The Optimal Use of Government Purchases for Stabilization

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May 12, 2016

Abstract

This paper explores whether and how government purchases should contribute to stabilization. The analysis relies on a matching model in which there is scope for stabilization because the unemployment rate may be inefficiently high or low. We find that optimal government purchases depart from the Samuelson [1954] level in proportion to the elasticity of substitution between public and private purchases, the unemployment gap, and the unemployment multiplier. The formula implies that optimal government purchases (i) remain at the Samuelson level if the unemployment rate is efficient, the multiplier is zero, or the elasticity of substitution is zero; (ii) completely fill the unemployment gap, as in Keynesian theory, if the elasticity of substitution is infinite; and (iii) deviate from the Samuelson level without completely filling the unemployment gap otherwise. Furthermore, we find that the optimal response of government purchases to an increase in unemployment is strongest for a moderate multiplier; the response is weaker with a higher multiplier because fewer government purchases are required to fill the unemployment gap, and with a lower multiplier because government purchases crowd out household purchases more. The results hold whether government purchases are financed with lump-sum or distortionary taxation.

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1. Introduction

Determining the optimal amount of government spending has always been an important question in public economics and macroeconomics. In the wake of the Great Recession, the question has become especially relevant. Indeed, with the zero lower bound constraining monetary policies worldwide, governments have turned away from their preferred stabilization tool—monetary policy—to alternative policies such as government purchases.

Several inroads have been made in answering the question. In public economics the literature has characterized the optimal provision of public goods and services when productive efficiency is respected. This literature has built upon the work of Samuelson [1954], who showed that in a productively efficient model, the optimal provision of public goods and services satisfies a simple formula: the marginal rate of substitution and marginal rate of transformation between public and private goods are equal. In macroeconomics the literature has measured the effects of government spending on economic activity—especially the output multiplier, defined as the change in GDP when government purchases increase by one dollar.[1]

In this paper we leverage these literatures to analyze the optimal provision of public goods and services when productive efficiency is not respected so that the economy may be too tight or too slack. Our objective is to propose a theory that could provide qualitative guidance to policymakers when they consider adjusting government purchases in response to unemployment fluctuations. We therefore develop an optimal government-purchases formula expressed with three estimable statistics: (i) the unemployment gap, which measures the gap between current and efficient unemployment rate; (ii) the unemployment multiplier, which measures the reduction in unemployment rate following an increase in government purchases by one percent of GDP ; and (iii) the elasticity of substitution between public and private services, which governs the reduction in the marginal rate of substitution between public and private services following an increase in government purchases. Government purchases are financed by taxes to balance the budget at all times, and we examine the effects of distortionary taxation on the formula.

To relax the assumption of productive efficiency made in the public-economics literature, we use a matching model for the analysis of optimal government purchases. The model embeds the public-good framework of Samuelson [1954] into the matching formalism developed by Michaillat

[1] See Kreiner and Verdelin [2012] and Ramey [2011] for excellent surveys of these literatures.

and Saez [2015].[2] There is one matching market where households sell private services to other households and public services to the government. Households derive utility from the private services that they purchase and from the public services purchased by the government. In equilibrium there is some unemployment: sellers are unable to sell all the labor services that they could produce. Absent any policy, productive efficiency is usually not respected in a matching model: the unemployment rate may be inefficiently low—when the price of services is too low and too much labor is devoted to purchasing labor services—or inefficiently high—when the price of services is too high and too much labor is idle.

In our theory, we take as given all other potential policies that could affect economic activity. If these policies can fully stabilize the economy, the unemployment rate is always efficient and the Samuelson formula applies. In practice these policies are subject to constraints (such as the zero lower bound for monetary policy) or have side effects that make it suboptimal to use them to stabilize the economy perfectly. When the economy is not perfectly stabilized, we need to add a correction term to the Samuelson formula to obtain our formula for optimal government purchases. The correction term measures the effect of government purchases on welfare through their influence on unemployment. The formula implies that optimal government purchases are above the Samuelson level if and only if government purchases bring unemployment closer to its efficient level; this occurs for example if unemployment is inefficiently high and government purchases lower unemployment.

To move beyond these general results, we rework our formula to express it with estimable statistics. We find that optimal government purchases depart from the Samuelson level in proportion to the elasticity of substitution between public and private services, the unemployment gap, and the unemployment multiplier. The unemployment gap determines the effect of unemployment on welfare. The multiplier determines the effect of government purchases on unemployment.

The formula highlights the role of the social value of government purchases. Stimulus skeptics often warn that additional government spending could be wasteful. Our formula qualifies their intuition: it is true that at the limit where the elasticity of substitution between public and private services is zero, so that public workers dig holes at the margin, government purchases should

[2] Other papers use matching models to address public-economics questions. See for instance Boone and Bovenberg [2002], Hungerbühler et al. [2006], and Lehmann, Parmentier and Van Der Linden [2011] on redistributive income taxation, Hungerbühler and Lehmann [2009] on the minimum wage, and Landais, Michaillat and Saez [2010*a*] on unemployment insurance.

remain at the Samuelson level and not be used for stabilization; but in the more realistic case where the elasticity is positive, government purchases should depart from the Samuelson level and be used to bring the unemployment rate closer to it efficient level. We also qualify the view of stimulus advocates in the Keynesian tradition who argue that government purchases should entirely fill the unemployment gap. It is true that at the limit where the elasticity of substitution is infinite, so that public services are perfect substitute for private services, government purchases should completely fill the unemployment gap; but in the more realistic case where the elasticity is finite, government purchases should bring the unemployment rate closer to its efficient level without filling the unemployment gap entirely.

The formula also shows that it is the sign, not the level, of the unemployment multiplier that determines the optimal cyclicality of government purchases. When the unemployment rate is inefficiently high, it is optimal to increase government purchases above the Samuelson level as soon as the multiplier is positive, even though government purchases crowd out household purchases. Whether the output multiplier, which is the same as the unemployment multiplier with lump-sum taxes, is above or below unity plays no role for the optimal cyclicality of government purchases.

Another concern of stimulus skeptics is that more government purchases require higher taxes, which could depress output through labor-supply responses. To examine this, we extend our model and introduce endogenous labor supply and distortionary taxation. Our main result carries over in the two situations that we consider.

The first situation corresponds to the classical view in public economics, which is also followed in macroeconomics: distortionary taxation adds a marginal cost of public fund. Our formula remains valid as long as the Samuelson level of government purchases is reduced to account for the marginal cost of public fund. The relevant multiplier remains the unemployment multiplier, but this multiplier is no longer equal to the output multiplier. With distortionary taxation, the output multiplier is smaller than the unemployment multiplier because taxes depress output through supply-side responses, but not the unemployment rate. In fact the unemployment multiplier equals the output multiplier net of supply-side responses. This output multiplier net of supply-side responses can be estimated when government purchases are financed with debt. Hence, when the output multiplier is negative because of strong supply-side responses (as in Barro and Redlick [2011]), it remains optimal to increase government purchases above the Samuelson level when unemployment is too high.

The second situation corresponds to the modern view in the public economics literature: public spending is funded at the margin according to the benefits principle [Kaplow, 1996, 2004]. In that case, there are no supply-side responses at the margin so the Samuelson level remains the same as with lump-sum taxes and the unemployment multiplier remains equal to the output multiplier.

To gain further insights, we explicitly relate the departure of optimal government purchases from the Samuelson level to the *initial* unemployment shock and other estimable statistics. This explicit formula clarifies the relationship between the multiplier and the optimal response of government purchases to unemployment fluctuations. Stimulus advocates usually believe in large multipliers and argue that they justify large responses of government purchases in recessions; conversely, stimulus skeptics usually believe in small multipliers and argue that they justify small or no responses of government purchases in recessions. In fact, our formula shows that the relation between the multiplier and the increase in government purchases for a given increase in unemployment has a hump shape. For small multipliers, the optimal amount of government purchases is determined by the crowding out of household purchases by government purchases; a higher multiplier implies less crowding out and higher optimal government purchases. For large multipliers, it is optimal to fill the unemployment gap; a higher multiplier implies that fewer government purchases are required to fill the gap.

Our paper complements a small literature that studies the optimal use of government purchases for stabilization. Mankiw and Weinzierl [2011] study the question in a disequilibrium model.[3] In their model there is no crowding out of household's consumption by government consumption—the output multiplier is one—so the optimal government purchases entirely fill the output gap. We complement this paper by using a matching model and thus allowing for crowding out; we find that it is usually optimal to only partially fill the output gap. Woodford [2011, sections 5 and 6] and Werning [2012, section 6] study the question at the zero lower bound in New Keynesian models. They find that government purchases should be used for stabilization beyond the Samuelson level when the economy is too slack. We complement these papers in two ways: by showing that this general principle holds not only at the zero lower bound but any time that the economy is not perfectly stabilized; and by describing the optimal policy in terms of estimable statistics.

[3] Roberts [1982] also analyzes optimal government purchases in a disequilibrium model.

2. A Matching Model for the Analysis of Government Purchases

This section presents a matching model of the economy. The model uses the formalism developed by Michaillat and Saez [2015]. There is unemployment that may be inefficiently high or low. Unemployment is inefficient when the price of consumption relative to the asset used for saving is not at the efficient level. Monetary policy usually affects this relative price, but there are many reasons why monetary policy cannot perfectly control the relative price and perfectly stabilize the economy. Throughout the paper we take monetary policy as given and simply assume that stabilization may not be perfect.

There are two goods in the model: private services, purchased by households, and public services, purchases by the government. All the services are produced by self-employed households. Government purchases are valuable to households, as in Samuelson [1954]. They also contribute to macroeconomic activity because they affect the aggregate demand through several channels.

2.1. The Supply Structure

The model is dynamic and set in continuous time. The economy consists of a government and a measure 1 of identical households. Households are self-employed, producing and selling services on a matching market.[4] There are two types of services: private services and public services. Public and private services are produced using the same technology. The government purchases public services and households purchases private services. All services are sold on the same market at the same price.

Each household has a productive capacity of k; the productive capacity indicates the maximum amount of services that a household could sell at any point in time.[5] At time t, households sell $C(t)$ services to other households and $G(t)$ services to the government. The output of services is the sum of household purchases and government purchases:

$$Y(t) = C(t) + G(t).$$

[4] We assume that households cannot consume their own labor services. For simplicity, we abstract from firms and assume that all production directly takes place within households. Michaillat and Saez [2015] show how the model can be modified to include firms hiring workers on a labor market and selling their production on a product market.

[5] Here the capacity is exogenous. In Section 4 we assume that households choose the capacity to maximize their utility, and we study how taxation affects this decision.

The matching process prevents households from selling their entire capacity so $Y(t) < k$.

The services are sold through long-term relationships. The relationships separate at rate $s > 0$. Since $Y(t)$ services are committed to existing relationships at time t, the idle capacity of households is $k - Y(t)$. Since part of households' capacity is idle, there are some workers who are idle. The rate of unemployment, defined as the share of workers who are idle, is $u(t) = 1 - Y(t)/k$.

To purchase labor services, households and government advertise $v(t)$ vacancies at time t. The rate at which new long-term relationships are formed is given by a Cobb-Douglas matching function: $h(t) = \omega \cdot (k - Y(t))^{\eta} \cdot v(t)^{1-\eta}$, where $k - Y(t)$ is aggregate idle capacity, $v(t)$ is aggregate number of vacancies, $\omega > 0$ governs the efficacy of matching, and $\eta \in (0,1)$ is the elasticity of the matching function with respect to idle capacity.[6]

The market tightness is $x(t) \equiv v(t)/(k - Y(t))$. The market tightness is the ratio of the two arguments in the matching function: aggregate vacancies and aggregate idle capacity. With constant returns to scale in matching, the tightness determines the rates at which sellers and buyers enter into new long-term trading relationships. At time t, each of the $k - Y(t)$ units of idle productive capacity is sold at rate $f(x(t)) = h(t)/(k - Y(t)) = \omega \cdot x(t)^{1-\eta}$ and each of the $v(t)$ vacancies is filled at rate $q(x(t)) = h(t)/v(t) = \omega \cdot x(t)^{-\eta}$. The selling rate $f(x)$ is increasing in x and the buying rate $q(x)$ is decreasing in x; hence, when the tightness is higher, it is easier to sell services but harder to buy them. A useful result is that $f(x) = q(x) \cdot x$.

Technically, output is a state variable with law of motion $\dot{Y}(t) = f(x(t)) \cdot (k - Y(t)) - s \cdot Y(t)$. The term $f(x(t)) \cdot (k - Y(t))$ is the number of new relationships formed at t. The term $s \cdot Y(t)$ is the number of existing relationships separated at t. If $f(x)$ and s remain constant over time, output converges to the steady-state level

$$Y(x,k) = \frac{f(x)}{f(x) + s} \cdot k. \tag{1}$$

In practice, output reaches this steady-state level quickly because market flows are large. Throughout the paper, we therefore simplify the analysis by modeling output as a jump variable equal to its steady-state value defined by (1). The function $Y(x,k)$ is in $[0,k]$, increasing in x and k $[0,+\infty)$, with $Y(0,k) = 0$ and $\lim_{x\to+\infty} Y(x,k) = k$. The elasticity of $Y(x,k)$ with respect to x is $(1-\eta) \cdot u(x)$. Intuitively, when the market tightness is higher, it is easier to sell services so output is higher.

[6] The empirical evidence summarized by Petrongolo and Pissarides [2001] indicates that a Cobb-Douglas specification for the matching function fits the data well.

The unemployment rate is directly related to output: $u = 1 - Y/k$. Hence, the simplification also implies that the unemployment rate is given by

$$u(x) = \frac{s}{s + f(x)}. \tag{2}$$

The function $u(x)$ is in $[0,1]$, decreasing in x, with $u(0) = 1$ and $\lim_{x\to+\infty} u(x) = 0$. The elasticity of $u(x)$ with respect to x is $-(1-\eta)\cdot(1-u(x))$. Intuitively, when the market tightness is higher, it is easier to sell services so the unemployment rate is lower. In Appendix A, we show that in US data, the unemployment rate given by (2) and the actual employment rate are indistinguishable.[7]

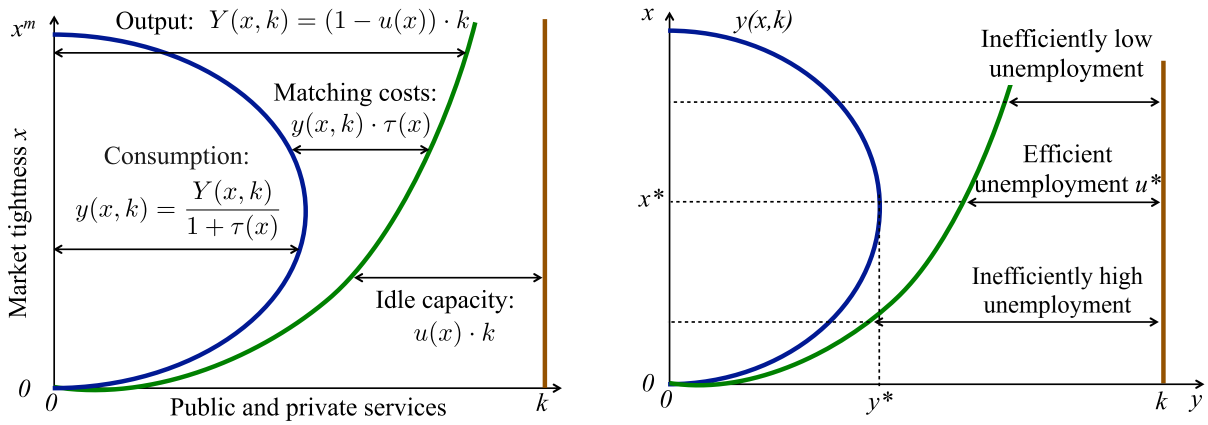
Advertising vacancies is costly. Posting one vacancy costs $\rho > 0$ services per unit time. Hence, a total of $\rho \cdot v(t)$ services are spent at time t on filling vacancies. These services represent the resources devoted by households and government to matching with appropriate providers of services. These resources devoted to matching are not consumed by households and do not enter their utility function. Hence, the consumption of private services, denoted by $c(t)$, equals household purchases net of the matching costs incurred by households, the consumption of public services, denoted by $g(t)$, equals government purchases net of the matching costs incurred by the government, and total consumption, denoted by $y(t) = c(t) + g(t)$, equals total output net of total matching costs.

Equation (1) and the definition of the market tightness imply that $s \cdot Y(t) = f(x(t)) \cdot (k - Y(t)) = q(x(t)) \cdot v(t)$. Hence, $y(t) = Y(t) - \rho \cdot v(t) = Y(t) - \rho \cdot s \cdot Y(t)/q(x(t))$. Consequently, $Y(t) = y(t) \cdot [1 + \tau(x(t))]$, where

$$\tau(x) \equiv \frac{\rho \cdot s}{q(x) - \rho \cdot s}. \tag{3}$$

Hence, consuming one service requires to purchase $1 + \tau$ services—one service for consumption plus τ services for matching. The matching wedge $\tau(x)$ is positive and increasing on $[0, x^m)$, where $x^m \in (0, +\infty)$ is defined by $q(x^m) = \rho \cdot s$. In addition, $\lim_{x\to x^m} \tau(x) = +\infty$. Intuitively, when the market tightness is higher, it is more difficult to match with a seller so the matching wedge is higher. The elasticity of $\tau(x)$ with respect to x is $\eta \cdot (1 + \tau(x))$. The logic implies that private consumption is related to household purchases by $c(t) = C(t)/[1 + \tau(x(t))]$ and public consumption to government purchases by $g(t) = G(t)/[1 + \tau(x(t))]$.

[7] We follow the approach of Hall [2005*a*] and Pissarides [2009], who simplify their analysis by ignoring the transitional dynamics of unemployment and assuming that unemployment is a jump variable that depends on tightness according to (2). Hall [2005*b*] provides empirical evidence showing that this simplification is accurate.



A. The supply structure B. Efficient and inefficient unemployment rates

Figure 1: The Matching Market

It is useful to write total consumption as a function of market tightness and capacity:

$$y(x,k) = \frac{1-u(x)}{1+\tau(x)} \cdot k. \tag{4}$$

This function $y(x,k)$ plays a central role in the analysis because it gives the amount of services that can be allocated to consumption for a given tightness. The expression (4) shows that consumption is below the capacity k because some services are not sold in equilibrium ($u(x) > 0$) and some services are used for matching instead of consumption ($\tau(x) > 0$). The function $y(x,k)$ is defined for $x \in [0, x^m]$ and $k \geq 0$, positive, with $y(0,k) = 0$ and $y(x^m,k) = 0$. The elasticity of $y(x,k)$ with respect to x is $(1-\eta) \cdot u(x) - \eta \cdot \tau(x)$. Hence, the elasticity of $y(x,k)$ is $1-\eta$ at $x = 0$ and $-\infty$ at $x = x^m$, and it is strictly decreasing in x. Therefore, the function $y(x,k)$ is strictly increasing for $x \leq x^*$ and strictly decreasing for $x \geq x^*$, where the tightness x^* is implicitly defined by $\partial y / \partial x|_k = 0$. The function $y(x,k)$ is therefore maximized at $x = x^*$.

DEFINITION 1. *The tightness and unemployment rate are* efficient *if they maximize total consumption y given the capacity k. We denote by x^* and u^* the efficient tightness and unemployment rate. The efficient tightness is characterized by*

$$\left.\frac{\partial y}{\partial x}\right|_k = 0 \qquad or \qquad (1-\eta) \cdot u(x^*) - \eta \cdot \tau(x^*) = 0. \tag{5}$$

The efficient unemployment rate is characterized by $u^* = u(x^*)$. The efficient tightness is the tight-

ness underlying the efficiency condition of Hosios [1990].

Figure 1 summarizes the model. Panel A depicts how consumption, output, and unemployment rate depend on market tightness in feasible allocations. Panel B depicts the function $y(x,k)$, the efficient tightness x^*, the efficient unemployment rate u^*, and situations in which tightness is inefficiently high and unemployment is inefficiently low ($x > x^*$, $u < u^*$), and situations in which tightness is inefficiently low and unemployment is inefficiently high ($x < x^*$, $u > u^*$). When unemployment is inefficiently high, too much of the economy's productive capacity is idle, and a marginal decrease in unemployment increases consumption. When unemployment is inefficiently low, too many resources are devoted to purchasing labor services, and a further decrease in unemployment reduces consumption.[8]

2.2. The Demand Structure and the Nature of the Equilibrium

We first present several concrete examples and then describe a generic demand structure. Following the methodology pioneered by Chetty [2006] in the case of optimal unemployment insurance, we will express the formula with sufficient statistics that summarize the relevant features of the demand structure. The statistics are *sufficient* in that they apply to a broad range of specifications of utility, price mechanism, and demand structure.[9] Our analysis will therefore apply to various demand structures for a given supply structure.

