The Role of Learning for Stock Prices and Business Cycles

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Abstract

The effects of financial frictions in business cycle models depend heavily on the underlying asset pricing theory. I examine the implications of learning-based asset pricing in a model in which firms face credit constraints that depend partly on their market value. Agents are learning about stock prices, but have conditionally model-consistent expectations otherwise. The model jointly matches key asset price and business cycle statistics, while the combination of financial frictions and learning produces powerful feedback between asset prices and real activity, adding substantial amplification of business cycle shocks. Patterns of predictability in agents' subjective forecast errors closely match survey data.

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Financial frictions are seen as a central mechanism by which asset prices interact with macroeconomic dynamics. Yet our understanding of this interaction remains incomplete, in part due to the inherent difficulty of modelling asset prices. Typical business cycle models still rely on an asset pricing theory based on rational expectations, time-separable preferences and moderate degrees of risk aversion. It is well

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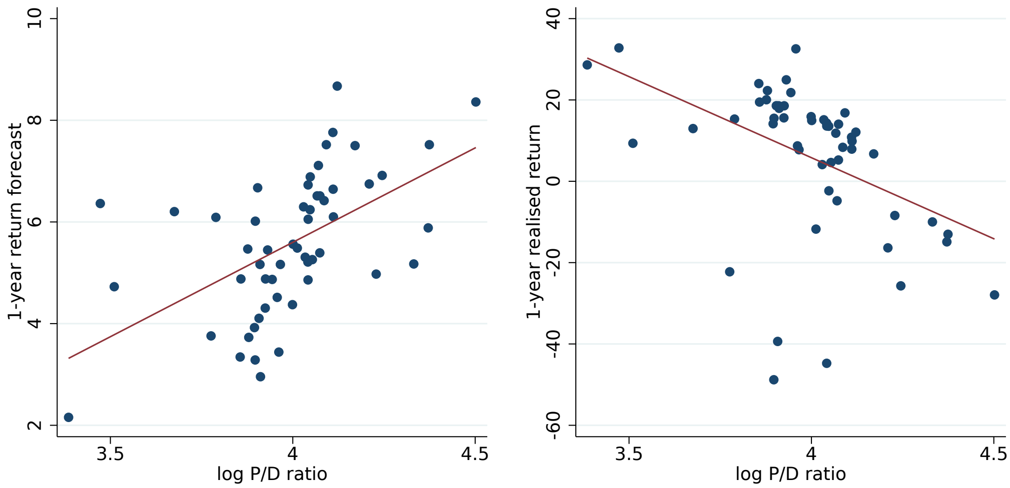
Figure 1: Return expectations and expected returns.



Expected nominal returns (left) are the mean response in the Graham-Harvey survey, realized nominal returns (right) and P/D ratio are from the S&P 500. Data period 2000Q32012Q4. Correlation coefficient for return forecasts $\rho = .54$, for realized returns $\rho = -.44$.

known that such a theory is inadequate for many empirical asset price regularities. At the same time, asset prices play a central role in macroeconomic dynamics in the presence of financial frictions. Conclusions drawn from models with financial frictions but without a good asset pricing theory are therefore questionable.

There is not a shortage of theories that aim to explain asset price dynamics. Most keep the rational expectations assumption and engineer preferences that deliver highly volatile discount factors. There are however compelling arguments for relaxing the rational expectations assumption instead. Measurements of expectations in surveys do not support the rational expectations hypothesis. The hypothesis implies, for example, that investors are fully aware of return predictability in the stock market, expecting lower returns when prices are high and vice versa. Instead, measured expectations imply they expect higher returns. This pattern has been documented extensively by Greenwood and Shleifer (2014) and is illustrated in Figure 1. The left panel plots the mean 12-month return expectation of the S&P500, as measured in the Graham-Harvey survey of American CFOs, against the value of the P/D ratio in the month preceding the survey. The correlation is strongly positive: Return expectations are more optimistic when stock valuations are high. This contrasts sharply with the actual return predictability in the right panel of the figure, where the correlation is strongly negative. Unless one rejects surveys as an unbiased measure of expectations, such a pattern cannot be reconciled with rational expectations.

