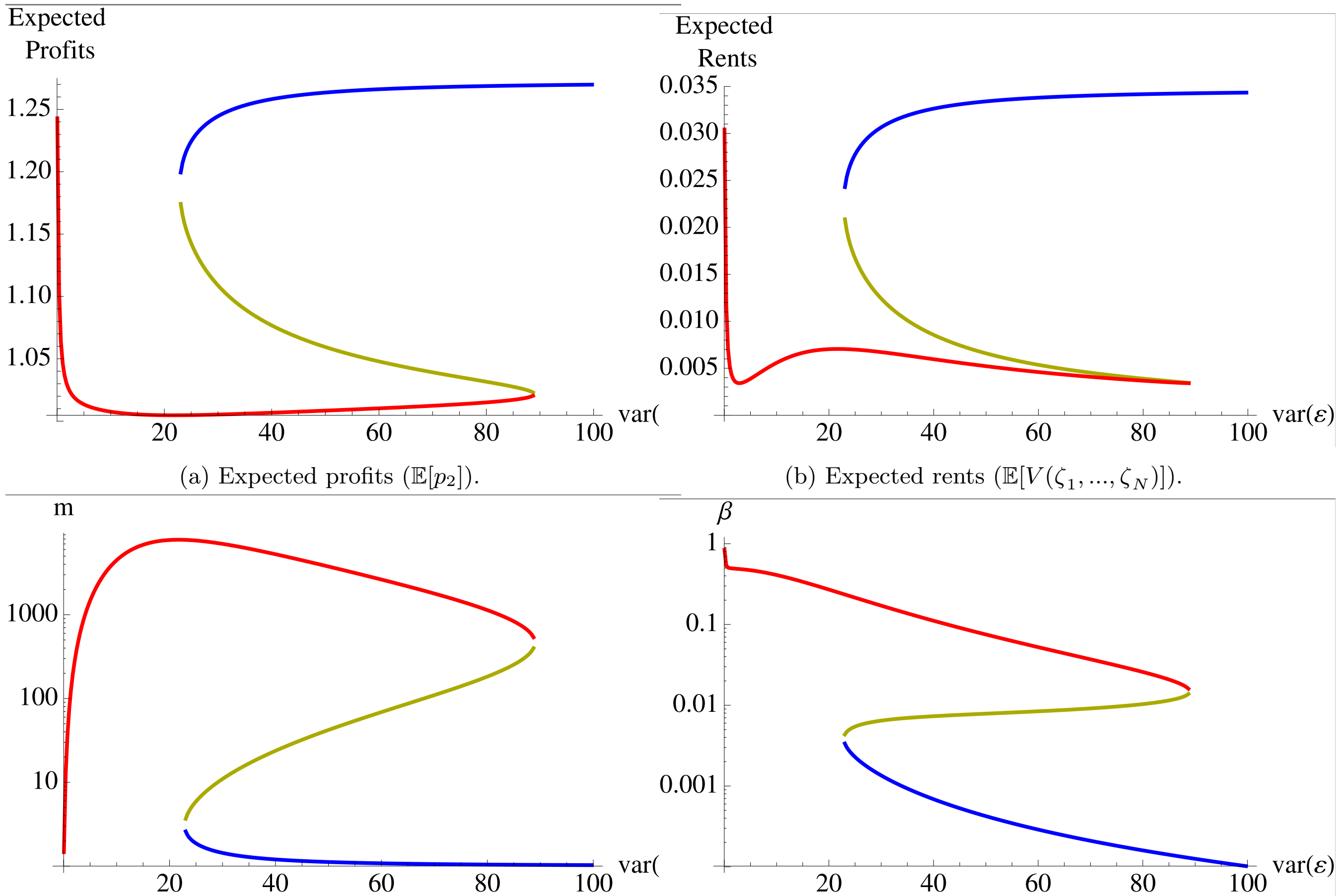
$$a = \frac{(var(\Delta\eta_i) + var(\bar{\eta}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\eta_i)var(\bar{\eta})var(\bar{\varphi})} \quad ; \quad b = \frac{1}{var(\Delta\eta_i)} \ ;$$
$$c = \frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})} \quad ; \quad d = \frac{1}{var(\Delta\varepsilon_i)}$$

In Figure 1d we plot the value of β for different values of $var(\varepsilon_i)$. The value of β gives the relative weight agents place on their signals. In Figure 1a and Figure 1b we plot the seller's profits and the buyers' rents. It is easy to check from the scale of the plots that the buyers' rents are small, and hence the total surplus is qualitatively the same as the seller's profits.

The different colors in the plot corresponds to the different roots of (19). The multiplicity of equilibria comes from a complementarity on how agents use their signals to determine their drop-out times. To illustrate the source of complementarity consider an information structure in which $var(\varepsilon_i)$ is large, and $corr(\eta_i, \eta_j)$ is non-negligible.

We begin by studying the equilibrium in blue. There is an equilibrium in which agents "almost" ignore their signals s_{i2} ($\beta \approx 0$). agent i by looking at the drop-out time of agent j learns η_j. Nevertheless, agent i does not learn anything about $\bar{\varphi}$ from the drop-out time of agent j. Hence, the information in agent's j drop-out time is not used by agent i to update his beliefs on θ_i. Hence, agent i predicts θ_i only using his private information. Yet, as $var(\varepsilon_i)$ is large, the agent places a small weight on s_{i2}. Hence, in equilibrium the weight on s_{i1} is much larger than

[21] The case $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0$ must be considered independently. If $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0$, then there is a unique equilibrium if and only if $b = 3ac$.



(a) Expected profits ($\mathbb{E}[p_2]$).

(b) Expected rents ($\mathbb{E}[V(\zeta_1, ..., \zeta_N)]$).

(c) Parameter m in log-scale.

(d) β in log-scale.

Figure 1: Expected outcome of ascending auction for $var(\bar{\varphi}) = 1$, $var(\eta_i) = (0.35)^2$, $corr(\eta_i, \eta_j) = 1/2$ and $N = 500$.

s_{i2}. This is the equilibrium plotted in blue in Figure 1.

If we restrict attention to the equilibrium in blue, the seller's profits and the buyers' rents increase with $var(\varepsilon_i)$. This is because as $var(\varepsilon_i)$ increases, agents place smaller weight on s_{i2} (β decreases). This leads to a higher surplus, which also induces a higher seller's profits and higher buyers' rents.

We now study the equilibrium in red in Figure 1. Suppose that all agents place a weight on s_{i2} that is non-negligible with respect to the weight on s_{i1}. In this case, when agent i observes the drop-out time of agent j he learns the realization of $\eta_j + \beta(\bar{\varphi} + \varepsilon_j)$. If agent i observes the drop-out time of many agents, then he can infer a signal almost the same $\bar{\eta} + \beta \cdot \bar{\varphi}$. That is, agent i knows that the drop-out time of other agents is determined by their taste shock and the value of the common shock. If agent i observes a high taste shock he expects that the drop-out time of other agents was driven by a high taste shock (as these are correlated). Yet, this makes agent i more pessimistic about $\bar{\varphi}$. This implies that when agent i observes a high taste shock, his overall expectations change a small amount. This is because agent i also interprets a high taste shock as a low common shock, which comes from the signal he obtains from the drop-out time of other agents. Hence, agent i reduces the weight that he places on s_{i1}. In relative terms, this implies that the weight agent i places on s_{i2} increases (that is, β increases).

If we restrict attention to the equilibrium in red, the seller's profits are quasi-convex in $var(\varepsilon_i)$. For a large enough $var(\varepsilon_i)$, the intuition is similar to the equilibrium in blue. That is, as $var(\varepsilon_i)$ increases, the weight on s_{i2} decreases sufficiently fast such that the total surplus increases. Nevertheless, for small values of $var(\varepsilon_i)$, the rate at which the weight on s_{i2} decreases is not fast enough to compensate for the fact that ε_i has a higher variance. This implies that the total surplus decreases because the correlation between the drop-out time of agents and the noise term ε_i increases.

It is interesting to note that the buyers' rents have an unusual shape with a hump. This is because as $var(\varepsilon_i)$ increases two things happen. The first thing that happens is that the total surplus decreases. The second thing that happens is that the level of payoff interdependence (m) increases. As the level of payoff interdependence increases, buyers get a higher share of the total surplus. Hence, there are values of $var(\varepsilon_i)$ for which the total surplus decreases but the buyers' rents increases. This leads to the unusual shape of the buyers' rents in Figure 1b.

We show that for large enough $var(\varepsilon_i)$ or small enough $var(\varepsilon_i)$ there is a unique equilibrium.

Corollary 3 (Uniqueness of Equilibrium)**.**
In the limits $var(\varepsilon_i) \to 0$ or $var(\varepsilon_i) \to \infty$ there is a unique equilibrium.

Corollary 3 shows that if the noise term is large enough or small enough, then there always exist a unique equilibrium. If $var(\varepsilon_i) \to 0$, then agents have complete information, and hence there is a unique equilibrium. If $var(\varepsilon_i) \to \infty$, then agents ignore s_{i2}, and it is as if agents have private values. Hence, there is a unique equilibrium.

Large Markets The model has an interesting discontinuity as we appproach large markets $(N \to \infty)$. Consider first a fixed N. If an agent could observe all signals $\{s_{i2}\}_{i\in N}$, then he would be able to compute the signal $\bar{s}_2 = \bar{\varphi} + \bar{\varepsilon}$. As $var(\varepsilon_i) \to \infty$:

$$var(\bar{\varepsilon}) = \frac{1}{N} var(\varepsilon_i) \to \infty.$$

Hence, even if an agent could observe all the signals $\{s_{i2}\}_{i\in N}$ the information about $\bar{\varphi}$ becomes negligible in the limit $var(\varepsilon_i) \to \infty$.

Consider now the case in which we first take the limit $N \to \infty$, and then we take the limit $var(\varepsilon_i) \to \infty$. Since we first take the limit $N \to \infty$, this is the same as imposing that $var(\bar{\varepsilon}) = 0$. This is because the average of many independent random variables converges to 0. Hence, by taking the limit $N \to \infty$ first and then $var(\varepsilon_i) \to \infty$ it is the same as first imposing $var(\bar{\varepsilon}) = 0$ and then taking the limit $var(\Delta\varepsilon_i) \to \infty$. In this case, the average signal observed by all players $\bar{s}_2 = \bar{\varphi} + \bar{\varepsilon} = \bar{\varphi}$. Hence, even in the limit the average signal of all players perfectly reveals $\bar{\varphi}$, even in the limit $var(\Delta\varepsilon) \to \infty$.

In this case, the result on uniqueness of equilibrium changes.

Corollary 4 (Multiplicity of Equilibria)**.**
If $var(\bar{\varepsilon}) = 0$ and we take the limit $var(\Delta\varepsilon_i) \to \infty$, then there are multiple equilibria if and only if:

$$var(\bar{\varphi}) \geq 4var(\Delta\eta_i)(1 + \frac{var(\Delta\eta_i)}{var(\bar{\eta})}). \tag{20}$$

Corollary 4 characterizes the distribution of payoff shocks under which there are multiple equilibria when $N \to \infty$ and $var(\Delta\varepsilon_i) \to \infty$. It is possible to check that, as the number of players increases, the possibility of multiple equilibria in the limit $var(\varepsilon_i) \to \infty$ also increases. This is because in the limit an agent that could observe all the signals $\{s_{i2}\}_{i \in N}$ could perfectly predict $\bar{\varphi}$ (for any value of $var(\varepsilon_i)$). Hence, in the limit $N \to \infty$ and $var(\varepsilon_i) \to \infty$, the model does not approach a model of private values.

Inequality (20) provides a bound on the size of $var(\bar{\varphi})$ such that there exists a public signal that would make agents place 0 weight on η_i. More precisely, if (20) is satisfed, then there exists κ such that $\mathbb{E}[\theta_i|\bar{s}, \eta_i] = \mathbb{E}[\theta_i|\bar{s}, \eta_i]$, with $\bar{s} = \kappa \cdot \bar{\varphi} + \bar{\eta}$. In the limit, in equilibrium agents place 0 weight on s_{i2} because $\varepsilon_i \to \infty$ (hence this signal is not informative). On the other hand, an agent learns $\bar{s}$ from the drop-out time of other agents. Hence, agents also place 0 weight on η_i. In the limit agents place 0 weight on both of their private signals.

4.4 Impact of Public Information

We now study the impact of public information on the equilibrium outcome. We assume agents have access to two public signals (in addition to the signals in (16)):

$$\bar{s}_3 = \bar{\eta} + \bar{e}_3 \text{ and } \bar{s}_4 = \bar{\varphi} + \bar{e}_4,$$

where $\bar{\varepsilon}_3$ and $\bar{\varepsilon}_4$ are independent of all random variables defined so far. Hence, agent i observes the signals $(s_{i1}, s_{i2}, \bar{s}_3, \bar{s}_4)$. The signal $\bar{s}_4$ is additional information about the common shock $(\bar{\varphi})$, and hence this can be seen as disclosing additional information about the good. On the other hand, $\bar{s}_3$ is a signal on the average taste shock of agents. This can be interpreted as allowing each agent to observe the other agents that are in the auction to get an estimate of their taste shock.[22] We study how the outcome of the equilibrium changes with the precision of $var(\bar{\varepsilon}_3)$ and $var(\bar{\varepsilon}_4)$.

We begin by comparing how the expected surplus changes with the precision of the public signals.

Proposition 2 (Comparative Statics of Public Signals: Surplus)**.**
If the ascending auction has a unique equilibrium, then the total surplus is decreasing in $var(\bar{e}_3)$ and $var(\bar{e}_4)$. In the limit:

$$\lim_{var(\varepsilon_3)\to 0} S(\zeta_1, ..., \zeta_N) = \lim_{var(\varepsilon_4)\to 0} S(\zeta_1, ..., \zeta_N) = \max\{\mathbb{E}[exp(\theta_1)|\mathbf{s}_1, ..., \mathbf{s}_N], ..., \mathbb{E}[exp(\theta_N)|\mathbf{s}_1, ..., \mathbf{s}_N]\}.$$

Proposition 2 shows that the surplus increases with the precision of the public signals. In Proposition 2 we require that the ascending auction has a unique equilibrium. If the ascending auction has three equilibria then the result holds for two of the equilibria, while the comparative static is reversed for the third equilibrium. In the limit in which one of the public signals is arbitrarily precise, the equilibrium approaches the first best.

The intuition on why the surplus is decreasing in $var(\bar{e}_4)$ is simple. As the public information about $\bar{\varphi}$ is more precise, an agent needs to place less weight on their private signal s_{i2} to predict $\bar{\varphi}$. This implies that the correlation between the drop-out time of an agent and the realization of the noise term ε_i decreases. Hence, the surplus increases.

The mechanism by which $var(\bar{e}_3)$ impacts efficiency is more subtle. It is transparent to see that a change in $var(\bar{e}_3)$ does not change an agent's expectation of his own payoff shock conditional only on his private information. That is:

$$\mathbb{E}[\theta_i|s_{i1}, s_{i2}, \bar{s}_3] = \mathbb{E}[\theta_i|s_{i1}, s_{i2}].$$

Nevertheless, $\bar{s}_3$ allows an agent to extract more information from the drop-out time of another agent. That is, agent i can use $\bar{s}_3$ to get a better prediction about $\bar{\varphi}$ from the drop-out time of

[22] All the results follow through in the same way if instead of having a public signal $\bar{s}_3 = \bar{\eta} + \bar{e}_3$ each agent i observes $N-1$ private signals on the payoff shocks of agents $j \neq i$. That is, if agent i observes signals $s_{i3} = \eta_\ell + \varepsilon_{i\ell 3}$ for all $\ell \neq i$.

agent j. Hence, this is an indirect mechanism by which agent i gets a more precise information about $\bar{\varphi}$. This ultimately decreases the weight agent places on s_{i2}, and hence it increases the surplus.

We now study the impact of the public signals on the seller's profits.

Proposition 3 (Comparative Statics of Public Signals: Profits)**.**
In the limit in which one of the public signals becomes arbitrarily precise:

$$\lim_{var(\varepsilon_3)\to 0} p_2 = 0 \quad \textit{and} \quad \lim_{var(\varepsilon_4)\to 0} p_2 = \max{}^{(2)}\{\mathbb{E}[exp(\theta_1)|\mathbf{s}_1,...,\mathbf{s}_N],...,\mathbb{E}[exp(\theta_N)|\mathbf{s}_1,...,\mathbf{s}_N]\},$$

where $\max^{(2)}\{\cdot\}$ *denotes the second order statistic (that is, the second maximum).*

Proposition 3 shows that, as the public signal about $\bar{\varphi}$ becomes arbitrarily precise ($var(\varepsilon_4) \to 0$), the outcome approaches the equilibrium under complete information. The intuition is that in the limit, agent ignore their private signal s_{i2}, and hence the only private signal they observe is $s_{i1} = \eta_i$. Hence, in the limit the equilibrium is as if agents observe one-dimensional signals.

The impact of the public signal about $\bar{\eta}$ on profits is different. As $\bar{\eta}$ becomes common knowledge ($var(\bar{\varepsilon}_3) \to 0$), the equilibrium profits becomes arbitrarily close to 0.[23] To illustrate the intuition, consider the case $N = 2$ (the intuition is similar to Corollary 2). As $var(\bar{\varepsilon}_3) \to 0$ it is almost common knowledge who is the agent with the highest valuation. If agent i is almost sure he does not have the highest payoff shock ($\Delta\eta_i < \Delta\eta_j$), then agent i knows he will win the object only if agent j observed a very bad signal about $\bar{\varphi}$. Hence, agent i knows that he will win the object only if $\bar{\varphi}$ is very low. Anticipating this, agent i drops out of the auction arbitrarily fast.

4.5 Discussion: Failure of Linkage Principle

The key insight from Section 4.4 is that the public signals has two effects. First, it decreases the winner's curse from one of the signals agents observe. This increases the profits. The second effect of public signals is that it increases the *interim* asymmetries between players. That is, players have better information about the difference in valuations between each other. This increases the winner's curse from another common shock, which decreases the profits. To illustrate this, we consider a slightly different information structure.

[23]Note that Proposition 3 makes no assumption on the variance of the shock ε_i or $\Delta\theta_i$. Hence, if $\varepsilon_i \approx 0$ and $var(\bar{\varepsilon}_4) \to 0$ then shocks are arbitrary close to common knowledge. Clearly, the result also holds for any distribution of payoff shocks.

Consider a model in which we modify s_{i1} as follows:

$$s'_{i1} = \eta_i + \varepsilon'_{i1},$$

where ε'_{i1} is a noise term independent of all other random variables in the model and independent across agents. To make the argument simpler, assume that the variance of ε'_{i1} is small. In this case, if agents observe *only* s'_{i1}, this would constitute a classic model with interdependent values (it is equivalent to Example 1). Hence, if agents observe only s'_{i1}, then the profits would be strictly increasing in $var(\bar{\varepsilon}_3)$ (this is direct from Lemma 4). That is, a public signal would increase profits because it decreases the payoff interdependence from the common factor $\bar{\eta}$.