To introduce an aggregate demand in the model, we need to introduce an asset that is in fixed supply and assume that households derive utility from holding this asset. Thus households spend part of their labor income on services and save part of it using the asset. Household hold the asset to smooth consumption and because they derive utility from holding it. We present one illustrative

[8] In our model output, Y, is proportional to the employment rate, $1-u$; hence, when output is 1 percent below trend, the employment rate is 1 percent below trend and the unemployment rate is slightly less than 1 percentage point above trend. This property seemingly contradicts Okun's law; Okun [1963] found that in US data for 1954–1962, output was 3 percent below trend when the unemployment rate was 1 percentage point above trend. The relationship between output gap and unemployment gap has evolved over time, however. In Appendix B, we estimate Okun's law for the entire 1951–2014 period and for the recent 1994–2014 period. We find that when the unemployment rate is 1 percentage point above trend, output is 1.8 percent below trend in the 1951–2014 period and 1.3 percent below trend in the 1994–2014 period.

[9] Sufficient statistics provide conceptual clarity and a link between theory and empirics. See Chetty [2009] for a discussion of the advantages of using sufficient statistics. See Farhi and Gabaix [2015] for a recent paper using the approach described here. In their theory of public economics with behavioral agents, Farhi and Gabaix consider a specific supply structure, borrowed from price theory, but a general demand structure that can capture many different behavioral biases (for instance, misperception of taxes, bounded rationality, hyperbolic discounting, projection bias). They obtain a formula expressed with sufficient statistics that summarize different biases.

model in which the asset are nominal bonds. We then show that the demand structure would remain similar if we replaced nominal bonds by other assets such as real bonds, money, or land.

In all the examples we assume that the government's budget is balanced at all times using a lump-sum tax $T(t) = G(t)$ levied on households.[10]

An Example: Nominal Bonds. This demand structure is borrowed from Michaillat and Saez [2014]. The asset that we introduce are nominal bonds, which are traded on a perfectly competitive market. They are issued and purchased by households, and they have a price of 1 in terms of money. Here money only plays the role of a unit of account. The rate of return is the nominal interest rate $i(t)$. At time t, a household holds $B(t)$ bonds. Bonds are in zero net supply, so in equilibrium, the bond market clears and $B(t) = 0$. The nominal interest rate is determined by the central bank, which sets an interest rate $i(t) = i(g(t))$. We allow the interest rate to depend on public consumption to allow for an interaction between monetary and fiscal policy.

The price of services in terms of money is $p(t)$. The inflation rate is $\pi(t) \equiv \dot{p}(t)/p(t)$. In the economy there are two goods—services and bonds—and hence one relative price (public and private services have the same price). The price of bonds relative to services is determined by the real interest rate, $r(t) = i(t) - \pi(t)$. Once the nominal interest rate, $i(t)$, is determined by the central bank, it is the inflation rate, $\pi(t)$, that determines the real interest rate. On a Walrasian market, the inflation rate would be determined such that supply equals demand on the market for services. On a matching market, things are different: we specify a general price mechanism that determines the inflation rate: $\pi(t) = \pi(g(t))$.[11] Given the inflation rate, the price of services moves according to $\dot{p}(t) = \pi(t) \cdot p(t)$. The initial price $p(0)$ is given. Given the inflation rate and nominal interest rate, the market tightness adjusts such that supply equals demand on the market for services.[12]

The law of motion of the representative household's real wealth $b(t) \equiv B(t)/p(t)$ is

$$\dot{b}(t) = (1 - u(x(t))) \cdot k - (1 + \tau(x(t))) \cdot c(t) + (i(t) - \pi(t)) \cdot b(t) - T(t). \tag{6}$$

Here, $b(t)$ is the household's real wealth, $(1 - u(x(t))) \cdot k$ is labor income, $c(t)$ is private consump-

[10] When households are Ricardian—in the sense that they do not view government bonds as net wealth because such bonds need to be repaid with taxes later on—financing government purchases with debt is economically equivalent to maintaining budget balance using a lump-sum tax [Barro, 1974]. Hence, our analysis would remain valid if the government used debt financing and households were Ricardian.

[11] In equilibrium all variables at time t depends solely on $g(t)$ so this is the most general mechanism possible.

[12] See the discussion in Michaillat and Saez [2015].

tion, $(1+\tau(x(t)))\cdot c(t)$ is household purchases, $i(t)-\pi(t)$ is the real interest rate, and $T(t)$ is the lump-sum tax paid to the government.

The household derives utility from consuming $c(t)$ private services and $g(t)$ public services, and from holding $b(t)$ units of real wealth. Its instantaneous utility function is separable: $\mathscr{U}(c(t),g(t))+\mathscr{V}(b(t))$. The function $\mathscr{U}$ is strictly increasing in its two arguments and concave. The function $\mathscr{V}$ is strictly increasing and concave. Assuming utility for real wealth is a simple way to introduce an aggregate demand in the economy.[13] The household's utility at time 0 is

$$\int_0^{+\infty} e^{-\delta\cdot t}\cdot\left[\mathscr{U}(c(t),g(t))+\mathscr{V}(b(t))\right]dt, \tag{7}$$

where $\delta>0$ is the subjective discount rate.

The household takes $b(0)=0$ and the paths of $x(t)$, $g(t)$, $i(t)$, $\pi(t)$, and $T(t)$ as given. It chooses the paths of $c(t)$ and $b(t)$ to maximize (7) subject to (6) and to a standard no-Ponzi condition. To solve the household's problem, we set up the current-value Hamiltonian:

$$\begin{aligned}\mathscr{H}(t,c(t),b(t)) &= \mathscr{U}(c(t),g(t))+\mathscr{V}(b(t))\\ &\quad +\lambda(t)\cdot\left[(1-u(x(t)))\cdot k-(1+\tau(x(t)))\cdot c(t)+(i(t)-\pi(t))\cdot b(t)-T(t)\right]\end{aligned}$$

with control variable $c(t)$, state variable $b(t)$, and current-value costate variable $\lambda(t)$. The necessary conditions for an interior solution to this maximization problem are $\partial\mathscr{H}/\partial c=0$, $\partial\mathscr{H}/\partial b=\delta\cdot\lambda(t)-\dot{\lambda}(t)$, and the appropriate transversality condition. The first-order conditions imply

$$\frac{\partial\mathscr{U}}{\partial c}(c(t),g(t))=\lambda(t)\cdot(1+\tau(x(t))) \tag{8}$$

$$\mathscr{V}'(b(t))=(\delta-i(t)+\pi(t))\cdot\lambda(t)-\dot{\lambda}(t). \tag{9}$$

We can now describe the equilibrium for a given public consumption, g. An equilibrium consists of paths for $[x(t), c(t), y(t), b(t), i(t), \pi(t), \lambda(t)]_{t=0}^{\infty}$ that satisfy $c(t)+g=y(t)$, $y(t)=y(x(t))$, $b(t)=0$, $i(t)=i(g)$, $\pi(t)=\pi(g)$, and equations (8) and (9). The first condition is the resource constraint, the second the equality of supply and demand on the market for services, the third the

[13] Other papers that include real wealth in the utility function include Kurz [1968] (in a growth model), Carroll [2000] (in a life-cycle model), Bakshi and Chen [1996] (in an asset-pricing model), and Michaillat and Saez [2014] (in a business-cycle model). See Michaillat and Saez [2014] for more details and some evidence in favor of this assumption.

equality of supply and demand on the market for bonds, the fourth monetary policy, the fifth the price mechanism on the market for services, and the sixth and seventh the first-order conditions of the household's utility-maximization problem.

The equilibrium can be represented as a dynamical system of dimension 1. The only variable in the system is the costate variable $\lambda(t)$. All the other variables can be recovered from $\lambda(t)$. The variable $\lambda(t)$ satisfies the following differential equation:

$$\dot{\lambda}(t) = (\delta - i + \pi) \cdot \lambda(t) - \mathscr{V}'(0).$$

Note that the real interest rate $r = i - \pi$ is constant over time. The steady-state value of the costate variable is $\lambda = \mathscr{V}'(0)/(\delta - i + \pi) > 0$. Since $\delta - (i - \pi) > 0$, we infer that the dynamical system is a source. As there is no state variable, our source system jumps from one steady state to the other in response to unexpected permanent shocks, and there are no transitional dynamics.

For a given g, the equilibrium is always in steady state. The welfare can just be measured by the instantaneous welfare, $\mathscr{U}(c,g) + \mathscr{V}(0)$. For the policy analysis, we need to relate c to g. As we have $c = y(x,k) - g$, we need to relate x to g. We can do that using the property that supply equals demand on the market for services.

In steady state, the desired amount of private consumption is

$$\frac{\partial \mathscr{U}}{\partial c}(c,g) = \frac{(1 + \tau(x)) \cdot \mathscr{V}'(0)}{\delta - i + \pi}. \tag{10}$$

This equation is the usual consumption Euler equation modified by the utility of wealth and evaluated in steady state. The demand for saving arises in part from intertemporal consumption-smoothing considerations and in part from the utility provided by wealth. The equation implies that at the margin, the household is indifferent between consuming and holding real wealth. The equation determines the level of aggregate demand.

Let $c(x,g,i,\pi)$ be the private consumption implicitly defined by (10). The constraint that $y = y(x,k)$ can be written as

$$c(x,g,i(g),\pi(g)) + g = y(x,k), \tag{11}$$

which determines equilibrium tightness $x(g)$. The objective of the government will be to maximize $\mathscr{U}(y(x(g),k) - g, g)$ where $x(g)$ is the tightness defined by (11).

Other Examples. We could build other examples that yield similar demand structures.

For instance, we could replace nominal bonds by real bonds. In that case the model would be real instead of nominal. The real interest would be determined on the market for services. The price mechanism determining the real interest rate would impose $r(t) = r(g(t))$. All the results from the example with nominal bonds obtain with real bonds, once $i(t) - \pi(t)$ is replaced by $r(t)$.

Another way to introduce aggregate demand would be to replace nominal bonds with land. The supply of land is fixed at l_0, and households derive utility from holding $l(t)$ units of land.[14] A justification for deriving utility form land is that land provides a number of services, such as natural resources or housing. Land is traded on a perfectly competitive market. The price of services in terms of land is $p(t)$. We specify a general price mechanism that determines the price of services: $p(t) = p(g(t))$. The law of motion of the household's holding of land is $\dot{l}(t) = p(t) \cdot (1 - u(x(t))) \cdot k - p(t) \cdot (1 + \tau(x(t))) \cdot c(t) - T(t)$. The properties of the model are similar. As earlier, the equilibrium immediately converges to steady state. In steady state the desired private consumption $c(x, g, p)$ is given by $\partial \mathscr{U} / \partial c = p \cdot (1 + \tau(x)) \cdot \mathscr{V}'(l_0)/\delta$. In that case, equilibrium tightness $x(g)$ is implicitly defined by $c(x, g, p(g)) + g = y(x, k)$.

Finally, a classical way to introduce an aggregate demand is to use money as an asset and assume that households derive utility from holding real money balances.[15] A justification for deriving utility from money is that money is useful to conduct transactions. The supply of money is fixed at M_0. At time t, a household holds $M(t)$ units of money. In equilibrium, the money market clears and $M(t) = M_0$. The price of services in terms of money is $p(t)$. We specify a general price mechanism that determines the price of services: $p(t) = p(g(t))$. The law of motion of the household's real wealth $m(t) \equiv M(t)/p(t)$ simply is $\dot{m}(t) = (1 - u(x(t))) \cdot k - (1 + \tau(x(t))) \cdot c(t) - \pi(t) \cdot m(t) - T(t)$. The properties of the model are similar. As earlier, the equilibrium immediately converges to steady state. In steady state the desired private consumption $c(x, g, p)$ is given by $\partial \mathscr{U} / \partial c = (1 + \tau(x)) \cdot \mathscr{V}'(M_o/p)/\delta$. Equilibrium tightness $x(g)$ is implicitly defined by $c(x, g, p(g)) + g = y(x, k)$.

[14] Other models in which land is an asset that enters households' utility function include Kocherlakota [2013], Liu, Wang and Zha [2013], and Iacoviello [2005].

[15] Following the work of Sidrauski [1967], many business-cycle models with money in the utility function have been developed. Two prominent examples are Barro and Grossman [1971] and Blanchard and Kiyotaki [1987].

Summary. In these four examples, the equilibrium immediately converges to steady state. In the steady-state equilibrium, tightness is a function of the provision of public services: $x = x(g)$. The tightness function $x(g)$ is determined by households' utility function, the price mechanism, and the supply structure. It summarizes everything that the government needs to know about the demand structure in the economy. With knowledge of the tightness function $x(g)$ and the supply structure of the economy, the government will determine the optimal provision of public services. The optimal g maximizes $\mathscr{U}(c,g) = \mathscr{U}(y(x(g),k) - g, g)$. The utility over the asset does not enter welfare because the asset is in fixed supply.

In general the price mechanism on the matching market does not guarantee efficiency. With random search, prices are determined in a situation of bilateral monopoly, so there are no market forces that ensure that the inflation rate, the real interest rate, or the price of services relative to land or money maintain the tightness $x(g)$ at x^* (see the discussion in [Pissarides, 2000, Chapter 8]). With directed search, market forces would ensure that prices adjust to maintain the tightness at x^*. But as soon as price-adjustment costs are combined with directed search, as proposed by Michaillat and Saez [2016], prices adjust sluggishly and the tightness usually departs from x^*. Since the price mechanism does not generally maintain the tightness at its efficient level on a matching market, policies are useful to stabilize the economy—that is, bring tightness closer to its efficient level.

Other policies could stabilize the economy, but we will focus on situations where these policies are unable to keep the tightness at its efficient level.[16] In our main example monetary policy could adjust the nominal interest rate to bring the economy to efficiency, but constraints such as the zero lower bound on nominal interest rates may prevent monetary policy to achieve this goal perfectly. In the other examples taxes could affect relative prices and bring the economy closer to efficiency, but the taxes may be difficult to implement or have costs not modeled here that make it suboptimal to bring the tightness to its efficient level.

3. Optimal Government-Purchases Formulas

This section describes optimal government purchases in the model of Section 2. We obtain several formulas that determine the optimal level of government purchases implicitly, and a formula that

[16] We take the same approach as Eggertsson and Woodford [2006] and Farhi and Werning [2013]. Eggertsson and Woodford study optimal monetary policy taking fiscal policy as given and assuming that fiscal policy fails to eliminate the decline in interest rate that led to a liquidity trap. Farhi and Werning study macroprudential policies assuming that monetary policy and tax instruments are unable to undo price rigidities and achieve the first-best allocation.

does so explicitly. These formulas provide qualitative principles about the optimal level of government purchases when the unemployment rate may not be efficient. In this section government purchases are financed by nondistortionary taxes; in Section 4 we will introduce distortionary taxes and revisit the analysis. In this section the insights are qualitative; in Section 5 we will calibrate the formulas to assess the order of magnitude of the optimal response of government purchases to realistic unemployment fluctuations.

3.1. Implicit Formulas

The welfare of an equilibrium is $\mathcal{U}(c,g)$. We assume that the utility function $\mathcal{U}$ is strictly increasing in c and g, concave, and homothetic. Since $c = y(x,k) - g$, welfare can be written as $\mathcal{U}(y(x,k)-g,g)$. Given a tightness function $x(g)$, the government chooses g to maximize $\mathcal{U}(y(x(g),k)-g,g)$. We assume that the welfare function $g \mapsto \mathcal{U}(y(x(g),k)-g,g)$ is well behaved: it admits a unique extremum and the extremum is a maximum.[17] Under this assumption, first-order conditions are not only necessary but also sufficient to describe the optimum of the government's problem.

The first-order condition of the government's problem is

$$0 = \frac{\partial \mathcal{U}}{\partial g} - \frac{\partial \mathcal{U}}{\partial c} + \frac{\partial \mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}. \tag{12}$$

Reshuffling the terms in the optimality condition and dividing the condition by $\partial \mathcal{U}/\partial c$ yields the formula for optimal government purchases:

Proposition 1. *In the model described in Section 2 optimal government purchases satisfy*

$$1 = MRS_{gc} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}, \tag{13}$$

where

$$MRS_{gc} \equiv \frac{\partial \mathcal{U}/\partial g}{\partial \mathcal{U}/\partial c}$$

is the marginal rate of substitution between public and private services.

[17] We know that $x \mapsto y(x,k)$ has a unique extremum and this extremum is a maximum. Furthermore, $\mathcal{U}$ is concave. Therefore, we need $g \mapsto x(g)$ to be well behaved in order for $g \mapsto \mathcal{U}(y(x(g),k)-g,g)$ to satisfy the assumption.

As is standard in optimal tax formulas, formula (13) characterizes the optimal level of government purchases implicitly. If the formula holds, then g maximizes welfare. If the right-hand side of (13) is above 1, a marginal increase in g raises welfare; conversely, if the right-hand side is below 1, a marginal increase in g reduces welfare.

The following lemma is useful for the analysis of formula (13).

LEMMA 1. *Since $\mathscr{U}$ is concave and homothetic, the marginal rate of substitution MRS_{gc} is a decreasing function of $g/c = G/C$.*

Proof. Since $\mathscr{U}$ is homothetic, it can be written $\mathscr{U}(c,g) = \mathscr{N}(n(c,g)))$ where the function $\mathscr{N}$ is increasing and the function n is strictly increasing in c and g, concave, and homogeneous of degree 1. Since n is homogeneous of degree 1, its partial derivatives $\partial n/\partial c$ and $\partial n/\partial g$ are homogeneous of degree 0. Combining these properties, we find

$$MRS_{gc} = \frac{\frac{\partial n}{\partial g}(c,g) \cdot \mathscr{N}'(w(c,g)))}{\frac{\partial n}{\partial c}(c,g) \cdot \mathscr{N}'(w(c,g)))} = \frac{\frac{\partial n}{\partial g}\left(1, \frac{g}{c}\right)}{\frac{\partial n}{\partial c}\left(\frac{c}{g}, 1\right)}.$$

We see that MRS_{gc} is a function of g/c. Furthermore, as n is strictly increasing in c and g, $\partial n/\partial g > 0$ and $\partial n/\partial c > 0$. And as n is concave, $\partial n/\partial g$ is decreasing in its second argument while $\partial n/\partial c$ is decreasing in its first argument. We conclude that MRS_{gc} is decreasing in $g/c = G/C$. □

The following definition introduces the classical formula of Samuelson [1954] for the provision of public services, which we use as a benchmark in the analysis of formula (13).

DEFINITION 2. *In the model described in Section 2, the* Samuelson formula *is*

$$1 = MRS_{gc}. \tag{14}$$

It imposes that the marginal utilities from public and private services be equal. The Samuelson levels *of G/C and $G/Y = (G/C)/(1+G/C)$, denoted by $(G/C)^* > 0$ and $(G/Y)^* > 0$, are the levels satisfying the Samuelson formula.*

Proposition 1 establishes that in a matching model the Samuelson formula is not valid as is but needs to be corrected. The optimal government-purchases formula in a matching model, given by (13), adds the correction $(\partial y/\partial x) \cdot (dx/dg)$ to the Samuelson formula. The correction term is the product of the effect of government purchases on tightness, dx/dg, and the effect of tightness on consumption, $\partial y/\partial x$; it is positive if and only if increasing government purchases raises total consumption. Given the relation between consumption and unemployment rate (Figure 1), the correction term is positive when an increase in government purchases brings unemployment toward its efficient level.

The correction term can be traced back to equation (12). An increase in government purchases affects welfare through three channels: (1) it raises public consumption (the first term in (12)); (2) for a given amount of total consumption, it reduces one-for-one private consumption (the second term in (12)); and (3) it affects the amount of economic activity and thus total consumption (the third term in (12)). The correction term measures the effect of government purchases on welfare through the third channel.

