R&D Networks: Theory, Empirics and Policy Implications☆

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Abstract

We study a structural model of R&D alliance networks where firms jointly form R&D collaborations to lower their production costs while competing on the product market. We provide a complete Nash equilibrium characterization, derive an efficiency analysis and determine the optimal R&D subsidy program that maximizes welfare. We then structurally estimate our model using a unique panel of R&D collaborations and annual company reports. We use our estimates to analyze the impact of R&D subsidy programs, and study how temporal changes in the network affect the optimal R&D policy.

Key words: R&D networks, innovation, spillovers, optimal subsidies, industrial policy
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1. Introduction

R&D partnerships have become a widespread phenomenon characterizing technological dynamics, especially in industries with a rapid technological development such as, for instance, the pharmaceutical, chemical and computer industries [cf. Hagedoorn, 2002; Powell et al., 2005; Roijakkers and Hagedoorn, 2006]. In those industries, firms have become more specialized in specific domains of a technology and tend to combine their knowledge with the knowledge of other firms that are specialized in different technological domains [Powell et al., 1996; Weitzman, 1998]. The increasing importance of R&D collaborations has spurred research for theoretical models studying these relationships, and for empirical tests of these models.

In this paper, we consider a general model of competition à la Cournot, where firms choose both R&D expenditures and output levels. Firms can reduce their costs of production by investing in R&D as well as by establishing R&D collaborations with other firms. An important – and realistic – innovation of our framework is to study the equilibrium outcomes in which firms can establish R&D collaborations with both competing firms in their own sector and firms in other sectors. In this model, R&D collaborations can be represented by a network. This allows us to write the profit function of each firm as a function of two matrices, $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ is the adjacency matrix of the network capturing all direct R&D collaborations, while $\mathbf{B}$ is a competition matrix that links every firm in the product market. Due to these two matrices and thus, these two opposing effects of technology spillovers and competition, all firms indirectly interact with all other firms. To illustrate this point, consider, for example, the car manufacturing sector. The price of a car is determined by the demand for cars and the competition with other car-producing firms. However, these firms have R&D collaborations not only with other car manufacturing firms but also with firms from other sectors (e.g. services or ICT).[1] As a result, the price of cars will also be indirectly influenced by firms from other industries.

We characterize the Nash equilibrium of our model for any type of R&D collaboration network (i.e. any matrix $\mathbf{A}$) as well as for any type of competition structure between firms (i.e. any matrix $\mathbf{B}$). We show that there exists a key trade off faced by firms between the *technology (or knowledge) spillover effect* of R&D and the *product rivalry effect* of competition. The former effect captures the *positive* impact of R&D collaborations on output and profits (through the matrix $\mathbf{A}$) while the latter captures the *negative* impact of competition and market stealing effects (through the matrix $\mathbf{B}$).

We show that the Nash equilibrium can be characterized by the fact that firms produce their goods proportionally to their Katz-Bonacich centrality [Bonacich, 1987], a well-known measure in the sociology literature that determines how central each firm is in the network, and also the degree of competition in the product market (see Proposition 1). In particular, a firm that is central in the R&D network will not always produce the highest output because the optimal output choice will also depend on the competition intensity the firm faces in the product market.

We also provide an efficiency analysis with an explicit expression for total welfare determined by producer and consumer surplus. We further derive lower and upper bounds on welfare that depend on the parameters as well as the topology of the R&D network (see Proposition 2). Moreover, we study which network is efficient (i.e. maximizing welfare) among all possible network configurations

[1] Many carmakers have realized that the next-generation vehicles cannot be built without more input from telecoms and software experts, while technology companies could benefit from a traditional car producing partner to help with industrial scale production, retail and repair. This generates incentives for carmakers and technology firms to cooperate more closely. See e.g. “Apple, BMW in courtship with an eye on car collaboration.” Reuters Technology (2015, Jul. 31th). Retrieved from `http://www.reuters.com/`.

(*network design*). We find that the complete graph is efficient when externalities from collaboration or competition are weak, but this may no longer be the case when they are high. In particular, in the presence of stronger externalities through R&D spillovers and competition, the star network generates higher welfare than the complete network. This happens when the welfare gains through concentration, which enter the welfare function through the Herfindahl index, dominate the welfare gains through maximizing total output.

Due to the existence of externalities through technology spillovers that are not internalized in the R&D decisions of firms, the social benefits of R&D are typically substantially greater than the private returns [cf. Jones and Williams, 1998; Spence, 1984]. This creates an environment where government funding programs that aim at fostering firms' R&D activities can be welfare improving. We analyze the optimal design of such R&D subsidy policy programs (where a planner can subsidize the firms' R&D effort costs) that take into account the network externalities in our model. We derive an exact formula for any type of network and competition structure that determines the optimal amount of subsidies per unit of R&D effort that should be given to each firm. We discriminate between homogeneous subsidies (see Proposition 3), where each firm obtains the same amount of subsidy per unit of R&D and targeted subsidies (see Proposition 4), where subsidies can be firm specific.

We then bring the model to the data by using a unique panel dataset of R&D collaborations and annual company reports over different sectors, countries and years. Using a structural econometric approach we estimate the first-order conditions of the model by testing the trade-off firms are facing between the *technology (or knowledge) spillover effect* of R&D collaborations and the *product rivalry effect* of competition mentioned above. In terms of identification strategy, we use firm and time fixed effects (as we have a panel of firms), an instrumental variables (IV) strategy and a network formation model. As predicted by the theoretical model, we find that the spillover effect has a *positive* and significant impact on output and profit while the competition effect has a *negative* and significant impact. We also show that the net effect of R&D collaborations is positive.

Following our theoretical results, we then empirically determine the optimal subsidy policy, both for the homogenous case where all firms receive the same subsidy per unit of R&D, and for the targeted case, where the subsidy per unit of R&D may vary across firms. The targeted subsidy program turns out to have a much higher impact on total welfare, as it can improve welfare by up to 140%, while the homogeneous subsidies can improve total welfare only by up to 4%.[2] We then empirically rank firms according to the welfare-maximizing subsidies that they receive by the planner. We find that the firms that should be subsidized the most are not necessarily the ones that have the highest market share, the largest number of patents or are the most central firms in the R&D network. Indeed these measures can only partially explain the ranking of firms that we find, as the market share is more related to the product market rivalry effect, while the R&D and patent stocks are more related to the technology spillover effect, and both enter into the computation of the the optimal subsidy program.

Finally, we compare our firm-specific optimal subsidies with those that are actually provided by government agencies such as EUREKA, a European intergovernmental organization that aims at fostering R&D in Europe.[3] We observe that the ranking of our optimal subsidy policy does not necessarily

[2]We find that the effect of targeted R&D subsidy programs can be large. Similarly Acemoglu and Akcigit [2006] find that the gain from size-dependent intellectual property right (IPR) policies can be substantial. Moreover, Akcigit [2009] finds a welfare rise by 65% from a uniform subsidy, and an additional 9% welfare gain from a size-dependent two-level R&D subsidy.

[3]See `http://www.eurekanetwork.org/`.

reflect the ranking of the actual subsidies implemented by EUREKA. However, this discrepancy is not surprising, as current public funding instruments such as EUREKA do not take into account network effects stemming from R&D collaborations that determine our optimal subsidy policy.

The rest of the paper is organized as follows. In the following Section 2, we compare our contribution to the existing literature. In Section 3, we develop a model of firms competing in the product market with technology sharing R&D collaborations that allow them to reduce their production costs. We characterize the Nash equilibrium of this game and show under which conditions it exists, is unique and interior. Section 4 determines welfare and investigates the optimal network structure of R&D collaborations. Section 5 discusses optimal R&D subsidies. Section 6 describes the data. Section 7 is divided into three parts. In Section 7.1, we define the econometric specification of our model while, in Section 7.2, we highlight our identification strategy. The estimation results are given in Section 7.3. Section 8 provides different robustness checks. The policy results of our empirical analysis are given in Section 9. Finally, Section 10 concludes the paper. All proofs can be found in Appendix A. The network definitions and characterizations used throughout the paper are given in the supplementary Appendix B, the Herfindahl concentration index is discussed in the supplementary Appendix C, an analysis in terms of Bertrand competition is performed in supplementary Appendix D. Supplementary Appendix E provides a theoretical model of intra- and interindustry collaborations while supplementary Appendix F provides a theoretical model of direct and indirect technology spillovers. Supplementary Appendix G gives a detailed description of how we construct and combine our different datasets for the empirical analysis.

2. Related Literature

Our paper lies at the intersection of different strands of the literature, and we would like to expose them in order to highlight our contribution.

Our theoretical model analyzes a game with strategic complementarities where firms decide about production and R&D effort by taking the network of complementarities as given. Thus, it belongs to a particular class of games known as *games on networks* [cf. Jackson and Zenou, 2015].[4] Compared to this literature, where a prominent paper is that of Ballester et al. [2006], we re-interpret their model in terms of R&D networks and extend their framework to account for competition between firms not only within the same product market but also between different product markets (see Proposition 1). This yields very general results that can encompass any possible network of collaborations and any possible interaction structure of competition between firms. We also provide an explicit welfare characterization, provide lower and upper bounds and determine which network maximizes total welfare in certain parameter ranges (see Proposition 2). To the best of our knowledge, this is one of the first papers to provide such an analysis.[5]

We also provide a policy analysis of R&D subsidies that consists in subsidizing firms' R&D efforts. We are able to determine the optimal subsidy level both when it is homogenous across firms (Proposition 3) and when it is targeted to specific firms (Proposition 4). We are not aware of any other studies of subsidy policies in the context of R&D networks.[6]

[4]The economics of networks is a growing field. For recent surveys of the literature, see Jackson [2008] and Jackson et al. [2016].

[5]An exception is the recent paper by Belhaj et al. [2016], who study network design in a game on networks with strategic complements, but neglect competition effects (global substitutes).

[6]There are papers that look at subsidies in industries with technology spillovers but the R&D network is not explicitly

In the industrial organization literature, there is a long tradition of models that analyze product and price competition with R&D collaborations, first pioneered by Arrow [1962]. One of their main insights is that the incentives to invest in R&D are reduced by the presence of such technology spillovers. This raised the interest in R&D cooperation as a means of internalizing spillovers. The seminal works by D'Aspremont and Jacquemin [1988] and Suzumura [1992] focus on the direct links between firms in the R&D collaboration process.

In this literature, however, there is no explicit network of R&D collaborations. The first paper that provides an explicit analysis of R&D networks is that by Goyal and Moraga-Gonzalez [2001].[7] The authors introduce a strategic Cournot oligopoly game in the presence of externalities induced by a network of R&D collaborations. Benefits arise in these collaborations from sharing knowledge about a cost-reducing technology. However, by forming collaborations, firms also change their own competitive position in the market as well as the overall market structure. Thus, there exists a two-way flow of influence from the market structure to the incentives to form R&D collaborations and, in turn, from the formation of collaborations to the market structure. Westbrock [2010] extends their framework to analyze welfare and inequality in R&D collaboration networks, but abstracts from R&D investment decisions.

Even though we do not study network formation as, for example, in Goyal and Moraga-Gonzalez [2001], compared to these papers, we are able to provide results for all possible networks with an arbitrary number of firms and a complete characterization of equilibrium output and R&D effort choices in multiple interdependent markets. We also determine policies related to network design and optimal R&D subsidy programs.

From an econometric perspective, there has recently been a significant progress in the literature on identification and estimation of social network models (see Blume et al. [2011] and Chandrasekhar [2016], for recent surveys). In his seminal work, Manski [1993] introduces a linear-in-means social interaction model with endogenous effects, contextual effects, and correlated effects. Manski shows that the linear-in-means specification suffers from the "reflection problem" and the different social interaction effects cannot be separately identified. Bramoullé et al. [2009] generalize Manski's linear-in-means model to a general social network model, whereas the endogenous effect is represented by the average outcome of the peers in the network. They provide conditions for the identification of the general social network model using the characteristics of an indirect connection as an instrument for the endogenous effect assuming that the network (and its adjacency matrix) is exogenous. However, if the adjacency matrix is endogenous, i.e., if there exists some unobservable factor that could affect both link formation and outcomes, then the above identification strategy will fail. Here, as we have a panel dataset where the network changes over time (whereas in many applications, the network is observed only at a single point in time; [see e.g. Bramoullé et al., 2009; Calvó-Armengol et al., 2009]), we adopt a similar identification strategy using instruments, but with both *firm and time fixed effects* to attenuate the potential endogeneity of the adjacency matrix. Then, we go even further by explicitly modeling the network formation process of R&D collaborations. Indeed, we add a first stage, where we explain an R&D collaboration between two firms by whether these two firms had an R&D collaboration or a common collaborator in the past, whether they are technologically close in terms of their patent portfolios and and whether they are geographically close. We then carry out

modelled. See e.g. Acemoglu et al. [2012]; Akcigit [2009]; Bloom et al. [2002]; Hinloopen [2001]; Leahy and Neary [1997]; Spencer and Brander [1983].

[7] See also Dawid and Hellmann [2014] and Goyal and Joshi [2003].

our instrumental variable (IV) estimation strategy described above using IVs based on the *predicted adjacency matrix* derived from the first stage.

There is also a large empirical literature on technology spillovers [see e.g. Bloom et al., 2013; Einiö, 2014; Griffith et al., 2004; Griliches, 1995], and R&D collaborations [see e.g. Hanaki et al., 2010; Powell et al., 2005]. Moreover, there is an extensive literature that estimates the effect of R&D subsidies on private R&D investments and other measures of innovative performance (for a survey, see Klette et al. [2000]). Methodologically, our paper belongs to a small but growing literature using structural empirical models to study the economics of innovation (see also the seminal works of Levin and Reiss [1988] and Griliches et al. [1986]) and the effects of R&D spillovers and technology diffusion [e.g. Eaton and Kortum, 2002; Takalo et al., 2013a].

There exist several papers that empirically study the impact of R&D subsidies on private R&D investments [e.g. Bloom et al., 2002; Feldman and Kelley, 2006; Takalo et al., 2013b]. However, to the best of our knowledge, our paper is the first that provides a ranking of firms in terms of R&D subsidies. Indeed, in our framework we are able to determine analytically the optimal R&D subsidy to each firm that maximizes total welfare, and this allows us to provide a ranking of all firms in our data. We show, in particular, that the highest subsidized firms are not necessarily those with the largest market share, a larger number of patents or the highest (betweenness, eigenvector or closeness) centrality in the network of R&D collaborations. We find, however, that larger firms should receive higher subsidies than smaller firms, as they generate more spillovers despite the fact that they lead to more competition. This result is in line with that of Bloom et al. [2013] who also find that smaller firms generate lower social returns to R&D because they operate more in technological niches.[8]

Further, observe that, contrary to Acemoglu et al. [2012] and Akcigit [2009], we do not focus on entry and exit but instead incorporate the network structure of R&D collaborating firms.[9] This allows us to take into account the R&D spillover effects of incumbent firms, which are typically ignored in studies of the innovative activity of incumbent firms versus entrants. Therefore, we see our analysis as complementary to that of Acemoglu et al. [2012], and we show that R&D subsidies can trigger considerable welfare gains when technology spillovers through R&D alliances are incorporated.

3. The Model

We consider a general Cournot oligopoly game where a set $\mathcal{N} = \{1, \ldots, n\}$ of firms is partitioned in $M \geq 1$ heterogeneous product markets.[10] We also allow for consumption goods to be imperfect substitutes (and thus differentiated products) by adopting the consumer utility maximization approach of Singh and Vives [1984]. We first consider the demand q_i, for the good produced by firm i in market $\mathcal{M}_m$, $m = 1, \ldots, M$. A representative consumer in market $\mathcal{M}_m$ obtains the following gross utility

[8] Schumpeter [1942] already argued that large firms are the most important contributors for generating innovations in an economy as only they would possess the required resources for setting up R&D laboratories and departments.

[9] Similar to our setup, Akcigit [2009] evaluates the effects of a size-dependent R&D subsidy on heterogeneous firms, and finds that the optimal size-dependent R&D subsidy policy does considerably better than an optimal uniform (size-independent) policy. However, differently to us Akcigit [2009] finds that the optimal (welfare-maximizing) policy provides higher subsidies to smaller firms. The difference between Akcigit [2009] and our framework is that he focusses on entry and exit while we incorporate technology spillovers thorough an explicit R&D network, in which concentration on large firms can induce large welfare gains. Moreover in Akcigit [2009] firms tend to lose their R&D capabilities with firm age and size, while we do not make this assumption.

[10] In the empirical analysis carried out in Section 6, we identify the market in which a firm operates by its primary 4-digit Standard Industrial Classification (SIC) code. As a result, a market corresponds to a particular industry or sector.

from consumption of the goods $(q_i)_{i\in\mathcal{M}_m}$

$$\bar{U}_m((q_i)_{i\in\mathcal{M}_m}) = \alpha_m \sum_{i\in\mathcal{M}_m} q_i - \frac{1}{2}\sum_{i\in\mathcal{M}_m} q_i^2 - \frac{\rho}{2}\sum_{i\in\mathcal{M}_m}\sum_{j\in\mathcal{M}_m, j\neq i} q_i q_j.$$

In this formulation, the parameter α_m captures the market size or the heterogeneity in products, whereas $\rho \in (0,1]$ measures the degree of substitutability between products. In particular, $\rho \to 1$ depicts a market of perfectly substitutable goods, while $\rho \to 0$ represents the case of local monopolies.

The consumer maximizes net utility $U_m = \bar{U}_m - \sum_{i\in\mathcal{M}_m} p_i q_i$, where p_i is the price of good i. This gives the inverse demand function for firm i

$$p_i = \bar{\alpha}_i - q_i - \rho \sum_{j\in\mathcal{M}_m, j\neq i} q_j, \tag{1}$$

where $\bar{\alpha}_i = \sum_{m=1}^{M} \alpha_m \mathbb{1}_{\{i\in\mathcal{M}_m\}}$. In the model, we will study both the general case where $\rho > 0$ but also the special case where $\rho = 0$. The latter case is when firms are local monopolies so that the price of the good produced by each firm i is only determined by its quantity q_i (and the size of the market) and not by the quantities of other firms, i.e. $p_i = \bar{\alpha}_i - q_i$.

Firms can reduce their production costs by investing in R&D as well as by establishing an R&D collaboration with another firm.[11] The amount of this cost reduction depends on the R&D effort e_i of firm i and the R&D efforts of the firms that are collaborating with i, i.e., R&D collaboration partners. Given the effort level $e_i \in \mathbb{R}_+$, the marginal cost c_i of firm i is given by[12,13]

$$c_i = \bar{c}_i - e_i - \varphi \sum_{j=1}^{n} a_{ij} e_j, \tag{2}$$

The network G can be represented by a symmetric $n \times n$ *adjacency matrix* $\mathbf{A}$. Its elements $a_{ij} \in \{0,1\}$ indicate whether there exists a link between nodes i and j.[14] In the context of our model, $a_{ij} = 1$ if firms i and j set up an R&D collaboration (0 otherwise) and $a_{ii} = 0$. In Equation (2), the total cost reduction for firm i stems from its own research effort e_i and the research effort of all other collaborating firms, i.e., *knowledge spillovers*, which is captured by the term $\sum_{j=1}^{n} a_{ij} e_j$, where $\varphi \geq 0$ is the marginal cost reduction due to the neighbor's R&D effort.[15] We assume that R&D effort is costly. In particular, the cost of R&D effort is an increasing function, exhibits decreasing returns, and is given by $\frac{1}{2}e_i^2$ [cf. e.g. D'Aspremont and Jacquemin, 1988]. Firm i's profit is then given by

$$\pi_i = (p_i - c_i)q_i - \frac{1}{2}e_i^2. \tag{3}$$

[11] For example, Bernstein [1988] finds that R&D spillovers decrease the unit costs of production for a sample of Canadian firms.

[12] The specification of marginal costs follows Goyal and Moraga-Gonzalez [2001] and generalizes earlier studies such as that by D'Aspremont and Jacquemin [1988] and Leahy and Neary [1997] where spillovers are assumed to take place between all firms in the industry and no distinction between collaborating and non-collaborating firms is made.

[13] We assume that the R&D effort independent marginal cost $\bar{c}_i$ is large enough such that marginal costs, c_i, are always positive for all firms $i \in \mathcal{N}$. See Equation (37) in the proof of Proposition 1 in Appendix A for a precise lower bound on $\bar{c}_i$.

[14] See supplementary Appendix B.1 for more definitions and characterizations of networks.

[15] In Equation (76) in supplementary Appendix F we present an extension of the model where firms benefit from both, direct technology spillovers between collaborating firms and indirect technology spillovers between non-collaborating firms. It is therefore important to note that we can generalize the model to capture potential technology spillovers between firms which are not necessarily engaged in an R&D collaboration.

Inserting marginal cost from Equation (2) and inverse demand from Equation (1) into Equation (3) gives the following strictly quasi-concave profit function for firm i

$$
\begin{aligned}
\pi_i &= (\bar{\alpha}_i - q_i - \rho \sum_{j \in \mathcal{M}_m, j \neq i} q_j - \bar{c}_i + e_i + \varphi \sum_{j=1}^{n} a_{ij} e_j) q_i - \frac{1}{2} e_i^2 \\
&= (\bar{\alpha}_i - \bar{c}_i) q_i - q_i^2 - \rho \sum_{j=1}^{n} b_{ij} q_i q_j + q_i e_i + \varphi q_i \sum_{j=1}^{n} a_{ij} e_j - \frac{1}{2} e_i^2, \qquad (4)
\end{aligned}
$$

where $b_{ij} \in \{0, 1\}$ indicates whether firms i and j operate in the same market or not, and let $\mathbf{B}$ be the $n \times n$ matrix whose ij-th element is b_{ij}. In Equation (4), we can write $\sum_{j \in \mathcal{M}_m, j \neq i} q_j = \sum_{j=1}^{n} b_{ij} q_j$ since $i \in \mathcal{M}_m$ and $b_{ij} = 1$ indicates that $j \in \mathcal{M}_m$. Let $\mathbf{B}$ be the $n \times n$ matrix whose ij-th element is b_{ij}. $\mathbf{B}$ captures which firms operate in the same market and which firms do not. Consequently, $\mathbf{B}$ can be written as a block diagonal matrix with zero diagonal and blocks of size $|\mathcal{M}_m|$, $m = 1, \ldots, M$. An illustration can be found below:[16]

$$
\mathbf{B} = \left(\begin{array}{cccc|cccc|c}
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots \\
1 & 0 & \cdots & \vdots & \vdots & & & \vdots & \\
\vdots & \vdots & \ddots & 1 & \vdots & & & \vdots & \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & \cdots \\
\hline
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & \\
\vdots & & & \vdots & 1 & 0 & \cdots & \vdots & \\
\vdots & & & \vdots & \vdots & \vdots & \ddots & 1 & \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & \\
\hline
\vdots & & & \vdots & & & & & \ddots
\end{array} \right)_{n \times n}
$$

We consider quantity competition among firms à la Cournot.[17] The next proposition establishes the Nash equilibrium where each firm i simultaneously chooses *both* its output, q_i, and its R&D effort, e_i, in an arbitrary network $\mathbf{A}$ of R&D collaborations and arbitrary competition matrix $\mathbf{B}$.[18]

Proposition 1. *Consider the n–player simultaneous move game with payoffs given by Equation (4) and strategy space in $\mathbb{R}_+^n \times \mathbb{R}_+^n$. Denote by $\mu_i \equiv \bar{\alpha}_i - \bar{c}_i$ for all $i \in \mathcal{N}$, $\boldsymbol{\mu}$ the corresponding $n \times 1$ vector with components μ_i, $\phi \equiv \varphi/(1-\rho)$, $\rho \in (0, 1]$, $\varphi \geq 0$, $|\mathcal{M}_m|$ the size of market m for $m = 1, \ldots, M$, $\mathbf{I}_n$ the $n \times n$ identity matrix, $\mathbf{u}$ the $n \times 1$ vector of ones and $\lambda_{\mathrm{PF}}(\mathbf{A})$ the largest eigenvalue of $\mathbf{A}$. Denote also by $\underline{\mu} = \min_i \{\mu_i \mid i \in \mathcal{N}\}$ and $\overline{\mu} = \max_i \{\mu_i \mid i \in \mathcal{N}\}$, with $0 < \underline{\mu} < \overline{\mu}$.*

(i) Let the firms' output levels be bounded from above and below such that $0 \leq q_i \leq \bar{q}$ for all $i \in \mathcal{N}$. Then a Nash equilibrium always exists. Further, if either $\rho = 0$, $\varphi = 0$ or[19]

$$
\rho + \varphi < \left(\max \left\{ \lambda_{\mathrm{PF}}(\mathbf{A}), \max_{m=1,\ldots,M} \{|\mathcal{M}_m| - 1\} \right\} \right)^{-1} \qquad (5)
$$

then the Nash equilibrium is unique.

[16]The observed competition matrix $\mathbf{B}$ from our data is shown in Figure 8 in the empirical Section 7.

[17]In supplementary Appendix D we show that the same functional forms for best response quantities and efforts can be obtained for price setting firms under Bertrand competition as we find them in the case of Cournot competition.

[18]See supplementary Appendix B.3 for a precise definition of the Bonacich centrality used in the proposition.

[19]A weaker bound can be obtained requiring that $\varphi \lambda_{\mathrm{PF}}(\mathbf{A}) + \rho \lambda_{\mathrm{PF}}(\mathbf{B}) < 1$. See also Figure A.2 in the proof of Proposition 1 in Appendix A.

(ii) If in addition

$$\rho \max_{m=1,\ldots,M}\{|\mathcal{M}_m|-1\} < 1-\varphi\lambda_{PF}(\mathbf{A}), \tag{6}$$

holds then there exists a unique interior Nash equilibrium with output levels, $0 < q_i < \bar{q}$ *for all* $i \in \mathcal{N}$*, and a large enough production capacity* $\bar{q}$*, given by*

$$\mathbf{q} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}. \tag{7}$$

(iii) Assume that there exists only a single market so that $M = 1$*. Let the* $\boldsymbol{\mu}$*-weighted Bonacich centrality be given by* $\mathbf{b}_{\boldsymbol{\mu}}(G,\phi) \equiv (\mathbf{I}_n - \phi\mathbf{A})^{-1}\boldsymbol{\mu}$*. If*

$$\phi\lambda_{\mathrm{PF}}(\mathbf{A}) + \frac{n\rho}{1-\rho}\left(\frac{\overline{\mu}}{\underline{\mu}}-1\right) < 1, \tag{8}$$

holds, then there exists a unique interior Nash equilibrium with output levels given by

$$\mathbf{q} = \frac{1}{1-\rho}\left(\mathbf{b}_{\boldsymbol{\mu}}(G,\phi) - \frac{\rho\,\|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1}{1+\rho(\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1-1)}\mathbf{b}_{\mathbf{u}}(G,\phi)\right). \tag{9}$$

(iv) Assume a single market (i.e., $M = 1$*) and that* $\mu_i = \mu$ *for all* $i \in \mathcal{N}$*. If* $\phi\lambda_{\mathrm{PF}}(\mathbf{A}) < 1$*, then there exists a unique interior Nash equilibrium with output levels given by*

$$\mathbf{q} = \frac{\mu}{1+\rho(\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1-1)}\mathbf{b}_{\mathbf{u}}(G,\phi). \tag{10}$$

(v) Assume a single market (i.e., $M = 1$*),* $\mu_i = \mu$ *for all* $i \in \mathcal{N}$ *and that goods are non-substitutable (i.e.,* $\rho = 0$*). If* $\varphi < \lambda_{\mathrm{PF}}(\mathbf{A})^{-1}$*, then the unique equilibrium quantities are given by* $\mathbf{q} = \mu\mathbf{b}_{\mathbf{u}}(G,\varphi)$*.*

(vi) Let $\mathbf{q}$ *be the unique Nash equilibrium quantities in any of the above cases (i) to (v), then for all* $i \in \mathcal{N} = \{1,\ldots,n\}$ *the equilibrium profits are given by*

$$\pi_i = \frac{1}{2}q_i^2, \tag{11}$$

and the equilibrium efforts are given by

$$e_i = q_i. \tag{12}$$

The existence of an equilibrium stated in case (i) of the proposition follows from the equivalence of the associated first order conditions with a bounded linear complementarity problem (LCP) [Byong-Hun, 1983].[20] Further, a unique solution is guaranteed to exist if $\rho = 0$ or when the matrix $\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A}$ is positive definite. The condition for the latter is stated in Equation (5) in case (ii) of the proposition. The subsequent parts of the proposition state the Nash equilibrium starting from the most general case where firms can operate and have links in any market (case (ii)) to the case where all firms operate in the same market (case (iii)) and where they have the same fixed cost of production and no product heterogeneity (case (iv)) and, finally, when goods are not substitutable (case (v)). Indeed, it is easily verified (see Appendix A; proof of Proposition 1) that the first-order condition with respect

[20]This is the linear version of the mixed complementarity problem analyzed in Simsek et al. [2005]. For a detailed discussion and analysis of LCP see Cottle et al. [1992].

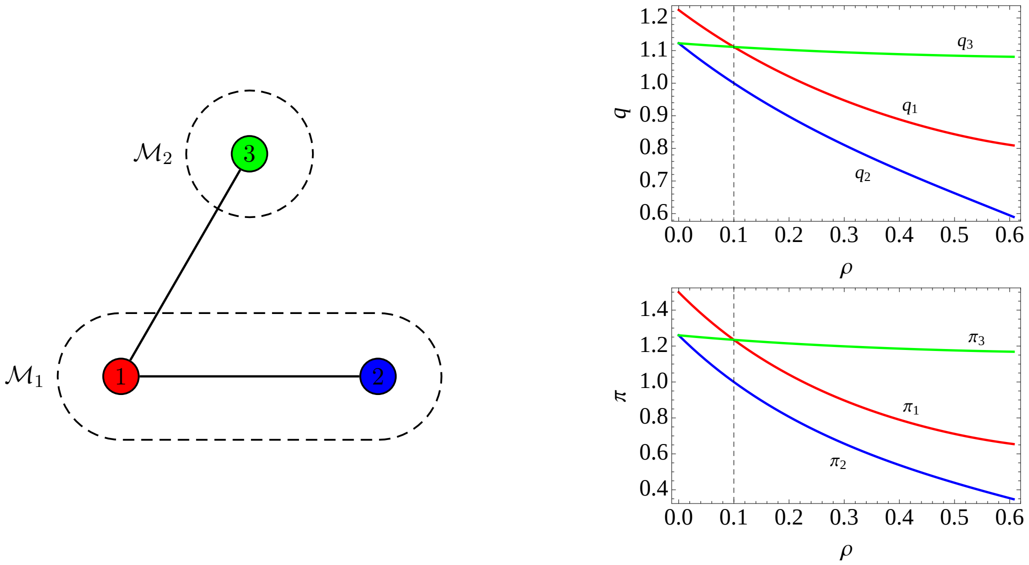


Figure 1: Equilibrium output from Equation (14) and profits for the three firms with varying values of the competition parameter $0 \leq \rho \leq \frac{1}{2}\left(\sqrt{2}-2\varphi\right)$, $\mu = 1$ and $\varphi = 0.1$. Profits of firms 1 and 3 intersect at $\rho = \varphi$ (indicated with a dashed line).

to R&D effort e_i is given by Equation (12),[21] while the first-order condition with respect to quantity q_i leads to

$$q_i = \mu_i - \rho \sum_{j=1}^{n} b_{ij} q_j + \varphi \sum_{j=1}^{n} a_{ij} q_j, \tag{13}$$

or, in matrix form, $\mathbf{q} = \boldsymbol{\mu} - \rho \mathbf{B}\mathbf{q} + \varphi \mathbf{A}\mathbf{q}$. In terms of the literature on games on networks [Jackson and Zenou, 2015], this proposition generalizes the results of Ballester et al. [2006] and Calvó-Armengol et al. [2009] for the case of local competition in different markets and choices of both effort and quantity. This proposition provides a total characterization of an interior Nash equilibrium as well as its existence and uniqueness in a very general framework when different markets and different products are considered. If we consider case (i), the new conditions are Equations (5) and (6), which guarantee the existence, uniqueness and interiority of the Nash equilibrium solutions in the most general case. In case (ii) where all firms operate in the same market, in order to obtain a unique interior solution, only the condition in Equation (8) is required, which generalizes the usual condition $\phi\lambda_{\mathrm{PF}}(\mathbf{A}) < 1$ given, for example, in Ballester et al. [2006]. In fact, the condition in Equation (8) imposes a more stringent requirement on $\rho, \varphi, \mathbf{A}$ as the left-hand side of the inequality is now augmented by $\frac{n\rho}{1-\rho}\left(\frac{\overline{\mu}}{\underline{\mu}} - 1\right) \geq 0$. That is, everything else equal, the higher the discrepancy $\overline{\mu}/\underline{\mu}$ of marginal payoffs at the origin, the lower is the level of network complementarities $\phi\lambda_{\mathrm{PF}}(\mathbf{A})$ that are compatible with a unique and interior Nash equilibrium.

More generally, the key insight of Proposition 1 is the interaction between the *network effect*, through the adjacency matrix $\mathbf{A}$, and the *market effect*, through the competition matrix $\mathbf{B}$ and this is why the first-order condition with respect to q_i given by Equation (13) takes both of them into account. To better understand this result, consider the following simple example where firms 1 and 2 as well as firms 1 and 3 have R&D collaborations. Suppose that there are two markets where firms 1 and 2 operate in the same market $\mathcal{M}_1$ while firm 3 operates alone in market $\mathcal{M}_2$ (see Figure 1).

[21] The proportional relationship between R&D effort levels and output in Equation (12) has been confirmed in a number of empirical studies [see e.g. Cohen and Klepper, 1996a,b; Klette and Kortum, 2004].

Then, the adjacency matrix $\mathbf{A}$ and the competition matrix $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume that firms are homogeneous such that $\mu_i = \mu$ for $i = 1, 2, 3$. Using Proposition 1, the equilibrium output is given by

$$\mathbf{q} = \mu(\mathbf{I} - \varphi\mathbf{A} + \rho\mathbf{B})^{-1}\mathbf{u} = \frac{\mu}{1 - 2\varphi^2 + 2\varphi\rho - \rho^2} \begin{pmatrix} 1 + 2\varphi - \rho \\ (\varphi + 1)(1 - \rho) \\ (1 + \rho)(1 + \varphi - \rho) \end{pmatrix}. \tag{14}$$

Profits are equal to $\pi_i = q_i^2/2$ for $i = 1, 2, 3$. The condition for an interior equilibrium is $\rho + \varphi < 1/\sqrt{2}$. Figure 1 shows an illustration of equilibrium outputs and profits for the three firms with varying values of the competition parameter $0 \leq \rho \leq \frac{1}{2}\left(\sqrt{2} - 2\varphi\right)$, $\mu = 1$ and $\varphi = 0.1$. We see that firm 1 has higher profits due to having the largest number of R&D collaborations when competition is weak (ρ is low compared to φ). However, when ρ increases, its profits decrease and become smaller than the profit of firm 3 when $\rho > \varphi$. This result highlights the key trade off faced by firms between the *technology (or knowledge) spillover effect* and the *product rivalry effect* of R&D [cf. Bloom et al., 2013] since the former increases with φ, which captures the intensity of the spillover effect while the latter increases with ρ, which indicates the degree of competition in the product market.

To better understand these two effects, consider the case of a single market ($M = 1$), where all three firms compete with each other in the same market so that[22]

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

If $\varphi/(1 - \rho) < 1/\sqrt{2}$, then the unique equilibrium output will be given by

$$\mathbf{q} = \frac{\mu}{1 - 2\varphi^2 + 4\varphi\rho + \rho - 2\rho^2} \begin{pmatrix} 1 + 2\varphi - \rho \\ 1 + \varphi - \rho \\ 1 + \varphi - \rho \end{pmatrix}. \tag{15}$$

Since there is only one market, the position in the network will determine which firm will produce the most and have the highest profit. As firm 1 is the most central firm in the network and has the highest Bonacich centrality, it has the highest profit. This is also immediately apparent from Equation (15). In other words, when $M = 1$, only the *technology (or knowledge) spillover effect* is of importance and the position in the network is the only determinant of output and profit. However, we saw that this was not the case in the previous example with two markets because, as compared to firm 3, even if firm 1 had the highest Bonacich centrality, it was competing with firm 2 on the product market while firm 3 had no competitor on its market. In other words, there is now a trade off between the position in the network (*technology (or knowledge) spillover effect*) and the position in the product market

[22] It is easily verified that, in this case, $\mathbf{B} = (\mathbf{u}\mathbf{u}^\top - \mathbf{I}_n)$ where $\mathbf{u} = (1, \ldots, 1)^\top$ is an n-dimensional vector of ones.

(*product rivalry effect*). We have seen that, depending on the values of ρ and φ, firm 1 can have a higher or lower output and profit than firm 3.

4. Welfare

We next turn to analyzing welfare in the economy. We will consider different cases from general to more specific ones. Inserting the inverse demand from Equation (1) into net utility U_m of the consumer in market $\mathcal{M}_m$ shows that

$$U_m = \frac{1}{2}\sum_{i\in\mathcal{M}_m} q_i^2 + \frac{\rho}{2}\sum_{i\in\mathcal{M}_m}\sum_{j\in\mathcal{M}_m, j\neq i} q_i q_j.$$

For given quantities, the consumer surplus is strictly increasing in the degree ρ of substitutability between products. In the special case of non-substitutable goods, when $\rho \to 0$, we obtain $U_m = \frac{1}{2}\sum_{i\in\mathcal{M}_m} q_i^2$, while in the case of perfectly substitutable goods, when $\rho \to 1$, we get $U_m = \frac{1}{2}\left(\sum_{i\in\mathcal{M}_m} q_i\right)^2$. The total consumer surplus is then given by $U = \sum_{m=1}^{M} U_m$. The producer surplus is given by aggregate profits $\Pi = \sum_{i=1}^{n}\pi_i$. As a result, total welfare is equal to $W = U + \Pi$.