On the other hand, if agents observe s'_{i1} and s_{i2}, then the profits would continue to be increasing in $var(\bar{\varepsilon}_3)$.[24] This is because a more precise information about $\bar{\eta}$ increases the interim differences between agents. This in turn, increases the payoff interdependence from the common factor $\bar{\varphi}$. Hence, a public signal decreases the payoff interdependence from one signal, and increases the payoff interdependence from the other signal.

5 General Multidimensional Signals

We now study an ascending auction, but allow for arbitrary multidimensional Gaussian signals. We keep the auction the same as the one described in Section 2. We extend the methodology in Section 4 to allow for asymmetric information structures. The idea remains the same as in Section 4. That is, we first compute an equilibrium statistic, and then compute the equilibrium as if agents observe only the equilibrium statistic. We later show that the analysis can be extended to a larger class of mechanisms.

5.1 Information Structure

We first study a model in which agents observe one-dimensional signals. In contrast to Section 3, we allow signals and payoff shocks to be asymmetrically distributed. We keep the information structure the same as in Section 2.2, but allow for arbitrary information structures. That is, we allow for any distribution of signals and fundamentals $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{(J+1)N}$ as long as the distribution is jointly Gaussian.

[24]This can be seen by Proposition 3 and a continuity argument.

5.2 One-Dimensional Signals

We begin by studying one-dimensional signals. If agents observe one-dimensional signals, and the average crossing condition is satisfied, then the ascending auction has an ex-post equilibrium that is efficient (see Krishna (2003)). The average crossing condition is defined as follows.

Definition 2 (Average Crossing Condition)**.**
The one-dimensional information structure $(s_1, ..., s_N, \theta_1, ..., \theta_N)$ satisfies the average crossing condition if for all $\mathcal{A} \subset \{1, ..., N\}$, and for all $i, j \in \mathcal{A}$ with $i \neq j$:

$$\frac{\partial \mathbb{E}[\theta_i|s_1, ..., s_N]}{\partial s_j} \leq \frac{1}{|\mathcal{A}|} \sum_{k \in N} \frac{\partial \mathbb{E}[\theta_k|s_1, ..., s_N]}{\partial s_j}$$

The average crossing condition guarantees that the impact of agent i's signal on agent j's valuation is not to high. The comparison is done with respect to the average impact that agent i's signal has on any group of agents that contains i. The average crossing condition is the extension of (6) to asymmetric environments. In fact, in symmetric environments (5) is equivalent to the average crossing condition.

To characterize the equilibrium we assume that agents are ordered as follows:

$$\mathbb{E}[\theta_i|s_1, ..., s_N] > ... > \mathbb{E}[\theta_N|s_1, ..., s_N]. \tag{21}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the signals of all agents. To characterize the equilibrium we define $\tilde{s}_1 \in \mathbb{R}$ as follows:

$$\tilde{s}_1 \triangleq \quad \arg\min_{s' \in \mathbb{R}} \quad \mathbb{E}[\theta_1|s', s_2, ..., s_N] \tag{22}$$

$$\text{subject to} \quad \forall i \in N, \quad \mathbb{E}[\theta_1|s', s_2, ..., s_N] \geq \mathbb{E}[\theta_i|s', s_2, ..., s_N] \tag{23}$$

$\tilde{s}_1$ is the signal that yields the lowest expected payoff shock to agent 1, but keeping the expected payoff shock of agent 1 above the expected payoff shocks of other agents.

Lemma 7 (Equilibrium for One-Dimensional Signals)**.**
The ascending auction has a Nash equilibrium in which agent 1 wins the object and pays a price:

$$p_2 = \mathbb{E}[\theta_1|\tilde{s}_1, s_2, ..., s_N]. \tag{24}$$

The ascending auction has an equilibrium in which the agent with the highest expected valuation wins the object. The price paid for the object is the expected valuation of the winner of the object, but evaluated at the minimum signal this agent could have observed and still win the object.

5.3 Equilibrium Statistic

We define an equilibrium statistic of the signals an agent observes.

Definition 3 (Equilibrium Statistic)**.**
We say random variables $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ *are an equilibrium statistic of* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{J \cdot N}$ *if:*

$$\forall i \in N, \quad \mathbb{E}[\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]. \tag{25}$$

The definition of an equilibrium statistic is the natural extension of Definition 1, but allowing for general J-dimensional signals. As before, in order to make the notation more compact, we often denote an equilibrium statistic by:

$$(\zeta_1, ..., \zeta_N) \triangleq (\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N). \tag{26}$$

5.4 Equilibrium Characterization

For the characterization of the equilibrium, we assume that the equilibrium statistic satisfies the average crossing condition. In the online appendix we provide sufficient conditions on the individual signals that an agent observes that guarantees that the equilibrium statistic satisfies the average crossing condition (See Section 11.3). For the equilibrium characterization we assume that agents are ordered as follows:

$$\mathbb{E}[\theta_i | \zeta_1, ..., \zeta_N] > ... > \mathbb{E}[\theta_N | \zeta_1, ..., \zeta_N]. \tag{27}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the equilibrium statistic of all agents. This is the natural extension of (18) when agents are asymmetric.

To characterize the equilibrium we define $\tilde{\zeta}_1 \in \mathbb{R}$ as follows:

$$\tilde{\zeta}_1 \triangleq \quad \arg\min_{\zeta' \in \mathbb{R}} \quad \mathbb{E}[\theta_1 | \zeta', \zeta_2, ..., \zeta_N] \tag{28}$$

$$\text{subject to} \quad \forall i \in N, \quad \mathbb{E}[\theta_1 | \zeta', \zeta_2, ..., \zeta_N] \geq \mathbb{E}[\theta_i | \zeta', \zeta_2, ..., \zeta_N] \tag{29}$$

$\tilde{\zeta}_1$ is the analogous of $\tilde{s}_i$, but using the equilibrium statistic.

Theorem 3 (Equilibrium for Multidimensional Signals)**.**
The ascending auction has a Nash equilibrium in which agent 1 *wins the object and pays a price:*

$$p_2 = \mathbb{E}[\theta_1 | \tilde{\zeta}_1, \zeta_2, ..., \zeta_N].$$

Proposition 3 shows that it is possible to characterize the equilibria in multidimensional environments by computing the equilibrium as if agents observed only the equilibrium statistic.

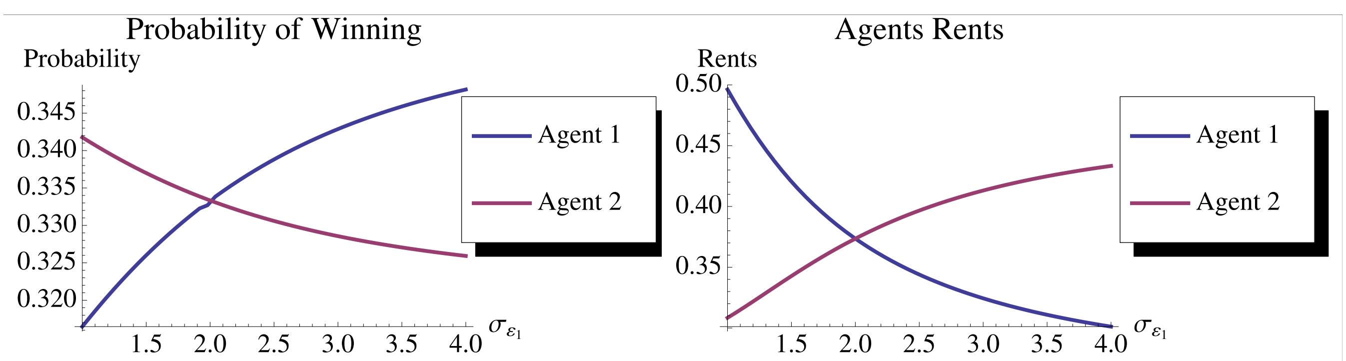
5.5 Example: Asymmetrically Informed Agents

We use Theorem 3 to compute the outcome of an ascending auction when agents are differentially informed. We do not compute the equilibrium analytically, but just illustrate the outcome numerically. The objective is to provide a specific application in which Theorem 3 is used, and to illustrate novel insights that arise when we allow for asymmetries.

We assume there are 3 agents $i \in \{1, 2, 3\}$ and the distribution of payoff shocks is symmetric. The information structure is as in (16), but modify the variance of the error term in signals s_{i2}. We assume that:

$$var(\varepsilon_1) \neq var(\varepsilon_2) \text{ and } var(\varepsilon_2) = var(\varepsilon_3)$$

That is, agent 1 has different precision on his information about the common shock $\bar{\varphi}$ than agent 2 and agent 3. Since agent 2 and agent 3 are symmetric, we compare agent 1 and agent 2.



(a) Expected probability with which an agent wins the good. (b) Expected rents of agents.

Figure 2: Outcome of ascending auctions under different precisions of information on common shocks ($\sigma_{\varepsilon_2} = 2$, $\sigma_\eta = 1$, $\rho_{\eta\eta} = 0$, $\sigma_{\bar{\varphi}} = 3$.)

In Figure 2a we plot the probability that agent 1 and agent 2 win the auction. In Figure 2b we plot the expected rents of agent 1 and agent 2. Note that the payoff environment is symmetric and hence all the differences are due to the asymmetry in the information structure. We can see that more precise information from agent 1 increases his rents, but reduces the rents of agent 2 and agent 3. Interestingly, a higher rent is associated with a lower probability of winning. That is, when agent 1 has more precise private information he gets a higher rents despite the fact that it reduces his probability of winning.

5.6 Existence of Equilibrium Statistic

The equilibrium statistic is the fundamental object that allow us to characterize the equilibrium in multidimensional environments. We now provide further results on the equilibrium statistic. We first prove that an equilibrium statistic exists.

Proposition 4 (Existence)**.**
If the variance covariance matrix $var(\mathbf{s}_1, ..., \mathbf{s}_N)$ has full rank, then an equilibrium statistic $(\zeta_1, ..., \zeta_N) \in \mathbb{R}^N$ exists.[25] *Additionally, if the information structure is symmetric, then there exists a symmetric equilibrium statistic.*

Proposition 4 guarantees the existence of equilibrium statistic for generic information structures. The uniqueness of the equilibrium statistic is clearly not guaranteed (see Section 4.3). In the appendix we further develop the characterization and properties of the equilibrium statistic. We show the following. The equilibrium statistic can be found by solving a bilinear system of equations. We provide sufficient conditions on signals, such that the equilibrium statistic satisfies (5) and, and hence such that the existence of Nash equilibrium in the ascending auction is satisfied. We prove that in symmetric environments there are at most $2J-1$ symmetric equilibrium statistic. Since this analysis is outside the main purpose of the paper, we relegate it to the appendix.

6 Other Mechanisms

We now extend the methodology to accommodate for other mechanisms. We show that it is possible to characterize a class of equilibria in which agents behave "as if" agents observe only their equilibrium statistic. Hence, the equilibrium is the same as characterizing the equilibrium

[25] We have not been able to find an example of an information structure in which an equilibrium statistic does not exist. In the natural cases in which $var(\mathbf{s}_1, ..., \mathbf{s}_N)$ does not have full rank (like a public signal), the equilibrium statistic still exists.

when agents observe one-dimensional signals. Importantly, the definition of equilibrium statistic does not change, as this depends only on the information structure and not on the payoff environment.

6.1 General Games

We consider a game with N players. Player $i \in N$ takes action $a_i \in A_i$, where A_i is assumed to be a metric space. The payoff of player $i \in \{1, ..., N\}$ depends on the realization of his payoff shock $\theta_i \in \mathbb{R}$ and the action taken by all players:

$$\mathbf{a} \triangleq (a_1, ..., a_N).$$

The payoff of player i is denote by $u_i(\theta_i, \mathbf{a})$. We denote by $(a_i', \mathbf{a}_{-i})$ the action profile:

$$(a_i', \mathbf{a}_{-i}) = (a_1, ..., a_{i-1}, a_i', a_{i+1}, ..., a_N).$$

We keep the information structure the same as in Section 5. The definition of equilibrium statistic is the same as in Definition 3.

We distinguish between the payoff environment and the information structure. This is because we want to compute the set of equilibria for a fixed payoff environment, but under a different information structures. The actions available to each agent and the utility functions are called the payoff environment and are denoted by P. The joint distribution of signals and fundamentals is the information structure and is denoted by $\mathcal{I}$. The game is defined by the payoff environment and the information structure $(P, \mathcal{I})$. Given an equilibrium statistic $(\zeta_1, ..., \zeta_N) \in \mathbb{R}^N$ we define the reduced form information structure $\hat{\mathcal{I}}$ by the information structure in which agent i observes *only* ζ_i.

In game $(P, \mathcal{I})$, a strategy profile for agent i is defined by a function $\alpha_i : \mathbb{R}^J \to A_i$. In game $(P, \hat{\mathcal{I}})$ a strategy for player i is a functions $\hat{\alpha}_i : \mathbb{R} \to A_i$. We denote by $(\boldsymbol{\alpha}(\mathbf{s}))$ the strategy profile given by:

$$(\boldsymbol{\alpha}(\mathbf{s})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_N(\mathbf{s}_N)).$$

We denote by $(a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i}))$ the strategy profile in which all agents play according to $(\boldsymbol{\alpha}(\mathbf{s}))$ except for player i, and player i takes action a_i' for all realizations of the signals he observes. That is,

$$(a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_{i-1}(\mathbf{s}_{i-1}), a_i', \alpha_{i+1}(\mathbf{s}_{i+1}), ..., \alpha_N(\mathbf{s}_N)).$$

6.2 Solution Concepts

We now provide three different equilibrium concepts. All these equilibrium concepts are standard in the literature, nevertheless we explicitly provide the definitions to make the comparisons more transparent. We first define a Nash equilibrium in the game.

Definition 4 (Nash equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ *forms a Nash equilibrium if for all players* $i \in N$*, for all signals realizations* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$*, and for all actions* $a_i' \in A_i$*:*

$$\mathbb{E}[u_i(\theta_i, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_i] \geq \mathbb{E}[u_i(\theta_i, (a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})))|\mathbf{s}_i]. \tag{30}$$

A Nash equilibrium is a strategy profile such that the action of each player is optimal taking as given the strategy by other players. We define an ex-post equilibrium.

Definition 5 (Ex-post Equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ *forms an ex-post equilibrium if for all players* $i \in N$*, for all signals realizations* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$*, and for all actions* $a_i' \in A_i$*:*

$$\mathbb{E}[u_i(\theta_i, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_1, ..., \mathbf{s}_N] \geq \mathbb{E}[u_i(\theta_i, (a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})))|\mathbf{s}_1, ..., \mathbf{s}_N]. \tag{31}$$

In an ex-post equilibrium, agent i's action is optimal even if he knew the realization of the signals of all other agents. In contrast to a Nash equilibrium, the information set with respect to which the action needs to be optimal is augmented. It is transparent to see that, if a strategy profile is an ex-post equilibrium, then it is also a Nash equilibrium.

Finally, we define a posterior equilibrium.

Definition 6 (Posterior Equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ *forms a posterior equilibrium if for all players* $i \in N$*, for all signals realizations* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$*, and for all actions* $a_i' \in A_i$*:*

$$\mathbb{E}[u_i(\theta_i, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_i, \boldsymbol{\alpha}(\mathbf{s})] \geq \mathbb{E}[u_i(\theta_i, (a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})))|\mathbf{s}_i, \boldsymbol{\alpha}(\mathbf{s})]. \tag{32}$$

In a posterior equilibrium, the strategy of agent i remains optimal even if he knew the actions taken by all other agents.[26] The difference between posterior equilibria and ex-post equilibria is

[26] The definition of posterior equilibria is due to Green and Laffont (1987).

the amount of information with respect to which a strategies is optimal. That is, the difference lies on the conditioning variables in (31) and (32). The action taken by player i is less informative than the signal agent i observes. Hence, if an equilibrium is an ex-post equilibrium, then it is also a posterior equilibrium. If a strategy profile is a posterior equilibrium, then the strategy profile is also a Nash equilibrium.

6.3 General Characterization of Equilibria

We now show how to compute posterior equilibria in game $(P, \mathcal{I})$. We do this by providing an equivalence between ex-post equilibria in game $(P, \hat{\mathcal{I}})$ and posterior equilibria in game $(P, \mathcal{I})$.

Theorem 4 (Equivalence)**.**
If $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ *is an equilibrium statistic and strategies profile* $\{\hat{\alpha}_i\}_{i\in N}$ *is an ex-post equilibrium in game* $(P, \hat{\mathcal{I}})$*, then the following strategy profile* $\{\alpha_i\}_{i\in N}$ *is a posterior equilibrium in game* $(P, \mathcal{I})$ *:*

$$\alpha_i(\mathbf{s}_i) = \hat{\alpha}_i(\boldsymbol{\beta}_i \cdot \mathbf{s}_i). \tag{33}$$

Proposition 4 shows that equilibria can be computed using a two step procedure. The first step is to find the one-dimensional equilibrium statistic using (25). The second step is to compute a posterior equilibrium as if agents observed only the equilibrium statistic. If a mechanism has an ex post equilibrium when agents observe one-dimensional signals, then in general this equilibrium is easy to characterize.

6.4 Mechanisms that have an Ex-Post Equilibrium

There is a large class of trading mechanisms that have an ex-post equilibrium when agents observe one-dimensional signals. These mechanisms include classic trading mechanisms, as well as mechanisms proposed by recent papers. In most of the recent literature the property of having an ex-post equilibria when agents observe one-dimensional signals is viewed as a desirable property. We briefly provide an overview of some of the mechanisms that have an ex-post equilibria when agents observe one-dimensional signals.

There are classic trading mechanisms that have an ex-post equilibrium when agents observe one-dimensional signals. For example, multi-unit ascending auctions (see for example, Ausubel (2004) or Perry and Reny (2005)) and generalized VCG mechanism (see for example, Dasgupta and Maskin (2000)). Additionally, supply function competition has an ex-post equilibria when agents are symmetric (see for example Klemperer and Meyer (1989) or Vives (2011)).[27] Many recent papers study novel mechanisms that have an ex-post equilibria when agents observe one-dimensional signals. Ausubel, Crampton, and Milgrom (2006) propose the Combinatorial Clock Auction that is meant to auction many related items. Sannikov and Skrzypacz (2014) study a variation of supply function equilibria in which each agents can condition on the quantity bought by other agents. Kojima and Yamashita (2014) study a variation of a double auction that improves upon the standard double auction along several dimensions. Finally, Hashimoto (2016) proposes a generalized random priority mechanism with budgets. All the mechanisms previously mentioned have an ex-post equilibria when agents observe one-dimensional signals. Hence, our analysis can be applied to understand what happens when agents observe multidimensional signals.