The third channel only operates when a change in economy activity has a first-order effect on welfare—that is, when productive efficiency is not satisfied. In a neoclassical model the third channel would not operate as productive efficiency is satisfied. Higher output leads to higher consumption and lower leisure. Higher consumption generates a first-order welfare gain and lower leisure a first-order welfare loss. The loss exactly offsets the gain since marginal product of labor and marginal rate of substitution between leisure and private consumption are equal. Hence, changes in output have no first-order effect on welfare. In a matching model productive efficiency may not be satisfied—it is only satisfied for $u = u^*$. Hence, the third channel usually operates.

Formula (13) describes when optimal government purchases are above and below the Samuelson level. Given that MRS_{gc} is decreasing in G/C and $G/Y = (G/C)/(1+G/C)$, the ratio of government purchases to output, G/Y, should be at the Samuelson level if the correction term is zero, above it if the correction term is positive, and below it if the correction term is negative. Hence, the government-purchases ratio should be at the Samuelson level when the unemployment rate is efficient and when government purchases have no effect on unemployment. When the unemployment rate is inefficient and government purchases have an effect on unemployment, however, the Samuelson formula is invalid. Government purchases should depart from the Samuelson level to push unemployment toward its efficient level.

Sometimes the provision of public services is justified even if public services are not as valuable as private services. Imagine that one public service is not valued as much as one private service: $\partial \mathscr{U}/\partial g < \partial \mathscr{U}/\partial c$ or $MRS_{gc} < 1$ for any (c, g). In that case the Samuelson formula implies no public services. But in a matching model, it may be optimal to provide some public services for stabilization. This happens when the correction term evaluated at $g = 0$ is greater than $1 - MRS_{gc}(0)$. This requires a large effect of public consumption on tightness and of tightness on total consumption.

The structure of our formula—the Samuelson formula plus a correction term arising from stabilization motives—is similar to the structure of the formulas obtained by Woodford [2011, equation (45)], Nakamura and Steinsson [2014, equation (27)], and Farhi and Werning [2012, equation (27)] in other models. The structure—a formula from public economics plus a correction capturing stabilization motives—is also similar to the structure of the optimal unemployment-insurance formulas derived by Landais, Michaillat and Saez [2010*a*, equation (23)] and Landais, Michaillat and Saez [2010*b*, equation (10)] in matching models of the labor market.

Formula (13) is useful to describe the economic forces at play, but it remains abstract. We rework the formula to express it with estimable statistics. We proceed in three steps: we work first on $1 - MRS_{gc}$, then on $\partial y / \partial x$, and finally on dx/dg.

Definition 3. *The* elasticity of substitution between public and private services ε *is defined by*

$$\frac{1}{\varepsilon} = -\frac{d\ln(MRS_{gc})}{d\ln(G/C)}. \tag{15}$$

Throughout we refer to the elasticity evaluated at $(G/C)^*$.

The elasticity of substitution is positive: $\varepsilon > 0$. At the limit $\varepsilon \to 0$, public services provided by the government are perfect complement for the private services purchased by households on the market. This means that a certain number of public services are needed for an economy of a given size, but beyond that number, additional public services have zero value—additional public workers "dig holes". A utility function with $\varepsilon = 0$ is the Leontief utility function $\mathscr{U}(c,g) = \min\{(1-\gamma)\cdot c, \gamma \cdot g\}$—in which case the Samuelson level is $(G/C)^* = (1-\gamma)/\gamma$. At the limit $\varepsilon \to +\infty$, the public services provided by the government are perfect substitutes for the private services purchased by households on the market. This means that households are equally happy if a unit of services is used for private or public services. A utility function with $\varepsilon \to +\infty$ is the linear utility function $\mathscr{U}(c,g) = c + g$—in which case any ratio G/C satisfies the Samuelson formula.

LEMMA 2. *The term* $1-MRS_{gc}$ *can be approximated as follows:*

$$1-MRS_{gc} \approx \frac{1}{\varepsilon} \cdot \frac{G/C-(G/C)^*}{(G/C)^*}. \tag{16}$$

The approximation is valid up to a remainder that is $O((G/C-(G/C)^*)^2)$. *Furthermore* $1-MRS_{gc}$ *and its approximation have the same sign: negative for* $G/C<(G/C)^*$, *zero for* $G/C=(G/C)^*$, *and positive for* $G/C>(G/C)^*$.

Proof. Lemma 1 establishes that MRS_{gc} is a function of G/C only. We obtain equation (16) from a first-order Taylor approximation of $MRS_{gc}(G/C)$ at $(G/C)^*$. We also use the facts that $MRS_{gc}((G/C)^*)=1$ (Definition 2) and that at $(G/C)^*$, $dMRS_{gc}/d(G/C)=1/(\varepsilon \cdot (G/C)^*)$ (Definition 3). The rest of the proposition follows from the property that MRS_{gc} is strictly decreasing in G/C (Lemma 1). □

Lemma 2 shows that $1-MRS_{gc}$ depends on the relative deviation of G/C from the Samuelson level, $(G/C)^*$, and on the elasticity of substitution between public and private services, ε. There are two interesting special cases. When $\varepsilon \to +\infty$, $MRS_{gc}=1$ for any G/C. This happens when the public services provided by the government substitute perfectly for the private services purchased by households. When $\varepsilon \to 0$, $MRS_{gc} \to 0$ for $G/C>(G/C)^*$. This happens when public services beyond the Samuelson level have zero value, so additional public workers "dig holes".

Next, we express the term $\partial y/\partial x$ from formula (13) with estimable statistics:

LEMMA 3. *The term* $\partial \ln(y)/\partial \ln(x)$ *can be approximated as follows:*

$$\frac{\partial \ln(y)}{\partial \ln(x)} \approx \frac{u-u^*}{1-u^*}. \tag{17}$$

The approximation is valid up to a remainder that is $O((u-u^*)^2)$. *Furthermore* $\partial \ln(y)/\partial \ln(x)$ *and its approximation have the same sign: negative for* $u<u^*$, *zero for* $u=u^*$, *and positive for* $u>u^*$.

Proof. We write the term $\partial \ln(y)/\partial \ln(x)$ as a function of u:

$$\frac{\partial \ln(y)}{\partial \ln(x)} = (1-\eta) \cdot u - \eta \cdot \tau(u).$$

The function $\tau(u)$ is defined by $\tau(u) = \tau(x(u))$, where $\tau(x)$ is defined by (3) and $x(u) = u^{-1}(u)$ is the inverse of the function $u(x)$ defined by (2). The derivative of $\tau(u)$ is

$$\tau'(u) = \tau'(x) \cdot x'(u) = \frac{\tau'(x)}{u'(x(u))} = \frac{\eta \cdot (1+\tau) \cdot \tau / x}{-(1-\eta) \cdot (1-u) \cdot u/x} = -\frac{\eta \cdot (1+\tau) \cdot \tau}{(1-\eta) \cdot (1-u) \cdot u}.$$

At $u = u^*$, $(1-\eta) \cdot u^* = \eta \cdot \tau(u^*)$ so the derivative simplifies to

$$\tau'(u^*) = -\frac{1+\tau(u^*)}{1-u^*}.$$

and

$$-\eta \cdot \tau'(u^*) = \frac{\eta + \eta \cdot \tau(u^*)}{1-u^*} = \frac{\eta + (1-\eta) \cdot u^*}{1-u^*} = \eta + \frac{u^*}{1-u^*}.$$

Thus the derivative of $\partial \ln(y)/\partial \ln(x)$ at u^* is $(1-\eta) - \eta \cdot \tau'(u^*) = 1/(1-u^*)$. Furthermore, by definition, $\partial \ln(y)/\partial \ln(x) = 0$ at u^*. A first-order Taylor expansion of $\partial \ln(y)/\partial \ln(x)$ at u^* therefore yields (17). □

Since $1 - u^* \approx 1$ in reality, Lemma 3 shows that the elasticity of total consumption with respect to tightness is approximately equal to the unemployment gap—the gap between the current unemployment rate u and the efficient unemployment rate u^*. Hence, the unemployment gap is a useful measure of the productive inefficiency of the economy: for instance, if the unemployment gap is 10 percentage points and tightness increases by 20 percent, then total consumption increases by 2 percent; if the unemployment gap is only 5 percentage points and tightness increases by the same amount, then total consumption only increases by 1 percent.

Finally, we work on the term dx/dg from formula (13). We first introduce a multiplier:

DEFINITION 4. *The* unemployment multiplier *is*

$$M = -\frac{Y}{1-u} \cdot \frac{du}{dG}. \tag{18}$$

Formally, the unemployment multiplier measures the percent increase of the employment rate, $1-u$, when government purchases increase by 1 percent of GDP. In practice, the unemployment multiplier admits a simpler interpretation. Since $1-u \approx 1$ in reality, $M \approx -du/(dG/Y)$ so the multiplier approximately measures the decrease of the unemployment rate when government purchases increase by 1 percent of GDP. The unemployment multiplier is closely related to the standard output multiplier dY/dG:

LEMMA 4. *When a change in government purchases and the associated change in taxes do not distort the capacity k supplied by households, as in the model of Section 2, the unemployment multiplier M equals the output multiplier dY/dG.*

Proof. Consider a change in government purchases dG that does not distort the capacity k supplied by households. This change leads to a change du in unemployment and, since $Y = (1-u)\cdot k$, to a change $dY = -du\cdot k$ in output. Hence, $dY/dG = k\cdot(-du/dG) = [Y/(1-u)]\cdot(-du/dG) = M$. □

The unemployment multiplier measures the response of unemployment to purchases of public services, G, but the welfare analysis studies the response of welfare to the consumption of public services, g. We therefore adjust the unemployment multiplier to measure the response of unemployment to a change in g. The adjusted multiplier will enter the optimal government-purchases formulas that we develop because it is closely related to the term dx/dg from formula (13).

DEFINITION 5. *The* adjusted multiplier *is*

$$m = \frac{M}{1 - \frac{G}{Y}\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot M}. \tag{19}$$

LEMMA 5. *The adjusted multiplier satisfies*

$$m = -\frac{y}{1-u}\cdot\frac{du}{dg}. \tag{20}$$

Thus, the term $(Y/G)\cdot(d\ln(x)/d\ln(g))$ can be expressed as

$$\frac{Y}{G}\cdot\frac{d\ln(x)}{d\ln(g)} = \frac{1}{1-\eta}\cdot\frac{m}{u}. \tag{21}$$

Proof. As $G = (1+\tau(x(g)))\cdot g$ and the elasticity of $1+\tau(x)$ with respect to x is $\eta\cdot\tau$, we find that

$$\frac{d\ln(G)}{d\ln(g)} = 1 + \eta\cdot\tau\cdot\frac{d\ln(x)}{d\ln(g)}.$$

Next, as the elasticity of $1-u(x)$ with respect to x is $(1-\eta)\cdot u$, we find that

$$\frac{d\ln(1-u)}{d\ln(G)}\cdot\frac{d\ln(G)}{d\ln(g)} = (1-\eta)\cdot u\cdot\frac{d\ln(x)}{d\ln(g)}.$$

Combining the last two equations to eliminate $d\ln(x)/d\ln(g)$, and noting that write $M = (Y/G) \cdot (d\ln(1-u)/d\ln(G))$, we get

$$\frac{d\ln(G)}{d\ln(g)} = \frac{1}{1 - \frac{G}{Y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot M}.$$

This implies that

$$m = \frac{d\ln(G)}{d\ln(g)} \cdot M = \frac{g}{G} \cdot M \cdot \frac{dG}{dg} = -\frac{Y/(1+\tau(x))}{1-u} \cdot \frac{du}{dG} \cdot \frac{dG}{dg} = -\frac{y}{1-u} \cdot \frac{du}{dg},$$

which establishes (20).

Since the elasticity of $1-u(x)$ with respect to x is $(1-\eta) \cdot u$, we find that

$$(1-\eta) \cdot u \cdot \frac{d\ln(x)}{d\ln(g)} = \frac{d\ln(1-u)}{d\ln(g)} = -\frac{g}{1-u} \cdot \frac{du}{dg}.$$

We obtain (21) by combining the above equation with (20) and using $Y/G = y/g$. □

Formally, the adjusted multiplier measures the percent increase of the employment rate, $1-u$, when public consumption increases by 1 percent of total consumption. In practice, $m \approx -du/(dg/y)$ so the adjusted multiplier approximately measures the decrease of the unemployment rate when public consumption increases by 1 percent of total consumption.

The adjusted multiplier, m, is larger than, but not very different from, the unemployment multiplier, M. In the United States, the government-purchases ratio G/Y is quite small, below 0.3; the unemployment multiplier is surely positive but below 1 [Ramey, 2013]; the term $[\eta/(1-\eta)] \cdot \tau/u$ is below 1 from bad to normal times; accordingly, in bad and normal times, $M < m < M/0.7 = 1.4 \times M$.

Using the results from Lemmas 2, 3, and 5, we can rewrite formula (13) in terms of estimable statistics:

PROPOSITION 2. *In the model described in Section 2, optimal government purchases satisfy*

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{\varepsilon}{1-\eta} \cdot m \cdot \frac{u-u^*}{u^*}. \tag{22}$$

Proof. We start from (13) and approximate $1 - MRS_{gc}$ using (16), approximate $\partial y/\partial x$ using (17), and rewrite dx/dg using (21). □

Formula (22) relates the optimal level of government purchases (G/C) to several estimable statistics. Two of these statistics can be measured using a large body of empirical work: the elasticity of the matching function with respect to idle capacity (η) and the adjusted multiplier (m). The existing empirical work has developed methods to estimate the statistics and provides estimates of these or related statistics (for surveys, see Petrongolo and Pissarides [2001] and Ramey [2011]). Two other statistics reflect the utility that households derive from public services: the Samuelson level ($(G/C)^*$) and the elasticity of substitution between public and private services (ε). These statistics reflect the value that society places on public services. They should not necessarily be estimated; rather, they should be considered as an input into the policy design. The last statistic is the unemployment gap ($u - u^*$). While there are not many papers measuring the gap, the recent work of Landais, Michaillat and Saez [2010*b*] and Michaillat and Saez [2016] develops methods to measure it in US data by combining measures of unemployment with measures of resources devoted to matching. Section 5 presents estimates of all these statistics and discusses the implications for optimal government purchases.

Like formula (13), formula (22) is implicit because its right-hand-side is endogenous to the policy. While we cannot read the optimal level of G/C off the formula, we know that for G/C to be optimal, the formula must hold. This property allows us to derive several qualitative properties of the optimal government purchases.

First, formula (22) shows that the welfare-maximizing level of government purchases generally depends on a government-purchases multiplier, confirming an intuition that macroeconomists have had for a long time. For instance, Woodford [2011, equation (45)] and Nakamura and Steinsson [2014, equation (27)] show that the output multiplier dY/dG affects the optimal level of government purchases in a New Keynesian model. In our theory, however, it is not the output multiplier but the unemployment multiplier that enters the formula. Indeed, the unemployment multiplier enters in the formula through the adjusted multiplier (m).

With lump-sum taxation, output and unemployment multipliers are the same (see Lemma 4) so the output multiplier matters for optimal government purchases. But with distortionary taxation, things are different. In Section 4 we will show that with distortionary taxation, optimal government purchases still depend on the unemployment multiplier, and the unemployment multiplier is equal to the output multiplier net of supply-side responses, so the output multiplier is not directly relevant for optimal government purchases.

The sign of the unemployment multiplier determines the cyclicality of optimal government purchases. If the unemployment multiplier is zero ($M = m = 0$), government purchases should remain at the Samuelson level. If the unemployment multiplier is positive ($M > 0$ so $m > 0$), the government-purchases ratio should be above the Samuelson level ($G/Y > (G/Y)^*$) when unemployment is inefficiently high ($u > u^*$), below the Samuelson level when unemployment is inefficiently low ($u < u^*$), and at the Samuelson level when unemployment is efficient ($u = u^*$). If the unemployment multiplier is negative, the government-purchases ratio should be below the Samuelson ratio when unemployment is inefficiently high, above the Samuelson level when unemployment is inefficiently low, and at the Samuelson level when unemployment is efficient. While the multiplier generally matters for the optimal level of government purchases, it is irrelevant when the unemployment rate is efficient: in that case, government purchases should simply be the Samuelson level.

At first glance, formula (22) seems to suggest that a larger multiplier requires larger government purchases when the unemployment rate is inefficiently high (for a given $u - u^*$, a larger M and thus m seems to indicate that the optimal G/C is larger). This logic is probably why people who believe in large multipliers also advocate for large stimulus spending in recessions whereas people who believe in small multipliers also oppose stimulus spending. But this logic is incorrect because formula (22) is implicit so that u on the right-hand side responds to G/C on the left-hand side. As a consequence, the multiplier also determines how strongly unemployment responds to government purchases; thus, the link between the size of the unemployment multiplier and the optimal level of stimulus spending is more complicated. Below, Proposition 3 makes the formula explicit and Proposition 4 shows how the optimal level of stimulus spending depends on the size of the multiplier.

Second, formula (22) shows that the elasticity of substitution between public and private services is also important to determine the optimal level of government purchases. The elasticity plays a role because it determines how quickly the marginal value of public services fades when government purchases increase. Mankiw and Weinzierl [2011] are concerned that the multiplier is not the right statistics to measure the effect of government purchases on welfare because an increase in government purchases also affects the composition of households' consumption. They are right, and formula (22) formalizes their intuition by showing that it is the product of the multiplier by the elasticity of substitution that links optimal government purchases to the unemployment gap. Apart from the discussion in Mankiw and Weinzierl [2011], the role of the elasticity of substitution has largely been neglected in previous work.

There are two interesting special cases regarding the elasticity of substitution between public and private services. The first special case is $\varepsilon \to 0$. As discussed above, this would be a situation in which we need a certain amount of public services for an economy of a given size, but beyond that number, additional public services have zero value (public workers "dig holes"). The formula says that with $\varepsilon = 0$ the ratio G/C should stay at the Samuelson ratio $(G/C)^*$, irrespective of the unemployment rate and multiplier. Increasing G/C beyond $(G/C)^*$ is never optimal because g is useless at the margin for G/C beyond $(G/C)^*$ and c is always crowded out by g.

The second special case is $\varepsilon \to +\infty$. This would be a situation in which the public services provided by the government perfectly substitute for the private services purchased by households. The formula says that with $\varepsilon \to +\infty$ government purchases should completely fill the unemployment gap such that $u = u^*$. This result holds even if the multiplier is very small and government purchases severely crowd out household purchases. Indeed, with $\varepsilon \to +\infty$, only the total amount of consumption matters for welfare—the composition of consumption does not—so government purchases should simply maximize total consumption, or equivalently bring unemployment to u^*.

Thus, we recover a result from the Keynesian theory when $\varepsilon \to \infty$. Keynesian theory recommends that government purchases be used to fill the output gap. This result can be formally derived in the Keynesian regime of a disequilibrium model, as in the recent paper by Mankiw and Weinzierl [2011]. In the Keynesian regime, there is no crowding out of household purchases by government purchases. If there is some value for government purchases, such that $MRS_{gc} > 0$, government purchases should be increased until the output gap is filled and the product market clears. It is optimal to use government purchases to fill entirely the output gap, as in our model with $\varepsilon \to \infty$. The logic behind the Keynesian result and the result in our model with $\varepsilon \to \infty$ are different, however: in Keynesian theory it is optimal to fill the output gap entirely because there is no crowding out of private services by public services; in our model with $\varepsilon \to \infty$ there is crowding out but the composition of consumption does not matter so it is also optimal to fill the gap entirely.

In reality, it is likely that public services have some value at the margin without being perfect substitutes for private services; that is, $\varepsilon > 0$ but $\varepsilon < +\infty$. In that case the optimal government-purchases ratio, G/Y, departs from the Samuelson level, $(G/Y)^*$, without completely filling the unemployment gap. This means that it is generally not optimal to fill the unemployment gap completely: even with optimal government purchases, the unemployment rate remains inefficient.

3.2. An Explicit Formula

While formulas (13) and (22) are useful to derive several qualitative properties of optimal government purchases, we cannot use these formula to answer more quantitative questions such as "What is the optimal departure of government purchases from the Samuelson level if the unemployment rate suddenly rises to 9%?" and "How does the amplitude of the departure depend on the value of the multiplier?". We cannot answer these questions with formulas (13) and (22) because they describe the optimal policy implicitly: for instance, G/C is implicitly defined by (22) because the right-hand side of (22) is endogenous to G/C. This is a typical limitation of sufficient-statistics formulas in public economics, for which the sufficient-statistics approach has often been criticized [Chetty, 2009]. Here we develop an explicit formula that addresses this limitation and provide several quantitative insights about optimal government purchases.