Based on such observations, Adam, Marcet and Nicolini (2015) have developed an asset pricing theory based on learning. The interpretation of price dynamics there is quite different: Stock prices fluctuate not because of variations in discount factors and expected compensation for risk, but because of variations in subjective expectations about prices and returns. The deviation of subjective from rational expectations is a natural measure of "price misalignments." In an endowment economy, this approach is able to explain the most common asset price puzzles remarkably well.

In this paper, I examine the implications of a learning-based asset pricing theory for the business cycle. I construct a model of firm credit frictions in which agents are learning about price growth in the stock market, as in Adam, Marcet and Nicolini (2015). At the same time, the model has a "financial accelerator" mechanism in which asset prices play a key role. Firms are subject to credit constraints, the tightness of which depends on their market value. The constraints emerge from a limited commitment problem in which defaulting firms can be restructured and resold as opposed to being liquidated. It provides a mechanism by which high stock market valuations translate into easier access to credit.

Deviating from the rational expectations hypothesis is not without problems. One needs to explicitly spell out the entire belief formation process, filling many degrees of freedom. The existing learning literature often suffers from a lack of transparency in this respect, or abandons expected utility maximization altogether in favor of more reduced-form equilibrium conditions. To address this problem, I develop a restriction on expectations that I call "conditionally model-consistent expectations." It can be seen as a refinement of the "internal rationality" requirement developed by Adam and Marcet (2011). Agents continue to maximize a well-defined stable objective function with coherent and time-consistent beliefs about the variables affecting their decisions. They can entertain arbitrary beliefs about one relative price in the economy, which in this paper is the price of equity. But their beliefs about any other variable must be consistent with the equilibrium conditions of the model. When agents endowed with these expectations evaluate their forecast errors, they find that their forecasting rules cannot be improved upon conditional on their subjective belief about stock prices. In this sense this is a minimal departure from rational expectations. What's more, spelling out a belief system for stock prices and then imposing conditionally model-consistent expectations is all that is needed to obtain a unique dynamic equilibrium. This allows me to parsimoniously incorporate asset price learning into any forward-looking model.

The model jointly matches key business cycle and asset price moments under learning. By contrast, the rational expectations version of the model cannot match asset price moments well, and also needs larger shocks to match the volatility of the business cycle. This points to a large amount of endogenous amplification under learning, and suggests that the relatively weak quantitative strength of the financial accelerator effect in many standard models—as discussed in (Cordoba and Ripoll, 2004)—is in part due to weak endogenous asset price volatility. Under learning, a positive feedback loop emerges between asset prices and the production side of the economy. When beliefs of learning investors are more optimistic, their demand for stocks increases. This raises firm valuations and relaxes credit constraints, in turn allowing firms to move closer to their profit optimum. Firms are able to pay higher dividends to their shareholders, raising stock prices further and propagating investor optimism.

I then compare the subjective forecasts of agents in the model with actual forecasts in survey data. Even though agents only learn about stock prices in the model, their expectational errors spill over into forecasts of other variables. For example, when agents are too optimistic about asset prices, they also become too optimistic about the tightness of credit constraints and therefore over-predict future investment. The model-generated expectations replicate remarkably well the predictability of forecast errors for several predictors and across a range of forecast variables.

In a series of sensitivity checks, I show that nominal rigidities greatly enhance the amplification effects of the feedback loop under learning. This is in part due to their ability to generate comovement in macroeconomic aggregates following shifts in borrowing constraints. The interest rate rule followed by the monetary authority also plays a role in the amplification mechanism. In particular, a positive response of interest rates to stock price growth is able to greatly reduce endogenous volatility by effectively stabilizing asset price expectations.