It is worth mentioning some mechanisms that do not have an ex post equilibria when agents observe one-dimensional signals. Two classic examples are Cournot competition and first price auction. In Cournot competition an agent tries to anticipate the quantity submitted by other agents, as these quantities will ultimately determine the equilibrium price. In a first price auction agents try to anticipate the bid of other agents. It is interesting to compare this with supply function equilibria and English auction. In supply function equilibria an agent can condition the quantity they buy on the equilibrium price, and hence he does not need to anticipate the demands submitted by other agents. Nevertheless, agents learn from the equilibrium price. Analogously, in an English auction an agent can condition on the drop-out time of other agents, and hence an agent does not need to anticipate the bids of other agents.

Lambert, Ostrovsky, and Panov (2014) study a static version of a Kyle (1985) trading model. In their paper they allow agents to observer arbitrary multidimensional signals. Although we share the common motivation of understanding trading mechanisms in informationally rich environments, both papers are methodologically different. A static version of a Kyle (1985) trading model does not have an ex-post equilibrium when agents observe one-dimensional signals, and hence the methodology developed in this paper is not useful to study a trading model as in Kyle (1985). For the same reason, the methodology developed in Lambert, Ostrovsky, and Panov (2014) is not useful to study the models we study in this paper.

Finally, we highlight that the posterior equilibria characterized in Theorem 4 are *not* an ex-post equilibrium.It is not difficult to check that in the model studied in Section 4 there are

[27] In symmetric environments the price aggregates all relevant information, and hence equilibria is privately revealing (see Vives (2011)). See Rostek and Weretka (2012) for a model with asymmetric agents in which there is no ex-post equilibria even when agents observe one-dimensional signals.

realization of signals in which agent i wins the object, but agent i pays a price that is higher than the expected valuation of agent i conditional on all signals. Yet, we note that the impossibility theorem in Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) does not apply to our model because we study an environment in which signals enter linearly in the expectation of a payoff shock. That is, this is a separable type-space.[28] The impossibility result in Jehiel and Moldovanu (2001) does apply to our model, and hence it is impossible to implement efficient outcomes. The sources of inefficiencies in an ascending auction have been explained in Section 4.

6.5 Spectrum Auctions and Multiplicity of Equilibria

One of the implications of Theorem 4 is that the multiplicity of equilibria explained in Section 4.3 also applies to a class of ascending auctions widely used in the auction of spectrum licenses. A common feature of these auctions is that the outcome of the auction varies widely from market to market, even when these markets are very similar. We discuss how our results provide a new mechanism that may help explain these difference in the outcome of these auctions. We begin by briefly discussing the evidence.

In the early 1990's the US auctioned radio spectrum frequencies for services such as mobile phones. The auction mechanisms was a simultaneous English auction. It is well established that the market outcomes varied greatly across different markets. For example, this was reported by *The Economist*:

> " In 14 different auctions since 1994, the FCC has attracted winning bids worth $23 billion. But suddenly it has all gone wrong. An auction last month of frequencies suitable for wireless data transmission, which was expected to raise $1.8 billion, produced only $13.6m." *The Economist*, May 17, 1997, p. 86.

An explanation for these large differences is that bidders' successfully colluded in some of these auctions (see for example Cramton and Schwartz (2002) or *The Economist* article previously cited).

In the early 2000's several European countries auctioned 3G mobile telecommunication licenses. As in the american auction of radio frequencies, the outcome greatly varied. Consider the following stark differences:

[28]See Jehiel, Meyer-ter Vehn, and Moldovanu (2008) for more results on implementation of ex post equilibrium in seprable environments.

> "There were enormous differences in the revenues from the European "third generation" (3G, or "UMTS") mobile-phone license auctions, from 20 Euros per capita in Switzerland to 650 Euros per capita in the UK, though the values of the licenses sold were similar. " Klemperer (2002)

An explanation for the large differences has been attributed to differences in the number of competitors (see Klemperer (2002)). Although there was a lack of competition in some of the auctions, some of the most successful auctions (from the seller's perspective) translated into large losses for the winners of these auctions (see *The Economist*, September 2, 2004). We can see that there was a large uncertainty about the value of these licenses among the bidders.

In recent years there has been a surge in using Combinatorial Clock Auction (see Ausubel, Crampton, and Milgrom (2006)). Levin and Skrzypacz (2014) shows that this auction format has an efficient ex-post equilibrium, albeit it also has other ex-post equilibria. As in the previous examples, the auctions that used this format also varied widely in the outcome:

> " Either bidding data or summary reports are publicly available for several CCA sales of radio spectrum licenses. This evidence suggests a striking degree of heterogeneity across bidders and across auctions." Levin and Skrzypacz (2014)

An explanation for these differences proposed Levin and Skrzypacz (2014) comes from the multiple ex-post equilibria when agents observe one-dimensional signals.

Of course, the auctions previously discussed had several differences among each others, and each of the previous explanations was probably an important factor. Nevertheless, our analysis shows that these class of mechanisms may naturally lead to multiple equilibria, regardless of the details of the mechanism. This provides an additional mechanisms by which ascending auctions in similar markets may yield very different outcomes. This may reinforce mechanisms previously discussed. It is precisely the fact that agents learn from each other's bidding that induces multiple equilibria. The multiplicity of equilibria shows that small differences in the information structure can have large impact on the outcome.

7 On the use of Gaussian Signals

The previous sections provide an analysis of an English auction, and other mechanisms, when agents observe multidimensional Gaussian signals. We have not being able to extend the analysis to general non-Gaussian signals. In our view, the novel predictions that arise with multidimensional signals are "unlikely to disappear" if one considers non-Gaussian multidimensional signals. Nevertheless, the existence of equilibria in environments with non-Gaussian multidimensional signals is of independent interest. This is specially the case considering that Jackson (2009) provide an example of a non-Gaussian multidimensional environment in which there is no equilibria.

In this section we discuss to what extent the intuitions in our analysis should extend to non-Gaussian signals. We argue that, with signals that have finite support, there is a natural difficulty in constructing equilibria in an ascending auction. This difficulty does not arise with signals that have continuous support. Hence, the non-existence result in Jackson (2009) is a result of the finite-support information structure. Additionally, we show that the analysis of Wilson (1998) with diffuse priors can be extended to non-Gaussian information structures. Hence, a priori there is reasonable hope that our analysis can be extended to non-Gaussian signals.

Finally, we highlight that the use of Gaussian signals in our model is no different than the use of Gaussian signals in the rational expectations equilibrium (see Grossman and Stiglitz (1980) or Hellwig (1980) for seminal contributions). The difficulty encountered in this literature in extending their models to allow for non-Gaussian signals suggests that extending our analysis to non-Gaussian signals is a non-trivial task. A noteworthy exception is Breon-Drish (2015) that extends models of trading with noise traders to distributions in the exponential family. This also suggests that it may be possible to extend our analysis to non-Gaussian signals.[29] Throughout the discussion we highlight additional connections that arise between our model and the literature on rational expectations equilibrium.

7.1 Finite Support Information Structures

We begin by discussing signals that have finite support. In particular, we study the example in Jackson (2009). By understanding the example in more detail it is possible to gain further intuitions on where the construction of equilibria fails, and why this may be determined by the finite support of the information structure.

We provide the example in Jackson (2009). We keep all the assumptions as in Section 4, but random variables are not Gaussian.[30] The payoff shock continues to be decomposed as a taste shock and a common shock (as in (15)). The taste show η_i and common shock φ are uniformly

[29]We hope to use the insights from this paper to extend our analysis to non-Gaussian signals. Nevertheless, the multidimensional signals impose an additional difficulty that makes the application of the techniques therein non-straightforward.

[30]Jackson (2009) assumes that the utility function is linear in θ_i. Taking the exponential of θ_i (as in (1)) is obviously inconsequential.

distributed with finite support:

$$\eta_i \in \{0, \kappa, 2\kappa, ..., 1\} \ ; \ \varphi \in \{0, \frac{1}{2}\}. \tag{34}$$

We assume that $1/\kappa$ is not divisible by 4. Equivalently, for all $x \in \mathbb{N}$, $x \cdot \kappa \neq \frac{1}{4}$. Agents observe two signals as follows:

$$s_i = \eta_i \ ; \ s_2 = |\varphi - \varepsilon_i|, \tag{35}$$

where ε_i takes the value 1/4 with probability m and with probability $(1-m)$ takes the value 0. This implies that agent i can infer φ by observing s_{i2} with probability $(1-m)$, and with probability m signal s_{i2} is non-informative. The shocks ε_i are independently distributed across agents.

Lemma 8 (Jackson (2009))**.**
There exists κ and m small enough such that, if agents' information structure is described by (34) *and* (35)*, the ascending auction does not have an equilibrium*

Lemma 8 is a result from Jackson (2009). The result shows that the ascending auction does not have an equilibrium in an environment that combines a classic model of common and private values.

We show that the information structure described by (34) and (35) does have an equilibrium statistic. Consider the following one-dimensional statistic:

$$\zeta_i = s_1 + s_2 \tag{36}$$

We show that (36) is an equilibrium statistic.

Proposition 5 (Existence of Equilibrium Statistic)**.**
If agents information structure described by (34) *and* (35) *then* (36) *is an equilibrium statistic. Moreover, the equilibrium statistic satisfies:*

$$\mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N] = \mathbb{E}[\theta_i|\mathbf{s}_1, ..., \mathbf{s}_N].$$

Proposition 36 shows that an equilibrium statistic exists. Moreover, the equilibrium statistics aggregate the information in all the signals of of all agents. This is because agent i by observing ζ_j can infer perfectly the realization of the signal s_{i2}. Hence, the information is not "muddled" as in the equilibrium statistic with Gaussian structures. Nevertheless, this is purely an artifact of the finite support. The fact that the equilibrium statistic aggregates all the information also suggests that the equilibrium statistic might not satisfy the single crossing property. Otherwise, the implemented allocation by the English auction would be efficient, and this would be a contradiction with Jehiel and Moldovanu (2001).

Finally, note that the difference between finite support signals and continuous signals has also appeared in the rational expectations equilibrium. Radner (1979) shows that in an economy with a finite number of states, a rational expectations equilibrium is fully revealing. That is, the price reveals all the information agents have. Nevertheless, as in an English auction, this is an artifact of the signals with finite support. This suggests that the English auction will not have an equilibrium for any generic multidimensional information structure that has finite support.

7.2 Non-Gaussian Signals with Diffuse Priors

We keep all the assumptions as in Section 4, but random variables are not Gaussian. The payoff shock continues to be decomposed as a taste shock and a common shock (as in (15)). Agents observe their taste shock and they observe a noisy signal about the shock $\bar{\varphi}$ (as in (16)). We assume that the utility of agent i if he wins the object is equal to θ_i (without taking the exponential as in (1)).

We assume the following distributions over random variables. We assume that $\bar{\varphi}$ and $\bar{\eta}$ are independent of each other, and each is drawn according to a diffuse prior. We specify later how we compute the expectations under the diffuse prior, but for now it is enough for the reader to think of these random variables as being drawn from a Gaussian distribution $\mathcal{N}(0, \sigma^2)$ under the limit $\sigma \to \infty$. The random variables ε_i are independent of all other random variables in the model and independent across agents, but drawn according to any arbitrary distribution that has finite variance. The random variables $\Delta\eta_i$ are drawn according to any arbitrary distribution that has finite variance, but they satisfy that $\sum_{i \in N} \Delta\eta_i = 0$.[31]

The intuition of the model is as follows. Conditional on observing s_{i1}, agents know η_i. Nevertheless, they do not know the realization of $\bar{\eta}$ and $\Delta\eta_i$ separately. Hence, agent i's beliefs is that the distribution of η_j is centered around s_{i1}, with a distribution equal to the distribution of $\Delta\eta_j$. Similarly, after observing s_{i2}, agent i's beliefs is that $\bar{\varphi}$ is centered around s_{i2} with a distribution equal to the distribution of $\Delta\varepsilon_i$.

[31] It is irrelevant whether the shocks $\{\Delta\eta_i\}_{i \in N}$ add up to 0, as it is always possible to redefine $\bar{\eta}$. We make this assumption just to be consistent with the notation.

Expectations. Since $\bar{\varphi}$ and $\bar{\eta}$ are drawn according to a diffuse prior, we specify how we compute the expectations. We begin by assuming that the expectation under diffuse priors has the following properties. Consider random variables $(z_1, .., z_J)$ that can be written as follows:

$$z_j = \bar{\varphi} + \sum_{i \in N} c_{\varepsilon ij} \Delta\varepsilon_i + \sum_{i \in N} c_{\eta ij} \Delta\eta_i,$$

where $\{c_{\epsilon i}, c_{\eta i}\}$ are constant. We assume the following:

$$\mathbb{E}[\bar{\varphi}|z_1, .., z_J, \eta_i] = \mathbb{E}[\bar{\varphi}|z_1, .., z_J]; \tag{37}$$

$$\mathbb{E}[\bar{\varphi}|z_1, .., z_J] + K = \mathbb{E}[\bar{\varphi}|z_1 + K, .., z_J + K]. \tag{38}$$

The interpretation of these conditions is as follows. (37) states that, conditional on $(z_1, .., z_J)$, the signal η_i has no information about $\bar{\varphi}$. This is because $(z_1, .., z_J)$ are independent of $\bar{\eta}$. Since $\bar{\eta}$ is drawn according to a diffuse prior, η_i contains on information about $\bar{\varphi}$ or $(z_1, .., z_J)$. Hence, η_i is not used in the expectation. (38) states that the expectation is translation invariant. Since $\bar{\varphi}$ is drawn according to a diffuse prior, the expectation of $\bar{\varphi}$ conditional on $(z_1, .., z_J)$, does not depend on the levels.

From (38), it is direct that the expectation can be computed as follows. Let $\mathcal{C}$ be the set of all functions $m : \mathbb{R}^N \to \mathbb{R}$ that are translation invariant (as in (38))) and such that $m(0, ..., 0) = 0$. Then, $\mathbb{E}[\bar{\varphi}|z_1, .., z_J]$ can be found solving the following minimization problem:

$$\mathbb{E}[\bar{\varphi}|z_1, .., z_J] \in \arg\min_{m \in \mathcal{C}} var(m(z_1 - \bar{\varphi}, ..., z_N - \bar{\varphi})).$$

That is, $\mathbb{E}[\bar{\varphi}|z_1, .., z_J]$ is found by minimizing the variance of the errors in the prediction.

Equilibrium Statistic. We now prove that the information structure has an equilibrium statistic.

Proposition 6 (Equilibrium Statistic)**.**
The information structure has an equilibrium statistic $\zeta_i = s_{i1} + s_{i2}$. Moreover, the equilibrium statistic satisfies:

$$\mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N] = \mathbb{E}[\bar{\varphi}_i|\zeta_1, ..., \zeta_N]. \tag{39}$$

Proposition 6 shows that an equilibrium statistic exists. Moreover, agent i's expectation of his payoff shock θ_i conditional on all the equilibrium statistics is as if agents had common values.

This is because the expectation of θ_i conditional on the equilibrium statistic of all agents is the same as the expectation of $\bar{\varphi}$ conditional on the equilibrium statistic of all agents.

The intuition is the same as in Section 4.3. A positive shock η_i has two effects. First, agent i is more optimistic about his payoff shock θ_i because his taste shock is high. Second, agent i becomes more pessimistic about $\bar{\varphi}$. This is because agent i learns $s_{i2} + \eta_j$ from the drop-out of agent j. Hence, if η_i is high, this means η_j is high, which means that s_{i2} must be low. Interestingly, with diffuse priors these two effects offset each other. Hence, in equilibrium agents bid as if they had common values.

Finally, we show that an equilibrium exists.

Corollary 5 (Existence of Equilibrium)**.**
The ascending auction has a Nash equilibrium. In equilibrium agents bid as if they observed only their equilibrium statistic ζ_i.

Corollary 5 is a straightforward conclusion of Proposition 6. First, and equilibrium statistic exists. Second, agents bid as if they had common values. Hence, the single crossing property is also satisfied. Hence, we can directly apply Milgrom and Weber (1982). The result shows that it is possible to aspire to characterize the set of equilibria with multidimensional signals when signals are not Gaussian. Nevertheless, the analysis here relies on the diffuse priors, and generalizing this is material for future work.

7.3 Higher Order Beliefs

There is an additional difficulty when one works with utility functions that are not linear θ_i, which is worth explaining. In the definition of an equilibrium statistic (Definition 3) we only make reference to an agent's expectation of his own payoff shock. Nevertheless, if the utility of agent i is not linear in θ_i, then it is necessary to keep track of the higher order beliefs of agent i about θ_i. In a Gaussian environment an agent's expectation of his own payoff shock completely determines the complete conditional distribution of the payoff shock. Nevertheless, this is not true for non-Gaussian environments.

Consider the following example. There is a single agent that solves the decision problem:

$$x^* \in \arg\max_{x \in \mathbb{R}} \mathbb{E}[-exp(-x \cdot \theta + \frac{\kappa}{2}x^2)|s_1, s_2],$$

where κ is a constant. We assume θ is normally distributed with 0 mean and a variance of 1.

The signals are as follows:

$$s_1 = \theta + \varepsilon \text{ and } s_2 = var(\varepsilon),$$

where ε is a noise term. That is, the second signal informs the agent about the variance of the noise term in the first signal. Using standard formulas for updating with Gaussian random variables, it is possible to check that:

$$x^* = \frac{2}{(1+s_2)\kappa + s_2} s_1.$$

Hence, x^* is a function of s_1 and s_2.

The important thing to note is that x^* depends on s_1 and s_2, but the way these two signals are used depends on κ. That is, by construction the decision problem requires s_1 and s_2 to be projected into a one-dimensional object. Nevertheless, this projection depends on the parameter of the game κ, and not only on the information structure. Hence, we can see that the construction in Definition 3 will not work in non-Gaussian environments as agents also need to keep track of their higher order beliefs about θ_i— in this example, the beliefs about the variance. In general, the way the signals are projected into a one-dimensional object does depend on the payoff structure of the game. Of course, an exception to this is the case in which the utility function of agents u_i is linear in θ_i. If $u_i(\theta_i, \mathbf{a}) = \theta_i \cdot v(\mathbf{a})$ for some $v(\cdot)$, then the decision of an agent only depends on his expectation of θ_i.

7.4 Non-Gaussian Equilibria with Gaussian Signals

Finally, it is worth discussing the possibility of non-Gaussian equilibria in environments with Gaussian information structures. Since there may be multiple equilibria that preserve the Gaussian beliefs, it is also possible to construct equilibria in which agents randomize between these equilibria. For example, by providing a public signal that is only noise, agents can use this as a coordination device to randomize between equilibria. These equilibria driven by a public signal would not preserve the Gaussian beliefs, and hence cannot be characterized by our methodology. This is the simplest example of a non-Gaussian equilibrium. Yet, these class of equilibria do not provide any additional economic insight. We do not know if there are additional equilibria that can provide meaningful new economic insights in environments.

8 Conclusions

In this paper we provide new predictions of ascending auctions. Nevertheless, there are several other potential applications for the same methodology. We discuss some of the directions open for future work.