Inserting profits as a function of output from Equation (11) leads to

$$W(G) = \sum_{i=1}^{n} q_i^2 + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} b_{ij} q_i q_j = \mathbf{q}^\top\mathbf{q} + \frac{\rho}{2}\mathbf{q}^\top\mathbf{B}\mathbf{q}.$$

To gain further insights, we will assume in the following that there is only a single market (with $M = 1$, $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = 1$ for all $i, j \in \mathcal{N}$) and make the homogeneity assumption that $\mu_i = \mu$ for all $i \in \mathcal{N}$. Then, welfare can be written as follows

$$W(G) = \frac{2-\rho}{2}\|\mathbf{q}\|_2^2 + \frac{\rho}{2}\|\mathbf{q}\|_1^2,$$

where $\|\mathbf{q}\|_p \equiv (\sum_{i=1}^{n} q_i^p)^{\frac{1}{p}}$ is the L^p-norm of $\mathbf{q}$. Further, note that the Herfindahl-Hirschman industry concentration index is given by [cf. Hirschman, 1964; Tirole, 1988][23]

$$H = \sum_{i=1}^{n}\left(\frac{q_i}{\sum_{j=1}^{n} q_j}\right)^2 = \frac{\|\mathbf{q}\|_2^2}{\|\mathbf{q}\|_1^2}, \tag{16}$$

and denoting total output by $Q = \|\mathbf{q}\|_1$, we can write welfare as follows

$$W(G) = \frac{1}{2}\|\mathbf{q}\|_1^2\left((2-\rho)\frac{\|\mathbf{q}\|_2^2}{\|\mathbf{q}\|_1^2} + \rho\right) = \frac{Q^2}{2}\left((2-\rho)H + \rho\right). \tag{17}$$

One can show that total output Q is largest in the complete graph [cf. Ballester et al., 2006]. However, as welfare depends on both, output Q and industry concentration H, it is not obvious that the complete graph (where $H = 1/n$ is small) is also maximizing welfare. As the following proposition illustrates, we can conclude that the complete graph is welfare maximizing (i.e. efficient) when externalities are weak, but this may no longer be the case when ρ or φ are high.

Proposition 2. *Assume that $\mu_i = \mu$ for all $i = 1, \ldots, n$, and let ρ, μ, φ and ϕ satisfy the restrictions*

[23] For more discussion of the Herfindahl index in the Nash equilibrium see the supplementary Appendix C.

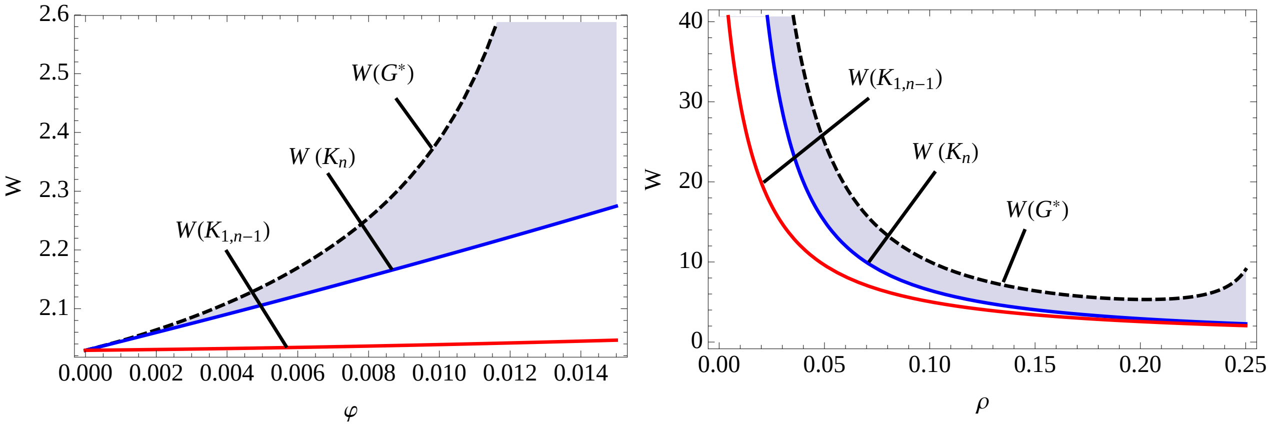


Figure 2: (Left panel) The upper and lower bounds of Equation (18) with $n = 50$, $\rho = 0.25$ for varying values of φ. (Right panel) The upper and lower bounds of Equation (18) with $n = 50$, $\varphi = 0.015$ for varying values of ρ.

of Proposition 1. Denote by $\mathcal{G}^n$ *the class of graphs with* n *nodes,* $K_n \in \mathcal{G}^n$ *the complete graph,* $K_{1,n-1} \in \mathcal{G}^n$ *the star network, and let the efficient graph be denoted by* $G^* = \mathrm{argmax}_{G\in\mathcal{G}^n} W(G)$*.*

(i) Welfare of the efficient graph G^* *can be bounded from above and below as follows:*

$$\frac{\mu^2 n(2+(n-1)\rho)}{2(1+(n-1)(\rho-\varphi))^2} \leq W(G^*) \leq \frac{\mu^2 n\left((1-\rho)^2(2+(n-1)\rho)-n(n-1)^2\rho\varphi^2\right)}{2((1+(n-1)(\rho-\varphi))^2\left((1-\rho)^2-(n-1)^2\varphi^2\right)}. \tag{18}$$

(ii) In the limit of independent markets, when $\rho \to 0$*, the complete graph is efficient,* $K_n = G^*$*.*

(iii) In the limit of weak R&D spillovers, when $\varphi \to 0$*, the complete graph is efficient,* $K_n = G^*$*.*

(iv) There exists a $\varphi^*(n,\rho) > 0$ *(which is decreasing in* ρ*) such that* $W(K_n) < W(K_{1,n-1})$ *for all* $\varphi > \varphi^*(n,\rho)$*, and the complete graph is not efficient,* $K_n \neq G^*$*.*

The upper and lower bounds of case (i) in Proposition 2 on welfare can be seen in Figure 2. The bounds indicate that welfare is typically increasing in strength of technology spillovers, φ, and decreasing in the degree of competition, ρ, at least when these are not too high. The figure is also consistent with cases (ii) and (iii), where it is shown that for weak spillovers the complete graph is efficient. However, Proposition 2, case (iv), shows that in the presence of stronger externalities through R&D spillovers and competition, the star network generates higher welfare than the complete network. This happens when the welfare gains through concentration, which enter the welfare function through the Herfindahl index H in Equation (17), dominate the welfare gains through maximizing total output Q.

While total output Q (and total R&D) is increasing with the degree of competition, measured by ρ (*Schumpeterian effect*; see e.g. Aghion et al. [2014]), this may not necessarily hold for welfare. This is illustrated in the right panel in Figure 3 where welfare for the star is shown for varying values of ρ. The presence of externalities through R&D spillovers and business stealing effects through market competition in highly centralized networks can thus give rise to a non-monotonic relationship between competition and welfare [cf. Aghion et al., 2005]. The centralization of the network structure, however, seems to be important for this result, as for example in a regular graph (such as the complete graph) welfare is decreasing monotonically with increasing ρ.[24]

[24]Decreasing welfare with increasing competition is a feature not only of the standard Cournot model (without exter-

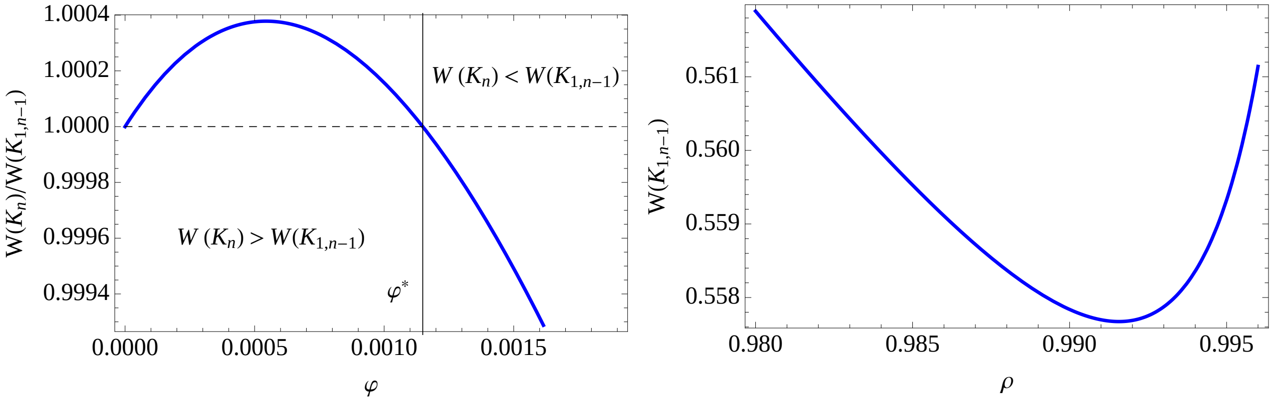


Figure 3: (Left panel). The ratio of welfare in the complete graph, K_n, and the star, $K_{1,n-1}$, for $n = 10$, $\rho = 0.981$ and varying values of φ ($< ((1-\rho)/\lambda_{\mathrm{PF}}(K_n) = 0.002$) (Right panel) Welfare in the star, $K_{1,n-1}$, with varying values of ρ for $n = 10$ and $\varphi = 0.001$ ($< (1-\rho)/\lambda_{\mathrm{PF}}(K_{1,n-1})$ for all values of ρ considered).

An exact and exhaustive characterization of the efficient network in the presence of competition (i.e. when $\rho > 0$) is extremely complicated and thus remains an unresolved issue. Because of the difficulty of the optimal network design problem – both from a theoretical and practical point of view – we next turn to an alternative, less demanding policy, where we attempt to induce welfare gains by subsidizing firms' R&D efforts.

5. The R&D Subsidy Policy

Because of the externalities generated by R&D activities, market resource allocation will typically not be socially optimal. Policy can resolve this market failure through R&D subsidy programs. In order to foster innovative activities and economic growth, governments in numerous countries have introduced R&D support programs aimed at increasing the R&D effort in the private sector.[25] Moreover, national governments in a number of countries subsidize the R&D activities of domestic firms, particularly in industries where foreign and domestically owned firms are in competition for international markets. Such programs are, for example, the EUREKA program in the European Union or the SPIR program in the United States.

To better understand R&D policies in collaboration networks, we extend our framework by considering an optimal R&D subsidy program that reduces the firms' R&D costs. For our analysis, we first assume that all firms obtain a homogeneous subsidy per unit of R&D effort spent. Then, we proceed by allowing the social planner to differentiate between firms and implement firm-specific R&D subsidies.[26]

nalities) but also of many traditional models in the literature including Aghion and Howitt [1992], and Grossman and Helpman [1991].

[25] Public R&D grants covered about 7.5% of private R&D in the OECD countries in 2004 [OECD, 2012]. R&D tax credits are another commonly used fiscal incentive for R&D investment. As of 2006, 32 states in the U.S. provided a tax credit on general, company-funded R&D [cf. Wilson, 2009]. For an overview of R&D tax credits see Bloom et al. [2002].

[26] We would like to emphasize that, as we have normalized the cost of R&D to one in the profit function of Equation (3), the absolute values of R&D subsidies are not meaningful in the subsequent analysis, but rather relative comparisons across firms are.

5.1. Homogeneous R&D Subsidies

An active government is introduced that can provide a subsidy, $s \in [0, \bar{s}]$ per unit of R&D effort. It is assumed that each firm receives the same per unit R&D subsidy. The profit of firm i with an R&D subsidy can then be written as:[27]

$$\pi_i = (\bar{\alpha} - \bar{c}_i)q_i - q_i^2 - \rho q_i \sum_{j \neq i} b_{ij} q_j + q_i e_i + \varphi q_i \sum_{j=1}^{n} a_{ij} e_j - \frac{1}{2} e_i^2 + s e_i. \tag{19}$$

This formulation follows Hinloopen [2000, 2001] and Spencer and Brander [1983], where each firm i receives a subsidy per unit of R&D.[28] The government (or the planner) is here introduced as an agent that can set subsidy rates on R&D effort in a period before the firms spend on R&D. The assumption that the government can pre-commit itself to such subsidies and thus can act in this leadership role is fairly natural. As a result, this subsidy will affect the levels of R&D conducted by firms, but not the resolution of the output game. In this context, the optimal R&D subsidy $s^* \in [0, \bar{s}]$, $\bar{s} > 0$, determined by the planner is found by maximizing total welfare $W(G, s)$ less the cost of the subsidy $s \sum_{i=1}^{n} e_i$, taking into account the fact that firms choose output and effort for a given subsidy level by maximizing profits in Equation (19). If we define net welfare as $\overline{W}(G, s) \equiv W(G, s) - s \sum_{i=1}^{n} e_i$, the social planner's problem is given by

$$s^* = \arg\max_{s \in [0, \bar{s}]} \overline{W}(G, s).$$

The following proposition derives the Nash equilibrium quantities and efforts and the optimal subsidy level that solves the planner's problem.

Proposition 3. *Consider the n–player simultaneous move game with profits given by Equation (19) where firms choose quantities and efforts in the strategy space in $\mathbb{R}_+^n \times \mathbb{R}_+^n$. Further, let μ_i, $i \in \mathcal{N}$ be defined as in Proposition 1.*

(i) If Equation (5) holds, then the matrix $\mathbf{M} = (\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A})^{-1}$ exists, and the unique interior Nash equilibrium in quantities with subsidies (in the second stage) is given by

$$\mathbf{q} = \tilde{\mathbf{q}} + s \mathbf{r}, \tag{20}$$

where $\tilde{\mathbf{q}} = \mathbf{M} \boldsymbol{\mu}$ and $\mathbf{r} = \mathbf{M}(\mathbf{u} + \varphi \mathbf{A} \mathbf{u})$. The equilibrium profits are given by

$$\pi_i = \frac{q_i^2 + s^2}{2}, \tag{21}$$

and efforts are given by $e_i = q_i + s$ for all $i = 1, \ldots, n$.

(ii) Assume that goods are not substitutable, i.e. $\rho = 0$. Then if $\sum_{i=1}^{n} (1 + 2r_i(1 - r_i)) \geq 0$, the optimal subsidy level (in the first stage) is given by

$$s^* = \frac{\sum_{i=1}^{n} \tilde{q}_i (2r_i - 1)}{\sum_{i=1}^{n} (1 - 2r_i (1 - r_i))},$$

[27] Similar to Section 3 we assume that the R&D effort independent marginal cost $\bar{c}_i$ is large enough such that marginal costs, c_i, are always positive for all firms $i \in \mathcal{N}$. See Equation (56) in the proof of Proposition 3 in Appendix A for a precise lower bound on $\bar{c}_i$.

[28] Leahy and Neary [1997] have also investigated subsidies to production in a similar framework.

provided that $0 < q_i < \bar{q}$ *for all* $i = 1, \dots, n$ *and* $0 < s^* < \bar{s}$.

(iii) Assume that goods are substitutable, i.e. $\rho > 0$. *Then if*

$$\sum_{i=1}^{n} \left(1 + 2r_i(1 - r_i) - \rho \sum_{j=1}^{n} b_{ij} r_i r_j \right) \geq 0,$$

the optimal subsidy level (in the first stage) is given by

$$s^* = \frac{\sum_{i=1}^{n} \left(\tilde{q}_i(2r_i - 1) + \frac{\rho}{2} \sum_{j=1}^{n} b_{ij}(\tilde{q}_i r_j + \tilde{q}_j r_i) \right)}{\sum_{i=1}^{n} \left(1 + r_i \left(2(1 - r_i) - \rho \sum_{j=1}^{n} b_{ij} r_j \right) \right)},$$

provided that $0 < q_i < \bar{q}$ *for all* $i = 1, \dots, n$ *and* $0 < s^* < \bar{s}$.

In part (i) of Proposition 3, we solve the second stage of the game where firms decide their output given the homogenous subsidy s. In parts (ii) and (iii) of the proposition, we solve the first stage when the planner optimally determines the subsidy per R&D effort when goods are not substitutable, i.e. $\rho = 0$, and when they are substitutable ($\rho > 0$). The proposition then determines the exact value of the optimal subsidy to be given to the firms embedded in a network of R&D collaborations in both cases. Interestingly, the optimal subsidy depends on the vector $\mathbf{r} = \mathbf{Mu} + \varphi \mathbf{MAu}$, where $\mathbf{Mu}$ is the Nash equilibrium output in the homogeneous firms case (see also Equation (7)) and the vector $\mathbf{d} = \mathbf{Au}$ determines the *degree* (i.e. number of links) of each firm.

5.2. Targeted R&D Subsidies

We now consider the case where the planner can discriminate between firms by offering different subsidies. In other words, we assume that each firm i, for all $i = 1, \dots, n$, obtains a subsidy $s_i \in [0, \bar{s}]$ per unit of R&D effort. The profit of firm i can then be written as:[29]

$$\pi_i = (\bar{\alpha} - \bar{c}_i)q_i - q_i^2 - \rho q_i \sum_{j \neq i} b_{ij} q_j + q_i e_i + \varphi q_i \sum_{j=1}^{n} a_{ij} e_j - \frac{1}{2} e_i^2 + s_i e_i. \tag{22}$$

As above, the optimal R&D subsidies $\mathbf{s}^*$ are then found by maximizing welfare $W(G, \mathbf{s})$ less the cost of the subsidy $\sum_{i=1}^{n} s_i e_i$, when firms are choosing output and effort for a given subsidy level by maximizing profits in Equation (22). If we define net welfare as $\overline{W}(G, \mathbf{s}) \equiv W(G, \mathbf{s}) - \sum_{i=1}^{n} e_i s_i$, then the solution to the social planner's problem is given by

$$\mathbf{s}^* = \arg\max_{\mathbf{s} \in [0, \bar{s}]^n} \overline{W}(G, \mathbf{s}).$$

The following proposition derives the Nash equilibrium quantities and efforts (second stage) and the optimal subsidy levels that solve the planner's problem (first stage).

Proposition 4. *Consider the* n*–player simultaneous move game with profits given by Equation (19) where firms choose quantities and efforts in the strategy space in* $\mathbb{R}_+^n \times \mathbb{R}_+^n$. *Further, let* μ_i, $i \in \mathcal{N}$ *be defined as in Proposition 1.*

[29] To guarantee non-negative marginal costs see also Footnote 27.

(i) If Equation (5) holds, then the matrix $\mathbf{M} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}$ *exists, and the unique interior Nash equilibrium in quantities with subsidies (in the second stage) is given by*

$$\mathbf{q} = \tilde{\mathbf{q}} + \mathbf{R}\mathbf{s}, \tag{23}$$

where $\mathbf{R} = \mathbf{M}(\mathbf{I}_n + \varphi\mathbf{A})$, $\tilde{\mathbf{q}} = \mathbf{M}\boldsymbol{\mu}$, *equilibrium efforts are given by* $e_i = q_i + s_i$ *and profits are given by*

$$\pi_i = \frac{q_i^2 + s_i^2}{2}, \tag{24}$$

for all $i = 1, \ldots, n$.

(ii) Assume that goods are not substitutable, i.e. $\rho = 0$. *Then if the matrix* $\mathbf{H} \equiv \mathbf{I}_n + 2\left(\mathbf{I}_n - \mathbf{R}^\top\right)\mathbf{R}$ *is positive definite, the optimal subsidy levels (in the first stage) are given by*

$$\mathbf{s}^* = \mathbf{H}^{-1}(2\mathbf{R} - \mathbf{I}_n)\tilde{\mathbf{q}},$$

provided that $0 < q_i < \bar{q}$ *and* $0 < s_i^* < \bar{s}$ *for all* $i = 1, \ldots, n$.

(iii) Assume that goods are substitutable, i.e. $\rho > 0$. *Then, if the matrix* $\mathbf{H} \equiv \mathbf{I}_n + 2\left(\mathbf{I}_n - \mathbf{R}^\top\left(\mathbf{I}_n + \frac{\rho}{2}\mathbf{B}\right)\right)\mathbf{R}$ *is positive definite, the optimal subsidy levels (in the first stage) are given by*

$$\mathbf{s}^* = 2\left(\mathbf{H} + \mathbf{H}^\top\right)^{-1}\left(2\mathbf{R}^\top\left(\mathbf{I}_n + \frac{\rho}{2}\mathbf{B}\right) - \mathbf{I}_n\right)\tilde{\mathbf{q}},$$

provided that $0 < q_i < \bar{q}$ *and* $0 < s_i^* < \bar{s}$ *for all* $i = 1, \ldots, n$.

As in the previous proposition, in part (i) of Proposition 4, we solve for the second stage of the game where firms decide their output given the *targeted* subsidy s_i. In parts (ii) and (iii), we solve the first stage of the model when the planner optimally decides the targeted subsidy per R&D effort when goods are substitutable (i.e. $\rho > 0$), and when they are not (i.e. $\rho = 0$). We are able to determine the exact value of the optimal subsidy to be given to each firm embedded in a network of R&D collaborations in both cases.[30] We will use the results of these two propositions below to empirically study subsidies in the presence of R&D collaborations between firms in our dataset.

In the following sections we will test the different parts of our theoretical predictions. First, we will test Proposition 1 and try to disentangle between the *technology (or knowledge) spillover effect* and the *product rivalry effect* of R&D. Second, once the parameters of the model have been estimated, we will use Propositions 3 and 4, respectively, to determine which firms should be subsidized, and how large their subsidies should be.

6. Data

To obtain a comprehensive picture of R&D alliances, we use data on interfirm R&D collaborations stemming from two sources that have been widely used in the literature [cf. Schilling, 2009]. The first one is the Cooperative Agreements and Technology Indicators (CATI) database [cf. Hagedoorn, 2002].

[30] Note that when the condition for positive definiteness is not satisfied then we can sill use parts (ii) or (iii) of Proposition 4, respectively, as a candidate for a welfare improving subsidy program. However, there might exist other subsidy programs which yield even higher welfare gains.

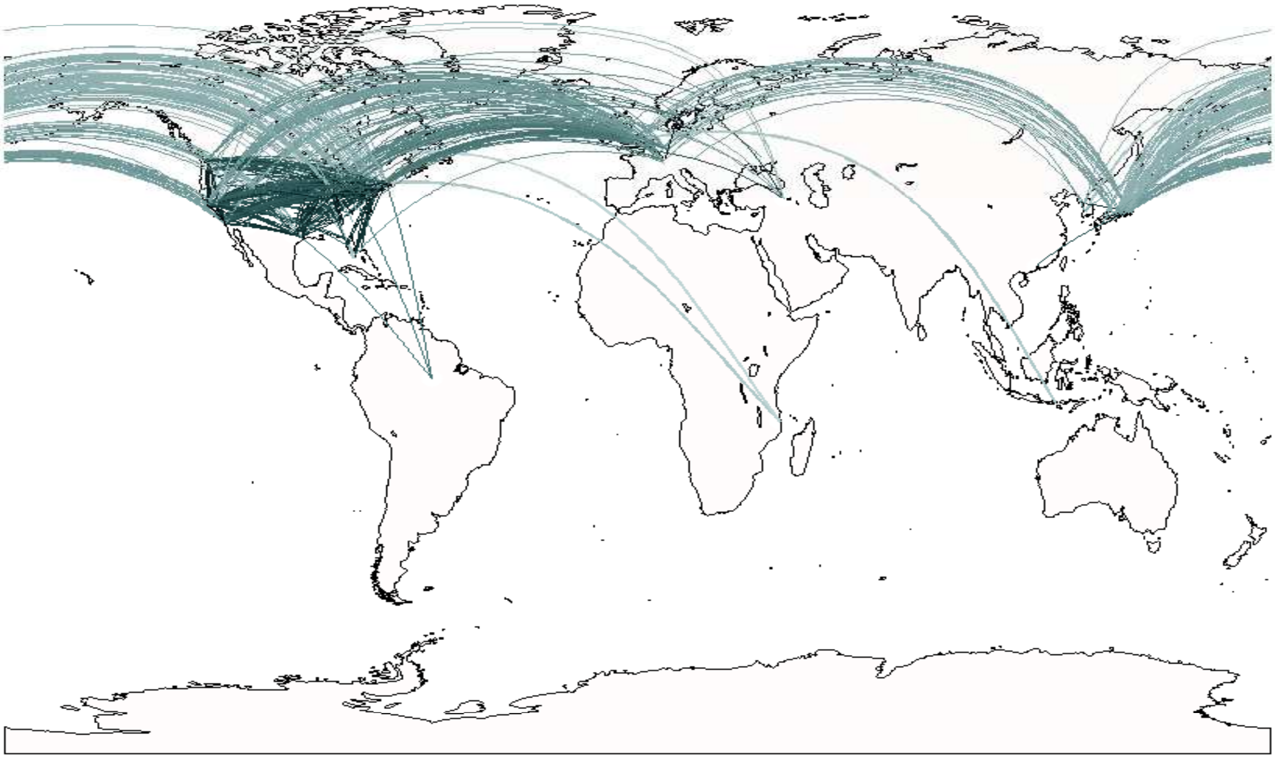


Figure 4: The locations (at the city level) of firms and their R&D alliances in the combined CATI-SDC databases. See also Figure G.6 in the supplementary Appendix G.

This database only records agreements for which a combined innovative activity or an exchange of technology is at least part of the agreement. The second source is the Thomson Securities Data Company (SDC) alliance database. SDC collects data from the U. S. Securities and Exchange Commission (SEC) filings (and their international counterparts), trade publications, wires, and news sources. We include only alliances from SDC that are classified explicitly as R&D collaborations.[31] Supplementary Appendix G.1 provides more information about the different R&D collaboration databases used for this study.

We then merged the CATI database with the Thomson SDC alliance database. For the matching of firms across datasets we used the name matching algorithm developed as part of the NBER patent data project [Atalay et al., 2011; Trajtenberg et al., 2009].[32] The merged datasets allow us to study patterns in R&D partnerships in several industries, both domestically and internationally, in different regions of the world over an extended period of several decades.[33] Figure 4 shows the locations (at the city level) of firms and their R&D alliances in the combined CATI-SDC database.

The systematic collection of inter-firm alliances started in 1987 and ended in 2006 for the CATI database. However, information about alliances prior to 1987 is available in both databases, and we use all information available starting from the year 1970 and ending in 2006.[34] We construct the R&D alliance network by assuming that an alliance lasts 5 years [similar to e.g. Rosenkopf and Padula,

[31] Schilling [2009] compares different alliance databases, including CATI and SDC that we are using for this study, and suggests a combination of both to obtain a good coverage of R&D collaborations across sectors.

[32] See https://sites.google.com/site/patentdataproject. We thank Enghin Atalay and Ali Hortacsu for making their name matching algorithm available to us.

[33] See supplementary Appendix G.4 for more information about the geographic dispersion and coverage across countries of our R&D alliance data.

[34] As explained below, we require at least two consecutive years of observations for each firm in the sample for our panel data analysis. This restriction forced us to discard all years prior to 1970. Further, Fama and French [1992] note that Compustat suffers from a large selection bias prior to 1962.

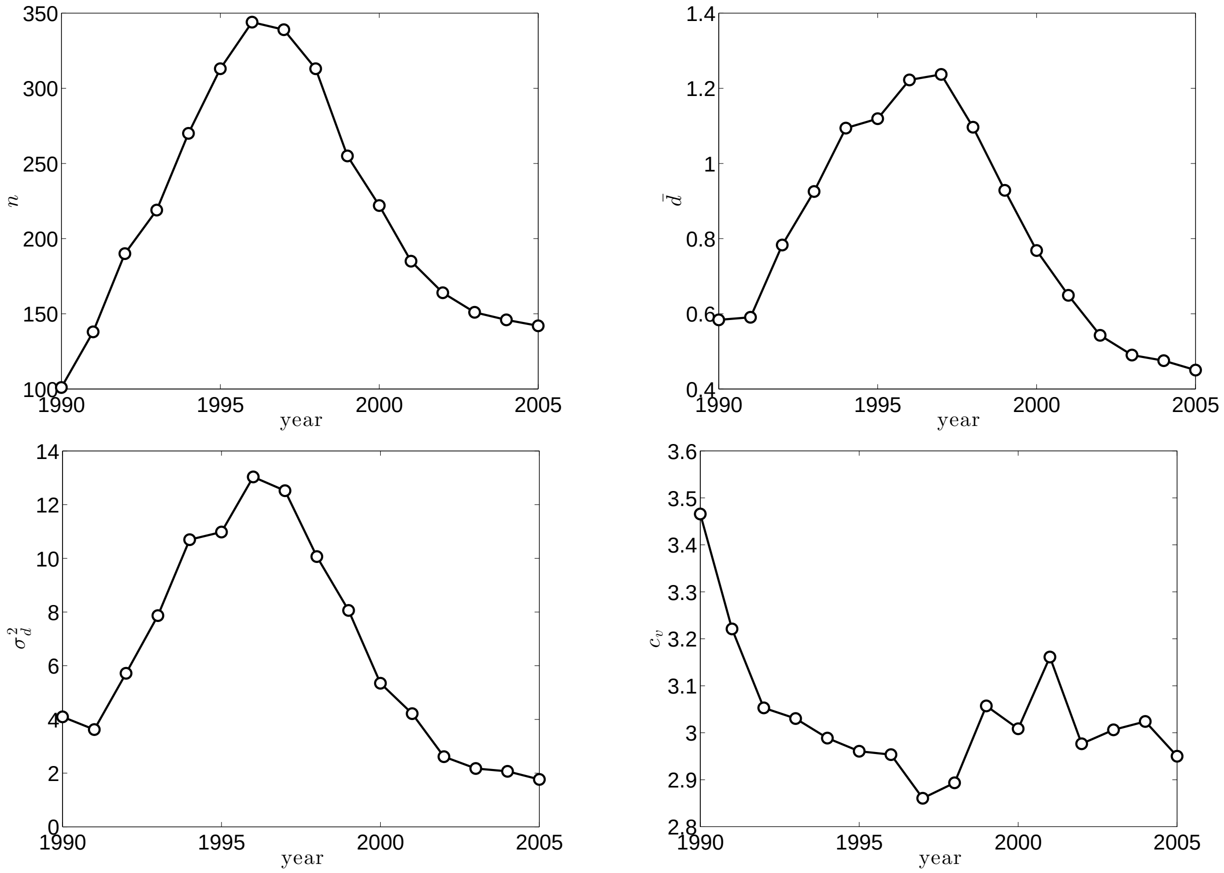


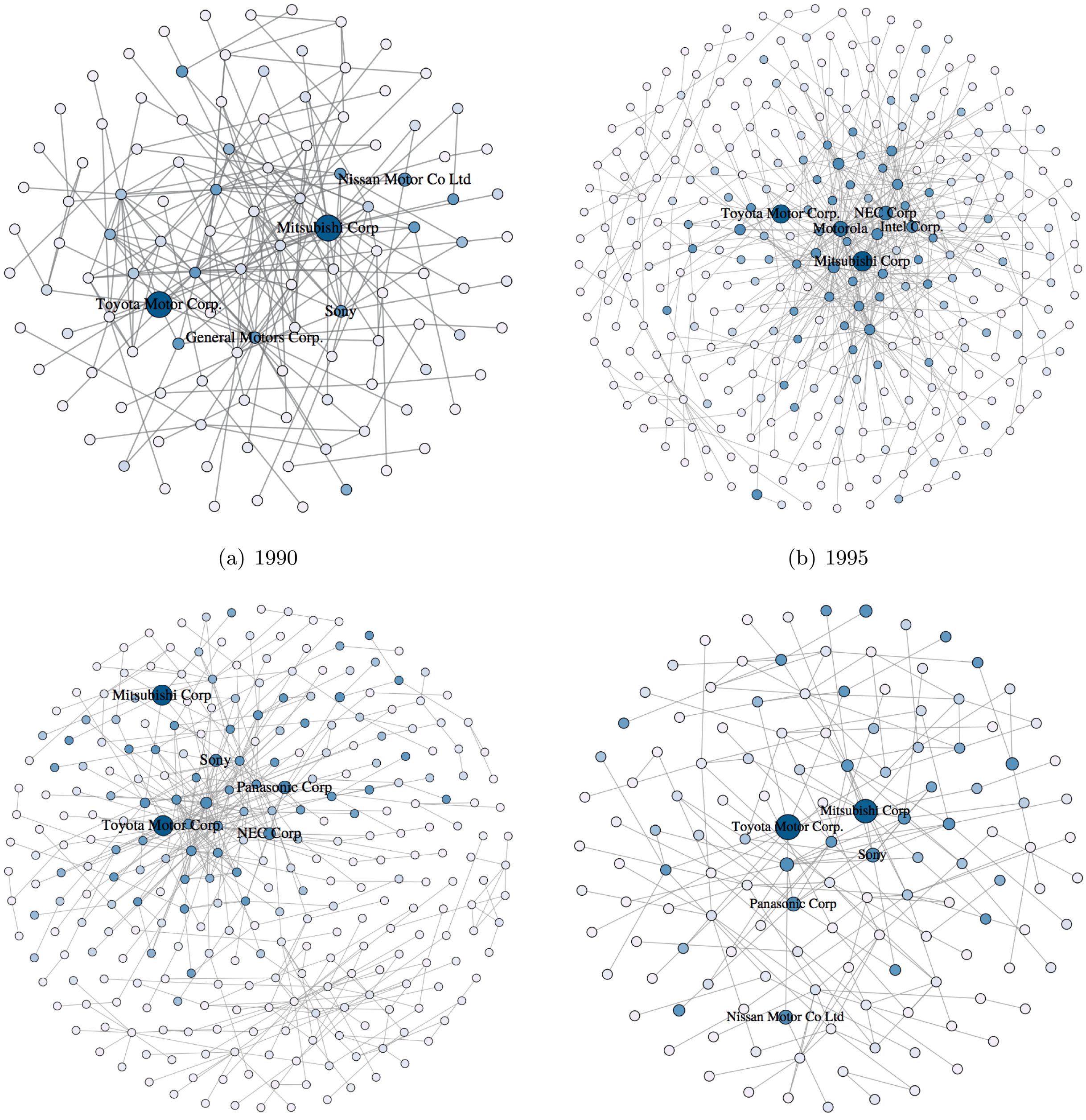
Figure 5: The number of firms, n, participating in an alliance, the average degree, $\bar{d}$, the degree variance, σ_d^2, and the degree coefficient of variation, $c_v = \sigma_d/\bar{d}$.

2008].[35] In the robustness section below (Section 8.1), we will test our model for different durations of an alliance.

Some firms might be acquired by other firms due to mergers and acquisitions (M&A) over time, and this will impact the R&D collaboration network [cf. e.g. Hanaki et al., 2010]. We account for M&A activities by assuming that an acquiring firm inherits all the R&D collaborations of the target firm. We use two complementary data sources to obtain comprehensive information about M&As. The first is the Thomson Reuters' SDC M&A database, which has historically been the reference database for empirical research in the field of M&As. The second database for M&As is Bureau van Dijk's Zephyr database, which is an alternative to the SDC M&As database. A comparison and more detailed discussion of the two M&As databases can be found in the supplementary Appendix G.2 and Bena et al. [2008].

Figure 5 shows the number of firms, n, participating in an alliance in the R&D network, the average degree, $\bar{d}$, the degree variance, σ_d^2, and the degree coefficient of variation, $c_v = \sigma_d/\bar{d}$, over the years 1990 to 2005. It can be seen that there are very large variations over the years in the number of firms having an R&D alliance with other firms. Starting from 1990, we observe a strong increase (due to the IT boom) followed by a steady decline from 1997 onwards. Both, the average number of alliances

[35] Rosenkopf and Padula [2008] use a five-year moving window assuming that alliances have a five-year life span, and state that the choice of a five-year window is consistent with extant alliance studies [e.g. Gulati and Gargiulo, 1999; Stuart, 2000] and conforms to Kogut [1988] finding that the normal life span of most alliances is no more than five years. Moreover, Harrigan [1988] studies 895 alliances from 1924 to 1985 and concludes that the average life-span of the alliance is relatively short, 3.5 years, with a standard deviation of 5.8 years and 85 % of these alliances last less than 10 years. Park and Russo [1996] focus on 204 joint ventures among firms in the electronic industry for the period 1979–1988. They show that less than half of these firms remain active beyond a period of five years and for those that last less than 10 years (2/3 of the total), the average lifetime turns out to be 3.9 years.



(a) 1990 (b) 1995

(c) 2000 (d) 2005

Figure 6: Network snapshots of the largest connected component for the years (a) 1990, (b) 1995, (c) 2000 and (d) 2005. Nodes' sizes and shades indicate their targeted subsidies (see Section 9). The names of the 5 highest subsidized firms are indicated in the network.

per firm (captured by the average degree $\bar{d}$), as well as the degree variance σ_d^2 follow a similar pattern. In contrast, the degree coefficient of variation, c_v, has decreased almost monotonically over the years.

In Figure 6 exemplary plots of the largest connected component in the R&D network for the years 1990, 1995, 2000 and 2005 are shown.[36] In 1990, the giant component had a core-periphery structure with many R&D interactions between firms from different sectors. If we look at the same picture in 2000, the core-periphery structure seems less obvious and two cores and a periphery seem to emerge, where there are only few interactions between firms of different sectors in one of the cores.[37] This may indicate more specialization in R&D alliance partnerships.

The combined CATI-SDC database provides the names for each firm in an alliance, but does not contain balance sheet information. We thus matched the firms' names in the CATI-SDC database with

[36]See supplementary Appendix B.1 for the definition of a connected component.
[37]See also Figure G.1 in supplementary Appendix G.1.

Table 1: Summary statistics computed across the years 1990 to 2005.

Variable	Obs.	Mean	Std. Dev.	Min.	Max.	Compustat Mean
Sales $[10^6]$	48,331	43,370.78	1,671,827.00	1.78×10^{-8}	1.67×10^8	1,085.05
Empl.	35,384	13,070.48	44,995.95	1	1.8×10^6	4,322.08
Capital $[10^6]$	48,836	690,876.80	4.13×10^7	1.75×10^{-8}	4.95×10^9	663.44
R&D Exp. $[10^6]$	30,329	67.71	279.89	3.37×10^{-4}	6,621.19	14.71
R&D Exp. / Empl.	23,249	43,123.52	964,622.00	0.56	1.07×10^8	4,060.12
R&D Stock $[10^6]$	18,083	289.40	1,198.21	1.07×10^{-3}	22,292.97	33.13
Num. Patents	18,543	1,836.69	9,440.84	1	208,251.00	14.39

Notes: Values for sales, capital and R&D expenses are in U.S. dollars with 1983 as the base year. Compustat means are computed across all firms in the combined Compustat U.S. and Global fundamentals annual databases over all non-missing observations. U.S. dollar translation rates for foreign currencies have been taken directly from Compustat yearly average exchange rates.

the firms' names in Standard & Poor's Compustat U.S. and Global annual fundamentals databases, as well as Bureau van Dijk's Osiris database, to obtain information about their balance sheets and income statements [see e.g. Dai, 2012]. Compustat and Osiris only contain firms listed on the stock market, so they typically exclude smaller firms. However, they should capture the most R&D intensive firms, as R&D is typically concentrated in publicly listed firms [cf. e.g. Bloom et al., 2013]. Supplementary Appendix G.3 provides additional details about the accounting databases used in this study.

For the purpose of matching firms across databases, we again use the above mentioned name matching algorithm. We could match roughly 26% of the firms in the alliance data. From our match between the firms' names in the alliance database and the firms' names in the Compustat and Osiris databases, we obtained a firm's sales and R&D expenditures. Individual firms' output levels are computed from deflated sales using 2-SIC digit industry-country-year specific price deflators from the OECD-STAN database [cf. Gal, 2013]. Furthermore, we use information on R&D expenditures to compute R&D capital stocks using a perpetual inventory method with a 15% depreciation rate [following Bloom et al., 2013; Hall et al., 2000]. Considering only firms with non-missing observations on sales, output and R&D expenditures we end up with a sample of $1,431$ firms and a total of $1,174$ collaborations over the years 1970 to 2006.[38]

The empirical distributions for output $P(q)$ (using a logarithmic binning of the data with 100 bins) and the degree distribution $P(d)$ are shown in Figure 7. Both are highly skewed, indicating a large degree of inequality in the number of goods produced as well as the number of R&D collaborations. Industry totals are computed across all firms in the Compustat U.S. and Global fundamentals databases (without missing observations). Basic summary statistics can be seen in Table 1. The table shows that the R&D collaborating firms in our sample are typically larger and have higher R&D expenditures than the average across all firms in the Compustat database. This is consistent with previous studies which found that cooperating firms tend to be larger and more R&D intensive [cf. e.g. Belderbos et al., 2004].

