Directed search with phantom vacancies*

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Abstract

When vacancies are filled, the ads that were posted are generally not withdrawn, creating phantom vacancies. The existence of phantoms implies that older job listings are less likely to represent true vacancies than are younger ones. We assume that job seekers direct their search based on the age of the job listing to equalize the probability of matching across listing age. Forming a match with a vacancy of age a creates a phantom of age a and thus creates a negative informational externality that affects all vacancies of age a or older. The magnitude of this externality decreases with a. The directed search behavior of job seekers leads them to over-apply to younger listings. We calibrate the model using US labor market data. The contribution of phantoms to overall frictions is large, but, conditional on the existence of phantoms, the social planner cannot improve much on the directed search allocation.

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1 Introduction

> I am currently on the job hunt and I had a question about applying to jobs online. You know how most websites will tell you the job has been posted 1 day ago, 28 days ago, etc. For some reason, I have concluded that I need to apply to a job the first week they post the position to have the best chances of being hired. Although I heard that it can take up to a month for the company to hire anyone for the position, I feel that applying to a job that was posted 3 weeks ago isn't that promising. What is your take on this situation?
>
> AskaManager.com

This quote illustrates two important points about job search. First, some of the information available to job seekers is out of date. Some advertised jobs have already been filled. These ads are for job openings that no longer exist; they are ads for phantom vacancies. Second, job seekers know that some advertised jobs are phantoms, and they adjust their application behavior accordingly. As jobs advertised in older listings are more likely to be phantoms, workers may decide to apply to more recently posted listings, but at the same time responding to older ads is also an option since fewer applicants are likely to be pursuing those jobs. Workers thus face a problem of directed search, namely, how to allocate their applications optimally across job listings of different ages.

Are phantoms a significant problem in real-world labor markets? Do matches fail to form because workers can't locate appropriate job listings or because the jobs that workers apply for are already taken? To the extent workers apply for jobs that are already filled, and casual empiricism suggests this is often the case, phantoms are important.[1] More formally, there is evidence from online job sites that phantoms are important. Using Craigslist data, Chéron and Decreuse (2016) show that the distribution of job listings by age over one month (the time at which Craigslist destroys ads) is uniform by week. This implies that ads are not withdrawn as soon as the corresponding jobs are filled; instead, job listings persist for some time as phantoms.[2]

In this paper, we build a model of directed search with phantoms and explore their im-

[1] Of course, phantoms are not solely a feature of online job markets. Word-of-mouth information can also be obsolete. Phantoms are also important in other markets with search frictions, for example, the market for rental housing, Airbnb (reference), etc.

[2] An alternative interpretation is that ads that are not withdrawn are postings by employers with multiple positions to fill. In principle, if not necessarily in practice, Craigslist does not allow ads for multiple positions. (See their FAQ section.) Also, in the Craiglist data, close to 60% of new listings are for jobs that were not advertised in the previous month. Among the remaining 40%, many ads are simply reposted because the previous posting did not generate a match.

plications for equilibrium and efficiency. In equilibrium, worker directed search satisfies an indifference condition – the expected payoffs associated with applying to ads of different ages must be equalized. However, even though workers direct their search optimally, we show that equilibrium is not constrained efficient. From the collective perspective, workers over-apply to younger listings. When a match forms, a phantom is added to the market, and the cost of the phantom is greater the younger is the ad that was posted for the job. That is, the social cost of a young phantom is greater than that of an old phantom. To see this, consider a match formed with a vacancy that was posted one month ago. This generates a one-month old phantom. This phantom can only affect the job seekers who pursue listings that are older than one month. However, a match formed with a newly posted vacancy generates a phantom that can affect all job seekers during its lifetime.

We calibrate our model to US data for the period 2000 - 2008. We show that the gains achieved by the constrained efficient allocation are modest. This is because, like job seekers, the social planner cannot distinguish between real vacancies and phantoms, and search strategies that produce more matches also produce more phantoms. Nonetheless, we find that the contribution of phantoms to overall labor market frictions and unemployment is large. In our baseline calibration, phantoms account for over 70% of frictional unemployment and more than one third of overall unemployment. This reflects the fact that applying for jobs that have already been filled is a quantitatively important driver of unemployment.

We also consider three extensions to our model. We first examine what happens when job listings are renewed after a limit age or at random intervals of time. Vacancy renewal means that a listing for an unfilled vacancy is replaced by a new one of age zero. This concentrates the distribution of vacancies at younger ages, and job seekers react by applying to even younger listings as compared with the allocation without renewal. The qualitative conclusion stays the same: directed search is not constrained efficient.

Second, we consider a complementary reason for why job seekers are concerned with the age of job listings. Some jobs may be lemons, i.e., jobs that no one wants to accept. Because no one accepts these jobs, they stay in the distribution of job postings until they naturally disappear. The proportion of job listings that are lemons increases with age, and this composition effect pushes job seekers to direct their search towards younger listings. Lemons generate an additional inefficiency: job seekers who direct their search towards younger listings increase the proportion of lemons in the stock of job listings. However, a key difference between lemons and phantoms is that matching does not produce lemons. As a result, the quantitative contribution of lemons to overall unemployment is small unless the fraction of lemons in new listings is very large.

Finally, we note that fixed-wage contracts cannot internalize the age-dependent informational externality. One could imagine firms posting sophisticated contracts advertising a wage that varies with the length of time it takes to fill the vacancy. Specifically, we consider Nash bargaining over the wage. The value of a vacancy falls with age because workers are less likely to apply for older vacancies. Thus the match surplus increases with vacancy age, and the wage does the same. Workers then have a stronger incentive to apply for older jobs even though the job-finding rate falls with age. In practice, of course, we do not observe age-dependent wages. Allowing wages to vary with vacancy age is simple a device we use to get a better uderstanding of the externality caused by phantoms.

Our paper is related to four strands of literature. First, the paper closest to ours is Chéron and Decreuse (2016). Their paper introduces the concept of phantom vacancies and shows how phantoms lead to an aggregate matching function even in the absence of other coordination frictions in the market. The authors then embed this matching function in the Diamond-Mortensen-Pissarides equilibrium search model and examine how the dynamics associated with phantom creation and destruction affect the business cycle properties of that model. One thing they do not do, however, is allow workers to direct their search by job listing age. Our paper is thus a natural complement to Chéron and Decreuse (2016).

Second, there is a literature on inefficiencies in search equilibrium that come from composition externalities, e.g., Albrecht, Navarro and Vroman (2010) and Chéron, Hairault and Langot (2011). These are models with worker heterogeneity in which individual decisions (to participate in the labor market, to form or dissolve matches) affect the distribution of worker types across the pool of unemployment. A related compositional effect obtains in our model. A worker who successfully targets a job listing of a particular age changes the distributions by age of both vacancies and phantoms but does not take the effect of these changes on other job seekers into account. Our paper differs from earlier work on composition externalities in that individual job search decisions have dynamic effects. A worker who directs his or her search towards ads of age a affects not only the composition of listings (real vacancies versus phantoms) at that age but also, with a lag, the composition of vacancies at all ages greater than a. In this sense, ours is a dynamic compositional externality.

Third, our paper is related to the literature on stock-flow matching, e.g., Coles and Smith (1998), in which job seekers initially search through the entire stock of vacancies but thereafter only look at the flow of new vacancies. The connection is that both our model and the stock-flow model predict that younger vacancies have a better chance of being filled quickly than do older vacancies, albeit for different reasons.[3]

[3]Using data from SnagAJob, Faberman and Kudlyak (2014) examine how worker application behavior varies with elapsed duration of unemployment. They find (p. xx) that “The fraction of applicants to a newly-posted vacancy rises with duration, consistent with a stock-flow model, but it does so only slightly, representing only

Finally, there is a substantial literature based on data culled from job search engines. Examples include Faberman and Kudlyak (2014) using data from SnagAJob, Marinescu (2015) using data from CareerBuilder, and Banfi and Villena Roldán (2016) using data from trabajando.com The main focus of these papers is on documenting empirical regularities in these new data sources. Our paper provides a complementary theoretical framework.

The outline of the rest of our paper is as follows. In the next section, we lay out our model of a labor market with phantoms. We then solve for the equilibrium directed search allocation, the random search allocation and the constrained efficient allocation in Section 3. We show that neither the directed search allocation nor the random search allocation is constrained efficient. In Section 4, we present our baseline calibration. Section 5 is devoted to our three extensions and Section 6 concludes.

2 The Model

We focus on the stationary state of a continuous time model. Calendar time is denoted by t. At each time, there is a fixed measure of jobs K that can be either vacant or occupied.[4] The measure of vacancies is $\mathsf{v}(t)$. They differ in age $a \geq 0$. There is also a continuum of workers of size one. Each worker can be either unemployed or employed. The measure of unemployed is $\mathsf{u}(t)$. By construction, we have $\mathsf{v}(t) + 1 - \mathsf{u}(t) = K$.

Employed workers and jobs separate at exogenous Poisson rate λ. Newly unemployed workers join the pool of unemployed and immediately start job search. Newly destroyed jobs join the pool of vacancies at age zero.

All jobs produce the flow output $y \equiv 1$ and pay the same wage w. The search market is segmented by listing age a. At time t, in each submarket a, $u(a,t)$ unemployed workers try to match with $v(a,t)$ vacancies. The matching process is frictional. On top of the usual search frictions, information persistence about former vacancies creates an additional friction. Each time a match is formed and the corresponding ad is not withdrawn, a phantom is created. The flow of new matches in submarket a is

$$M(a,t) = \pi(a,t) m(u(a,t), v(a,t) + p(a,t)), \tag{1}$$

where $p(a,t)$ is the measure of phantoms and $\pi(a,t) = \dfrac{v(a,t)}{v(a,t)+p(a,t)}$ is the ratio of vacancies to job listings (vacancies + phantoms) in submarket a (the "nonphantom proportion"). We

17% of applications during a job seeker's sixth month of search." This suggests that while stock-flow matching may take place, it cannot explain all of the patterns that we see in the data.

[4] We assume a fixed measure of jobs to abstract from the well-known congestion and thick-market externalities that result from vacancy creation in order to focus on the externalities associated with phantoms.

define $\theta(a,t) \equiv \dfrac{v(a,t)+p(a,t)}{u(a,t)}$ as the tightness in the submarket corresponding to ads of age a. The numerator is composed of the total measure of age-a ads, thus including both phantoms and vacancies.

The function m is strictly concave, has constant returns to scale, and is such that $m(0, v{+}p) = m(u, 0) = 0$, $\lim_{u\to 0} m_1(u, v+p) = \lim_{v+p\to 0} m_2(u, v+p) = \infty$. Its elasticity with respect to job listings is $\alpha(\theta) = \theta m_2(1,\theta)/m(1,\theta)$. We assume that $\alpha'(\theta) \leq 0$ for all $\theta \geq 0$. Although this assumption is not necessary for our results, it is a sufficient condition to establish particular results and we indicate this where relevant. The flow of new matches has two components: the contact rate m multiplied by π, the proportion of contacts where a vacancy rather than a phantom is involved. Phantoms impede the search process: by increasing p at given v and u, the flow of new matches, M, is reduced.

Unemployed workers spread themselves over the different submarkets with total unemployment at time t of $\mathsf{u}(t) = \int_0^\infty u(a,t)da$. Similarly, the total measure of vacancies at time t is $\mathsf{v}(t) = \int_0^\infty v(a,t)da$ and the total measure of phantoms is $\mathsf{p}(t) = \int_0^\infty p(a,t)da$. Unemployment obeys the law of motion:

$$d\mathsf{u}(t)/dt = -\int_0^\infty M(a,t)da + \lambda(1-\mathsf{u}), \tag{2}$$

and the stocks of vacancies and phantoms evolve according to

$$\partial v(a,t)/\partial a + \partial v(a,t)/\partial t = -M(a,t), \tag{3}$$

$$\partial p(a,t)/\partial a + \partial p(a,t)/\partial t = M(a,t) - \delta p(a,t), \tag{4}$$

with $\delta > 0$, $v(0,t) = \lambda(1-\mathsf{u}(t))$, and $p(0,t) = 0$. The measure of vacancies decreases across age by the measure of new matches, while phantoms depreciate at rate δ.

The law of motion for vacancies assumes that vacancies cannot be renewed or refreshed. That is, a job that became available at date t and which is still available at date $t+a$ is identified as having age a, and nothing can be done to signal that this vacancy is still active rather than a phantom. Section 5 analyzes the case where vacancies can be renewed, i.e., relisted as new vacancies.

The vacancy proportion of ads, $\pi(a,t)$, evolves according to

$$\partial \pi(a,t)/\partial a + \partial \pi(a,t)/\partial t = \pi(a,t)(-\frac{M(a,t)}{v(a,t)} + \delta(1-\pi(a,t))), \tag{5}$$

with $\pi(0,t) = 1$.

In steady state, calendar time does not affect any of the variables. Thus $d\mathsf{u}/dt = d\mathsf{v}/dt = 0$, $\partial v(a,t)/\partial t = \partial p(a,t)/\partial t = 0$, $u(a,t) = u(a)$, and $M(a,t) = M(a)$. Hereafter we refer to variables without mentioning calendar time again. A dot over a variable denotes the derivative of the variable with respect to age a, i.e., $\dot{x} \equiv x'(a)$.

For later use, we define the job-finding rate by listing age as $\mu(a) = M(a)/u(a)$, and the job-filling rate by listing age as $\eta(a) = M(a)/v(a)$. Finally, ϕ_v, ϕ_{v+p}, and ϕ_u denote the density functions of vacancies, total job listings (vacancies + phantoms) and unemployment by age. By definition, we have $\phi_v(a) = v(a)/\mathsf{v}$, $\phi_{v+p}(a) = (v(a) + p(a))/(\mathsf{v} + \mathsf{p})$ and $\phi_u(a) = u(a)/\mathsf{u}$ for all $a \geq 0$.

To close the model, we need to define how the job seekers allocate themselves across the different submarkets. This will differ depending on whether search is random or directed by listing age.