We assume that the unemployment rate is initially at an inefficient level $u_0 \neq u^*$. As government purchases change, unemployment endogenously responds. By describing this endogenous response, we obtain the following explicit formula:

PROPOSITION 3. *Consider the model described in Section 2, and assume that the economy is at an equilibrium $[(G/C)^*, u_0]$, where the unemployment rate u_0 may be inefficient. Then optimal government purchases satisfy*

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{\varepsilon}{1-\eta} \cdot \frac{m}{1 + z \cdot \frac{\varepsilon}{1-\eta} \cdot m^2} \cdot \frac{u_0 - u^*}{u^*} \tag{23}$$

where the constant z is defined by

$$z = (G/Y)^* \cdot [1 - (G/Y)^*] \cdot \frac{1-u^*}{u^*}. \tag{24}$$

the adjusted multiplier m is evaluated at $[(G/C)^*, u^*]$ so

$$m = \frac{M}{1 - (G/Y)^* \cdot M}. \tag{25}$$

Once optimal government purchases are in place, the unemployment rate is

$$u \approx u^* + \frac{1}{1 + z \cdot \frac{\varepsilon}{1-\eta} \cdot m^2} \cdot (u_0 - u^*) \,. \tag{26}$$

The approximations in (23) *and* (26) *are valid up to a remainder that is* $O((u_0 - u^*)^2 + (G/C - (G/C)^*)^2)$.

Proof. The complete proof is in Appendix C; here we present a simplified version that conveys the logic. First, we recognize that $u \approx u_0 + \alpha \cdot (G/C - (G/C)^*)$, where α is the marginal effect of G/C on u. Plugging this expression into (22) and re-arranging, we obtain

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{\varepsilon}{1-\eta} \cdot \frac{m}{1 - \frac{\varepsilon}{1-\eta} \cdot m \cdot \alpha \cdot \frac{(G/C)^*}{u^*}} \cdot \frac{u_0 - u^*}{u^*}.$$

The proof in Appendix C shows that $\alpha = -m \cdot z \cdot u^*/(G/C)^*$, which yields (23). As α measures the effect of government purchases on unemployment, it is not surprising that it is proportional to the adjusted multiplier, m. To obtain (26), we simply divide (22) by (23). □

Formula (23) links the departure of the optimal government purchases from the Samuelson level to the initial unemployment gap, measured before government purchases have been adjusted from the Samuelson level, and the same estimable statistics as in formula (22): the elasticity of substitution between public and private services (ε), the elasticity of the matching function with respect to idle capacity (η), and the adjusted multiplier (m). The formula is explicit because the right-hand side is not endogenous to the policy. If the left-hand and right-hand sides of the formula do not match, it is sufficient to adjust the left-hand side to obtain the optimal policy—it was not possible to do that with formula (22) because the right-hand side was endogenous to the policy.

Since formula (23) explicitly gives the optimal policy from estimable statistics, the formula can be applied by policymakers to determine the optimal response of government purchases to a shock that leads to an inefficient unemployment rate. To apply the formula, policymakers need measures of η, ε, and m obtained in normal times and a measure of $u_0 - u^*$, the unemployment gap before any policy reaction. Section 5 uses formula (23) and a range of estimates for η, ε, and m to compute the optimal response of government purchases to a shock that raises the unemployment rate by 50% (for instance, from 6% to 9%).

Formula (23) is obtained from formula (13) using first-order approximations. Proposition 3 shows that formula (23) is valid up to a remainder that is $O((u_0 - u^*)^2 + (G/C - (G/C)^*)^2)$. In practice, government purchases are never far from the Samuelson level so $(G/C - (G/C)^*)^2$ remains small. On the other hand, the unemployment rate displays large fluctuations, so there is a risk that $(u_0 - u^*)^2$ is large and formula (23) inaccurate. To alleviate this concern, Section 6 calibrates and simulates a model and shows that formula (23) provides a good approximation of (13) even far from the efficient unemployment rate.

One particular reason why the first-order approximations used to obtain formula (23) may be inaccurate is that the unemployment multiplier may be higher when the unemployment rate is higher. For instance, Auerbach and Gorodnichenko [2012, 2013], Nakamura and Steinsson [2014], and Jorda and Taylor [2016] find that multipliers are higher when the unemployment rate is high and output is low. The variations of the unemployment multiplier only have second-order effects in formula (23)—accounting for the variations of the multiplier when u deviates from u^* would only add a term that is $O((u - u^*)^2)$ to the formula. But this second-order term could be large if the unemployment rate is far from efficiency and if the unemployment multiplier responds strongly to the unemployment rate. Section 6 calibrates and simulates a model in which the multiplier is countercyclical to gauge the quality of formula (23) when the unemployment multiplier depends on the unemployment rate. The simulations suggest that formula (23) provides a good approximation to the exact formula (13) even when the unemployment multiplier is sharply countercyclical.

In the policy debates accompanying recessions, people who believe in large multipliers usually advocate for large stimulus spending while people who believe in small multipliers oppose stimulus spending. The link between multiplier and size of the stimulus package is justified either by a basic "bang-for-the-buck" calculation looking at the effect of the first dollar of stimulus spending on output or by a more sophisticated implicit formula similar to (13).[18] The following proposition, based on formula (23), shows that while the result that larger multipliers make stimulus spending more desirable is valid for small multipliers, it is not valid any more when multipliers are above a certain threshold:

PROPOSITION 4. *Assume that the unemployment multiplier, M, is positive. If the initial unemploy-*

[18] Mankiw and Weinzierl [2011, p.212] and Eggertsson and Krugman [2012, p.1492] discuss the "bang-for-the - buck" argument. Nakamura and Steinsson [2014, p. 788] derive an implicit formula similar to (13) and note that "The extent of extra desirable spending will depend on the size of the multiplier, with a larger multiplier implying that more spending is desirable."

ment rate is inefficiently high ($u_0 > u^*$), optimal government purchases are above the Samuelson level ($G/C > (G/C)^*$). The amplitude of the departure from the Samuelson level is increasing in the elasticity of substitution between public and private services, ε. Furthermore, the amplitude is 0 when $M = 0$, increasing in M for $M \in [0, M^\dagger]$, maximized at $M = M^\dagger$, decreasing in M for $M \in [M^\dagger, (Y/G)^*]$, and 0 for $M \to (Y/G)^*$. The maximizing unemployment multiplier is

$$M^\dagger = \frac{1}{\left(\frac{G}{Y}\right)^* + \sqrt{\frac{z \cdot \varepsilon}{1-\eta}}} \tag{27}$$

and the maximal amplitude is

$$\frac{(G/C)^\dagger - (G/C)^*}{(G/C)^*} = \frac{1}{2} \cdot \sqrt{\frac{\varepsilon}{(1-\eta) \cdot z}} \cdot \frac{u_0 - u^*}{u^*}. \tag{28}$$

Proof. The proposition is based on formula (23) with $u_0 - u^* > 0$. For $M \geq 0$ and thus $m \geq 0$, the amplitude of the response is 0 at $m = 0$, increasing in m for $m \in [0, m^\dagger]$, maximized at $m = m^\dagger$, decreasing in m for $m \in [m^\dagger, \infty)$, and 0 for $m \to \infty$. The amplitude is maximized at $m^\dagger = \sqrt{(1-\eta)/(z \cdot \varepsilon)}$, where (28) holds. All the results are translated in terms of M using the fact that in (23), m is evaluated at $[(G/C)^*, u^*]$, so $m = M/[1 + (G/Y)^* \cdot [\eta/(1-\eta)] \cdot M)]$ or equivalently $M = 1/[(1/m) + (G/Y)^*]$. □

The main result from the proposition is that a higher unemployment multiplier does not necessarily imply a stronger response of government purchases to fluctuations in unemployment. Instead, the optimal increase in government purchases above the Samuelson level after an increase in unemployment above its efficient level is a hump-shaped function of the unemployment multiplier. The unemployment multiplier that calls for the strongest response is given by (27). The strongest possible response of government purchases to a given increase in unemployment is given by (28). The value (28) gives an upper bound on the optimal size of stimulus packages for a given increase in unemployment. This value is useful for policy as empirical research has not yet reached a consensus on the precise value of the multiplier.

There is a simple intuition behind this apparently surprising result. Consider first a small unemployment multiplier: $M \to 0$. We can neglect the feedback effect of G on u because the multiplier is small so $u \approx u_0$. Hence, the application of formula (22) yields

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \varepsilon \cdot m \cdot \frac{u_0 - u^*}{u^*}. \tag{29}$$

From this formula, it is clear that when $M \to 0$, $[G/C - (G/C)^*]/(G/C)^*$ increases with m and thus M.[19] The intuition is that for small multipliers, the optimal amount of government purchases is determined by the crowding out of household purchases by government purchases; a higher multiplier means less crowding out and thus higher optimal government purchases.

Consider next a large multiplier, $M \to Y/G$ and $m \to +\infty$. With such a large multiplier, G/C remains constant as G increases.[20] Since the marginal rate of substitution between public and private services only depends on G/C, increasing G fills the unemployment gap without changing the marginal rate of substitution. The optimal policy therefore is to maintain G/C at $(G/C)^*$ while entirely filling the unemployment gap $u_0 - u^*$. Equation (A11) indicates that filling the unemployment gap requires

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \frac{1}{z \cdot m} \cdot \frac{u_0 - u^*}{u^*} \tag{30}$$

Clearly, when m is large, $[G/C - (G/C)^*]/(G/C)^*$ decreases with m and accordingly with M.[21] The intuition is that if the multiplier is high, government purchases are a potent policy that can bring the economy close to the efficient tightness without distorting the allocation of output between public and private services. As the multiplier rises, fewer government purchases are required to bring tightness to its efficient level.

If the unemployment multiplier is negative (which is unlikely, as we will discuss in Section 5) and if the initial unemployment rate is inefficiently high ($u_0 > u^*$), optimal government purchases are below the Samuelson level ($G/C < (G/C)^*$). However, the amplitude of the departure depends on ε and the amplitude of the multiplier, $|M|$, in the same way as in Proposition 4.

[19] Note that the explicit formula (23) simplifies to (29) when $M \to 0$ and thus $m \to 0$.

[20] As $C + G = Y$, we have $dC/dG = dY/dG - 1$ and hence $d\ln(G/C)/d\ln(G) = (Y/C) \cdot (1 - d\ln(Y)/d\ln(G)) = (Y/C) \cdot (1 - (G/Y) \cdot M)$. Therefore, $d\ln(G/C)/d\ln(G) \to 0$ when $M \to Y/G$.

[21] Note that the explicit formula (23) simplifies to (30) when $m \to +\infty$.

4. Government Purchases Financed by a Distortionary Income Tax

So far we have abstracted from distortionary taxation. In this section we extend the model by introducing an endogenous productive capacity and a distortionary income tax used to finance government purchases. In this extended model, an increase in government purchases affects both the aggregate demand, as before, and the aggregate supply, because it requires an increase in taxes that affects the capacity supplied by households.

We consider two types of distortionary income tax: a linear income tax, and an income tax implemented following the benefit principle. Using a linear income tax was the traditional approach to taxation in public economics and remains the standard approach in macroeconomics. Using the benefit principle is the modern approach to taxation in public economics. We find that with both approaches to taxation the results derived above remain valid.

4.1. The Traditional Approach to Taxation

The representative household supplies the productive capacity $k(t)$ at some utility cost. The instantaneous utility function in (7) becomes $\mathscr{U}(c(t),g(t)) - \mathscr{W}(k(t)) + \mathscr{V}(b(t))$, where the function $\mathscr{W}$ is strictly increasing in k and convex. The choice of $k(t)$ can be seen as a labor-supply decision at each instant t. Since timing is not ambiguous, we simplify notation by dropping the time index t.

In this subsection, we follow an approach to taxation that is traditional in public economics and widely used in macroeconomics. We assume that the government government uses a linear income tax τ^L. The household's labor income in (6) becomes $(1-\tau^L)\cdot Y(x,k) = (1-\tau^L)\cdot(1-u(x))\cdot k$. To finance public consumption g, the tax rate must be $\tau^L = (1+\tau(x))\cdot g/Y = g/y$.

The household chooses k at each instant t to maximize utility. For all the demand structures that we considered in Section 2, the first-order condition with respect to k is

$$\mathscr{W}'(k) = \lambda\cdot(1-\tau^L)\cdot(1-u(x)),$$

where λ is the costate variable associated with the law of motion of real wealth in the household's Hamiltonian. Combining this equation with the first-order condition with respect to c, given by (8), we obtain

$$MRS_{kc} = (1-\tau^L)\cdot\frac{1-u(x)}{1+\tau(x)}, \tag{31}$$

where

$$MRS_{kc} \equiv \frac{\mathscr{W}'(k)}{\frac{\partial \mathscr{U}}{\partial c}(c,g)} \tag{32}$$

is the marginal rate of substitution between labor and private consumption. Equation (31) describes the optimal supply of capacity by households. The supply decision is distorted by the income tax: a higher τ^L implies a lower k. In fact, equation (31) implicitly defines a function $k(g)$ that describes how the aggregate productive capacity responds to a change in government purchases and the associated change in the income tax rate. Since the income tax is distortionary, the function $k(g)$ is decreasing in g.

As with nondistortionary taxation, the model converges immediately to steady state with distortionary taxation. The welfare of an equilibrium is $\mathscr{U}(c,g) - \mathscr{W}(k)$. Given a tightness function $x(g)$ and a capacity function $k(g)$, the government chooses g to maximize $\mathscr{U}(y(x(g),k(g))-g,g) - \mathscr{W}(k(g))$. As above, we assume that the welfare function $g \mapsto \mathscr{U}(y(x(g),k(g))-g,g) - \mathscr{W}(k(g))$ is well behaved such that first-order conditions are necessary and sufficient to describe the optimum of the government's problem. For an interior solution, the first-order condition of the problem is

$$0 = \frac{\partial \mathscr{U}}{\partial g} - \frac{\partial \mathscr{U}}{\partial c} + \frac{d\mathscr{W}}{dk} \cdot \frac{dk}{dg} + \frac{\partial \mathscr{U}}{\partial c} \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} + \frac{\partial \mathscr{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}. \tag{33}$$

Dividing the condition by $\partial \mathscr{U} / \partial c$, we obtain

$$1 = MRS_{gc} - \left(MRS_{kc} - \frac{\partial y}{\partial k}\right) \cdot \frac{dk}{dg} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Households' optimal choice of capacity, described by (31), implies that $MRS_{kc} = (1-\tau^L) \cdot (\partial y / \partial k)$. With the linear income tax, formula (13) is therefore altered as follows:

Proposition 5. *In the model described in Section 4.1, optimal government purchases satisfy*

$$1 = MRS_{gc} + \tau^L \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}. \tag{34}$$

Formula (34) is not the same as formula (13); however, the two formulas have the same structure once the Samuelson formula is modified to account for distortionary taxation.

DEFINITION 6. *In the model described in Section 4.1, the* modified Samuelson formula *is*

$$1 = MRS_{gc} + d, \tag{35}$$

where the statistic

$$d \equiv \tau^L \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} < 0 \tag{36}$$

is evaluated in the equilibrium where government purchases are set optimally. The Samuelson levels *of G/C and G/Y, denoted $(G/C)^*$ and $(G/Y)^*$, are the levels satisfying the modified Samuelson formula.*

The modified Samuelson formula can be written $MRS_{gc} = 1 - d$, where $d < 0$ is given by (36). As discussed by Ballard and Fullerton [1992], the statistic $1 - d > 1$ is the marginal cost of funds. It is more than 1 because the linear income tax distorts the supply of productive capacity so obtaining one public service costs more than on private service. Because the marginal cost of funds is higher with a distortionary tax, the modified Samuelson formula implies a lower level of government purchases than the regular formula. The modified Samuelson formula was developed by Atkinson and Stern [1974] and Stiglitz and Dasgupta [1971] to describe the optimal provision of public goods with a linear income tax. Following these two papers, a large literature has analyzed the optimal provision of public goods with this traditional approach to taxation. Ballard and Fullerton [1992] and Kreiner and Verdelin [2012] survey the literature, and Feldstein [1997], Slemrod and Yitzhaki [2001], and Kleven and Kreiner [2006] are recent contributions to it.

With a linear income tax, our optimal government-purchases formula can be written as the modified Samuelson formula plus a correction equal to $(\partial y/\partial x) \cdot (dx/dg)$. The correction term remains the same whether taxes are nondistortionary, as in formula (13), or distortionary, as in formula (34). Although the Samuelson level of government purchases is different with a distortionary linear income tax, the deviation of optimal government purchases from the Samuelson level remains the same as in Section 3:

PROPOSITION 6. *In the model described in Section 4.1, the deviation of optimal government purchases from the Samuelson level satisfies* (22).

Proof. Proposition 5 and Definition 6 imply that the Samuelson level of government purchases is defined by $MRS_{gc}((G/C)^*) = 1 - d$, while the optimal level of government purchases satisfies

$MRS_{gc}(G/C) = 1 - d - (\partial y/\partial x) \cdot (dx/dg)$. Hence, $MRS_{gc}((G/C)^*) - MRS_{gc}(G/C) = (\partial y/\partial x) \cdot (dx/dg)$. We obtain $MRS_{gc}((G/C)^*) - MRS_{gc}(G/C) = (1/\varepsilon) \cdot [(G/C - (G/C)^*)/(G/C)^*]$ using the same logic as in Lemma 2. Lemma 3 remains valid so we write $\partial y/\partial x$ as in (17). Lemma 5 also remains valid so we write dx/dg as in (21). Combining these results yields formula (22). □

Formula (22) remains valid when a distortionary linear income tax is used to finance government purchases. Hence, the deviation of optimal government purchases from the Samuelson level remains the same. The Samuelson level of government purchases changes, however: is lower when taxes are distortionary because tax distortions make government purchases more costly.

In the formula the unemployment multiplier M (which determines m) is a *policy elasticity*, in the sense of Hendren [2015]: it measures the change in u for a change in G accompanied by the change in taxes that maintains a balanced government budget. In Section 3 taxes are not distortionary, so the unemployment multiplier should be measured using a policy reform in which taxation is not distortionary. Here taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the increase in income tax following the increase in government purchases distorts the labor supply.

An important difference with the distortionary linear income tax is that the output multiplier, dY/dG, cannot be used to determine how government purchases should deviate from the Samuelson level. With nondistortionary taxation, we have seen that the deviation of optimal government purchases from the Samuelson level is determined by the unemployment multiplier, M, which turns out to be the same as the output multiplier. With distortionary taxation, the deviation of optimal government purchases from the Samuelson level is still determined by the unemployment multiplier (as the adjusted multiplier, m, can still be calculated using the unemployment multiplier and equation (19)); but it cannot be computed using the output multiplier because the unemployment and output multipliers are no longer equal. With distortionary taxation, the unemployment multiplier equals the output multiplier net of the supply-side response:

LEMMA 6. *In the model described in Section 4.1, the linear income tax is distortionary, so the unemployment multiplier M is no longer equal to the output multiplier dY/dG:*

$$M = \frac{dY}{dG} - \frac{Y}{k} \cdot \frac{dk}{dG} > \frac{dY}{dG}. \tag{37}$$

Proof. Output is $Y = (1-u)\cdot k$ so

$$\frac{d\ln(Y)}{dG} = \frac{d\ln(1-u)}{dG} + \frac{d\ln(k)}{dG}$$
$$\frac{dY}{dG} = -\frac{Y}{1-u}\cdot\frac{du}{dG} + \frac{Y}{k}\cdot\frac{dk}{dG} = M + \frac{Y}{k}\cdot\frac{dk}{dG}.$$

Since taxes are distortionary, $dk/dG < 0$ and $M > dY/dG$. □

The implication from Lemma 6 is that what matters is not the output multiplier but the unemployment multiplier, which equals the output multiplier net of the supply-side response $(Y/k)\cdot(dk/dG)$. The supply-side response $(Y/k)\cdot(dk/dG)$ measures the percentage change in labor supply when government purchases increase by 1 percent of GDP. As the linear income tax is distortionary, the supply-side response is negative and the unemployment multiplier is larger than the output multiplier.