7. Econometric Analysis

7.1. Econometric Specification

In this section, we introduce the econometric equivalent to the equilibrium quantity produced by each firm given in Equation (13). Our empirical counterpart of the marginal cost c_{it} of firm i from Equation

[38] See the supplementary Appendix G for a discussion about the representativeness of our data sample, and Section 8.4 for a discussion about the impact of missing data on our estimation results.

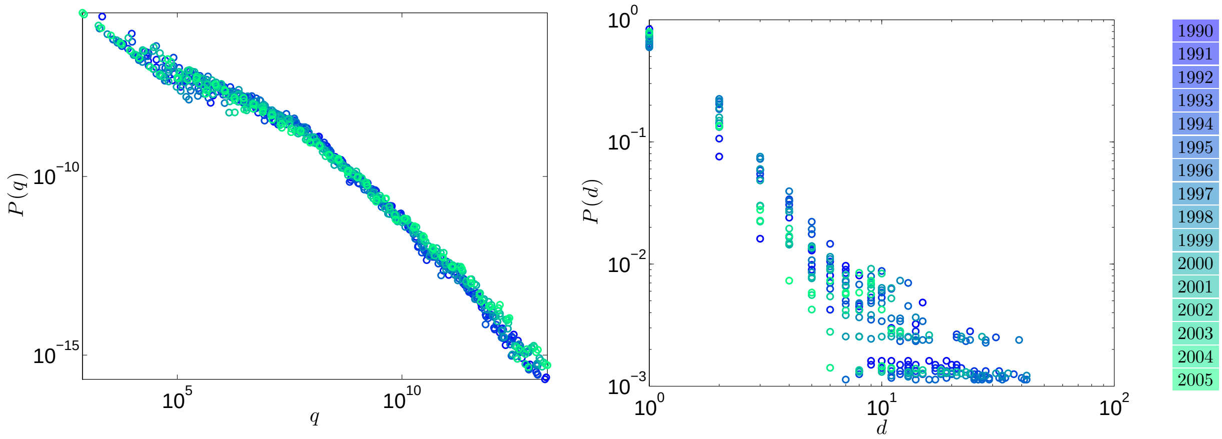


Figure 7: Empirical output distribution $P(q)$ and the distribution of degree $P(d)$ for the years 1990 to 2005. The data for output has been logarithmically binned and non-positive data entries have been discarded. Both distributions are highly skewed.

(2) at period t has a fixed cost equal to $\bar{c}_{it} = \eta_i^* - \epsilon_{it} - x_{it}\beta$, and thus we get

$$c_{it} = \eta_i^* - \varepsilon_{it} - \beta x_{it} - e_{it} - \varphi \sum_{j=1}^{n} a_{ij,t} e_{jt}, \tag{25}$$

where x_{it} is a measure for the productivity of firm i, η_i^* captures the unobserved (to the econometrician) time-invariant characteristics of the firms, and ε_{it} captures the remaining unobserved (to the econometrician) characteristics of the firms.

Following Equation (1), the inverse demand function for firm i is given by

$$p_{it} = \bar{\alpha}_m + \bar{\alpha}_t - q_{it} - \rho \sum_{j=1}^{n} b_{ij} q_{jt}, \tag{26}$$

where $b_{ij} = 1$ if i and j are in the same market and zero otherwise. In this equation, $\bar{\alpha}_m$ indicates the market-specific fixed effect and $\bar{\alpha}_t$ captures the time fixed effect due to exogenous demand shifters that affect consumer income, number of consumers (population), consumer taste and preferences and expectations over future prices of complements and substitutes or future income.

Denote by $\kappa_t \equiv \bar{\alpha}_t$ and $\eta_i \equiv \bar{\alpha}_m - \eta_i^*$. Observe that κ_t captures the time fixed effect while η_i, which includes both $\bar{\alpha}_m$ and η_i^*, captures the firm fixed effect. Then, proceeding as in Section 3 (see, in particular the proof of Proposition 1), adding subscript t for time and using Equations (25) and (26), the econometric model equivalent to the best-response quantity in Equation (13) is given by

$$q_{it} = \varphi \sum_{j=1}^{n} a_{ij,t} q_{jt} - \rho \sum_{j=1}^{n} b_{ij} q_{jt} + \beta x_{it} + \eta_i + \kappa_t + \epsilon_{it}. \tag{27}$$

Observe that the econometric specification in Equation (27) has a similar specification as the product competition and technology spillover production function estimation in Bloom et al. [2013] where the estimation of φ will give the intensity of the *technology (or knowledge) spillover effect* of R&D, while the estimation of ρ will give the intensity of the *product rivalry effect*. However, as opposed to these authors, we explicitly take into account the technology spillovers stemming from R&D collaborations by using a network approach.

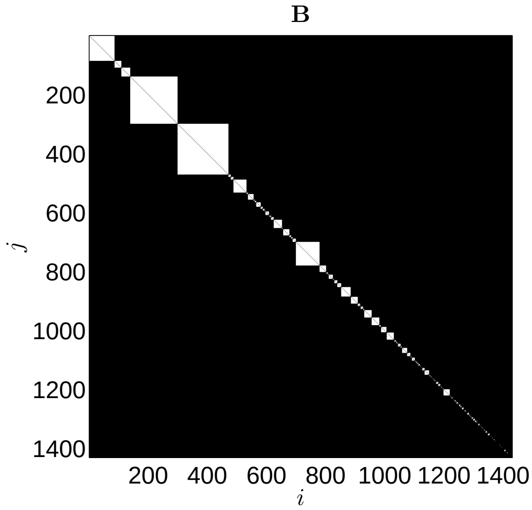


Figure 8: The empirical competition matrix $\mathbf{B} = (b_{ij})_{1 \leq i,j \leq n}$ measured by 4-digit level industry SIC codes.

In vector-matrix form, we can write Equation (27) as

$$\mathbf{q}_t = \varphi \mathbf{A}_t \mathbf{q}_t - \rho \mathbf{B} \mathbf{q}_t + \mathbf{X}_t \beta + \boldsymbol{\eta} + \kappa_t \mathbf{u}_n + \boldsymbol{\epsilon}_t, \tag{28}$$

where $\mathbf{q}_t = (q_{1t}, \cdots, q_{nt})^\top$, $\mathbf{A}_t = [a_{ij,t}]$, $\mathbf{B} = [b_{ij}]$, $\mathbf{X}_t = (x_{1t}, \cdots, x_{nt})^\top$, $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n)^\top$, $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \cdots, \epsilon_{nt})^\top$, and $\mathbf{u}_n$ is an n-dimensional vector of ones.

For the T periods, Equation (28) can be written as

$$\mathbf{q} = \varphi \mathrm{diag}\{\mathbf{A}_t\} \mathbf{q} - \rho (\mathbf{I}_T \otimes \mathbf{B}) \mathbf{q} + \mathbf{X} \beta + \mathbf{u}_T \otimes \boldsymbol{\eta} + \boldsymbol{\kappa} \otimes \mathbf{u}_n + \boldsymbol{\epsilon}, \tag{29}$$

where $\mathbf{q} = (\mathbf{q}_1^\top, \cdots, \mathbf{q}_T^\top)^\top$, $\mathbf{X} = (\mathbf{X}_1^\top, \cdots, \mathbf{X}_T^\top)^\top$, $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_T)^\top$, and $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1^\top, \cdots, \boldsymbol{\epsilon}_T^\top)^\top$. All vectors are of dimension $(nT \times 1)$, where T is the number of years available in the data.

In terms of data, our main variables will be measured as follows. Output q_{it} is calculated using sales divided by the country-year-industry price deflator from the OECD-STAN database [cf. Gal, 2013]. The network data stems from the combined CATI-SDC databases and we set $a_{ij,t} = 1$ if there exists an R&D collaboration between firms i and j in the last s years before time t, where s is the duration of an alliance.[39] The exogenous variable x_{it} is the firm's time-lagged R&D stock at the time $t-1$. Finally, we measure b_{ij} as in the theoretical model so that $b_{ij} = 1$ if firms i and j are the same industry (measured by the industry SIC codes at the 4-digit level) and zero otherwise. The empirical competition matrix $\mathbf{B}$ can be seen in Figure 8. The block-diagonal structure indicating different markets is clearly visible.

7.2. Identification Issues

We adopt a structural approach in the sense that we estimate the first-order condition of the firms' profit maximization problem in terms of output and R&D effort, which lead to Equations (27) and (28). The best-response quantity in Equation (28) then corresponds to a higher-order Spatial Auto-Regressive (SAR) model with two spatial lags, $\mathbf{A}_t \mathbf{q}_t$ and $\mathbf{B} \mathbf{q}_t$ [cf. Lee and Liu, 2010].

There are several potential identification problems in the estimation of Equation (27) or (28). We face, actually, three sources of potential bias[40] arising from (i) *correlated or common-shock effects*, (ii)

[39] For the benchmark estimation results reported in Table 2, we set $s = 5$. We report estimation results with different lengths of alliance durations in Tables 5 and 6, and find that the results are robust.

[40] It should be clear that there is no exogeneous contextual effect (and thus no reflection problem) in Equation (27).

simultaneity of q_{it} and q_{jt}, and (*iii*) *endogenous network formation* (selection).

7.2.1. Correlated or Common-Shock Effects

This is standard problem that arises in estimating network effects due to the fact that there may be common environmental factors that affect the behavior of members of the same network in a similar manner. To deal with correlated or common-shock effects, we add both time, κ_t, and firm, η_i, fixed effects to our outcome Equation (27).

7.2.2. Simultaneity of Product Quantities

We use instrumental variables when estimating our outcome Equation (27) to deal with the issue of simultaneity of q_{it} and q_{jt}. Indeed, the output of firm i at time t, q_{it}, is a function of the total output of all firms collaborating in R&D with firm i at time t, i.e. $\bar{q}_{a,it} \equiv \sum_{j=1}^{n} a_{ij,t} q_{jt}$, and the total output of all firms that operate in the same market as firm i, i.e. $\bar{q}_{b,it} \equiv \sum_{j=1}^{n} b_{ij} q_{jt}$. Due the feedback effect, q_{jt} also depends on q_{it} and, thus, $\bar{q}_{a,it}$ and $\bar{q}_{b,it}$ are endogenous.

To deal with this issue, we instrument $\bar{q}_{a,it}$ by the time-lagged total R&D stock of all firms with an R&D collaboration with firm i, i.e. $\sum_{j=1}^{n} a_{ij,t} x_{it}$, and instrument $\bar{q}_{b,it}$ by the time-lagged total R&D stock of all firms that operate in the same industry as firm i, i.e. $\sum_{j=1}^{n} b_{ij} x_{it}$. The rationale for this IV solution is as follows. The total R&D stock of R&D collaborators and product competitors of firm i *directly* affects the total output of these firms but only *indirectly* affects the output of firm i through the total output of these same firms. As a result, we believe that the time-lagged R&D stock of these firms is a valid instrument (i.e. satisfies the exclusion restriction condition) for the output of each firm.

More formally, to estimate Equation (29), first we transform it with the projector $\mathbf{J} = (\mathbf{I}_T - \frac{1}{T}\mathbf{u}_T\mathbf{u}_T^\top) \otimes (\mathbf{I}_n - \frac{1}{n}\mathbf{u}_n\mathbf{u}_n^\top)$. The transformed Equation (29) is

$$\mathbf{Jq} = \varphi \mathbf{J}\text{diag}\{\mathbf{A}_t\}\mathbf{q} - \rho \mathbf{J}(\mathbf{I}_T \otimes \mathbf{B})\mathbf{q} + \mathbf{JX}\beta + \mathbf{J}\boldsymbol{\epsilon}, \tag{30}$$

where the firm and time fixed effects $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ have been cancelled out.[41] Let $\mathbf{Q}_1 = \mathbf{J}[\text{diag}\{\mathbf{A}_t\}\mathbf{X}, (\mathbf{I}_T \otimes \mathbf{B})\mathbf{X}, \mathbf{X}]$ denote the IV matrix and $\mathbf{Z} = \mathbf{J}[\text{diag}\{\mathbf{A}_t\}\mathbf{q}, (\mathbf{I}_T \otimes \mathbf{B})\mathbf{q}, \mathbf{X}]$ denote the matrix of regressors in Equation (30). As there is a single exogenous variable in Equation (30), the model is just-identified. The IV estimator of parameters $(\varphi, \rho, \beta)^\top$ is given by $(\mathbf{Q}_1^\top \mathbf{Z})^{-1}\mathbf{Q}_1^\top \mathbf{q}$. With the estimated $(\varphi, \rho, \beta)^\top$, one can recover $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ by the least squares dummy variable method.

Finally, to allow for potential correlation in unobservables across firms collaborating in R&D (stemming, for example, from unobserved R&D subsidies), the standard deviation of the IV estimator is estimated by the spatial heteroskedasticity and autocorrelation consistent (HAC) estimator suggested by Kelejian and Prucha [2007]. It is a standard error that is consistent (or robust) when the error term ϵ_{it} is heteroskedastic and spatially correlated.

7.2.3. Endogenous Network Formation

Obviously, the above IV-based identification strategy is valid only if the R&D alliance matrix, $\mathbf{A}_t = [a_{ij,t}]$, is exogenous. In constrast, $\mathbf{A}_t$ is endogenous if there exists an *unobservable factor* that affects

[41] For unbalanced panels, the firm and time fixed effects can be eliminated by a projector given in Wansbeek and Kapteyn [1989].

both the outputs, q_{it} and q_{jt}, and the R&D alliance, indicated by $a_{ij,t}$. If the unobservable factor is firm-specific, then it is captured by the firm fixed-effect η_i. If the unobservable factor is time-specific, then it is captured by the time fixed-effect κ_t. Therefore, the fixed effects in the panel data model are helpful for attenuating the potential endogeneity of $\mathbf{A}_t$.

However, it may still be that there are some unobservable firm-specific factors that do vary over time and that affect the possibility of R&D collaborations and thus make the matrix $\mathbf{A}_t = [a_{ij,t}]$ endogeneous. To deal with this issue, we run a two-stage IV estimation where, in the first stage, we estimate a link formation model, and, in the second stage, we employ the IV strategy explained above using IVs based on the predicted adjacency matrix from the first stage link formation regression.

Let us now explain the first stage, i.e. the link formation model. We estimate a logistic regression model with corresponding log-odds ratio [cf. Cameron and Trivedi, 2005]:

$$\log\left(\frac{\mathbb{P}\left(a_{ij,t}=1 \mid (\mathbf{A}_\tau)_{\tau=1}^{t-s-1}, f_{ij,t-s-1}, l_{ij}\right)}{1-\mathbb{P}\left(a_{ij,t}=1 \mid (\mathbf{A}_\tau)_{\tau=1}^{t-s-1}, f_{ij,t-s-1}, l_{ij}\right)}\right)$$
$$= \gamma_0 + \gamma_1 \max_{\tau=1,\ldots,t-s-1} a_{ij,\tau} + \gamma_2 \max_{\substack{\tau=1,\ldots,t-s-1\\ k=1,\ldots,n}} a_{ik,\tau}a_{kj,\tau} + \gamma_3 f_{ij,t-s-1} + \gamma_4 f_{ij,t-s-1}^2 + \gamma_5 l_{ij}, \qquad (31)$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 are parameters governing the formation of R&D collaborations. In this model, $\max_{\tau=1,\ldots,t-s-1} a_{ij,\tau}$ is a dummy variable, which is equal to 1 if firms i and j had an R&D collaboration before time $t-s$ (where s is the duration of an alliance) and 0 otherwise; $\max_{\tau=1,\ldots,t-s-1;k=1,\ldots,n} a_{ik,\tau}a_{kj,\tau}$ is a dummy variable, which is equal to 1 if firms i and j had a common R&D collaborator before time $t-s$, and 0 otherwise; $f_{ij,t-s-1}$ and $f_{ij,t-s-1}^2$ are the time-lagged technological proximities of firms i and j (cf. e.g. Sec. 3.5 in Nooteboom et al. [2006] and Powell and Grodal [2006]), measured here by either the Jaffe or the Mahalanobis patent similarity indices at time $t-s-1$;[42] and l_{ij} is a dummy variable, which is equal to 1 if firms i and j are located in the same city, and 0 otherwise.[43]

The rationale for this IV solution is as follows. Take, for example, the dummy variable, which is equal to 1 if firms i and j had a common R&D collaborator before time $t-s$, and 0 otherwise. This means that, if firms i and j had a common collaborator in the past (i.e. before time $t-s$), then they are more likely to have an R&D collaboration today, i.e. $a_{ij,t}=1$, but, conditional on the firm and time fixed effects, this should not affect *directly* the output of firm i today (at time t). As a result, the past common collaborator of these firms should not be correlated with the error term in the outcome Equation (27) (i.e. satisfies the exclusion restriction condition).[44] A similar argument can be made for the other variables in Equation (31). As a result, using IVs based on the predicted adjacency matrix $\widehat{\mathbf{A}}_t$ should alleviate the concern of invalid IVs due to the endogeneity of the adjacency matrix $\mathbf{A}_t$.

[42] We matched the firms in our alliance data with the owners of patents recorded in the Worldwide Patent Statistical Database (PATSTAT). This allowed us to obtain the number of patents and the patent portfolio held for about 36% of the firms in the alliance data. From the firms' patents, we then computed their technological proximity following Jaffe [1986] as $f_{ij}^{\mathrm{J}} = \frac{\mathbf{P}_i^\top \mathbf{P}_j}{\sqrt{\mathbf{P}_i^\top \mathbf{P}_i}\sqrt{\mathbf{P}_j^\top \mathbf{P}_j}}$, where $\mathbf{P}_i$ represents the patent portfolio of firm i and is a vector whose k-th component P_{ik} counts the number of patents firm i has in technology category k divided by the total number of technologies attributed to the firm. As an alternative measure for technological similarity we also use the Mahalanobis proximity index f_{ij}^{M} introduced in Bloom et al. [2013]. Supplementary Appendix G.5 provides further details about the match of firms to their patent portfolios and the construction of the technology proximity measures f_{ij}^k, $k \in \{\mathrm{J}, \mathrm{M}\}$.

[43] Observe that the predictors for the link-formation probability are either time-lagged or predetermined so the IVs constructed with $\widehat{\mathbf{A}}_t$ are less likely to suffer from any endogeneity issues.

[44] Note that a similar strategy has been used by Graham [2014] for modeling network formation.

Table 2: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression of Equation (28). Model A does not include firm fixed effects (f.e.), while Model B introduces also firm fixed effects. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	Model A		Model B	
φ	0.0071*	(0.0042)	0.0088**	(0.0040)
ρ	0.0037***	(0.0014)	0.0064***	(0.0023)
β	0.1598***	(0.0336)	0.1589***	(0.0339)
# firms	1431		1431	
# obs.	19448		19448	
F statistic	670.717		366.561	
time f.e.	yes		yes	
firm f.e.	no		yes	

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

Formally, let $\mathbf{Q}_2 = \mathbf{J}[\text{diag}\{\widehat{\mathbf{A}}_t\}\mathbf{X}, (\mathbf{I}_T \otimes \mathbf{B})\mathbf{X}, \mathbf{X}]$ denote the IV matrix based on the predicted R&D alliance matrix and $\mathbf{Z} = [\text{diag}\{\mathbf{A}_t\}\mathbf{q}, (\mathbf{I}_T \otimes \mathbf{B})\mathbf{q}, \mathbf{X}]$ denote the matrix of regressors in Equation (30). Then, the estimator of the parameters $(\varphi, \rho, \beta)^\top$ with IVs based on the predicted adjacency matrix is given by $(\mathbf{Q}_2^\top \mathbf{Z})^{-1}\mathbf{Q}_2^\top \mathbf{q}$.

To summarize, we use the following step-wise procedure to implement our estimation method:

Step 1: Estimate the link formation model of Equation (31). Use the estimated model to predict links. Denote the predicted adjacency matrix by $\widehat{\mathbf{A}}_t$ and its elements by $\widehat{a}_{ij,t}$.

Step 2: Estimate the outcome Equation (27) using $\sum_{j=1}^n \widehat{a}_{ij,t}x_{jt}$ and $\sum_{j=1}^n b_{ij}x_{jt}$ as IVs for $\sum_{j=1}^n a_{ij,t}q_{jt}$ and $\sum_{j=1}^n b_{ij,t}q_{jt}$, respectively.

7.3. Estimation Results

7.3.1. Main results

Table 2 reports the parameter estimates of Equation (28) with time fixed effects only (Model A) and both time and firm fixed effects (Model B). In these regressions, we do not introduce the link formation Equation (31). We see that, in both models, we obtain the expected signs, that is the *technology (or knowledge) spillover effect* (estimate of φ) always has a *positive* impact on own output while the *product rivalry effect* (estimate of ρ) always has *negative* impact on own output. Indeed, the more a given firm collaborates with other firms in R&D, the higher is its output production. This indicates that R&D by allied firms in the network is associated with higher product value and indicate that there exist strategic complementarities between own and allied firms levels of production. However, conditional on technology spillovers, the more firms that compete in the same market, the lower is the production of the good by the given firm. As in Bloom et al. [2013], this table shows that the magnitude of the first effect (technology spillover) is much higher than that of the second effect (product rivalry). Keeping all other firms' output levels constant, suppose that firm j is both a collaboration partner of firm i and operates in the same market as firm i. Then, we find that the net effect of firm j increasing its output by one unit is captured by the difference of the two effects. As the technology spillover effect is much higher than the rivalry effect, we find that the net returns to

R&D collaborations are strictly positive. Furthermore, this table also shows that a firm's productivity captured by its own time-lagged R&D stock has a positive and significant impact on own output.[45] Finally, the first stage F statistics for both models are well above the conventional benchmark for weak IVs [cf. Stock and Yogo, 2005].

7.3.2. Endogenous Network Formation

As stated above, we also consider IVs based on the predicted R&D alliance matrix, i.e. $\widehat{\mathbf{A}}_t\mathbf{X}_t$ following Kelejian and Piras [2014] to estimate Equation (28). We obtain the predicted link-formation probability $\hat{a}_{ij,t}$ from the logit regression of $a_{ij,t}$ on whether firms i and j collaborated before time $(t-s)$, where s is the duration of an alliance, whether firms i and j shared a common collaborator before time $(t-s)$, the time-lagged technological proximity of firms i and j represented by $f_{ij,t-s}$ and $f_{ij,t-s}^2$, and whether firms i and j are located in the same city (see Equation (31)).

The logit regression result of Equation (31), using either the Jaffe or Mahalanobis patent similarity measure, is reported in Table 3. The estimated coefficients are all statistically significant with expected signs. Interestingly, having a past collaboration or a past common collaborator or being established in the same city increases the probability that two firms have an R&D collaboration today. Furthermore, being technology close (measured by either the Jaffe or the Mahalanobis patent similarity measure) in the past also increases the chance of having an R&D collaboration today, even though this relationship is concave.

Consider now the estimation of Equation (27) with IVs based on the predicted adjacency matrix. The estimates are reported in Table 4. We find that the estimates of both the technology spillovers and the product rivalry effect are still significant and have the correct signs. Compared to Table 2, the estimate of the technology spillovers (i.e. the estimation of φ) has, however, a higher value.

8. Robustness Checks

8.1. Time Span of Alliances

We here analyze the impact of considering different time spans (other than 5 years as in the previous section) for the duration of an alliance. The estimation results for alliance durations ranging from 3 to 7 years are shown in Table 5. We find that the estimates are robust over the different durations considered.

Our assumption that the time span is constant for all alliances may seem restrictive. As a further robustness check, we randomly draw a life span for each alliance from an exponential distribution with the mean ranging from 3 to 7 years. The estimation results are shown in Table 6. We find that the estimates are still robust.

8.2. Intra- versus Inter-industry Collaborations

So far, we have assumed that network effects or knowledge spillovers were the same whether they were intra- or inter-industry collaborations. In the real-world, the knowledge spillovers between two firms in the same industry (say Volvo and Honda in the car manufacturing sector) may be different

[45] An alternative specification with the number of patents instead of the R&D stock as explanatory variable for the productivity of the firm yielded results with the same signs and significance except for the R&D spillover coefficient which became insignificant under this specification.

Table 3: Link regression estimation results. Technological similarity, P, is measured either using the Jaffe or the Mahalanobis patent similarity measures. The estimation is based on the observed alliances in the years 1971–2006.

	Jaffe	Mahalanobis
Past collaboration	0.5253***	0.5119***
	(0.0111)	(0.0114)
Common collaborator	0.2429***	0.2457***
	(0.0114)	(0.0112)
P_{ij}	16.0279***	5.8735***
	(0.4740)	(0.1870)
P_{ij}^2	-23.3989***	-2.6928***
	(1.2043)	(0.2149)
Location	1.3000***	1.2884***
	(0.0783)	(0.0784)
McFadden's R^2	0.0927	0.0938
time f.e.	yes	yes
firm f.e.	yes	yes

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

Table 4: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression of Equation (28) using the predicted network from a logistic regression model with estimates in Table 3. Technological similarity is measured either using the Jaffe or the Mahalanobis patent similarity measures. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	Jaffe		Mahalanobis	
φ	0.0129**	(0.0065)	0.0166***	(0.007)
ρ	0.0052***	(0.0017)	0.0041***	(0.0018)
β	0.1424***	(0.0288)	0.1276***	(0.0288)
# firms	1431		1431	
# obs.	19448		19448	
F statistic	510.045		468.473	
time f.e.	yes		yes	
firm f.e.	yes		yes	

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

Table 5: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression of Equation (28) including time fixed effects and firm fixed effects assuming different (fixed) durations of an alliance ranging from 3 to 7 years. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	3 years	4 years	5 years	6 years	7 years
φ	0.0085	0.0097*	0.0088**	0.0065*	0.0027
	(0.0060)	(0.0050)	(0.0040)	(0.0035)	(0.0033)
ρ	0.0073***	0.0066***	0.0064***	0.0068***	0.0080***
	(0.0024)	(0.0024)	(0.0023)	(0.0024)	(0.0025)
β	0.1692***	0.1607***	0.1589***	0.1654***	0.1817***
	(0.0335)	(0.0339)	(0.0339)	(0.0357)	(0.0362)
# firms	1431	1431	1431	1431	1431
# obs.	19448	19448	19448	19448	19448
F statistic	308.685	337.182	366.561	390.186	404.885
time f.e.	yes	yes	yes	yes	yes
firm f.e.	yes	yes	yes	yes	yes

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

Table 6: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression of Equation (28) including time fixed effects and firm fixed effects assuming different (random) durations of an alliance following an exponential distribution with mean ranging from 3 to 7 years. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	3 years	4 years	5 years	6 years	7 years
φ	0.0291***	0.0001	0.0021	0.0071**	0.0104***
	(0.0064)	(0.0066)	(0.0049)	(0.0032)	(0.0030)
ρ	0.0032	0.0089***	0.0085***	0.0072***	0.0053**
	(0.0025)	(0.0027)	(0.0024)	(0.0022)	(0.0023)
β	0.1159***	0.1937***	0.1875***	0.1688***	0.1513***
	(0.0326)	(0.0373)	(0.0343)	(0.0296)	(0.0317)
# firms	1431	1431	1431	1431	1431
# obs.	19448	19448	19448	19448	19448
F statistic	358.174	330.276	410.859	419.976	371.786
time f.e.	yes	yes	yes	yes	yes
firm f.e.	yes	yes	yes	yes	yes

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

Table 7: Parameter estimates (with spatial HAC standard errors in parenthesis) from a fixed effects panel regression with time dummies of Equation (33). Model C does not include firm fixed effects (f.e.), while Model D introduces also firm fixed effects. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	Model C		Model D	
φ_1	0.0175*	(0.0091)	0.0182**	(0.0092)
φ_2	0.0036	(0.0040)	0.0052	(0.0039)
ρ	0.0031**	(0.0015)	0.0054**	(0.0026)
β	0.1421***	(0.0369)	0.1425***	(0.0380)
# firms	1431		1431	
# obs.	19448		19448	
F statistic	260.724		161.349	
time f.e.	yes		yes	
firm f.e.	no		yes	

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

than between two firms from different industries (for example, between Volvo and Toshiba in the car manufacturing and ICT sectors, respectively) [cf. Bernstein, 1988].[46] The rationale is that the involved firms might differ in the similarity of their areas of technological competences and knowledge domains depending on whether the collaborating firms operate in the same or in different industries [cf. Nooteboom et al., 2006; Powell and Grodal, 2006].[47]

In this section, we extend our empirical model of Equation (27) by allowing for intra-industry technology spillovers to differ from inter-industry spillovers. The generalized model is given by[48]

$$q_{it} = \varphi_1 \sum_{j=1}^{n} a_{ij,t}^{(1)} q_{jt} + \varphi_2 \sum_{j=1}^{n} a_{ij,t}^{(2)} q_{jt} - \rho \sum_{j=1}^{n} b_{ij} q_{jt} + \beta x_{it} + \eta_i + \kappa_t + \epsilon_{it}, \tag{32}$$

where $a_{ij,t}^{(1)} = a_{ij,t} b_{ij}$, $a_{ij,t}^{(2)} = a_{ij,t}(1 - b_{ij})$, and the coefficients φ_1 and φ_2 capture the intra-industry and the inter-industry technology spillover effect, respectively. In vector-matrix form, we have:

$$\mathbf{q}_t = \varphi_1 \mathbf{A}_t^{(1)} \mathbf{q}_t + \varphi_2 \mathbf{A}_t^{(2)} \mathbf{q}_t - \rho \mathbf{B} \mathbf{q}_t + \mathbf{X}_t \beta + \boldsymbol{\eta} + \kappa_t \mathbf{u}_n + \boldsymbol{\epsilon}_t. \tag{33}$$

The parameter estimates of Equation (33) are given in Table 7. We observe that the signs and statistical significance of the estimates of ρ and β remain the same as before. Interestingly, the intra-industry R&D spillover coefficient is significantly positive, while the inter-industry R&D spillover coefficient is insignificant. This highlights the importance of technology spillovers from firms in the same industry.

[46] Bernstein [1988] studies the effects of intra- and interindustry R&D spillovers on the costs structure of production of Canadian firms and finds that such spillovers decrease the unit costs of production. However, no distinction between collaborating and competing firms are made in this study.

[47] This specification also allows for testing the possibility that allied firms which operate in the same market might form a collusive agreement and thus affect each other's quantity levels differently than firms operating in different markets [cf. Goeree and Helland, 2012].

[48] The theoretical foundation of Equation (32) can be found in supplementary Appendix E.

Table 8: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression with time fixed effects and firm fixed effects of Equation (35). Technological similarity is measured either using the Jaffe or the Mahalanobis patent similarity measures. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	Jaffe		Mahalanobis	
φ	0.0082**	(0.0041)	0.0083**	(0.0040)
χ	0.0044***	(0.0016)	0.0030***	(0.0008)
ρ	0.0056***	(0.0022)	0.0050**	(0.0022)
β	0.1527***	(0.0323)	0.1495***	(0.0323)
# firms	1431		1431	
# obs.	19448		19448	
F statistic	262.370		249.477	
time f.e.	yes		yes	
firm f.e.	yes		yes	

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

8.3. Direct and Indirect Technology Spillovers

In this section, we extend our empirical model of Equation (27) by allowing for both, direct (between collaborating firms) and indirect (between non-collaborating firms) technology spillovers. The generalized model is given by[49]

$$q_{it} = \varphi \sum_{j=1}^{n} a_{ij,t} q_{jt} + \chi \sum_{j=1}^{n} f_{ij,t} q_{jt} - \rho \sum_{j=1}^{n} b_{ij} q_{jt} + \beta x_{it} + \eta_i + \kappa_t + \epsilon_{it}, \tag{34}$$

where $f_{ij,t}$ are weights characterizing alternative channels for technology spillovers than R&D collaborations (measured by the technological proximity between firms; see Bloom et al. [2013]), and the coefficients φ and χ capture the direct and the indirect technology spillover effect, respectively. In vector-matrix form, we then have:

$$\mathbf{q}_t = \varphi \mathbf{A}_t \mathbf{q}_t + \chi \mathbf{F}_t \mathbf{q}_t - \rho \mathbf{B} \mathbf{q}_t + \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\eta} + \kappa_t \mathbf{u}_n + \boldsymbol{\epsilon}_t. \tag{35}$$

The results of a fixed-effect panel regression of Equation (35) are shown in Table 8.[50] Both spillover coefficients, φ and χ, are positive and significant. From Equation (34), the total technology spillover effect is given by $(\varphi a_{ij,t} + \chi f_{ij,t})$. Suppose $a_{ij,t} = 1$, $a_{ik,t} = 0$, and $f_{ij,t} = f_{ik,t}$ for firms i, j and k. Then the total technology spillover effect between firms i and j given $(\varphi + \chi f_{ij,t})$ is stronger than that between firms i and k given by $\chi f_{ik,t}$.

We will use the the estimated parameters in the last column in Table 8 for our policy analysis in Section 9, where we allow for both direct and indirect technology spillovers.

[49]The theoretical foundation of Equation (34) can be found in supplementary Appendix F.
[50]We do not include the link formation model of Equation (31) in this regression because we have shown in Table 4 that it does not change significantly the estimates of the parameters.

Table 9: Parameter estimates from a panel regression of Equation (35) with a random subsample of the firms under different sampling rates. The dependent variable is output obtained from deflated sales. The empirical mean and standard deviation (in parentheses) of the estimates from 500 random subsamples are reported. The estimation is based on the observed alliances in the years 1971–2006.

	Sampling Rate		
	90%	80%	70%
φ	0.0097	0.0097	0.0081
	(0.0060)	(0.0116)	(0.0294)
ρ	0.0072	0.0086	0.0100
	(0.0048)	(0.0088)	(0.0144)
β	0.1596	0.1633	0.1726
	(0.0289)	(0.0470)	(0.0990)
time f.e.	yes	yes	yes
firm f.e.	yes	yes	yes

8.4. Sampled Networks

The balance sheet data we used for the empirical analysis covers only publicly listed firms. It is now well known that the estimation with sampled network data could lead to biased estimates [see, e.g. Chandrasekhar and Lewis, 2011]. To investigate the direction and magnitude of the bias due to the sampled network data, we conduct a limited simulation experiment. In the experiment, we randomly drop 10%, 20%, and 30% of the firms (and the R&D alliances associated with the dropped firms) in our data (corresponding to the sampling rate of 90%, 80%, and 70%). For each sampling rate, we randomly draw 500 subsamples and re-estimate Equation (28) for each subsample. We report the empirical mean and standard deviation of the estimates for each sampling rate in Table 9. As the sampling rate reduces, the standard deviation of the estimates increases while the mean remains roughly the same. This simulation result alleviates the concern on the estimation bias due to sampling.

As a further robustness check, we also report the estimation result of Equation (28) using only U.S. firms in Table 10, where the data coverage is better. We find that the estimates are still robust.

9. R&D Subsidy Policies

With our estimates from the previous section we are now able to empirically determine the optimal subsidy policy, both for the homogenous case, where all firms receive the same subsidy per unit of R&D (see Proposition 3), and for the targeted case, where the subsidy per unit of R&D may vary across firms (see Proposition 4).[51]

9.1. Empirical Results for Optimal R&D Subsidies

In Figure 9, in the top panel, we calculate the optimal homogenous subsidy times R&D effort over time, using the subsidies in the year 1990 as the base level (top left panel), and the percentage increase

[51] Additional details about the numerical implementation of the optimal subsidies program can be found in supplementary Appendix H.

Table 10: Parameter estimates (with spatial HAC standard errors in parenthesis) from a panel regression of Equation (28) for U.S. firms only. Model A' does not include firm fixed effects (f.e.), while Model B' introduces also firm fixed effects. The dependent variable is output obtained from deflated sales. The estimation is based on the observed alliances in the years 1971–2006.

	Model A'		Model B'	
φ	0.0097***	(0.0034)	0.0094***	(0.0035)
ρ	0.0050***	(0.0014)	0.0184***	(0.002)
β	0.0027***	(0.0002)	0.0027***	(0.0002)
# firms	1199		1199	
# obs.	16939		16939	
time f.e.	yes		yes	
firm f.e.	no		yes	

*** Statistically significant at 1% level.
** Statistically significant at 5% level.
* Statistically significant at 10% level.