The results in the paper can be naturally extended to other information structures. Hence, there are plenty of open questions in terms of how the information impacts the outcome of an ascending auction. For example, in Section 5.5 we studied the impact of differential information on the equilibrium outcome. Although this was a illustrative example, it is possible to analyze the problem analytically and provide general guidelines to understand how information impacts the outcome. Of course, there are other natural source of asymmetries to studies, for example the case in which payoff shocks are asymmetrically distributed.

In the companion paper Heumann (2016), we use this methodology to study the limits of information aggregation in large markets. In particular, we study a continuum of traders trading a divisible asset via supply function competition. We study to what extent there is a limit to the amount of information that can be aggregate by prices. As agents observe multidimensional signals, but the supply function an agent submits is measurable with respect to a one-dimensional equilibrium statistic, there is a natural limit to the amount of information that can be aggregated by the price. The key aspect of the analysis is to understand the efficiency of the information revealed by the equilibrium statistic. We study whether the use of information by agents is optimal, and how can taxes increase or decrease the amount of information revealed by prices in equilibrium.

We believe it is also natural to apply our methodology to study other mechanisms. As we have explained in Section 6.3, the same methodology can be applied to many other trading mechanisms. Although the equilibrium statistic does not change, the characterization for one-dimensional signals does change. Since impact of the distribution of a one-dimensional information structure is different for different mechanisms, the impact of multidimensional signals on these mechanisms will be different than in an ascending auction.

9 Appendix: Proofs of Results in Main Text

Proof Theorem 1 First, note that for any pair of jointly normal random variables (x, y):

$$\mathbb{E}[exp(x)|y] = exp(\mathbb{E}[x|y] + \frac{1}{2}var(x|y)). \tag{40}$$

It is easy to check that:

$$var(\theta_i|s_1, s_2, ..., s_N) = (1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}).$$

Hence,

$$\begin{aligned} \mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] \quad = \quad & exp\Bigg(\mathbb{E}[\theta_2|s_2, s_2, ..., s_N] & (41) \\ & +\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg) & (42) \end{aligned}$$

It is important to note that, for any $i \in \{1, ..., N\}$ the errors of the prediction $\mathbb{E}[\theta_i|s_1, ..., s_N]$ are Gaussian. That is, $\theta_i - \mathbb{E}[\theta_i|s_1, ..., s_N]$ is a Gaussian random variables. Hence, we can use (40) to compute (41). Nevertheless, $\mathbb{E}[\theta_2|s_1, ..., s_N]$ is not a Gaussian random variable as this is the second maximum over N random variables. Hence, to compute $\mathbb{E}[\mathbb{E}[exp(\theta_2)|s_1, s_2, ..., s_N]]$ we cannot use (40).

Rewriting (3) explicitly for $i = 1$:

$$\mathbb{E}[\theta_2|s_1, s_2, ..., s_N] = \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_2 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s}.$$

Now, replacing s_1 by s_2 in the previous expression, we get:

$$\begin{aligned}
\mathbb{E}[\theta_2|s_2, s_2, ..., s_N] &= \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s} \\
&\quad +(\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)} - \frac{1}{N}\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)} + \frac{1}{N}\frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})})(s_2 - s_1) \\
&= \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s} \\
&\quad +(1 - \frac{1}{N} + \frac{1}{N}m)\Bigg((\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_2 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s}) \\
&\quad -(\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s})\Bigg) \\
&= \mathbb{E}[\theta_1|s_1, ..., s_N] + (1 - \frac{1}{N} + \frac{1}{N}m)(\mathbb{E}[\theta_2|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_1|s_1, s_2, ..., s_N]) \\
&= \mathbb{E}[\theta_1|s_1, ..., s_N] + (\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N]).
\end{aligned}$$

Replacing into (41), we get:

$$\begin{aligned}
\mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \\
&\quad +(\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N]) \\
&\quad +\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg).
\end{aligned}$$

Note that:

$$\begin{aligned}
S(s_1, ..., s_N) = \mathbb{E}[exp(\theta_1)|s_1, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \\
&\quad +\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg)
\end{aligned}$$

Hence,

$$\mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] = S(s_1, ..., s_N) \times exp\Bigg((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N])\Bigg).$$

Similarly, we can compete the buyers' rents:

$$\begin{aligned} V(s_1,...,s_N) &= S(s_1,...,s_N) - \mathbb{E}[exp(\theta_2)|s_2,s_2,...,s_N] \\ &= S(s_1,...,s_N) \times \left(1 - exp\left((\frac{1-m}{N}-1)(\mathbb{E}[\theta_1|s_1,s_2,...,s_N] - \mathbb{E}[\theta_2|s_1,s_2,...,s_N])\right)\right). \end{aligned}$$

Hence, we prove the result. ■

Proof Theorem 2 This is a direct corollary of Theorem 4 and Lemma 3 (in the proof of Lemma 3 we show that the equilibrium of the ascending auction is an ex-post equilibrium). ■

Proof Theorem 3 This is a direct corollary of Theorem 4 and Lemma 3 (in the proof of Lemma 7 we show that the equilibrium of the ascending auction is an ex-post equilibrium). ■

Proof Theorem 4. By the construction of the equilibrium statistic it is clear that for any equilibrium statistic, the joint distribution of the random variables $(\theta_1,...,\theta_N,\mathbf{s}_1,...,\mathbf{s}_N,\zeta_1,...,\zeta_N)$ are jointly normally distributed. We first provide the main steps of the proof and then explain each step in detail. If $\hat{\alpha}_i:\mathbb{R}\to A_i$ is an ex-post equilibrium of game $(P,\hat{\mathcal{I}})$, then:

$$\Rightarrow \quad \forall i\in N, \forall \boldsymbol{\zeta}\in\mathbb{R}^N,\ \forall a_i'\in A_i, \quad \mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\boldsymbol{\zeta}] \geq \mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\boldsymbol{\zeta}] \tag{43}$$

$$\Rightarrow \quad \forall i\in N, \forall \boldsymbol{\zeta}\in\mathbb{R}^N,\ \forall \mathbf{s}_i\in\mathbb{R}^J,\ \forall a_i'\in A_i, \quad \mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\boldsymbol{\zeta},\mathbf{s}_i] \geq \mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\boldsymbol{\zeta},\mathbf{s}_i] \tag{44}$$

$$\begin{aligned} \Rightarrow \quad \forall i\in N, \forall \boldsymbol{\zeta}\in\mathbb{R}^N,\ \forall \mathbf{s}_i\in\mathbb{R}^J,\ \forall a_i'\in A_i, \quad & \mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)] \\ & \geq \mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)] \end{aligned} \tag{45}$$

$$\begin{aligned} \Rightarrow \quad \forall i\in N, \forall (\mathbf{s}_1,...,\mathbf{s}_N)\in\mathbb{R}^{J\cdot N},\ \forall a_i'\in A_i, \quad & \mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\boldsymbol{\beta}_1\cdot\mathbf{s}_1),...,\hat{\alpha}_N(\boldsymbol{\beta}_N\cdot\mathbf{s}_N)] \\ & \geq \mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\boldsymbol{\beta}_1\cdot\mathbf{s}_1),...,\hat{\alpha}_N(\boldsymbol{\beta}_N\cdot\mathbf{s}_N)] \end{aligned} \tag{46}$$

$$\Rightarrow \alpha^*:\mathbb{R}^J\to M \text{ defined by } \alpha^*(\mathbf{s_i}) = \hat{\alpha}(\zeta_i) = \hat{\alpha}(\boldsymbol{\beta}_\mathbf{i}\cdot\mathbf{s}_i) \text{ is a posterior equilibrium of game } G \tag{47}$$

Step (43) This is by definition of ex-post equilibria in game $(P,\hat{\mathcal{I}})$.

Step (44) First, note that the expectations are over random variable θ_i. Hence, we need to prove that:

$$\forall\zeta\in\mathbb{R}^N,\ \forall\mathbf{s}_i\in\mathbb{R}^J,\ \theta_i|_{\boldsymbol{\zeta}} = \theta_i|_{\boldsymbol{\zeta},\mathbf{s}_i}.$$

That is, the distribution of θ_i conditional on $\boldsymbol{\zeta}$ is the same same as the conditional distribution of θ_i conditional on $\boldsymbol{\zeta}$ and $\mathbf{s}_i$. As the random variables are normally distributed, it suffices to prove that:

$$\forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J,\ \mathbb{E}[\theta_i|\boldsymbol{\zeta}] = \mathbb{E}[\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i]; \tag{48}$$

$$\forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J, var(\theta_i|\boldsymbol{\zeta}){=}var(\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i) \tag{49}$$

(48) is true by the definition of an equilibrium statistic. (49) is true because the variables are jointly Gaussian and hence:

$$var(\theta_i|\boldsymbol{\zeta}) = var(\theta_i) - var(\mathbb{E}[\theta_i|\boldsymbol{\zeta}]){=}var(\theta_i) - var(\mathbb{E}[\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i]) = var(\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i)$$

Step (45) Note that $\hat{\alpha}_i(\zeta_i)$ is measurable with respect to ζ_i. Hence,

$$\mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}, \mathbf{s}_i] = \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \boldsymbol{\zeta}, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)]; \tag{50}$$

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}, \mathbf{s}_i] = \mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \boldsymbol{\zeta}, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)]. \tag{51}$$

That is, we can add $\hat{\alpha}_i(\zeta_i)$ as conditioning variable. Hence, we can write (44) as follows:

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \boldsymbol{\zeta}, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)] \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \boldsymbol{\zeta}, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)].$$

Taking expectation of the previous equation conditional on $(\mathbf{s}_i, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N))$ and using the law of iterated expectations:

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)] \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)].$$

Hence, we prove the step.
Step (46) This is using that $\boldsymbol{\beta}_i \cdot \mathbf{s}_i = \zeta_i$, hence the inequality obviously holds.
Step (47) Is just by the definition of posterior equilibria.
Hence, we prove the result.$\blacksquare$
Proof Proposition 1 We can write the expected surplus as follows:

$$\mathbb{E}[S(s_1, ..., s_N)] = \mathbb{E}[\mathbb{E}[\exp(\theta_1)|s_1, ..., s_N]] = \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1 + \bar{\theta})|\bar{s}, \Delta s_1, ..., \Delta s_N]].$$

Since the common component of the variables are independent of the idiosyncratic component of the random variables, we have:

$$\mathbb{E}[S(s_1, ..., s_N)] = \mathbb{E}[\mathbb{E}[\exp(\bar{\theta})|\bar{s}]]\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]].$$

Using the law of iterated expectations:

$$\mathbb{E}[\mathbb{E}[\exp(\bar{\theta})|\bar{s}]] = exp(\frac{1}{2}var(\bar{\theta})).$$

Hence,

$$\mathbb{E}[S(s_1, ..., s_N)] = exp(\frac{1}{2}var(\bar{\theta})) \times \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]].$$

Clearly the $\mathbb{E}[S(s_1, ..., s_N)]$ does not depend on m or $corr(\bar{s}, \bar{\theta})$. We need to prove that (53) is increasing in $corr(\Delta s_i, \Delta\theta_i)$.

We now use a coupling argument to prove that, for all s'_i, s_i such that $corr(\Delta s'_i, \Delta\theta_i) > corr(\Delta s_i, \Delta\theta_i)$, then:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s'_1, ..., \Delta s'_N]] > \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]].$$

Since $corr(\Delta s'_i, \Delta\theta_i) > corr(\Delta s_i, \Delta\theta_i)$, we assume that s'_i is strictly more informative than s_i in a Blackwell sense. That is, we assume that s_i can be written as follows:

$$s_i = s'_i + \varepsilon_{i2}, \tag{52}$$

where ε_{i2} is a noise term independent of $\Delta\theta_i$ and s'_i. Of course, for two arbitrary signals s'_i, s_i (52) might not be satisfied. Nevertheless, this does not matter for the argument because to compute:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s'_1, ..., \Delta s'_N]]$$

what matters is the joint distribution of $(\Delta s'_i, \Delta\theta_i)$. Hence, if we prove it for signal s'_i that satisfies (52) we will have proven it for all signals.

For each $i \in N$, we define random variables:

$$\Delta\varphi_i \triangleq \mathbb{E}[\Delta\theta_i|\Delta s_1, ..., \Delta s_N] \text{ and } \Delta e_i \triangleq \Delta\theta_i - \Delta\varphi.$$

Clearly, $\Delta\varphi_i$ is independent of Δe_i. On the other hand, we can write:

$$\begin{aligned}\mathbb{E}[\Delta\theta_i|\Delta s_i'] &= \mathbb{E}[\Delta\theta_i|\Delta s_i', \Delta s_i] \\ &= \mathbb{E}[\Delta\theta_i|\Delta s_i] + \mathbb{E}[\Delta\theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i]|\Delta s_i', \Delta s_i] \\ &= \Delta\varphi_i + \mathbb{E}[\Delta e_i|\Delta s_i', \Delta s_i].\end{aligned}$$

Define,

$$\Delta\varphi_i' \triangleq \mathbb{E}[\Delta\theta_i|\Delta s_i'] \text{ and } \Delta q_i' \triangleq \mathbb{E}[\Delta e_i|\Delta s_i', \Delta s_i] \text{ and } \Delta e_i' \triangleq \Delta\theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i'].$$

Note that:

$$\begin{aligned}\Delta e_i' &= \theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i'] = \theta_i - (\Delta\varphi_i + \Delta q_i') \\ &= = (\theta_i - \Delta\varphi_i) - \Delta q_i' = \Delta e_i - \Delta q_i'\end{aligned}$$

Clearly, $\Delta e_i'$ is independent of $\Delta\varphi_i'$. Additionally, note that:

$$\mathbb{E}[q_i'\Delta\varphi_i] = \mathbb{E}[\mathbb{E}[\Delta e_i|\Delta s_i, \Delta s_i']\Delta\varphi_i]] = \mathbb{E}[\mathbb{E}[\Delta\varphi_i\Delta e_i|\Delta s_i', \Delta s_i]] = \mathbb{E}[\Delta\varphi_i\Delta e_i] = 0.$$

The first equality is by definition of $\Delta q_i'$. The second equality is using that $\Delta\varphi_i$ is measurable with respect to Δs_i. The third equality is using the law of iterated expectations. The fourth equality is just using that the errors from the expectations are uncorrelated with the expectation.Hence, q_i' is independent of $\Delta\varphi_i$.

We denote by $\Delta\hat{e}$ a random variable that has the same variance as Δe_i, but is independent of all random variables. Similarly, for each random variable previously defined, we denote by a hat over the variable a typical variable that has the same distribution but is independent of all other random variables. For example, $\Delta\hat{q}'$ is a random variable that has the same variance as $\Delta q_i'$, but is independent of all random variables.

It is clear to see that:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]] = \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\}+\Delta e_k)]] = \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\}+\Delta\hat{e})]], \tag{53}$$

with k satisfying that $\Delta\varphi_k = \max\{\Delta\varphi_1, ..., \Delta\varphi_N\}$. The first equation is obtained by explicitly writing down the expectation. The second equality comes from the fact that we can replace Δe_k

with $\Delta\hat{e}$ because Δe_k is independent of $\Delta\varphi_k$ (hence, the distribution of the random variables does not change).

We can write an equation analogous to (53) for s'_i as follows:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s'_1, ..., \Delta s'_N]] = \mathbb{E}[\exp(\max\{\Delta\varphi_1 + q'_1, ..., \Delta\varphi_N + q'_N\} + \Delta\hat{e}')]]. \tag{54}$$

Clearly, we have that:

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s'_1, ..., \Delta s'_N]] &= \mathbb{E}[\exp(\max\{\Delta\varphi_1 + \Delta q'_1, ..., \Delta\varphi_N + \Delta q'_N\} + \Delta\hat{e}')]] \\
&< \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\} + \Delta\hat{q}' + \Delta\hat{e}')]] \\
&= \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s'_1, ..., \Delta s'_N]].
\end{aligned}$$

The inequality comes from the fact that the right hand side corresponds to the left hand side, but ignoring the realization of the random variables $\Delta q'_i$, and replacing this with a typical realization $\hat{q}'$. Hence, we prove the result.$\blacksquare$.

Proof Proposition 2 The analysis of the equilibrium with public signals is equivalent to redefining the variance of the common shocks. To formalize the argument define:

$$\bar{\eta}' \triangleq \bar{\eta} - \mathbb{E}[\bar{\eta}|s_3] \text{ and } \bar{\varphi}' \triangleq \bar{\varphi} - \mathbb{E}[\varphi|s_4].$$

Note that the equilibrium analysis is equivalent to a model in which the payoff shock of agents is:

$$\theta'_i = \Delta\eta_i + \bar{\eta}' + \bar{\varphi}',$$

and agents observe signals:

$$s'_{i1} = \Delta\eta_i + \bar{\eta}' \text{ and } s'_{i2} = \bar{\varphi}' + \varepsilon_i.$$

Hence, the model is equivalent to redefining $var(\bar{\eta})$ and $var(\bar{\varphi})$.

Define the polynomial:

$$p(\beta) \triangleq \frac{-1}{var(\Delta\varepsilon_i)} + \beta\frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})} + \beta^2\frac{-1}{var(\Delta\eta_i)} + \beta^3\frac{(var(\Delta\eta_i) + var(\bar{\eta}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\eta_i)var(\bar{\eta})var(\bar{\varphi})}. \tag{55}$$

It is easy to check that, $p(\beta)$ is decreasing $var(\bar{\eta})$ and $var(\bar{\varphi})$. If $p(\beta)$ has a unique root $p(\beta^*) = 0$, then $p(\beta^*)$ is increasing at β^*. Hence, if $p(\beta)$ has a unique root, then this root is increasing in

$var(\bar{\eta})$ and $var(\bar{\varphi})$. This implies that the equilibrium statistic:

$$\zeta_i = s_{i1} + \beta s_{i2},$$

satisfies that $corr(\Delta\zeta_i, \Delta\theta_i)$ is decreasing in $var(\bar{\eta})$ and $var(\bar{\varphi})$. Looking at Proposition 1, this implies that the total surplus $\mathbb{E}[S(\zeta_1, ..., \zeta_N)]$ is decreasing in $var(\bar{\eta})$ and $var(\bar{\varphi})$.

For the limit, note that in the limit $var(\bar{\eta}) \to 0$ or $var(\bar{\varphi}) \to 0$ the polynomial $p(\beta)$ has a unique root $p(\beta^*) = 0$, and $\beta^* \to 0$. Hence, in the limit the equilibrium statistic satisfies $\zeta_i \to \eta_i'$. Hence, in the limit $corr(\Delta\zeta_i, \Delta\theta_i) \to 1$. Hence, in the limit the equilibrium approaches the first best. Hence, we prove the result. ■

Proof Proposition 3 The proof is similar to the proof of Proposition 2. We use all the definitions and arguments therein, and extend them to show the results on profits.