The empirical evidence presented by Barro and Redlick [2011] suggests that the output multiplier is negative (around -0.5) while the output multiplier net of the supply-side response (estimated using deficit-financed public spending) is positive (around 0.5). The empirical evidence presented by Ramey [2013] also suggests that the unemployment multiplier is positive (also around 0.5). Furthermore, the unemployment multiplier is necessarily positive in the class of models studied here, whereas the output multiplier can be negative if the supply-side response is strong enough. When the output multiplier is negative but the unemployment multiplier positive, our formula suggests that optimal government purchases are above the Samuelson level when unemployment is inefficiently high. That is, government purchases and the taxes used to finance them should be countercyclical.

These results are surprising: when taxes are so distortionary that they make the output multiplier negative, we find that it is optimal to increase taxes in order to finance higher government purchases when unemployment is inefficiently high. But how can more distortionary taxes and thus lower output be desirable in an already depressed economy? The intuition is the following. The Samuelson formula (35) states that a certain level of distortionary taxation is optimal to fund government purchases when the economy is efficient. From this level, by the envelope theorem, any small increase in taxes and government purchases has zero effect on welfare—even though it might have a first-order negative effect on output—as costs and benefits net out. Hence, in a slack economy, the relevant first-order effect is whether increasing taxes and government purchases reduces inefficient slack. Such slack is measured by the unemployment rate and not by the level of output. That is why it is the unemployment multiplier and not the output multiplier that matters.

Following the same steps as for Proposition 3, we can show that the explicit formula (23) remains valid with a distortionary linear income tax:

Proposition 7. *Consider the model described in Section 4.1, and assume that the economy is at an equilibrium* $[(G/C)^{*}, u_0]$*, where the unemployment rate* u_0 *may be inefficient. Then the deviation of optimal government purchases from the Samuelson level satisfies* (23) *and the unemployment rate once optimal government purchases are in place satisfies* (26).

As explained in Lemma 6, with a distortionary linear income tax, it is the unemployment multiplier (M) and not the output multiplier (dY/dG) that must be used to compute the adjusted multiplier (m) in formulas (23) and (26).

4.2. The Modern Public-Economics Approach to Taxation

With homogenous households, the government should raise revenue with a lump-sum tax to avoid distortions. With heterogenous households, on the other hand, a distortionary tax system may be justified. If the government values redistribution, a uniform lump-sum tax is not desirable [Diamond and Mirrlees, 1971]. Even if the government does not value redistribution, people with low income might not be able to pay the uniform lump-sum tax. But in our analysis households are homogenous so the welfare calculus in Proposition 5 is biased. It includes the cost of distortionary taxation (a lower labor supply) but not its benefits (such as redistribution).

For this reason the modern public-economics literature argues that the optimal provision of public goods is orthogonal to labor-supply distortions. This result seems to contradict formula (35). If taxes are distortionary, shouldn't public spending be less desirable? If the government uses a distortionary tax schedule $T(k)$ for some unspecified reasons, it is true that financing government purchases creates a deadweight loss proportional to the labor-supply response. In addition, if an increase in government purchases is bundled with an inefficient tax increase, it seems unappealing. But if the increase is bundled with a tax increase reducing distortions (an increase in the lump-sum component of the tax schedule with a reduction of the marginal tax rate), it seems appealing. Because the distortions exist for reasons unrelated to the provision of public goods, the *benefit principle* argues that an increase in government purchases should be financed by a tax increase that does not change labor-supply distortions. The benefit principle was first discussed by Hylland and Zeckhauser [1979], and its general principles established by Kaplow [1996, 1998].

The modern approach to taxation in public economics consists in using a nonlinear tax schedule $T(k)$ and tax reforms following the benefit principle. The modern approach is surveyed and compared to the traditional approach in Kaplow [2004] and Kreiner and Verdelin [2012]. The result of the modern approach are also consistent with the approach taken by Christiansen [1981], Boadway and Keen [1993], Sandmo [1998], Coate [2000], Jacobs [2013], and others, who study the optimal provision of public goods when taxes are optimally set to satisfy redistributive objectives.[22]

In this subsection, we follow the modern approach to taxation. We assume that government purchases are financed by a nonlinear tax schedule $T(k)$. With that tax, the household's labor income in (6) becomes $(1-u(x))\cdot(k-T(k))$. We apply the benefit principle whereby a change in government purchases is financed by a tax change designed to leave individual utility unchanged. A Pareto improvement is possible if the reform generates a government budget surplus or deficit: a surplus could be redistributed back to households, thus creating a Pareto improvement; with a deficit, the opposite of the proposed reform would create a surplus and hence make a Pareto improvement possible. Hence, the formula for optimal government purchases is obtained when the reform leaves the government budget balanced. We find that formula (13) remains valid:

Proposition 8. *In the model described in Section 4.2, optimal government purchases satisfy* (13).

Proof. Starting from an equilibrium $[c,g,y,x,k]$, we implement a small change dg. We follow the benefit principle: the change dg is funded by a change in tax $dT(k)$ designed to keep the household's choice of k unchanged.

The choice of k is given by an equation similar to (31):

$$\mathscr{W}'(k) = \frac{\partial \mathscr{U}}{\partial c}(y(x,k)-g,g)\cdot\frac{1-u(x)}{1+\tau(x)}\cdot\left[1-T'(k)\right].$$

We can adjust $dT'(k)$ for all k such that the household does not change k after the changes dg and

[22] See also Kaplow [2008] for a textbook treatment of the modern approach, and Weinzierl [2015] for a recent paper on the topic.

dx. For any k, $dT'(k)$ must satisfy

$$\mathscr{W}'(k) = \frac{\partial \mathscr{U}}{\partial c}(y(x+dx,k)-g-dg, g+dg) \cdot \frac{1-u(x+dx)}{1+\tau(x+dx)} \cdot \left[1-T'(k)-dT'(k)\right].$$

The government budget's must be balanced in the new equilibrium, which requires $g+dg = (T(k)+dT(k)) \cdot (1-u(x+dx))/(1+\tau(x+dx))$. This condition is satisfied by adjusting the lump-sum component of the tax schedule by the appropriate amount $dT(0)$. With this change $dT(k)$ in tax, the supply of capacity is unaffected by the change dg in government purchases and the government's budget stays balanced.

Thanks to the change $dT(k)$, the household does not change his choice of k after the change dg so that $dk = 0$. Taking a first-order expansion around the initial equilibrium, we obtain

$$\frac{\partial \mathscr{U}}{\partial c} \cdot \left[\left(1-\frac{T(k)}{k}\right) \cdot \frac{\partial y}{\partial x} \cdot dx - y \cdot \frac{dT(k)}{k}\right] + \frac{\partial \mathscr{U}}{\partial g} \cdot dg = 0.$$

Dividing by $\partial \mathscr{U}/\partial c$ and re-arranging yields

$$\frac{T(k)}{k} \cdot \frac{\partial y}{\partial x} \cdot dx + y \cdot \frac{dT(k)}{k} = MRS_{gc} \cdot dg + \frac{\partial y}{\partial x} \cdot dx.$$

The equation gives the effect of the reform on the government budget balance $R = y(x,k) \cdot T(k)/k - g$:

$$dR = \frac{T(k)}{k} \cdot \frac{\partial y}{\partial x} \cdot dx + y(x,k) \cdot \frac{dT(k)}{k} - dg = (MRS_{gc} - 1) \cdot dg + \frac{\partial y}{\partial x} \cdot dx.$$

(We used again $dk = 0$.) At the optimum, $dR = 0$ so equation (13) holds. □

Proposition 8 establishes that formula (13) also applies when labor supply is endogenous and the benefit principle applies. The supply responses are orthogonal to the proper level of government purchases in Samuelson's theory as in this generalization. This finding mirrors the result that the Samuelson formula is valid with endogenous labor supply [Kaplow, 1996; Samuelson, 1954].

Since formula (13) remains valid under the benefit principle, and since capacity k is not distorted at the margin by a change in government purchases, we can show that all the other results from Section 3 hold:

DEFINITION 7. *In the model described in Section 4.2, the* Samuelson formula *is* (14).

Proposition 9. *In the model described in Section 4.2, the deviation of optimal government purchases from the Samuelson level satisfies* (22).

Lemma 7. *In the model described in Section 4.2, marginal changes in government purchases are financed by marginal changes in the tax schedule that are not distortionary, so the unemployment multiplier M equals the output multiplier dY/dG.*

Proposition 10. *Consider the model described in Section 4.2, and assume that the economy is at an equilibrium $[(G/C)^*, u_0]$, where the unemployment rate u_0 may be inefficient. Then the deviation of optimal government purchases from the Samuelson level satisfies* (23) *and the unemployment rate once optimal government purchases are in place satisfies* (26).

As discussed above, in formula (22) the unemployment multiplier M (which determines m) is a policy elasticity in the sense of Hendren [2015]: it measures the change in u for a change in G accompanied by the change in taxes that maintains a balanced government budget. In Section 4.1 taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the increase in income tax following the increase in government purchases distorts the labor supply. Here we apply the benefit principle when financing government purchases so the tax schedule is adjusted to keep the capacity the same. Accordingly the policy reform used to measure the unemployment multiplier needs to leave labor supply unchanged (e.g., deficit finance government purchases).

5. Optimal Response to a 50% Increase in Unemployment

In this section we calibrate our formula with US data to assess how government purchases should respond when the unemployment rate rises 50% above its efficient level. We especially analyze how the optimal response depends on the value of the unemployment multiplier.

The starting point of the analysis is an economy in which the unemployment rate is efficient and government purchases are at the Samuelson level: $u = u^*$ and $G/C = (G/C)^*$. A shock hits the economy and raises the unemployment rate to an inefficient level $u_0 > u^*$. As an illustration, we assume that u_0 is 50% above u^*. If the efficient unemployment rate in the United States is around 6% (its average value over the 1951–2014 period), the analysis looks at a shock that raises the unemployment rate by 3 percentage points to 9%. This is roughly the magnitude of the shock observed at the beginning of the Great Recession in the United States: the US unemployment rate rose from 6% to 9% between September 2008 and April 2009.

To assess the optimal response of government purchases to the increase in unemployment, we rely on formula (23). The formula gives the optimal response of government purchases as a function of a few estimable statistics. An advantage of the formula is that it applies whether the increase in government purchases is financed by a lump-sum tax, a linear income tax, or a nonlinear income tax adjusted following the benefit principle.

To apply formula (23), we need estimates of the following statistics: the unemployment multiplier M, the elasticity of substitution between public and private services ε, the elasticity of the selling rate $1-\eta$, the efficient unemployment rate u^*, and the Samuelson level $(G/Y)^*$. The statistics ε and $1-\eta$ appear directly in the formula. The statistics M, u^*, and $(G/Y)^*$ do not appear directly but are used to determine the value of the statistics m and z.

The unemployment multiplier can be estimated directly by measuring the response of the unemployment rate (in percentage points) when government purchases increase by 1% of GDP. Ramey [2013, Section 4.2] uses several of the main identification schemes used in the literature to estimate the unemployment multiplier. All the specifications give an unemployment multiplier between 0.2 and 0.5, except one specification that gives a multiplier of 1. The unemployment multiplier can also be estimated indirectly using estimates of the output multiplier. We found that if government purchases are financed by a lump-sum tax, the unemployment multiplier is the same as the output multiplier, dY/dG. Under the standard assumption that households are Ricardian, lump-sum taxation is equivalent to deficit financing. Thus, estimates of the output multiplier obtained when the increase in government purchases is financed by deficit spending can be used as estimates of the unemployment multiplier. Ramey [2011] discusses the estimates of the output multiplier in the literature. Table 1 in Ramey [2011] shows that in aggregate analyzes on postwar US data, the output multiplier is between 0.6 and 1.6 when the increase in government purchases is financed by deficit spending. If households are not Ricardian, deficit spending could stimulate output beyond what a balanced-budget increase in G would achieve [Romer and Bernstein, 2009]. In that case, the range 0.6–1.6 may overstate the value of the output multiplier when government purchases are financed by lump-sum taxes. Overall, the unemployment multiplier is likely to fall in the 0.2–1.6 range, probably toward to lower end of the range. Given the uncertainty about the exact estimate of the multiplier, we compute the optimal response of government purchases for all the multiplier values between 0 and 2.

The elasticity of substitution between public and private services describes how households value additional public services. Different utility functions over private and public services yield different elasticities. A Leontief utility function has an elasticity of 0. A Cobb-Douglas utility function has an elasticity of 1. A linear utility function has an elasticity of $+\infty$. As we do not have any estimate of the elasticity of substitution, we consider several values: $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$.

The elasticity $1 - \eta$ describes how the matching function depends on idle capacity and vacancies. We use the estimate $\eta = 2/3$ obtained by Landais, Michaillat and Saez [2010*b*]. They estimate η using US data for the 1951–2014 period. This estimate is in line with the results of the vast empirical literature estimating matching functions. In their survey, Petrongolo and Pissarides [2001] conclude that most estimates of η fall between 0.5 and 0.7.

We need a measure of government purchases to compute $(G/Y)^*$. Using employment data constructed by the BLS from the Current Employment Statistics (CES) survey, we measure G/C as the ratio of employment in the government industry to employment in the private industry. We measure G/C with employment data to be consistent with our other statistics obtained from labor market data. The average of G/C over the 1951–2014 period is 19.9%, so the average of $G/Y = (G/C)/(1+G/C)$ is 16.6%. We assume that the government determines average government purchases using the Samuelson formula so we set $(G/Y)^* = 16.6\%$. If unemployment is efficient on average, it is optimal to determine average government purchases with the Samuelson formula.

There are several ways to estimate the efficient unemployment rate u^*. One way is to assume that unemployment is efficient on average. The assumption that the economy is efficient on average has a long tradition: macroeconomists have made it at least since Okun [1963] did in his famous paper that proposed Okun's law. Taking the average of the unemployment rate constructed by the BLS from the CPS over the 1951–2014 period, we find that $u^* = 5.9\%$. An alternative would be to use the microfounded estimate of the efficient unemployment rate proposed by Michaillat and Saez [2016]. In US data, they find that $u^* = 4.7\%$. The choice of u^* does not much influence on the quantitative results, so we stick with the traditional estimate $u^* = 5.9\%$.

Panel A in Figure 2 displays the results obtained by calibrating formula (23). The graph displays the optimal $G/Y - (G/Y)^*$ for $(u_0 - u^*)/u^* = 50\%$ as a function of the unemployment multiplier. Since we use $u^* = 5.9\%$, the shock considered here leads to an increase in unemployment by $5.9 \times 0.5 = 3$ percentage points to $u_0 = 5.9 + 3 = 8.9\%$. The results described here illustrate and complement the theoretical results from Section 3.

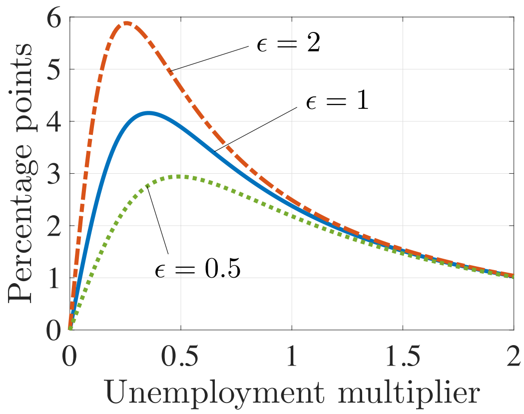
A first observation is that government purchases should remain at the Samuelson level if the unemployment multiplier is 0. For positive unemployment multipliers, government purchases should rise above the Samuelson level.

A second observation is that even with a small multiplier of 0.2, government purchases should increase significantly above the Samuelson level in response to the unemployment increase. With an elasticity of substitution of 0.5, the government-purchases ratio G/Y should increase by 2 percentage points; for an elasticity of substitution of 1, it should increase by 3.5 points; and for an elasticity of substitution of 2, it should increase by 5.7 points. Thus, optimal government purchases are markedly countercyclical even for small positive multipliers.

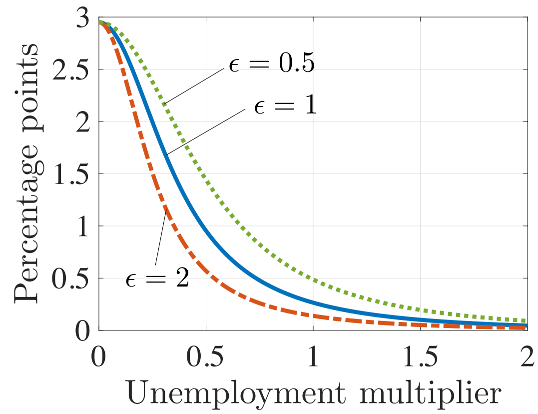
A third observation is that the optimal increase in government purchases rises with the elasticity of substitution between public and private services. For instance, with an unemployment multiplier of 0.5, the government-purchases ratio should increase by 2.9 percentage points with a low elasticity of 0.5, by 3.9 points with an elasticity of 1, and by 4.6 points with a high elasticity of 2. Hence, the elasticity of substitution significantly influences the optimal response of government purchases to unemployment fluctuations.

A fourth observation is that the optimal increase in government purchases does not rise monotonically with the unemployment multiplier. Instead, it is a hump-shaped function of the multiplier. It is true that the optimal increase in government purchases rises with the multiplier for low values of the multiplier. For instance, with an elasticity of substitution of 1, the government-purchases ratio should increase by 2.1 percentage points with a low multiplier of 0.1, but it should increase by 3.5 points with a higher multiplier of 0.2 and by 4.1 points with an even higher multiplier of 0.3. However, the optimal increase in the government-purchases ratio diminishes for higher values of the multiplier. For instance, with the same elasticity of substitution of 1, the government-purchases ratio should only increase by 2.4 percentage points with a multiplier of 1, by 1.5 points with a multiplier of 1.5, and by 1 point with a multiplier of 2. As discussed above, with large multipliers the optimal policy is to fill the unemployment gap $u_0 - u^*$, and the larger the multiplier the smaller the amount of additional government purchases required to fill the gap.

A last observation is that the value of the elasticity of substitution becomes less important when the multiplier is larger. Since the optimal policy is to fill the unemployment gap for large multipliers, the elasticity of substitution has little influence on the policy decision. Indeed, for an



A. Optimal deviation $G/Y - (G/Y)^*$



B. Unemployment gap $u - u^*$ with the optimal G/Y

Figure 2: Optimal Deviation of the Government-Purchases Ratio from the Samuelson Level With an Initial Unemployment Rate 50% above the Efficient Unemployment Rate

Notes: Initially, government purchases are at the Samuelson level $(G/Y)^*$ and the unemployment rate u_0 is 50% above the efficient unemployment rate. This level of government purchases is not optimal. Panel A displays $G/Y - (G/Y)^*$ where G/Y is the optimal ratio of government purchases to output. The values of $G/Y - (G/Y)^*$ are computed using formula (23), ε equal to 0.5, 1, and 2, $\eta = 0.46$, $(G/Y)^* = 16.6\%$, $u^* = 5.9\%$, M between 0 and 2, and the fact that $[G/Y - (G/Y)^*]/(G/Y)^* \approx [1 - (G/Y)^*] \cdot [G/C - (G/C)^*]/(G/C)^*$. (This is a first-order Taylor approximation of $G/Y = (G/C)/(1 + G/C)$ around $(G/C)^*$.) Panel B displays the unemployment gap $u - u^*$ once government purchases are adjusted from the Samuelson level to their optimal level. The values of $u - u^*$ are computed using formula (26).

unemployment multiplier above 1, the optimal levels of government purchases for $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$ are nearly indistinguishable.

The unemployment rate that prevails in equilibrium once government purchases have been adjusted to their optimal level is given by (26). Panel B of Figure 2 displays the deviation $u - u^*$ of the unemployment rate from its efficient level after optimal government purchases have been implemented. Initially the unemployment rate is 3 percentage points above the efficient unemployment rate. With the optimal policy, the unemployment rate falls below that initial level.

A first observation is that for small values of the multiplier the unemployment rate barely falls below its initial level. This is the case even though government purchases increase significantly. With a low multiplier of 0.2, the unemployment rate only falls by 0.4 percentage points with an elasticity of substitution of 0.5, by 0.7 points with an elasticity of substitution of 1, and by 1.1 points with an elasticity of substitution of 2.

A second observation is that despite the hump-shaped pattern of the increase in government purchases, the unemployment rate decreases with the unemployment multiplier.