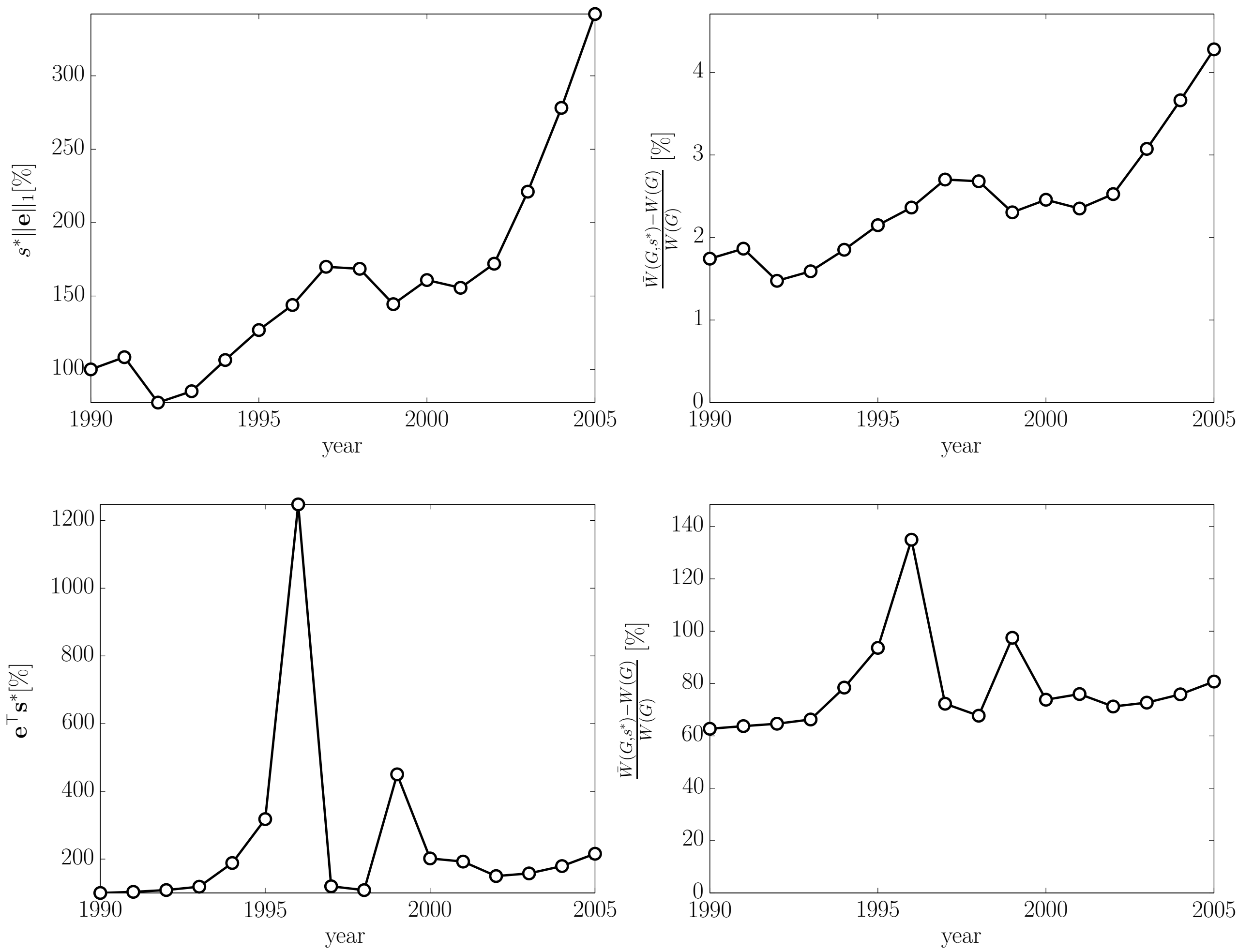
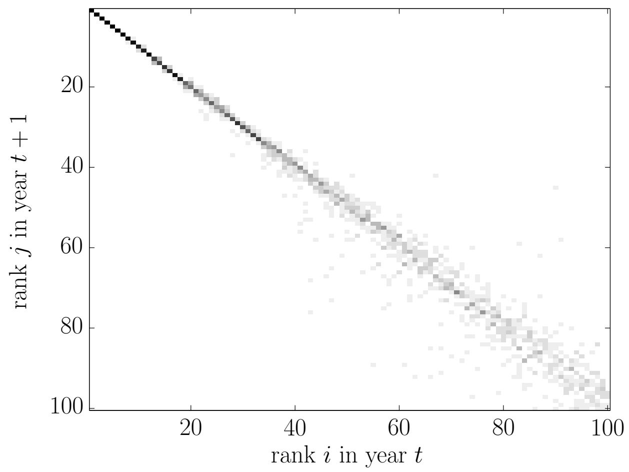


Figure 9: (Top left panel) The total optimal subsidy payments, $s^*\|\mathbf{e}\|_1$, in the homogeneous case over time, using the subsidies in the year 1990 as the base level. (Top right panel) The percentage increase in welfare due to the homogeneous subsidy, s^*, over time. (Bottom left panel) The total subsidy payments, $\mathbf{e}^\top\mathbf{s}^*$, when the subsidies are targeted towards specific firms, using the subsidies in the year 1990 as the base level. (Bottom right panel) The percentage increase in welfare due to the targeted subsidies, $\mathbf{s}^*$, over time.



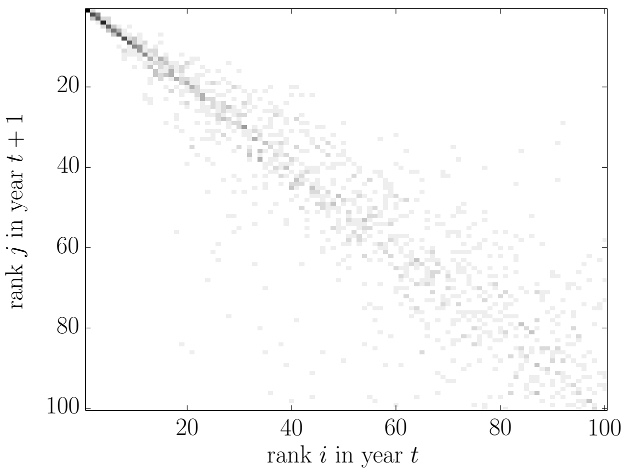


Figure 10: The transition matrix T_{ij} from the rank i in year t to the rank j in year $t+1$ for the homogeneous subsidies ranking (left panel) and the targeted subsidies ranking (right panel) for the first 100 ranks.

in welfare due to the homogenous subsidy over time (top right panel). The total subsidized R&D effort more than trippled over the time between 1990 and 2005. In terms of welfare, the highest increase (around 4 %) is obtained in the year 2005, while the increase in welfare in 1995 is smaller (below 2 %). The bottom panel of Figure 9 does the same exercise for the targeted subsidy policy. The total expenditures on the targeted subsidies are typically higher than the ones for the homogeneous subsidies, and they can also vary by several orders of magnitude. The targeted subsidy program also turns out to have a much higher impact on total welfare, as it can improve welfare by up to 140 %, while the homogeneous subsidies can improve total welfare only by up to 4 %.[52] Moreover, the optimal subsidy levels show a strong variation over time. Both the homogeneous and the aggregate targeted subsidy seem to follow a cyclical trend, similar to the strong variation we have observed for the number of firms participating in R&D collaborations in a given year in Figure 5. This cyclical trend is also reminiscent of the R&D expenditures observed in the empirical literature on business cycles [cf. Galí, 1999].

We can compare the optimal subsidy level predicted from our model with the R&D tax subsidies actually implemented in the United States and selected other countries between 1979 to 1997 [see Bloom et al., 2002; Impullitti, 2010]. While these time series typically show a steady increase of R&D subsidies over time, they do not seem to incorporate the cyclicality that we obtain for the optimal subsidy levels. Our analysis thus suggests that policy makers should adjust R&D subsidies to these cycles.

We next proceed by providing a ranking of firms in terms of targeted subsidies.[53] Such a ranking can guide a planner who wants to maximize total welfare by introducing an R&D subsidy program, identify which firms should receive the highest subsidies, and how high these subsidies should be. The ranking of the first 25 firms by their optimal subsidy levels in 1990 can be found in Table 11 while the one for 2005 is shown in Table 12.[54] We see that the ranking of firms in terms of subsidies does not correspond to other rankings in terms of network centrality, patent stocks or market share.

There is also volatility in the ranking since many firms that are ranked in the top 25 in 1990 are no

[52] Note that similarly large welfare effects of firm-specific R&D subsidies can be found in Akcigit [2009].

[53] Relatedly, Takalo et al. [2013a] analyze the welfare effects of targeted R&D subsidies using project-level data from Finland.

[54] The network statistics shown in these tables correspond to the full CATI-SDC network dataset, prior to dropping firms with missing accounting information. See supplementary Appendix G.1 for more details about the data sources and construction of the R&D alliances network.

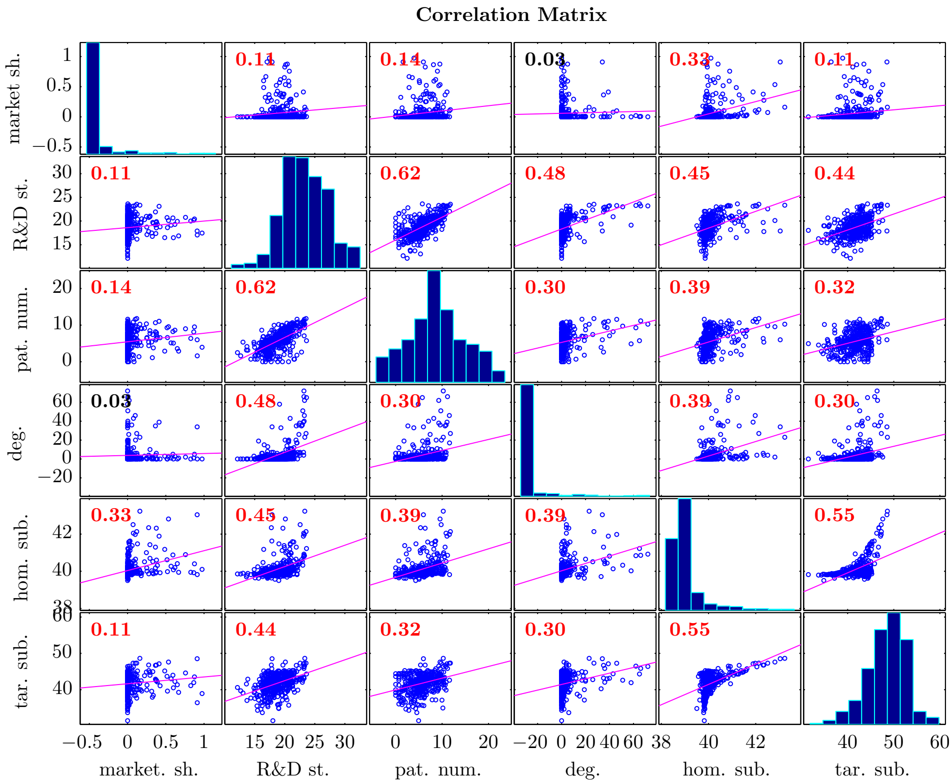


Figure 11: Pair correlation plot of market shares, R&D stocks, the number of patents, the degree, the homogeneous subsidies and the targeted subsidies (cf. Table 12), in the year 2005. The Spearman correlation coefficients are shown for each scatter plot. The data have been log and square root transformed to account for the heterogeneity in the data.

longer there in 2005 (for example *Isuzu Motors Ltd.*, *Kajima Corp.*, *Suzuki Motor Corp.*, etc.). Figure 12 shows the change in the ranking of the 25 highest subsidized firms (Table 11) from 1990 to 2005. Figure 10 shows the transition probability T_{ij} from a rank i in year t to a rank j in year $t+1$ for the first 100 ranks, both for the homogeneous subsidies as well as the targeted subsidies. We observe that in both cases the subsidy rankings are quite stable over time (with the homogeneous subsidies being slightly more stable than the targeted subsidies), where most transitions occur along the diagonal of T_{ij}. There is a larger variation at the bottom right corner of T_{ij} and less variation at the top left corner, showing that the upper ranks are more stable than the lower ranks.

A comparison of market shares, R&D stocks, the number of patents, the degree (i.e. the number of R&D collaborations), the homogeneous subsidy and the targeted subsidy shows a high correlation between the R&D stock and the number of patents, with a (Spearman) correlation coefficient of 0.62 for the year 2005. A slightly weaker correlation can also be found for the homogeneous subsidy and the targeted subsidy, with a correlation coefficient of 0.55 for the year 2005. The corresponding pair correlation plots for the year 2005 can be seen in Figure 11. We also find that highly subsidized firms tend to have a larger R&D stock, and also a larger number of patents, degree and market share. However, these measures can only partially explain the ranking of the firms, as the market share is more related to the product market rivalry effect, while the R&D and patent stocks are more related to the technology spillover effect, and both enter into the computation of the optimal subsidy program.

Observe that our subsidy rankings typically favor larger firms as they tend to be better connected in the R&D network than small firms.[55] This adds to the discussion of whether large or small firms

[55] We further find a significant correlation between market share and the optimal (homogeneous) subsidy levels of 0.47 in the year 1990 and 0.42 in the year 2005. See also Figure 11.

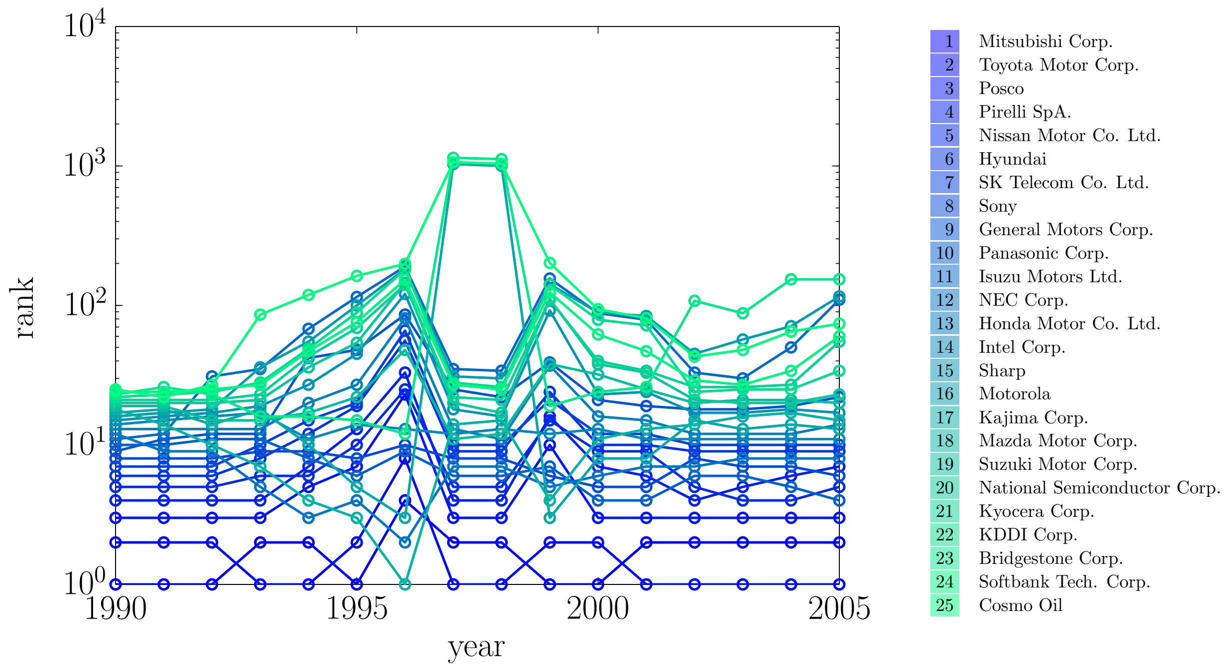


Figure 12: Change in the ranking of the 25 highest subsidized firms (Table 11) from 1990 to 2005.

are contributing more to the innovativeness of an economy [cf. Mandel, 2011],[56] by adding another dimension along which larger firms can have an advantage over small ones, namely by creating R&D spillover effects that contribute to the overall productivity of the economy.[57] While studies such as Spencer and Brander [1983] and Acemoglu et al. [2012] find that R&D should often be taxed rather than subsidized, we find in line with e.g. Hinloopen [2001] that R&D subsidies can have a significantly positive effect on welfare. As argued by Hinloopen [2001], the reason why our results differ from those of Spencer and Brander [1983] is that we take into account the consumer surplus when deriving the optimal R&D subsidy. Moreover, in contrast to Acemoglu et al. [2012], we do not focus on entry and exit but incorporate the network of R&D collaborating firms. This allows us to take into account the R&D spillover effects of incumbent firms, which are typically ignored in studies of the innovative activity of incumbent firms versus entrants. Therefore, we see our analysis as complementary to that of Acemoglu et al. [2012], and we show that R&D subsidies can trigger considerable welfare gains when technology spillovers through R&D alliances are incorporated.

9.2. EUREKA R&D Funding

We can compare our firm-specific optimal subsidies with those that are actually provided by government agencies. For this purpose we have matched the firms in our dataset with the firms that have obtained R&D subsidies from EUREKA, a European intergovernmental organization for promoting market-driven industrial R&D.[58] A ranking of the first 10 firms according to our optimal subsidy policy considering only those firms that received funding from EUREKA is shown in Table 13. We observe that the ranking of our subsidy policy does not necessarily reflect the ranking of the actual

[56] See also "Big and clever. Why large firms are often more inventive than small ones." The Economist (2011, Dec. 17th). Retrieved from http://www.economist.com.

[57] Our findings regarding the pro-welfare effect of R&D conducted by large firms is in line with the results obtained by Bloom et al. [2013], where it is noted that "...smaller firms generate lower social returns to R&D because they operate more in technological niches."

[58] See http://www.eurekanetwork.org/. From our sample of the EUREKA database we consider all participants and their fundings up to and including the year 2005.

Table 11: Subsidies ranking for the year 1990 for the first 25 firms.

Firm	Share [%][a]	num pat.	$\mathbf{d}$	$\mathbf{v}_{\mathrm{PF}}$	Betweenness[b]	Closeness[c]	$\mathbf{q}$ [%][d]	hom. sub. [%][e]	tar. sub. [%][f]	SIC[g]	Country	Rank
Mitsubishi Corp.	31.5745	112	146	0.1143	0.0016	0.0531	6.1218	36.1451	342.8523	5099	JPN	1
Toyota Motor Corp.	4.7957	5148	48	0.0575	0.0003	0.0432	6.1801	39.1549	337.3762	3711	JPN	2
Posco	3.6121	632	0	0.0000	0.0000	0.0000	4.0463	28.4283	134.9814	3312	KOR	3
Pirelli SpA.	19.3624	2122	3	0.0000	0.0000	0.0098	3.3010	23.2172	90.0464	3357	ITA	4
Nissan Motor Co. Ltd .	3.1112	10715	4	0.0037	0.0000	0.0226	2.6492	18.3758	59.9585	3711	JPN	5
Hyundai	0.7881	222	0	0.0000	0.0000	0.0000	2.4339	17.4950	48.6254	3714	KOR	6
SK Telecom Co. Ltd.	1.0245	0	0	0.0000	0.0000	0.0000	2.3513	16.9492	45.2517	4812	KOR	7
Sony	6.2852	33183	48	0.0864	0.0003	0.0445	1.7165	8.4601	30.6815	3600	JPN	8
General Motors Corp.	9.2732	76644	88	0.1009	0.0007	0.0493	1.3490	3.5846	23.9254	3711	USA	9
Panasonic Corp.	11.2383	28916	6	0.0085	0.0000	0.0307	1.5412	10.2029	21.0898	3600	JPN	10
Isuzu Motors Ltd.	0.7929	732	15	0.0134	0.0000	0.0328	1.3238	5.3905	20.6465	3711	JPN	11
NEC Corp.	21.3726	7023	16	0.0168	0.0001	0.0297	1.4552	8.4467	20.2827	7373	JPN	12
Honda Motor Co. Ltd.	2.2440	14661	0	0.0000	0.0000	0.0000	1.4072	10.2719	17.0261	3711	JPN	13
Intel Corp.	9.3900	1132	67	0.1260	0.0003	0.0468	0.8830	0.3639	12.5580	3674	USA	14
Sharp	10.1892	8441	23	0.0459	0.0001	0.0398	1.0321	4.1459	12.3360	3651	JPN	15
Motorola	14.1649	21454	70	0.1186	0.0004	0.0442	0.8233	0.3639	10.7215	3663	USA	16
Kajima Corp.	12.2738	130	2	0.0031	0.0000	0.0266	1.0236	5.7348	10.5653	1600	JPN	17
Mazda Motor Corp.	1.4160	1345	2	0.0000	0.0000	0.0077	0.9841	7.2973	8.5694	3711	JPN	18
Suzuki Motor Corp.	0.6314	306	7	0.0086	0.0000	0.0314	0.9017	6.6169	7.2878	3711	JPN	19
National Semiconductor Corp.	4.0752	1642	43	0.0943	0.0001	0.0440	0.6877	1.1953	6.7442	3674	USA	20
Kyocera Corp.	7.6596	420	21	0.0355	0.0001	0.0400	0.7076	2.1659	6.5366	3674	JPN	21
KDDI Corp.	1.2755	0	0	0.0000	0.0000	0.0000	0.8889	6.8600	6.4668	4812	JPN	22
Bridgestone Corp.	16.6357	584	0	0.0000	0.0000	0.0000	0.8521	6.1894	6.2809	3011	JPN	23
Softbank Technology Inc.	0.4859	0	6	0.0116	0.0000	0.0280	0.7826	5.1603	5.6957	7370	USA	24
Cosmo Oil	0.9155	82	0	0.0000	0.0000	0.0000	0.7909	6.2752	5.0828	2911	JPN	25

[a] Market share in the primary 4-digit SIC sector in which the firm is operating. In case of missing data the closest year with sales data available has been used.
[b] The normalized betweenness centrality is the fraction of all shortest paths in the network that contain a given node, divided by $(n-1)(n-2)$, the maximum number of such paths.
[c] The closeness centrality of node i is computed as $\frac{2}{n-1}\sum_{j=1}^{n} 2^{-\ell_{ij}(G)}$, where $\ell_{ij}(G)$ is the length of the shortest path between i and j in the network G [Dangalchev, 2006], and the factor $\frac{2}{n-1}$ is the maximal centrality attained for the center of a star network.
[d] The relative output of a firm i follows from Proposition 1.
[e] The homogeneous subsidy for each firm i is computed as $e_i^* s^*$, relative to the average subsidy $\frac{1}{n}\sum_{j=1}^{n} e_j^* s^*$ (see Proposition 3).
[f] The targeted subsidy for each firm i is computed as $e_i^* s_i^*$, relative to the average subsidy $\frac{1}{n}\sum_{j=1}^{n} e_j^* s_j^*$ (see Proposition 4).
[g] The primary 4-digit SIC code of a firm in the database.

Table 12: Subsidies ranking for the year 2005 for the first 25 firms.

Firm	Share [%][a]	num pat.	$\mathbf{d}$	$\mathbf{v}_{\mathrm{PF}}$	Betweenness[b]	Closeness[c]	$\mathbf{q}$ [%][d]	hom. sub.[%][e]	tar. sub. [%][f]	SIC[g]	Country	Rank
Toyota Motor Corp.	3.9805	28546	23	0.0086	0.0002	0.0200	2.7489	25.1026	183.0215	3711	JPN	1
Mitsubishi Corp.	44.6576	1430	34	0.1153	0.0014	0.0354	2.5375	20.7279	165.0369	5099	JPN	2
Posco	2.4162	14149	2	0.0000	0.0000	0.0001	1.6921	16.5427	66.5605	3312	KOR	3
Panasonic Corp.	7.1748	131552	27	0.1028	0.0003	0.0289	1.2933	8.3740	49.2358	3600	JPN	4
Nissan Motor Co. Ltd.	1.7840	18103	4	0.0017	0.0000	0.0135	1.3789	12.1997	48.6179	3711	JPN	5
Sony	6.0302	105398	46	0.2440	0.0006	0.0367	1.2433	7.1886	47.6294	3600	JPN	6
Pirelli SpA.	6.2500	2210	3	0.0075	0.0000	0.0174	1.3752	13.5718	44.0368	3357	ITA	7
NEC Corp.	9.6029	63461	39	0.1605	0.0004	0.0323	1.1539	6.5636	41.1072	7373	JPN	8
Hyundai	1.1898	377	4	0.0013	0.0000	0.0129	1.0965	10.3519	29.1985	3714	KOR	9
SK Telecom Co. Ltd.	0.5513	571	6	0.0158	0.0001	0.0204	1.0794	10.0714	28.4479	4812	KOR	10
Sharp	6.1651	41919	35	0.1632	0.0004	0.0310	0.8753	3.6386	26.4370	3651	JPN	11
Microsoft Corp.	10.9732	10639	62	0.1814	0.0020	0.0386	0.7780	2.2984	22.0781	7372	USA	12
Intel Corp.	5.0169	28513	72	0.2410	0.0011	0.0359	0.7401	2.4992	19.6558	3674	USA	13
Honda Motor Co. Ltd.	1.8747	51624	4	0.0132	0.0001	0.0184	0.8354	7.6938	17.9881	3711	JPN	14
Motorola	6.6605	70583	66	0.1598	0.0017	0.0356	0.5714	2.3445	11.4966	3663	USA	15
Sun Microsystems	3.7442	14605	36	0.1052	0.0007	0.0337	0.5498	1.4905	11.3862	3571	USA	16
Idemitsu Kosan Co. Ltd.	0.3710	5571	2	0.0182	0.0000	0.0193	0.6153	5.0598	10.4617	2911	JPN	17
Ford Motor Co.	3.6818	27452	7	0.0015	0.0000	0.0139	0.5688	4.0007	10.1741	3711	USA	18
Mazda Motor Corp.	0.5525	4598	1	0.0001	0.0000	0.0070	0.5006	4.8802	6.6312	3711	JPN	20
General Motors Corp.	3.9590	90652	19	0.0067	0.0002	0.0193	0.4462	3.0574	6.5786	3711	USA	22
Data General Corp.	0.0000	1426	0	0.0000	0.0000	0.0000	0.4146	1.0912	6.5380	3570	USA	23
Dell	18.9098	80	2	0.0190	0.0000	0.0216	0.4009	0.8659	6.3830	3571	USA	24
JFE Holdings Inc.	2.6457	14	4	0.0087	0.0000	0.0197	0.4718	3.9523	6.3103	3312	JPN	25
Microelectronics Technology Corp.	0.1943	33	0	0.0000	0.0000	0.0000	0.3899	0.9866	5.8774	3572	TWN	29
Logitech International SA	0.7606	507	0	0.0000	0.0000	0.0000	0.3839	0.9356	5.7684	3577	CHE	32
Dentsu Inc.	12.1309	319	3	0.0000	0.0001	0.0030	0.4382	3.2251	5.6989	7311	JPN	33

[a] Market share in the primary 4-digit SIC sector in which the firm is operating. In case of missing data the closest year with sales data available has been used.

[b] The normalized betweenness centrality is the fraction of all shortest paths in the network that contain a given node, divided by $(n-1)(n-2)$, the maximum number of such paths.

[c] The closeness centrality of node i is computed as $\frac{2}{n-1}\sum_{j=1}^{n} 2^{-\ell_{ij}(G)}$, where $\ell_{ij}(G)$ is the length of the shortest path between i and j in the network G [Dangalchev, 2006], and the factor $\frac{2}{n-1}$ is the maximal centrality attained for the center of a star network.

[d] The relative output of a firm i follows from Proposition 1.

[e] The homogeneous subsidy for each firm i is computed as $e_i^* s^*$, relative to the average subsidy $\frac{1}{n}\sum_{j=1}^{n} e_j^* s^*$ (see Proposition 3).

[f] The targeted subsidy for each firm i is computed as $e_i^* s_i^*$, relative to the average subsidy $\frac{1}{n}\sum_{j=1}^{n} e_j^* s_j^*$ (see Proposition 4).

[g] The primary 4-digit SIC code of a firm in the database.

Table 13: Optimal subsidies ranking for the year 2005 including the first 10 firms which also received funding trough EUREKA.

Firm	hom. sub.[%][a]	tar. sub. [%][b]	EUREKA sub. [%][c]	SIC[d]	Country	Rank[e]
Renault	1.4859	0.5354	0.0009	3711	FRA	238
TRW Inc. (ZF Friedrichshafen)	1.1668	0.4041	0.0114	3714	GER	273
Tandberg Data ASA	0.7445	0.3805	0.0019	3572	NOR	283
L'Oreal SA	1.2102	0.1314	0.0023	2844	FRA	405
Sydkraft AB	1.2817	0.1109	0.0004	4911	SWE	432
Carraro Spa.	0.9030	0.0923	0.0022	3714	ITA	458
SDL Inc.	1.0302	0.0144	0.0000	7371	GBR	624
York International Corp.	0.8501	0.0004	0.0001	3585	GBR	774
H Lundbeck A/S	0.8138	0.0000	0.0001	2834	DNK	1088
Riber SA	0.8444	0.0000	0.1728	3679	FRA	1252

[a] The homogeneous subsidy for each firm i is computed as $e_i^* s^*$, relative to the total homogeneous subsidies $\sum_{j=1}^n e_j^* s^*$ (see Proposition 3).

[b] The targeted subsidy for each firm i is computed as $e_i^* s_i^*$, relative to the total targeted subsidies $\sum_{j=1}^n e_j^* s_j^*$ (see Proposition 4).

[c] The EUREKA subsidies comprise the total accumulated contribution to project costs (relative to the total funds across all firms) in a given year, where all project costs involving a particular firm are considered. For more detailed information see http://www.eurekanetwork.org/.

[d] The primary 4-digit SIC code according to Compustat U.S. and Global fundamentals databases.

[e] The rank corresponds to the ranking of Table 12.

subsidies implemented by EUREKA. For example, *Riber SA* received funding of 0.17 % of the overall (accumulated) funding in the year 2005 and is ranked 1252st according to our optimal subsidy policy, while *Renault* received funding of only 0.0009 % of the overall funding while being ranked 238th, far above *Riber SA*. However, this discrepancy is not surprising, as current public funding instruments such as EUREKA do not take into account network effects and externalities stemming from R&D collaborations that determine our optimal subsidy policy.

10. Conclusion

In this paper, we have developed a model where firms jointly form R&D collaborations (networks) to lower their production costs while at the same time competing on the product market. We have highlighted the positive role of the network in terms of technology spillovers and the negative role of product rivalry in terms of market competition. We have also determined the importance of targeted subsidies on the total welfare of the economy.

Using a panel of R&D alliance networks and annual reports, we have then tested our theoretical results and first showed that the magnitude of the technology spillover effect is much higher than that of the product rivalry effect, indicating that the latter dominates the former so that the net returns to R&D collaborations are strictly positive. We have also identified the firms that should be subsidised the most. Finally, we have drawn some policy conclusions about optimal R&D subsidies from the results obtained over different sectors, as well as their temporal variation.

We believe that the methodology developed in this paper offers a fruitful way of analyzing the existence of R&D spillovers and their policy implications in terms of firms' subsidies across and within different industries. We also believe that putting forward the role of networks in terms of R&D collaborations is important to understanding the different aspects of these markets.

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Appendix

A. Proofs

We first state a lemma that will be needed for the proof of Proposition 1.

Lemma 1. *Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric, real matrices and assume that the inverse $\mathbf{A}^{-1}$ exists and is non-negative and also that $\mathbf{B}$ is non-negative. Provided that $\lambda_{\max}(\mathbf{A}^{-1}\mathbf{B}) < 1$ we have that*

(i) the following series expansion exists

$$(\mathbf{A}+\mathbf{B})^{-1} = \sum_{k=0}^{\infty}(-1)^k(\mathbf{A}^{-1}\mathbf{B})^k\mathbf{A}^{-1},$$

(ii) for any $\mathbf{x} \in \mathbb{R}^n_+$ we have that $\mathbf{A}^{-1}\mathbf{B}\mathbf{x} < \mathbf{x}$, and

(iii) if also $\mathbf{A}^{-1}\mathbf{x} > \mathbf{0}$ then $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{x} > \mathbf{0}$.

Proof of Lemma 1 (i) Notice that

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^{-1} &= (\mathbf{A}(\mathbf{I}_n + \mathbf{A}^{-1}\mathbf{B}))^{-1} \\ &= (\mathbf{I}_n + \mathbf{A}^{-1}\mathbf{B}))^{-1}\mathbf{A}^{-1} \\ &= \sum_{k=0}^{\infty}(-1)^k(\mathbf{A}^{-1}\mathbf{B})^k\mathbf{A}^{-1},\end{aligned}$$

where the Neumann series expansion for $(\mathbf{I}_n + \mathbf{A}^{-1}\mathbf{B}))^{-1}$ can be applied if $\lambda_{\max}(\mathbf{A}^{-1}\mathbf{B}) < 1$.

(ii) Observe that $\lambda_{\max}(\mathbf{A}^{-1}\mathbf{B}) < 1$ is equivalent to $\mathbf{A}^{-1}\mathbf{B}\mathbf{x} < \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n_+$. To see this consider an orthonormal basis of $\mathbb{R}^n$ spanned by the eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$. Then we can write $\mathbf{x} = \sum_{i=1}^n c_i\mathbf{v}_i$ with suitable coefficients $c_i = \mathbf{x}^\top\mathbf{v}_i/(\mathbf{v}_i^\top\mathbf{v}_i)$ and $\mathbf{A}^{-1}\mathbf{B}\mathbf{v}_i = \lambda_i\mathbf{v}_i$. It then follows that

$$\mathbf{A}^{-1}\mathbf{B}\mathbf{x} = \sum_{i=1}^n c_i\lambda_i\mathbf{v}_i \le \lambda_{\max}(\mathbf{A}^{-1}\mathbf{B})\sum_{i=1}^n c_i\mathbf{v}_i = \lambda_{\max}(\mathbf{A}^{-1}\mathbf{B})\mathbf{x}.$$

Hence, if $\lambda_{\max}(\mathbf{A}^{-1}\mathbf{B}) < 1$ it must hold that $\mathbf{A}^{-1}\mathbf{B}\mathbf{x} < \mathbf{x}$.

(iii) We can write the series expansion of the inverse as follows

$$(\mathbf{A}+\mathbf{B})^{-1}\mathbf{x} = \sum_{k=0}^{\infty}(-1)^k(\mathbf{A}^{-1}\mathbf{B})^k\mathbf{A}^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{x} + \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{x} - \ldots .$$

By assumption we have that $\mathbf{A}^{-1}\mathbf{x} \ge \mathbf{0}$. Then denote by $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{x} \ge \mathbf{0}$. Then the first two terms in the series can be written as

$$(\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{B})\mathbf{A}^{-1}\mathbf{x} = (\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{B})\tilde{\mathbf{x}} > 0,$$

where the inequality follows from part (ii) of the lemma. Next, consider the third and fourth terms in the series expansion

$$(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B})\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}(\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{B})\tilde{\mathbf{x}} \ge 0,$$

where the inequality follows again from the fact that $(\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{B})\tilde{\mathbf{x}} > 0$ from part (ii) of the lemma and the assumption that $\mathbf{A}^{-1}$ and $\mathbf{B}$ are non-negative matrices. We can then iterate by induction to show the desired claim.

□

Proof of Proposition 1 We start by providing a condition on the marginal cost $\bar{c}_i$ such that all firms choose an interior R&D effort level. The marginal cost of firm i from Equation (2) can be written as

$$c_i = \max\left\{0, \bar{c}_i - e_i - \varphi\sum_{j=1}^n a_{ij}e_j\right\}. \tag{36}$$

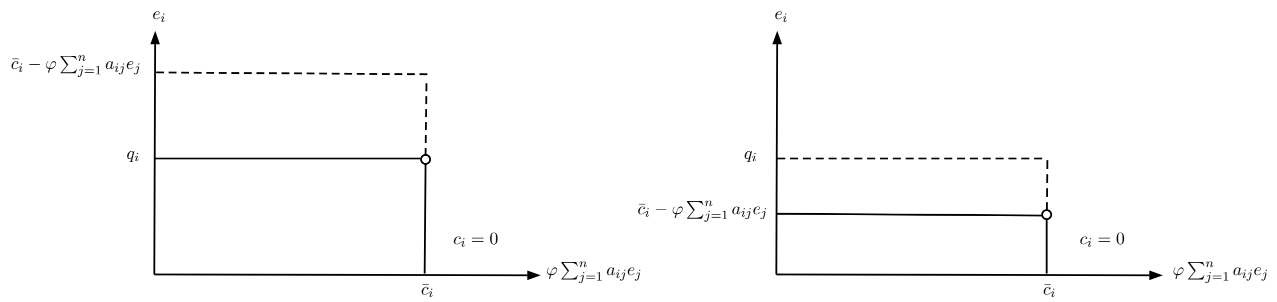

Figure A.1: The best response effort level, e_i, of firm i for $q_i < \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j$ (left panel) and $q_i > \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j$ (right panel).

The profit function of Equation (3) can then be written as

$$\pi_i = (p_i - c_i) q_i - \frac{1}{2} e_i^2 = \begin{cases} p_i q_i - \frac{1}{2} e_i^2, & \text{if } \bar{c}_i \leq e_i + \varphi \sum_{j=1}^n a_{ij} e_j, \\ (p_i - \bar{c}_i + e_i + \varphi \sum_{j=1}^n a_{ij} e_j) q_i - \frac{1}{2} e_i^2, & \text{otherwise.} \end{cases}$$

It is clear that when $\bar{c}_i \leq \varphi \sum_{j=1}^n a_{ij} e_j$ the profit of firm i is decreasing with e_i, and hence, firm i sets $e_i = 0$. On the other hand, if $\bar{c}_i > \varphi \sum_{j=1}^n a_{ij} e_j$ then for all $0 \leq e_i < \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j$ we have that

$$\frac{\partial \pi_i}{\partial e_i} = q_i - e_i = 0,$$

so that we obtain $e_i = q_i$. Moreover, when $q_i > \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j$ then the effort of firm i is given by $e_i = \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j$. It then follows that the best response effort level of firm i is given by

$$e_i = \begin{cases} 0, & \text{if } \bar{c}_i < \varphi \sum_{j=1}^n a_{ij} e_j, \\ \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j, & \text{if } \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j \leq q_i, \\ q_i, & \text{if } \bar{c}_i - \varphi \sum_{j=1}^n a_{ij} e_j > q_i. \end{cases}$$

An illustration of the best response effort level, e_i, of firm i can be seen in Figure A.1. Note that with $q_i \in [0, \bar{q}]$ we must have that $0 \leq e_i \leq q_i \leq \bar{q}$, and therefore

$$\max_{i \in \mathcal{N}} \left\{ e_i + \varphi \sum_{j=1}^n a_{ij} e_j \right\} \leq \bar{q}(1 + \varphi(n-1)).$$

Hence, requiring that

$$\min_{i \in \mathcal{N}} \bar{c}_i > \bar{q}(1 + \varphi(n-1)), \tag{37}$$

implies that the best response effort level of firm i is given by

$$e_i = q_i, \tag{38}$$

and the marginal cost is given by $c_i = \bar{c}_i - e_i - \varphi \sum_{j=1}^n a_{ij} e_j = \bar{c}_i - q_i - \varphi \sum_{j=1}^n a_{ij} q_j$ for all $i \in \mathcal{N}$. For the remainder of the proof we assume that this conditions is satisfied.