3 Random search, directed search, and constrained-efficient allocations

We describe three different allocations. Each one is associated with a particular distribution of job seekers over listing ages.

3.1 Random search allocation

Before we derive the directed search allocation, we start, as a baseline, with the allocation obtained when workers do not observe the job listing age. Thus search is random and the ratio of vacancies and phantoms to job seekers is constant over age.

For all $a \geq 0$,

$$\theta(a) = (v(a) + p(a))/u(a) = \theta. \tag{6}$$

Vacancies and phantoms evolve according to

$$\begin{aligned} \dot{v} &= -v\frac{m(1,\theta)}{\theta}, \\ \dot{p} &= v\frac{m(1,\theta)}{\theta} - \delta p, \end{aligned}$$

with $v(0) = v_0 = \lambda(1 - \mathsf{u})$ and $p(0) = 0$. This gives two ordinary differential equations in v and p.

The solution is

$$v(a) = v_0 \exp(-am(1,\theta)/\theta), \tag{7}$$

$$p(a) = \sigma v_0 \left[\exp\left(-\delta a\right) - \exp(-am(1,\theta)/\theta)\right], \tag{8}$$

with $\sigma = \frac{m(1,\theta)}{\theta}\left[\frac{m(1,\theta)}{\theta} - \delta\right]^{-1}$. Using the random search property (6), the unemployment rate by listing age is

$$u(a) = \frac{v_0}{\theta}\left[\sigma \exp\left(-\delta a\right) + (1-\sigma)\exp\left(-am(1,\theta)/\theta\right)\right]. \tag{9}$$

The overall unemployment rate is $\mathsf{u} = \int_0^\infty u(a)da$. Using $v_0 = \lambda(1-\mathsf{u})$, we obtain

$$\mathsf{u} = \frac{\lambda + (\lambda/\delta)m(1,\theta)/\theta}{m(1,\theta) + \lambda + (\lambda/\delta)m(1,\theta)/\theta}. \tag{10}$$

Equation (10) is the Beveridge curve. It highlights the role played by phantoms. Holding tightness constant, unemployment decreases with δ, the rate at which phantoms depreciate. Phantoms create a negative informational externality that reduces the efficiency of the matching process. As this externality is entirely due to job creation, its magnitude increases with the rate at which vacancies are filled $m(1,\theta)/\theta$. Thus it decreases with tightness.

To close the model, we use the resource constraint $1 - \mathsf{u} + \mathsf{v} = K$.

Proposition 1 (Random search allocation) There is a unique level of labor market tightness, $\theta^{\text{rs}} > 0$, which is implicitly defined by

$$\frac{\lambda\theta + m(1,\theta)}{m(1,\theta) + \lambda + (\lambda/\delta)m(1,\theta)/\theta} = K. \tag{11}$$

In the random search allocation, the following properties hold for all $a \in \mathbb{R}_+$:

(i) $\theta(a) = \theta^{\text{rs}}$;

(ii) the law of motion of the nonphantom proportion is

$$\begin{aligned} \frac{\dot{\pi}}{\pi} &= -\frac{m(1,\theta^{\text{rs}})}{\theta^{\text{rs}}} + \delta(1-\pi) < 0, \\ \pi(0) &= 1; \end{aligned} \tag{12}$$

(iii) the job-filling rate $\eta(a)$ is constant, whereas the nonphantom proportion $\pi(a)$ and the job-finding rate $\mu(a)$ strictly decrease.

The proof is given in Appendix A.

In the random search allocation, the job-filling rate is $\eta(a) = m(1,\theta)/\theta$ for all $a \geq 0$. It is constant over listing age. That is, the chance of a vacancy being filled does not change with age. However, the job-finding rate does vary with listing age. The job-finding rate is $\mu(a) = m(1,\theta)\pi(a)$, and the nonphantom proportion $\pi(a)$ decreases with listing age.

In the random search allocation, tightness decreases with the phantom death rate, i.e., $d\theta^{\text{rs}}/d\delta < 0$. This depreciation rate governs the quantitative impact of the negative informational externality that affects the matching process. When δ decreases, the phantom stock increases, all else equal. Thus the ratio of vacancies plus phantoms to unemployment goes up.

The corollary is that unemployment increases when δ decreases. To see this, we write the resource constraint as:

$$u = 1 - K/(1 + \lambda\theta/m(1, \theta)).$$

The phantom death rate δ only affects the right-hand side through changes in θ. A decrease in δ implies that $m(1, \theta^{\text{rs}})/\theta^{\text{rs}}$ goes down and so unemployment must rise.

3.2 Directed search allocation

When search is directed, workers observe listing age and choose which submarket to enter. A submarket a is open if and only if $u(a) > 0$. We denote by Ω the set of open submarkets. All open submarkets must yield the same expected payoff. Thus the job-finding rate $\mu(a) = M(a)/u(a) = \pi(a)m(1, \theta(a))$ must be the same across open submarkets. Given that $v(a) > 0$ for all $a \geq 0$, the properties of the meeting function m imply that $u(a) > 0$ for all $a \geq 0$. If not, a worker entering a non-open submarket would immediately find a job. Thus all submarkets are open in equilibrium and so $\Omega = \mathbb{R}_+$.

Since $\pi(0) = 1$ the fact that $\mu(a)$ is constant across age implies for all $a \geq 0$ that

$$\pi(a)m(1, \theta(a)) = m(1, \theta(0)). \tag{13}$$

This equation defines a strictly decreasing relationship between $\theta(a)$ and $\pi(a)$; the relationship is parameterized by $\theta(0)$. Unlike the random search allocation, tightness is not constant in the directed search allocation. It strictly increases with $\theta(0)$ and strictly decreases with $\pi(a)$.

Differentiating (13) with respect to age gives:

$$-\alpha(\theta)\frac{\dot{\theta}}{\theta} = \frac{\dot{\pi}}{\pi}. \tag{14}$$

Using (5), (13) and (14), we can characterize the derivatives of π and θ as

$$\dot{\pi} = \pi(-\frac{m(1, \theta(0))}{\theta(a)} + \delta(1 - \pi)), \tag{15}$$

$$-\alpha(\theta)\dot{\theta} = (1 - \pi)\left(\delta\theta - m(1, \theta) - m(1, \theta(0)\right), \tag{16}$$

The two differential equations can be solved given $\pi(0) = 1$ and $\theta(0)$. Note, however, that the latter value can only be found once the equilibrium is solved.

The stationary number of unemployed balances inflows and outflows. That the exit rate from unemployment does not vary with listing age considerably simplifies the calculation. Indeed, outflows are $\int_0^\infty M(a)da = m(1, \theta(0))\mathsf{u}$. Thus $\mathsf{u} = \frac{\lambda}{\lambda + m(1, \theta(0))}$.

The number of vacancies by age follows. Since $\dot{v} = -vm(1,\theta)/\theta$, we have $v(a) = v(0) \times \exp\left[-\int_0^a \frac{m(1,\theta(b))}{\theta(b)} db\right]$. Finally, the resource constraint is

$$1 - \mathsf{u} + \int_0^\infty v(a)da = K. \tag{17}$$

Proposition 2 (Directed search allocation) Let $\theta^{\text{ds}} : (0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ be implicitly defined by $\theta^{\text{ds}}(\pi, \theta_0) = \theta$ solution of

$$\pi m(1,\theta) = m(1,\theta_0). \tag{18}$$

In a directed search allocation, the following properties hold for all $a \in \mathbb{R}_+$:

(i) $\theta(a) = \theta^{\text{ds}}(\pi(a), \theta(0))$, which implies that

$$\alpha(\theta)\frac{\dot{\theta}}{\theta} + \frac{\dot{\pi}}{\pi} = 0; \tag{19}$$

(ii) the motion of the vacancy proportion is

$$\begin{aligned} \frac{\dot{\pi}}{\pi} &= -\frac{m(1,\theta)}{\theta} + \delta(1-\pi) < 0, \\ \pi(0) &= 1; \end{aligned} \tag{20}$$

(iii) the job-filling rate $\eta(a)$ and the vacancy proportion $\pi(a)$ strictly decrease, tightness $\theta(a)$ strictly increases and the job-finding rate $\mu(a)$ is constant.

The proof is given in Appendix A.

The differential equation in part (i) describes tightness by listing age given initial condition $\theta(0)$. The resource constraint (17) then determines $\theta(0)$.

The job-filling rate is $\eta(a) = \dfrac{m(1,\theta(a))}{\theta(a)}$, which decreases with listing age, while the job-finding rate, $\mu(a) = m(1,\theta(0))$, is constant with respect to listing age. When there are no phantoms, the job-finding rate is also constant across age. In that case, the reason is simply that ratio of vacancies to unemployed is constant across age. Accounting for phantoms while allowing for directed search gives the same result but for a different reason. Workers who take into account obsolete information apply to younger listings. Thus tightness increases over age and the job-filling rate decreases with age.

Tightness does not increase without bounds. Let $\theta_\infty = \lim_{a\to\infty} \theta(a)$ be the asymptotic tightness. Setting $\dot{\theta} = \dot{\pi} = 0$ in equations (15) and (16) gives $(\delta - \eta_\infty)(m_\infty/m_0 - 1) = \eta_\infty$, where $m_\infty = m(1,\theta_\infty)$ and $\eta_\infty = m_\infty/\theta_\infty$. It follows that $\delta > \eta_\infty > 0$. The asymptotic vacancy proportion is $\pi_\infty = \frac{\delta - \eta_\infty}{\delta} = \frac{m_0}{m_\infty} \in (0,1)$. Thus θ monotonically increases to $\theta_\infty << \infty$ and π monotonically decreases to $\pi_\infty > 0$.

3.3 Social Planner Problem

The planner observes the age of job listings and decides on the allocation of the unemployed across the different submarkets. Like job seekers, the planner cannot distinguish between vacancies and phantoms.

Given our focus on steady-state allocations, we assume the discount rate is equal to zero. Thus the constrained efficient allocation minimizes the unemployment rate.

The constrained efficient allocation solves the following optimal control problem:

$$\max_{u(.)} -\int_0^\infty u(a)da \tag{$*$}$$

subject to

$$\begin{aligned}
\dot{v} &= -m(\pi(a)u(a), v(a)), && (c1)\\
v(0) &= \lambda\left(1-\int_0^\infty u(a)da\right), && (c2)\\
\dot{p} &= m(\pi(a)u(a), v(a)) - \delta p(a), && (c3)\\
p(0) &= 0, && (c4)\\
\lambda\left(1-\int_0^\infty u(a)da\right) &= \int_0^\infty m(\pi(a)u(a), v(a))da, && (c5)\\
1-\int_0^\infty u(a)da + \int_0^\infty v(a)da &= K. && (c6)
\end{aligned}$$

The planner is subject to the evolution of vacancies over age ($c1$)–($c2$), the evolution of phantoms over age ($c3$)–($c4$), the inflow-outflow constraint ($c5$), and the resource constraint ($c6$).

Proposition 3 (Efficient allocation) Let $\theta^{\text{eff}} : (0,1] \times [0,\infty) \to [0,\infty)$ be implicitly defined by

$$\theta^{\text{eff}}(\pi, \theta_0) = \theta = \frac{1-[1-\alpha(\theta)]\pi}{\alpha(\theta_0)\pi}\frac{1-\alpha(\theta_0)}{1-\alpha(\theta)}\theta_0. \tag{21}$$

In the efficient allocation, the following properties hold for all $a \in \mathbb{R}_+$:

(i) the optimal tightness is $\theta(a) = \theta^{\text{eff}}(\pi(a), \theta(0))$, which implies that

$$-\frac{\dot{\alpha}}{1-\alpha} + \alpha(\theta)\frac{\dot{\theta}}{\theta} + \frac{\dot{\pi}}{\pi} = -(1-\alpha)(1-\pi)\frac{\dot{\theta}}{\theta} < 0; \tag{22}$$

(ii) the law of motion of the vacancy proportion is

$$\begin{aligned}
\frac{\dot{\pi}}{\pi} &= -\frac{m(1,\theta)}{\theta} + \delta(1-\pi) < 0, && (23)\\
\pi(0) &= 1; && (24)
\end{aligned}$$

(iii) tightness $\theta(a)$ strictly increases, the vacancy proportion $\pi(a)$, the job-filling rate $\eta(a)$ and the job-finding rate $\mu(a)$ strictly decrease.

The proof is given in Appendix A.

Part (i) shows the policy function, i.e. the dependence of optimal tightness – the control – vis-à-vis the vacancy proportion – the key state variable. Optimal tightness decreases with the vacancy proportion. Like the directed search allocation, having few phantoms relative to vacancies attracts the job seekers. Note that the policy function θ^{eff} is implicitly defined. When the meeting technology is Cobb-Douglas, the elasticity α is constant and the policy function reduces to $\theta^{\text{eff}}(\pi, \theta_0) = \frac{1-(1-\alpha)\pi}{\alpha\pi}\theta_0$. Part (i) also displays the associated ordinary differential equation describing the motion of optimal tightness. This ODE turns out to be useful because it can easily be compared to the equivalent ODE in the directed search allocation (see below Proposition 4).

Part (ii) describes the motion of the vacancy proportion. When combined with the policy function, equation (23) implies that the vacancy proportion decreases with age. Therefore optimal tightness increases with age as in the directed search allocation. This is why part (iii) tells that the job-filling rate η falls with age. More interestingly, part (iii) also shows that the job-finding rate decreases with age. Unlike the other properties shown by part (iii), this property requires the assumption $\alpha'(\theta) \leq 0$. When $\alpha'(\theta) > 0$, we cannot exclude cases where μ increases with age.

Proposition 4 (Constrained efficiency) Let $\delta < \infty$; that is, suppose there are phantoms. Then

(i) The random search allocation is not constrained efficient.

(ii) The directed search allocation is generically constrained inefficient.

The proof is given in Appendix A.