A third observation is that the equilibrium unemployment rate decreases with the elasticity of substitution. The reason is that the increase in government purchases rises with the elasticity of substitution. The effect of the elasticity of substitution is substantial: with a multiplier of 0.5, the unemployment rate falls to 7.8% with an elasticity of substitution of 0.5, to 7.3% with an elasticity of substitution of 1, and to 6.8% with an elasticity of substitution of 2.

A last observation is that, with an unemployment multiplier above 1, the stabilization of the unemployment rate is almost perfect. Irrespective of the elasticity of substitution, the unemployment rate achieved with the optimal policy is less than 0.5 percentage points above the efficient unemployment rate. We know from (22) that the stabilization cannot be perfect: since $G/Y > (G/Y)^*$, the optimality condition (22) imposes that $u > u^*$. But with large multipliers, government purchases are so potent that the gap between u and u^* is negligible. With a multiplier of 2, the unemployment rate is less than 0.1 percentage point above its efficient level for all the elasticities of substitution considered.

6. Simulations

In this section we simulate a matching model calibrated to empirical evidence for the US economy. The model that we use is the model with real bonds presented in Section 2. Aggregate demand shocks are parameterized by the marginal utility of wealth; with higher marginal utility of wealth, households desire to save more and consume less, which depresses aggregate demand. The amplitude of the fluctuations in tightness and unemployment are governed by the rigidity of the interest rate. The rigidity of the interest rate also governs the size of the multiplier.

The simulation has several purposes. First, it shows how the statistics used in our formulas, especially the unemployment multiplier, are related to the parameters of a specific model. Second, it demonstrates that the class of matching models on which the analysis relies provides a good description of the business cycle. In response to aggregate-demand shocks, the model generates realistic countercyclical fluctuations in the unemployment rate and in the unemployment multiplier. Third, it establishes that the explicit formula for optimal government purchases is accurate, even though is obtained using first-order approximations.

6.1. Calibration

We calibrate the model to US data for 1951–2014. The calibration ensures that the two statistics at the heart of our formulas—the elasticity of substitution between public and private services, ε, and the unemployment multiplier, M— match the empirical evidence. As discussed in Section 5, reasonable estimates of the elasticity of substitution and multiplier are $\varepsilon = 1$ and $M = 0.5$.

We specify the utility function as follows:

$$\mathscr{U}(c,g) = \left[(1-\gamma)\cdot c^{\frac{\varepsilon-1}{\varepsilon}} + \gamma \cdot g^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{38}$$

where $\gamma \in (0,1)$ indicates the value of public services relative to private services, and $\varepsilon > 0$ is the elasticity of substitution. We set $\varepsilon = 1$, obtaining a Cobb-Douglas utility function $\mathscr{U}(c,g) = c^{\gamma} \cdot g^{1-\gamma}$.

We need to specify an interest rate schedule $r(x,g)$. A common assumption in the matching literature is that prices are efficient—they maintain the economy at the efficient unemployment rate u^*.[23] The efficient real interest rate is the interest rate r^* that ensures that unemployment is efficient. Hence, using equation (10), we have:

$$r^* = \delta - (1+\tau(x^*)) \cdot \frac{\mathscr{V}'(0)}{\frac{\partial \mathscr{U}}{\partial c}(y^* - g, g)}. \tag{39}$$

If the interest rate is continuously at r^*, the unemployment rate is always at u^*, and the Samuelson formula holds.

In practice, however, the economy experiences business cycles with fluctuations in unemployment. To describe these variations, we assume that the interest rate is not as flexible as the efficient interest rate. We consider an interest-rate schedule of the form

$$r(g) = \delta - \mu \cdot \frac{\mathscr{V}'(0)^{1-\alpha}}{\frac{\partial \mathscr{U}}{\partial c}(y^* - g, g)^{1-\beta}}. \tag{40}$$

The parameter $\mu > 0$ governs the level of the real interest rate. The parameter $\alpha \in [0,1]$ measures the rigidity of the real interest rate with respect to aggregate demand shocks: if $\alpha = 1$, the real

[23] Another typical assumption is that prices are determined by bargaining. Bargained prices usually have similar properties to efficient prices [Michaillat and Saez, 2015].

interest rate does not respond at all to aggregate demand shocks; if $\alpha = 0$, the real interest rate responds as much to aggregate demand shocks as the efficient real interest rate. When $\alpha = 0$ aggregate demand shocks are absorbed by the real interest rate, but when $\alpha > 0$ aggregate demand shocks generate fluctuations in tightness. The parameter $\beta \in [0,1]$ measures the rigidity of the real interest rate with respect to changes in the marginal utility of private services: if $\beta = 1$, the real interest rate does not respond at all to shocks to the marginal utility; if $\beta = 0$, the real interest rate responds as much to shocks to the marginal utility as the efficient real interest rate.

Appendix D shows that when the unemployment rate is efficient and government purchases are optimal, the unemployment multiplier simplifies to

$$M = \frac{\beta}{(G/Y)^* + \varepsilon \cdot [1 - (G/Y)^*]},$$

where β is the coefficient of the interest-rate schedule (40). When $\beta = 0$, government purchases shocks are absorbed by the real interest rate, the aggregate demand does not depend on government purchases, and the multiplier is zero. When $\beta > 0$, however, the multiplier is positive. With $\varepsilon = 1$, we set $\beta = 0.5$ to match $M = 0.5$ in the average state.

The rest of the calibration is standard and relegated to Appendix E. Since the values of the elasticity of substitution and unemployment multiplier are uncertain, Appendix F presents additional simulations targeting $\varepsilon = 0.5$, $\varepsilon = 2$, $M = 0.2$, and $M = 1$.

6.2. *Results*

We simulate our model under aggregate demand shocks. The economy jumps from one steady-state equilibrium to another in response to unexpected permanent shocks. Hence, we represent the business cycle as a succession of steady states. We simulate business cycles generated by aggregate demand shocks by computing a collection of steady states parameterized by different values for the marginal utility of wealth, $\mathscr{V}'(0)$. In each case, we perform two simulations: one in which the government-purchases ratio G/Y remains constant at 16.6%, its average value for 1951–2014, and one in which G/Y is at its optimal level, given by (22).

Figure 3 displays the results of the simulations. Each steady state is indexed by a marginal utility of wealth $\mathscr{V}'(0) \in [0.97, 1.03]$. On the one hand, the steady states with low $\mathscr{V}'(0)$ represent booms: they have a relatively low interest rate and low unemployment. On the other hand, the

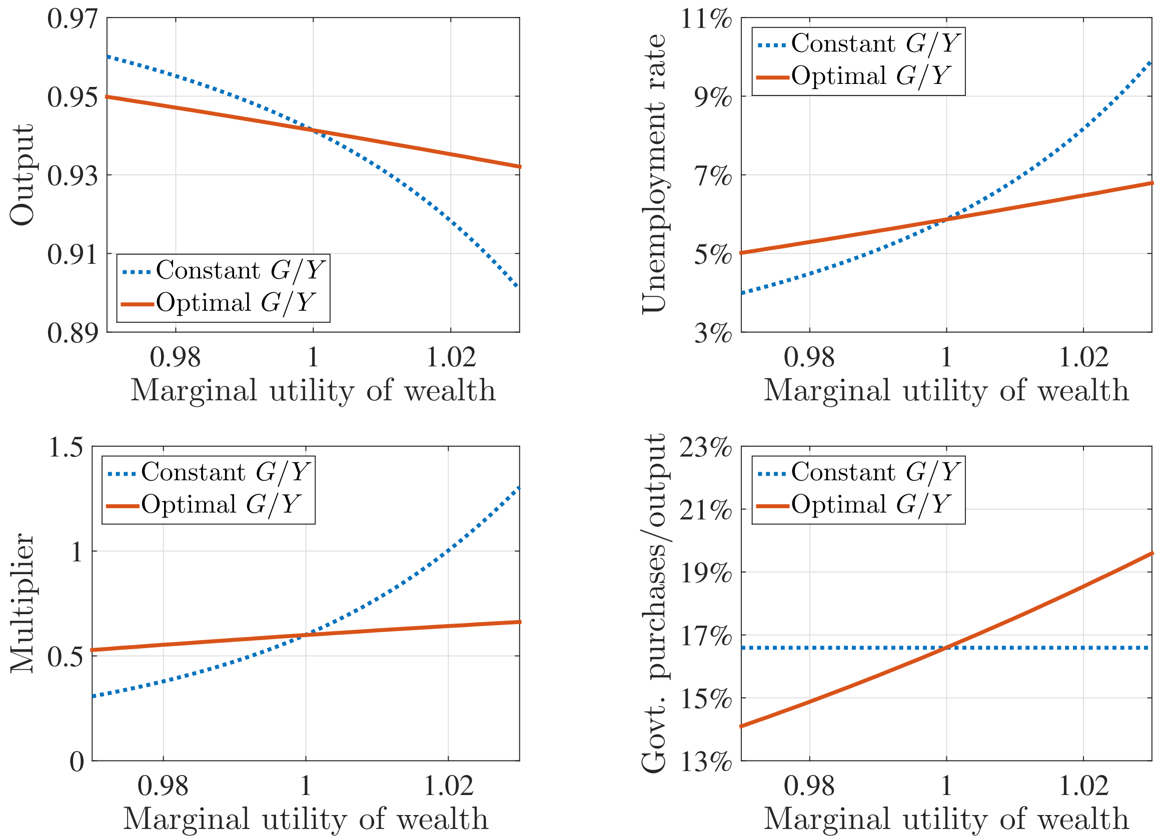


Figure 3: Simulations Under Aggregate Demand Shocks

steady states with high $\mathscr{V}'(0)$ represent slumps: they have a relatively high interest rate and high unemployment. Unemployment rises from 4.0% to 9.9%, and output falls accordingly, when $\mathscr{V}'(0)$ increases from 0.97 to 1.03 and G/Y remains constant.

On average the unemployment multiplier is 0.5, matching empirical evidence. The multiplier is sharply countercyclical, increasing from 0.3 to 1.3 when the unemployment rate increases from 4.0% to 9.9%. This sharp increase of the multiplier when the unemployment rate is high and output is low is consistent with the empirical evidence provided by Auerbach and Gorodnichenko [2012, 2013], Nakamura and Steinsson [2014], Jorda and Taylor [2016], and others that multipliers are higher when the unemployment rate is higher or output is lower.[24] The mechanism behind the countercyclicality of the multiplier is described by Michaillat [2014]. The size of the multiplier depends on the extent of crowding-out of household purchases by government purchases; the

[24] Contrary to this evidence, Owyang, Ramey and Zubairy [2013] and Ramey and Zubairy [2014] find that multipliers in the United States are not necessarily larger when there is more slack in the economy.

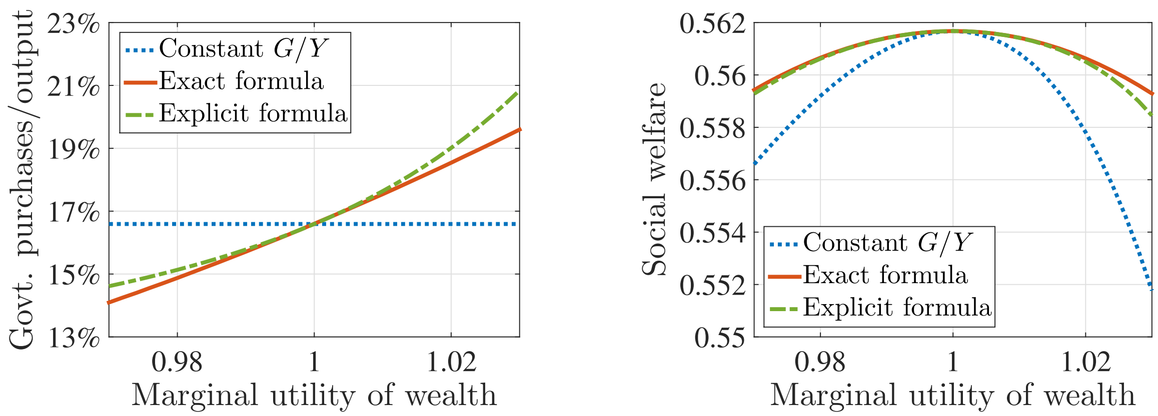


Figure 4: Accuracy of the Explicit Formula

crowding-out is determined by the amplitude of the increase in tightness. When unemployment is high, the government needs to advertise few vacancies to purchase additional services because the matching process is congested by sellers of services. Moreover, the idle capacity is so large that the vacancies posted and services purchased by the government have little influence on tightness. Consequently, when unemployment is high, the increase in tightness is small and crowding-out is weak after an increase in government purchases.

The model is calibrated so that the unemployment rate is efficient when $\mathscr{V}'(0) = 1$. Hence, the unemployment rate is inefficiently high when $\mathscr{V}'(0) > 1$ and inefficiently low when $\mathscr{V}'(0) < 1$. Since the multiplier is positive, G/Y should be more generous than the Samuelson ratio when $\mathscr{V}'(0) > 1$ and less generous when $\mathscr{V}'(0) < 1$. Indeed, the optimal G/Y is markedly countercyclical, increasing from 14.1% when $\mathscr{V}'(0) = 0.97$ to 19.6% when $\mathscr{V}'(0) = 1.03$.

Of course, the unemployment rate responds to the adjustment of G/Y from its original level of 16.6% to its optimal level. When $\mathscr{V}'(0) > 1$, the optimal G/Y is higher than 16.6% so the unemployment rate is below its original level: at $\mathscr{V}'(0) = 1.03$ the unemployment rate falls by 3.1 percentage points from 9.9% to 6.8%. When $\mathscr{V}'(0) < 1$, the optimal G/Y is below 16.6% so the unemployment rate is above its original level: at $\mathscr{V}'(0) = 0.97$ the unemployment rate increases by 1 percentage point from 4.0% to 5.0%.

Although the multiplier varies, which could be a source of inaccuracy, the explicit formula (23) is quite accurate. Figure 4 compares the ratios G/Y obtained with the exact formula (13) and with the explicit formula (23). As expected, at $\mathscr{V}'(0) = 1$, the two formulas gives the same G/Y. The approximation is less precise when the initial unemployment rate is further away from its efficient level, but it remains satisfactory: at $\mathscr{V}'(0) = 0.97$, the exact formula gives $G/Y = 14.1\%$ while the explicit formula gives $G/Y = 14.6\%$; at $\mathscr{V}'(0) = 1.03$, the exact formula gives $G/Y = 19.6\%$ while the explicit formula gives $G/Y = 20.8\%$. Despite these discrepancies, the social welfare values resulting from the two formulas are nearly identical.

7. Conclusion

We present a highly simplified theory of optimal government purchases. The implication of the theory is that as long as the unemployment multiplier and elasticity of substitution between public and private services are nonzero, government purchases should depart form the classical Samuelson level and contribute to stabilization by bringing the unemployment rate closer to its efficient level. The unemployment multiplier can be measured by directly from the effect of government purchases on the unemployment rate, or indirectly from the effect of government purchases on output net of the supply-side responses caused by distortionary taxation. Any available estimate of the unemployment multiplier is positive, and the elasticity of substitution is positive except if marginal public workers dig holes, so optimal government purchases are in all likelihood above the Samuelson level when unemployment is inefficiently high and below the Samuelson level when unemployment is inefficiently low.

Government purchases are therefore particularly useful for macroeconomic stabilization whenever monetary policy is constrained. This situation arises when the zero lower bound on nominal interest rates is binding (for instance, in the US, Japan, and the European Union in the aftermath of the Great Recession). The situation is also relevant for members of a monetary union, which face a fixed monetary policy but can tailor government purchases and taxes to local conditions (for instance, a country in the eurozone or a state in the United States subject to local shocks). In all these situations, adjusting government purchases as described in the paper could improve stabilization.

We focus on budget-balanced government purchases for two reason. First, it is conceptually useful to separate government purchases from government debt (we discuss optimal debt policy in Michaillat and Saez [2014]). Second, budget-balanced government purchases are empirically relevant. Indeed, US states cannot run deficits, eurozone countries face severe limits on their budget deficits, and often countries find it difficult to increase their budget deficit markedly in recessions.

The theory provides several insights that challenge existing views about the use of government purchases for stabilization. First, the cutoff value of the multiplier that justifies an increase in government purchases in slumps is 0, and not 1 as in macroeconomic models in which government purchases are wasteful. With any positive unemployment multiplier, it is optimal to increase government purchases above the Samuelson level when the unemployment rate is inefficiently high, even though government purchases crowd out household purchases.

Second, unlike in Keynesian theory, it is usually not optimal to completely fill the unemployment gap. Completely filling the unemployment gap is only optimal if public consumption is perfect substitute for private consumption. If, realistically, public consumption is an imperfect substitute for private consumption, the optimal government purchase policy only partially fills the unemployment gap.

Third, unlike what the "bang-for-the-buck" logic suggest, it is not necessarily optimal to respond more strongly to an increase in unemployment when the multiplier is larger. The relation between the size of the unemployment multiplier and the amplitude of the optimal increase in government purchases after an increase in unemployment is not increasing but hump-shaped, with a peak at an unemployment multiplier around 0.5 in a US calibration. Optimal government purchases increase less for multipliers above 0.5 because when multipliers are large, a higher multiplier means that fewer government purchases are required to fill the unemployment gap. Optimal government purchases increase less for multipliers below 0.5 because when multipliers are small, a smaller multiplier means that government purchases crowd out household purchases more and are therefore less desirable.

Fourth, when taxes are distortionary, it is not the standard output multiplier but the unemployment multiplier that should be used in our optimal government-purchases formula. Since the unemployment multiplier equals the output multiplier net of the supply-side response caused by distortionary taxation, it is well possible to have a negative output multiplier and a positive unemployment multiplier as shown empirically by Barro and Redlick [2011]. In that situation, a negative output multiplier does not mean that government purchases should decrease after an increase in unemployment; to the contrary, since the unemployment multiplier remains positive, and despite the negative output multiplier, government purchases should increase after an increase in unemployment.

Fifth, there is another statistic that has been neglected but that is as important as the multiplier to determine the optimal response of government purchases: the elasticity of substitution between public and private services. On the one hand, if the elasticity of substitution is zero, government purchases should remain at the Samuelson level. On the other hand, if the elasticity of substitution is infinite, government purchases should perfectly stabilize unemployment. In between these two polar cases, the deviation of the optimal government-purchases from the Samuelson level when unemployment deviates from its efficient level is larger for larger elasticities of substitution.

Our analysis is limited on several aspects. These limitations could be addressed in future work to gain a more complete understanding of the use of government purchases for stabilization. It would be interesting to include dynamic aspects to the analysis, such as dynamic smoothing of distortionary taxes over time and the use of debt, as in Barro [1979]. Applying the framework to study optimal government investment on infrastructure projects could also be interesting.

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Appendix A: The Rapidity of the Transitional Dynamics of the Unemployment Rate

In the analysis, we abstract from transitional dynamics for the unemployment rate and output. In this appendix we use US data for 1951–2014 to validate this approximation. We argue here that these transitional dynamics are quantitatively negligible. Given that output and unemployment rate are linearly related in the model ($Y = 1 - u$), it suffices to make the case for one of the two variables. Using the same labor market data as in Section 5, we focus on the unemployment rate.

We begin by constructing a time series for the selling rate f_t. We measure one unit of service by one job. The selling rate therefore is a selling rate. We assume that unemployed workers find a job according to a Poisson process with arrival rate f_t. Under this assumption, the monthly selling rate satisfies $f_t = -\ln(1 - F_t)$, where F_t is the monthly job-finding probability. We construct a time series for F_t following the method developed by Shimer [2012]. We use the relationship

$$F_t = 1 - \frac{u_{t+1} - u^s_{t+1}}{u_t},$$

where u_t is the number of unemployed persons at time t and u^s_t is the number of short-term unemployed persons at time t. We measure u_t and u^s_t in the data constructed by the BLS from the CPS. The number of short-term unemployed persons is the number of unemployed persons with zero to four weeks duration, adjusted as in Shimer [2012] for the 1994–2014 period. Panel A of Figure A1 displays the monthly selling rate and its trend. The selling rate averages 56% between 1951 and 2014.