The remainder of the paper is structured as follows. Section 1 briefly discusses the related literature. Section 2 presents a simplified version of the model that permits an analytic solution. It shows that credit frictions or asset price learning alone does not generate either amplification of shocks or interesting asset price dynamics, although their combination does. The full model is presented in Section 3. Section 4 contains the quantitative results. Section 5 contains sensitivity checks, including the effects of different interest rate rules. Section 6 concludes.

1 Related literature

The early literature on financial frictions emphasized their role for amplyfing business cycle shocks (Kiyotaki and Moore, 1997), but the quantitative importance of the "financial accelerator" mechanism is often found to be small (Quadrini, 2011). The more recent literature on financial frictions instead emphasizes shocks to borrowing constraints as independent drivers of the business cycle (e.g. Jermann and Quadrini, 2012), or alternatively introduces direct shocks to asset prices. Xu, Wang and Miao (2013) develop a model in which borrowing limits depend on stock market valuations through a credit friction similar to that in my model. They prove the existence of rational liquidity bubbles and introduce a shock that governs the size of this bubble. Liu, Wang and Zha (2013) use a similar framework with land prices instead of stock prices. This paper takes a different approach by going back to the question of financial frictions as an amplification mechanism. Learning endogenously generates volatility in asset prices, and interacts with financial frictions to form a feedback loop that amplifies standard business cycle shocks.

Besides learning, there exist of course other asset pricing theories to address asset pricing puzzles, including habit, long-run risk, and disaster risk. Each of them has been shown to be compatible with standard business cycle moments in production economies (Boldrin, Christiano and Fisher, 2001, for habit; Tallarini Jr., 2000, for long-run risk; and Gourio, 2012, for disaster risk). But each of them also has its problems (e.g. Lettau and Uhlig 2000; Epstein, Farhi and Strzalecki 2013), and none of them is able to explain the disconnect between statistically predicted returns and expectations of returns in surveys. There is currently no consensus among economists which asset pricing theory is "best", and the goal of this paper is not to set up a horse race between learning and alternative asset pricing theories. Rather, the goal is to examine in detail the interaction of learning-based asset pricing and financial frictions.

There are a number of papers that study models of financial frictions in combination with learning (Caputo, Medina and Soto, 2010; Milani, 2011; Gelain, Lansing and Mendicino, 2013). Their approach consists of two steps: first, derive the linearized equilibrium conditions of the economy under rational expectations; second, replace all terms involving expectations with parameterized forecast functions, and update the parameters every period. Such an approach certainly produces very rich dynamics, but is problematic on several grounds. First, such parameterized expectation equations often do not correspond anymore to intertemporal optimization problems. Second, the analysis of these models is often complex and lack transparency. Here, I develop a more transparent and parsimonious approach. Beliefs are restricted to be conditionally model-consistent and agents make optimal choices given a consistent set of beliefs. This preserves much of the intuition of a rational expectations model, and at the same time allows for "spillovers" of forecast errors on asset prices into other forecasts. The approach also differs from that of Fuster, Hebert and Laibson (2012). There, agents learn only about exogenous variables, and endogenous outcomes can therefore not feed back into beliefs. In this model, agents learn about endogenous prices, with a feedback loop between beliefs and real activity.

Finally, the paper relates to the research on survey data on expectations. It is well known that expectations measured in surveys fail to conform to the rational expectations hypothesis because forecast errors are statistically predictable (e.g. Croushore, 1997). Coibion and Gorodnichenko (2015) interpret forecast error predictability as evidence in favour of rational inattention models. The model in this paper produces similar predictability statistics in a learning model.

2 Simplified model

In this section, I construct a simplified version of the model which illustrates the interaction between credit frictions and learning about asset prices. For the sake of brevity, I omit a formal description of the model, which can be found in the appendix. The key insight is that neither learning nor financial frictions alone generate sizable amplification of business cycle shocks or asset price volatility in a production economy, while in combination they do.