Randomizing over the different job listings without accounting for their age is not efficient. This result is not surprising: the efficient allocation correlates tightness to the vacancy proportion, whereas random search implies that tightness is constant over listing age. However, the result differs from standard search models where applying for the different jobs with equal probability is efficient. The reason here is, of course, that older job listings have a larger probability of being phantoms.

More interestingly, the directed search allocation is also inefficient. Comparing equations (19) and (22) reveals two differences. The first one is unrelated to phantoms. In the directed search allocation, what matters is the job-finding rate $\pi m(1, \theta)$, which must be constant over listing age. This implies that its growth rate $\dot{\pi}/\pi + \alpha\dot{\theta}/\theta = 0$. In the efficient allocation, what matters is the marginal productivity of vacancies $(1-\alpha)\pi m(1, \theta)$, i.e., the job-finding rate times the elasticity $1-\alpha$. This elasticity may vary with age, which the term $-\dot{\alpha}/(1-\alpha) = -\dot{\theta}\alpha'/(1-\alpha)$ captures in the left-hand side of equation (22).

The second difference is due to the presence of an intertemporal externality that is internalized in the efficient allocation. The corresponding term lies on the right-hand side of equation (22). This term is proportional to the change in the social marginal value of phantoms. An increase in tightness at a given age translates into more matches at this age, which fuels the phantom stock. This effect reduces the social marginal value of vacancies at all ages, which explains why this term is negative.

The magnitude of the informational externality decreases with age. Having one more phantom reduces the nonphantom proportion more when this proportion is large than when it is small. Therefore the planner allocates fewer job seekers to recent listings than in the directed search allocation. This explains why the job-finding rate decreases with age.

3.4 Efficient wage schedule

Workers do not account for the negative informational externality associated with how they direct their search. The distribution of job seekers over listing age is generally inefficient as a result. If the wage were conditioned on vacancy age, it could alter this distribution, thereby internalizing this externality. We now discuss the property of such efficient wages. This allows us to provide an alternative measure of the informational externality induced by directed search. Of course, we again emphasize that we do not actually see age-dependent wage schedules.

In the directed search allocation, suppose that a vacancy of age a pays wage $w(\pi(a))$ when the worker obtains the corresponding job. Let $\omega(\pi) = w(\pi)/w(1)$ be the *wage schedule*. The value of having a job paying $w(a)$ is $W(a)$, whereas the value of job search is U:

$$\begin{aligned} rW(a) &= w(a) + \lambda[U - W(a)], \\ rU &= \mu(a)[W(a) - U]. \end{aligned}$$

The no-arbitrage condition between possible submarkets implies that

$$\pi(a)m(1, \theta(a))[W(a) - U] = m(1, \theta_0)[W(0) - U], \tag{25}$$

which is equivalent to

$$\frac{\pi m(1, \theta(\pi))}{r + \lambda + \pi m(1, \theta(\pi))}\omega(\pi) = \frac{m(1, \theta_0)}{r + \lambda + m(1, \theta_0)}. \tag{26}$$

This redefines the implicit function θ^{ds}, already presented in Proposition 2, relating tightness to the vacancy proportion. The new function involves the wage schedule ω, i.e. $\theta^{\text{ds}} = \theta^{\text{ds}}(\pi, \omega)$.

When $r \to 0$, we obtain

$$[1 - u(\pi)]\omega(\pi) = 1 - u(1), \tag{27}$$

where $u(\pi) = \lambda/(\lambda + \pi m(1, \theta(\pi))$ is the unemployment rate at vacancy proportion π, i.e., unemployment for job seekers who only search job listings where the vacancy proportion is π. Given any wage schedule, the employment probability $1 - u(\pi)$ adjusts so that the product of this employment probability and the wage schedule equals the employment probability when there are no phantoms, i.e., when workers search for new vacancies.

An *efficient wage schedule* $\omega^{\text{eff}} : [0, 1] \to \mathbb{R}$ is such that tightness is the same in the directed search and in the efficient allocations, that is, $\theta^{\text{eff}}(\pi) = \theta^{\text{ds}}(\pi, \omega^{\text{eff}})$ for all $\pi \in [0, 1]$. The function ω^{eff} satisfies:

$$\omega^{\text{eff}}(\pi) = \frac{m(1, \theta(0))}{\lambda + m(1, \theta(0))} \frac{\lambda + m(\pi, \pi\theta^{\text{eff}}(\pi))}{m(\pi, \pi\theta^{\text{eff}}(\pi))}. \tag{28}$$

We can also characterize it as follows:

$$\ln \omega^{\text{eff}}(\pi) = \ln[1 - u^{\text{eff}}(1)] - \ln[1 - u^{\text{eff}}(\pi)] \approx u^{\text{eff}}(\pi) - u^{\text{eff}}(1), \tag{29}$$

when u^{eff} is sufficiently small. The log efficient wage schedule at vacancy proportion π compensates for the unemployment rate differential between vacancy proportions π and one.

The efficient wage schedule satisfies $\omega^{\text{eff}}(1) = 1$ because $\theta^{\text{eff}}(1) = \theta(0)$. Moreover

$$\begin{aligned} \frac{\dot{\omega}^{\text{eff}}}{\omega^{\text{eff}}} &= \frac{\lambda}{\lambda + \pi m(1, \theta^{\text{eff}}(\pi))} \left[(1 - \alpha)(1 - \pi) \frac{\dot{\theta}^{\text{eff}}}{\theta^{\text{eff}}} - \frac{\dot{\alpha}}{1 - \alpha} \right] \\ &= u^{\text{eff}}(\pi) \left[(1 - \alpha)(1 - \pi) \frac{\dot{\theta}^{\text{eff}}}{\theta^{\text{eff}}} - \frac{\dot{\alpha}}{1 - \alpha} \right]. \end{aligned} \tag{30}$$

The growth rate of the efficient wage schedule is proportional to the unemployment rate at vacancy proportion π. It linearly increases with the growth rate of tightness $\dot{\theta}/\theta$ and the phantom proportion $1 - \pi$. It also accounts for the change in elasticity $1 - \alpha$, which affects the marginal productivity of vacancies.

Given the relationship between $\dot{\theta}/\theta$ and $\dot{\pi}/\pi$, we can also write

$$\frac{\dot{\omega}^{\text{eff}}}{\omega^{\text{eff}}} = -u^{\text{eff}}(\pi) \left[\frac{(1 - \alpha)(1 - \pi)}{\alpha + (1 - \alpha)(1 - \pi)} \frac{\dot{\pi}}{\pi} + \frac{\alpha}{\alpha + (1 - \alpha)(1 - \pi)} \frac{\dot{\alpha}}{1 - \alpha} \right], \tag{31}$$

which shows that the growth rate of the efficient wage schedule varies inversely with the growth rate of the vacancy proportion.

The efficient wage schedule specifies the wage growth required to internalize the externality caused by phantoms. However, it does not specify the wage level. In our model with a fixed number of jobs, the wage level redistributes welfare between firms and workers without affecting employment probabilities. Therefore the wage level is indeterminate and only wage growth matters.

4 Calibrations

Suppose the meeting technology is Cobb-Douglas, i.e., $m(u, v+p) = m_0 u^{1-\alpha}(v+p)^{\alpha}$, where $m_0 > 0$ and $\alpha \in (0,1)$. We use JOLTS and other BLS data for the period 2000-2008. Over this period, the monthly probability of finding a job was about $\mu_m = 0.4$. Thus $1 - \exp(-\mu) = 0.4$ and $\mu = -\ln(1-0.4) \approx 0.511$. The monthly job loss probability was $\lambda_m = 0.03$. This gives $\lambda = -\ln(1-0.03) \approx 0.03$. The corresponding stationary unemployment rate is $\mathsf{u} = \lambda/(\lambda+\mu) \approx 5.6\%$. In the JOLTS dataset, firms report how many vacancies they have. Thus the dataset provides a measure of v. The mean ratio of vacancies to unemployed was about $x = \mathsf{v}/\mathsf{u} = 0.5$.

Davis et al (2013) argue that the JOLTS dataset underestimates the actual number of vacancies because of time aggregation. However, limiting the job seekers to the unemployed also underestimates the actual number of job seekers. To compare our assumptions with what we see in the data, we compute the average duration needed by employers to fill their jobs. This duration is $d = \int_0^{\infty} \exp\left[-\int_0^a \eta(s)\right] ds$. By construction, $v(a) = v(0)\exp\left[-\int_0^a \eta(s)\right]$. Therefore $d = \int_0^{\infty} v(s)ds/v(0) = \mathsf{v}/v(0) = x\mathsf{u}/[\lambda(1-\mathsf{u})]$. It follows that $d \approx 0.989$, i.e., vacancies are filled in less than a month on average. This is roughly the upper bound of the 14 to 25 day interval suggested by Davis et al.[5]

As for the phantoms, we assume that phantoms reach one month with probability 0.5. Thus $\delta_m = 0.5$ and so $\delta = -\ln(1-0.5) \approx 0.7$, i.e. phantoms live for 1.4 months on average. As noted in our introduction, Chéron and Decreuse (2016) use Craigslist data and examine the distribution of job listings by age. The support of this distribution is one month, as Craigslist destroys ads after 30 days. Chéron and Decreuse show that the distribution is uniform by week, which implies that employers let their ads reach the termination age. The value of δ that implies that phantoms live for one month on average is 1.0. However, this value leads to underestimate the phantom proportion. Furthermore, ads are likely to persist longer on alternative websites or through mouth-to-mouth communication. This is why we choose to target $\delta_m = 0.5$. We consider alternative values in Section 4.2.

As for α, the aggregate matching technology is $M = m_0 \int_0^{\infty} \pi(a)u(a)\theta(a)^{\alpha} da$. Parameter α is not the elasticity of the job-finding rate with respect to tightness. The reason is that increasing tightness directly raises the contact rate, which is captured by α, but also fuels the phantom stock, which reduces the direct effect. Moreover, α does not have an empirical counterpart because we do not measure the job-finding rate at each listing age. Therefore we

[5] Their months count 26 working days. Davis et al. obtain their estimates from a statistical model in which the job-filling rate is constant over the month, whereas our directed search model predicts that the job-filling rate strongly decreases with age. Further research is needed to understand how the consideration of phantom vacancies affects such estimates at the firm level.

need to compute the aggregate elasticity of the matching function with respect to v/u. This elasticity depends on the vacancy proportion at each age and on the allocation of job seekers across submarkets. Intuitively, this elasticity should be larger than α: having more vacancies reduces the nonphantom proportion, thereby decreasing the number of matches (see Chéron and Decreuse (2016) for the random search case). We set $\alpha = 0.2$. This implies that the long-run elasticity of the matching function is about 0.4, a value close to Shimer (2005). We consider alternative values in Section 4.2.

We set the scale parameter m_0 so that the unemployment rate of the directed search allocation is equal to u. Finally, we use the resource constraint to set $K = 0.972$. The interpretation is that if there were no frictions in the labor market – no meeting frictions and no phantoms –, 2.8% of the workforce would nonetheless remain unemployed.

Table 1 gives the set of parameters in the baseline calibration.

m_0	α	δ	λ	K	μ	u
1.091	0.2	0.7	0.03	0.972	0.511	0.056

Table 1: Parameter values in the baseline calibration

4.1 Baseline calibration

We describe the directed search allocation in the baseline calibration and compare it with the random search and directed search allocations. These latter allocations are found by using the same parameter set, but changing the rule allocating job seekers to the different listing ages.

By construction the unemployment rate is $\mathsf{u}^{\mathrm{ds}} = 5.6\%$ in the directed search allocation, whereas it is $\mathsf{u}^{\mathrm{eff}} = 5.4\%$ in the efficient allocation and $\mathsf{u}^{\mathrm{rs}} = 5.9\%$ in the random search allocation.

Figure 1 displays the density of job seekers by listing age in the three different allocations: random search, directed search and efficient search. Figure 1 shows dramatic differences in search behavior. In the random search allocation, the density slightly decreases over listing age: it halves in two months. In the directed search allocation, the density has a spike for newly created vacancies and then strongly decreases. The efficient allocation reveals an intermediate behavior: the spike is much less pronounced for new vacancies. Figure 2 shows the resulting tightness by listing age in the three allocations. Tightness is constant under random search, whereas it strictly increases with listing age in the other cases. Because job seekers are concerned about phantoms, they direct their search towards younger listings, and thus the ratio of advertised jobs to searchers increases with age.

To understand why the directed search allocation is not constrained efficient, it is useful to look at the job-finding rate (shown in Figure 3) and the vacancy proportion (shown in Figure 4).