(Part 1: $var(\varepsilon_4) \to 0$**)**. We first prove that:

$$\lim_{var(\varepsilon_4)\to 0} p_2 = \mathbb{E}[exp(\theta_2)|s_1, ..., s_N].$$

As we showed in Proposition 2, in the limit $var(\varepsilon_4) \to 0$ we have that $\beta \to 0$. Hence, agents behave as if they observe only $s_{i1}' = \eta_i'$. Hence, in the limit $var(\varepsilon_4) \to 0$ we have that $m \to 1$. The limit on m is easy to check as in the limit $var(\bar{\eta}') > 0$ and $var(\Delta\eta_i)$ are well defined, and hence the limit of m is well defined. Hence, in the limit agents behave as if they had private values. Hence:

$$\lim_{var(\varepsilon_4)\to 0} p_2 = \max{}^{(2)}\{\mathbb{E}[exp(\theta_1)|\mathbf{s}_1, ..., \mathbf{s}_N], ..., \mathbb{E}[exp(\theta_N)|\mathbf{s}_1, ..., \mathbf{s}_N]\}.$$

(Part 2: $var(\varepsilon_3) \to 0$**).** We now prove that:

$$\lim_{var(\varepsilon_3)\to 0} p_2 = 0.$$

The proof is more subtle than Part 1 because in the limit $var(\varepsilon_3) \to 0$ two things happen simultaneously. First, $\beta \to 0$ and second $var(\bar{\eta}') \to 0$. Hence, when we look at the limit of m we have that $cov(\bar{\zeta}, \bar{\theta}') \to 0$ and $var(\bar{\zeta}') \to 0$. Hence, the limit of m cannot be immediately calculated.

To calculate the limits we calculate the speed at which different terms converge to 0. We say

$x(var(\bar{\eta}'))$ is of order $var(\bar{\eta}')^k$ if:

$$\lim_{var(\bar{\eta}')\to 0} \frac{x(var(\bar{\eta}'))}{var(\bar{\eta}')^{\ell}} = \begin{cases} \infty & \ell > k; \\ 0 & \ell < k. \end{cases}$$

We denote this by $x = O(var(\bar{\eta}')^k)$.

Now, note that, in the limit $var(\bar{\eta}') \to 0$, we must have that any root of $p(\beta)$ (defined in (55)) is of order $var(\bar{\eta}')^{1/3}$. If β is of an order bigger than this, then the polynomial $p(\beta)$ is greater than 0. If β is of an order smaller than $var(\bar{\eta}')^{1/3}$, then $p(\beta)$ is negative.

Hence, the equilibrium statistic in the limit $var(\varepsilon_3) \to 0$ satisfies that $\beta = O(var(\bar{\eta}')^{1/3})$. Hence, in the limit $var(\varepsilon_3) \to 0$,

$$var(\bar{\zeta}) = var(\bar{\eta}' + \beta(\bar{\varphi}' + \bar{\varepsilon})) = var(\bar{\eta}' + O(var(\bar{\eta}')^{1/3})(\bar{\varphi}' + \bar{\varepsilon})) = O(var(\bar{\eta}')^{2/3}).$$

On the other hand,

$$cov(\bar{\zeta}, \bar{\theta}') = cov(\bar{\eta}' + O(var(\bar{\eta}')^{1/3})\bar{\varphi}', \bar{\eta}' + \bar{\varphi}') = O(var(\bar{\eta}')^{1/3}).$$

Hence, in the limit $var(\varepsilon_3) \to 0$:

$$\frac{cov(\bar{\zeta}, \bar{\theta}')}{var(\bar{\zeta})} = O(var(\bar{\eta}')^{-1/3}) \to \infty.$$

This implies that in the limit in the limit $var(\varepsilon_3) \to 0$, $m \to \infty$. Looking at Theorem 1, this implies that in the limit $var(\varepsilon_3) \to 0$:

$$p_2 \to 0.$$

Hence, we prove the result. ■

Proof Proposition 4 The proof has several steps and hence we first provide an overview of the proof. In the first part we define a sequence of functions, such that for each set of vectors $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_{i-1}\mathbf{s}_{i-1}, \boldsymbol{\beta}_{i-1}\mathbf{s}_{i-1}, ..., \boldsymbol{\beta}_N\mathbf{s}_N)$ that agent i might learn, the function assigns the vector $\boldsymbol{\beta}_i$ that corresponds to the expectation, but with a small "punishment" for making the first component small. In the second step we prove that this function has a fixed point. In the third step we prove that the sequence of fixed points of the functions converges to an equilibrium statistic.

(Step 1) We define function $v_i^k : \mathbb{R}^{N(J-1)} \to \mathbb{R}^J$ as follows:

$$v_i^k(\boldsymbol{\beta}_{-i}) \triangleq \arg\min_{\boldsymbol{\beta}_i \in \mathbb{R}^J} \Bigg(\min_{m_{i\ell} \in \mathbb{R}^N} \mathbb{E}[\Big(\theta_i - \sum_{\ell \in N} m_{i\ell} \boldsymbol{\beta}_\ell \mathbf{s}_\ell\Big)^2 | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] \tag{56}$$

$$-\frac{1}{k} \min\{\log(\beta_{i1})), 0\}\Bigg), \text{ subject to: } m_{ii} \in \{-1, 1\} \tag{57}$$

where $\boldsymbol{\beta}_{-i} \triangleq (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_{i-1}, \boldsymbol{\beta}_{i+1}, ..., , \boldsymbol{\beta}_N)$. To interpret v_i^k, note that in the limit $k \to \infty$ this is just the regular expectation. That is, for all $\boldsymbol{\beta}_{-i}$ there exists $m_{i\ell} \in \mathbb{R}^N$ such that:

$$\mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = m_{ii} \cdot v_i^\infty(\boldsymbol{\beta}_{-i}) \cdot \mathbf{s}_i + \sum_{\ell \neq i} m_{i\ell} \boldsymbol{\beta}_\ell \mathbf{s}_\ell.$$

In this case,

$$\mathbb{E}[\Big(\theta_i - (m_{ii} \cdot v_i^\infty(\boldsymbol{\beta}_{-i}) \cdot \mathbf{s}_i + \sum_{\ell \neq i} m_{i\ell} \boldsymbol{\beta}_\ell \mathbf{s}_\ell)\Big)^2 | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = 0$$

On the other hand, for $k \neq \infty$, there is a small punishment for making the first component of $\boldsymbol{\beta}_i$ small. Finally, note that in (56) the term $\boldsymbol{\beta}_i \cdot \mathbf{s}_i$ is multiplied by m_{ii} so there is no loss of generality in considering vectors $\boldsymbol{\beta}_i$ in which the first component is greater or equal than 0.

For a fixed k, we can find a $\epsilon > 0$ small enough such that:

$$\forall i \in N, \forall \boldsymbol{\beta}_{-i} \in \mathbb{R}^{(N-1)J}, \quad |v_{i1}^k(\boldsymbol{\beta}_{-i})| > \epsilon.$$

That is, agent i places a weight at least ϵ on his signal s_{i1}, for any information he might learn from the signals of other players. That is, there is a fixed lower bound on the first component $|v_{i1}^k(\boldsymbol{\beta}_{-i})| > \epsilon$.

We can make ϵ small enough such that, for all $\boldsymbol{\beta}_{-i} \in \mathbb{R}^{(N-1)J}$, $||v_i^k(\boldsymbol{\beta}_{-i})|| < 1/\epsilon$. That is, the weights that are placed on each signal $||v_i^k(\boldsymbol{\beta}_{-i})||$ are bounded from above. This just comes from the fact that if $|\boldsymbol{\beta}_i|| \to \infty$, then the objective function of (56) goes to $+\infty$. Hence, this obviously cannot be a solution.

We define:

$$\mathcal{R} \triangleq \{v \in \mathbb{R}^J : \ v_1 > \epsilon \text{ and } ||v|| < 1/\epsilon.\}$$

That is, $\mathcal{R}$ is the set of vectors such that the first component has size of at least ϵ and the modulus is at most $1/\epsilon$. Note that for any $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \mathcal{R}$ and any $\lambda \in [0,1]$:

$$\lambda\beta_1 + (1-\lambda)\beta_1' \geq \epsilon;$$

$$||\lambda\boldsymbol{\beta} + (1-\lambda)\boldsymbol{\beta}'|| \leq \lambda||\boldsymbol{\beta}|| + (1-\lambda)||\boldsymbol{\beta}'|| \leq 1/\epsilon.$$

Clearly, $\mathcal{R}$ is also a closed and bounded set. Hence, $\mathcal{R}$ is a compact and convex set. Note that the range of v_i^k is $\mathcal{R}$.

Henceforth we assume that the domain is of v_i^k is $\mathcal{R}^{N-1}$.

(Step 2) We define function $\mathbf{v}^k : \mathcal{R}^N \to \mathcal{R}^N$, where

$$\mathbf{v}^k(\boldsymbol{\beta}) \triangleq (v_1^k(\boldsymbol{\beta}_{-1}), ..., v_N^k(\boldsymbol{\beta}_{-N})).$$

We prove that there exists $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k) \in \mathcal{R}^N$, such that such that $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k) \in \mathbb{R}^{N \cdot J}$ is a fixed point of the function $\mathbf{v}^k$. First note that $\mathbf{v}^k : \mathcal{R}^N \to \mathcal{R}^N$, with $\mathcal{R}^N$ being convex and compact. Also, note that v_i^k is continuous. This comes from the fact that the variance covariance matrix $(\mathbf{s}_1, ..., \mathbf{s}_N)$ has full rank and $\mathcal{R}$ does not include 0. Hence, by Brower's fix point theorem, $\mathbf{v}^k$ has a fixed point. We denote that fixed point by $(\boldsymbol{\beta}_1^k, ..., \boldsymbol{\beta}_N^k)$.

(Step 3) We now define the sequence:

$$(\boldsymbol{\nu}_1^k, ..., \boldsymbol{\nu}_N^k) \triangleq (\frac{\boldsymbol{\beta}_1^k}{||\boldsymbol{\beta}_1^k||}, ..., \frac{\boldsymbol{\beta}_N^k}{||\boldsymbol{\beta}_N^k||}).$$

Clearly, $(\boldsymbol{\nu}_1^k, ..., \boldsymbol{\nu}_N^k)$ has a convergent subsequence. We define the limit by $(\boldsymbol{\nu}_1^\infty, ..., \boldsymbol{\nu}_N^\infty)$. It is clear that:

$$\lim_{k\to\infty} \mathbb{E}[\left(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\nu}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^k \cdot \mathbf{s}_N]\right)^2 |\mathbf{s}_i, \boldsymbol{\nu}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^k \cdot \mathbf{s}_N] = 0.$$

All variance covariance matrices have full rank, and hence:

$$\mathbb{E}[\left(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\nu}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^k \cdot \mathbf{s}_N]\right)^2 |\mathbf{s}_i, \boldsymbol{\nu}_1^k \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^k \cdot \mathbf{s}_N]$$

is a continuous function. Hence,

$$\mathbb{E}[\left(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\nu}_1^\infty \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^\infty \cdot \mathbf{s}_N]\right)^2 |\mathbf{s}_i, \boldsymbol{\nu}_1^\infty \cdot \mathbf{s}_1, ..., \boldsymbol{\nu}_N^\infty \cdot \mathbf{s}_N] = 0.$$

Hence, $(\boldsymbol{\nu}_1^\infty, ..., \boldsymbol{\nu}_N^\infty)$ is an equilibrium statistic of the information structure. Hence, we prove that there exists an equilibrium statistic.

(Symmetric Information Structures.) Clearly, if the information structure is symmetric, then we can repeat the argument using symmetric equilibrium statistic. That is, instead of considering $\boldsymbol{\beta}_{-i} \triangleq (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_{i-1}, \boldsymbol{\beta}_{i+1}, ..., , \boldsymbol{\beta}_N)$, one needs to consider $\boldsymbol{\beta}_{-i}$ in which $\boldsymbol{\beta}_j = \boldsymbol{\beta}_\ell$, for all $j, \ell \neq i$.

In this case, the equilibrium statistic found will be symmetric. Hence, we prove the result.■

Proof Proposition 5. First, note that:

$$\mathbb{E}[\theta_i|\mathbf{s}_1, ..., \mathbf{s}_N] = \begin{cases} \eta_i & \text{if } \exists j \in \mathbb{N} \text{ such that } s_{j2} = 0 \\ \eta_i + \frac{1}{2} & \text{if } \exists j \in \mathbb{N} \text{ such that } s_{j2} = \frac{1}{2} \\ \eta_i + \frac{1}{4} & \text{otherwise} \end{cases}$$

We now check that $\mathbb{E}[\theta_i|\mathbf{s}_1, ..., \mathbf{s}_N] = \mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N]$. We first provide the expectation, and then explain the different cases separately.

$$\mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N] = \begin{cases} \zeta_i & \text{if } \zeta_i \in \{0, \kappa, ..., 1\} \text{ or } \zeta_i - \frac{1}{2} \in \{0, \kappa, ..., 1\} \\ \zeta_i - \frac{1}{4} & \text{if } (\zeta_i - \frac{1}{4}) \in \{0, \kappa, ..., 1\} \text{ and } \exists j \in N \text{ such that } \zeta_j \in \{0, \kappa, ..., 1\} \\ \zeta_i + \frac{1}{4} & \text{if } (\zeta_i - \frac{1}{4}) \in \{0, \kappa, ..., 1\} \text{ and } \exists j \in N \text{ such that } \zeta_j - \frac{1}{2} \in \{0, \kappa, ..., 1\} \\ \zeta_i & \text{ otherwise} \end{cases}$$

The expectation of agent i conditional on all equilibrium statistics has 4 cases. The first case is that $\zeta_i \in \{0, \kappa, ..., 1\}$ or $\zeta_i - \frac{1}{2} \in \{0, \kappa, ..., 1\}$. This can only happen if agent i observed $s_{i2} = 0$ or $s_{i2} = 1/2$. In this case agent i knows the realization of $\bar{\varphi}$, and hence the expectation is equal to ζ_i. The rest of the cases imply $\zeta_i \notin \{0, \kappa, ..., 1\}$ and $\zeta_i - \frac{1}{2} \notin \{0, \kappa, ..., 1\}$, which implies that $s_{i2} = 1/4$. Note that $1/\kappa$ is not divisible by 4, and hence this implies that $\zeta_i - \frac{1}{4} \in \{0, \kappa, ..., 1\}$. If some agent $j \in N$ observed $s_{j2} = 0$, then this implies that $\zeta_j \in \{0, \kappa, ..., 1\}$. This implies that $\bar{\varphi} = 0$. Hence, $\theta = \eta_i = \zeta_i - 1/4$. Note that we are in the case in which $s_{i2} = 1/4$. The case in which some $j \in N$ satisfies $\zeta_j - 1/2 \in \{0, \kappa, ..., 1\}$ is completely analogous. Finally, if all agents observe $s_{i2} = 1/4$, then no agent knows the realization of $\bar{\varphi}$. This is the last case.

Hence, we prove the result.■

Proof Proposition 6 We need to prove that:

$$\mathbb{E}[\theta_i|s_{i1}, s_{i2}, \zeta_1, ..., \zeta_N] = \mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N].$$

We note that:

$$\begin{aligned}
\mathbb{E}[\theta_i|\eta_i, s_{i2}, \zeta_1, ..., \zeta_N] &= \eta_i + \mathbb{E}[\bar{\varphi}|\eta_i, s_{i2}, \zeta_1, ..., \zeta_N] && (58)\\
&= \eta_i + \mathbb{E}[\bar{\varphi}|\eta_i, s_{i2}, \zeta_1 - \eta_i, ..., \zeta_N - \eta_i] && (59)\\
&= \eta_i + \mathbb{E}[\bar{\varphi}|s_{i2}, \zeta_1 - s_{i1}, ..., \zeta_N - s_{i1}] && (60)\\
&= \mathbb{E}[\bar{\varphi}|s_{i2} + \eta_i, \zeta_1, ..., \zeta_N] && (61)\\
&= \mathbb{E}[\bar{\varphi}|\zeta_1, ..., \zeta_N] && (62)
\end{aligned}$$

The explanations are as follows. (58) is just using the decomposition of the payoff shock as in (15). (59) is from the fact that the conditioning variables in both equations are the same, expect in (59) we subtracted η_i from all signals. (60) is from (37). (61) is from (38). (62) is from the definition of equilibrium statistic in (39). Hence, we prove the result.■

Proof Lemma 1 First, note that:

$$\mathbb{E}[\theta_i|s_1, ..., s_N] = \mathbb{E}[\theta_i|s_i, \bar{s}] = \mathbb{E}[\theta_i|\Delta s_i, \bar{s}].$$

The first equation is by symmetry, as conditioning on $(s_1, ..., s_N)$ must be the same as conditioning just on $(s_i, \bar{s})$ (see Lemma 10 in the appendix). The second equation is using that $(\Delta s_i, \bar{s})$ is a linear combination of signals $(s_i, \bar{s})$.

Note that $(\Delta s_i, \Delta\theta_i)$ are independent of $(\bar{s}, \bar{\theta})$ (see Lemma 9 in the appendix). Hence,

$$\mathbb{E}[\theta_i|\Delta s_i, \bar{s}] = \mathbb{E}[\bar{\theta}|\bar{s}] + \mathbb{E}[\Delta\theta_i|\Delta s_i] = \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_i + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s}.$$

Hence, we prove the result. ■

Proof Lemma 2 This is by the definition of variance covariance matrix and using Lemma 9 (in the appendix) to show some of the covariances are 0.

Additionally, the following covariances are equal to 0 by symmetry (see Lemma 9):

$$cov(\bar{\theta}, \Delta\theta_i) = cov(\bar{s}, \Delta\theta_i) = cov(\bar{\theta}, \Delta s_i) = cov(\bar{s}, \Delta s_i) = 0.$$

Hence, $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$ is determined by 6 coefficients. Hence, the coefficients in (7) completely determine the variance covariance matrix of $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$. Hence, we prove the result.■

Proof Lemma 3 It is clear that the drop-out prices defined by (10) is a feasible outcome in an ascending auction. We just need to show this is an equilibrium. In fact, we show this is an ex-post equilibrium (see Definition 5).

We first prove that the strategy of agent 1 (that is, agent that observes s_1) the strategy the leads to (10) is optimal. We first provide the main steps of the proof and then explain each step:

$$
\begin{aligned}
\mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \qquad (63)\\
&\quad +(\frac{1-m}{N}-1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N])\\
&\quad +\frac{1}{2}((1-corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1-corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg).\\
&\leq exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \qquad (64)\\
&\quad +\frac{1}{2}((1-corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1-corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg)\\
&= \mathbb{E}[exp(\theta_1)|s_1, s_2, ..., s_N] \qquad (65)
\end{aligned}
$$

Equality (63) is calculated explicitly in the proof of Theorem 1. Inequality (64) is using that $m \in [-(N-1), \infty)$ and $\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N] > 0$ by construction. Equality (65) is also calculated explicitly in Theorem 1. Clearly for agent 1 it is optimal to win the auction. As he cannot modify the price he pays, his action is optimal.