2.1 Model setup

The economy consists of a representative household and a representative firm. There are two physical goods, labor and a consumption/investment good. The household is risk-neutral with discount rate β. It inelastically supplies one unit of labor and also holds the debt and equity claims on the firm. Debt claims pay a gross real interest rate R. For the debt market to be in equilibrium, the interest rate has to equal $R = 1/\beta$. Equity claims trade at a price P_t and pay dividends D_t. For the stock market to be in equilibrium, the following Euler equation has to hold:

$$P_t = \beta \mathbb{E}_t^{\mathcal{P}} \left[P_{t+1} + D_{t+1} \right]. \tag{1}$$

Note that the expectation operator is evaluated under the probability measure $\mathcal{P}$. Agents use this measure when forming their expectations to solve their optimization problems. Under learning, the distribution of outcomes expected under $\mathcal{P}$ does not necessarily coincide with the distribution induced in equilibrium.

The representative firm operates a constant returns to scale technology in capital K_{t-1} and labor L_t. Labor is hired at the competitive wage rate w_t. The capital stock depreciates at the rate δ and must be financed entirely with borrowed funds. The firm faces a leverage constraint by which its debt claims cannot exceed a fraction ξ of its total market value (equity and debt). For simplicity, all earnings are assumed to be paid out as dividends every period and the number of shares outstanding is fixed at unity. The firm maximizes expected dividends as follows:

$$\mathbb{E}_t^{\mathcal{P}} D_{t+1} = \max_{K_t, L_{t+1}} \mathbb{E}_t^{\mathcal{P}} \left[K_t^{\alpha} \left(A_{t+1} L_{t+1} \right)^{1-\alpha} - w_{t+1} L_{t+1} + (1-\delta) K_t - R K_t \right] \tag{2}$$

$$\text{s.t. } 0 \leq K_t \leq \frac{\xi}{1-\xi} P_t \tag{3}$$

Note that the borrowing constraint (3) relates the level of the capital stock K_t (equal to the value of the firm's outstanding debt) to the value of its equity. The microfoundation of this constraint is discussed in the next section. For the labor market to be in equilibrium, the wage rate w_t has to adjust such that the firm demands $L_t = 1$. Define the rationally expected return on capital as:

$$R^k\left(K_t, A_t\right) = \alpha \left(\frac{A_t}{K_t}\right)^{1-\alpha} \mathbb{E}_t\left[\varepsilon_{t+1}^{1-\alpha}\right] + 1 - \delta. \tag{4}$$

Finally, the only exogenous shock in the model is a permanent innovation to productivity, which evolves as:

$$\log A_t = \log A_{t-1} + \varepsilon_t,\ \varepsilon_t \sim iid\,\mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right). \tag{5}$$

2.2 Rational expectations equilibrium

I first describe the equilibrium under rational expectations.[1] Start with the case $\xi = 1$. In this limiting case, the borrowing constraint (3) can never bind. The firm invests up to the efficient level where the expected return on capital equals the interest rate. As a result, capital is simply proportional to productivity: $K_t/A_t = K^*$ for some fixed value K^*.

Once we introduce financial frictions by setting $\xi < 1$, how much amplification do we get? The answer is: none. For all values of ξ strictly below one, the borrowing constraint is always binding, and the equilibrium is characterized by the following two equations:

$$P_t = A_t\bar{P} \quad = \quad A_t\frac{\left(R^k\left(\bar{K},1\right) - R\right)\bar{K}}{R-1} \tag{6}$$

$$K_t = A_t\bar{K} \quad = \quad \frac{\xi}{1-\xi}P_t \tag{7}$$

The first equation pins down the stock market value of the firm, which depends on the capital stock through expected dividends in the enumerator. These dividends depend on capital through the size of the firm and the rate of return on capital. The second equation determines the capital stock that can be reached by exhausting the borrowing constraint that depends on the stock market value. In the unique equilibrium, the capital stock is proportional to productivity, just as was the case when $\xi = 1$.