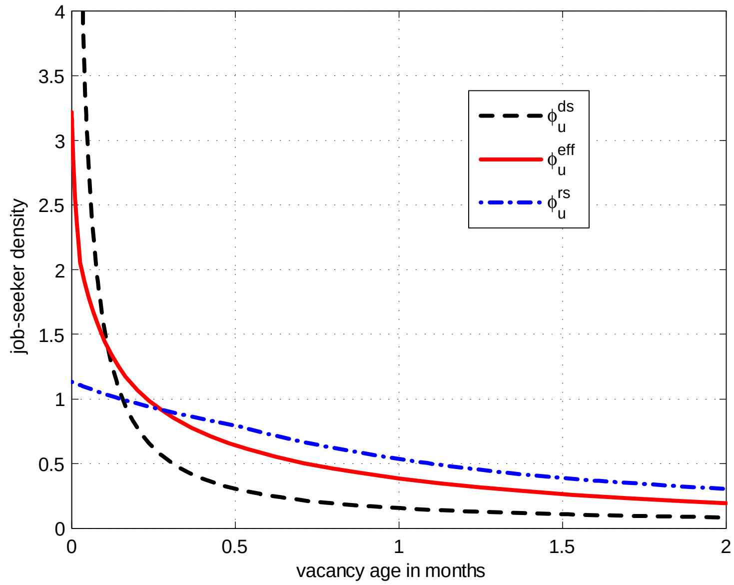

Figure 1: Density of job seekers by listing age (in months)

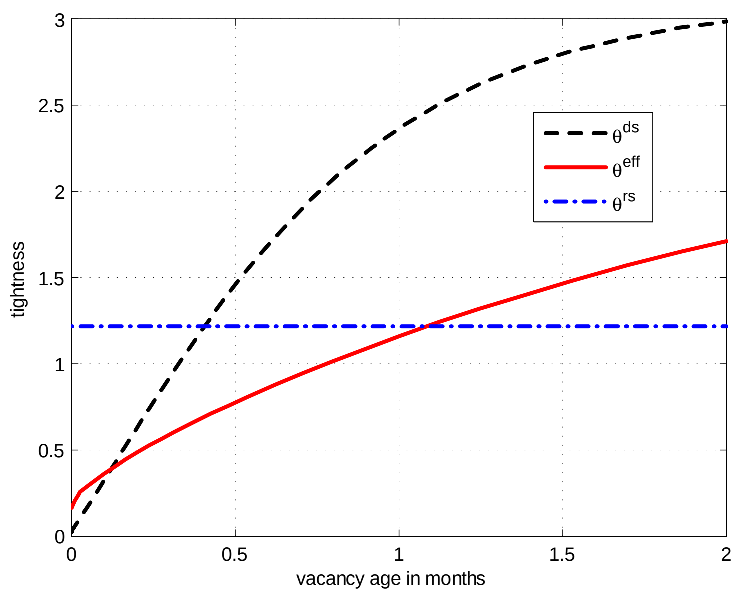

Figure 2: Tightness by listing age (in months)

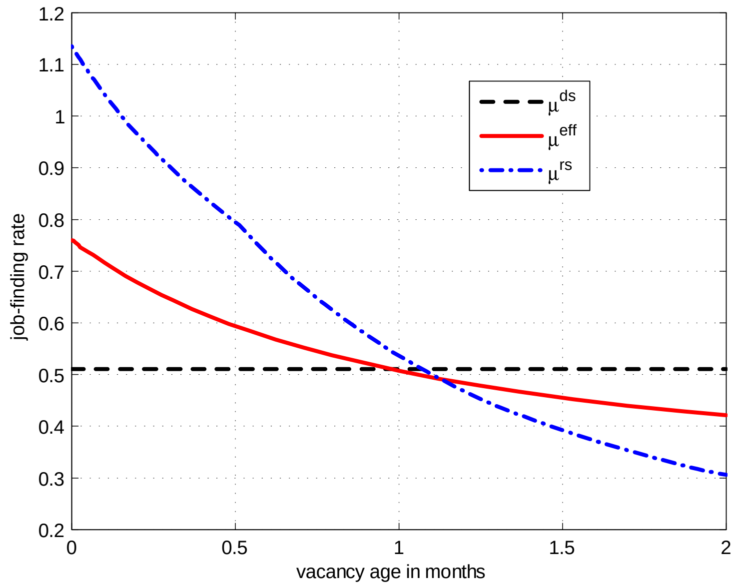

Figure 3: Job-finding rate by listing age (in months)

The job-finding rate is by definition constant under directed search, and strictly decreases over age in the other allocations. In these latter cases, reallocating individual search effort towards younger job listings would increase the individual odds of finding a job. This individual gain would also lead to a social gain in the random search allocation. However, it would decrease welfare in the efficient allocation. Figure 4 shows why. When job seekers direct their search towards more recent postings, the vacancy proportion decreases very rapidly when a is low. As phantoms haunt the search process for some time, such a strong decline in the vacancy proportion persists and affects all listing ages. The efficient allocation accounts for this effect. The pace of phantom creation is smoother over listing age, and the vacancy proportion declines less rapidly.

Directed search leads job seekers to over-apply to young job postings. Reallocating some job seekers from younger listings to older ones leads to efficiency gains. The reason is that directed search imposes a negative informational externality that persists across different listing ages. Applying to job listings of a given age, say a, sometimes means finding jobs of that age. This creates phantoms, and these phantoms then haunt the market. Because of the nature of directed search, phantoms only hurt agents who respond to job postings of age a or more. The magnitude of the externality thus decreases with listing age. When workers apply to newly created vacancies, they potentially affect all the other agents through phantom creation. When they direct their applications to old listings, they affect almost no one.

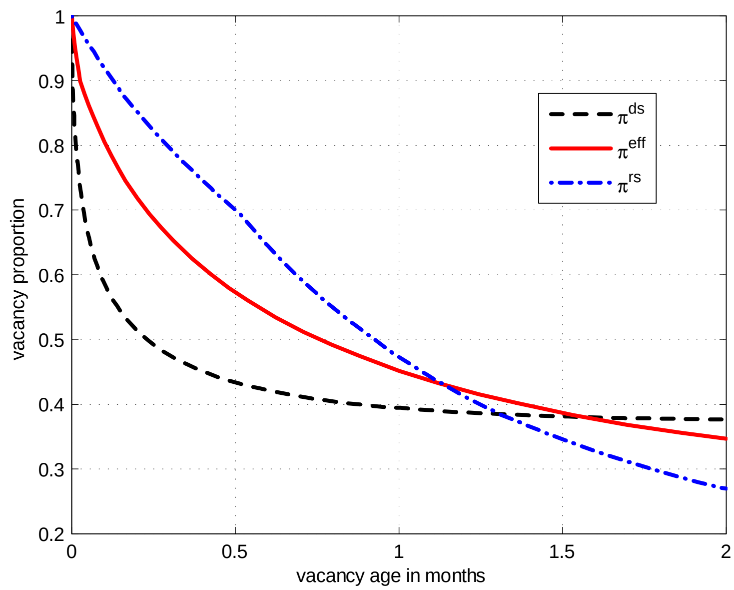

Figure 4: Nonphantom proportion by listing age (in months)

Quantitatively, the efficiency gains achieved by the efficient allocation are rather modest. Going from the worst (random search) to the best search method (efficient search) reduces the unemployment rate by 0.5 percentage points, i.e., a 10% decrease. The unemployment rate is only slightly affected by the way agents search for jobs. This is because any search strategy that produces more matches also increases the steady-state stock of phantoms. The vacancy proportion, therefore, does not vary much across allocations. In our baseline calibration, we have $\pi^{\text{rs}} = 42.9\%$, $\pi^{\text{ds}} = 40.7\%$, $\pi^{\text{eff}} = 39.1\%$.

This, however, does not mean that phantom vacancies do not have a large quantitative impact. In our calibration, removing all phantoms from the market leads to large employment gains. The corresponding allocation obtains as $\delta \to \infty$. The job-finding rate is 0.809 and the corresponding unemployment rate is 0.036.

We can compute the contribution of phantoms to unemployment. If there were no frictions, employment would be determined by the short side of the market, and the unemployment rate would be $\min(1 - K, 0)$. In our calibration, $K = 0.972 < 1$, so the nonfrictional unemployment rate is equal to 0.028. Under directed search the unemployment rate is $\mathsf{u}^{\text{ds}} = 0.056$. It follows that matching frictions account for half the unemployment rate. Without phantoms, the unemployment rate would decrease to 0.036. Hence, phantoms account for $(0.056 - 0.036)/(0.056 - 0.028) \approx 71\%$ of overall frictions and $0.02/0.056 \approx 36\%$ of unemployment.

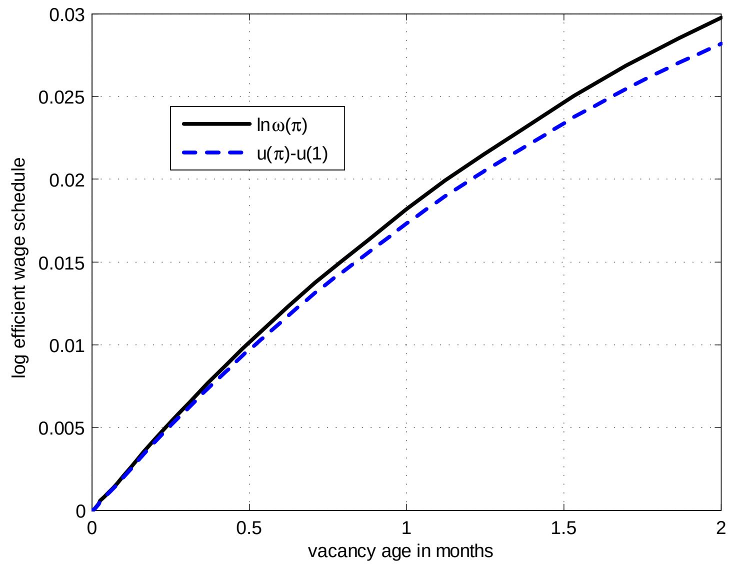

Figure 5: Efficient wage schedule in the baseline calibration

Figure 5 shows the efficient wage schedule in the baseline calibration. It also plots its approximation by the unemployment rate differential $u(\pi) - u(1)$. The wage must increase by 2% over the first month, and 3% over two months. This compensates for the change in job-finding rate.

4.2 Alternative parameterizations

We examine how the magnitude of inefficiency varies when we let parameters α and δ take different values.For each value of α and δ, we re-calibrate the directed search allocation by adjusting m_0 so that the predicted unemployment rate matches the US unemployment rate over the period 2000-2008. The two other allocations are then found by changing the rule $\theta(a)$ that assigns job seekers to listing ages.

Figure 6 depicts the unemployment rate as a function of α. By construction, the directed search unemployment rate u^{ds} does not change with α. We have $\mathsf{u}^{\text{rs}} > \mathsf{u}^{\text{ds}} > \mathsf{u}^{\text{eff}}$ for all $\alpha < 1$. Random search, therefore, always does worse than directed search. Figure 6 shows that the unemployment rate differential between the directed search and the efficient allocation decreases with α. The three allocations coincide when $\alpha = 1$. In this case, the matching technology is $M(a) = m_0 v(a)$ for all $a \in \mathbb{R}_+$ and phantoms do not affect it.

To evaluate what the various values of α mean, it is useful to relate them to the corresponding long-run elasticity of the job-finding rate with respect to $\mathbf{x} = \mathsf{v}/\mathsf{u}$., which is what is commonly

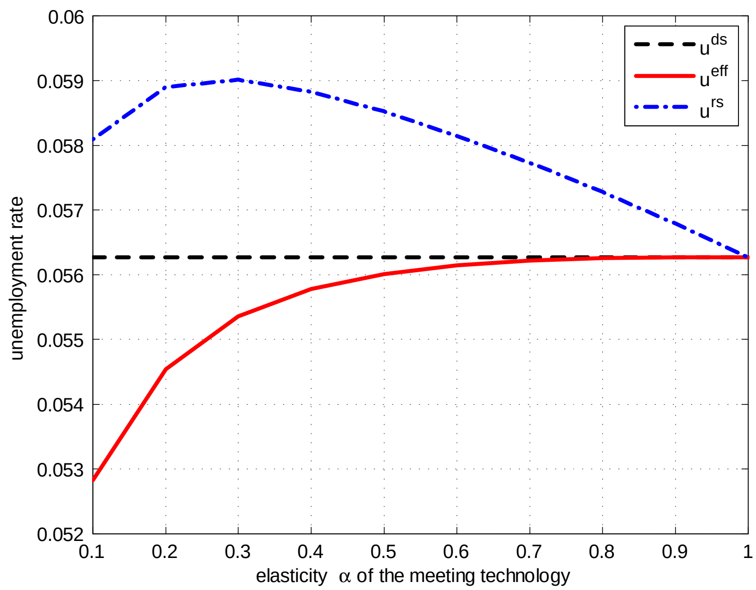

Figure 6: Unemployment rates in the three allocations as functions of parameter α

reported in the literature. The difference arises because α is the elasticity of the meeting or contact function with respect to θ and when θ changes it also changes the nonphantom proportion, π, as well as the distribution of searchers across submarkets. Figure 7 plots the long-run elasticity with respect to $\mathbf{x}$ against α. We compute the long-run elasticity as follows: Let $\ln x_\alpha$ and $\ln \mu_\alpha$ denote, respectively, the log vacancy-to-unemployed ratio and the log job-finding rate of the directed search allocation conditional on α. By construction, $\ln x_\alpha = \ln x = -\ln 2$ and $\ln \mu_\alpha = \ln 511 - \ln 1000$. We then decrease the total number of jobs K by one percentage point and recompute the directed search allocation. Let $\ln x_-$ and $\ln \mu_-$ denote, respectively, the associated log tightness and log job-finding rate. We similarly consider a one-percentage-point increase in the number of jobs and compute $\ln x_+$ and $\ln \mu_+$. The long-run elasticity of the matching function is approximated by

$$\frac{d\ln M}{d\ln x} \approx \frac{1}{2}\left(\frac{\ln \mu_+ - \ln \mu_\alpha}{\ln x_+ - \ln x_\alpha} + \frac{\ln \mu_- - \ln \mu_\alpha}{\ln x_- - \ln x_\alpha}\right). \tag{32}$$

As intuition suggests, the long-run elasticity is larger than α. It is around Shimer's (2005) value of 0.4 when α lies between 0.1 and 0.2.

Figure 8 depicts the unemployment rate as a function of the phantom death rate δ. Here again random search performs worse than directed search. The unemployment rate differential between these two allocations and the efficient one decreases with δsince this parameter reduces the phantom proportion at all ages.