We next provide the proofs for the different parts of the proposition:

(i) The first derivative of the profit function with respect to the output q_i of firm i is given by

$$\frac{\partial \pi_i}{\partial q_i} = \bar{\alpha}_i - \bar{c}_i - 2 q_i - \rho \sum_{j=1}^n b_{ij} q_j + e_i + \varphi \sum_{j=1}^n a_{ij} e_j.$$

Inserting the optimal R&D efforts, $e_i = q_i$, then gives

$$\frac{\partial \pi_i}{\partial q_i} = (\bar{\alpha}_i - \bar{c}_i) - q_i - \rho \sum_{j=1}^n b_{ij} q_j + \varphi \sum_{j=1}^n a_{ij} q_j.$$

A Nash equilibrium is a vector $\mathbf{q} \in [0, \bar{q}]^n$ that satisfies the following system of equations: $\frac{\partial \pi_i}{\partial q_i} = 0, \quad \forall i \in \mathcal{N}$ such that $0 < q_i < \bar{q}$, $\frac{\partial \pi_i}{\partial q_i} < 0, \quad \forall i \in \mathcal{N}$ such that $q_i = 0$ and $\frac{\partial \pi_i}{\partial q_i} > 0, \quad \forall i \in \mathcal{N}$ such that $q_i = \bar{q}$. In the following we denote by $\mu_i \equiv \bar{\alpha}_i - \bar{c}_i$. Then the Nash equilibrium output levels q_i can be found from the solution to the following equations

$$\begin{aligned} q_i = 0, \quad &\text{if} \quad -\mu_i + q_i + \rho \sum_{j=1}^n b_{ij} q_j - \varphi \sum_{j=1}^n a_{ij} q_j > 0, \\ q_i = \mu_i - \rho \sum_{j=1}^n b_{ij} q_j + \varphi \sum_{j=1}^n a_{ij} q_j, \quad &\text{if} \quad -\mu_i + q_i + \rho \sum_{j=1}^n b_{ij} q_j - \varphi \sum_{j=1}^n a_{ij} q_j = 0, \\ q_i = \bar{q}, \quad &\text{if} \quad -\mu_i + q_i + \rho \sum_{j=1}^n b_{ij} q_j - \varphi \sum_{j=1}^n a_{ij} q_j < 0. \end{aligned} \tag{39}$$

The problem of finding a vector $\mathbf{q}$ such that the conditions in (39) are satisfied is known as the bounded linear complementarity problem (LCP) [Byong-Hun, 1983].[59] The corresponding best response function $f_i : [0, \bar{q}]^{n-1} \to [0, \bar{q}]$ can be written compactly as follows:

$$f_i(\mathbf{q}_{-i}) \equiv \max \left\{ 0, \min \left\{ \bar{q}, \mu_i - \rho \sum_{j=1}^n b_{ij} q_j + \varphi \sum_{j=1}^n a_{ij} q_j \right\} \right\}. \tag{40}$$

Since $[0, \bar{q}]^{n-1}$ is a convex compact subset of $\mathbb{R}^{n-1}$ and f is a continuous function on this set, a solution to the fixed point equation $q_i - f(\mathbf{q}_{-i}) = 0$ is guaranteed to exist by Brouwer's fixed point theorem.

Observe that the bounded LCP in (39) is equivalent to the Kuhn-Tucker optimality conditions of the following quadratic programming (QP) problem with box constraints [cf. Byong-Hun, 1983]:

$$\min_{\mathbf{q} \in [0, \bar{q}]^n} \left\{ -\boldsymbol{\mu}^\top \mathbf{q} + \frac{1}{2} \mathbf{q}^\top (\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A}) \mathbf{q} \right\}. \tag{41}$$

An alternative proof for the existence of an equilibrium then follows form the Frank-Wolfe Theorem [Frank and Wolfe, 1956].[60]

Moreover, a unique solution is guaranteed to exist if $\rho = 0$ or when the matrix $\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A}$ is positive definite. The case of $\rho = 0$ has been analyzed in Belhaj et al. [2014]. The authors show that a unique equilibrium exists when output levels are bounded for any value of the spillover parameter φ. In the following we will provide sufficient conditions for positive definiteness (and thus uniqueness) when $\rho > 0$.

Consider first the case of $\varphi = 0$. The matrix $\mathbf{I}_n + \rho \mathbf{B}$ is positive definite if and only if all its eigenvalues are positive. The smallest eigenvalue of $\mathbf{I}_n + \rho \mathbf{B}$ is given by $1 + \rho \lambda_{\min}(\mathbf{B})$. Then, all eigenvalues are positive if $\lambda_{\min}(\mathbf{B}) > -\frac{1}{\rho}$. The matrix $\mathbf{B}$ has elements $b_{ij} \in \{0, 1\}$ and can be written as a block diagonal matrix $\mathbf{B} \equiv \sum_{m=1}^M (\mathbf{u}_m \mathbf{u}_m^\top - \mathbf{D}_m)$, with $\mathbf{u}_m$ being an $n \times 1$ zero-one vector with elements $(\mathbf{u}_m)_i = 1$ if $i \in \mathcal{M}_m$ and $(\mathbf{u}_m)_i = 0$ otherwise for all $i = 1, \ldots, n$. Moreover, $\mathbf{D}_m = \text{diag}(\mathbf{u}_m)$ is

[59] This is the linear version of the mixed complementarity problem analyzed in Simsek et al. [2005] and is similar to the problem studied in Bloch and Quérou [2013]. For a detailed discussion and analysis of LCP see Cottle et al. [1992].

[60] The Frank-Wolfe Theorem states that if a quadratic function is bounded below on a nonempty polyhedron, then it attains its infimum.

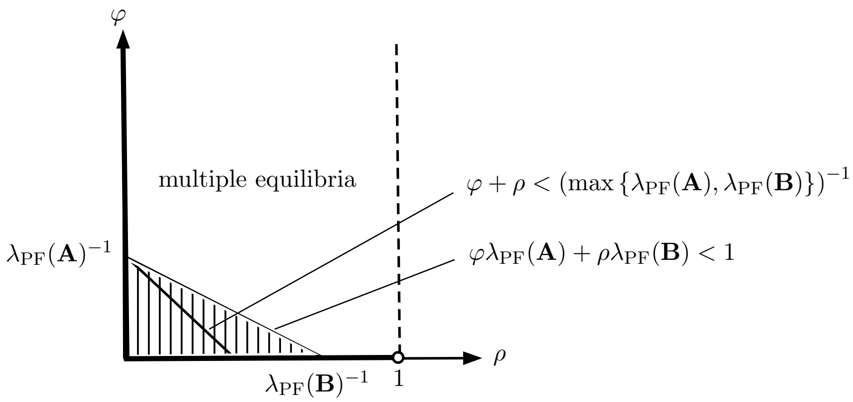

Figure A.2: Illustration of the parameter regions where an equilibrium is unique, or multiple equilibria can exist.

the diagonal matrix with diagonal entries given by $\mathbf{u}_m$. Since $\mathbf{B}$ is a block diagonal matrix with zero diagonal and blocks of size $|\mathcal{M}_m|$, $m = 1, \ldots, M$, the spectrum (set of eigenvalues) of $\mathbf{B}$ is given by $\{|\mathcal{M}_1| - 1, |\mathcal{M}_2| - 1, ..., |\mathcal{M}_M| - 1, -1, \ldots, -1\}$. Hence, the smallest eigenvalue of $\mathbf{B}$ is -1 and the condition for positive definiteness becomes $-1 > -\frac{1}{\rho}$, or equivalently, $\rho < 1$, which holds by assumption.

Next we consider the case of $\varphi > 0$. The matrix $\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A}$ is positive definite if its smallest eigenvalue is positive, that is when $\lambda_{\min}(\rho\mathbf{B} - \varphi\mathbf{A}) + 1 > 0$. This is equivalent to $\lambda_{\mathrm{PF}}(\varphi\mathbf{A} + (-\rho)\mathbf{B}) < 1$. Since $\lambda_{\mathrm{PF}}(\varphi\mathbf{A} + (-\rho)\mathbf{B}) \leq \varphi\lambda_{\mathrm{PF}}(\mathbf{A}) + \rho\lambda_{\mathrm{PF}}(\mathbf{B})$,[61] a sufficient condition is then given by $(\rho + \varphi)\max\{\lambda_{\mathrm{PF}}(\mathbf{A}), \lambda_{\mathrm{PF}}(\mathbf{B})\} < 1$, or equivalently $\rho + \varphi < (\max\{\lambda_{\mathrm{PF}}(\mathbf{A}), \lambda_{\mathrm{PF}}(\mathbf{B})\})^{-1}$. We have that the largest eigenvalue of the matrix $\mathbf{B}$ is equal to the size of the largest market $|\mathcal{M}_m|$ minus one (as this is a block-diagonal matrix with all elements being one in each block and zero diagonal), so that a sufficient condition for invertibility (and thus uniqueness) is given by

$$\rho + \varphi < \left(\max\left\{\lambda_{\mathrm{PF}}(\mathbf{A}), \max_{m=1,\ldots,M}\{(|\mathcal{M}_m| - 1)\}\right\}\right)^{-1}.$$

Figure A.2 shows an illustration of the parameter regions where an equilibrium is unique, or multiple equilibria can exist.

When the matrix $\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A}$ is not positive definite, and we allow for $\rho > 0$, then the objective function in Equation (41) will be non-convex, and there might exist multiple equilibria. Computing these equilibria can be done via numerical algorithms for solving box-constrained non-convex quadratic programs [cf. e.g. Chen and Burer, 2012].[62]

(ii) We provide a characterization of the interior equilibrium, $0 < q_i < \bar{q}$ for all $i \in \mathcal{N}$. From the best response function in Equation (40) we get

$$q_i = \mu_i - \rho\sum_{j=1}^n b_{ij}q_j + \varphi\sum_{j=1}^n a_{ij}q_j. \tag{42}$$

In matrix-vector notation it can be written as $\mathbf{q} = \boldsymbol{\mu} - \rho\mathbf{B}\mathbf{q} + \varphi\mathbf{A}\mathbf{q}$ or, equivalently, $(\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})\mathbf{q} = \boldsymbol{\mu}$.

We have assumed that the matrix $\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A}$ is positive definite. This means that all its eigenvalues are positive. Moreover, is its real and symmetric, and thus has only real eigenvalues. A matrix is invertible, if its determinant is not zero. The determinant of a matrix is equivalent to the product of its eigenvalues. Hence, if a matrix has only positive real eigenvalues, then its determinant is not zero

[61] Let $\|\cdot\|$ be any matrix norm, including the spectral norm, which is just the largest eigenvalue. Then we have that $\|\sum_{i=1}^n \alpha_i\mathbf{A}_i\| \leq \sum_{i=1}^n |\alpha_i|\|\mathbf{A}_i\| \leq \left(\sum_{i=1}^n |\alpha_i|\right)\max_i\|\mathbf{A}_i\|$ by Weyl's theorem [cf. e.g. Horn and Johnson, 1990, Theorem 4.3.1].

[62] See also Equation (81) and below.

and it is invertible. When the inverse of $\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A}$ exists, we can write equilibrium quantities as

$$\mathbf{q} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}.$$

We have shown that there exists a unique equilibrium given by $\mathbf{q} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}$, but we have not yet shown that it is interior, i.e. $q_i > 0$, $\forall i \in \mathcal{N}$. Profits in equilibrium can be written as

$$\pi_i = (\bar{\alpha}_i - \bar{c}_i)q_i - \rho q_i \sum_{j=1}^{n} b_{ij}q_j + \varphi q_i \sum_{j=1}^{n} a_{ij}q_j - \frac{1}{2}q_i^2.$$

From Equation (42) it follows that

$$\begin{aligned}
\rho q_i \sum_{j=1}^{n} b_{ij}q_j - \varphi q_i \sum_{j=1}^{n} a_{ij}q_j &= ((\rho\mathbf{B} - \varphi\mathbf{A})\mathbf{q})_i \\
&= q_i((\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})\mathbf{q} - \mathbf{q})_i \\
&= q_i\left((\bar{\alpha}_i - \bar{c}_i) - q_i\right), \qquad (43)
\end{aligned}$$

so that we can write equilibrium profits as

$$\pi_i = (\bar{\alpha}_i - \bar{c}_i)q_i - q_i\left((\bar{\alpha}_i - \bar{c}_i) - q_i\right) - \frac{1}{2}q_i^2 = \frac{1}{2}q_i^2. \qquad (44)$$

(iii) We assume that all firms operate in the same market so that $M = 1$. The first-order condition for a firm i is given by Equation (42), which, when $M = 1$, can be written as

$$q_i = \mu_i - \rho\sum_{j \neq i} q_j + \varphi\sum_{j=1}^{n} a_{ij}q_j$$

Let us denote by $\hat{q}_{-i} \equiv \sum_{j\neq i} q_j$ the total output of all firms excluding firm i. The equation above is equivalent to

$$q_i = \mu_i - \rho\hat{q}_{-i} + \varphi\sum_{j=1}^{n} a_{ij}q_j$$

We can now define $\hat{q} \equiv \sum_{j\neq i} q_j + q_i$, which corresponds to the total output of all firms (including i). The equation above is now equivalent to

$$q_i = \mu_i - \rho\hat{q} + \rho q_i + \varphi\sum_{j=1}^{n} a_{ij}q_j,$$

or

$$q_i = \frac{1}{1-\rho}\mu_i - \frac{\rho}{1-\rho}\hat{q} + \frac{\varphi}{1-\rho}\sum_{j=1}^{n} a_{ij}q_j. \qquad (45)$$

Observe that even if firms are local monopolies (i.e. $\rho = 0$) this solution is still well-defined. Observe also that $1 - \rho > 0$ if and only if $\rho < 1$, which we assume throughout.

In matrix form, Equation (45) can be written as

$$\left(\mathbf{I}_n - \frac{\varphi}{1-\rho}\mathbf{A}\right)\mathbf{q} = \frac{1}{1-\rho}\boldsymbol{\mu} - \frac{\rho\hat{q}}{1-\rho}\mathbf{u},$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^\top$, and $\mathbf{u} = (1, \ldots, 1)^\top$. Denote $\phi = \varphi/(1-\rho)$. If $\phi\lambda_{\mathrm{PF}}(\mathbf{A}) < 1$, this is equivalent to

$$\mathbf{q} = \frac{1}{1-\rho}\left(\mathbf{I}_n - \phi\mathbf{A}\right)^{-1}\boldsymbol{\mu} - \frac{\rho\hat{q}}{1-\rho}\left(\mathbf{I}_n - \phi\mathbf{A}\right)^{-1}\mathbf{u}.$$

This equation is equivalent to

$$\mathbf{q} = \frac{1}{1-\rho}\left(\mathbf{b}_{\boldsymbol{\mu}}(G,\phi) - \rho\hat{q}\,\mathbf{b}_{\mathbf{u}}(G,\phi)\right), \tag{46}$$

where $\mathbf{b}_{\mathbf{u}}(G,\varphi/(1-\rho)) = (\mathbf{I}_n - \phi\mathbf{A})^{-1}\mathbf{u}$ is the unweighted vector of Bonacich centralities and $\mathbf{b}_{\boldsymbol{\mu}}(G,\varphi/(1-\rho)) = (\mathbf{I}_n - \phi\mathbf{A})^{-1}\boldsymbol{\mu}$ is the weighted vector of Bonacich centralities where the weights are the μ_i for $i = 1,\ldots,n$.[63]

We need now to calculate $\hat{q}$. Multiplying Equation (46) to the left by $\mathbf{u}^\top$, we obtain

$$(1-\rho)\,\hat{q} = \|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1 - \rho\hat{q}\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1,$$

where

$$\|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1 = \mathbf{u}^{\mathrm{T}}\mathbf{b}_{\boldsymbol{\mu}}(G,\phi) = \sum_{i=1}^{n} b_{\mu_i}(G,\phi) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]}\mu_j,$$

is the sum of the weighted Bonacich centralities and

$$\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1 = \mathbf{u}^\top\mathbf{b}_{\mathbf{u}}(G,\phi) = \sum_{i=1}^{n} b_{u,i}(G,\phi) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]}$$

is the sum of the unweighted Bonacich centralities. Solving this equation, we get

$$\hat{q} = \frac{\|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1}{(1-\rho)+\rho\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1}$$

Plugging this value of $\hat{q}$ into Equation (46), we finally obtain

$$q_i = \frac{1}{1-\rho}\left(b_{\boldsymbol{\mu},i}(G,\phi) - \frac{\rho\,\|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1}{1-\rho+\rho\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1}b_{\mathbf{u},i}(G,\phi)\right). \tag{47}$$

This corresponds to Equation (9) in the proposition.

In the following we provide conditions which guarantee that the equilibrium is always interior. For that, we would like to show that $q_i > 0$, $\forall i = 1,\ldots,n$. Using Equation (47), this is equivalent to

$$b_{\boldsymbol{\mu},i}(G,\phi) > \frac{\rho\,\|\mathbf{b}_{\boldsymbol{\mu}}(G,\phi)\|_1}{1-\rho+\rho\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1}b_{\mathbf{u},i}(G,\phi), \quad \forall i = 1,\ldots,n. \tag{48}$$

Denote by $\underline{\mu} = \min_i\{\mu_i \mid i \in N\}$ and $\overline{\mu} = \max_i\{\mu_i \mid i \in N\}$, with $\underline{\mu} < \overline{\mu}$. Then, $\forall i = 1,\ldots,n$, we have

$$\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]}\mu_j \le \overline{\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]} = \overline{\mu}\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1$$

and

$$b_{\boldsymbol{\mu},i}(G,\phi) = \sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]}\mu_j \ge \underline{\mu}\,b_{\mathbf{u},i}(G,\phi) = \sum_{j=1}^{n}\sum_{p=0}^{\infty}\phi^p a_{ij}^{[p]}\underline{\mu}$$

Thus, a sufficient condition for Equation (48) to hold is

$$\underline{\mu}\,b_{\mathbf{u},i}(G,\phi) > \frac{\rho\overline{\mu}\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1}{1-\rho+\rho\,\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|_1}b_{\mathbf{u},i}(G,\phi),$$

[63] A definition and further discussion of the Bonacich centrality is given in Appendix B.3.

or equivalently

$$\underline{\mu} > \frac{\rho \overline{\mu} \left\| \mathbf{b}_{\mathbf{u}}(G, \phi) \right\|_1}{1 - \rho + \rho \left\| \mathbf{b}_{\mathbf{u}}(G, \phi) \right\|_1},$$

or

$$1 - \rho > \rho \left\| \mathbf{b}_{\mathbf{u}}(G, \phi) \right\|_1 \left(\frac{\overline{\mu}}{\underline{\mu}} - 1 \right). \tag{49}$$

Next, observe that, by definition

$$\left\| \mathbf{b}_{\mathbf{u}}(G, \phi) \right\|_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=0}^{\infty} \phi^p a_{ij}^{[p]} = \sum_{p=0}^{\infty} \phi^p \mathbf{u}^\top \mathbf{A}^p \mathbf{u}. \tag{50}$$

We know that $\lambda_{\mathrm{PF}}(\mathbf{A}^p) = \lambda_{\mathrm{PF}}(\mathbf{A})^p$, for all $p \geq 0$.[64] Also, $\mathbf{u}^\top \mathbf{A}^p \mathbf{u}/n$ is the average connectivity in the matrix $\mathbf{A}^p$ of paths of length p in the original network $\mathbf{A}$, which is smaller than its spectral radius $\lambda_{\mathrm{PF}}(\mathbf{A})^p$ [Cvetkovic et al., 1995], i.e. $\mathbf{u}^\top \mathbf{A}^p \mathbf{u}/n \leq \lambda_{\mathrm{PF}}(\mathbf{A})^p$. Therefore, Equation (50) leads to the following inequality

$$\left\| \mathbf{b}_{\mathbf{u}}(G, \phi) \right\|_1 = \sum_{p=0}^{\infty} \phi^p \mathbf{u}^\top \mathbf{A}^p \mathbf{u} \leq n \sum_{p=0}^{\infty} \phi^p \lambda_{\mathrm{PF}}(\mathbf{A})^p = \frac{n}{1 - \phi \lambda_{\mathrm{PF}}(\mathbf{A})}.$$

A sufficient condition for Equation (49) to hold is thus

$$\phi \lambda_{\mathrm{PF}}(\mathbf{A}) + \frac{n\rho}{1-\rho} \left(\frac{\overline{\mu}}{\underline{\mu}} - 1 \right) < 1.$$

Clearly, this interior equilibrium is unique. This is the condition given in the proposition for case (iii).

(ii) We now show that we have an interior equilibrium with all firms producing at positive quantity levels, that is $\mathbf{q} = (\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A})^{-1} \boldsymbol{\mu} > \mathbf{0}$. To do this we will apply Lemma 1. Let $\mathbf{I}_n - \varphi \mathbf{A}$ be the matrix $\mathbf{A}$ in the lemma and $\rho \mathbf{B}$ the corresponding matrix $\mathbf{B}$. We have that both are real and symmetric, and that $\mathbf{B}$ is a non-negative matrix. Furthermore, provided that $\varphi < 1/\lambda_{\mathrm{PF}}(\mathbf{A})$, the inverse $\mathbf{A}^{-1}$ exists and is non-negative. Next, we need to show that $\lambda_{\mathrm{PF}}(\mathbf{A}^{-1}\mathbf{B}) < 1$, but this is equivalent to $\lambda_{\mathrm{PF}}((\mathbf{I}_n - \varphi \mathbf{A})^{-1} \rho \mathbf{B}) < 1$. Note that

$$\lambda_{\mathrm{PF}}((\mathbf{I}_n - \varphi \mathbf{A})^{-1} \rho \mathbf{B}) = \rho \lambda_{\mathrm{PF}}((\mathbf{I}_n - \varphi \mathbf{A})^{-1} \mathbf{B}) \leq \rho \lambda_{\mathrm{PF}}((\mathbf{I}_n - \varphi \mathbf{A})^{-1}) \lambda_{\mathrm{PF}}(\mathbf{B}) = \frac{\rho \lambda_{\mathrm{PF}}(\mathbf{B})}{1 - \varphi \lambda_{\mathrm{PF}}(\mathbf{A})},$$

so that a sufficient condition is given by

$$\frac{\rho \lambda_{\mathrm{PF}}(\mathbf{B})}{1 - \varphi \lambda_{\mathrm{PF}}(\mathbf{A})} < 1,$$

which is implied by

$$\rho \lambda_{\mathrm{PF}}(\mathbf{B}) = \rho \max_{m=1,\ldots,M} \{(|\mathcal{M}_m| - 1)\} < 1 - \varphi \lambda_{\mathrm{PF}}(\mathbf{A}).$$

The lemma then implies that $(\mathbf{A} + \mathbf{B})^{-1} \mathbf{x} > \mathbf{0}$ for any vector $\mathbf{x} > \mathbf{0}$, and in particular for the vector $\boldsymbol{\mu}$, which is positive by assumption.

(iv) Assume that not only $M = 1$ but also $\mu_i = \mu$ for all $i = 1, \ldots, n$. If $\phi \lambda_{\mathrm{PF}}(\mathbf{A}) < 1$, the equilibrium condition in Equation (47) can be further simplified to

$$\mathbf{q} = \frac{\mu}{1 - \rho + \rho \| \mathbf{b}_{\mathbf{u}}(G, \phi) \|_1} \mathbf{b}_{\mathbf{u}}(G, \phi).$$

[64] Observe that the relationship $\lambda_{\mathrm{PF}}(\mathbf{A}^p) = \lambda_{\mathrm{PF}}(\mathbf{A})^p$, $p \geq 0$, holds true for both symmetric as well as asymmetric adjacency matrices $\mathbf{A}$ as long as $\mathbf{A}$ has non-negative entries, $a_{ij} \geq 0$.

It should be clear that the output is now always strictly positive.

(v) Assume that markets are independent and goods are non-substitutable (i.e., $\rho = 0$). If $\varphi < \lambda_{\mathrm{PF}}(\mathbf{A})^{-1}$, the equilibrium quantity further simplifies to $\mathbf{q} = \mu \mathbf{b}_{\mathbf{u}}(G, \phi)$, which is always strictly positive.

(vi) Finally, the equilibrium profit and effort follow from Equations (44) and (38).

$\square$

Proof of Proposition 2 (ii) Assuming that $\mu_i = \mu$ for all $i = 1, \ldots, n$, at the Nash equilibrium, and that $\rho = 0$, we have that $\mathbf{q} = \mu \mathbf{M}(G, \varphi)\mathbf{u}$, where we have denoted by $\mathbf{M}(G, \varphi) \equiv (\mathbf{I}_n - \varphi \mathbf{A})^{-1}$.[65] We then obtain $W(G) = \mathbf{q}^\top \mathbf{q} = \mu^2 \mathbf{u}^\top \mathbf{M}(G, \varphi)^2 \mathbf{u}$. Observe that the quantity $\mathbf{u}^\top \mathbf{M}(G, \varphi)\mathbf{u}$ is the walk generating function, $N_G(\varphi)$, of G that we defined in detail in Appendix B.2. Using the results of Appendix B.2, we obtain

$$\begin{aligned}
\mathbf{u}^\top \mathbf{M}(G, \varphi)^2 \mathbf{u} &= \mathbf{u}^\top \left(\sum_{k=0}^{\infty} \varphi^k \mathbf{A}^k \right)^2 \mathbf{u} \\
&= \mathbf{u}^\top \left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} \varphi^l \mathbf{A}^l \varphi^{k-l} \mathbf{A}^{k-l} \right) \mathbf{u} \\
&= \sum_{k=0}^{\infty} (k+1) \varphi^k \mathbf{u}^\top \mathbf{A}^k \mathbf{u} \\
&= N_G(\varphi) + \sum_{k=0}^{\infty} k \varphi^k \mathbf{u}^\top \mathbf{A}^k \mathbf{u}.
\end{aligned}$$

Alternatively, we can write

$$\sum_{k=0}^{\infty} (k+1) \varphi^k \mathbf{u}^\top \mathbf{A}^k \mathbf{u} = \sum_{k=0}^{\infty} (k+1) N_k \varphi^k = \frac{d}{d\varphi}(\varphi N_G(\varphi)),$$

so that

$$\mathbf{u}^\top \mathbf{M}(G, \varphi)^2 \mathbf{u} = \frac{d}{d\varphi}(\varphi N_G(\varphi)) = N_G(\varphi) + \varphi \frac{d}{d\varphi} N_G(\varphi).$$

In the k-regular graph G_k it holds that $N_G(\varphi) = \frac{n}{1-k\varphi}$ and $\frac{d}{d\varphi}(\varphi N_G(\varphi)) = N_G(\varphi) + \varphi \frac{d}{d\varphi} = N_G(\varphi) = \frac{n}{1-k\varphi} + \frac{nk\varphi}{(1-k\varphi)^2} = \frac{n}{1-k\varphi}\left(1 + \frac{k\varphi}{1-k\varphi}\right) = \frac{n}{(1-k\varphi)^2}$. Using the fact that the number of links in a k-regular graph is given by $m = \frac{nk}{2}$ we obtain a lower bound on welfare in the efficient graph given by $\frac{\mu^2 n}{(1-\frac{2m}{n}\varphi)^2} \leq W(G^*)$. This lower bound is highest for the complete graph K_n where $m = n(n-1)/2$, so that[66]

$$\frac{\mu^2 n}{(1-(n-1)\varphi)^2} \leq W(G^*).$$

[65] Note that there exists a relationship between the matrix $\mathbf{M}(G, \varphi)$ with elements $m_{ij}(G, \varphi)$ and the length of the shortest path $\ell_{ij}(G)$ between nodes i and j in the network G. Namely $\ell_{ij}(G) = \lim_{\varphi \to 0} \frac{\partial \ln m_{ij}(G,\varphi)}{\partial \ln \varphi} = \lim_{\varphi \to 0} \frac{\varphi}{m_{ij}(G,\varphi)} \frac{\partial m_{ij}(G,\varphi)}{\partial \varphi}$. See also Newman [2010, Chap. 6]. This means that the length of the shortest path between i and j is given by the relative percentage change in the weighted number of walks between nodes i and j in G with respect to a relative percentage change in φ in the limit of $\varphi \to 0$.

[66] Using Rayleigh's inequality, one can show that $\frac{d}{d\varphi}(\varphi N_G(\varphi)) \geq \frac{1}{\lambda_1} \frac{d}{d\varphi}$ [Van Mieghem, 2011, p. 51]. From this we can obtain a lower bound on welfare given by $W(G) \geq \mu^2 \frac{1}{\lambda_1} \frac{d}{d\varphi}(N_G(\varphi))$.

In order to derive an upper bound, observe that

$$\mathbf{u}^\top \mathbf{A}^k \mathbf{u} = \sum_{i=1}^n (\mathbf{u}^\top \mathbf{v}_i)^2 \lambda_i^k,$$
$$N_G(\varphi) = \sum_{i=1}^n \frac{(\mathbf{v}_i^\top \mathbf{u})^2}{1-\lambda_i \varphi},$$

so that we can write

$$\begin{aligned}
\mathbf{u}^\top \mathbf{M}(G,\varphi)^2 \mathbf{u} &= \sum_{i=1}^n \frac{(\mathbf{v}_i^\top \mathbf{u})^2}{1-\lambda_i\varphi} + \sum_{i=1}^n (\mathbf{u}^\top \mathbf{v}_i)^2 \sum_{k=0}^\infty k\varphi^k \lambda_i^k \\
&= \sum_{i=1}^n \frac{(\mathbf{v}_i^\top \mathbf{u})^2}{1-\lambda_i\varphi} + \sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2 \varphi\lambda_i}{(1-\varphi\lambda_i)^2} \\
&= \sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{1-\varphi\lambda_i}\left(1+\frac{\varphi\lambda_i}{1-\varphi\lambda_i}\right) \\
&= \sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{(1-\varphi\lambda_i)^2}.
\end{aligned}$$

From the above it follows that welfare can also be written as

$$W(G) = \mu^2 \frac{d}{d\varphi}(\varphi N_G(\varphi)) = \mu^2 \sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{(1-\varphi\lambda_i)^2}.$$

This expression shows that gross welfare is highest in the graph where λ_1 approaches $1/\varphi$. We then can upper bound welfare as follows[67]

$$W(G) = \mu^2 \sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{(1-\varphi\lambda_i)^2} \leq \mu^2 \frac{\sum_{i=1}^n (\mathbf{u}^\top \mathbf{v}_i)^2}{(1-\varphi\lambda_1)^2} \leq \mu^2 \frac{n}{(1-\varphi\lambda_1)^2},$$

where we have used the fact that $N_G(0) = \sum_{i=1}^n (\mathbf{u}^\top \mathbf{v}_i)^2 = n$ so that $(\mathbf{u}^\top \mathbf{v}_1)^2 < n$. Note that the largest eigenvalue λ_1 is upper bounded by the largest eigenvalue of the complete graph K_n, where it is equal to $n-1$. In this case, upper and lower bounds coincide, and the efficient graph is therefore complete, that is $K_n = \mathrm{argmax}_{G \in \mathcal{G}^n} W(G)$.

(i) Welfare can be written as

$$W(G) = \frac{2-\rho}{2}\frac{\mu^2}{\rho^2} \frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2 \mathbf{u} + \frac{\rho}{2-\rho}(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2}{\left(\frac{1-\rho}{\rho} + \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}\right)^2}.$$

For the k-regular graph G_k we have that

$$\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = \frac{n}{1-(k-1)\phi},$$
$$\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u} = \frac{n}{(1-(k-1)\phi)^2},$$

[67] An alternative proof uses the fact that $\lambda_1 \geq \left(\frac{N_k(G)}{n}\right)^{\frac{1}{k}}$ [cf. Van Mieghem, 2011, p. 47], so that $\frac{d}{d\varphi}(\varphi N_G(\varphi)) = \sum_{k=0}^\infty \varphi^k (k+1) N_k(\varphi) \leq n \sum_{k=0}^\infty (\lambda_1\varphi)^k (k+1) = n \sum_{k=0}^\infty (\lambda_1 \varphi)^k + n \sum_{k=0}^\infty k(\lambda_1\varphi)^k = n\left(\frac{1}{1+\varphi\lambda_1} + \frac{\varphi\lambda_1}{(1+\varphi\lambda_1)^2}\right) = \frac{n}{(1+\varphi\lambda_1)^2}$.

and welfare is given by

$$W(G_k) = \frac{\mu^2 n((n-1)\rho + 2)}{2(\rho(k\phi + n - 1) - k\phi + 1)^2}.$$

As $k = 2m/n$ this is

$$W(G_k) = \frac{\mu^2 n^3((n-1)\rho + 2)}{2(2m(\rho - 1)\phi + (n-1)n\rho + n)^2}.$$

Together with the definition of the average degree $\bar{d} = \frac{2m}{n}$ this gives us the lower bound on welfare for all graphs with m links. For the complete graph K_n we get

$$\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = \frac{n}{1-(n-1)\phi},$$
$$\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u} = \frac{n}{(1-(n-1)\phi)^2},$$

so that we obtain for welfare in the complete graph

$$W(K_n) = \frac{\mu^2 n(2 + (n-1)\rho)}{2((n-1)\rho(\phi+1) - (n-1)\phi + 1)^2}.$$

Using the fact that $\phi = \frac{\varphi}{1-\rho}$ we can write this as follows

$$W(K_n) = \frac{\mu^2 n(2 + (n-1)\rho)}{2((n-1)\rho - (n-1)\varphi + 1)^2}.$$

This gives us the lower bound on welfare $W(K_n) \leq W(G^*)$. To obtain an upper bound, note that welfare can be written as

$$W(G) = \frac{\mu^2}{2\rho^2} \frac{(2-\rho)\frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} + \rho}{\frac{\left(\frac{1-\rho}{\rho} + \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}\right)^2}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2}}.$$

Next, observe that

$$\frac{\left(\frac{1-\rho}{\rho} + \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}\right)^2}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} = \left(1 + \frac{1-\rho}{\rho}\frac{1}{\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}}\right)^2 \geq \left(1 + \frac{1-\rho}{\rho}\frac{1-\lambda_1\phi}{n}\right)^2,$$

where we have used the fact that $\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = N_G(\phi) \leq \frac{n}{1-\lambda_1\phi}$. This implies that

$$W(G) \leq \frac{\mu^2}{2\rho^2} \frac{(2-\rho)\frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} + \rho}{\left(1 + \frac{1-\rho}{\rho}\frac{1-\lambda_1\phi}{n}\right)^2} \tag{51}$$

Next, observe that the Herfindahl industry concentration index is defined as $H = \sum_{i=1}^n s_i^2$, where the market share of firm i is given by $s_i = \frac{q_i}{\sum_{j=1}^n q_j}$ [cf. e.g. Tirole, 1988]. Using our equilibrium characterization from Equation (10) we can write

$$H(G) = \sum_{i=1}^n \left(\frac{q_i}{\sum_{j=1}^n q_j}\right)^2 = \frac{\sum_{i=1}^n b_i(G,\phi)^2}{\left(\sum_{j=1}^n b_j(G,\phi)\right)^2} = \frac{\mathbf{b}(G,\phi)^\top \mathbf{b}(G,\phi)}{(\mathbf{u}^\top \mathbf{b}(G,\phi))^2} = \frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2}. \tag{52}$$

The upper bound for welfare can then be written more compactly as follows

$$W(G) \leq \frac{\mu^2}{2\rho^2} \frac{(2-\rho)H(G) + \rho}{\left(1 + \frac{1-\rho}{\rho}\frac{1-\lambda_1\phi}{n}\right)^2}. \tag{53}$$

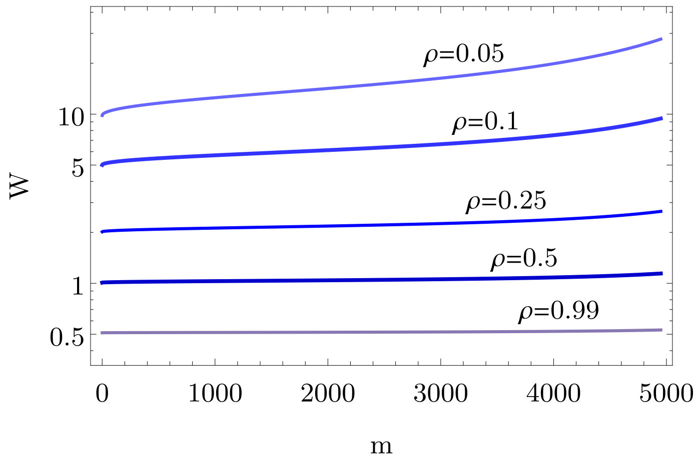

Figure A.3: The RHS in Equation (54) with varying values of $m \in \{0, 1, \ldots, n(n-1)/2\}$ for $n = 100$, $\varphi = 0.9(1-\rho)/n$ and $\rho \in \{0.05, 0.1, 0.25, 0.5, 0.99\}$.

Further, we have that

$$H(G) = \frac{\mathbf{u}^\top \mathbf{M}^2(G,\phi)\mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} = \frac{\frac{d}{d\phi}(\phi N_G(\phi))}{N_G(\phi)^2} = \frac{\sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{(1-\phi\lambda_i)^2}}{\left(\sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{1-\phi\lambda_i}\right)^2} \leq \frac{\frac{1}{1-\phi\lambda_1}\sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{1-\phi\lambda_i}}{\left(\sum_{i=1}^n \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{1-\phi\lambda_i}\right)^2}$$
$$= \frac{1}{(1-\phi\lambda_1)N_G(\phi)} \leq \frac{1}{(1-\phi\lambda_1)(n+2m\phi)} \leq \frac{1}{(1-\phi\sqrt{\frac{2m(n-1)}{n}})(n+2m\phi)},$$

where we have used the fact that $N_G(\phi) \geq n + 2m\phi$ for $\phi \in [0, 1/\lambda_1)$, and the upper bound $\lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}$ [cf. Van Mieghem, 2011, p. 52]. Inserting into the upper bound in Equation (51) and substituting $\phi = (1-\rho)/\varphi$ gives

$$W(G^*) \leq \frac{\mu^2 n^2}{2} \frac{\rho + (2-\rho)\frac{(1-\rho)^2}{(n(1-\rho)+2m\varphi)\left(1-\rho-\varphi\sqrt{\frac{2m(n-1)}{n}}\right)}}{\left(1+(n-1)\rho-\varphi\sqrt{\frac{2m(n-1)}{n}}\right)^2}. \tag{54}$$

The RHS in Equation (54) is increasing in m (see Figure A.3) and attains its maximum at $m = n(n-1)/2$, where we get

$$W(G^*) \leq \frac{\mu^2 n\left((\rho-1)^2((n-1)\rho+2) - (n-1)^2 n\rho\varphi^2\right)}{2((n-1)\rho - n\varphi + \varphi + 1)^2\left((\rho-1)^2 - (n-1)^2\varphi^2\right)}.$$

(iii) Assuming that $\mu_i = \mu$ for all $i = 1, \ldots, n$, we have that

$$\mathbf{q} = \frac{\mu}{1+\rho(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}-1)}\mathbf{M}(G,\phi)\mathbf{u},$$

with $\mathbf{M}(G,\phi) \equiv (\mathbf{I}_n - \phi\mathbf{A})^{-1}$, and we can write

$$W(G) = \frac{\mu^2}{2(1+\rho(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}-1))^2}\left((2-\rho)\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u} + \rho(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2\right).$$

Using the fact that $\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = N_G(\phi)$ and $\mathbf{u}^\top \mathbf{M}(G,\phi)^2\mathbf{u} = \frac{d}{d\phi}(\phi N_G(\phi))$, we then can write welfare in terms of the walk generating function $N_G(\phi)$ as

$$W(G) = \frac{\mu^2}{2(1+\rho(N_G(\phi)-1))^2}\left((2-\rho)\frac{d}{d\phi}\left(\phi N_G(\phi)\right) + \rho N_G(\phi)^2\right).$$

Next, observe that

$$N_G(\phi) = N_0 + N_1\phi + N_2\phi^2 + O(\phi^3),$$

and consequently

$$\frac{d}{d\phi}\left(\phi N_G(\phi)\right) = N_0 + 2N_1\phi + 3N_2\phi^2 + O(\phi^3).$$

Inserting into welfare gives

$$W(G) = \frac{\mu^2 N_0((N_0-1)\rho+2)}{2((N_0-1)\rho+1)^2} - \frac{\mu^2 N_1(\rho-1)((N_0-1)\rho+2)}{((N_0-1)\rho+1)^3}\phi + O(\phi)^2.$$

Using the fact that $N_0 = n$ and $N_1 = 2m$ we get

$$W(G) = \frac{\mu^2 n((n-1)\rho+2)}{2((n-1)\rho+1)^2} + \frac{2\mu^2 m(1-\rho)(2+(n-1)\rho)}{(1+(n-1)\rho)^3}\phi + O(\phi)^2.$$

Up to terms linear in ϕ this is an increasing function of m, and hence is largest in the complete graph K_n.