Next, we construct the separation rate following the method developed by Shimer [2012]. The separation rate s_t is implicitly defined by

$$u_{t+1} = \left(1 - e^{-f_t - s_t}\right) \cdot \frac{s_t}{f_t + s_t} \cdot h_t + e^{-f_t - s_t} \cdot u_t,$$

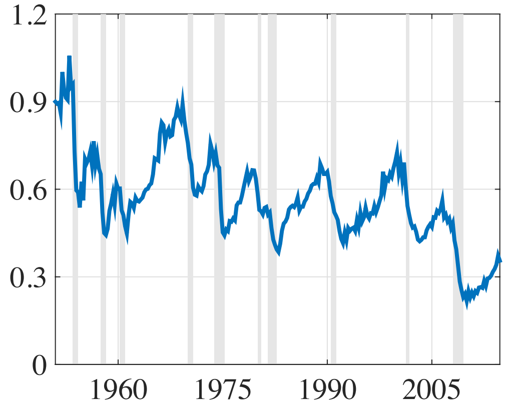
where h_t is the number of persons in the labor force at time t, u_t is the number of unemployed persons at time t, and f_t is the monthly selling rate. We measure u_t and h_t in the data constructed by the BLS from the CPS, and we use the series that we have just constructed for f_t. Panel B of Figure A1 displays the monthly separation rate and its trend. The separation rate averages 3.3% between 1951 and 2014.[25]

Finally, we compare the actual unemployment rate and the steady-state unemployment rate

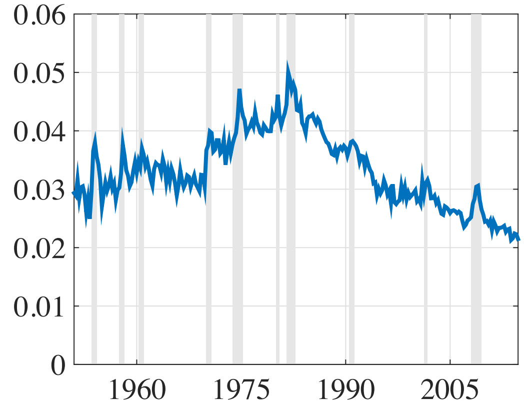
$$u_t = \frac{s_t}{f_t + s_t}. \tag{A1}$$

The two series, displayed in Panel C of Figure A1, are almost identical. Since the actual unemploy-

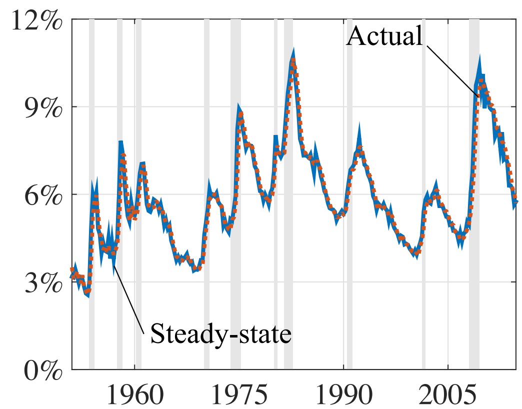
[25] One concern is that increases in government purchases cannot be undone because long-term relationships created by the government, especially employment relationships, are effectively permanent. It is true that government relationships separate more slowly than private relationships: using data constructed by the BLS from the JOLTS for 2000–2014, we find that the average monthly separation rate is 3.9% for jobs in the private sector and 1.4% for jobs in the government sector. Nevertheless, the separation rate for government relationships remains sizable. If no new relationships were created by the government, the level of government purchases would rapidly decrease: with a hiring freeze, US government employment would fall by $1 - \exp(-0.014 \cdot 12) = 15\%$ in one year.



A. job-finding rate (per month)



B. Separation rate (per month)



C. Actual and steady-state unemployment rates

Figure A1: The Irrelevance of Transitional Dynamics For the Unemployment Rate

Notes: Panel A: The selling rate f is constructed from CPS data following the methodology of Shimer [2012]. The series $\overline{f}$ is the low-frequency trend of f produced using a HP filter with smoothing parameter 10^5. Panel B: The separation rate s is constructed from CPS data following the methodology of Shimer [2012]. The series $\bar{s}$ is the low-frequency trend of s produced using a HP filter with smoothing parameter 10^5. Panel C: The actual unemployment rate is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the BLS from the CPS. The steady-state unemployment rate is computed using (A1): $u_t = s_t/(f_t + s_t)$; this rate abstracts from transitional dynamics. The two series are almost identical showing that transitional dynamics are quantitatively unimportant. The shaded areas represent the recessions identified by the NBER.

ment rate barely departs from its steady-state level, the transitional dynamics of the unemployment rate are unimportant.[26]

Appendix B: Okun's Law in the United States for 1951–2014

In this appendix, we revisit Okun's law in the United States for 1951–2014. Okun's law is a statistical relationship between deviations of output from trend and deviations of unemployment from trend. It was first proposed by Okun [1963], who found that in US data for 1954–1962, output was 3 percent below trend when the unemployment rate was 1 percentage point above trend.

We estimate Okun's law in US data for the 1951–2014 period and for the recent 1994–2014 period. We find that the relationship between output gap and unemployment gap has evolved over time.

We measure output Y by the real GDP constructed by the BEA as part of the NIPA. We produce the trend Y^* of output using a HP filter with smoothing parameter 10^5. Panel A of Figure A2 displays Y and Y^*. We use the unemployment rate u by unemployment rate constructed by the BLS from the CPS. We produce the trend u^* of unemployment using a HP filter with smoothing parameter 10^5.. Panel B of Figure A2 displays u and u^*.

Okun's law is the following linear relationship:

$$\frac{Y_t - Y_t^*}{Y_t^*} = -\chi \cdot (u_t - u_t^*) \tag{A2}$$

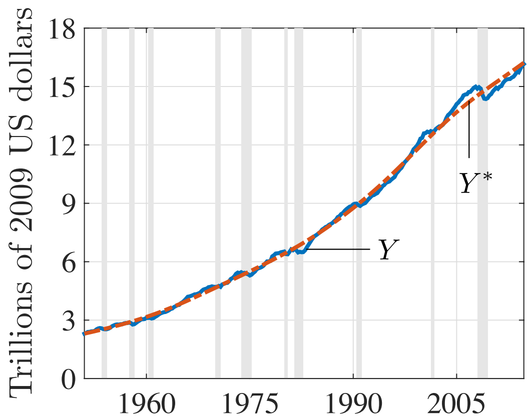
The coefficient χ was estimated around 3 by Okun [1963] in US data for 1954–1962. Regressing $(Y_t - Y_t^*)/Y_t^*$ on $(u_t - u_t^*)$ with ordinary least squares, we estimate a coefficient of 1.8 on the 1951–2014 period and a coefficient of 1.3 on the 1994–2014 period. Panels C and D illustrate Okun's law for the two periods. We conclude that when the unemployment rate is 1 percentage point above trend, output is 1.8 percent below trend in the 1951–2014 period and 1.3 percent below trend in the 1994–2014 period. In our model, we have $Y(t) = (1-u(t)) \cdot k$ and hence $dY/Y = -du/(1-u)$ which leads to an Okun's coefficient of $1/(1-u) \approx 1.06$ as the average unemployment rate is $u = 5.9\%$. The model coefficient is below the empirical coefficient of 1.3 for the recent 1994–2014 period, but not extremely far from it.

Appendix C: Proof of Proposition 3

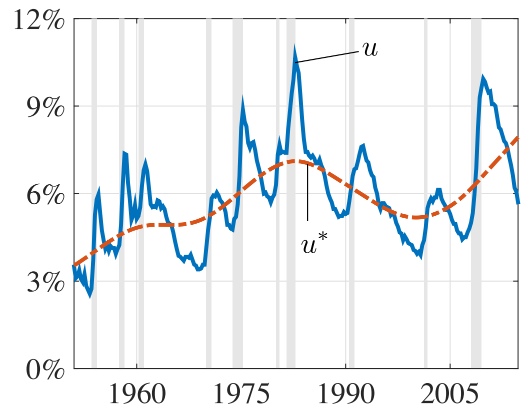
In a feasible allocation, all the variables are functions of g and x. Equivalently, they can be defined as functions of G/C and x. Focusing on feasible allocations, we can therefore write formula (13) as $\Omega(G/C, x) = 0$ where

$$\Omega(G/C, x) = 1 - MRS_{gc} - \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

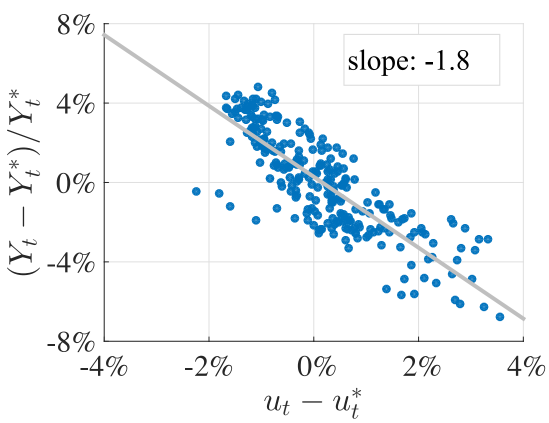
[26]Panel C of Figure A1 is similar to Figure 1 in Hall [2005*b*]. Even though we use different measures of the job-finding and separation rates and a longer time period, Hall's conclusion that transitional dynamics are irrelevant remains valid.



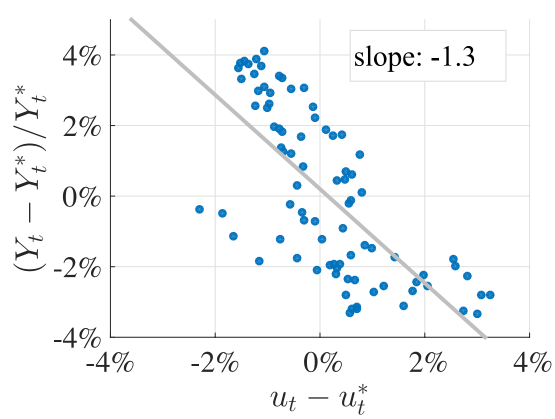
A. Output



B. Unemployment rate



C. Okun's law for 1951–2014



D. Okun's law for 1994–2014

Figure A2: Okun's Laws in the United States, 1951–2014

Notes: Panel A: Output Y is seasonally adjusted quarterly real GDP in chained 2009 dollars constructed by the BEA as part of the NIPA. The series Y^* is the low-frequency trend of Y produced using a HP filter with smoothing parameter 10^5. Panel B: The unemployment rate u is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the BLS from the CPS. The series u^* is the low-frequency trend of u produced with a HP filter with smoothing parameter 10^5. The shaded areas in Panels A and B represent the recessions identified by the NBER. Panel C: The series Y, Y^*, u, and u^* used to construct the scatter plot are from Panels A and B. These series cover the 1951–2014 period. The plot also displays the regression line used to estimate the coefficient χ in (A2). Panel D: This plot is obtained from the scatter plot in Panel C by restricting the data to the 1994–2014 period.

By definition, $MRS_{gc}=1$ at $(G/C)^*$ and $\partial y/\partial x=0$ at x^* so $\Omega((G/C)^*,x^*)=0$. Hence the first-order Taylor expansion of Ω around $[(G/C)^*,x^*]$ is

$$\Omega(G/C,x)=\frac{\partial\Omega}{\partial(G/C)}\cdot[G/C-(G/C)^*]+\frac{\partial\Omega}{\partial x}\cdot(x-x^*)+O(\|\mathbf{w}\|^2) \tag{A3}$$

where the derivatives are evaluated at $[(G/C)^*,x^*]$ and $\mathbf{w}\equiv[G/C-(G/C)^*,x-x^*]\in\mathbb{R}^2$ and $\|\cdot\|$ is any norm on $\mathbb{R}^2$.

The first step to computing the partial derivatives of Ω is to compute the partial derivatives of $1-MRS_{gc}$ at $[(G/C)^*,x^*]$. With homothetic preferences, MRS_{gc} is a function of G/C only and $\partial MRS_{gc}/\partial x=0$. Furthermore, by definition of ε,

$$-\frac{\partial MRS_{gc}}{\partial(G/C)}=\frac{1}{\varepsilon}\cdot\frac{MRS_{gc}((G/C)^*)}{(G/C)^*}=\frac{1}{\varepsilon}\cdot\frac{1}{(G/C)^*}.$$

The second step to computing the partial derivatives of Ω is to compute the partial derivatives of $(\partial y/\partial x)\cdot(dx/dg)$ at $[(G/C)^*,x^*]$. Consumption $y(x,k)$ is a function of x only, so $y'(x)$ is a function of x only. We therefore have

$$\begin{aligned}\frac{\partial(y'(x)\cdot x'(g))}{\partial(G/C)}&=y'(x^*)\cdot\frac{\partial x'(g)}{\partial(G/C)}=0\\ \frac{\partial(y'(x)\cdot x'(g))}{\partial x}&=y''(x^*)\cdot x'(g^*)+y'(x^*)\cdot\frac{\partial x'(g)}{\partial x}=y''(x^*)\cdot x'(g^*).\end{aligned} \tag{A4}$$

From this we infer that

$$\frac{\partial\Omega}{\partial(G/C)}=\frac{1}{\varepsilon}\cdot\frac{1}{(G/C)^*}. \tag{A5}$$

It only remains to compute $y''(x^*)$ and $x'(g^*)$.

The elasticity of $u(x)$ is $-(1-\eta)\cdot(1-u(x))$ and the elasticity of $\tau(x)$ is $\eta\cdot(1+\tau(x))$ so the elasticity of $y(x)=(1-u(x))/(1+\tau(x))$ is $(1-\eta)\cdot u(x)-\eta\cdot\tau(x)$ and hence

$$y'(x)=\frac{y(x)}{x}\cdot(1-\eta)\cdot u(x)\cdot\underbrace{\left(1-\frac{\eta}{1-\eta}\cdot\frac{\tau(x)}{u(x)}\right)}_{\equiv z(x)}. \tag{A6}$$

Since $z(x^*)=0$, we have

$$y''(x^*)=\frac{y(x^*)}{x^*}\cdot(1-\eta)\cdot u(x^*)\cdot z'(x^*).$$

Using the elasticity of τ and u, we infer that the elasticity of τ/u is $\eta\cdot(1+\tau(x))+(1-\eta)\cdot(1-u(x))=1+\eta\cdot\tau(x)-(1-\eta)\cdot u(x)$. At $x=x^*$, $\eta\cdot\tau(x^*)-(1-\eta)\cdot u(x^*)=0$ so the elasticity of τ/u is 1 and

$$z'(x^*)=-\frac{\eta}{1-\eta}\cdot\frac{\tau(x^*)/u(x^*)}{x^*}=-\frac{1}{x^*}.$$

We conclude that

$$y''(x^*) = -\frac{y(x^*)}{(x^*)^2} \cdot (1-\eta) \cdot u(x^*). \tag{A7}$$

In equilibrium, $Y = Y(x,k)$, $x = x(g)$, and $G = (1+\tau(x(g))) \cdot g$. We can therefore differentiate Y in two different ways:

$$\frac{d\ln(Y)}{d\ln(g)} = \frac{d\ln(Y)}{d\ln(G)} \cdot \frac{d\ln(G)}{d\ln(g)} = \frac{d\ln(Y)}{d\ln(x)} \cdot \frac{d\ln(x)}{d\ln(g)}.$$

As the elasticity of $1+\tau(x)$ with respect to x is $\eta \cdot \tau$ and $G = (1+\tau(x)) \cdot g$, we find that

$$\frac{d\ln(G)}{d\ln(g)} = 1 + \eta \cdot \tau \cdot \frac{d\ln(x)}{d\ln(g)}.$$

Since the elasticity of $Y(x,k)$ with respect to x is $(1-\eta) \cdot u$, we conclude that

$$\frac{d\ln(Y)}{d\ln(G)} \cdot \left(1 + \eta \cdot \tau(x) \cdot \frac{d\ln(x)}{d\ln(g)}\right) = (1-\eta) \cdot u(x) \cdot \frac{d\ln(x)}{d\ln(g)}.$$

Some algebra yields

$$x'(g) = \frac{x}{g} \cdot \frac{1}{(1-\eta) \cdot u} \cdot \frac{(G/Y) \cdot (dY/dG)}{1 - \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot (G/Y) \cdot (dY/dG)}. \tag{A8}$$

Using the results that $(1-\eta) \cdot u(x^*) = \eta \cdot \tau(x^*)$ and $G/Y = g/y$, we find that at $[(G/C)^*, x^*]$

$$x'(g^*) = \frac{x^*}{y(x^*)} \cdot \frac{1}{(1-\eta) \cdot u^*} \cdot m,$$

where m is defined by (19). Combining this equation with (A7) as showed by (A4), we conclude that

$$\frac{\partial \Omega}{\partial x} = \frac{m}{x^*}. \tag{A9}$$

The combination of (A3), (A5), and (A9) yields (22).

In a feasible allocation, all the variables can be expressed as a function of G/C and x; in particular, $g = y \cdot (g/y) = y \cdot (G/Y) = y(x) \cdot (G/C)/(1+G/C)$. Among all the feasible allocations, the equilibrium allocations satisfy the constraint $x = x(g)$, or equivalently $g = x^{-1}(x)$. An equilibrium allocation must therefore satisfy

$$\Lambda(G/C, x) \equiv x^{-1}(x) - y(x) \cdot \frac{G/C}{1+G/C} = 0.$$

By definition, $[(G/C)^*, x_0]$ is an equilibrium so $\Lambda((G/C)^*, x_0) = 0$. Accordingly, the first-order Taylor expansion of Λ around $[(G/C)^*, x_0]$ is

$$\Lambda(G/C, x) = \frac{\partial \Lambda}{\partial (G/C)} \cdot [G/C - (G/C)^*] + \frac{\partial \Lambda}{\partial x} \cdot (x - x_0) + O(\|\mathbf{w}\|^2), \tag{A10}$$

where the derivatives are evaluated at $[(G/C)^*, x_0]$ and $\mathbf{w} \equiv [G/C - (G/C)^*, x - x_0] \in \mathbb{R}^2$.

Using (A8), we infer that at $[(G/C)^*, x_0]$,

$$\frac{dx^{-1}}{dx} = \frac{y(x_0)}{x_0} \cdot (1-\eta) \cdot u(x_0) \cdot \frac{1 - \frac{\eta}{1-\eta} \cdot \frac{\tau(x_0)}{u(x_0)} \cdot (G/Y)^* \cdot (dY/dG)_0}{dY/dG}.$$

Here the multiplier $(dY/dG)_0$ is evaluated at $[(G/C)^*, x_0]$. Equation (A6) gives the expression for $y'(x_0)$. Last, simple algebra indicates yields

$$\frac{\partial\,[G/C/(1+G/C)]}{\partial(G/C)} = \frac{1}{(1+G/C)^2} = (C/Y)^2 = \frac{C}{Y} \cdot \frac{C}{G} \cdot \frac{G}{Y}.$$

Combining these results, we find that at $[(G/C)^*, x_0]$

$$\frac{\partial \Lambda}{\partial(G/C)} = y(x_0) \cdot (C/Y)^* \cdot \frac{(G/Y)^*}{(G/C)^*}, \qquad \frac{\partial \Lambda}{\partial x} = \frac{y(x_0)}{x_0} \cdot \frac{(1-\eta) \cdot u(x_0)}{m_0},$$

where m_0 is defined by (19) with G/Y and M evaluated at $[(G/C)^*, x_0]$.

In an equilibrium allocation, $\Lambda(G/C, x) = 0$. Using (A10), we infer that tightness in an equilibrium allocation is related to government purchases by

$$x - x_0 \approx \frac{-\partial\Lambda/\partial(G/C)}{\partial\Lambda/\partial x} \cdot \left(G/C - (G/C)^*\right),$$

where the approximation is valid up to a remainder that is $O((x-x_0)^2 + (G/C - (G/C)^*)^2)$. Combining this approximation with the expressions for the partial derivatives of Λ, we obtain

$$\frac{x - x_0}{x_0} \approx -z_0 \cdot m_0 \cdot \frac{G/C - (G/C)^*}{(G/C)^*}. \tag{A11}$$

where z_0 is defined by (24) with G/Y, C/Y and u evaluated at $[(G/C)^*, x_0]$.