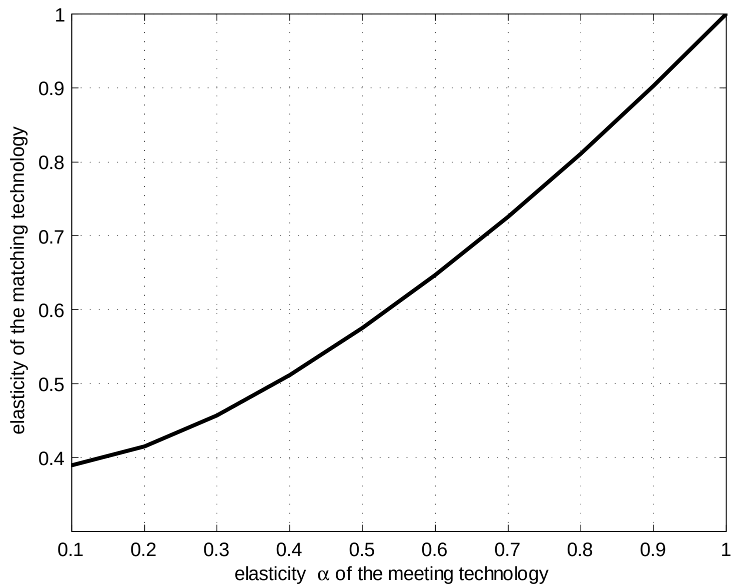

Figure 7: Long-run elasticity of the matching technology as a function of the elasticity α of the meeting technology

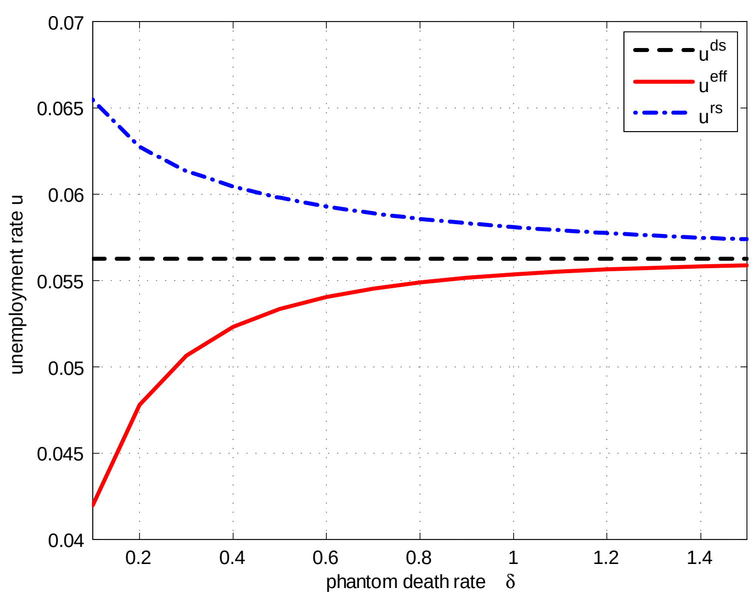

Figure 8: Unemployment rates in the three allocations as functions of parameter δ

5 Extensions

We examine three extensions: (i) vacancy renewal, (ii) lemons, and (iii) wages set by Nash bargaining.

5.1 Renewing offers

This extension has two goals. On the one hand, vacancy destruction often occurs at a deterministic date. For instance, websites typically destroy ads after a limit age is reached: one month on Craigslist and two months on Monster. Employers with unfilled jobs need to repost their vacancies. In addition, some websites offer employers the possibility to refresh vacancies before the age limit is reached. If the previous ad disappears, each vacancy is characterized by a single ad. If it does not, then a given vacancy may have several ads at one time, each corresponding to a specific age. The former effect is the pure effect of renewing vacancies. The latter effect mixes vacancy renewal with search intensity. We focus on the former, i.e., the pure renewal effect. In both cases, the vacancy is then relisted as a new one; that is, a is reset to zero.

We assume vacancy renewal automatically occurs after age A, and randomly happens at rate γ between 0 and A. This modifies the dynamics of vacancies as follows:

$$\dot{v} = -M - \gamma v, \tag{33}$$

$$v(0) = \lambda(1 - \mathsf{u}) + \gamma \mathsf{v} + v(A). \tag{34}$$

At each age $a \leq A$, the measure of vacancies decreases by the flow of matches formed with vacancies of that age and by the flow of vacancies that are refreshed. The inflow of new vacancies is then equal to newly destroyed employment relationships $\lambda(1-\mathsf{u})$ plus the numbers of randomly renewed vacancies $\gamma\mathsf{v}$ and deterministically renewed vacancies $v(A)$.

The rest of the model is unchanged. We thus proceed as in the previous section. We first derive the directed search allocation, and then construct the other possible allocations.

Under directed search, the allocation of job seekers across the different listing ages equalizes the job-finding rate at the different ages. Thus $\mu(a) = \pi(a)m(1, \theta(a)) = m(1, \theta(0))$. The nonphantom proportion is now affected by vacancy renewal:

$$\frac{\dot{\pi}}{\pi} = -m(1, \theta)/\theta + (\delta - \gamma)(1 - \pi), \tag{35}$$

$$\pi(0) = 1. \tag{36}$$

The additional term $-\gamma(1 - \pi)$ captures the idea that vacancy renewal tends to speed up the decline of the nonphantom proportion with listing age. Renewing a vacancy means destroying a vacancy to replace it by another one with age zero.

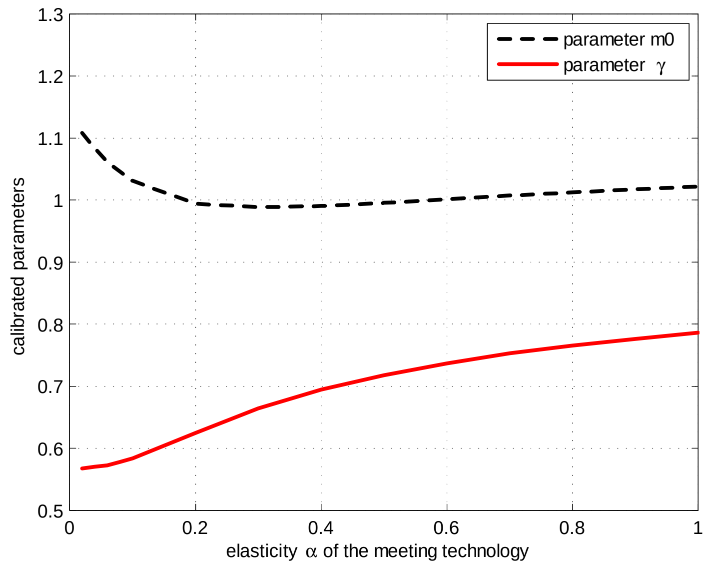

Figure 9: Parameters γ and m_0 as functions of the elasticity α of the meeting technology

To find the directed search allocation, we use the condition $\pi(a)m(1,\theta(a)) = m(1,\theta(0))$ to replace θ by a function of π and $\theta(0)$ and solve the Cauchy problem (35). Initial tightness $\theta(0)$ then solves the resource constraint $1 - \mathsf{u} + \mathsf{v} = K$ together with the equations of motion for $v(a)$ and $p(a)$ and the definition of tightness $u(a) = (v(a) + p(a))/\theta(a)$. Alternatively, the random search allocation results when $\theta(a) = \theta(0)$ for all $a \in [0, A]$, whereas Theorem 1 in the proof of Proposition 3 shows that efficient tightness is $\theta(a) = \theta^{\text{eff}}(\pi(a), \theta(0))$ for all $a \in [0, A]$, where the function θ^{eff} is defined in Proposition 2.

The model calibration involves two new parameters: the maximum lifetime of a vacancy A and the rate of vacancy renewal γ. We set $A = 2$, i.e., obsolete information cannot persist after two months. In the meantime, we set δ to 0, so that job listings deterministically live for two months. As for γ, we use information from Craigslist. Data from May-June 2015 that suggest that the proportion of newly created vacancies in a new cohort of vacancies is about 55% (see Appendix B). The corresponding theoretical moment is $\lambda(1 - \mathsf{u})/[\lambda(1 - \mathsf{u}) + \gamma\mathsf{v} + v(2)]$, i.e., the ratio of newly destroyed employment relationships to overall new job listings. We let α vary from 0 to one, and fix γ and m_0 so that $\lambda(1 - \mathsf{u})/[\lambda(1 - \mathsf{u}) + \gamma\mathsf{v} + v(2)] = 55\%$ and $\mathsf{u} = 0.0563$.

Figure 9 plots the resulting parameters γ and m_0 against the elasticity α. The rate of vacancy renewal lies between 0.5 and 0.8, indicating that job listings are renewed every ten days on average, until they are automatically refreshed after two months. The parameter m_0 is about 1.0.

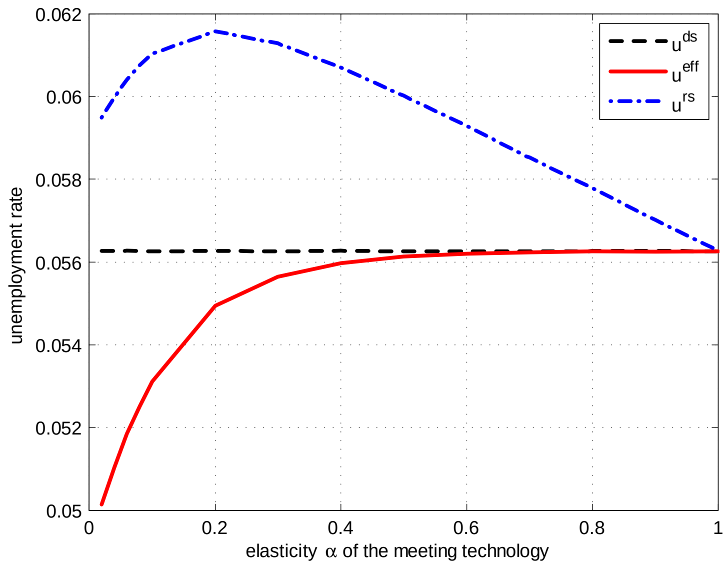

Figure 10: Unemployment rates in the three allocations as functions of parameter α

Figure 10 depicts the unemployment rate in the three different allocations. The result is very close to Figure 6. Directed search performs better than random search, and the unemployment differential between the directed search and the efficient allocations decreases with α. Comparing the two figures again, vacancy renewal increases the difference between the unemployment rates in the random search and directed search allocations. Random search is more costly with renewal: as seen in equation (35), with vacancy renewal, the nonphantom proportion declines more rapidly with age.

When $\alpha = 0.2$ as in the baseline calibration, we have $\mathsf{u}^{\text{eff}} = 0.055$, $\mathsf{u}^{\text{ds}} = 0.056$ and $\mathsf{u}^{\text{rs}} = 0.062$. The unemployment differential between the random search and the efficient allocations reaches 0.7 percentage points, 0.2 percentage points larger than in the baseline calibration. The corresponding nonphantom proportions are $\pi^{\text{eff}} = 35.5\%$, $\pi^{\text{ds}} = 34.8\%$, $\pi^{\text{rs}} = 44.9\%$. Without information obsolescence, i.e., $\delta \to \infty$, the unemployment rate is 0.038. In the directed search allocation, phantoms account for $(0.056 - 0.038)/\ (0.056 - 0.028) \approx 64\%$ of overall frictions and $0.018/0.056 \approx 32\%$ of unemployment.

Figure 11 plots the efficient wage schedule for the calibration with vacancy renewal. The increase in the efficient wage schedule is stronger than in the baseline case. The wage must increase by 4% in one month and 10% in two months. This is mostly due to the fact that job listings are destroyed after two months, whereas they may live forever in the baseline calibration.

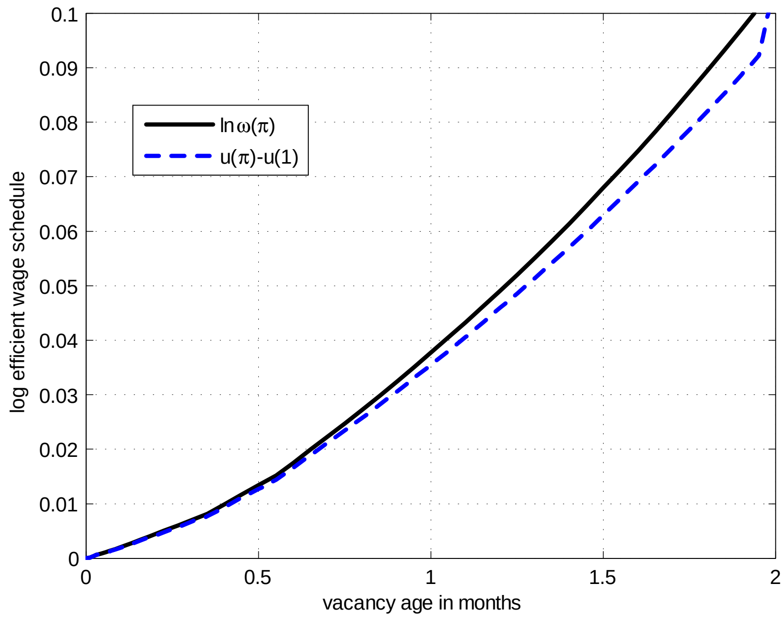

Figure 11: Efficient wage schedule in the calibration with vacancy renewal

5.2 Lemons vs phantoms

In our model, the only reason that workers pay attention to listing age is because they are concerned about phantoms. We now turn to another explanation based on the existence of jobs that are lemons. Like phantoms, lemons do not lead to profitable matches and so create congestion for job seekers. The longer a job listing stays posted, the more likely it is a lemon. Workers are therefore more inclined to search for young listings. We now examine this argument in our framework.

Let $l(a)$ be the number of lemons at age a, whereas $\mathsf{l} = \int_0^\infty l(a)da$ is the total number of lemons. Lemons impede matches in a similar fashion to phantoms. The number of matches at age a is now

$$M(a) = \frac{v(a)}{v(a) + p(a) + l(a)} m(u(a), v(a) + p(a) + l(a)). \tag{37}$$

We now denote $\pi(a) = v(a)/(v(a) + p(a) + l(a))$ and $\theta(a) = (v(a) + p(a) + l(a))/u(a)$

At each time there is an inflow of new lemons $l(0) = l_0$ and $\pi(0) = v(0)/(v(0) + l_0) < 1$. These lemons disappear at the same rate as phantom vacancies do; that is, $\dot{l} = -\delta l$. To ensure that the number of lemons stays constant, we suppose $l_0 = \delta \mathsf{l}$.