Financial frictions do not lead to any amplification or propagation of shocks in the rational expectations equilibrium. They have a *level* effect on output, capital, etc., but the *dynamics* of the model are identical for any value of ξ. Similarly, the behavior of asset prices is entirely independent of ξ. The stock price evolves simply as:

$$\log P_t = \log P_{t-1} + \varepsilon_t. \tag{8}$$

[1] In a rational expectations equilibrium, the distribution of outcomes under $\mathcal{P}$ coincides with the equilibrium distribution of outcomes. In that case, one can write $\mathbb{E}^{\mathcal{P}}\left[\cdot\right] = \mathbb{E}\left[\cdot\right]$.

Intuitively, with financial frictions, a shock to productivity raises asset prices just as much as to allow the firm to instantly adjust the capital stock proportionately. At the same time, stock returns are not volatile and unpredictable at all horizons. The complete irrelevance of financial frictions for the model dynamics is particular to the assumptions placed on the model, but it illustrates nevertheless why financial frictions often have small quantitative effects.

2.3 Learning equilibrium

I now describe the equilibrium under learning. Conceptually, the only difference to a rational expectations equilibrium is that the measure $\mathcal{P}$ under which agents evaluate expectations can differ from the actual distribution of model outcomes. Otherwise, agents continue to make optimal choices given their expectations are such that all markets clear—the equilibrium satisfies "internal rationality" (Adam and Marcet, 2011).

How, then, is the subjective belief system $\mathcal{P}$ defined? First of all, agents are not endowed with the knowledge of the equilibrium law of motion for stock prices (8).[2] Instead, under $\mathcal{P}$ agents believe that the stock price P_t evolves as follows:

$$\log P_t = \log P_{t-1} + \mu_t + \eta_t \tag{9}$$

$$\mu_t = \mu_{t-1} + \nu_t \tag{10}$$

$$\text{where } \begin{pmatrix} \eta_t \\ \nu_t \end{pmatrix} \sim iid\,\mathcal{N}\left(-\frac{1}{2}\begin{pmatrix} \sigma_\eta^2 \\ \sigma_\nu^2 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\nu^2 \end{pmatrix}\right). \tag{11}$$

This specification is identical to the one in Adam, Marcet and Nicolini (2015). With an appropriate prior, Bayesian updating of this belief system amounts to a simple Kalman filtering problem where the belief about μ_t is updated in the direction of the last forecast error: When agents see stock prices rising faster than they expected, they will also expect them to rise by more in the future.

In a forward-looking general equilibrium model such as this one, and even more so in the full model of the next section, there are many more expectations affecting the equilibrium other than those about P_t. Households need to forecast future dividends in order to determine their demand for stocks in (6). Firms need to forecast future productivity and wages in order to decide their demand for capital in (2). This leaves many degrees of freedom to be filled. My focus in this paper is to concentrate on the effects of stock price learning, while remaining as close as possible to rational expectations. To this end, I impose that agents know the true distribution of the exogenous shock ε_t, and that they have c*onditionally model-consistent expectations* with respect to the stock price P_t. To be precise, I impose the following:

[2] Intuitively, one can imagine that agents are unable to determine that the representative household is the only investor in the market, instead being unsure about the aggregate demand schedule for stocks and the resulting equilibrium price process.

Assumption. Let y_t be the collection of all endogenous model variables. Let g be the actual law of motion that recursively describes the equilibrium under learning: $y_t = g(y_{t-1}, \varepsilon_t)$, with the stock price evolving as $P_t = g_P(y_{t-1}, \varepsilon_t)$. Agents' subjective expectations about y_t under $\mathcal{P}$ are assumed to also follow a subjective recursive law of motion h: $y_t = h(y_{t-1}, \varepsilon_t, P_t)$, which is such that expected and realized outcomes coincide conditional on stock prices:

$$\mathbb{E}_{t-1}^{\mathcal{P}}[y_t \mid \varepsilon_t, P_t] = h(y_{t-1}, \varepsilon_t, g_P(y_{t-1,}, \varepsilon_t)) = g(y_{t-1}, \varepsilon_t) = y_t. \tag{12}$$

This way of restricting expectation formation is new to the learning literature. It implies that, while agents do not know the equilibrium pricing function g_P, they make the smallest possible expectational errors consistent with their subjective view about the evolution of stock prices. The subjective law of motion h can be derived from the equilibrium equations of the model, taking out the market clearing condition in the stock market (and that in the final goods market, by Walras' law) and replacing it with the subjective law of motion (9)–(11). The solution procedure is further discussed in Section 3.3.