(iv) Welfare can be written as

$$W(G) = \frac{\mu^2\left((\mathbf{u}^\top\mathbf{M}(G,\phi)\mathbf{u})^2\rho + \mathbf{u}^\top\mathbf{M}(G,\phi)^2\mathbf{u}(2-\rho)\right)}{2((\mathbf{u}^\top\mathbf{M}(G,\phi)\mathbf{u}-1)\rho+1)^2}.$$

For the complete graph we obtain

$$\begin{aligned}\mathbf{u}^\top\mathbf{M}(K_n,\phi)\mathbf{u} &= \frac{n}{1-(n-1)\phi},\\ \mathbf{u}^\top\mathbf{M}(K_n,\phi)^2\mathbf{u} &= \frac{n}{(1-(n-1)\phi)^2}.\end{aligned}$$

With $\phi = \frac{\varphi}{1-\rho}$ welfare in the complete graph is given by

$$W(K_n) = \frac{\mu^2 n((n-1)\rho+2)}{2((n-1)\rho - n\varphi + \varphi + 1)^2},$$

For the star $K_{1,n-1}$

$$\begin{aligned}\mathbf{u}^\top\mathbf{M}(K_{1,n-1},\phi)\mathbf{u} &= \frac{2(n-1)\phi+n}{1-(n-1)\phi^2},\\ \mathbf{u}^\top\mathbf{M}(K_{1,n-1},\phi)^2\mathbf{u} &= \frac{(n-1)n\phi^2+4(n-1)\phi+n}{((n-1)\phi^2-1)^2}.\end{aligned}$$

Inserting $\phi = \frac{\varphi}{1-\rho}$, welfare in the star is then given by

$$W(K_{1,n-1}) = \frac{\mu^2\left((n-1)\varphi^2(n(3\rho+2)-4\rho) - 4(n-1)(\rho-1)\varphi((n-1)\rho+2) + n(\rho-1)^2((n-1)\rho+2)\right)}{2\left(-2(n-1)\rho\varphi + (\rho-1)((n-1)\rho+1) + (n-1)\varphi^2\right)^2}. \tag{55}$$

Welfare of the star $K_{1,n-1}$ for varying values of ρ can be seen in Figure 3, right panel. For the ratio of welfare in the complete graph and the star we then obtain

$$\frac{W(K_n)}{W(K_{1,n-1})} = n(2+(n-1)\rho)\left(2(n-1)\rho\varphi+(1-\rho)((n-1)\rho+1)-(n-1)\varphi^2\right)^2$$
$$\times \frac{1}{(1+(n-1)\rho-(n-1)\varphi)^2\left((n-1)\varphi^2(n(3\rho+2)-4\rho)+4(n-1)(1-\rho)\varphi((n-1)\rho+2)+n(1-\rho)^2((n-1)\rho+2)\right)}.$$

This ratio equals one when $\varphi = \varphi^*(n,\rho)$, which is given by

$$\begin{aligned}\varphi^*(n,\rho) &= \frac{1}{6A(n-1)((n-1)\rho+n)} \\ &\times\left(\sqrt[3]{2}A^2+2A(n-1)(2-\rho(3(n-1)\rho+5))+2^{2/3}(n-1)\right) \\ &\times\left(6n^2-(n-1)(15(n-2)n+8)\rho^2+(n(3(n-16)n+76)-16)\rho-32n+8\right),\end{aligned}$$

where we have denoted by

$$\begin{aligned}A = &\left(-3(n-1)^2\left(n\left(3n\left(6n^2-33n+86\right)-248\right)+32\right)\right. \\ &\times\rho^2-27(n-2)(n-1)^4n\rho^4+(n-1)^3(9(n-2)n(3n-19)-32)\rho^3 \\ &\left.+3\sqrt{3}B-12n(n(5n(3(n-5)n+31)-153)+66)\rho-16n(n(n(9n-29)+33)-15)+96\rho-32\right)^{\frac{1}{3}},\end{aligned}$$

and

$$\begin{aligned}B = &\left((n-2)(n-1)^3n((n-1)\rho+n)^2\right. \\ &\times\left(27(n-2)(n-1)^3n\rho^6-2(n-1)^2(9(n-2)n(6n-19)-32)\rho^5\right. \\ &+(n-1)(n(n(2n(37n-526)+3283)-3046)+384)\rho^4+2(n(n(n(n(n+242)-1936)+4384)-3264)+448)\rho^3 \\ &\left.\left.+4((n-2)n(n(3n+302)-786)-256)\rho^2+24(n-2)(n(n+56)-12)\rho+16(n(n+34)-8)\right)\right)^{\frac{1}{2}}.\end{aligned}$$

We then have that $W(K_n) > W(K_{1,n-1})$ if $\varphi < \varphi^*(n,\rho)$ and $W(K_n) < W(K_{1,n-1})$ otherwise. An illustration can be seen in Figure 3, left panel.

□

Proof of Proposition 3 (i) We first introduce a lower bound on the effort independent marginal cost $\bar{c}_i$ such that the marginal cost c_i is strictly positive in equilibrium. We then must have that $\bar{c}_i > e_i + \varphi\sum_{j=1}^n a_{ij}e_j$ and the profit function of firm i can be written as Equation (19). The FOC of profits with respect to effort is

$$\frac{\partial\pi_i}{\partial e_i} = q_i - e_i + s = 0,$$

so that equilibrium effort is

$$e_i = q_i + s.$$

Requiring non-negative marginal cost then implies that $\bar{c}_i > q_i + s + \varphi\sum_{j=1}^n a_{ij}e_j$. A sufficient condition for this to hold for all firms $i \in \mathcal{N}$ is given by

$$\max_{i\in\mathcal{N}} \bar{c}_i > \bar{q}+\bar{s}+\varphi\sum_{j=1}^n a_{ij}(\bar{q}+\bar{s}) = (1+\varphi(n-1))(\bar{q}+\bar{s}). \tag{56}$$

The marginal change of profits with respect to output is given by

$$\frac{\partial\pi_i}{\partial q_i} = (\bar{\alpha}-\bar{c}_i)-2q_i-\rho\sum_{j\neq i}b_{ij}q_j+e_i+\varphi\sum_{j=1}^n a_{ij}e_j,$$

where we have denoted by $\mu_i \equiv \bar{\alpha} - \bar{c}_i$. Inserting equilibrium efforts gives

$$
\begin{aligned}
q_i = 0, \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s(1+\varphi d_i) > 0,\\
q_i = \mu_i - \rho\sum_{j\neq i} b_{ij}q_j + \varphi\sum_{j=1}^n a_{ij}q_j + s(1+\varphi d_i), \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s(1+\varphi d_i) = 0,\\
q_i = \bar{q}, \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s(1+\varphi d_i) < 0,
\end{aligned} \tag{57}
$$

where $d_i = \sum_{j=1}^n a_{ij}$ is the degree of firm i. The problem of finding a vector $\mathbf{q}$ such that the conditions in (57) hold is known as the bounded linear complementarity problem [Byong-Hun, 1983]. The corresponding best response function $f_i : [0,\bar{q}]^{n-1} \to [0,\bar{q}]$ can be written compactly as follows:

$$
f_i(\mathbf{q}_{-i}) \equiv \max\left\{0, \min\left\{\bar{q}, \mu_i + s(1+\varphi d_i) - \rho\sum_{j\neq i} b_{ij}q_j + \varphi\sum_{j=1}^n a_{ij}q_j\right\}\right\}. \tag{58}
$$

We observe that the firm's output is increasing with the subsidy s, and this increase is higher for firms with a larger number of collaborations, d_i. Existence and uniqueness follow under the same conditions as in the proof of Proposition 1.[68]

In the following we provide a characterization of the interior equilibrium. In vector-matrix notation we then can write for the interior output levels

$$
(\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})\mathbf{q} = \boldsymbol{\mu} + s\mathbf{u} + \varphi s\mathbf{A}\mathbf{u}.
$$

The equilibrium output can further be written as follows

$$
\mathbf{q} = \tilde{\mathbf{q}} + s\mathbf{r},
$$

where we have denoted by

$$
\begin{aligned}
\tilde{\mathbf{q}} &\equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu} = \mathbf{M}\boldsymbol{\mu}\\
\mathbf{r} &\equiv \varphi(\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\left(\frac{1}{\varphi}\mathbf{I}_n + \mathbf{A}\right)\mathbf{u} = \mathbf{M}\mathbf{u} + \varphi\mathbf{M}\mathbf{d},
\end{aligned}
$$

where $\mathbf{M} \equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}$. The vector $\tilde{\mathbf{q}}$ gives equilibrium quantities in the absence of the subsidy and is derived in Section 3. The vector $\mathbf{r}$ has elements r_i for $i = 1, \ldots, n$. Furthermore, equilibrium profits are given by

$$
\pi_i = \frac{1}{2}q_i^2 + \frac{1}{2}s^2,
$$

(ii) Net social welfare is given by

$$
\overline{W}(G,s) = W(G,s) - s\sum_{i=1}^n e_i = \sum_{i=1}^n \left(q_i^2 + \pi_i - se_i\right) = \sum_{i=1}^n q_i^2 - s\sum_{i=1}^n q_i - \frac{n}{2}s^2.
$$

[68] To see this simply replace μ_i with $\mu_i + s(1+\varphi d_i)$ in the proof of Proposition 1.

Using the fact that $q_i = \tilde{q}_i + sr_i$, where

$$\begin{aligned}\tilde{\mathbf{q}} &= (\mathbf{I}_n - \varphi\mathbf{A})^{-1}\boldsymbol{\mu} = \mathbf{M}\boldsymbol{\mu}\\ \mathbf{r} &= \varphi(\mathbf{I}_n - \varphi\mathbf{A})^{-1}\left(\frac{1}{\varphi}\mathbf{I}_n + \mathbf{A}\right)\mathbf{u} = \boldsymbol{\mu} + \varphi\mathbf{d},\end{aligned}$$

we can write net welfare as follows

$$\overline{W}(G,s) = \sum_{i=1}^{n}(\tilde{q}_i + r_i s)^2 - \sum_{i=1}^{n}(\tilde{q}_i + r_i s) - \frac{n}{2}s^2.$$

The FOC of net welfare $\overline{W}(G,s)$ is given by

$$\frac{\partial\overline{W}(G,s)}{\partial s} = 2\sum_{i=1}^{n}\tilde{q}_i\left(2r_i - 1\right) + s\sum_{i=1}^{n}\left(2r_i^2 - 2r_i - 1\right) = 0,$$

from which we obtain the optimal subsidy level

$$s^* = \frac{\sum_{i=1}^{n}\tilde{q}_i\left(1 - 2r_i\right)}{\sum_{i=1}^{n}\left(r_i\left(2r_i - 2\right) - 1\right)},$$

where the equilibrium quantities are given by Equation (20). For the second-order derivative we obtain

$$\frac{\partial^2\overline{W}(G,s)}{\partial s^2} = -\sum_{i=1}^{n}\left(-2r_i^2 + 2r_i + 1\right),$$

and we have an interior solution if the condition $\sum_{i=1}^{n}\left(-2r_i^2 + 2r_i + 1\right) \geq 0$ is satisfied.

(iii) Net welfare can be written as

$$\begin{aligned}\overline{W}(G,s) &= \frac{1}{2}\sum_{i=1}^{n}q_i^2 + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}b_{ij}q_iq_j + \sum_{i=1}^{n}\pi_i - s\sum_{i=1}^{n}e_i\\ &= \sum_{i=1}^{n}q_i^2 + \frac{n}{2}s^2 + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}b_{ij}q_iq_j - \sum_{i=1}^{n}(q_i + s)s.\end{aligned}$$

Using the fact that $q_i = \tilde{q}_i + sr_i$, where

$$\begin{aligned}\tilde{\mathbf{q}} &\equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}\\ \mathbf{r} &\equiv \varphi(\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\left(\frac{1}{\varphi}\mathbf{I}_n + \mathbf{A}\right)\mathbf{u},\end{aligned}$$

we can write net welfare as follows

$$\overline{W}(G,s) = \sum_{i=1}^{n}(\tilde{q}_i + r_i s)^2 - ns^2 + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}b_{ij}(\tilde{q}_i + sr_i)(\tilde{q}_j + sr_j) - \sum_{i=1}^{n}(\tilde{q}_i s + r_i s^2).$$

The FOC of net welfare $\overline{W}(G,s)$ is given by

$$\frac{\partial\overline{W}(G,s)}{\partial s} = \sum_{i=1}^{n}\left(2\tilde{q}_i r_i - \tilde{q}_i + \frac{\rho}{2}b_{ij}(\tilde{q}_i r_j + \tilde{q}_j r_i)\right) + s\sum_{i=1}^{n}\left(2r_i^2 - 2r_i - 1 + \rho\sum_{j=1}^{n}b_{ij}r_ir_j\right) = 0,$$

from which we obtain the optimal subsidy level

$$s^* = \frac{\sum_{i=1}^n \left(\tilde{q}_i(2r_i+1) + \frac{\rho}{2}\sum_{j=1}^n b_{ij}(\tilde{q}_i r_j + \tilde{q}_j r_i)\right)}{\sum_{i=1}^n \left(1 + r_i\left(2 - 2r_i - \rho\sum_{j=1}^n b_{ij} r_j\right)\right)},$$

where the equilibrium quantities are given by Equation (20). The second-order derivative is given by

$$\frac{\partial^2 \overline{W}(G,s)}{\partial s^2} = -\sum_{i=1}^n \left(-2r_i^2 + 2r_i + 1 - \rho\sum_{j=1}^n b_{ij} r_i r_j.\right).$$

Hence, the solution is interior if $\sum_{i=1}^n \left(-2r_i^2 + 2r_i + 1 - \rho\sum_{j=1}^n b_{ij} r_i r_j\right) \geq 0$.

□

Proof of Proposition 4 (i) Under the same conditions as in the proof of Proposition 3 we have that the marginal cost is non-negative. The FOC of profits from Equation (22) with respect to effort then is

$$\frac{\partial \pi_i}{\partial e_i} = q_i - e_i + s_i = 0,$$

so that equilibrium effort is

$$e_i = q_i + s_i.$$

The marginal change of profits with respect to output is given by

$$\frac{\partial \pi_i}{\partial q_i} = \mu_i - 2q_i - \rho\sum_{j\neq i} b_{ij} q_j + e_i + \varphi\sum_{j=1}^n a_{ij} e_j,$$

where we have denoted by $\mu_i \equiv \bar{\alpha} - \bar{c}_i$. Inserting equilibrium efforts gives

$$\begin{aligned}
q_i = 0, \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s_i - \varphi\sum_{j=1}^n a_{ij}s_j > 0,\\
q_i = \mu_i - \rho\sum_{j\neq i} b_{ij}q_j + \varphi\sum_{j=1}^n a_{ij}q_j + s_i + \varphi\sum_{j=1}^n a_{ij}s_j, \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s_i - \varphi\sum_{j=1}^n a_{ij}s_j = 0,\\
q_i = \bar{q}, \text{ if } & -\mu_i + q_i + \rho\sum_{j=1}^n b_{ij}q_j - \varphi\sum_{j=1}^n a_{ij}q_j - s_i - \varphi\sum_{j=1}^n a_{ij}s_j < 0.
\end{aligned} \tag{59}$$

The problem of finding a vector $\mathbf{q}$ such that the conditions in (59) hold is known as the bounded linear complementarity problem [cf. Byong-Hun, 1983]. The corresponding best response function $f_i : [0,\bar{q}]^{n-1} \to [0,\bar{q}]$ can be written compactly as follows:

$$f_i(\mathbf{q}_{-i}) \equiv \max\left\{0, \min\left\{\bar{q}, \mu_i - \rho\sum_{j\neq i} b_{ij}q_j + \varphi\sum_{j=1}^n a_{ij}q_j + s_i + \varphi\sum_{j=1}^n a_{ij}s_j\right\}\right\}. \tag{60}$$

We observe that the firm's output is increasing with the unit subsidy s_i of firm i, and the total amount of subsidies received by firms collaborating with firm i. Existence and uniqueness follow under the same conditions as in the proof of Proposition 1.[69]

In the following we assume that these conditions are met and we focus on the characterization of an

[69] To see this simply replace μ_i with $\mu_i + s_i + \varphi\sum_{j=1}^n a_{ij}s_j$ in the proof of Proposition 1.

interior equilibrium. In vector-matrix notation equilibrium output levels can be written as

$$(\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})\mathbf{q} = \boldsymbol{\mu} + \mathbf{s} + \varphi\mathbf{A}\mathbf{s}.$$

We then can write

$$\mathbf{q} = \tilde{\mathbf{q}} + \mathbf{R}\mathbf{s},$$

where we have denoted by

$$\tilde{\mathbf{q}} \equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu} = \mathbf{M}\boldsymbol{\mu},$$
$$\mathbf{R} \equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}(\mathbf{I}_n + \varphi\mathbf{A}) = \mathbf{M} + \varphi\mathbf{M}\mathbf{A},$$

with $\mathbf{M} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}$. The matrix $\mathbf{R}$ has elements r_{ij} for $1 \leq i, j \leq n$. Furthermore, one can show that equilibrium profits are given by

$$\pi_i = \frac{1}{2}q_i^2 + \frac{1}{2}s_i^2.$$

(ii) Net welfare can be written as follows

$$\overline{W}(G, \mathbf{s}) = \sum_{i=1}^{n}\left(\frac{q_i^2}{2} + \pi_i - s_i e_i\right) = \sum_{i=1}^{n} q_i^2 - \sum_{i=1}^{n} q_i s_i - \frac{1}{2}\sum_{i=1}^{n} s_i^2.$$

Using the fact that $q_i = \tilde{q}_i + r_{ij}s_j$, with $\tilde{\mathbf{q}} = (\mathbf{I}_n - \varphi\mathbf{A})^{-1}\boldsymbol{\mu} = \mathbf{M}\boldsymbol{\mu}$, and $\mathbf{R} = (\mathbf{I}_n - \varphi\mathbf{A})^{-1}(\mathbf{I}_n + \varphi\mathbf{A})$, where $\mathbf{R}$ is symmetric, i.e. $\mathbf{R}^\top = \mathbf{R}$, we can write net welfare as follows

$$\overline{W}(G, \mathbf{s}) = \sum_{i=1}^{n}\tilde{q}_i^2 - \sum_{i=1}^{n}\tilde{q}_i s_i - \frac{1}{2}\sum_{i=1}^{n} s_i^2 + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{ij}s_j\right)\left(2\tilde{q}_i + \sum_{j=1}^{n} r_{ij}s_j - s_i\right). \tag{61}$$

Equation (61) can be written in vector-matrix notation as follows

$$\overline{W}(G, \mathbf{s}) = \tilde{\mathbf{q}}^\top\tilde{\mathbf{q}} - \mathbf{s}^\top(\mathbf{I}_n - 2\mathbf{R})\tilde{\mathbf{q}} - \frac{1}{2}\mathbf{s}^\top\left(\mathbf{I}_n + 2(\mathbf{I}_n - \mathbf{R}^\top)\mathbf{R}\right)\mathbf{s}.$$

Denoting by $\mathbf{H} \equiv \mathbf{I}_n + 2(\mathbf{I}_n - \mathbf{R}^\top)\mathbf{R}$ and $\mathbf{c}^\top \equiv \tilde{\mathbf{q}}^\top(\mathbf{I}_n - 2\mathbf{R})$ we find that maximizing net welfare is equivalent to solving the following quadratic programming problem [cf. Lee et al., 2005; Nocedal and Wright, 2006]: $\min_{\mathbf{s}\in[0,\bar{s}]_+^n}\left\{\mathbf{c}^\top\mathbf{s} + \frac{1}{2}\mathbf{s}^\top\mathbf{H}\mathbf{s}\right\}$. The FOC for net welfare $\overline{W}(G, \mathbf{s})$ of Equation (61) yields the following system of linear equations

$$\frac{\partial\overline{W}(G, \mathbf{s})}{\partial\mathbf{s}} = -\tilde{\mathbf{q}}^\top(\mathbf{I}_n - 2\mathbf{R}) - \left(\mathbf{I}_n + 2(\mathbf{I}_n - \mathbf{R}^\top)\mathbf{R}\right)\mathbf{s} = \mathbf{0}.$$

This can be written as $\left(\mathbf{I}_n + 2(\mathbf{I}_n - \mathbf{R}^\top)\mathbf{R}\right)\mathbf{s} = (2\mathbf{R} - \mathbf{I}_n)\tilde{\mathbf{q}}$. When the conditions for invertibility of the matrix $\mathbf{H}$ are satisfied, it follows that the optimal subsidy levels can be written as

$$\mathbf{s}^* = \mathbf{H}^{-1}(2\mathbf{R} - \mathbf{I}_n)\tilde{\mathbf{q}}, \tag{62}$$

with $\tilde{\mathbf{q}} = (\mathbf{I}_n - \varphi\mathbf{A})^{-1}\boldsymbol{\mu} = \mathbf{b}_{\boldsymbol{\mu}}$. The second-order derivative (Hessian) is given by

$$\frac{\partial^2\overline{W}(G, \mathbf{s})}{\partial\mathbf{s}\partial\mathbf{s}^\top} = -\mathbf{H}.$$

Hence, we obtain a global maximum for the concave quadratic optimization problem if the matrix $\mathbf{H}$ is positive definite, which means that it is also invertible and its inverse is also positive definite.

(iii) In the case of interdependent markets, when goods are substitutable, net welfare can be written as

$$\begin{aligned}\overline{W}(G,\mathbf{s}) &= \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + \rho \sum_{i=1}^{n}\sum_{j\neq i}^{n} b_{ij} q_i q_j\right) + \sum_{i=1}^{n} \pi_i - \sum_{i=1}^{n} s_i e_i \\ &= \sum_{i=1}^{n} q_i^2 - \sum_{i=1}^{n} q_i s_i - \frac{1}{2}\sum_{i=1}^{n} s_i^2 + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} b_{ij} q_i q_j .\end{aligned}$$

Using the fact that $q_i = \tilde{q}_i + r_{ij} s_j$, with $\tilde{\mathbf{q}} \equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}$ and $\mathbf{R} \equiv (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}(\mathbf{I}_n + \varphi\mathbf{A})$, where $\mathbf{R}$ is in general not symmetric, unless $\mathbf{AB} = \mathbf{BA}$,[70] we can write net welfare as follows

$$\overline{W}(G,\mathbf{s}) = \tilde{\mathbf{q}}^\top \tilde{\mathbf{q}} + \frac{\rho}{2}\tilde{\mathbf{q}}^\top \mathbf{B}\tilde{\mathbf{q}} - \tilde{\mathbf{q}}^\top (\mathbf{I}_n - \rho\mathbf{BR} - 2\mathbf{R})\,\mathbf{s} - \frac{1}{2}\mathbf{s}^\top \left(\mathbf{I}_n + 2\left(\mathbf{I}_n - \frac{\rho}{2}\mathbf{R}^\top\mathbf{B} - \mathbf{R}^\top\right)\mathbf{R}\right)\mathbf{s}. \quad (63)$$

If we denote by

$$\mathbf{H} \equiv \mathbf{I}_n + 2\left(\mathbf{I}_n - \mathbf{R}^\top\left(\mathbf{I}_n + \frac{\rho}{2}\mathbf{B}\right)\right)\mathbf{R},$$

and $\mathbf{c}^\top \equiv \tilde{\mathbf{q}}^\top(\mathbf{I}_n - 2\mathbf{R} - \rho\mathbf{BR})$ we find that maximizing net welfare is equivalent to solving the following quadratic programming problem [cf. Lee et al., 2005; Nocedal and Wright, 2006]: $\min_{\mathbf{s}\in\mathbb{R}_+^n}\left\{\mathbf{c}^\top\mathbf{s} + \frac{1}{2}\mathbf{s}^\top\mathbf{H}\mathbf{s}\right\}$, where we can replace $\mathbf{H}$ with the symmetric matrix $\frac{1}{2}\left(\mathbf{H}^\top + \mathbf{H}\right)$ to obtain an equivalent problem. The FOC from Equation (63) is given by

$$\frac{\partial\overline{W}(G,\mathbf{s})}{\partial\mathbf{s}} = -\left(\mathbf{I}_n - \mathbf{R}^\top\left(\mathbf{I}_n + \frac{\rho}{2}\mathbf{B}\right)\right)\tilde{\mathbf{q}} - \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^\top\right)\mathbf{s}.$$

When the matrix $\mathbf{H} + \mathbf{H}^\top$ is invertible, the optimal subsidy levels can be written as

$$\mathbf{s}^* = 2\left(\mathbf{H} + \mathbf{H}^\top\right)^{-1}\left(2\mathbf{R}^\top\left(\mathbf{I}_n + \frac{\rho}{2}\mathbf{B}\right) - \mathbf{I}_n\right)\tilde{\mathbf{q}}, \quad (64)$$

where the equilibrium quantities in the absence of the subsidy are given by $\tilde{\mathbf{q}} = (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})^{-1}\boldsymbol{\mu}$. The second-order derivative (Hessian) is given by

$$\frac{\partial^2\overline{W}(G,\mathbf{s})}{\partial\mathbf{s}\partial\mathbf{s}^\top} = -\frac{1}{2}\left(\mathbf{H} + \mathbf{H}^\top\right).$$

Hence, we obtain a global maximum for the concave quadratic optimization problem if the matrix $\mathbf{H} + \mathbf{H}^\top$ is positive definite. Note that if this matrix is positive definite then it is also invertible and its inverse is also positive definite.

□

[70] While the inverse of a symmetric matrix is symmetric, the product of symmetric matrices is not necessarily symmetric.

Supplement to "R&D Networks: Theory, Empirics and Policy Implications"

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B. Definitions and Characterizations

B.1. Network Definitions

A *network* (*graph*) $G \in \mathcal{G}^n$ is the pair $(\mathcal{N}, \mathcal{E})$ consisting of a set of *nodes* (*vertices*) $\mathcal{N} = \{1, \ldots, n\}$ and a set of edges (*links*) $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ between them, where $\mathcal{G}^n$ denotes the family of undirected graphs with n nodes. A link (i, j) is *incident* with nodes i and j. The *neighborhood* of a node $i \in \mathcal{N}$ is the set $\mathcal{N}_i = \{j \in \mathcal{N} : (i, j) \in \mathcal{E}\}$. The *degree* d_i of a node $i \in \mathcal{N}$ gives the number of links incident to node i. Clearly, $d_i = |\mathcal{N}_i|$. Let $\mathcal{N}_i^{(2)} = \bigcup_{j \in \mathcal{N}_i} \mathcal{N}_j \setminus (\mathcal{N}_i \cup \{i\})$ denote the second-order neighbors of node i. Similarly, the k-th order neighborhood of node i is defined recursively from $\mathcal{N}_i^{(0)} = \{i\}$, $\mathcal{N}_i^{(1)} = \mathcal{N}_i$ and $\mathcal{N}_i^{(k)} = \bigcup_{j \in \mathcal{N}_i^{(k-1)}} \mathcal{N}_j \setminus \left(\bigcup_{l=0}^{k-1} \mathcal{N}_i^{(l)} \right)$. A *walk* in G of length k from i to j is a sequence $\langle i_0, i_1, \ldots, i_k \rangle$ of nodes such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and i_p and i_{p+1} are (directly) linked, that is $i_p i_{p+1} \in \mathcal{E}$, for all $0 \leq p \leq k - 1$. Nodes i and j are said to be *indirectly linked* in G if there exists a walk from i to j in G containing nodes other than i and j. A pair of nodes i and j is connected if they are either directly or indirectly linked. A node $i \in \mathcal{N}$ is *isolated* in G if $\mathcal{N}_i = \emptyset$. The network G is said to be *empty* (denoted by $\bar{K}_n$) when all its nodes are isolated.

A *subgraph*, G', of G is the graph of subsets of the nodes, $\mathcal{N}(G') \subseteq \mathcal{N}(G)$, and links, $\mathcal{E}(G') \subseteq \mathcal{E}(G)$. A graph G is *connected*, if there is a path connecting every pair of nodes. Otherwise G is disconnected. The *components* of a graph G are the maximally connected subgraphs. A component is said to be *minimally connected* if the removal of any link makes the component disconnected.

A *dominating set* for a graph $G = (\mathcal{N}, \mathcal{E})$ is a subset $\mathcal{S}$ of $\mathcal{N}$ such that every node not in $\mathcal{S}$ is connected to at least one member of S by a link. An *independent set* is a set of nodes in a graph in which no two nodes are adjacent. For example the central node in a star $K_{1,n-1}$ forms a dominating set while the peripheral nodes form an independent set.

Let $G = (\mathcal{N}, \mathcal{E})$ be a graph whose distinct positive degrees are $d_{(1)} < d_{(2)} < \ldots < d_{(k)}$, and let $d_0 = 0$ (even if no agent with degree 0 exists in G). Furthermore, define $\mathcal{D}_i = \{v \in \mathcal{N} : d_v = d_{(i)}\}$ for $i = 0, \ldots, k$. Then the set-valued vector $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is called the *degree partition* of G. Consider a *nested split graph* $G = (\mathcal{N}, \mathcal{E})$ and let $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ be its degree partition. Then the nodes $\mathcal{N}$ can be partitioned in independent sets $\mathcal{D}_i$, $i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$ and a dominating set $\bigcup_{i=\left\lfloor \frac{k}{2} \right\rfloor + 1}^{k} \mathcal{D}_i$ in the graph $G' = (\mathcal{N} \setminus \mathcal{D}_0, \mathcal{E})$. Moreover, the neighborhoods of the nodes are nested. In particular, for each node $v \in \mathcal{D}_i$, $\mathcal{N}_v = \bigcup_{j=1}^{i} \mathcal{D}_{k+1-j}$ if $i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$ if $i = 1, \ldots, k$, while $\mathcal{N}_v = \bigcup_{j=1}^{i} \mathcal{D}_{k+1-j} \setminus \{v\}$ if $i = \left\lfloor \frac{k}{2} \right\rfloor + 1, \ldots, k$.

In a *complete* graph K_n, every node is adjacent to every other node. The graph in which no pair of nodes is adjacent is the empty graph $\bar{K}_n$. A *clique* $K_{n'}$, $n' \leq n$, is a complete subgraph of the network G. A graph is *k-regular* if every node i has the same number of links $d_i = k$ for all $i \in \mathcal{N}$. The complete graph K_n is $(n-1)$-regular. The cycle C_n is 2-regular. In a *bipartite graph* there exists a partition of the nodes in two disjoint sets V_1 and V_2 such that each link connects a node in V_1 to a node in V_2. V_1 and V_2 are independent sets with cardinalities n_1 and n_2, respectively. In a complete bipartite graph K_{n_1,n_2} each node in V_1 is connected to each other node in V_2. The *star* $K_{1,n-1}$ is a complete bipartite graph in which $n_1 = 1$ and $n_2 = n - 1$.

The *complement* of a graph G is a graph $\bar{G}$ with the same nodes as G such that any two nodes of $\bar{G}$ are adjacent if and only if they are not adjacent in G. For example the complement of the complete graph K_n is the empty graph $\bar{K}_n$.

Let $\mathbf{A}$ be the symmetric $n \times n$ *adjacency matrix* of the network G. The element $a_{ij} \in \{0,1\}$ indicates if there exists a link between nodes i and j such that $a_{ij} = 1$ if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$. The k-th power of the adjacency matrix is related to walks of length k in the graph. In particular, $\left(\mathbf{A}^k\right)_{ij}$ gives the number of walks of length k from node i to node j. The *eigenvalues* of the adjacency matrix $\mathbf{A}$ are the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ has a nonzero solution vector $\mathbf{v}_i$, which is an *eigenvector* associated with λ_i for $i = 1, \ldots, n$. Since the adjacency matrix $\mathbf{A}$ of an undirected graph G is real and symmetric, the eigenvalues of $\mathbf{A}$ are real, $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. Moreover, if $\mathbf{v}_i$ and $\mathbf{v}_j$ are eigenvectors for different eigenvalues, $\lambda_i \neq \lambda_j$, then $\mathbf{v_i}$ and $\mathbf{v}_j$ are orthogonal, i.e. $\mathbf{v}_i^\top \mathbf{v}_j = 0$ if $i \neq j$. In particular, $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of $\mathbf{A}$. Since $\mathbf{A}$ is a real symmetric matrix, there exists an orthogonal matrix $\mathbf{S}$ such that $\mathbf{S}^\top \mathbf{S} = \mathbf{S}\mathbf{S}^\top = \mathbf{I}$ (that is $\mathbf{S}^\top = \mathbf{S}^{-1}$) and $\mathbf{S}^\top \mathbf{A}\mathbf{S} = \mathbf{D}$, where $\mathbf{D}$ is the diagonal matrix of eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{S}$ are the corresponding eigenvectors. The *Perron-Frobenius eigenvalue* $\lambda_{\mathrm{PF}}(G)$ is the *largest real eigenvalue* of $\mathbf{A}$ associated with G, i.e. all eigenvalues λ_i of $\mathbf{A}$ satisfy $|\lambda_i| \leq \lambda_{\mathrm{PF}}(G)$ for $i = 1, \ldots, n$ and there exists an associated nonnegative eigenvector $\mathbf{v}_{\mathrm{PF}} \geq 0$ such that $\mathbf{A}\mathbf{v}_{\mathrm{PF}} = \lambda_{\mathrm{PF}}(G)\mathbf{v}_{\mathrm{PF}}$. For a connected graph G the adjacency matrix $\mathbf{A}$ has a unique largest real eigenvalue $\lambda_{\mathrm{PF}}(G)$ and a positive associated eigenvector $\mathbf{v}_{\mathrm{PF}} > 0$. The largest eigenvalue $\lambda_{\mathrm{PF}}(G)$ has been suggested to measure the irregularity of a graph [Bell, 1992], and the components of the associated eigenvector $\mathbf{v}_{\mathrm{PF}}$ are a measure for the centrality of a node in the network. A measure $C_v : \mathcal{G} \to [0,1]$ for the centralization of the network G has been introduced by Freeman [1979] for generic centrality measures $\mathbf{v}$. In particular, the *centralization* C_v of G is defined as $C_v(G) \equiv \sum_{i \in G} (v_{i^*} - v_i) / \max_{G' \in \mathcal{G}^n} \sum_{j \in G'} (v_{j^*} - v_j)$, where i^* and j^* are the nodes with the highest values of centrality in the networks G, G', respectively, and the maximum in the denominator is computed over all networks $G' \in \mathcal{G}^n$ with the same number n of nodes. There also exists a relation between the number of walks in a graph and its eigenvalues. The number of closed walks of length k from a node i in G to herself is given by $\left(\mathbf{A}^k\right)_{ii}$ and the total number of closed walks of length k in G is $\mathrm{tr}\left(\mathbf{A}^k\right) = \sum_{i=1}^n \left(\mathbf{A}^k\right)_{ii} = \sum_{i=1}^n \lambda_i^k$. We further have that $\mathrm{tr}\left(\mathbf{A}\right) = 0$, $\mathrm{tr}\left(\mathbf{A}^2\right)$ gives twice the number of links in G and $\mathrm{tr}\left(\mathbf{A}^3\right)$ gives six times the number of triangles in G.

The *cores* of a graph are defined as follows: Given a network G, the induced subgraph $G_k \subseteq G$ is the k-core of G if it is the largest subgraph such that the degree of all nodes in G_k is at least k. Note that the cores of a graph are nested such that $G_{k+1} \subseteq G_k$. Cores can be used as a measure of centrality in the network G, and the largest k-core centrality across all nodes in the graph is called the *degeneracy* of G. Note that k-cores can be obtained by a simple pruning algorithm: at each step, we remove all nodes with degree less than k. We repeat this procedure until there exist no such nodes or all nodes are removed. We define the coreness of each node as follows: The coreness of node i, cor_i, is k if and only if $i \in G_k$ and $i \notin G_{k+1}$. We have that $\mathrm{cor}_i \leq d_i$. However, there is no other relation between the degree and coreness of nodes in a graph.

Finally, a *nested split graph* is a graph with a nested neighborhood structure such that the set of neighbors of each node is contained in the set of neighbors of each higher degree node [Cvetkovic and Rowlinson, 1990; Mahadev and Peled, 1995]. A nested split graph is characterized by a *stepwise adjacency matrix* $\mathbf{A}$, which is a symmetric, binary $(n \times n)$-matrix with elements a_{ij} satisfying the following condition: if $i < j$ and $a_{ij} = 1$ then $a_{hk} = 1$ whenever $h < k \leq j$ and $h \leq i$. Both, the complete graph, K_n, as well as the star $K_{1,n-1}$, are particular examples of nested split graphs. Nested split graphs are also the graphs which maximize the largest eigenvalue, $\lambda_{\mathrm{PF}}(G)$, [Brualdi and Solheid, 1986], and they are the ones that maximize the degree variance [Peled et al., 1999]. See for example König et al. [2014] for a discussion of further properties of nested split graphs.