Under directed search, workers spread themselves over the various listing ages so that the job-finding rate is the same at all ages. Thus $\pi(a)m(1, \theta(a)) = \pi(0)m(1, \theta(0))$. As in the phantom model, we have $\alpha\dot{\theta}/\theta = -\dot{\pi}/\pi$ with

$$\frac{\dot{\pi}}{\pi} = -\frac{m(1,\theta)}{\theta} + \delta(1-\pi). \tag{38}$$

Using this equation with the condition $\pi(a) = \pi(0)m(1,\theta(0))/m(1,\theta(a))$ leads to

$$\alpha\dot{\theta} = m(1,\theta) - \delta\theta + \pi(0)\delta\theta m(1,\theta_0)/m(1,\theta). \tag{39}$$

To find $\theta(0)$, we proceed as before. The unemployment rate is now $\mathsf{u} = \lambda/(\lambda + m(1,\theta(0)))$, whereas the number of vacancies is $\pi(0)\lambda(1-\mathsf{u})\int_0^\infty \exp[-\int_0^a m(1,\theta(s))/\theta(s)ds]da$. Finally, the resource constraint is $1 - \mathsf{u} + \mathsf{v} = K$, which gives

$$\frac{m(1,\theta(0))}{\lambda + m(1,\theta(0))}\left\{1 + \pi(0)\lambda\int_0^\infty \exp\left[-\int_0^a m(1,\theta(b))/\theta(b)db\right]da\right\} = K. \tag{40}$$

A directed search allocation is a function $\theta(a)$ solving the differential equation (39) such that the resource constraint (40) holds. The computation of equilibrium is slightly more demanding because there is an additional fixed-point problem. In particular, the tightness pattern is conditional on $\pi(0)$, which must satisfy $\pi(0) = \lambda(1-\mathsf{u})/(\lambda(1-\mathsf{u}) + l_0)$.

Alternatively, the random search allocation results when $\theta(a) = \theta(0)$ for all $a \in [0,\infty)$, whereas Theorem 1 in the proof of Proposition 3 shows that efficient tightness is $\theta(a) = \theta^{\text{eff}}(\pi(a), \theta(0), \pi(0))$ for all $a \in \mathbb{R}_+$, where:

$$\theta^{\text{eff}}(\pi, \theta_0, \pi_0) = \theta = \frac{1 - [1-\alpha(\theta)]\pi}{1 - [1-\alpha(\theta_0)]\pi_0}\frac{1-\alpha(\theta_0)}{1-\alpha(\theta)}\frac{\pi_0}{\pi}\theta_0. \tag{41}$$

We consider a parameterization similar to Section 4, Table 1. We set phantoms to 0 by having $\delta_{phantoms} \to \infty$. We choose l_0 so that the lemon proportion among new ads is 10% in the directed search allocation. We obtain $l_0 = 0.0032$. As for their death rate, we consider two cases $\delta_{lemons} = .5$ and $\delta_{lemons} = 3$. We need an explicit assumption about the way lemons are accounted for in the JOLTS dataset. Like phantoms, we suppose that firms do not declare lemons in the vacancy set. It follows that $x = \mathsf{v}/\mathsf{u} = 0.5$. The scale parameter m_0 adjusts so that the predicted job-finding rate of the directed search allocation equals the US unemployment rate. This gives $m_0 = 0.6836$.

We start with $\delta_{lemons} = 0.5$. Figure 12 shows the equilibrium tightness-listing age schedule in the three allocations. As lemons disappear at a slower rate than vacancies, the lemon proportion increases with age in the directed search allocation. Thus tightness must increase with age to compensate for the rise in the lemon proportion. As in the phantom case, market segmentation by age is associated with an informational externality that decreases with age. Searching for newly created vacancies increases the lemon proportion at all future ages, which is not taken into account by the job seekers when they spread themselves over the different market segments.

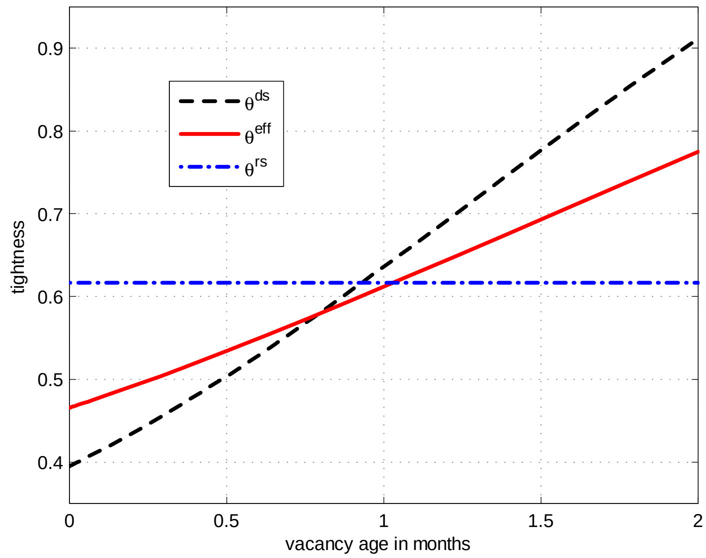

Figure 12: Tightness by listing age with lemons, death rate = 0.5

This is why the tightness-listing age schedule is less steep in the efficient allocation than in the directed search one.

The magnitude of the externality is small. The unemployment rate is 0.0563 in the directed search allocation, 0.0562 in the efficient allocation, and 0.567 in the random search allocation. Matching does not generate the kind of intertemporal frictions that we observed with phantoms. Thus the extent of additional frictions is limited by the lemon proportion among newly created vacancies. We limit this proportion to 10%, and we do not see much larger numbers as being realistic.

Then we set $\delta_{lemons} = 3$, which implies $m_0 = 0.6035$. Figure 13 reveals that tightness decreases with listing age. This arises because lemons disappear at a faster rate than vacancies. Thus the lemon proportion is larger at younger listing ages and job seekers search for older vacancies as a result. This case cannot arise in the phantom case because the phantom proportion among newly created vacancies is equal to 0. Thus the phantom proportion cannot decrease over time. When the phantom death rate is very large, the phantom proportion stays constant after a finite age.

5.3 Wage bargaining

To internalize the negative informational externality, the wage contract must be conditional on the vacancy age. In this section, we focus on Nash bargaining over the match surplus as a simple

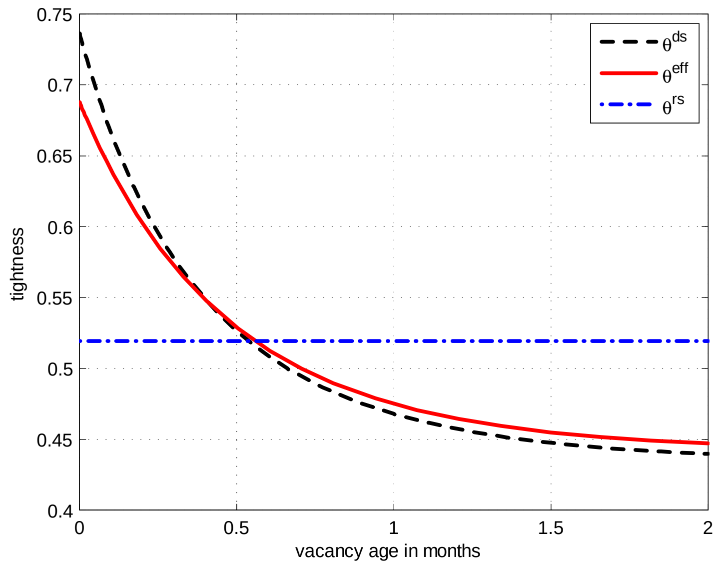

Figure 13: Tightness by listing age with lemons, death rate = 3

way to address this.

Once a worker and a firm meet, they bargain over a non-renegotiable wage $w(a)$. That is, the worker is paid this wage until the dissolution of the match. When the match ends, the worker joins the pool of unemployed and the job becomes a new vacancy, i.e. a vacancy of age 0.

With Nash bargaining, we have

$$rU = \max_{a\geq 0}\{\mu(a)[W(a)-U]\}, \tag{42}$$
$$rW(a) = w(a)+\lambda[U-W(a)], \tag{43}$$
$$rV(a) = \eta(a)[J(a)-V(a)]+V'(a), \tag{44}$$
$$rJ(a) = 1-w(a)+\lambda[V(0)-J(a)]. \tag{45}$$

Let $S(a) = W(a) - U + J(a) - V(a)$ be the match surplus when the firm and the worker meet:

$$rS(a) = 1-\lambda S(a)+\lambda[V(0)-V(a)]-rV(a)-rU. \tag{46}$$

It follows that $S'(a) = -V'(a)$ and $S(a) = S(0) + V(0) - V(a)$. The worker obtains the share $\nu \in (0, 1]$ of the match surplus, whereas the firm obtains the remaining share $1 - \nu$. Combining the various equations we obtain

$$rS(a) = 1-\lambda S(0)-\eta(a)(1-\nu)S(a)-rU+S'(a). \tag{47}$$

In equilibrium all submarkets are open and yield the same utility. This implies that $\pi(a)m(1, \theta(a))S(a) =$

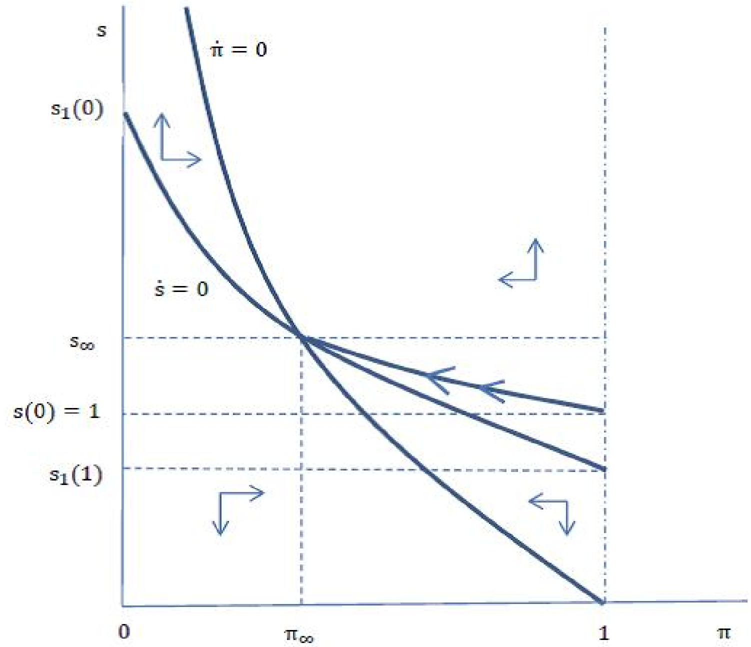

Figure 14: Phase diagram. The stable branch is indicated by arrows. There is a unique value of $S(0)$ such that the trajectory starting in $(1,1)$ converges to (π_∞, s_∞).

$m(1,\theta_0)S(0)$ for all $a \geq 0$. Taking the derivative of the left-hand side with respect to a gives

$$\alpha\frac{\dot{\theta}}{\theta} = -\left(\frac{\dot{\pi}}{\pi} + \frac{\dot{S}}{S}\right). \tag{48}$$

The percent change in tightness is proportional to the negative of the percent change in the nonphantom proportion plus the percent change in match surplus. Thus if match surplus goes up with age, then the tightness-age profile is less steep than in the fixed-wage allocation.

We now show that the surplus does increase with age. Let $s(a) = S(a)/S(0)$ for all $a \geq 0$ be the *normalized surplus*. Let also $\theta^{\mathrm{b}} : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be the implicit solution θ to $\pi m(1,\theta)s = m(1,\theta_0)$. Then the normalized surplus and the nonphantom proportion follow

$$\dot{s} = [r + (1-\nu)\eta(\theta^b(\pi, s, \theta(0))]s - [S(0)^{-1} - \lambda - \nu m(1, \theta(0))], \tag{49}$$

$$\dot{\pi} = (-\eta(\theta^b(\pi, s, \theta(0)) + \delta(1-\pi))\pi, \tag{50}$$

with $s(0) = \pi(0) = 1$. This system of ODEs is parameterized by $\theta(0)$ the initial tightness and $S(0)$ the initial match surplus. Initial tightness solves the resource constraint $1 - u + v = K$. As for the initial match surplus, we impose a transversality condition forbidding cases where S tends to 0 or infinity.

Figure 14 displays the phase diagram. The locus $\dot{s} = 0$ defines a first relationship between s and π, i.e., $s = s_1(\pi, \theta(0), S(0))$, with $[r + (1-\nu)\eta(\theta^b(\pi, s, \theta(0))]s = [S(0)^{-1} - \lambda - \nu m(1, \theta(0))]$. This makes sense only if $S(0) < [\lambda + \nu m(1, \theta(0))]^{-1}$. The locus is strictly decreasing with $rs_1(0) = [S(0)^{-1} - \lambda - \nu m(1, \theta(0))]$ and $0 < rs(1) < rs_1(0)$. The function s_1 strictly decreases with initial surplus $S(0)$. The locus $\dot{\pi} = 0$ defines a second relationship between s and π, i.e., $s = s_2(\pi, \theta(0), S(0))$, with $\eta(\theta^b(\pi, s, \theta(0) = \delta(1-\pi)$. This locus is also strictly decreasing, with $s_2(0) = \infty$ and $s_2(1) = 0$. The two loci intersect once in (s_∞, π_∞), with $s_\infty \in (s_1(1), s_1(0))$ and $\pi_\infty \in (0, 1)$. The limit normalized surplus s_∞ decreases with $S(0)$, whereas the limit nonphantom proportion π_∞ increases with it.

The trajectory starts in $s(0) = \pi(0) = 1$. The vector field shows that the trajectory generally diverges with s tending to infinity or (s, π) tending to $(0, 1)$. There is a unique value of $S(0)$ implying that (s, π) converges to (s_∞, π_∞). This value is such that $s(1) < 1$, a property holding on Figure 14. The normalized surplus increases and the nonphantom proportion decreases along the stable branch. Wage bargaining leads to decreasing match surplus with age. As explained above, this implies that tightness is less steep than in the directed search allocation.