In this simple model, the equilibrium under learning is easy to compute. It consists of the following three equations:

$$P_t = \frac{\left(R^k(K_t, A_t) - R\right)K_t}{R - \exp\left(\hat{\mu}_t + \frac{1}{2}\sigma_\mu^2\right)} \tag{13}$$

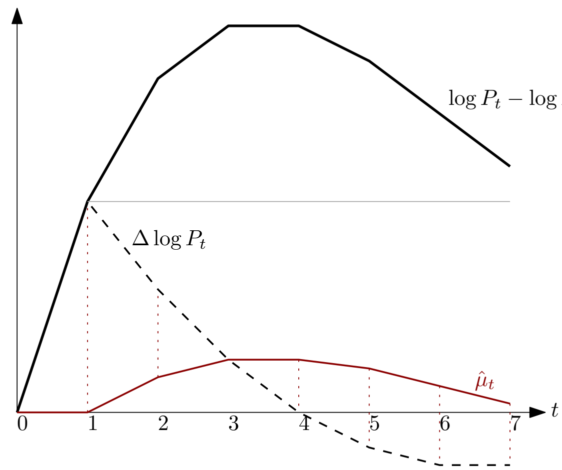
$$K_t = \frac{\xi}{1-\xi} P_t \tag{14}$$

$$\hat{\mu}_{t+1} = \hat{\mu}_t - \frac{\sigma_\nu^2}{2} + g\left(\log\frac{P_t}{P_{t-1}} - \hat{\mu}_t + \frac{\sigma_\eta^2 + \sigma_\nu^2}{2}\right) \tag{15}$$

The second equation is the borrowing constraint. The third equation is the updating equation of the Kalman filtering problem solved by an agent who believes (9)–(11), with $\hat{\mu}_t = \mathbb{E}_{t-1}^{\mathcal{P}}[\mu_t]$.[3] Some comment is necessary on the first equation. It is obtained by imposing equilibrium in the stock market, i.e. solving Equation (1) for P_t under the subjective belief measure $\mathcal{P}$. The expectation of future prices $\mathbb{E}_t^{\mathcal{P}} P_{t+1}$ is easily substituted with the perceived law of motion (9). The expectation of future dividends $\mathbb{E}_t^{\mathcal{P}} D_{t+1}$ is found by applying conditionally model-consistent expectations. These imply that i) all agents share the same belief system, so that the household expects dividend payments consistent with the firm's expected optimal choice; ii) the firm itself correctly forecasts future productivity A_{t+1} and expects future wages w_{t+1}

[3] I effectively impose that forecasts of stock prices are updated only after equilibrium prices are determined. This "lagged belief updating" is common in the learning literature. It makes all feedback between forecasts and prices inter- rather than intratemporal. For further discussion see Adam, Beutel and Marcet (2014).

Figure 2: Stock price dynamics under learning.



that are consistent with labor market clearing. As a result, the one period-ahead expectation of dividends coincides with rational expectations.[4]