We first prove that the strategy of agent 2 (that is, agent that observes s_2) the strategy the leads ton (10) is optimal. That is, it is optimal for agent 2 to not win the object. The argument for agent j, with $j > 2$ is obviously the same as for agent 2.

On the other hand, for any agent j such that $s_j < s_1$, if he wins the object he will pay:

$$\mathbb{E}[exp(\theta_1)|s_1, s_1, s_2, ..., s_N].$$

That is, agent 2 pays the expected valuation of agent 1, assuming that the signal of agent 2 is equal to the signal of agent 1. This is the same as under the equilibrium, but replacing s_2 by s_1.

Using the previous argument:

$$\begin{aligned}
\mathbb{E}[exp(\theta_1)|s_1, s_1, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_2|s_2, ..., s_N] \\
&\quad -(\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N]) \\
&\quad +\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg). \\
&\geq exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \\
&\quad +\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg) \\
&= \mathbb{E}[exp(\theta_1)|s_1, s_2, ..., s_N]
\end{aligned}$$

The steps are the same as before, except in this case the inequality 66 goes the other way because there is a minus sign before the term. Clearly, generically the inequality will be strict. Hence, agent $j = 2$ cannot win the object and pay a price less than his valuation. Hence, we prove the result.■

Proof Lemma 4 Note that the model with public signal:

$$s^p = \bar{\theta} + \bar{\varepsilon}^p,$$

can be analyzed the same way as a model in which the common shock is given by:

$$\bar{\theta}' \triangleq \bar{\theta} - \mathbb{E}[\bar{\theta}|s^p],$$

and agents observe signals:

$$s_i' = s_i - \mathbb{E}[s_i|s^p].$$

As s^p is conditionally independent of s_i we have that:

$$\mathbb{E}[s_i|s^p] = \mathbb{E}[\mathbb{E}[s_i|s^p, \bar{\theta}]|s^p] = \mathbb{E}[\mathbb{E}[s_i|\bar{\theta}]|s^p] = \mathbb{E}[\frac{cov(s_i, \bar{\theta})}{var(\bar{\theta})}\bar{\theta}|s^p].$$

Note that by Proposition 1 and Corollary 1 the surplus and the seller's profits do no depend on $corr(\bar{s}, \bar{\theta})$. Clearly, the public signal does not change $corr(\Delta s_i, \Delta\theta_i)$. Hence, the surplus generated with s^p does not change. On the other hand, m' is given by:

$$m' = \frac{var(\Delta s_i')cov(\bar{\theta}', \bar{s}')}{var(\bar{s}')cov(\Delta\theta_i', \Delta s_i')}.$$

Clearly the idiosyncratic component of the random variables do not change with s^p, hence:

$$m' = \frac{var(\Delta s_i)cov(\bar{\theta}', \bar{s}')}{var(\bar{s}')cov(\Delta\theta_i, \Delta s_i)}.$$

Note that:

$$\begin{aligned}
\frac{cov(\bar{\theta}', \bar{s}')}{var(\bar{s}')} &= \frac{cov(\bar{\theta} - \mathbb{E}[\bar{\theta}|s^p], \bar{s} - \frac{cov(\bar{s},\bar{\theta})}{var(\bar{\theta})}\mathbb{E}[\bar{\theta}|s^p])}{var(\bar{s}) - var(\mathbb{E}[\bar{s}|s^p])} && (66)\\
&= \frac{cov(\bar{\theta} - \mathbb{E}[\bar{\theta}|s^p], \frac{cov(\bar{s},\bar{\theta})}{var(\bar{\theta})}\bar{\theta} - \frac{cov(\bar{s},\bar{\theta})}{var(\bar{\theta})}\mathbb{E}[\bar{\theta}|s^p])}{var(\bar{s}) - var(\mathbb{E}[\bar{s}|s^p])} && (67)\\
&= \frac{\frac{cov(\bar{s},\bar{\theta})}{var(\bar{\theta})}(var(\bar{\theta}) - var(\mathbb{E}[\bar{\theta}|s^p]))}{var(\bar{s}) - var(\mathbb{E}[\bar{s}|s^p])} && (68)\\
&= \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\frac{1 - \frac{var(\mathbb{E}[\bar{\theta}|s^p]))}{var(\bar{\theta})}}{1 - \frac{var(\mathbb{E}[\bar{s}|s^p])}{var(\bar{s})}} && (69)\\
&< \frac{cov(\bar{s}, \bar{\theta})}{var(\bar{s})} && (70)
\end{aligned}$$

The explanation is as follows. (66) is by construction of the random variable $\bar{\theta}'$ and s_i'. (67) is using that s^p is independent of s_i conditional on $\bar{\theta}$. Hence, the covariances can be written as follows

$$cov(\bar{\theta}, \bar{s}) = \frac{cov(\bar{s}, \bar{\theta})}{var(\bar{\theta})}cov(\bar{\theta}, \bar{\theta}) \text{ and } cov(s^p, \bar{s}) = cov(s^p, \mathbb{E}[\bar{s}|\bar{\theta}]) = \frac{cov(\bar{s}, \bar{\theta})}{var(\bar{\theta})}cov(s^p, \bar{\theta}).$$

(68) is using the collinearity of the covariance. (69) is re-arranging terms. (70) is by using the fact that $\bar{s}$ is independent of s^p conditional on $\bar{\theta}$, which implies:

$$\frac{var(\mathbb{E}[\bar{s}|s^p])}{var(\bar{s})} = (1 - corr(\bar{s}, s^p)^2) < (1 - corr(\bar{\theta}, s^p)^2)\frac{var(\mathbb{E}[\bar{\theta}|s^p]))}{var(\bar{\theta})}.$$

This implies that:

$$\frac{1 - \frac{var(\mathbb{E}[\bar{\theta}|s^p]))}{var(\bar{\theta})}}{1 - \frac{var(\mathbb{E}[\bar{s}|s^p])}{var(\bar{s})}} < 1.$$

Hence, $m' < m$. Hence, the seller's profits decrease with s^p. Hence, we prove the result.■

Proof Lemma 5 Using (82) for the statistic $\zeta_i = s_{i1} + \beta s_{i2}$ we get:

$$cov(\theta_i - \mathbb{E}[\Delta\theta_i|\Delta\zeta_i] - \mathbb{E}[\bar{\theta}|\bar{\zeta}], s_{i1}) = 0. \tag{71}$$

Writing the expectations explicitly:

$$\mathbb{E}[\Delta\theta_i|\Delta\zeta_i] = \mathbb{E}[\Delta\eta_i|\Delta\eta_i + \beta\Delta\varepsilon_i] = \frac{var(\Delta\eta_i)}{var(\Delta\eta_i) + \beta^2 var(\Delta\varepsilon_i)}(\Delta\eta_i + \beta\Delta\varepsilon_i);$$

$$\mathbb{E}[\bar{\theta}|\bar{\zeta}] = \mathbb{E}[\bar{\eta} + \bar{\varphi}|\bar{\eta} + \beta(\bar{\varphi} + \bar{\varepsilon})] = \frac{var(\bar{\eta}) + \beta var(\bar{\varphi})}{var(\bar{\eta}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon}))}(\bar{\eta} + \beta(\bar{\varphi} + \bar{\varepsilon})).$$

Rewriting (71) we get:

$$var(\Delta\eta_i)(1 - \frac{var(\Delta\eta_i)}{var(\Delta\eta_i) + \beta^2 var(\Delta\varepsilon_i)}) + var(\bar{\eta})(1 - \frac{var(\bar{\eta}) + \beta var(\bar{\varphi})}{var(\bar{\eta}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon}))}) = 0.$$

Rearranging terms:

$$\begin{aligned} 0 &= (var(\bar{\eta}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon})))var(\Delta\eta_i)\beta var(\Delta\varepsilon_i) \\ &\quad + (var(\Delta\eta_i) + \beta^2 var(\Delta\varepsilon_i))var(\bar{\eta})(\beta(var(\bar{\varphi}) + var(\bar{\varepsilon})) - var(\bar{\varphi})). \end{aligned}$$

Grouping up terms, we get:

$$\begin{aligned} 0 &= -var(\bar{\varphi})var(\bar{\eta})var(\Delta\eta_i) + \beta\Big(var(\Delta\eta_i)var(\bar{\eta})(var(\bar{\varphi}) + var(\bar{\varepsilon})) + var(\Delta\varepsilon_i)var(\bar{\eta})var(\Delta\eta_i)\Big) \\ &\quad -\beta^2 \cdot var(\Delta\varepsilon_i)var(\bar{\eta})var(\bar{\varphi}) \\ &\quad +\beta^3\Big((var(\bar{\varphi}) + var(\bar{\varepsilon}))var(\Delta\eta_i)var(\Delta\varepsilon_i) + var(\Delta\varepsilon_i)var(\bar{\eta})(var(\bar{\varphi}) + var(\bar{\varepsilon}))\Big). \end{aligned}$$

Simplifying terms we get:

$$\frac{-1}{var(\Delta\varepsilon_i)} + \frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})}\beta + \frac{-1}{var(\Delta\eta_i)}\beta^2 + \frac{(var(\Delta\eta_i) + var(\bar{\eta}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\eta_i)var(\bar{\eta})var(\bar{\varphi})}\beta^3 = 0$$

Hence, we prove the result. ■

Proof Lemma 6 It is a standard property of cubic polynomials that they have a unique root if and only if their discriminant is greater than 0. For (19) this reduces to the condition in Lemma 6. Hence, we prove the result.■

Proof Lemma 8. The example is the same as Jackson (2009). ■

Proof Lemma 7 We prove in two steps. First we prove that the equilibrium has an efficient ex-post equilibrium. Second we prove that the price paid is (24).

Step 1: The proof that the ascending auction has en efficient ex-post equilibrium is almost direct from Krishna (2003) (see Theorem 2 therein). The definition of average crossing condition in Krishna (2003) is slightly different than Definition 2, and hence we only need to prove that Definition 2 implies the definition of average crossing condition in Krishna (2003).

Using the normality of the information structure, it is easy to check that:

$$\mathbb{E}[exp(\theta_i)|s_1,...,s_N] = exp(\mathbb{E}[\theta_i|s_1,...,s_N] + \frac{var(\theta_i|s_1,...,s_N)}{2})$$

Hence,

$$\frac{\partial \mathbb{E}[exp(\theta_i)|s_1,...,s_N]}{\partial s_j} = exp(\theta_i)\frac{\partial \mathbb{E}[\theta_i|s_1,...,s_N]}{\partial s_j}.$$

For any signal realization $(s_1,...,s_N)$, we denote:

$$E(s_1,...,s_N) \triangleq \{i \in N : i \in \arg\max_{\ell \in N} \mathbb{E}[exp(\theta_\ell)|s_1,...,s_N].$$

Hence, for all $i,\ell \in E(s_1,...,s_N)$, $\mathbb{E}[exp(\theta_\ell)|s_1,...,s_N] = \mathbb{E}[exp(\theta_i)|s_1,...,s_N]$. Hence, for all $\mathcal{A} \subset E(s_1,...,s_N)$, and for all $i,j \in \mathcal{A}$ with $i \neq j$:

$$\frac{\partial \mathbb{E}[exp(\theta_i)|s_1,...,s_N]}{\partial s_j} \leq \frac{1}{|\mathcal{A}|}\sum_{k\in N}\frac{\partial \mathbb{E}[exp(\theta_k)|s_1,...,s_N]}{\partial s_j} \iff \frac{\partial \mathbb{E}[\theta_i|s_1,...,s_N]}{\partial s_j} \leq \frac{1}{|\mathcal{A}|}\sum_{k\in N}\frac{\partial \mathbb{E}[\theta_k|s_1,...,s_N]}{\partial s_j}.$$

Hence, if the information structure satisfies condition in Definition 2, then it also satisfies the average crossing condition in Krishna (2003). From Theorem 2 in Krishna (2003), the ascending auction has an ex-post equilibrium.

Step 2: It is easy to check that there is a direct mechanism that implements the outcome in Lemma 7. That is, there is a direct mechanism that has an ex-post equilibrium in which the agent with the highest valuation gets the object and pays (24). The proof can be found in Ausubel (1999), but the argument is essentially the same as Lemma 3. Hence, there exists a mechanism that has an ex-post equilibrium that implements an efficient allocation with payment (24).

Perry and Reny (1999) provides a revenue equivalence theorem for ex-post equilibria. That is, if two mechanisms implement the same allocation as an ex-post equilibrium, then the payments must be the same. Hence, (24) must also be the payment in the outcome of the ascending auction.

Hence, we prove the result.■

Proof Corollary 1 The seller's profits are given by:

$$\begin{aligned}\mathbb{E}[\pi_2] &= \mathbb{E}\Bigg[exp\Bigg((\frac{1-m}{N}-1)(\mathbb{E}[\theta_1|s_1,...,s_N]-\mathbb{E}[\theta_2|s_1,...,s_N])\Bigg)\mathbb{E}[\exp(\theta_1)|s_1,...,s_N]\Bigg]\\ &= \mathbb{E}\Bigg[exp\Bigg((\frac{1-m}{N}-1)(\mathbb{E}[\Delta\theta_1|\Delta s_1,...,\Delta s_N]-\mathbb{E}[\theta_2|\Delta s_1,...,\Delta s_N])\Bigg)\\ &\quad \times\mathbb{E}[\exp(\bar{\theta}+\Delta\theta_1)|\bar{s},\Delta s_1,...,\Delta s_N]\Bigg]\end{aligned}$$

Since the expectation does not depend on m, it is clear that $\mathbb{E}[\pi_2]$ is decreasing in m (note that by construction $\mathbb{E}[\Delta\theta_1|\Delta s_1,...,\Delta s_N] > \mathbb{E}[\theta_2|\Delta s_1,...,\Delta s_N]$). Additionally, the realization of the common component of random variables is independent of the realization of the idiosyncratic component of random variables. Hence, we can use the law of iterated expectations to take expectations over the common component of the signals. We get:

$$\begin{aligned}\mathbb{E}[\pi_2] &= \mathbb{E}\Bigg[exp\Bigg((\frac{1-m}{N}-1)(\mathbb{E}[\Delta\theta_1|\Delta s_1,...,\Delta s_N]-\mathbb{E}[\theta_2|\Delta s_1,...,\Delta s_N])\Bigg)\\ &\quad \times\mathbb{E}[\exp(\Delta\theta_1)|\bar{s},\Delta s_1,...,\Delta s_N]\Bigg]\times\exp\left(\frac{1}{2}corr(\bar{s},\bar{\theta})^2 var(\bar{\theta})\right).\end{aligned}$$

Since all the terms depend only on the realization of the idiosyncratic component of signals, we have that the expectation does not depend on $corr(\bar{s},\bar{\theta})$. The proof for the rents of the buyers is completely analogous, except for the fact that the rents are increasing in m. Hence, we prove the result.■

Proof Corollary ?? If $m < -(N-1)$, then $p_2 > S(s_1,...,s_N)$. Hence, the price paid is always bigger than the surplus generated. Hence, the buyers' are getting negative rents. Hence, this cannot be an equilibrum. Hence, we prove the result.■

Proof Corollary 2 Note that for the noise-free information structure:

$$corr(\Delta s_i,\Delta\theta_i) = corr(\bar{s},\bar{\theta}) = 1 \text{ and } m = \mu. \tag{72}$$

This is straightforward to check the result from (12) and (72). Hence, we prove the result. ■

Proof Corollary 3 We prove the result using the characterization in Lemma 6.

In the limit $var(\varepsilon_i) \to 0$, we have that $var(\Delta\varepsilon_i) \to 0$. In this case, $d \to \infty$ and $c \to \infty$. Clearly the term that dominates is $-4ac$.

In the limit $var(\varepsilon_i) \to 0$, we have that $var(\bar{\varepsilon}) \to \infty$. In this case, $a \to \infty$. Clearly the term that dominates is $-27a^2d^2$.

Proof Corollary 4 We prove the result using the characterization in Lemma 6.

By assuming $var(\bar{\varepsilon}) = 0$ and considering the limit:

$$\lim_{var(\Delta\varepsilon_i)\to\infty} 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = \frac{-4var(\Delta\eta_i)^2 - 4var(\Delta\eta_i)var(\bar{\eta}) + var(\bar{\eta})var(\bar{\varphi})}{var(\Delta\eta_i)^2 var(\bar{\eta}) var(\bar{\varphi})^3}.$$

By re-arranging terms we get the result. **Proof Corollary 5** The proof is a straightforward rewrite of Theorem 4 and using that when agents observe only the equilibrium it is as if they had common values. Note that, if the utility of agents is linear in θ_i then in the proof of Theorem 4 it clearly does not matter that the equilibrium statistics is non-Gaussian. Hence, Milgrom and Weber (1982) can be applied. Hence, we prove the result. ■

Proof Equation (6)**.** If $m \geq 0$, we clearly have that:

$$cov(\Delta\theta_i, \Delta s_i) > 0 \text{ and } cov(\bar{\theta}, \bar{s}) > 0.$$

On the other hand, for any $\theta_j \neq \theta_i$

$$|cov(\Delta\theta_i, \Delta s_i)| > |cov(\Delta\theta_j, \Delta s_i)|.$$

Hence, for any θ_i, θ_j, we have that:

$$cov(s_i, \theta_i) = |cov(\Delta\theta_i, \Delta s_i)| + |cov(\bar{\theta}, \bar{s})| > |cov(\Delta\theta_j, \Delta s_i)| + |cov(\bar{\theta}, \bar{s})| \geq cov(s_i, \theta_j).$$

Hence,

$$corr(\theta_i, s_i)^2 > corr(\theta_j, s_i)^2.$$

10 Online Appendix A: Additional Properties of Symmetric Information Structures

We now discuss the properties of the orthogonal decomposition in (2). To make the reading easier we repeat the definitions here:

$$\bar{\theta} \triangleq \frac{1}{N}\sum_{i\in N}\theta_i \ \ ; \ \ \Delta\theta_i = \theta_i - \bar{\theta} \ \ ; \ \ \bar{\mathbf{s}} \triangleq \frac{1}{N}\sum_{i\in N}\mathbf{s}_i \ \ ; \ \ \mathbf{\Delta s}_i = \mathbf{s}_i - \bar{\mathbf{s}}. \tag{73}$$

The fundamental property of the orthogonalization in (73) is that common variables (variables that have a bar over them) are orthogonal to idiosyncratic variables (variables preceded by a Δ). We formalize this in the following lemma.

Lemma 9 (Orthogonal Decomposition)**.**
If the information structure is symmetric then the random variables $(\Delta\theta_i, \mathbf{\Delta s}_i)$ *are independent of* $(\bar{\theta}, \bar{\mathbf{s}})$.

Lemma 9 shows that in symmetric environments the random variables that are common to all agents (variables that have an over-bar) are orthogonal to idiosyncratic random variables (variables preceeded by a Δ). Lemma 9 also shows that in symmetric environments the information structure is completely determined by the variance covariance matrices of random variables $(\Delta\theta_i, \mathbf{\Delta s}_i)$ and $(\bar{\theta}, \bar{\mathbf{s}})$.