In the particular case where $\nu = 1$, the motion of the normalized surplus does not depend on π. The transversality condition implies that $\dot{s} = 0$ for all $a \geq 0$. Thus $s(a) = 1$ and $S(a) = S(0) = 1/[r + \lambda + \nu A\theta(0)^\alpha]$ for all $a \geq 0$. It follows that the wage is constant over time and the wage bargaining allocation coincides with the directed search allocation. When $\nu = 1$, the value of having a vacancy coincides with the value of holding a filled job. Therefore the match surplus cannot change over age.

When $\nu < 1$, the model cannot be solved analytically. We turn to the Cobb-Douglas case and set all parameters as in the baseline calibration displayed in Section 4, Table 1. We compute the wage bargaining allocation for different values of ν. Figure 15 plots the resulting unemployment rate as a function of ν. It also shows the unemployment rates in the directed search and the efficient allocations.

The relationship between the unemployment rate and the bargaining power is U-shaped. As explained above, the unemployment rate coincides with the directed search one when $\nu = 1$. The unemployment rate matches the efficient one for values of ν around 0.15. It is actually lower by 0.01 percentage point due to the numerical precision of the simulations. There is a value of the bargaining power that decentralizes the efficient allocation. This result is not particularly surprising: efficiency requires that older vacancies pay more than younger ones and this is precisely what wage bargaining achieves.

Nash bargaining can help to internalize the age dependent informational externality. If submarket tightness increases with a, then vacancies become less valuable as they age. This

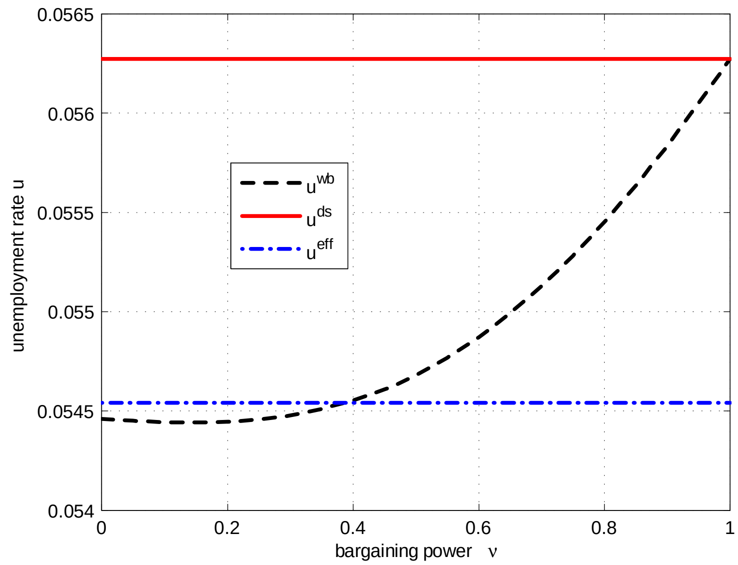

Figure 15: Unemployment rate as a function of bargaining power

implies that match surplus increases with age, and, since the worker gets a fixed fraction of the surplus, this also means that older vacancies pay higher wages. Workers thus have a stronger incentive to direct their search towards older job listings, thus ameliorating the externality caused by phantoms.

5.4 The Chicago question

This section addresses the following question: "If phantoms are so important, why don't job search platforms change the way they are designed so as to minimize the problem?" It takes time to identify and internalize negative externalities. Though we can find many complaints on the web about obsolete information on job boards (see the supplementary appendix of Chéron and Decreuse, 2016), such customer dissatisfaction has not been directly addressed yet.

However, search engines for job listings compete over pricing schemes and services offered to employers and job seekers. They provide different packages and some of them may *involuntarily* affect the nonphantom proportion on the website. The case of indeed.com is here enlightening. Indeed was launched in 2004 and, according to Wikipedia, became in 2010 the most high-traffic job website in the US. Indeed applies pay-per-click pricing. Employers only pay when someone clicks to view their jobs. This costs between $0.25 and $1.50 per click.

Indeed advertises this pricing strategy as follows: "Pay per click means you only pay when we deliver results – and you control what you pay". There is another reason why the pay-per-click strategy can increase market shares: this reduces the phantom job proportion on the website. Employers have incentive to suppress obsolete ads to avoid paying for additional clicks. In the vocabulary of our paper, the phantom death rate should be large on this website.

Having a small amount of obsolete information is a competitive advantage. Applicants are more likely to hang around the website and job ads receive more applications. Employers do not need to refresh their ads as often as they do on alternative websites. They trust the website as a result and this may contribute to explaining the success story of this job board.

6 Conclusion

To be added

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APPENDIX

A Proofs

Proof of Proposition 1:

Part (i). We have $\mathsf{v} = \int_0^\infty v(a)da$. Thus $\mathsf{v} = \lambda(1-\mathsf{u})\theta/m(1,\theta)$. The resource constraint $1-\mathsf{u}+\mathsf{v} = K$ implies that $(1-\mathsf{u})(1+\lambda\theta/m(1,\theta)) = K$. When combined with (10), this equation gives (11). To prove existence, let $\psi : [0,\infty) \to (-\infty,\infty)$ be such that

$$\psi(\theta) = m(1,\theta)(1-K) + \lambda\theta - K\frac{\lambda}{\delta}\frac{m(1,\theta)}{\theta} - K\lambda. \tag{51}$$

The steady-state tightness, if any, is such that $\psi(\theta_{rs}) = 0$. The properties of the function m imply that $\lim_{\theta\to 0}\psi(\theta) < 0$ and $\lim_{\theta\to\infty}\psi(\theta) > 0$. By continuity, there is $\theta_{rs} > 0$ such that $\psi(\theta_{rs}) = 0$. Moreover,

$$\psi'(\theta) = \frac{1}{\theta}\left[\alpha(\theta)m(1,\theta)(1-K) + \lambda\theta + K\frac{\lambda}{\delta}(1-\alpha(\theta))m(1,\theta)\right]. \tag{52}$$

Using the fact that $\psi(\theta_{rs}) = 0$, we have

$$\psi'(\theta_{rs}) = \frac{1}{\theta_{rs}}\left[\alpha(\theta_{rs})K\lambda + (1-\alpha(\theta_{rs}))\lambda\theta_{rs} + K\frac{\lambda}{\delta}m(1,\theta_{rs})/\theta_{rs}\right] > 0. \tag{53}$$

It follows that θ_{rs} is unique.

Part (ii) results from equation (5) with $\partial\pi(a,t)/\partial t = 0$ and $\theta(a) = \theta^{\mathrm{rs}}$ for all $a \geq 0$.

Part (iii). The recruitment rate $\eta(a) = m(1,\theta^{\mathrm{rs}})/\theta^{\mathrm{rs}}$ for all $a \geq 0$. The Cauchy problem (12) is of the form $\dot{\pi} = f(\pi)$. As $\dot{\pi}(0) = f(1) < 0$, we have $\dot{\pi}(a) < 0$ for all $a \geq 0$ by continuity of f. Lastly, $\mu(a) = \pi(a)m(1,\theta^{\mathrm{rs}})$ for all $a \geq 0$, which implies it strictly decreases with a.

Proof of Proposition 2:

Part (i). That $\theta(a) = \theta^{\mathrm{ds}}(\pi(a),\theta(0))$ results from the text before Proposition 2. The ODE (19) follows by differentiating with respect to age a. We now prove existence. At given $\theta(0)$, equation (19) determines a unique function $\theta(a,\theta_0)$. This function strictly increases with a. Let $\psi : [0,\infty) \to (-\infty,\infty)$ be such that

$$\psi(\theta_0) = \frac{m(1,\theta_0)}{\lambda + m(1,\theta_0)}\left\{1 + \lambda\int_0^\infty \exp\left[-\int_0^a \frac{m(1,\theta(b,\theta_0))}{\theta(b,\theta_0)}db\right]da\right\} - K. \tag{54}$$

We have $\lim_{\theta\to 0}\psi(\theta) = -K$, whereas $\lim_{\theta\to\infty}\psi(\theta) = +\infty$. Thus there is $\theta_0 > 0$ such that $\psi(\theta_0) = 0$.

Part (ii) results from equation (5) with $\partial\pi(a,t)/\partial t = 0$.

Part (iii). The job-finding rate is constant as a result of the directed search assumption. As for the nonphantom proportion π, we have

$$\begin{aligned}\frac{\dot{\pi}}{\pi} &= -\frac{m(1,\theta^{\mathrm{ds}}(\pi))}{\theta^{\mathrm{ds}}(\pi)} + \delta(1-\pi) < 0,\\ \pi(0) &= 1,\end{aligned}$$

where the dependence of θ^{ds} vis-à-vis $\theta(0)$ has been neglected. This law of motion is of the kind $\dot{\pi} = f(\pi)$. As $\dot{\pi}(0) = f(1) < 0$, we have $\dot{\pi}(a) < 0$ for all $a \geq 0$ by continuity of f. The ODE implies that $\dot{\theta}$ has the sign of $-\dot{\pi}$; therefore $\dot{\theta} > 0$. Lastly, we have $\eta(a) = m(1,\theta(a))/\theta(a)$, which strictly decreases with a because $m(1,\theta)/\theta$ strictly decreases with θ.

Proof of Proposition 3:

Preamble.—We consider a generalized problem where vacancies last at most $A > 0$. Then they are renewed. Renewal also occurs randomly at constant rate γ. There are also lemons, jobs that nobody wants. Each cohort of new vacancies contain l_0 lemons. They depreciate over age at rate $\delta > 0$, just like phantoms. The basic model obtains when A tends to infinity, and $\gamma = l_0 = 0$. The model with vacancy renewal requires A finite and $l_0 = 0$. The model with lemons involves $\gamma = 0$ and A tends to infinity. Though A may be infinite, we will use the notation $[0, A]$ to denote the set of possible job ages.

Planners' problem in raw form.—The objective of the planner is to minimize unemployment u. The resource constraint is $1 - \mathsf{u} + \mathsf{v} = K$. Therefore minimizing u is equivalent to minimizing v. The planner distributes the u unemployed over the different submarkets. As $\theta(a) = [v(a) + p(a) + l(a)]/u(a)$ for all $a \in [0, A]$, this is equivalent to choosing the function θ. Finally, the initial number of vacancies is $v(0) = \lambda(1 - \mathsf{u}) + \gamma\mathsf{v} + v(A)$.

The planner's problem is the following

$$\max_{\theta(.)} -\int_0^A v(a)da \tag{$*$}$$

subject to

$$\begin{aligned}\dot{v} &= -\eta(\theta)v - \gamma v,\\ \dot{p} &= \eta(\theta)v - \delta p,\\ v(0) &= \lambda\left(K - \int_0^A v(a)da\right) + \gamma\int_0^A v(a)da + v(A),\\ 1 - \int_0^A [(v(a)+p(a)+l(a))/\theta(a)]da + \int_0^A v(a)da &= K,\\ p(0) &= 0,\\ l(a) &= l_0 e^{-\delta a}.\end{aligned}$$

There are two state variables v and p, and one control variable θ.

We now prove the following result.

Theorem 1 Let $\theta^{\text{eff}}(\pi, \theta_0, \pi_0)$ be implicitly defined by

$$\theta = \frac{1-(1-\alpha(\theta))\pi}{1-(1-\alpha(\theta_0))\pi_0} \frac{(1-\alpha(\theta_0))\pi_0}{(1-\alpha(\theta))\pi} \theta_0 \tag{55}$$

The optimal control is such that $\theta(a) = \theta^{\text{eff}}(\pi(a), \theta(0), \pi(0))$, where the law of motion of the nonphantom proportion is

$$\begin{aligned} \frac{\dot{\pi}}{\pi} &= -m(1,\theta)/\theta + (\delta - \gamma)(1-\pi), \\ \pi(0) &= v(0)/(v(0) + l_0). \end{aligned} \tag{56}$$

Planner's problem in standard form.—To solve (∗), we transform the two integral constraints into differential equations with associated relevant boundary conditions. Let $x_1(a) = \int_0^a v(s)ds$ and $x_2(a) = \int_0^a \{[v(s) + p(s) + l(s)]/\theta(s) - v(s)\}ds$ for all $a \in [0, A]$. The planner's problem is now the following:

$$\max_{\theta(.)} - \int_0^A v(a)da \tag{$*$}$$

subject to

$$\begin{aligned} \dot{v} &= -\eta(\theta)v - \gamma v, \\ \dot{p} &= \eta(\theta)v - \delta p, \\ \dot{x}_1 &= v, \\ \dot{x}_2 &= (v + p + l_0 e^{-\delta a})/\theta - v, \\ \lambda K - v(0) + v(A) + (\gamma - \lambda)x_1(A) &= 0, \\ K - 1 + x_2(A) &= 0, \\ p(0) &= x_1(0) = x_2(0) = 0. \end{aligned}$$

Solving.—Let $H : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian such that:

$$H(y, \sigma, \theta) = -v - \sigma_1[\eta(\theta) + \gamma]v + \sigma_2[\eta(\theta)v - \delta p] + \kappa_1 v + \kappa_2[(v + p + l_0 e^{-\delta a})/\theta - v],$$

where σ_1 and σ_2 are the costates associated to the state variables v and p, and κ_1 and κ_2 are the costates associated to the artefact variables x_1 and x_2, $y = (v, p, x_1, x_2)$ and $\sigma = (\sigma_1, \sigma_2, \kappa_1, \kappa_2)$. We define $\phi_1(y(0), y(A)) = \lambda K - v(0) + v(A) + (\gamma - \lambda)x_1(A)$, $\phi_2(y(0), y(A)) = K - 1 + x_2(A)$, $\phi_3(y(0), y(A)) = p(0)$, $\phi_4(y(0), y(A)) = x_1(0)$, $\phi_5(y(0), y(A)) = x_2(0)$.

We now introduce the main result that we need, a version of Theorem 11.1 in Hestenes (1966), which accounts for equality constraints on the state variables at the beginning and at the end of possible ages.