B.2. Walk Generating Functions

Denote by $\mathbf{u} = (1, \ldots, 1)^\top$ the n-dimensional vector of ones and define $\mathbf{M}(G,\phi) = (\mathbf{I}_n - \phi\mathbf{A})^{-1}$. Then, the quantity $N_G(\phi) = \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u}$ is the *walk generating function* of the graph G [cf. Cvetkovic et al., 1995]. Let N_k denote the number of walks of length k in G. Then we can write N_k as follows

$$N_k = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{[k]} = \mathbf{u}^\top \mathbf{A}^k \mathbf{u},$$

where $a_{ij}^{[k]}$ is the ij-th element of $\mathbf{A}^k$. The walk generating function is then defined as

$$N_G(\phi) \equiv \sum_{k=0}^{\infty} N_k \phi^k = \mathbf{u}^\top \left(\sum_{k=0}^{\infty} \phi^k \mathbf{A}^k \right) \mathbf{u} = \mathbf{u}^\top \left(\mathbf{I}_n - \phi \mathbf{A} \right)^{-1} \mathbf{u} = \mathbf{u}^\top \mathbf{M}(G, \phi) \mathbf{u}.$$

For a k-regular graph G_k, the walk generating function is equal to

$$N_{G_k}(\phi) = \frac{n}{1 - k\phi}.$$

For example, the cycle C_n on n nodes (see Figure B.1, left panel) is a 2-regular graph and its walk generating function is given by $N_{C_n}(\phi) = \frac{1}{1-2\phi}$. As another example, consider the star $K_{1,n-1}$ with n nodes (see Figure B.1, middle panel). Then the walk generating function is given by

$$N_{K_{1,n-1}}(\phi) = \frac{n + 2(n-1)\phi}{1 - (n-1)\phi^2}.$$

In general, it holds that $N_G(0) = n$, and one can show that $N_G(\phi) \geq 0$. We further have that

$$\mathbf{M}(G, \phi) = (\mathbf{I}_n - \phi \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k = \sum_{k=0}^{\infty} \phi^k \mathbf{S} \boldsymbol{\Lambda}^k \mathbf{S}^\top,$$

where $\boldsymbol{\Lambda} \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix containing the eigenvalues of the real, symmetric matrix $\mathbf{A}$, and $\mathbf{S}$ is an orthogonal matrix with columns given by the orthogonal eigenvectors of $\mathbf{A}$ (with $\mathbf{S}^\top = \mathbf{S}^{-1}$), and we have used the fact that $\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^\top$ [Horn and Johnson, 1990]. The eigenvectors $\mathbf{v}_i$ have the property that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ and are normalized such that $\mathbf{v}_i^\top \mathbf{v}_i = 1$. Note that $\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^\top$ is equivalent to $\mathbf{A} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$. It then follows that

$$\mathbf{u}^\top \mathbf{M}(G, \phi) \mathbf{u} = \mathbf{u}^\top \mathbf{S} \sum_{k=0}^{\infty} \phi^k \boldsymbol{\Lambda}^k \mathbf{S}^\top \mathbf{u},$$

where

$$\mathbf{S}^\top \mathbf{u} = \left(\mathbf{u}^\top \mathbf{v}_1, \ldots, \mathbf{u}^\top \mathbf{v}_n \right)^\top,$$

and

$$\boldsymbol{\Lambda}^k = \begin{pmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \lambda_n^k \end{pmatrix} = \lambda_1^k \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \left(\frac{\lambda_2}{\lambda_1}\right)^k & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \left(\frac{\lambda_n}{\lambda_1}\right)^k \end{pmatrix}.$$

We then can write

$$\mathbf{u}^\top \mathbf{M}(G, \phi) \mathbf{u} = \sum_{k=0}^{\infty} \phi^k \lambda_1^k \left(\mathbf{u}^\top \mathbf{v}_1, \ldots, \mathbf{u}^\top \mathbf{v}_n \right) \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \left(\frac{\lambda_2}{\lambda_1}\right)^k & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \left(\frac{\lambda_n}{\lambda_1}\right)^k \end{pmatrix} \left(\mathbf{u}^\top \mathbf{v}_1, \ldots, \mathbf{u}^\top \mathbf{v}_n \right)^\top,$$

which gives

$$\begin{aligned}
\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} &= \sum_{k=0}^{\infty} \phi^k \lambda_1^k \left((\mathbf{u}^\top \mathbf{v}_1)^2 + \left(\frac{\lambda_2}{\lambda_1}\right)^k (\mathbf{u}^\top \mathbf{v}_2)^2 + \ldots + \left(\frac{\lambda_n}{\lambda_1}\right)^k (\mathbf{u}^\top \mathbf{v}_n)^2 \right) \\
&= \sum_{i=1}^{n} (\mathbf{u}^\top \mathbf{v}_i)^2 \sum_{k=0}^{\infty} \phi^k \lambda_i^k \\
&= \sum_{i=1}^{n} \frac{(\mathbf{u}^\top \mathbf{v}_i)^2}{1-\phi\lambda_i}.
\end{aligned}$$

The above computation also shows that

$$N_k = \mathbf{u}^\top \mathbf{A}^k \mathbf{u} = \sum_{i=1}^{n} (\mathbf{u}^\top \mathbf{v}_i)^2 \lambda_i^k.$$

Hence, we can write the walk generating function as follows

$$N_G(\phi) = \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = \sum_{k=0}^{\infty} N_k \phi^k = \sum_{i=1}^{n} \frac{(\mathbf{v}_i^\top \mathbf{u})^2}{1-\lambda_i \phi}.$$

If λ_1 is much larger than λ_j for all $j \geq 2$, then we can approximate

$$N_G(\phi) \approx (\mathbf{u}^\top \mathbf{v}_1)^2 \sum_{k=0}^{\infty} \phi^k \lambda_1^k = \frac{(\mathbf{u}^\top \mathbf{v}_1)^2}{1-\phi\lambda_1}.$$

Moreover, there exists the following relationship between the largest eigenvalue λ_{PF} of the adjacency matrix and the number of walks of length k in G [cf. Van Mieghem, 2011, p. 47]

$$\lambda_{\mathrm{PF}}(G) \geq \left(\frac{N_k(G)}{n}\right)^{\frac{1}{k}},$$

and, in particular,

$$\lim_{k\to\infty} \left(\frac{N_k(G)}{n}\right)^{\frac{1}{k}} = \lambda_{\mathrm{PF}}(G).$$

Hence, we have that $n\lambda_{\mathrm{PF}}(G)^k \geq N_k(G)$, and

$$N_G(\phi) = \sum_{k=0}^{\infty} N_k \phi^k \leq n \sum_{k=0}^{\infty} (\lambda_{\mathrm{PF}}(G)\phi)^k = \frac{n}{1-\phi\lambda_{\mathrm{PF}}(G)}. \tag{65}$$

To derive a lower bound, note that for $\phi \geq 0$, $N_G(\phi)$ is increasing in ϕ, so that $N_G(\phi) \geq N_0 + \phi N_1 + \phi^2 N_2$. Using the fact that $N_0 = n$, $N_1 = 2m = n\bar{d}$ and $N_2 = \sum_{i=1}^n d_i^2 = n(\bar{d}^2 + \sigma_d^2)$, we then get the lower bound

$$N_G(\phi) \geq n + 2m\phi + n(\bar{d}^2 + \sigma_d^2)\phi^2. \tag{66}$$

Finally, Cvetkovic et al. [1995, p. 45] have found an alternative expression for the walk generating function given by

$$N_G(\phi) = \frac{1}{\phi}\left((-1)^n \frac{c_{\mathbf{A}^c}\left(-\frac{1}{\phi}-1\right)}{c_{\mathbf{A}}\left(\frac{1}{\phi}\right)} - 1 \right),$$

where $c_{\mathbf{A}}(\phi) \equiv \det(\mathbf{A} - \phi \mathbf{I}_n)$ is the characteristic polynomial of the matrix $\mathbf{A}$, whose roots are the eigenvalues of $\mathbf{A}$. It can be written as $c_{\mathbf{A}}(\phi) = \phi^n - a_1\phi^{n-1} + \ldots + (-1)^n a_n$, where $a_1 = \mathrm{tr}(\mathbf{A})$ and $a_n = \det(\mathbf{A})$. Furthermore, $\mathbf{A}^c = \mathbf{u}\mathbf{u}^\top - \mathbf{I}_n - \mathbf{A}$ is the complement of $\mathbf{A}$, and $\mathbf{u}\mathbf{u}^\top$ is an $n \times n$ matrix of ones. This is a convenient expression for the walk generating function, as there exist fast algorithms to compute the characteristic polynomial [Samuelson, 1942].

B.3. Bonacich Centrality

In the following we introduce a network measure capturing the centrality of a firm in the network due to Katz [1953] and later extended by Bonacich [1987]. Let $\mathbf{A}$ be the symmetric $n \times n$ *adjacency matrix* of the network G and λ_{PF} its largest real eigenvalue. The matrix $\mathbf{M}(G,\phi) = (\mathbf{I}-\phi\mathbf{A})^{-1}$ exists and is non-negative if and only if $\phi < 1/\lambda_{\mathrm{PF}}$.[71] Then

$$\mathbf{M}(G,\phi) = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k. \tag{67}$$

The Bonacich centrality vector is given by

$$\mathbf{b}_{\mathbf{u}}(G,\phi) = \mathbf{M}(G,\phi) \cdot \mathbf{u}, \tag{68}$$

where $\mathbf{u} = (1,\ldots,1)^\top$. We can write the Bonacich centrality vector as

$$\mathbf{b}_{\mathbf{u}}(G,\phi) = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k \cdot \mathbf{u} = (\mathbf{I} - \phi\mathbf{A})^{-1} \cdot \mathbf{u}.$$

For the components $b_{\mathbf{u},i}(G,\phi)$, $i = 1,\ldots,n$, we get

$$b_{\mathbf{u},i}(G,\phi) = \sum_{k=0}^{\infty} \phi^k (\mathbf{A}^k \cdot \mathbf{u})_i = \sum_{k=0}^{\infty} \phi^k \sum_{j=1}^{n} \left(\mathbf{A}^k\right)_{ij}. \tag{69}$$

The sum of the Bonacich centralities is then exactly the walk generating function we have introduced in Section B.2

$$\sum_{i=1}^{n} b_{\mathbf{u},i}(G,\phi) = \mathbf{u}^\top \mathbf{b}_{\mathbf{u}}(G,\phi) = \mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u} = N_G(\phi).$$

Moreover, because $\sum_{j=1}^{n} \left(\mathbf{A}^k\right)_{ij}$ counts the number of all walks of length k in G starting from i, $b_{\mathbf{u},i}(G,\phi)$ is the number of all walks in G starting from i, where the walks of length k are weighted by their geometrically decaying factor ϕ^k. In particular, we can decompose the Bonacich centrality as follows

$$b_i(G,\rho) = \underbrace{b_{ii}(G,\phi)}_{\text{closed walks}} + \underbrace{\sum_{j\neq i} b_{ij}(G,\phi)}_{\text{out-walks}}, \tag{70}$$

where $b_{ii}(G,\phi)$ counts all closed walks from firm i to i and $\sum_{j\neq i} b_{ij}(G,\phi)$ counts all the other walks from i to every other firm $j \neq i$. Similarly, Ballester et al. [2006] define the *intercentrality* of firm $i \in \mathcal{N}$ as

$$c_i(G,\phi) = \frac{b_i(G,\phi)^2}{b_{ii}(G,\phi)}, \tag{71}$$

where the factor $b_{ii}(G,\phi)$ measures all closed walks starting and ending at firm i, discounted by the factor ϕ, whereas $b_i(G,\phi)$ measures the number of walks emanating at firm i, discounted by the factor ϕ. The intercentrality index hence expresses the ratio of the (square of the) number of walks leaving a firm i relative to the number of walks returning to i.

We give two examples in the following to illustrate the Bonacich centrality. The graphs used in these examples are depicted in Figure B.1. First, consider the star $K_{1,n-1}$ with n nodes (see Figure B.1, middle panel) and assume w.l.o.g. that 1 is the index of the central node with maximum degree.

[71] The proof can be found e.g. in Debreu and Herstein [1953].

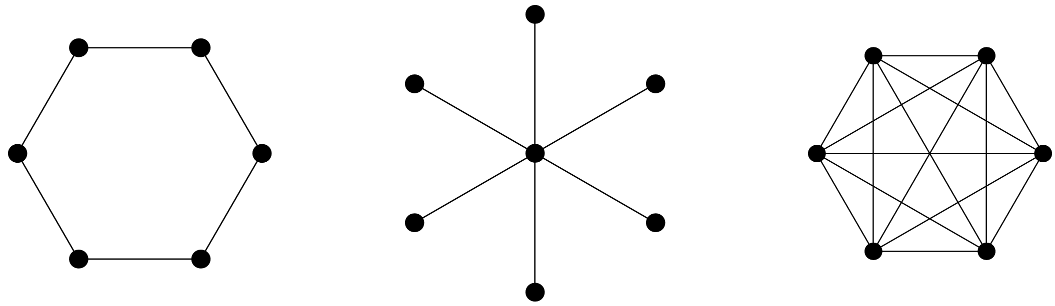

Figure B.1: Illustration of a cycle C_6, a star $K_{1,6}$ and a complete graph, K_6.

We now compute the Bonacich centrality for the star $K_{1,n-1}$. We have that

$$\begin{aligned}
\mathbf{M}(K_{1,n-1},\phi) = (\mathbf{I}-\phi\mathbf{A})^{-1} &= \begin{pmatrix} 1 & -\phi & \cdots & \cdots & -\phi \\ -\phi & 1 & 0 & & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & & \ddots & & \vdots \\ \vdots & \vdots & & & 0 \\ -\phi & 0 & \cdots & 0 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{1-(n-1)\phi^2} \begin{pmatrix} 1 & \phi & \cdots & \cdots & \phi \\ \phi & 1-(n-2)\phi^2 & \phi^2 & & \phi^2 \\ \vdots & \phi^2 & \ddots & \ddots & \vdots \\ & & \ddots & & \vdots \\ \vdots & \vdots & & & \phi^2 \\ \phi & \phi^2 & \cdots & \phi^2 & 1-(n-2)\phi^2 \end{pmatrix}.
\end{aligned}$$

Since $\mathbf{b} = \mathbf{M}\cdot\mathbf{u}$ we then get

$$\mathbf{b}(K_{1,n-1},\phi) = \frac{1}{1-(n-1)\phi^2}\left(1+(n-1)\phi, 1+\phi, \ldots, 1+\phi\right)^\top. \tag{72}$$

Next, consider the complete graph K_n with n nodes (see Figure B.1, right panel). We have

$$\begin{aligned}
\mathbf{M}(K_n,\phi) = (\mathbf{I}-\phi\mathbf{A})^{-1} &= \begin{pmatrix} 1 & -\phi & \cdots & \cdots & -\phi \\ -\phi & 1 & -\phi & & -\phi \\ \vdots & -\phi & \ddots & \ddots & \vdots \\ & & \ddots & & \vdots \\ \vdots & \vdots & & & -\phi \\ -\phi & -\phi & \cdots & -\phi & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{1-(n-2)\phi-(n-1)\phi^2} \begin{pmatrix} 1-(n-2)\phi & \phi & \cdots & \cdots & \phi \\ \phi & 1-(n-2)\phi & \phi & & \phi \\ \vdots & \phi & \ddots & \ddots & \vdots \\ & & \ddots & & \vdots \\ \vdots & \vdots & & & \phi \\ \phi & \phi & \cdots & \phi & 1-(n-2)\phi \end{pmatrix}.
\end{aligned}$$

With $\mathbf{b} = \mathbf{M} \cdot \mathbf{u}$ we then have that

$$\mathbf{b}(K_n, \phi) = \frac{1}{1-(n-1)\phi} (1, \ldots, 1)^\top . \tag{73}$$

The Bonacich matrix of Equation (67) is also a measure of structural similarity of the firms in the network, called *regular equivalence*. Leicht et al. [2006] define a similarity score b_{ij}, which is high if nodes i and j have neighbors that themselves have high similarity, given by $b_{ij} = \phi \sum_{k=1}^{n} a_{ik} b_{kj} + \delta_{ij}$. In matrix-vector notation this reads $\mathbf{M} = \phi \mathbf{A}\mathbf{M} + \mathbf{I}$. Rearranging yields $\mathbf{M} = (\mathbf{I} - \phi \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k$, assuming that $\phi < 1/\lambda_{\mathrm{PF}}$. We hence obtain that the similarity matrix $\mathbf{M}$ is equivalent to the Bonacich matrix from Equation (67). The average similarity of firm i is $\frac{1}{n}\sum_{j=1}^{n} b_{ij} = \frac{1}{n} b_{\mathbf{u},i}(G,\phi)$, where $b_{\mathbf{u},i}(G,\phi)$ is the Bonacich centrality of i. It follows that the Bonacich centrality of i is proportional to the average regular equivalence of i. Firms with a high Bonacich centrality are then the ones which also have a high average structural similarity with the other firms in the R&D network.

The interpretation of eingenvector-like centrality measures as a similarity index is also important in the study of correlations between observations in principal component analysis and factor analysis [cf. Rencher and Christensen, 2012]. Variables with similar factor loadings can be grouped together. This basic idea has also been used in the economics literature on segregation [e.g. Ballester and Vorsatz, 2013].

There also exists a connection between the Bonacich centrality of a node and its coreness in the network (see Appendix B.1). The following result, due to Manshadi and Johari [2010], relates the Nash equilibrium to the k-cores of the graph: If $\mathrm{cor}_i = k$ then $b_i(G,\phi) \geq \frac{1}{1-\phi k}$, where the inequality is tight when i belongs to a disconnected clique of size $k+1$. The coreness of networks of R&D collaborating firms has also been studied empirically in Kitsak et al. [2010] and Rosenkopf and Schilling [2007]. In particular, Kitsak et al. [2010] find that the coreness of a firm correlates with its market value. We can easily explain this from our model because we know that firms in higher cores tend to have higher Bonacich centrality, and therefore higher sales and profits (cf. Proposition 1).

C. Herfindahl Index

Denoting by $\mathbf{x} \equiv \mathbf{M}(G,\phi)\mathbf{u} = \mathbf{b}_{\mathbf{u}}(G,\phi)$, we can write the Herfindahl index of Equation (16) in the Nash equilibrium as follows[72]

$$H(G) = \frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2 \mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} = \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x}\|_1^2} = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} |x_i|\right)^2} = \gamma(\mathbf{x})^{-1},$$

which is the inverse of the *participation ratio* $\gamma(\mathbf{x})$. The participation ratio $\gamma(\mathbf{x})$ measures the number of elements of $\mathbf{x}$ which are dominant. We have that $1 \leq \gamma(\mathbf{x}) \leq n$, where a value of $\gamma(\mathbf{x}) = n$ corresponds to a fully homogenous case, while $\gamma(\mathbf{x}) = 1$ corresponds to a fully concentrated case (note that, if all x_i are identical then $\gamma(\mathbf{x}) = n$, while if one x_i is much larger than all others we have $\gamma(\mathbf{x}) = 1$). Moreover, $\gamma(\mathbf{x})$ is scale invariant, that is, $\gamma(\alpha\mathbf{x}) = \gamma(\mathbf{x})$ for any $\alpha \in \mathbb{R}_+$. The participation ratio $\gamma(\mathbf{x})$ is further related to the *coefficient of variation* $c_v(\mathbf{x}) = \frac{\sigma(\mathbf{x})}{\mu(\mathbf{x})}$, where $\sigma(\mathbf{x})$ is the standard deviation and $\mu(\mathbf{x})$ the mean of the components of $\mathbf{x}$, via the relationship $c_v(\mathbf{x})^2 = \frac{n}{\gamma(\mathbf{x})} - 1$. This implies that

$$H(G) = \frac{\mathbf{u}^\top \mathbf{M}(G,\phi)^2 \mathbf{u}}{(\mathbf{u}^\top \mathbf{M}(G,\phi)\mathbf{u})^2} = \frac{c_v(\mathbf{x})^2 + 1}{n} \sim \frac{c_v(\mathbf{x})^2}{n}.$$

Hence, the Herfindhal index is maximized for the graph G with the highest coefficient of variation in the components of the Bonacich centrality $\mathbf{b}_{\mathbf{u}}(G,\phi)$. Finally, as for small values of ϕ the Bonacich centrality becomes proportional to the degree, the variance of the Bonacich centrality will be determined by the variance of the degree. It is known that the graphs that maximize the degree variance are nested split graphs [cf. Peled et al., 1999].

[72]See also Equation (52).

D. Bertrand Competition

In the case of price setting firms we obtain from the profit function in Equation (3) the FOC with respect to price p_i for firm i

$$\frac{\partial \pi_i}{\partial p_i} = (p_i - c_i)\frac{\partial q_i}{\partial p_i} - q_i = 0.$$

When $i \in \mathcal{M}_m$, then observe that from the inverse demand in Equation (1) we find that

$$q_i = \frac{\alpha_m(1-\rho_m) - (1-(n_m-2)\rho_m)p_i + \rho_m \sum_{j\in\mathcal{M}_m, j\neq i} p_j}{(1-\rho)(1+(n_m-1)\rho_m)},$$

where $n_m \equiv |\mathcal{M}_m|$. It then follows that

$$\frac{\partial q_i}{\partial p_i} = -\frac{1-(n_m-2)\rho_m}{(1-\rho_m)(1+(n_m-1)\rho_m)}.$$

Inserting into the FOC with respect to p_i gives

$$q_i = -\frac{1-(n_m-2)\rho_m}{(1-\rho_m)(1+(n_m-1)\rho_m)}(p_i - c_i).$$

Inserting Equations (1) and (2) yields

$$\begin{aligned} q_i &= \frac{(1-(n_m-2)\rho_m)(\alpha_m - \bar{c}_i)}{\rho_m(4-(2-\rho_m)n_m - \rho_m)} - \frac{1-(n_m-2)\rho_m}{4-(2-\rho_m)n_m-\rho_m} \sum_{j\in\mathcal{M}_m, j\neq i} q_j \\ &+ \frac{(1-(n_m-2)\rho_m)}{\rho_m(4-(2-\rho_m)n_m-\rho_m}e_i + \frac{(1-(n_m-2)\rho_m)\varphi}{\rho_m(4-(2-\rho_m)n_m-\rho_m}\sum_{j=1}^n a_{ij}e_j. \end{aligned}$$

The FOC with respect to R&D effort is the same as in the case of perfect competition, so that we get $e_i = q_i$. Inserting equilibrium effort and rearranging terms gives

$$\begin{aligned} q_i &= \frac{(1-(n_m-2)\rho_m)(\alpha_m - \bar{c}_i)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)} \\ &- \frac{\rho_m(1-(n_m-2)\rho_m)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)} \sum_{j\in\mathcal{M}_m, j\neq i} q_j \\ &+ \frac{\varphi(1-(n_m-2)\rho_m)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)} \sum_{j=1}^n a_{ij}q_j. \end{aligned}$$

If we denote by

$$\begin{aligned} \mu_i &\equiv \frac{(1-(n_m-2)\rho_m)(\alpha_m - \bar{c}_i)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)}, \\ \rho &\equiv \frac{\rho_m(1-(n_m-2)\rho_m)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)}, \\ \lambda &\equiv \frac{\varphi(1-(n_m-2)\rho_m)}{\rho_m(4-(2-\rho_m)n_m-\rho_m) - 1(1-(n_m-2)\rho_m)}. \end{aligned}$$

Then we can write equilibrium quantities as follows

$$q_i = \mu_i - \rho\sum_{j=1}^n b_{ij}q_j + \lambda\sum_{j=1}^n a_{ij}q_j. \tag{74}$$

Observe that the reduced form Equation (74) is identical to the Cournot case in Equation (42).

E. Intra- versus Interindustry Collaborations: Theory

We extend our model by allowing for intra-industry technology spillovers to differ from inter-industry spillovers. The profit of firm $i \in \mathcal{N}$ is still given by $\pi_i = (p_i - c_i)q_i - \frac{1}{2}e_i^2$, where the inverse demand is $p_i = \bar{\alpha}_i - q_i - \rho \sum_{j=1}^n b_{ij} q_j$. The main change is in the marginal cost of production, which is now equal to

$$c_i = \bar{c}_i - e_i - \varphi_1 \sum_{j=1}^n a_{ij}^{(1)} e_j - \varphi_2 \sum_{j=1}^n a_{ij}^{(2)} e_j,$$

where we have introduced two matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ with elements $a_{ij}^{(1)}$ and $a_{ij}^{(2)}$, respectively, indicating a collaboration within the same industry (with the superscript (1)) or across different industries (with the superscript (2)). Note that we can write $\mathbf{A}^{(1)} = \mathbf{A} \circ \mathbf{B}$ and $\mathbf{A}^{(2)} = \mathbf{A} \circ (\mathbf{U} - \mathbf{B})$, with the matrix $\mathbf{B}$ having elements $b_{ij} \in \{0, 1\}$ indicating whether firms i and j operate in the same market or not, $\mathbf{U}$ being a matrix of all ones, and $\circ$ denotes the Hadamard elementwise matrix product.[73] Inserting this marginal cost of production into the profit function gives

$$\pi_i = (\bar{\alpha}_i - \bar{c}_i)q_i - q_i^2 - \rho q_i \sum_{j=1}^n b_{ij} q_j + q_i e_i + \varphi_1 q_i \sum_{j=1}^n a_{ij}^{(1)} e_j + \varphi_2 q_i \sum_{j=1}^n a_{ij}^{(2)} e_j - \frac{1}{2} e_i^2.$$

As above, from the first-order condition with respect to R&D effort, we obtain $e_i = q_i$. Inserting this optimal effort into the first-order condition with respect to output, we obtain

$$q_i = \bar{\alpha}_i - \bar{c}_i - \rho \sum_{j=1}^n b_{ij} q_j + \varphi_1 \sum_{j=1}^n a_{ij}^{(1)} q_j + \varphi_2 \sum_{j=1}^n a_{ij}^{(2)} q_j.$$

Denoting by $\mu_i \equiv \bar{\alpha}_i - \bar{c}_i$, we can write this as

$$q_i = \mu_i - \rho \sum_{j=1}^n b_{ij} q_j + \varphi_1 \sum_{j=1}^n a_{ij}^{(1)} q_j + \varphi_2 \sum_{j=1}^n a_{ij}^{(2)} q_j. \tag{75}$$

If the matrix $\mathbf{I}_n + \rho \mathbf{B} - \varphi_1 \mathbf{A}^{(1)} - \varphi_2 \mathbf{A}^{(2)}$ is invertible, this gives us the equilibrium quantities

$$\mathbf{q} = (\mathbf{I}_n + \rho \mathbf{B} - \varphi_1 \mathbf{A}^{(1)} - \varphi_2 \mathbf{A}^{(2)})^{-1} \boldsymbol{\mu}.$$

Let us now write the econometric equivalent of Equation (75). Proceeding as in Section 7.1, using Equations (25) and (26) and introducing time t, we get

$$\mu_{it} = \mathbf{x}_{it}^\top \boldsymbol{\beta} + \eta_i + \kappa_t + \epsilon_{it}.$$

Plugging this value of μ_{it} into Equation (75), we obtain

$$q_{it} = \varphi_1 \sum_{j=1}^n a_{ij,t}^{(1)} q_{jt} + \varphi_2 \sum_{j=1}^n a_{ij,t}^{(2)} q_{jt} - \rho \sum_{j=1}^n b_{ij} q_{jt} + \mathbf{x}_{it}^\top \beta + \eta_i + \kappa_t + \epsilon_{it},$$

where $a_{ij,t}^{(1)} = a_{ij,t} b_{ij}$ and $a_{ij,t}^{(2)} = a_{ij,t}(1 - b_{ij})$. This is Equation (32) in Section 8.2.

F. Direct and Indirect Technology Spillovers: Theory

We extend our model by allowing for direct (between collaborating firms) and indirect (between non-collaborating firms) technology spillovers. The profit of firm $i \in \mathcal{N}$ is still given by $\pi_i = (p_i - c_i)q_i - \frac{1}{2}e_i^2$,

[73] Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. The Hadamard product of $\mathbf{A}$ and $\mathbf{B}$ is defined by $[\mathbf{A} \circ \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} [\mathbf{B}]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, i.e. the Hadamard product is simply an element-wise multiplication.

where the inverse demand is $p_i = \bar{\alpha}_i - q_i - \rho \sum_{j=1}^{n} b_{ij} q_j$. The main change is in the marginal cost of production, which is now equal to[74]

$$c_i = \bar{c}_i - e_i - \varphi \sum_{j=1}^{n} a_{ij} e_j - \chi \sum_{j=1}^{n} w_{ij} e_j, \tag{76}$$

where w_{ij} are weights characterizing alternative channels for technology spillovers than R&D collaborations (representing for example a patent cross-citation, a flow of workers, or technological proximity measured by the matrix P_{ij} introduced in Footnote 42). Inserting this marginal cost of production into the profit function gives

$$\pi_i = (\bar{\alpha}_i - \bar{c}_i) q_i - q_i^2 - \rho q_i \sum_{j=1}^{n} b_{ij} q_j + q_i e_i + \varphi q_i \sum_{j=1}^{n} a_{ij} e_j + \chi q_i \sum_{j=1}^{n} w_{ij} e_j - \frac{1}{2} e_i^2.$$

As above, from the first-order condition with respect to R&D effort, we obtain $e_i = q_i$. Inserting this optimal effort into the first-order condition with respect to output, we obtain

$$q_i = \bar{\alpha}_i - \bar{c}_i - \rho \sum_{j=1}^{n} b_{ij} q_j + \varphi \sum_{j=1}^{n} a_{ij} q_j + \chi \sum_{j=1}^{n} w_{ij} q_j.$$

Denoting by $\mu_i \equiv \bar{\alpha}_i - \bar{c}_i$, we can write this as

$$q_i = \mu_i - \rho \sum_{j=1}^{n} b_{ij} q_j + \varphi \sum_{j=1}^{n} a_{ij} q_j + \chi \sum_{j=1}^{n} w_{ij} q_j. \tag{77}$$

If the matrix $\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A} - \chi \mathbf{W}$ is invertible, this gives us the equilibrium quantities

$$\mathbf{q} = (\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A} - \chi \mathbf{W})^{-1} \boldsymbol{\mu}.$$

Let us now write the econometric equivalent of Equation (77). Proceeding as in Section 7.1, using Equations (25) and (26) and introducing time t, we get

$$\mu_{it} = \mathbf{x}_{it}^{\top} \boldsymbol{\beta} + \eta_i + \kappa_t + \epsilon_{it}.$$

Plugging this value of μ_{it} into Equation (77), we obtain

$$q_{it} = \varphi \sum_{j=1}^{n} a_{ij,t} q_{jt} + \chi \sum_{j=1}^{n} w_{ij,t} q_{jt} - \rho \sum_{j=1}^{n} b_{ij} q_{jt} + \mathbf{x}_{it}^{\top} \beta + \eta_i + \kappa_t + \epsilon_{it}.$$

This is Equation (34) in Section 8.3.

G. Data

In the following appendices we give a detailed account on how we constructed our data sample. In Appendix G.1 we describe the two raw datasources we have used to obtain information on R&D collaborations between firms. In Appendix G.2 we explain how we complemented these data with information about mergers and acquisitions, while Appendix G.3 explains how we supplemented the alliance information with firms' balance sheet statements. Moreover, Appendix G.4 discusses the geographic distribution of the firms in our data sample. Finally, Appendix G.5 provides the details on how we complemented the alliance data with the firms patent portfolios and computed their technological proximities.

[74] See also Eq. (1) in Goyal and Moraga-Gonzalez [2001].

G.1. R&D Network

To get a comprehensive picture of alliances we use data on interfirm R&D collaborations stemming from two sources which have been widely used in the literature [cf. Schilling, 2009]. The first is the Cooperative Agreements and Technology Indicators (CATI) database [cf. Hagedoorn, 2002]. The database only records agreements for which a combined innovative activity or an exchange of technology is at least part of the agreement. Moreover, only agreements that have at least two industrial partners are included in the database, thus agreements involving only universities or government labs, or one company with a university or lab, are disregarded. The second is the Thomson Securities Data Company (SDC) alliance database. SDC collects data from the U. S. Securities and Exchange Commission (SEC) filings (and their international counterparts), trade publications, wires, and news sources. We include only alliances from SDC which are classified explicitly as research and development collaborations. A comparative analysis of these two databases (and other alternative databases) can be found in Schilling [2009].

We then merged the CATI database with the Thomson SDC alliance database. For the matching of firms across datasets we adopted the name matching algorithm developed as part of the NBER patent data project [Trajtenberg et al., 2009] and developed further by Atalay et al. [2011].[75] From the firms in the CATI database and the firms in the SDC database we could match 21% of the firms appearing in both databases. Considering only firms without missing observations on sales, output and R&D expenditures (see also Appendix G.3 below on how we obtained balance sheet and income statement information), gives us a sample of $1,431$ firms and a total of $1,174$ collaborations over the years 1970 to 2006.[76] The average degree of the firms in this sample is 1.64 with a standard deviation of 5.64 and the maximum degree is 76 attained by *Motorola Inc.*. Figure G.1 shows the largest connected component of the R&D collaboration network with all links accumulated up to the year 2005 (see Appendix B.1). The figure indicates two clusters appearing which are related to the different industries in which firms are operating.

Figure G.2 shows the average clustering coefficient, C, the relative size of the largest connected component, $\max_{\{H \subseteq G\}} |H|/n$, the average path length, ℓ, and the eigenvector centralization C_v (relative to a star network of the same size) over the years 1990 to 2005 (see Wasserman and Faust [1994] and Appendix B.1 for the definitions). We observe that the network shows the highest degree of clustering and the largest connected component around the year 1997, an average path length of around 5, and a centralization index C_v between 0.3 and 0.6. Moreover, comparing our subsample and the original network (where firms have not been dropped because of missing accounting information) we find that both exhibit similar trends over time. This seems to suggest that the patterns found in the subsample are representative for the overall patterns in the data. Further, the clustering coefficient and the size of the largest connected component exhibit a similar trend as the number of firms and the average number of collaborations that we have seen already in Figure 5.

Figure G.3 shows the degree distribution, $P(d)$, the average nearest neighbor connectivity, $k_{\text{nn}}(d)$, the clustering degree distribution, $C(d)$, and the component size distribution, $P(s)$ across different years of observation [cf. e.g. König, 2011]. The degree distribution decays as a power law, the average nearest neighbor degree is increasing with the degree, indicating an assortative network, the clustering degree distribution is decreasing with the degree and the component size distribution indicates a large connected component (see also Figure G.1) with smaller components decaying as a power law.

Figure G.4 and Tables 14 and 15 illustrate the industrial composition of our sample of R&D collaborating firms at the main 2-digit and 4-digit standard industry classification (SIC) levels, respectively. At the 2-digit level, the chemicals and pharmaceuticals sectors make up for the largest fraction (22.08%) of firms in our data, followed by electronic equipment and business services. This is similar to the sectoral decomposition provided in Schilling [2009], who identifies the biotech and information technology sectors as the most prominent in the CATI and SDC R&D collaboration databases.

[75] See `https://sites.google.com/site/patentdataproject`. We would like to thank Enghin Atalay and Ali Hortacsu for sharing their name matching algorithm with us.

[76] This is the sample that we have used for our empirical analysis in Section 7.

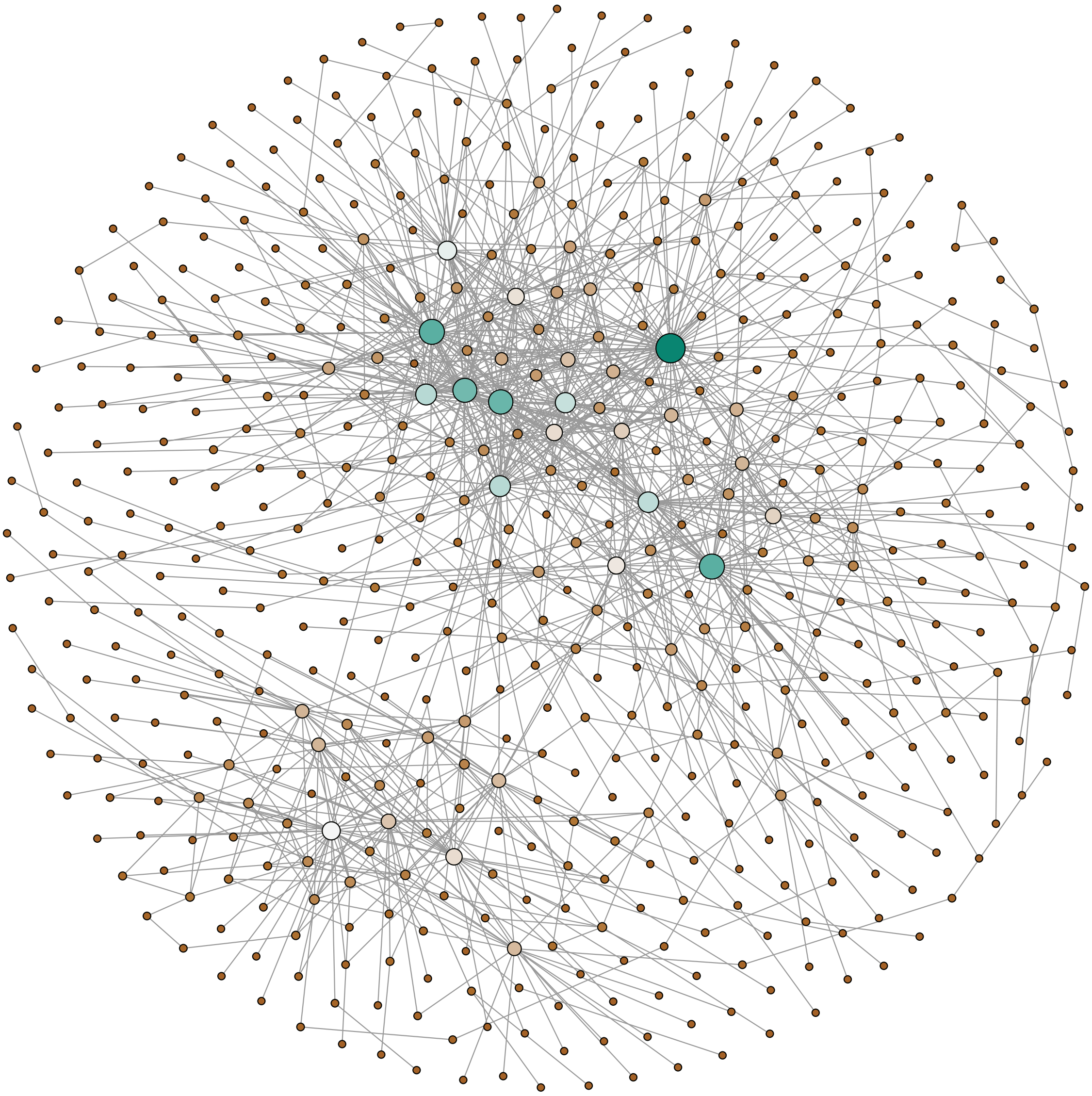

Figure G.1: The largest connected component of the R&D collaboration network with all links accumulated until the year 2005. The nodes' colors indicate sectors according to 4-digit SIC codes while the nodes' sizes indicate the number of collaborations of a firm.