We now show that in symmetric environments the information about the average equilibrium is sufficient for the information in all equilibrium statistics.

Lemma 10 (Symmetric Equilibrium Statistic)**.**
The random variables $(\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N)$ *are an equilibrium statistic if and only if:*

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}]. \tag{74}$$

Proposition 10 shows that in symmetric environments, the average sufficient statistic $\boldsymbol{\beta}\cdot\bar{\mathbf{s}}$ is sufficient for the information in all equilibrium statistics $(\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N)$. Hence, when we analyze symmetric environments all the information agent i observes is summarized by $\boldsymbol{\beta}\cdot\bar{\mathbf{s}}$. This allow us to provide a more succinct characterization of the equilibrium statistic.

11 Online Appendix B: Properties and Computation of Equilibrium Statistic

11.1 Characterization of Equilibrium Statistic

There is an equivalent description of (25) which makes transparent how equilibrium statistics can be computed in applications.

Proposition 7 (Characterization of Equilibrium Statistic)**.**
The random variables $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^{NJ}$ *are an equilibrium statistic if and only if, there exists* $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$*, such that:*

$$\forall i \in N, \forall j \in J \qquad cov(\theta_i, s_{ij}) - \sum_{\ell=1}^{N} m_{i\ell} \cdot cov(\boldsymbol{\beta}_\ell \cdot s_\ell, s_{ij}) = 0, \tag{75}$$

$$\forall i \in N, \forall j \in N \backslash \{i\}, \qquad cov(\theta_i, \boldsymbol{\beta}_j \cdot \mathbf{s}_j) - \sum_{\ell=1}^{N} m_{i\ell} \cdot cov(\boldsymbol{\beta}_\ell \cdot \mathbf{s}_\ell, \boldsymbol{\beta}_j \cdot \mathbf{s}_j) = 0, \tag{76}$$

Proposition 7 provides a set of equations that are necessary and sufficient to find equilibrium statistics. There are $N^2 + J \cdot N$ variables corresponding to $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$ and $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{NJ}$. It is easy to check that (75) defines $J \cdot N$ equations, while (76) corresponds to $N \times (N-1)$ equations. There are less equation than variables because the equilibrium statistics are not uniquely defined. If $\boldsymbol{\beta}_i \cdot \mathbf{s}_i$ is an equilibrium statistic and we multiply this equilibrium statistic by any number, then this would still be an equilibrium statistic. That is, by looking at Definition 3 we can see that we can multiply the equilibrium statistic by any scalar.[32]

The set of equations are bi-linear in the variables. That is, for a fixed values of $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$, (75) defines $J \cdot N$ linear equations on $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N) \in \mathbb{R}^{NJ}$. Hence, solving for equilibrium statistic is computationally simple. The characterization of symmetric equilibrium statistics in symmetric environments can be further simplified. We explain how the characterization can be simplified in the appendix (see Section 11.2). The characterization in Proposition 7 for the special case of the symmetric information structure (16), yields the cubic equation (19).

Finally, we interpret the coefficients $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$. These coefficients are the weights that agents place on their own equilibrium statistic and the equilibrium statistic of other agents when taking the expectation $\mathbb{E}[\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]$. Consider an equilibrium statistic $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$

[32] In order to get a system of equations that has the same number of equations than unknowns any normalization on the $\boldsymbol{\beta}_i$ works. For example, to derive (19) we imposed that the first component of $\boldsymbol{\beta}$ is equal to one (that is, $\beta_{i1} = 1$).

and parameters $(m_{i\ell})_{i,\ell\in N^2} \in \mathbb{R}^{N^2}$ that solve for (75) and (76), then:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \sum_{\ell\in N} m_{i\ell}\boldsymbol{\beta}_\ell \cdot \mathbf{s}_\ell \tag{77}$$

The proof of (77) can be found in the proof of Proposition 7. This implies that the parameters $(m_{i\ell})_{i,\ell\in N^2} \in \mathbb{R}^{N^2}$ are directly related to the average crossing condition.

Lemma 11 (Average Crossing Condition for Equilibrium Statistic)**.**
The equilibrium statistic $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$ *satisfies the average crossing condition, if and only if, the parameters* $(m_{i\ell})_{i,\ell\in N^2} \in \mathbb{R}^{N^2}$ *that solve for* (75) *and* (76) *satisfy that for all* $\mathcal{A} \subset \{1, ..., N\}$*, and for all* $i, j \in \mathcal{A}$ *with* $i \neq j$*:*

$$m_{ij} \leq \frac{1}{|\mathcal{A}|}\sum_{k\in N} m_{kj}$$

Lemma 11 provides a simple way to compute the average crossing condition from the characterization in Proposition 7.

Independent Signals. A common assumption in the mechanism literature is that agents' signals are independently distributed. In particular, Dasgupta and Maskin (2000) show that in a generalized VCG mechanism, when agents receive independent signals it is possible to calculate the equilibrium of the game using a one-dimensional sufficient statistic. If the signals of agents are independently distributed, then the equilibrium statistic is equal to $\zeta_i = \mathbb{E}[\theta_i|\mathbf{s}_i]$. That is, the equilibrium statistic of agent i is agent i's expectation of his own payoff shock conditional only on his private information. To check this, note that for any equilibrium statistic $(\zeta_1, ..., \zeta_N)$:

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \zeta_1, ..., \zeta_N] = \mathbb{E}[\theta_i|\mathbf{s}_i] + \sum_{j\neq i}\mathbb{E}[\theta_i|\zeta_j].$$

Hence, the analysis can be reduced to a one-dimensional problem by taking the expected payoff shock of each agent conditional only on his private information. The fundamental aspect of our analysis is to understand how the information agents learn from the drop-out time of other agents modifies the equilibrium statistic, and how this leads to novel predictions.

11.2 Equilibrium Statistic in Symmetric Environments

We now discuss how the characterization of the equilibrium statistic can be simplified in symmetric environments. In order to provide additional intuitions on the equilibrium statistic we compare this with the expectation of an agent of his own payoff shock conditional *only* on his private information. Note that a equilibrium statistic corresponds to a particular linear combination of the signals. Hence, we can compare the equilibrium statistic directly with the expectation of an agent of his own payoff shock conditional *only* on his private information. This allows us to illustrate how the use of private signals by agent i is modified by the information in the actions of players different than i. For example, this allow us to understand how in an ascending auction the information agent i learns from the drop-out time of other agents modifies how agent i uses his signals to determine his drop-out time.

An agent's expectation of his own payoff shock conditional *only* on his private information can be computed in closed form:[33]

$$\mathbb{E}[\theta_i|\mathbf{s}_i] = \mathbf{b}\cdot\mathbf{s}_i, \quad \text{with} \quad \mathbf{b} \triangleq (\Sigma_{\bar{s}\bar{s}} + \Sigma_{\Delta s \Delta s})^{-1}\cdot cov(\theta_i, \mathbf{s}_i). \tag{78}$$

We call $\mathbf{b}\cdot s_i$ agent i's *interim* expectation.[34] In an ascending auction, if agent i ignored the information from the drop-out price of other agents, then agent i would determine his own drop-out price using the one-dimensional statistic $\mathbf{b}\cdot\mathbf{s_i}$. The errors of the interim expectation of agent i are orthogonal to the information agent i observes:

$$\forall j \in J, \quad cov(\theta_i - \mathbb{E}[\theta_i|\mathbf{b}\cdot\mathbf{s}_i], \mathbf{s}_{ij}) = 0. \tag{79}$$

That is, each signal s_{ij} must be uncorrelated with the errors in the predictions of θ_i.

The one-dimensional statistic can be found using a similar logic, but it is necessary to disentangle common and idiosyncratic components. We can compute the one-dimensional statistic in "almost" closed form. For this, we first define a function $\Phi : \mathbb{R}^J \to \mathbb{R}$:

$$\Phi(\boldsymbol{\beta}) \triangleq \frac{\left(\boldsymbol{\beta}\cdot((N-1)cov(\bar{\theta},\bar{\mathbf{s}}) - cov(\Delta\theta_i, \boldsymbol{\Delta}\mathbf{s}_i)) - (N-1)\boldsymbol{\beta}\cdot var(\bar{\mathbf{s}})\cdot\boldsymbol{\beta} + \boldsymbol{\beta}\cdot var(\boldsymbol{\Delta}\mathbf{s}_i)\cdot\boldsymbol{\beta}\right)}{((N-1)^2\boldsymbol{\beta}\cdot var(\bar{\mathbf{s}})\cdot\boldsymbol{\beta} + \boldsymbol{\beta}\cdot var(\boldsymbol{\Delta}\mathbf{s}_i)\cdot\boldsymbol{\beta})}. \tag{80}$$

The equilibrium statistic can be computed as follows.

[33]Properties of Bayesian updating with Gaussian random variables are standard in the literature. For example, see Hogg, McKean, and Craig (2005).

[34]Throughout this section, we denote by $\mathbf{b}\in\mathbb{R}^J$ the linear combination of signals in agent i's *interim* expectation, while $\boldsymbol{\beta}\in\mathbb{R}^J$ denotes the linear combination of signals of the one-dimensional equilibrium statistic.

Corollary 6 (Equilibrium Use of Information)**.**
The random variables $(\boldsymbol{\beta} \cdot s_1, ..., \boldsymbol{\beta} \cdot s_N) \in \mathbb{R}^N$ *forms an equilibrium statistic if and only if, there exists* $(m, \tilde{m}) \in \mathbb{R}^2$ *such that* $m = \Phi(\boldsymbol{\beta} \cdot \tilde{m}^{-1})$ *and:*

$$\boldsymbol{\beta} = \tilde{m} \cdot \Big((1 + (N-1)m)var(\bar{\mathbf{s}}) + (1-m)var(\boldsymbol{\Delta}\mathbf{s}_i)\Big)^{-1} \cdot cov(\theta_i, \mathbf{s}_i) \tag{81}$$

As in Proposition 7 the equilibrium statistics are not uniquely defined, in this case the parameter $\tilde{m}$ is just a rescaling parameter that can be normalized to 1. Hence, Corollary 6 allows us to compute the set of linear Nash equilibrium in closed form up to a one-dimensional parameter (m). That is, given the equilibrium value of m, the equilibrium value of $\boldsymbol{\beta}$ is determined in closed form by (81).

It is remarkable that the only difference between a equilibrium statistic $(\boldsymbol{\beta})$ and the *interim* expectation $(\mathbf{b})$ comes from the re-weighting of the variance of the common component of signals $(var(\bar{\mathbf{s}}, \bar{\mathbf{s}}))$ and idiosyncratic component of signals $(var(\boldsymbol{\Delta}\mathbf{s}, \boldsymbol{\Delta}\mathbf{s}))$. The re-weighting of common and idiosyncratic component is the only modification needed to incorporate that an agent's expectation of his own payoff shock is conditioned on the equilibrium actions of other players. As before, we can interpret the equilibrium statistic in terms of the covariance between the error in the expectations of agents and the signals an agent observes.

Proposition 8 (Characterization of One-Dimensional Statistic)**.**
The random variables $(\boldsymbol{\beta} \cdot s_1, ..., \boldsymbol{\beta} \cdot s_N) \in \mathbb{R}^N$ *forms an equilibrium statistic if and only if:*

$$\forall j \in J, \quad cov(\theta_i - \mathbb{E}[\Delta\theta_i | \boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i] - \mathbb{E}[\bar{\theta} | \boldsymbol{\beta} \cdot \bar{\mathbf{s}}], s_{ij}) = 0. \tag{82}$$

Proposition 8 shows that the equilibrium statistic is computed in a similar way than the *interim* expectations under normal random variables. In contrast to (79), common and idiosyncratic component of the random variables in (82) are separated in the expectation.

Proposition 8 suggests how agents use their private information and the information from the actions of other players in an equilibrium. The information in the actions of other players allows an agent to place different weight on the common and idiosyncratic component of the equilibrium statistic. This allows agents to use the common component of signals to predict the common component of their payoff shock, and use the idiosyncratic component of signals to predict the idiosyncratic component of the payoff shock.

Finally, we bound the number of equilibrium statistics in symmetric environments.

Proposition 9 (Bound on Number of Equilibria).
For any symmetric information structure, there are at most $(2J-1)$ *symmetric equilibrium statistic in which* $||\boldsymbol{\beta}|| = 1$.

Lemma 9 provides a bound on the number of symmetric equilibrium statistic. The additional constraint that $||\boldsymbol{\beta}|| = 1$ is to eliminate the multiplicity of equilibrium statistic that arise because we can multiply them by a scalar. In other words, this is equivalent to looking for equilibrium statistic that satisfy (81) with $m \tilde{=} 1$. Although for $J = 2$ we know that this bound is tight, we have not found a general class of examples that proves that the bound is tight for all J.

11.3 Single Crossing Property

In a symmetric environment, the equilibrium statistic must satisfy (5) in order for an equilibrium to exists. In asymmetric environment the equilibrium statistic must satisfy the average crossing condition. For specific examples (as in Section 4) it is easy to check whether the equilibrium statistic satisfy the average crossing condition. We now study conditions on the signals such that the equilibrium statistic satisfies (5) in symmetric environments.

We provide sufficient conditions on symmetric information structures such that (5) is satisfied. We first define the monotonicity condition.

Definition 7 (Multidimensional Monotonicity Condition).
A symmetric information structure satisfies the multidimensional monotonicity condition if:

$$i, \ell, \in N, \quad \forall j, k \in J, \quad cov(\bar{s}_k, \bar{s}_j | \bar{\theta}) = cov(\Delta s_{ik}, \Delta s_{ij} | \Delta\theta) = 0 \text{ and } corr(s_{ij}, \theta_i)^2 \geq corr(s_{ij}, \theta_\ell)^2. \tag{83}$$

The multidimensional monotonicity condition corresponds to two condition on the information structure. It is required that all signals satisfy (6). Second, we require that signals are conditionally independent. We now show that the multidimensional monotonicity condition is sufficient to guarantee that the equilibrium statistic satisfies the one-dimensional single crossing property.

Proposition 10 (Single Crossing Condition)**.**
If the monotonicity condition is satisfied, then every symmetric equilibrium statistic satisfies (5)*.*

Proposition 10 provides sufficient conditions such that every equilibrium statistic satisfies (6). The more restrictive condition in (83) is the conditional independence of signals, as this is a condition that is satisfied only in non-generic information structures. Nevertheless, for every information structure that satisfies (83), there is an open set of information structures that yield equilibrium statistic that satisfy (6). We can apply Proposition 10 to guarantee the existence of symmetric equilibrium.

Corollary 7 (Existence of Equilibrium)**.**
If an information is symmetric and satisfies the monotonicity condition, then the ascending auction has a symmetric equilibrium.

Corollary 7 uses the monotonicity condition and the existence of equilibrium statistic to guarantee the existence of equilibrium in the ascending auction. One would like to generalize Proposition 10 to asymmetric environments. Nevertheless, we have not been able to find general conditions that guarantee that every equilibrium statistic satisfies the average crossing condition. If an asymmetric information structure is "close" to a symmetric information structure that satisfies the monotonicity condition, then the information structure will have an equilibrium statistic that satisfies the average crossing condition. This comes from the fact (75) and (76) are a bi-linear system of equations and hence in general the set of solutions will change continuously as one changes the parameters of the information structure.

12 Online Appendix C: Proofs of Results in Appendix

Proof Proposition 7 (Only If) Suppose that the random variables $(\boldsymbol{\beta} \cdot s_1, ..., \boldsymbol{\beta} \cdot s_N) \in \mathbb{R}^N$ forms an equilibrium statistic, then by definition:

$$\forall i \in N, \quad \mathbb{E}[\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]. \tag{84}$$

We know that there exists $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$ such that:

$$\mathbb{E}[\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \sum_{\ell \in N} m_{i\ell} \boldsymbol{\beta}_\ell \mathbf{s}_\ell.$$

Hence, there exists $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$ such that:

$$\forall i \in N, \quad \mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \sum_{\ell \in N} m_{i\ell} \cdot \boldsymbol{\beta}_\ell \mathbf{s}_\ell. \tag{85}$$

Multiplying by s_{ij}, we have (note that s_{ij} can be brought into the expectation as the expectation conditions on s_{ij}):

$$\forall i \in N, \quad \mathbb{E}[s_{ij}\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = s_{ij} \sum_{\ell \in N} m_{i\ell} \boldsymbol{\beta}_\ell \mathbf{s}_\ell.$$

Taking expectation and using the law of iterated expectations:

$$\forall i \in N, \quad \mathbb{E}[s_{ij}\theta_i] = \sum_{\ell \in N} m_{i\ell} \mathbb{E}[s_{ij} \cdot \boldsymbol{\beta}_\ell \mathbf{s}_\ell].$$

Using that all random variables have zero mean:

$$\forall i \in N, \quad cov(s_{ij}, \theta_i) = \sum_{\ell \in N} m_{i\ell} cov(s_{ij}, \boldsymbol{\beta}_\ell \mathbf{s}_\ell).$$

Repeating the argument for all $j \in J$, we get all equations in (75). Repeating the argument using $\boldsymbol{\beta}_\ell \cdot \mathbf{s}_\ell$ we get all equations in (76). Hence, we prove sufficiency.

(If) Suppose that there exists $(m_{i\ell})_{i,\ell \in N^2} \in \mathbb{R}^{N^2}$ such that (75) and (76) are satisfied. Now, define random variable:

$$z \triangleq \sum_{\ell \in N} m_{i\ell} \cdot \boldsymbol{\beta}_\ell \mathbf{s}_\ell.$$

First, note that z is measurable with respect to $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$ and hence it is also measurable with respect to $(\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$. Second note that for all $i \in N$:

$$\forall j \in J, \quad cov(\mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] - z, s_{ij}) = 0; \tag{86}$$

$$\forall \ell \in N, \quad cov(\mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] - z, \boldsymbol{\beta}_\ell \cdot \mathbf{s}_\ell) = 0. \tag{87}$$

To check (86) note the following.

$$\begin{aligned} cov(\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] - z, s_{ij}) &= \mathbb{E}[s_{ij}\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]] - \mathbb{E}[s_{ij} z] \quad (88) \\ &= \mathbb{E}[s_{ij}\theta_i] - \mathbb{E}[s_{ij} \sum_{\ell \in N} m_{i\ell} \cdot \boldsymbol{\beta}_\ell \mathbf{s}_\ell] \quad (89) \\ &= cov(s_{ij}, \theta_i) - \sum_{\ell \in N} m_{i\ell} \cdot cov(s_{ij}, \boldsymbol{\beta}_\ell \mathbf{s}_\ell) \quad (90) \\ &= 0 \quad (91) \end{aligned}$$

(88) is from the fact that all variables have a mean of 0 and hence the covariance is the same as the expectation of the product of the random variables. (89) is bringing s_{ij} inside the expectation and using the law of iterated expectations. (90) is the using that all variables have a mean of 0. Finally, (87) can be proved in an analogous way.