Theorem 2 (*Maximum principle*) Suppose $\theta^*(.)$ is optimal for the optimization problem $(*)$ and let x^* be the corresponding trajectory of the state variables. Then there exists $\sigma^* : [0, A] \to \mathbb{R}^4$ and $\rho^* \in \mathbb{R}^5$ such that for all $a \in [0, A]$:

A. Maximization principle

$$H(y^*(a), \sigma^*(a), \theta^*(a)) = \max_{\theta \geq 0} H(y^*(a), \sigma^*(a), \theta) \tag{57}$$

B. Adjoint equations

$$\sigma_i^*(a) = -\frac{\partial H(y^*(a), \sigma^*(a), \theta^*(a))}{\partial y_i(a)}, i = 1, 4 \tag{58}$$

C. Transversality conditions

$$\sigma_i^*(0) = \sum_{j=1,\dots,5} \rho_j^* \frac{\partial \phi_j}{\partial y_i(0)}, i = 1, 3, 4 \tag{59}$$

$$\sigma_i^*(A) = -\sum_{j=1,\dots,5} \rho_j^* \frac{\partial \phi_j}{\partial y_i(A)}, i = 1, 3, 4, \tag{60}$$

$$\sigma_2^*(A) p(A) = 0. \tag{61}$$

The transversality conditions feature the Lagrange multiplier ρ^*, which is associated with the constraints $\phi_i(.) = 0$, $i = 1, ..., 5$. In practice, we neglect the trivial constraints $\phi_3(.) = \phi_4(.) = \phi_5(.) = 0$. There are no constraints involving the initial or final number of phantoms. The associated transversality condition is $\sigma_2(A)p(A) = 0$. As long as A remains finite, $p(A) > 0$ and so this constraint implies $\sigma_2(A) = 0$. The constraint remains valid when A tends to infinity. However, $\lim_{A\to\infty} p(A) = 0$ and the constraint does not imply $\lim_{A\to\infty} \sigma_2(A) = 0$.

Applying Theorem 2.—Hereafter we neglect the star * notation. We first focus on B and C, which do not involve particular discussions. This gives

$$\begin{aligned}
\frac{\partial H}{\partial v} &= -1 + (\sigma_2 - \sigma_1)\eta(\theta) - \sigma_1\gamma + \kappa_1 + \kappa_2(\theta^{-1} - 1) = -\dot{\sigma}_1, && (62)\\
\frac{\partial H}{\partial p} &= -\delta\sigma_2 + \kappa_2/\theta = -\dot{\sigma}_2, && (63)\\
\frac{\partial H}{\partial x_i} &= 0 = -\dot{\kappa}_i,\ i = 1, 2, && (64)\\
\sigma_1(0) &= -\rho_1, && (65)\\
\sigma_1(A) &= -\rho_1, && (66)\\
\sigma_2(A)p(A) &= 0, && (67)\\
\kappa_1(A) &= \rho_1(\lambda - \gamma), && (68)\\
\kappa_2(A) &= -\rho_2. && (69)
\end{aligned}$$

It follows that $\kappa_1(a) = \rho_1(\lambda - \gamma)$ and $\kappa_2(a) = -\rho_2$ for all $a \in [0, A]$. Note that $\sigma_1(0) = \sigma_1(A)$: the marginal value of vacancies is the same at the beginning and at the end of the interval of possible ages. This property is crucial because it helps us to guess the solution.

We now turn to the maximization principle A. Suppose that the first-order condition is necessary. Then we have

$$\frac{\partial H}{\partial \theta} = (\sigma_2 - \sigma_1)\eta'(\theta)v - \kappa_2\frac{v + p + l_0e^{-\delta a}}{\theta^2} = 0. \tag{70}$$

Re-arranging terms, we obtain

$$-(\sigma_2 - \sigma_1)(1 - \alpha(\theta))\pi m(1, \theta) = \kappa_2 \tag{71}$$

Taking the second derivative of the Hamiltonian gives

$$\begin{aligned}
\frac{\partial^2 H}{\partial \theta^2} &= (\sigma_2 - \sigma_1)\eta''(\theta)v + 2\kappa_2\frac{v + p + l}{\theta^3}\\
&= \frac{v + p + l}{\theta^3}\left[(\sigma_2 - \sigma_1)\frac{\theta\eta''(\theta)}{\eta'(\theta)}\frac{\theta\eta'(\theta)}{\eta(\theta)}\theta\eta(\theta)\pi + 2\kappa_2\right]\\
&= \frac{v + p + l}{\theta^3}\left[-(\sigma_2 - \sigma_1)\frac{\theta\eta''(\theta)}{\eta'(\theta)}(1 - \alpha(\theta))\pi m(1, \theta) + 2\kappa_2\right].
\end{aligned}$$

Once evaluated in the proposed maximum, we have

$$\frac{\partial^2 H}{\partial \theta^2} = \kappa_2\frac{v + p + l}{\theta^3}\left[\frac{\theta\eta''(\theta)}{\eta'(\theta)} + 2\right]. \tag{72}$$

But $\theta\eta''(\theta)/\eta'(\theta) = m_{\theta\theta}(1, \theta)/\eta'(\theta) - 2 > -2$. Thus the candidate satisfies the second-order condition provided that $\kappa_2 = -\rho_2 < 0$, which implies $\rho_2 > 0$. By virtue of (71), this also implies $\sigma_2 - \sigma_1 < 0$ for all $a \in [0, A]$.

Putting things together.—We differentiate equation (71) with respect to age. This gives

$$(\dot{\sigma}_2 - \dot{\sigma}_1)(1-\alpha(\theta))\pi m(1,\theta) + \rho_2 \left[\frac{-\dot{\alpha}}{1-\alpha} + \frac{\dot{\pi}}{\pi} + \alpha \frac{\dot{\theta}}{\theta} \right] = 0. \tag{73}$$

From (62) and (63), we obtain

$$\dot{\sigma}_1 = 1 + \rho_1(\gamma - \lambda) - \rho_2 + \gamma\sigma_1 - \rho_2 \frac{1-(1-\alpha)\pi}{(1-\alpha)\pi\theta}, \tag{74}$$

$$\dot{\sigma}_2 = \delta\sigma_2 + \rho_2/\theta. \tag{75}$$

Integrating with the condition $\sigma_2(A) = 0$ gives $\sigma_2(a) = -\rho_2 \int_a^A e^{-\delta(s-a)}/\theta(s)ds < 0$ for all $a \in [0, A]$. The social marginal value of phantoms is of course negative. When phantoms have a finite lifetime, σ_2 tends to 0 as the job listing age approaches A. Otherwise, the motion of σ_2 depends on the motion of θ.

Optimal control.—As said above, we have $\sigma_1(0) = \sigma_1(A) = -\rho_1$. The natural conjecture is that $\sigma_1(a) = -\rho_1$ for all $a \in [0, A]$. Therefore $\dot{\sigma}_1 = 0$ and equation (74) implies

$$z \equiv \frac{1 - \rho_1\lambda - \rho_2}{\rho_2} = \frac{1-(1-\alpha)\pi}{(1-\alpha)\pi\theta} \tag{76}$$

for all a. In particular, this equation is true for $a = 0$:

$$z = \frac{1-(1-\alpha(\theta(0)))\pi(0)}{(1-\alpha(\theta(0)))\pi(0)\theta(0)}, \tag{77}$$

The corresponding control is then given by $\theta(a) = \theta^{\text{eff}}(\pi(a), \theta(0), \pi(0))$. The conjecture satisfies all the optimality conditions provided $1 - \rho_1\lambda - \rho_2 > 0$ and $\rho_2 > 0$. Computing the motion of $\pi = v/(v + p + l)$ from those of v, p and l gives equation (56).

To conclude, note that the implicit function θ^{eff} always exists and is uniquely defined. Indeed, consider the function $g(\theta) = \theta - \frac{1-(1-\alpha(\theta))\pi}{1-(1-\alpha(\theta_0))\pi_0} \frac{(1-\alpha(\theta_0))\pi_0}{(1-\alpha(\theta))\pi}\theta_0$. We have $\lim_{\theta\to 0} g(\theta) < 0$ and $\lim_{\theta\to\infty} g(\theta) > 0$ because $\alpha'(\theta) \leq 0$ implies $\lim_{\theta\to+\infty}(1-\alpha(\theta))\theta = +\infty$. Therefore θ^{eff} exists by continuity of g. Moreover, $g'(\theta^{\text{eff}}) = 1 - \frac{\theta^{\text{eff}}\alpha'(\theta^{\text{eff}})}{1-\alpha(\theta^{\text{eff}})}[1-(1-\alpha(\theta^{\text{eff}}))\pi]^{-1}$. Since $\frac{\theta\alpha'(\theta)}{1-\alpha(\theta)} = \alpha(\theta) + \frac{\theta m_{\theta\theta}(1,\theta)}{m_\theta(1,\theta)} \frac{\alpha(\theta)}{1-\alpha(\theta)} < \alpha(\theta)$, we have $g'(\theta^{\text{eff}}) > 1 - \alpha(\theta^{\text{eff}})[1-(1-\alpha(\theta^{\text{eff}}))\pi]^{-1} = (1-\alpha(\theta^{\text{eff}}))(1-\pi)[1-(1-\alpha(\theta^{\text{eff}}))\pi]^{-1} > 0$. Therefore θ^{eff} is unique.

Proposition 2.—Parts (i) and (ii) are particular cases of Theorem 1. Namely, $\gamma = 0$, $l_0 = 0$, $A \to \infty$ and $\pi(0) = 1$. Equation (22) follows by differenciation of θ^{eff} with respect to a. Part (iii) We have

$$\frac{\dot{\pi}}{\pi} = (-\frac{m(1,\theta^{\text{eff}}(\pi))}{\theta^{\text{eff}}(\pi)} + \delta)(1-\pi),$$

$$\pi(0) = 1,$$

where the dependence of θ^{eff} vias-à-vis $\theta(0)$ has been neglected. This has the form $\dot{\pi} = h(\pi)$. As $\dot{\pi}(0) = h(1) < 0$, $\dot{\pi}(a) < 0$ for all $a \geq 0$. Moreover $\partial\theta^{\text{eff}}/\partial\pi < 0$, which implies that $\theta(a)$ increases with a. In turn, this implies that $\eta(a) = m(1, \theta(a))/\theta(a)$ decreases with a. Lastly, equation (22) implies that the growth rate of the job-finding rate is

$$\alpha(\theta)\frac{\dot{\theta}}{\theta} + \frac{\dot{\pi}}{\pi} = -(1-\alpha)(1-\pi)\frac{\dot{\theta}}{\theta} + \frac{\theta\alpha'(\theta)}{1-\alpha}\frac{\dot{\theta}}{\theta}, \tag{78}$$

which is negative when $\alpha'(\theta) \leq 0$.

Proof of Proposition 4:

Part (i). Efficient tightness strictly increases with listing rate, whereas tightness is constant in the random search allocation.

Part (ii). Equation (78) shows that the growth rate of the job-finding rate is strictly negative for $a > 0$ (it is equal to 0 at $a = 0$). This differs from the directed search allocation where the job-finding rate is constant.

B Setting parameters with Craigslist

We study the 524,948 job listings advertised on Craigslit's website as of June 10 2015. Each ad is characterized by an ID number, a job type, a city, and a date of creation. Sometimes the job listing is a repost from a previous one. The ad description then contains the ID number of the initial ad. All ads are automatically destroyed after one month. Therefore all job listings were created between May 10 and June 10 2015.

We make assumptions to compute vacancy renewal and detect multiple ads for the same job. For this purpose, all job listings are called *daughters*, whereas reposted ads are called *mothers*. A mother is also a daughter when it is less than 30 days old. A daughter has one mother at most and has *sisters* when all siblings share the same mother. A daughter is an *orphan* when it has no mother. By definition, a mother who is also a daughter may have sisters.

We suppose that a daughter advertises for a new job if and only if this is an orphan. The if part means that potential employers who already advertised for a given job use the repost function when they create a new ad for the same job. This excludes the unlikely cases where employers lost details of the initial ad and edited a new one as a result. The only if part means that daughters with a mother cannot advertise for a new job. Thus we forbid cases where an employer uses a previous ad to describe a new position. This assumption is very conservative, as casual evidence shows that multiple-daughter families are posted by large employers possibly advertising for many jobs in the same month.

There are two reasons why ads are reposted on Craigslist. The first reason is the ad reaches 30 days and the jobs stays unfilled. The second reason is the ad is reposted within the month. To set the renewal rate in our model, we compute the proportion of orphans among ads that were posted during the last 24 hours of the sample. This proportion is 56.6%. It does not vary much across cities and job types. In the directed search allocation, this proportion is $\lambda(1-\mathsf{u})/(\lambda(1-\mathsf{u})+\gamma\mathsf{v}+v(1))$. Therefore we fix γ and m_0 so that the theoretical proportion equals the empirical one and the unemployment rate is 5.63%. In practice, we consider a longer horizon because the ad can persist longer on alternative websites, or through mouth-to-mouth communication. Thus we compute $v(2)$ instead of $v(1)$. Given the calibration method, this choice implies that the calibrated renewal rate is larger with $v(2)$ than with $v(1)$. For instance, when $\alpha = 0.2$ we obtain $\gamma = 0.5$ when considering $v(2)$ and $\gamma = 0$ when considering $v(1)$.

Our model forbids the coexistence of multiple ads for the same job. On Craigslist, unlike alternative websites like Monster.com, reposting does not destroy the initial ad. Instead two ads of different ages coexist for the same job. We can detect multiple ads for the same job by computing the proportion of daughters with at least one sister. This proportion is 28.5%: about three ads over four are alone to describe the corresponding job. This figure likely overestimates the real frequency of multiple ads for the same job. For instance, public institutions that advertise their positions under the heading 'Government jobs' typically use a single mother for different jobs.

Alternatively, we could consider that these daughters with the same mother actually advertise for different jobs. The only source of renewal would be deterministic and this would dramatically reduce the proportion of newly created jobs in the last 24-hour cohort of job listings. Going further requires to know better the particularities of these ads.