Formula (22) describes how government purchases are related to tightness in a feasible allocation in which government purchases are optimal. In an equilibrium, starting from an inefficient allocation $[(G/C)^*, x_0]$, tightness responds endogenously to government purchases as described by (A11). In an equilibrium in which government purchases are optimal, both (22) and (A11) are satisfied simultaneously. Substituting x in (22) using (A11) and doing a bit of algebra, we find that the optimal level of government purchases approximately satisfy

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \underbrace{\frac{-\varepsilon^* \cdot m^*}{1 + \varepsilon^* \cdot m^* \cdot z_0 \cdot m_0 \cdot \frac{x_0}{x^*}}}_{\equiv \Gamma(x_0, x^*, (G/C)^*)} \cdot \frac{x_0 - x^*}{x^*},$$

where m^* is defined by (19) with G/Y and M evaluated at $[(G/C)^*, x^*]$ and ε^* is defined by (15) with G/C evaluated at $(G/C)^*$. This approximation is valid up to a remainder that is $O((x_0 - x^*)^2 + (G/C - (G/C)^*)^2)$. With a first-order Taylor approximation around $[x^*, x^*, (G/C)^*]$ we can write

$\Gamma(x_0, x^*, (G/C)^*) = \Gamma(x^*, x^*, (G/C)^*) + (\partial\Gamma/\partial x_0) \cdot (x_0 - x^*) + O((x_0 - x^*)^2)$. Hence we have

$$\frac{G/C - (G/C)^*}{(G/C)^*} \approx \Gamma(x^*, x^*, (G/C)^*) \cdot \frac{x_0 - x^*}{x^*}$$

up to a remainder that is $O((x_0 - x^*)^2 + (G/C - (G/C)^*)^2)$, which establishes (23).

Equation (26) is obtained by identifying (23) and (22), both of which hold in the equilibrium with optimal government purchases.

Appendix D: The Unemployment Multiplier in the Simulation Model

In this appendix we derive an expression for the unemployment multiplier in the simulation model described in Section 6. We use this expression for the calibration for he model. The expression also shows that the multiplier is higher when the unemployment rate is higher.

Proposition A1. *In the simulation model, the multiplier is obtained from*

$$\frac{d\ln(Y)}{d\ln(G)} = \left[1 - (1-\beta) \cdot \frac{1 - G/Y}{1 - G/Y^*} \cdot \frac{\kappa(G/(Y-G))}{\kappa(G/(Y^*-G))}\right] \cdot \left[1 + \varepsilon \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot \frac{C}{Y} \cdot \kappa\left(\frac{G}{C}\right)\right]^{-1}.$$

where the auxiliary function κ is defined by

$$\kappa(\theta) = 1 + \frac{1-\gamma}{\gamma} \cdot \theta^{\frac{1-\varepsilon}{\varepsilon}}. \tag{A12}$$

When the unemployment rate is efficient and government purchases are given by the Samuelson formula, the unemployment multiplier simplifies to

$$M = \frac{\beta}{(G/Y)^* + \varepsilon \cdot [1 - (G/Y)^*]}.$$

In addition, if the elasticity of substitution between public and private services is $\varepsilon = 1$, the unemployment multiplier simplifies to $M = \beta$.

Proof. The proof of the proposition proceeds in five steps.

Step 1. Using equation (38) and simple algebra, we write the marginal utility of consumption as

$$\frac{\partial \mathscr{U}}{\partial c}(c, g) = (1-\gamma) \cdot \mathscr{U}\left(1, \frac{G}{C}\right)^{\frac{1}{\varepsilon}}.$$

Furthermore, the elasticity of $\mathscr{U}(1,\theta)$ with respect to θ is $1/\kappa(\theta)$, where $\kappa(\theta)$ is given by (A12).

Step 2. Using the results from step 1 and equation (40), we rewrite the interest-rate schedule as

$$\delta - r = \frac{\mu}{(1-\gamma)^{1-\beta}} \cdot \mathscr{V}'(0)^{1-\alpha} \cdot \mathscr{U}\left(1, \frac{G}{Y^* - G}\right)^{-\frac{1-\beta}{\varepsilon}}.$$

The results from step 1 and more algebra imply that the elasticity of the interest-rate schedule is

$$\frac{d\ln(\delta-r)}{d\ln(G)} = -\frac{1-\beta}{\varepsilon}\cdot\frac{1}{\kappa\left(G/\left(Y^*-G\right)\right)}\cdot\frac{Y^*}{Y^*-G}.$$

Note that $\delta - r$ depends only on G and not on x.

Step 3. We implicitly define $C(G,x)$ as the solution of

$$\mathscr{U}\left(1,\frac{G}{C}\right)^{-\frac{1}{\varepsilon}} = \frac{1-\gamma}{\mathscr{V}'(0)}\cdot\frac{\delta-r(G)}{1+\tau(x)}.$$

The function $C(G,x)$ is the household purchases that satisfy the Euler equation (10) for government purchases G and a tightness x. The results from steps 1 and 2 and simple algebra imply that

$$\frac{\partial\ln(C)}{\partial\ln(x)} = -\varepsilon\cdot\eta\cdot\tau\cdot\kappa\left(\frac{G}{C}\right)$$
$$\frac{\partial\ln(C)}{\partial\ln(G)} = 1-(1-\beta)\cdot\frac{Y^*}{Y^*-G}\cdot\frac{\kappa(G/C)}{\kappa\left(G/\left(Y^*-G\right)\right)}.$$

Step 4. The equilibrium condition determining market tightness is

$$Y = C(G,x)+G.$$

We differentiate this equilibrium condition with respect to G:

$$\frac{d\ln(Y)}{d\ln(G)} = \frac{C}{Y}\cdot\left(\frac{\partial\ln(C)}{\partial\ln(G)}+\frac{\partial\ln(C)}{\partial\ln(x)}\cdot\frac{d\ln(x)}{d\ln(G)}\right)+\frac{G}{Y}$$

Equation (1) implies that

$$\frac{d\ln(x)}{d\ln(G)} = \frac{1}{(1-\eta)\cdot u}\cdot\frac{d\ln(Y)}{d\ln(G)}.$$

Using these equations and the elasticities from step 3, we obtain

$$\left[1+\frac{C}{Y}\cdot\varepsilon\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot\kappa\left(\frac{G}{C}\right)\right]\cdot\frac{d\ln(Y)}{d\ln(G)} = 1-(1-\beta)\cdot\frac{Y-G}{Y}\cdot\frac{Y^*}{Y^*-G}\cdot\frac{\kappa(G/(Y-G))}{\kappa\left(G/\left(Y^*-G\right)\right)}$$
$$\frac{d\ln(Y)}{d\ln(G)} = \frac{1-(1-\beta)\cdot\frac{1-G/Y}{1-G/Y^*}\cdot\frac{\kappa(G/(Y-G))}{\kappa(G/(Y^*-G))}}{1+\frac{C}{Y}\cdot\varepsilon\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot\kappa\left(\frac{G}{C}\right)}.$$

Step 5. When the unemployment rate is efficient, $Y = Y^*$ and $1 = [\eta/(1-\eta)]\cdot\tau/u$. Hence, the expression for the multiplier simplifies to

$$\frac{d\ln(Y)}{d\ln(G)} = \frac{\beta}{1+\varepsilon\cdot\frac{C}{Y}\cdot\kappa\left(\frac{G}{C}\right)}.$$

Furthermore, when government purchases satisfy the Samuelson formula, $G/C = (G/C)^* = [\gamma/(1-\gamma)]^{\varepsilon}$ so $\kappa(G/C) = (Y/G)^*$ and the multiplier simplifies to

$$M = \frac{dY}{dG} = \frac{\beta}{(G/Y)^* + \varepsilon \cdot [1-(G/Y)^*]}.$$

Finally, if $\varepsilon = 1$, then $dY/dG = \beta$. □

Appendix E: Complete Calibration of the Simulation Model

In this appendix we calibrate the model of Section 6 to US data for 1951–2014. The calibration is summarized in Table A1.

Values of variables targeted for calibration

We calibrate several parameters such that variables in the average state are equal to their average value measured in the data. We target an average unemployment rate $\overline{u} = 5.9\%$, an average market tightness $\overline{x} = 0.65$, an average government-purchases ratio $\overline{G/Y} = 16.6\%$. These average values come from the times series constructed in Section 5 and Appendix A. We also normalize the values of marginal utility of wealth in the average state to $\overline{\mathscr{V}'(0)} = 1$.

If the market is efficient on average, then (5) implies that $\overline{\tau} = \overline{u} \cdot (1-\eta)/\eta$, where $\overline{\tau}$ is the average value of the matching wedge and $\overline{u}$ the average value of the unemployment rate. We set $\eta = 0.46$ as estimated in the main text and $\overline{u} = 5.9\%$, and obtain $\overline{\tau} = 6.9\%$.

Values of calibrated parameters

We begin by calibrating the three parameters determining the statistics at the heart of our formulas. Based on the discussion in Section 5, we calibrate the model to obtain an elasticity of substitution between public and private services of 1, a elasticity of the selling rate of 0.54, and a multiplier in the average state of 0.5; hence, we set $\varepsilon = 1$, $\eta = 0.46$, and $\beta = 0.5$.[27]

Next, we calibrate parameters related to matching. We set the separation rate to its average value for 1951–2014: $s = 3.3\%$ (see Appendix A). To calibrate the matching efficacy, we exploit the relationship $\overline{u} \cdot f(\overline{x}) = s \cdot (1-\overline{u})$, which implies $\omega = s \cdot (\overline{x})^{\eta-1} \cdot (1-\overline{u})/\overline{u} = 0.67$. To calibrate the vacancy-filling cost, we exploit the relationship $\overline{\tau} = \rho \cdot s/[\omega \cdot (\overline{x})^{-\eta} - \rho \cdot s]$, which implies $\rho = \omega \cdot (\overline{x})^{-\eta} \cdot \overline{\tau}/[s \cdot (1+\overline{\tau})] = 1.6$.

Then, we calibrate the parameters of the utility function. We find that $MRS_{gc} = [\gamma/(1-\gamma)] \cdot (G/C)^{-1/\varepsilon}$. Given that $MRS_{gc}((G/C)^*) = 1$, we infer that $\gamma/(1-\gamma) = ((G/C)^*)^{1/\varepsilon}$. We assume that the average of the ratio G/C is the Samuelson ratio so $(G/C)^* = 19.9\%$. With $\varepsilon = 1$ and $(G/C)^* = 19.9\%$, we set $\gamma = 0.17$.

Last, we calibrate the parameters of the interest-rate schedule. For aggregate demand shocks to generate fluctuations, we need $\alpha > 0$. The value of α determines the elasticity of output to the marginal utility of wealth, $\mathscr{V}'(0)$. Since we do not know the amplitude of the fluctuations of $\mathscr{V}'(0)$, the exact value of α is irrelevant; we arbitrarily set $\alpha = 1$. Last, using (40) in the average state and

[27] Appendix D establishes the link between β and the multiplier.

Table A1: Calibration of the Simulation Model (Monthly Frequency)

Value	Description	Source
	Panel A. Values of variables targeted for calibration	
$\overline{u} = 5.9\%$	Unemployment rate	CPS, 1951–2014
$\overline{x} = 0.65$	Market tightness	Barnichon [2010], JOLTS, CPS, 1951–2014
$\overline{G/Y} = 16.6\%$	Government-purchases ratio	CES, 1951–2014
$\overline{\tau} = 6.8\%$	Matching wedge	Efficiency on average (see Appendix A)
$\overline{M} = 0.5$	Unemployment multiplier	Literature (see Section 5)
$\overline{\mathscr{V}'(0)} = 1$	Marginal utility of wealth	Normalization
	Panel B. Values of calibrated parameters	
$1 - \eta = 1/3$	Elasticity of the selling rate $f(x)$	Landais, Michaillat and Saez [2010*b*]
$s = 3.3\%$	Separation rate	CPS, 1951–2014
$\omega = 0.67$	Matching efficacy	Matches targets
$\rho = 1.6$	Matching cost	Matches targets
$\varepsilon = 1$	Elasticity of substitution	Section 5
$\gamma = 0.17$	Parameter of utility function	Matches $\overline{G/Y} = 16.6\%$
$\alpha = 1$	Parameter of interest-rate schedule	Normalization
$\beta = 0.5$	Parameter of interest-rate schedule	Matches $\overline{M} = 0.5$
$\mu = 1.4$	Parameter of interest-rate schedule	Matches targets

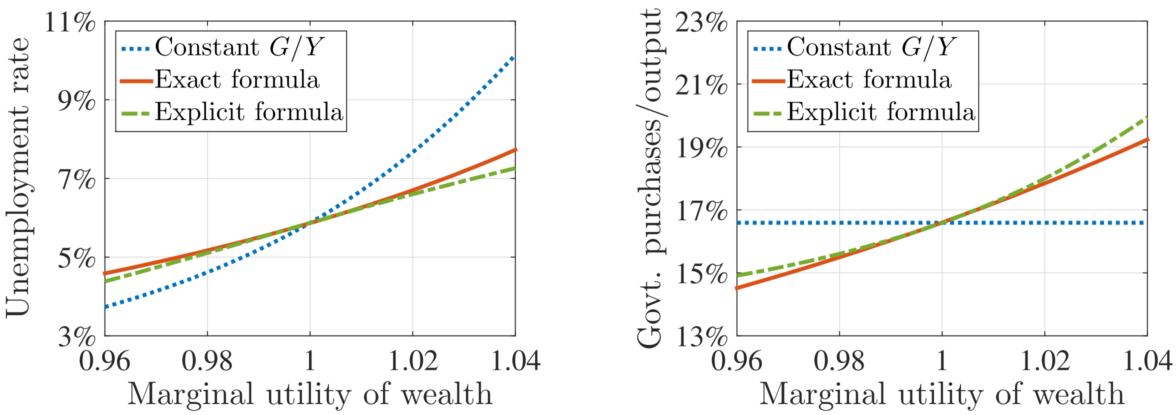
the expression (38) for $\mathscr{U}$, we find that $\mu = (\delta - r^*) \cdot \left[(1-\gamma) \cdot \mathscr{U}(1, (G/C)^*)^{1/\varepsilon}\right]^{1-\beta}$. The expression (39) implies that in the average state $\delta - r^* = (1 + \tau(x^*)) \Big/ \left[(1-\gamma) \cdot \mathscr{U}(1, (G/C)^*)^{1/\varepsilon}\right]$. Combining these expressions, we infer that $\mu = (1 + \overline{\tau}) / \left[(1-\gamma) \cdot \mathscr{U}(1, \overline{G/C})^{1/\varepsilon}\right]^{\beta}$. Using the calibrated values of all the parameters, we obtain $\mu = 1.4$.

Appendix F: Robustness of Simulation Results

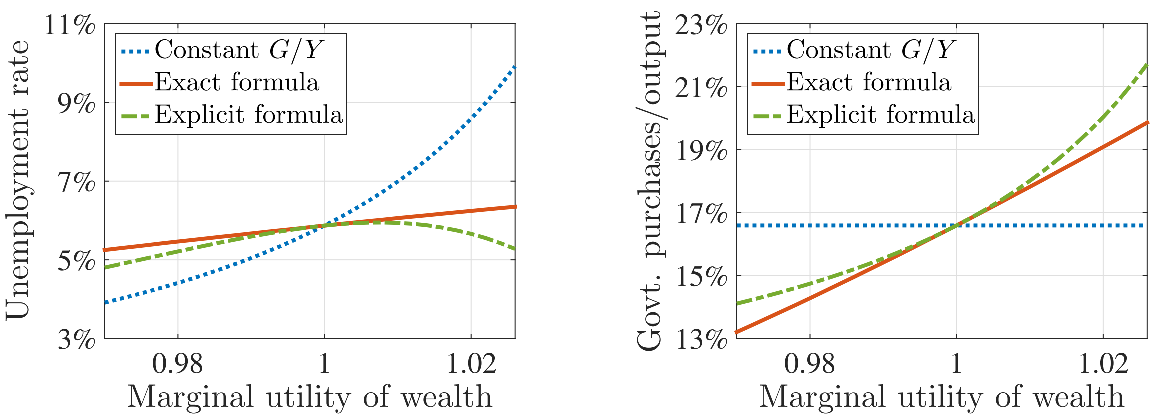
The simulation results in Section 5 are obtained for an elasticity of substitution between public and private services of $\varepsilon = 1$ and an average unemployment multiplier of $M = 0.5$. In this appendix we repeat the simulations for alternative values of ε and M.

Figure A3 displays simulations for $\varepsilon = 0.5$ and $\varepsilon = 2$. The figure shows that when ε is lower, the optimal government-purchases ratio responds less to fluctuations in unemployment, and consequently, fluctuations in unemployment are less attenuated. When $\varepsilon = 0.5$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.1\%$, which reduces the unemployment rate to 7.6%. But when $\varepsilon = 2$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.9\%$, which reduces the unemployment rate to 6.3%. The figure also shows that the explicit formula (23) is more accurate for lower values of ε.

Figure A4 displays simulations for $M = 0.2$ and $M = 1$. The figure shows that a higher value of M does not imply that optimal government purchases respond more strongly to a rise in unemployment; it does imply, however, that fluctuations in unemployment are attenuated. When $M = 0.2$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 19.7\%$, which only reduces the unemployment rate to 8.8%. But when $dY/dG = 1$ and the unemployment rate reaches 9.9%, optimal government purchases increase to $G/Y = 18.5\%$, which reduces the unemployment rate to 6.2%. The figure also shows that the explicit formula (23) is more accurate for lower values of M.

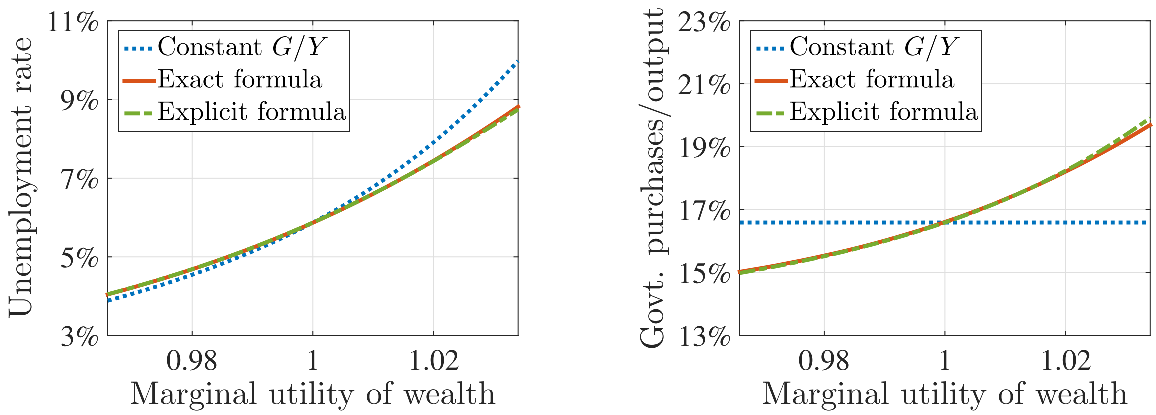


A. $\varepsilon = 0.5$ and $M = 0.5$

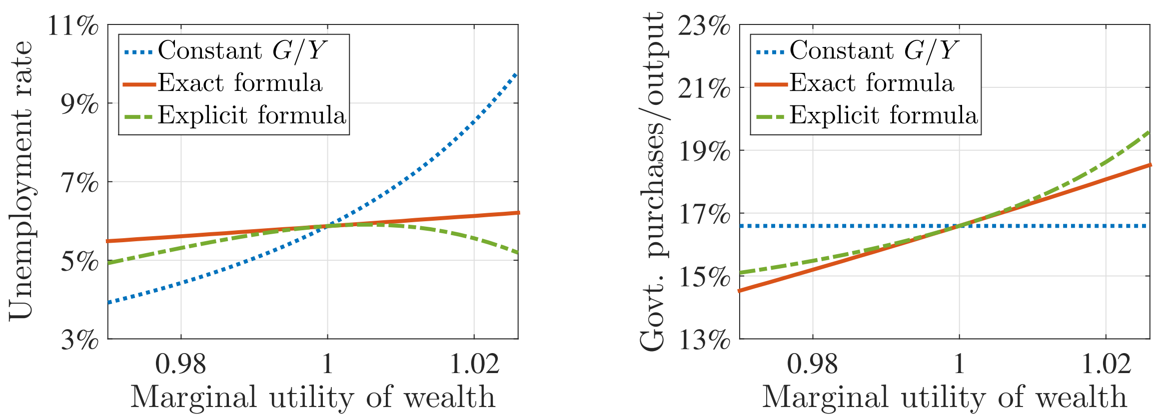


B. $\varepsilon = 2$ and $M = 0.5$

Figure A3: Simulations for Various Elasticities of Substitution



A. $\varepsilon = 1$ and $M = 0.2$



B. $\varepsilon = 1$ and $M = 1$

Figure A4: Simulations for Various Unemployment Multipliers