Using (86) we must have that:

$$z = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]. \quad (92)$$

Using that z is also measurable with respect to $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$:

$$\begin{aligned} z &= \mathbb{E}[z|\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] \\ &= \mathbb{E}[\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]|\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]] \\ &= \mathbb{E}[\theta_i|\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]. \end{aligned}$$

The first equation comes from the fact that z is measurable with respect to $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$, and hence z conditional on $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N)$ is equal to z. The second equation is using (92). The third equation is using the law of iterated expectations. Hence, using again (92):

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]$$

Hence, we prove necessity. Hence, we prove the result ■

Proof Proposition 8 (Only If) Let $(\boldsymbol{\beta} \cdot s_1, ..., \boldsymbol{\beta} \cdot s_N) \in \mathbb{R}^N$ be an equilibrium statistic. Using Lemma 10 we have that:

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{s}].$$

By construction of the expectation, we have that:

$$cov(\theta_i - \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}], \mathbf{s}_i) = 0.$$

Using the property of the equilibrium statistic:

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}].$$

Hence,

$$cov(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}], \mathbf{s}_i) = 0.$$

Using Lemma 9, we know that variables with a bar are orthogonal to variables preceded by a Δ and hence:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i] + \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \bar{\mathbf{s}}].$$

Hence,

$$cov(\theta_i - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta} \cdot \bar{\mathbf{s}}] - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i], \mathbf{s}_i) = 0.$$

Hence we prove sufficiency.

(If) Let $\boldsymbol{\beta}$ be such that:

$$cov(\theta_i - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta} \cdot \bar{\mathbf{s}}] - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i], \mathbf{s}_i) = 0.$$

Hence, we have that:

$$cov(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}], \mathbf{s}_i) = 0.$$

This implies that:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}]$$

Hence, we prove necessity. Hence, we prove the result.■

Proof Proposition 9 We look for equilibrium statistic that satisfy that $\tilde{m} = 1$. We define the matrix:

$$M = \Big((1 - (N-1)m)var(\bar{\mathbf{s}}) + (1+m)var(\boldsymbol{\Delta}\mathbf{s}_i)\Big).$$

We note that:

$$M^{-1} = \frac{1}{det(M)} M^A,$$

where M^A is the adjoint matrix of M. M is a square matrix of dimension J. It is easy to see that, the determinant of M is a polynomial of degree J in variable m. Each element of M^A is the determinant of a minor of M. Hence, each element of M^A has degree $J-1$ on variable m. Finally, we define:

$$\mathbf{v} \triangleq cov(\mathbf{s}_i, \theta_i) \cdot M^A,$$

and note that $\boldsymbol{\beta} = \mathbf{v}/det(M)$. Note that each element of $\mathbf{v}$ has degree $J-1$ on variable m.

We can solve for equilibria by replacing $\boldsymbol{\beta}$ in (80), we get:

$$\begin{aligned}
&m(((N-1)^2\mathbf{v} \cdot var(\bar{\mathbf{s}}) \cdot (\mathbf{v})^T + \mathbf{v} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \mathbf{v})) = \\
&\Bigg(det(M) \cdot \mathbf{v} \cdot ((N-1)cov(\bar{\theta}, \bar{\mathbf{s}}) - cov(\Delta\theta_i, \boldsymbol{\Delta}\mathbf{s}_i)) \\
&-(N-1)(\mathbf{v}) \cdot var(\bar{\mathbf{s}}) \cdot (\mathbf{v})^T + (\mathbf{v}) \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot (\mathbf{v})^T \Bigg).
\end{aligned}$$

The terms $((N-1)^2\mathbf{v} \cdot var(\bar{\mathbf{s}}) \cdot (\mathbf{v})^T$ have order $2J-2$ in m while $det(M) \cdot \mathbf{v}$ has order $2J-1$ in m. Hence, the whole polynomial is of order $2J-1$ in m. Hence, it has $2J-1$ solutions. Hence, there are at most $2J-1$ different m.

For a fixed m solving $\boldsymbol{\beta}$ is a linear system of equations. Hence, there are at most $2J-1$ solutions. Hence, we prove the result.■

Proof Proposition 10 We proceed in two steps. We first prove that all symmetric equilibrium statistic $\boldsymbol{\beta} \in \mathbb{R}^J$ have the same sign in all of its components. That is, for all $j \in J$, $\beta_j \geq 0$ or for all $j \in J$, $\beta_j \leq 0$. To prove this, we use Lemma 8. The second step is to prove that, if components of $\boldsymbol{\beta} \in \mathbb{R}^J$ have the same sign, then the equilibrium statistic satisfies (5).

Step 1: We proceed by contradiction. That is, we assume that there exists $j, k \in \mathbb{R}$ such that $\beta_k \cdot \beta_j < 0$. We proceed in sub-cases. Denote by $\boldsymbol{\beta}$ a vector that is the same as $\boldsymbol{\beta}$, but with all components having a positive sign.

Since the noise terms are independently distributed, it is clear to see that:

$$var(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}|\bar{\theta}) = var(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}|\bar{\theta}) \text{ and } var(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}|\Delta\theta_i) = var(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}|\Delta\theta_i).$$

Additionally, by the monotonicity condition, for all s_{ij}, $cov(\Delta\theta_i, \Delta s_{ij}), cov(\bar{\theta}, \bar{s}_j) \geq 0$. Hence,

$$|cov(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}, \bar{\theta})| > |cov(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}, \bar{\theta})| \text{ and } |cov(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}, \Delta\theta_i)| > |cov(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}, \Delta\theta_i)|.$$

Hence,

$$|corr(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}, \bar{\theta})| > |corr(\boldsymbol{\beta} \cdot \bar{\mathbf{s}}, \bar{\theta})| \text{ and } |corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}, \Delta\theta_i)| > |corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s_i}, \Delta\theta_i)|.$$

We now show that:

$$cov(\theta_i - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i] - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta} \cdot \bar{\mathbf{s}}], \boldsymbol{\beta} \cdot \mathbf{s}_i) > 0. \tag{93}$$

If we prove this, then this would be a contradiction with Proposition 8, as (82) cannot hold for the linear combination of signals given by $\boldsymbol{\beta}$. Hence, $\boldsymbol{\beta} \cdot \mathbf{s}_i$ is not an equilibrium statistic.

To prove (93), note that:

$$cov(\theta_i - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i] - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta} \cdot \bar{\mathbf{s}}], \boldsymbol{\beta} \cdot \mathbf{s}_i) = cov(\Delta\theta_i - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i], \boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i) + cov(\bar{\theta} - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta}\cdot\bar{\mathbf{s}}], \boldsymbol{\beta}\cdot\bar{\mathbf{s}}). \tag{94}$$

Yet, calculating the terms, we get:

$$\begin{aligned}
cov(\Delta\theta_i - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i], \boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i) &= \sqrt{var(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i)var(\Delta\theta_i)}\Big(|corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \Delta\theta_i)| - && (95)\\
& \quad |corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \Delta\theta_i)||corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i)|\Big) && (96)\\
&> \sqrt{var(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i)var(\Delta\theta_i)}\Big(|corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \Delta\theta_i)| - && (97)\\
& \quad |corr(\boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i, \Delta\theta_i)|\Big) && (98)\\
&> 0 && (99)
\end{aligned}$$

The same holds for the common variables, hence (93) must be satisfied.

Step 2: For any equilibrium statistic $\boldsymbol{\beta}$ all components are positive, and by the monotonicity condition:

$$cov(\Delta\theta_i, \boldsymbol{\beta} \cdot \boldsymbol{\Delta}\mathbf{s}_i) > 0 \text{ and } cov(\bar{\theta}, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}) > 0.$$

Hence, clearly $m \geq 0$. ■

Proof Lemma 9 Consider normal random variables $\{y_i\}_{i\in N}, \{z_i\}_{i\in N}$ symmetrically distributed. That is:

$$\forall i,k \in \{1,...,N\}, \quad \begin{pmatrix} y_i \\ y_k \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{yy}\sigma_y^2 \\ \rho_{yy}\sigma_\theta^2 & \sigma_y^2 \end{pmatrix} \right) \text{ and } \begin{pmatrix} z_i \\ z_k \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_z^2 & \rho_{zz}\sigma_z^2 \\ \rho_{zz}\sigma_z^2 & \sigma_z^2 \end{pmatrix} \right),$$

with an arbitrary correlation $corr(y_i, z_i)$. Define:

$$\bar{y} \triangleq \frac{1}{N}\sum_{i\in N} y_i \ ; \ \Delta y_i \triangleq y_i - \bar{y} \ ; \ \bar{z} \triangleq \frac{1}{N}\sum_{i\in N} z_i \ ; \ \Delta z_i \triangleq z_i - \bar{z}. \tag{100}$$

We prove that:

$$cov(\bar{y}, \Delta z_i) = 0. \tag{101}$$

We first provide the steps, and then explain each step.

$$\begin{aligned}
cov(\bar{y}, \Delta z_i) &= cov(\frac{1}{N}\sum_{k\in N} y_k, z_i - \frac{1}{N}\sum_{j\in N} z_j) && (102)\\
&= \frac{1}{N}\sum_{k\in N} cov(y_k, z_i) - \frac{1}{N^2}\sum_{k\in N}\sum_{j\in N} cov(y_k, z_j) && (103)\\
&= \frac{1}{N}(cov(y_i, z_i) + \sum_{k\neq i} cov(y_k, z_i)) - \frac{1}{N^2}\sum_{k\in N}(cov(y_k, z_k) + \sum_{j\neq k} cov(y_k, z_j)) && (104)\\
&= \frac{1}{N}(cov(y_i, z_i) + (N-1)cov(y_k, z_i)) - \frac{1}{N}(cov(y_k, z_k) + (N-1)cov(y_k, z_j)) && (105)\\
&= 0 && (106)
\end{aligned}$$

The explanation of each step is as follows. (102) is by construction of $\bar{y}$ and Δz_i. (103) is by the collinearity of the covariance. (104) is expanding terms. (105) is using symmetry. More specifically,

$$\forall i, k, \ell, j \in N, \quad \text{with } j \neq i \text{ and } k \neq \ell, \quad cov(z_i, y_j) = cov(z_k, y_\ell);$$

$$\forall i \in N, \quad cov(z_i, y_i) = cov(z_k, y_k).$$

(106) is trivially by checking both terms are the same.

Note that, it is clear from the proof that $cov(\bar{y}, \Delta y_i) = 0$ must also be satisfied. Since all random variables are Gaussian, if they have 0 covariance, they must be independent. Hence, we prove the result. ■

Proof Lemma 10 For any $\boldsymbol{\beta} \in \mathbb{R}^J$, there exists constants $m_{i\ell}$, c_{ij} and $\tilde{m}_{i\ell}$ such that the expectation can be written as follows:

$$\mathbb{E}[\theta_i | \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in N} m_{ij}\boldsymbol{\beta}\cdot\mathbf{s}_j;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_{ij}s_{ij} + \sum_{j\neq i} \tilde{m}_{ij}\boldsymbol{\beta}\cdot\mathbf{s}_j.$$

By symmetry, we can find $m, m', \tilde{m}, c_j$ such that:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = m\sum_{j\neq i} \boldsymbol{\beta}\cdot\mathbf{s}_j + m'\cdot\boldsymbol{\beta}\cdot\mathbf{s}_i;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_j s_{ij} + \sum_{j\neq i} \tilde{m}\cdot\boldsymbol{\beta}\cdot\mathbf{s}_j.$$

We can re-write both equations as follows:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = Nm\bar{s} - m\cdot s_i + m'\cdot\boldsymbol{\beta}\cdot\mathbf{s}_i;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_j s_{ij} + N\tilde{m}\cdot\boldsymbol{\beta}\cdot\bar{\mathbf{s}} - \tilde{m}s_i.$$

Hence, we have that in symmetric environments $\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N]$ is measurable with respect to $(\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}})$ and $\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N]$ is measurable with respect to $(\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}})$. That is:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}];$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}].$$

Hence, $(\boldsymbol{\beta}\cdot\mathbf{s}_1..., \boldsymbol{\beta}\cdot\mathbf{s}_N)$ is an equilibrium statistic if and only if

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}].$$

Hence, we prove the result. ■

Proof Lemma 11 The proof is direct from (77) and the definition of average crossing condition, it just corresponds to replacing the derivatives with the parameters $\{m_{i\ell}\}_{i,\ell\in N^2}$. ■

Proof Corollary 6 Let $(\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N)$ be an equilibrium statistic. We can then write Proposition 7 using the assumption of symmetry. We denote for all $i \neq j$, $m = m_{ij}$ and $\tilde{m} = m_{ii}$. We can write (75) for a typical $i \in N$ and (76) for some $\ell \neq i$. We can write these equations as follows:

$$cov(\theta_i, \mathbf{s}_i) - \tilde{m}\cdot cov(\boldsymbol{\beta}\cdot\mathbf{s}_i, \mathbf{s}_i) - (N-1)m\cdot cov(\boldsymbol{\beta}\cdot\mathbf{s}_\ell, \mathbf{s}_i) = 0, \tag{107}$$

$$cov(\theta_i, \boldsymbol{\beta}\cdot\mathbf{s}_\ell) - ((N-2)m + \tilde{m})\cdot cov(\boldsymbol{\beta}\cdot\mathbf{s}_j, \boldsymbol{\beta}\cdot\mathbf{s}_\ell) - m\cdot cov(\boldsymbol{\beta}\cdot\mathbf{s}_\ell, \boldsymbol{\beta}\cdot\mathbf{s}_\ell) = 0, \tag{108}$$

We now rewrite the covariances as follows:

$$cov(\mathbf{s}_i, \mathbf{s}_i) = var(\mathbf{s}_i) = var(\bar{\mathbf{s}}) + var(\boldsymbol{\Delta}\mathbf{s}_i);$$

$$cov(\mathbf{s}_i, \mathbf{s}_\ell) = cov(\bar{\mathbf{s}} + \boldsymbol{\Delta}\mathbf{s}_i, \bar{\mathbf{s}} + \boldsymbol{\Delta}\mathbf{s}_\ell) = var(\bar{\mathbf{s}}) + cov(\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\Delta}\mathbf{s}_\ell).$$

Note that by construction $\sum_{i \in N} \boldsymbol{\Delta}\mathbf{s}_i = 0$, and hence:

$$0 = cov(\Delta s_\ell, \sum_{i \in N} \boldsymbol{\Delta}\mathbf{s}_i) = var(\boldsymbol{\Delta}\mathbf{s}_\ell) + (N-1)cov(\boldsymbol{\Delta}\mathbf{s}_\ell, \boldsymbol{\Delta}\mathbf{s}_i),$$

with $i \neq \ell$. Hence,

$$cov(\mathbf{s}_i, \mathbf{s}_\ell) = var(\bar{\mathbf{s}}) - \frac{1}{N-1} var(\boldsymbol{\Delta}\mathbf{s}_i).$$

We can write (107) in vector form as follows:

$$cov(\theta_i, \mathbf{s}_i) - \tilde{m} \cdot \boldsymbol{\beta} \cdot (var(\bar{\mathbf{s}}) + var(\boldsymbol{\Delta}\mathbf{s}_i)) - (N-1)m \cdot \boldsymbol{\beta} \cdot (var(\bar{\mathbf{s}}) - \frac{1}{N-1} var(\boldsymbol{\Delta}\mathbf{s}_i)) = 0$$

Hence, (107) can be written as follows:

$$\boldsymbol{\beta} = \Big((\tilde{m} + (N-1)m) var(\bar{\mathbf{s}}) + (\tilde{m} - m) var(\boldsymbol{\Delta}\mathbf{s}_i) \Big)^{-1} \cdot cov(\theta_i, \mathbf{s}_i) \tag{109}$$

On the other hand, we can write (108) as follows:

$$m = \frac{\Big(\boldsymbol{\beta} \cdot ((N-1)cov(\bar{\theta}, \bar{\mathbf{s}}) - cov(\Delta\theta_i, \boldsymbol{\Delta}\mathbf{s}_i)) - \tilde{m} \cdot (N-1)\boldsymbol{\beta} \cdot var(\bar{\mathbf{s}}) \cdot \boldsymbol{\beta} + \tilde{m} \cdot \boldsymbol{\beta} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \boldsymbol{\beta} \Big)}{((N-1)^2 \boldsymbol{\beta} \cdot var(\bar{\mathbf{s}}) \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \boldsymbol{\beta})} \tag{110}$$

We make the following change of variables $\tilde{m}' = 1/\tilde{m}$ and $m' = m/\tilde{m}$. (109) and (110) can be written as follows:

$$\boldsymbol{\beta} = \tilde{m}' \Big((1 + (N-1)m') var(\bar{\mathbf{s}}) + (1 - m') var(\boldsymbol{\Delta}\mathbf{s}_i) \Big)^{-1} \cdot cov(\theta_i, \mathbf{s}_i) \tag{111}$$

$$\frac{m'}{\tilde{m}'} = \frac{\Big(\boldsymbol{\beta} \cdot ((N-1)cov(\bar{\theta}, \bar{\mathbf{s}}) - cov(\Delta\theta_i, \boldsymbol{\Delta}\mathbf{s}_i)) - \frac{1}{\tilde{m}'} \cdot (N-1)\boldsymbol{\beta} \cdot var(\bar{\mathbf{s}}) \cdot \boldsymbol{\beta} + \frac{1}{\tilde{m}'} \cdot \boldsymbol{\beta} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \boldsymbol{\beta} \Big)}{((N-1)^2 \boldsymbol{\beta} \cdot var(\bar{\mathbf{s}}) \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \boldsymbol{\beta})} \tag{112}$$

We can rewrite (112) as follows:

$$m' = \frac{\left(\frac{\boldsymbol{\beta}}{\tilde{m}'} \cdot ((N-1)cov(\bar{\theta}, \bar{\mathbf{s}}) - cov(\Delta\theta_i, \boldsymbol{\Delta}\mathbf{s}_i)) - (N-1)\frac{\boldsymbol{\beta}}{\tilde{m}'} \cdot var(\bar{\mathbf{s}}) \cdot \frac{\boldsymbol{\beta}}{\tilde{m}'} + \cdot \frac{\boldsymbol{\beta}}{\tilde{m}'} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \frac{\boldsymbol{\beta}}{\tilde{m}'} \right)}{((N-1)^2 \frac{\boldsymbol{\beta}}{\tilde{m}'} \cdot var(\bar{\mathbf{s}}) \cdot \frac{\boldsymbol{\beta}}{\tilde{m}'} + \frac{\boldsymbol{\beta}}{\tilde{m}'} \cdot var(\boldsymbol{\Delta}\mathbf{s}_i) \cdot \frac{\boldsymbol{\beta}}{\tilde{m}'})} \tag{113}$$

Hence, we prove the result.■

Proof Corollary 7 By Proposition 5.6 and equilibrium statistic exists. By Proposition 5.6 all equilibrium statistic satisfy (5). It is easy to check that this implies that the average crossing condition is satisfied. By Theorem 3, this implies that an equilibrium exists.

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