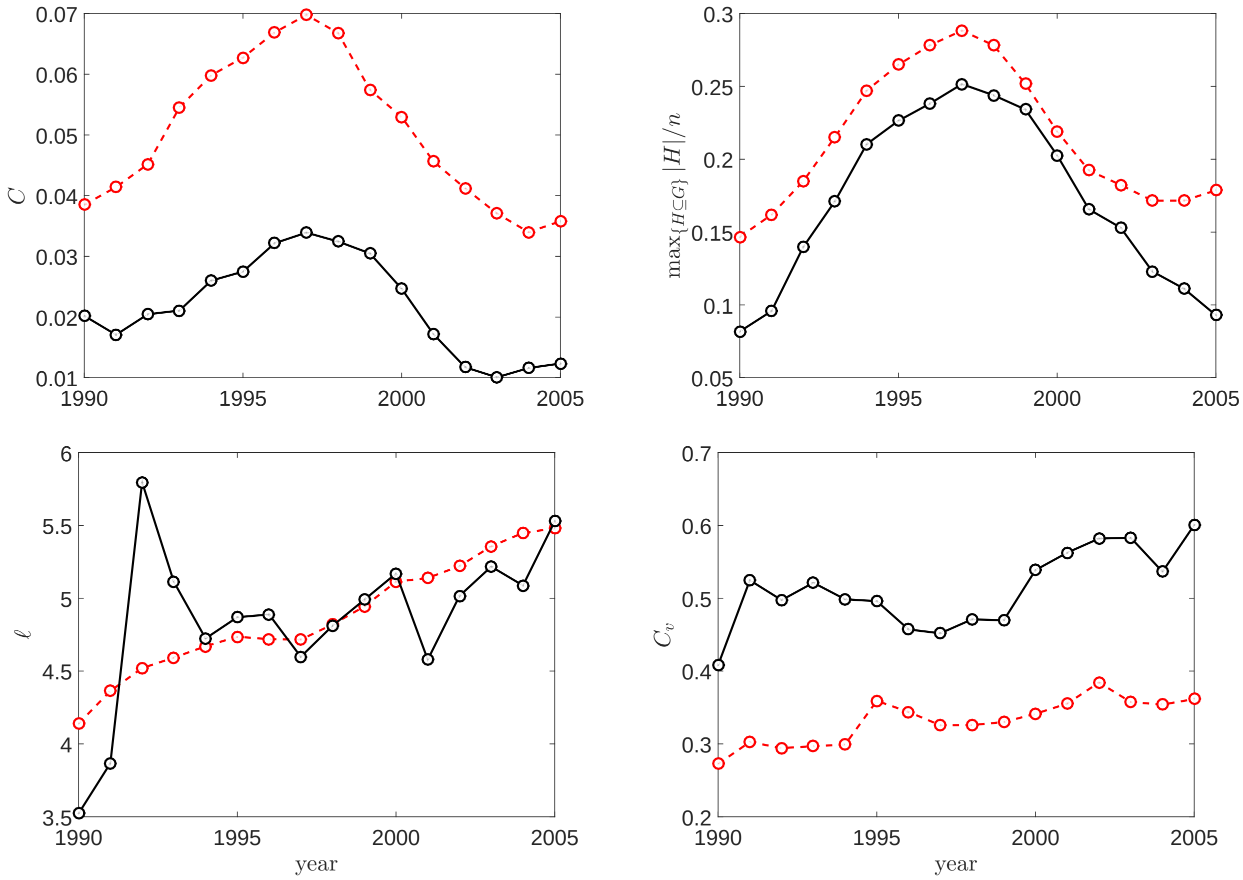

Figure G.2: The average clustering coefficient, C, the relative size of the largest connected component, $\max_{\{H\subseteq G\}} |H|/n$, the average path length, ℓ, and the eigenvector centralization C_v (relative to a star network of the same size) over the years 1990 to 2005 (see Appendix B.1). Dashed lines indicate the corresponding quantities for the original network (where firms have not been dropped because of missing accounting information), while solid lines indicate the subsample with 1,431 firms that we have used in the empirical Section 7.

Table 14: The 20 largest sectors at the 2-digit SIC level.

Sector	2-dig SIC	# firms	% of tot.	Rank
Chemical and Allied Products	28	316	22.08	1
Electronic and Other Electric Equipment	36	230	16.07	2
Business Services	73	215	15.02	3
Industrial Machinery and Equipment	35	181	12.65	4
Instruments and Related Products	38	170	11.88	5
Transportation Equipment	37	79	5.52	6
Primary Metal Industries	33	28	1.96	7
Engineering and Management Services	87	28	1.96	8
Fabricated Metal Products	34	19	1.33	9
Communications	48	19	1.33	10
Oil and Gas Extraction	13	14	0.98	11
Rubber and Miscellaneous Plastics Products	30	13	0.91	12
Miscellaneous Manufacturing Industries	39	13	0.91	13
Petroleum and Coal Products	29	12	0.84	14
Electric Gas and Sanitary Services	49	12	0.84	15
Food and Kindred Products	20	10	0.70	16
Paper and Allied Products	26	10	0.70	17
Health Services	80	9	0.63	18
Stone Clay and Glass Products	32	7	0.49	19
Wholesale Trade - Durable Goods	50	6	0.42	20

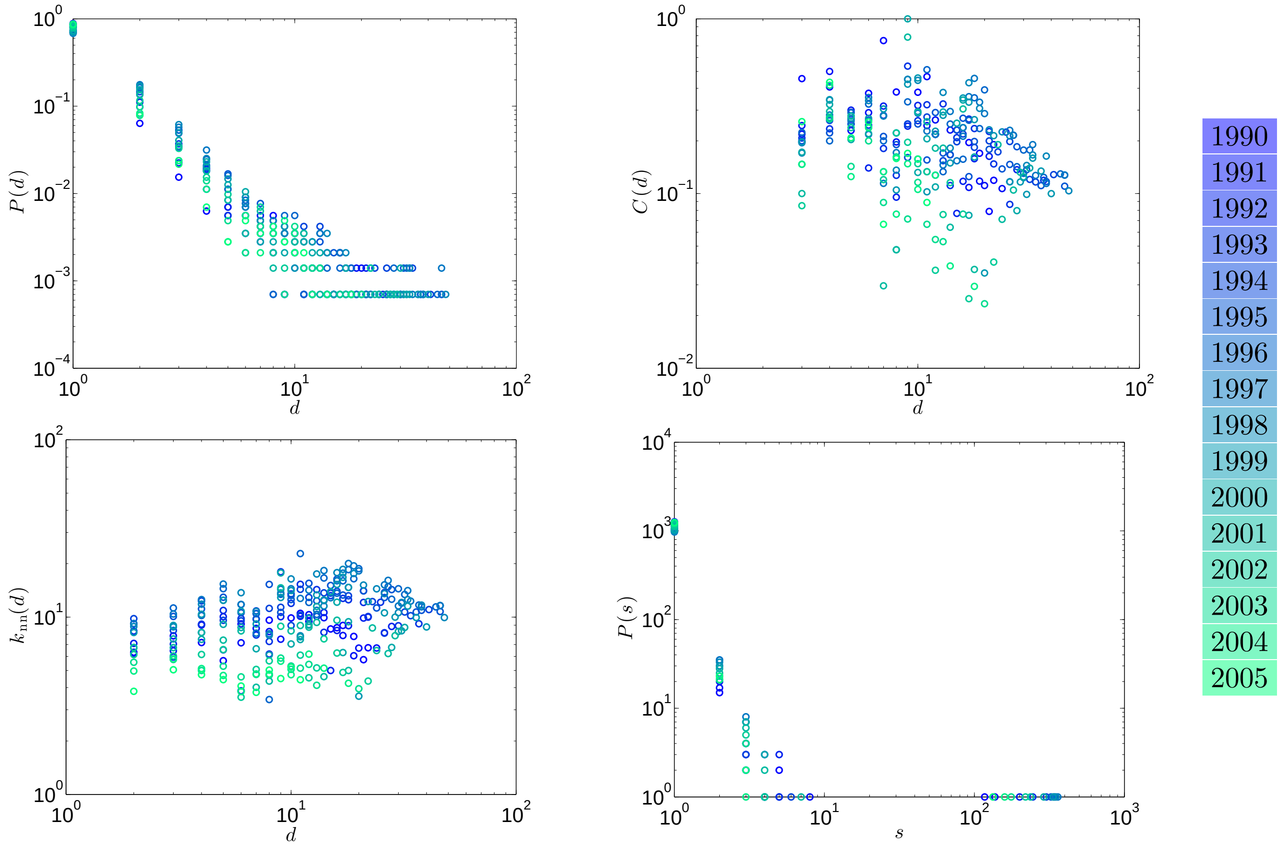

Figure G.3: The degree distribution, $P(d)$, the average nearest neighbor connectivity, $k_{\mathrm{nn}}(d)$, the clustering degree distribution, $C(d)$, and the component size distribution, $P(s)$.

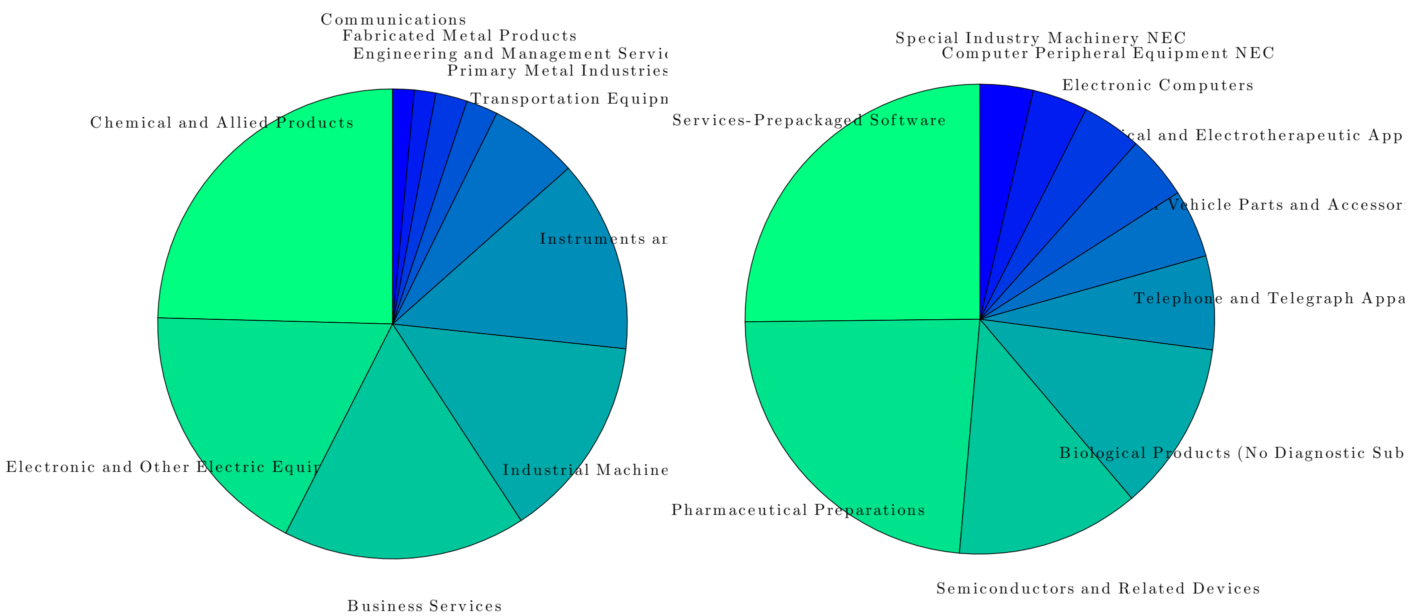

Figure G.4: The shares of the ten largest sectors at the 2-digit (left panel) and 4-digit (right panel) SIC levels. See also Tables 14 and 15, respectively.

Table 15: The 20 largest sectors at the 4-digit SIC level.

Sector	4-dig SIC	# firms	% of tot.	Rank
Services-Prepackaged Software	7372	172	12.02	1
Pharmaceutical Preparations	2834	160	11.18	2
Semiconductors and Related Devices	3674	86	6.01	3
Biological Products (No Diagnostic Substances)	2836	80	5.59	4
Telephone and Telegraph Apparatus	3661	44	3.07	5
Motor Vehicle Parts and Accessories	3714	32	2.24	6
Electromedical and Electrotherapeutic Apparatus	3845	30	2.10	7
Electronic Computers	3571	28	1.96	8
Computer Peripheral Equipment NEC	3577	26	1.82	9
Special Industry Machinery NEC	3559	25	1.75	10
In Vitro and In Vivo Diagnostic Substances	2835	24	1.68	11
Surgical and Medical Instruments and Apparatus	3841	23	1.61	12
Services-Computer Integrated Systems Design	7373	23	1.61	13
Laboratory Analytical Instruments	3826	22	1.54	14
Radio and TV Broadcasting and Communications Equipment	3663	21	1.47	15
Motor Vehicles and Passenger Car Bodies	3711	21	1.47	16
Household Audio and Video Equipment	3651	19	1.33	17
Instruments For Meas and Testing of Electricity and Elec Signals	3825	18	1.26	18
Computer Storage Devices	3572	17	1.19	19
Search Detection Navigation Guidance Aeronautical Sys	3812	15	1.05	20

Table 16 shows the 20 largest countries in terms of R&D collaborating firms in our dataset. The U.S. is clearly the dominant country where most of the firms in our sample are being headquartered.

G.2. Mergers and Acquisitions

Some firms might be acquired by other firms due to mergers and acquisitions (M&A) over time, and this will impact the R&D collaboration network [cf. Hanaki et al., 2010].

To get a comprehensive picture of the M&A activities of the firms in our dataset, we use two extensive datasources to obtain information about M&As. The first is the Thomson Reuters' Securities Data Company (SDC) M&A database, which has historically been the most widely used database for empirical research in the field of M&As. Data in SDC dates back to 1965 with a slightly more complete coverage of deals starting in the early 1980s. The second database with information about M&As is Bureau van Dijk's (BvD) Zephyr database, which is a recent alternative to the SDC M&As database. The history of deals recorded in Zephyr goes back to 1997. In 1997 and 1998 only European deals are recorded, while international deals are included starting from 1999. According to Huyghebaert and Luypaert [2010], Zephyr "covers deals of smaller value and has a better coverage of European transactions". A comparison and more detailed discussion of the two databases can be found in Bollaert and Delanghe [2015] and Bena et al. [2008].

We merged the SDC and Zephyr databases (with the above mentioned name matching algorithm; see also Atalay et al. [2011]; Trajtenberg et al. [2009]) to obtain information on M&As of $116,641$ unique firms. Using the same name matching algorithm we could identify 43.08% of the firms in the combined CATI-SDC alliance database that also appear in the combined SDC-Zephyr M&As database. We then account for the M&A activities of these matched firms when constructing the R&D collaboration network by assuming that an acquiring firm in a M&A inherits all the R&D collaborations of the target firm, and we remove the target firm form from the network.

G.3. Balance Sheet Statements

The combined CATI-SDC alliance database provides the names for each firm in an alliance, but it does not contain information about the firms' output levels or R&D expenses. We therefore matched the firms' names in the combined CATI-SDC database with the firms' names in Standard & Poor's Compustat U.S. and Global fundamentals annual databases and Bureau van Dijk (BvD)'s Osiris

Table 16: The 20 largest countries in terms of R&D collaborating firms in our dataset.

Name	Code	# firms	% of tot.	Rank
United States	USA	1189	83.09	1
Japan	JPN	148	10.34	2
France	FRA	17	1.19	3
Sweden	SWE	14	0.98	4
United Kingdom	GBR	11	0.77	5
Switzerland	CHE	10	0.70	6
Germany	DEU	6	0.42	7
Italy	ITA	6	0.42	8
Finland	FIN	5	0.35	9
Slovakia	SVK	5	0.35	10
Belgium	BEL	3	0.21	11
Netherlands	NLD	3	0.21	12
Norway	NOR	3	0.21	13
Canada	CAN	2	0.14	14
Denmark	DNK	2	0.14	15
India	IND	2	0.14	16
Australia	AUS	1	0.07	17
Austria	AUT	1	0.07	18
Greece	GRC	1	0.07	19
Hong Kong	HKG	1	0.07	20

database, to obtain information about their balance sheets and income statements.[77] These databases contain only firms listed on the stock market, so they typically exclude smaller private firms, but this is inevitable if one is going to use market value data. Nevertheless, R&D is concentrated in publicly listed firms, and our data sources thus cover most of the R&D activities in the economy [cf. e.g. Bloom et al., 2013]. Compustat contains financial data extracted from company filings.

Compustat North America is a database of U.S. and Canadian fundamental and market information on active and inactive publicly held companies. It provides more than 300 annual and 100 quarterly income statements, balance sheets and statement of cash flows. Compustat Global is a database of non-U.S. and non-Canadian companies and contains market information on more than 33,900 active and inactive publicly held companies with annual data history from 1987. The Compustat databases cover 99% of the world's total market capitalization with annual company data history available back to 1950.

Osiris is owned by Bureau van Dijk (BvD) and it contains a wide range of accounting and other items for firms from over 120 countries. Osiris contains financial information on globally listed public companies with coverage for up to 20 years on over 62,191 companies by major international industry classifications. It claims to cover all publicly listed companies worldwide. In addition, it covers major non-listed companies when they are primary subsidiaries of publicly listed companies, or in certain cases, when clients request information from a particular company.

For a detailed comparison and discussion of the Compustat and Osiris databases see Dai [2012] and Papadopoulos [2012].

For the matching of firms across datasets we adopted the name matching algorithm developed as part of the NBER patent data project [Atalay et al., 2011; Trajtenberg et al., 2009]. We could match 25.53% of the firms in the combined CATI-SDC database with the combined Compustat-Osiris database. For the matched firms we obtained their sales and R&D expenditures. U.S. dollar translation rates for foreign currencies have been taken directly from the Compustat yearly averaged exchange

[77] We chose to use two alternative database for firm level accounting data to get as much information as possible about balance sheets and income statements for the firms in the R&D collaboration database. The accounting databases used here are complementary, as Compustat features a greater coverage of large companies in more developed countries, while BvD Osiris contains a higher number of small firms from developing countries and tends to have a better coverage of European firms [cf. Dai, 2012].

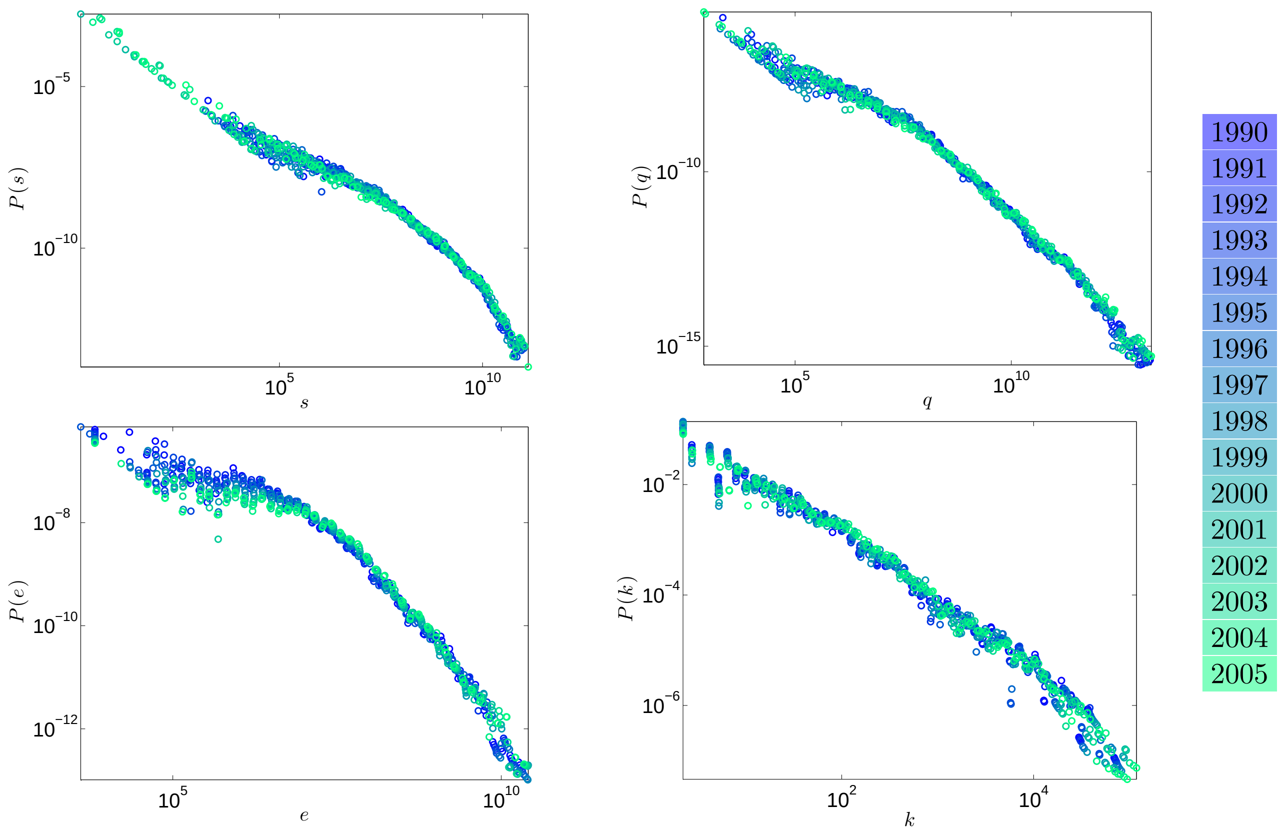

Figure G.5: The sales distribution, $P(s)$, the output distribution, $P(q)$, the R&D expenditures distribution, $P(e)$, and the patent stock distribution, $P(k)$, across different years ranging from 1990 to 2005 using a logarithmic binning of the data [McManus et al., 1987].

rates. We adjusted for inflation using the consumer price index of the Bureau of Labor Statistics (BLS), averaged annually, with 1983 as the base year. Individual firms' output levels are computed from deflated sales using 2-SIC digit industry-country-year specific price deflators from the OECD-STAN database [cf. Gal, 2013]. We then dropped all firms with missing information on sales, output and R&D expenditures. This pruning procedure left us with a subsample of $1,431$, on which the empirical analysis in Section 7 is based.[78]

The empirical distributions for sales, $P(s)$, output , $P(q)$, R&D expenditures, $P(e)$, and the patent stocks, $P(k)$, across different years ranging from 1990 to 2005 (using a logarithmic binning of the data with 100 bins [cf. McManus et al., 1987]) are shown in Figure G.5. All distributions are highly skewed, indicating a large degree of inequality in firms' sizes and patent activities.

G.4. Geographic Location and Distance

In order to determine the locations of the firms in our data we have added the longitude and latitude coordinates associated with the city of residence of each firm in our data. Among the matched cities in our dataset 93.67% could be geo-localized using ArcGIS [cf. e.g. Dell, 2009] and the Google Maps Geocoding API.[79] We then used Vincenty's algorithm to compute the distances between pairs of geo-localized firms [cf. Vincenty, 1975]. The mean distance, $\overline{d}$, and the distance distribution, $P(d)$, across collaborating firms are shown in Figure H.1, while Figures G.6 and G.7 show the locations (at the city level) of firms in the database (see also Figure 4). The largest distance between collaborating firms appears around the turn of the millennium, while the distance distribution is heavily skewed. We find that R&D collaborations tend to be more likely between firms that are close, showing that geography matters for R&D collaborations and spillovers, in line with previous empirical studies [cf. Lychagin

[78] Section 8.4 discusses how sensitive our empirical results are with respect to subsampling (i.e. missing data).
[79] See https://developers.google.com/maps/documentation/geocoding/intro.

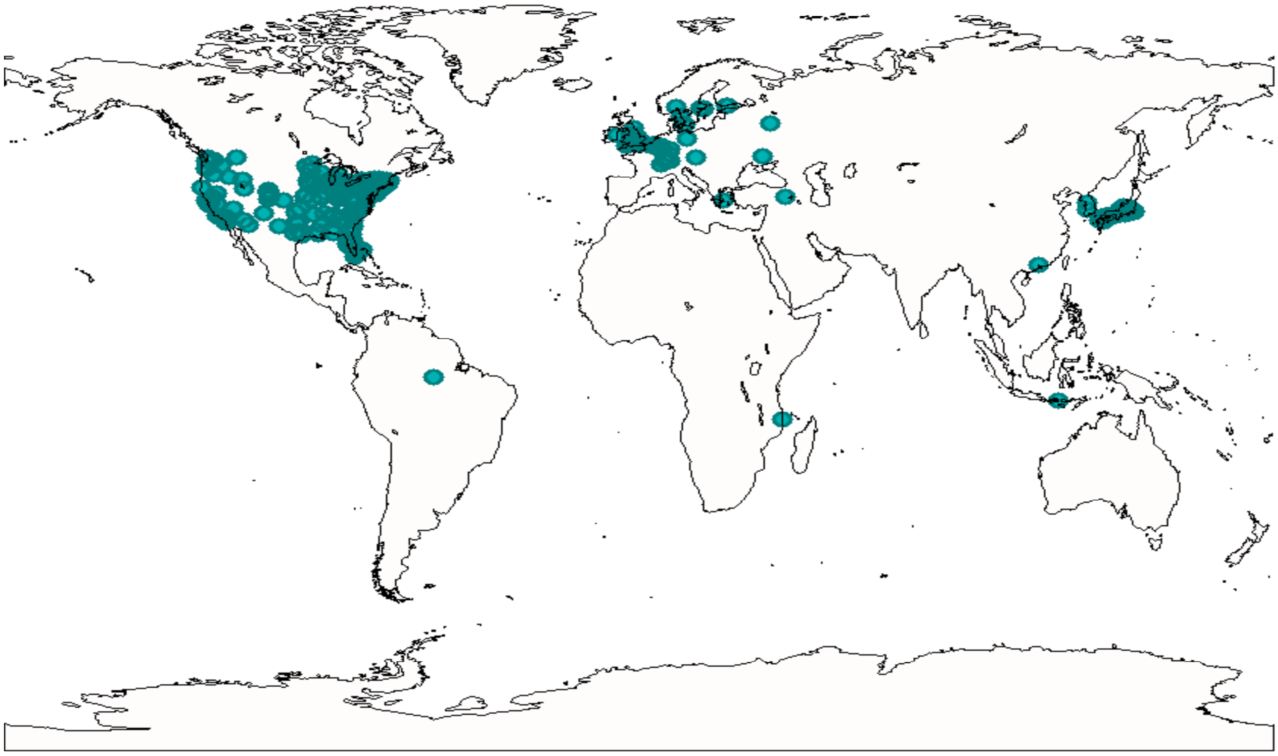

Figure G.6: The locations (at the city level) of firms in the combined CATI-SDC database.

et al., 2010].

G.5. Patents

We identified the patent portfolios of the firms in our dataset using the EPO Worldwide Patent Statistical Database (PATSTAT) [Hall et al., 2001; Jaffe and Trajtenberg, 2002]. The creation of this worldwide statistical patent database was initiated by the OECD task force on patent statistics. It includes bibliographic details on patents filed to 80 patent offices worldwide, covering more than 60 million documents. Hence filings in all major countries and at the World International Patent Office are covered. We matched the firms in our data with the assignees in the PATSTAT database using the above mentioned name matching algorithm [Atalay et al., 2011; Trajtenberg et al., 2009]. We only consider granted patents (or successful patents), as opposed to patents applied for, as they are the main drivers of revenue derived from R&D expenditures [cf. Copeland and Fixler, 2012]. Using our name matching algorithm we obtained matches for 36.05% of the firms in our data with patent information. The distribution of the number of patents is shown in Figure G.5. The technology classes were identified using the main international patent classification (IPC) numbers at the 4-digit level.

From the firms' patents, we then computed the technological proximity of firm i and j as

$$f_{ij}^{\mathrm{J}} = \frac{\mathbf{P}_i^\top \mathbf{P}_j}{\sqrt{\mathbf{P}_i^\top \mathbf{P}_i}\sqrt{\mathbf{P}_j^\top \mathbf{P}_j}}, \tag{78}$$

where, for each firm i, $\mathbf{P}_i$ is a vector whose k-th component, P_{ik}, counts the number of patents firm i has in technology category k divided by the total number of technologies attributed to the firm [cf. Bloom et al., 2013; Jaffe, 1989]. Thus, $\mathbf{P}_i$ represents the patent portfolio of firm i. We use the three-digit U.S. patent classification system to identify technology categories [Hall et al., 2001]. We denote by $\mathbf{F}^{\mathrm{J}}$ the $(n \times n)$ matrix with elements $(f_{ij}^{\mathrm{J}})_{1 \leq i,j \leq n}$.

We next consider the Mahalanobis technology proximity measure introduced by Bloom et al. [2013]. To construct this metric, we need to introduce some additional notation. Let N be the number of

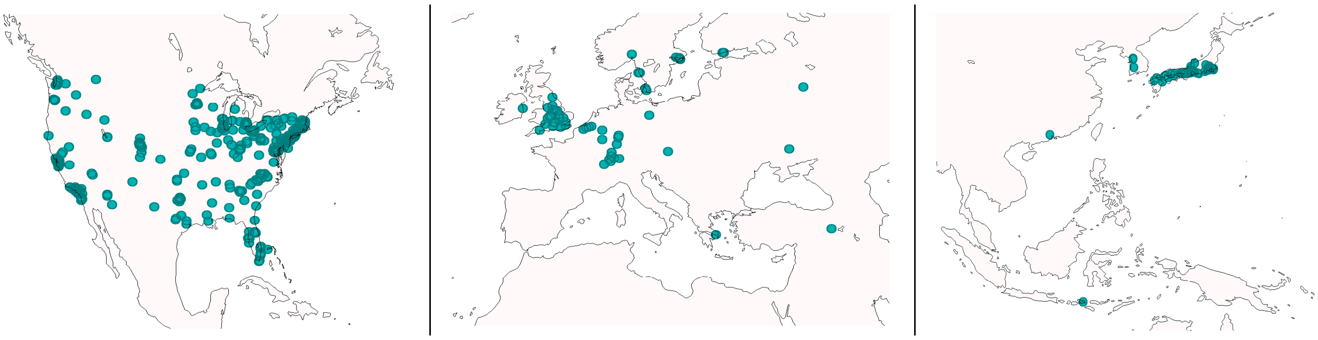

Figure G.7: The locations (at the city level) in the U.S., Europe and Asia of firms in the combined CATI-SDC database.

technology classes, n the number of firms, and let $\mathbf{T}$ be the $(N \times n)$ patent shares matrix with elements

$$T_{ji} = \frac{1}{\sum_{k=1}^{n} P_{ki}} P_{ji},$$

for all $1 \leq i \leq n$ and $1 \leq j \leq N$. Further, we construct the $(N \times n)$ normalized patent shares matrix $\tilde{\mathbf{T}}$ with elements

$$\tilde{T}_{ji} = \frac{1}{\sqrt{\sum_{k=1}^{N} T_{ki}^2}} T_{ji},$$

and the $(n \times N)$ normalized patent shares matrix across firms is defined by $\tilde{\mathbf{X}}$ with elements

$$\tilde{X}_{ik} = \frac{1}{\sqrt{\sum_{i=1}^{N} T_{ki}^2}} T_{ki}.$$

Let $\mathbf{\Omega} = \tilde{\mathbf{X}}^\top \tilde{\mathbf{X}}$. Then the $(n \times n)$ Mahalanobis technology similarity matrix with elements $(f_{ij}^{\mathrm{M}})_{1 \leq i,j \leq n}$ is defined as

$$\mathbf{F}^{\mathrm{M}} = \tilde{\mathbf{T}}^\top \mathbf{\Omega} \tilde{\mathbf{T}}. \tag{79}$$

Figure H.2 shows the average patent proximity across collaborating firms using the Jaffe metric f_{ij}^{J} of Equation (78) or the Mahalanobis metric f_{ij}^{M} of Equation (79). Both are monotonic increasing over almost all years of observations. This suggests that R&D collaborating firms tend to become more similar over time.

H. Numerical Algorithm for Computing Optimal Subsidies

The bounded linear complementarity problem (LCP) of Equation (59) is equivalent to the Kuhn-Tucker optimality conditions of the following quadratic programming (QP) problem with box constraints [cf. Byong-Hun, 1983]

$$\min_{\mathbf{q} \in [0, \bar{q}]^n} \left\{ -\boldsymbol{\nu}(\mathbf{s})^\top \mathbf{q} + \frac{1}{2} \mathbf{q}^\top \left(\mathbf{I}_n + \rho \mathbf{B} - \varphi \mathbf{A} \right) \mathbf{q} \right\}, \tag{80}$$

where $\boldsymbol{\nu}(\mathbf{s}) \equiv \boldsymbol{\mu} + (\mathbf{I}_n + \varphi \mathbf{A})\mathbf{s}$. Moreover, net welfare is given by

$$\overline{W}(G, \mathbf{s}) = \sum_{i=1}^{n} \left(\frac{q_i^2}{2} + \pi_i - s_i e_i \right) = \boldsymbol{\mu}^\top \mathbf{q} - \mathbf{q}^\top \left(\frac{\rho}{2} \mathbf{B} - \varphi \mathbf{A} \right) \mathbf{q} + \varphi \mathbf{q}^\top \mathbf{A} \mathbf{s} - \frac{1}{2} \mathbf{s}^\top \mathbf{A} \mathbf{s}.$$

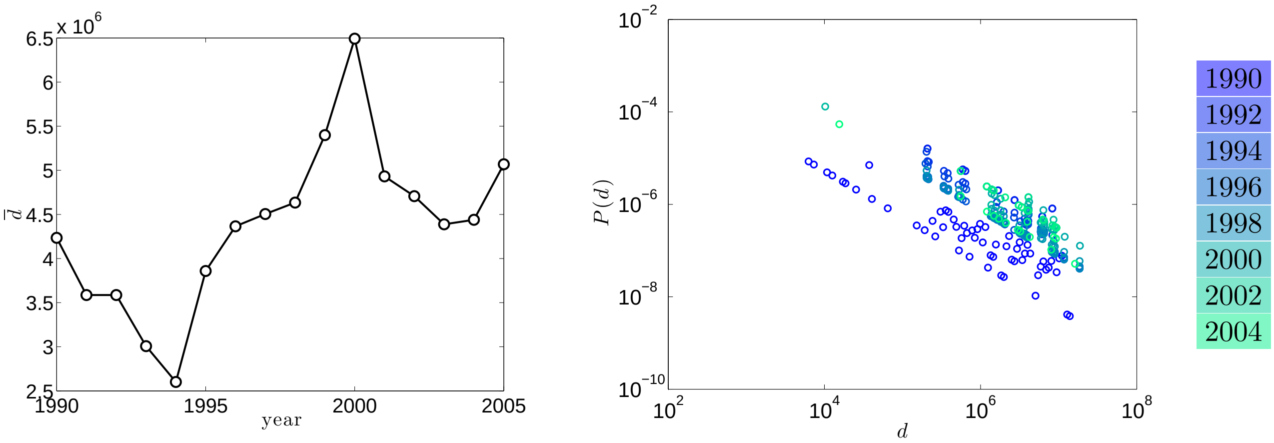

Figure H.1: The mean distance, $\overline{d}$, and the distance distribution, $P(d)$, across collaborating firms in the combined CATI-SDC database.

Finding the optimal subsidy program $\mathbf{s}^* \in [0,\bar{s}]^n$ is then equivalent to solving the following *bilevel optimization problem* [cf. Bard, 2013]

$$\begin{aligned} \max_{\mathbf{s}\in[0,\bar{s}]^n} \quad & \overline{W}(G,\mathbf{s}) = \boldsymbol{\mu}^\top \mathbf{q}^*(\mathbf{s}) - \mathbf{q}^*(\mathbf{s})^\top \left(\frac{\rho}{2}\mathbf{B} - \varphi\mathbf{A}\right)\mathbf{q}^*(\mathbf{s}) + \varphi \mathbf{q}^*(\mathbf{s})^\top \mathbf{A}\mathbf{s} - \frac{1}{2}\mathbf{s}^\top \mathbf{A}\mathbf{s} \\ \text{s.t.} \quad & \mathbf{q}^*(\mathbf{s}) = \min_{\mathbf{q}\in[0,\bar{q}]^n} \left\{ -\boldsymbol{\nu}(\mathbf{s})^\top \mathbf{q} + \frac{1}{2}\mathbf{q}^\top (\mathbf{I}_n + \rho\mathbf{B} - \varphi\mathbf{A})\,\mathbf{q} \right\}. \end{aligned} \tag{81}$$

The bilevel optimization problem of Equation (81) can be implemented in `MATLAB` following a two-stage procedure. First, one computes the Nash equilibrium output levels $\mathbf{q}^*(\mathbf{s})$ as a function of the subsidies $\mathbf{s}$ by solving a quadratic programming problem, for example using the `MATLAB` function `quadprog`, or the nonconvex quadratic programming problem solver with box constraints `QuadProgBB` introduced in Chen and Burer [2012].[80] Second, one can apply an optimization routine to this function calculating the subsidies which maximize net welfare $\overline{W}(G,\mathbf{s})$, for example using `MATLAB`'s function `fminsearch` (which uses a Nelder-Mead algorithm).

This bilevel optimization problem can be formulated more efficiently as a *mathematical programming problem with equilibrium constraints* (MPEC; see also Luo et al. [1996]). While in the above procedure the `quadprog` algorithm solves the quadratic problem with high accuracy for each iteration of the `fminsearch` routine, MPEC circumvents this problem by treating the equilibrium conditions as constraints. This method has recently been proposed to structural estimation problems following the seminal paper by Su and Judd [2012]. The MPEC approach can be implemented in `MATLAB` using a constrained optimization solver such as `fmincon`.[81]

Finally, to initialize the optimiziation algorithm we can use the theoretical optimal subsidies from Propositions 3 and 4, by setting the output levels of the firms which would produce at negative quantities under these policies to zero (if there are any), and then apply a bounded quadratic programming algorithm to determine the Nash equilibrium quantities under these subsidy policies.

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[80] However, in the data that we have analyzed in this paper the quadratic programming subproblem of determining the Nash equilibrium outptut levels always turned out to be convex, and therefore we always obtained a unique Nash equilibrium.

[81] Su and Judd [2012] further recommend to use the `KNITRO` version of `MATLAB`'s `fmincon` function to improve speed and accuracy.

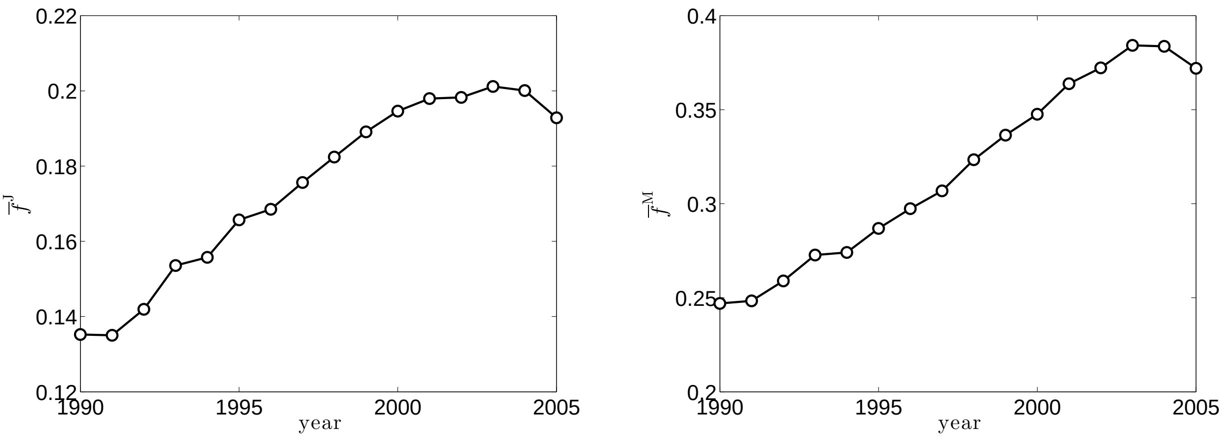

Figure H.2: The mean patent proximity across collaborating firms using the Jaffe metric f_{ij}^{J} of Equation (78) or the Mahalanobis metric f_{ij}^{M} of Equation (79).

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