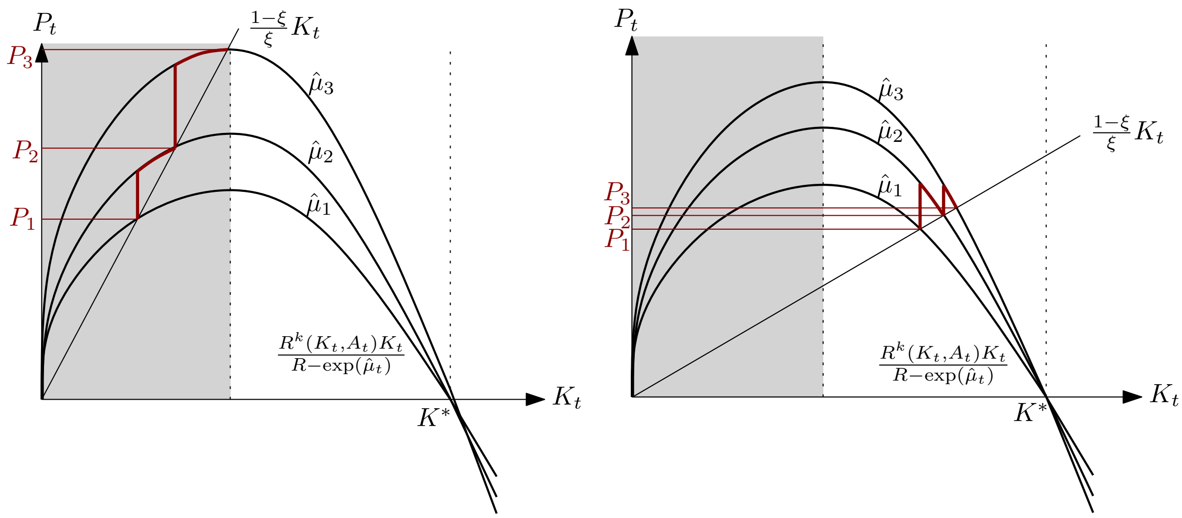
Figure 2 depicts the dynamics of stock prices after a positive productivity innovation $\varepsilon_1 > 0$. The initial shock at $t = 1$ raises stock prices and the capital stock proportionally to productivity through Equations (13) and (14), just as under rational expectations. Learning investors observe the rise in P_1 and are unsure whether it is due to a transitive shock ($\eta_1 > 0$) or a permanent increase in the growth rate of stock prices ($\nu_1 > 0$). They therefore revise their beliefs $\hat{\mu}_2$ upward in Equation (15). In the next period $t = 2$, the more optimistic beliefs increase the demand for stocks, and the market clearing price Equations (13) has to be higher, in turn relaxing credit constraints and fueling investment. Beliefs continue to rise in subsequent periods as long as observed asset price growth (dashed black line in Figure 2) is higher than the current belief $\hat{\mu}_t$ (solid red line). The differences between observed and expected price growth are the subjective forecast errors (dotted red lines). In the figure, the increase in prices and beliefs ends at $t = 3$, when the forecast error is zero. There is no need for a further belief revision. But in the absence of subsequent shocks, no change in $\hat{\mu}_t$ implies no change in the price P_t, so that realized asset price growth is zero at $t = 4$, at a time when agents expect strongly positive price growth. This triggers a downward revision in beliefs and an endogenous reversal in prices. Ultimately prices return to their steady-state level.

These learning dynamics lead to return volatility and predictability. To see this, it is convenient to look at the forward P/D ratio:

$$\frac{P_t}{\mathbb{E}_t^{\mathcal{P}} D_{t+1}} = \frac{1}{R - \exp\left(\hat{\mu}_t + \frac{1}{2}\sigma_\mu^2\right)}.$$

[4]However, the n-period ahead expectation $\mathbb{E}_t^{\mathcal{P}} D_{t+n}$ for $n \geq 2$ does not coincide with rational expectations, as it depends on the expectation of P_{t+n-1}.

Figure 3: Endogenous response of dividends.



(a) Amplification.

(b) Dampening.

The forward P/D ratio is directly related to the belief $\hat{\mu}_t$, and even small changes in this belief can have a large impact on the P/D ratio as the denominator is close to zero. Furthermore, since the system (13)–(14) is stationary, a high P/D ratio predicts a future decline in $\hat{\mu}_t$ and therefore falling prices and low returns.

Learning affects economic activity because it influences current stock prices and therefore the tightness of the borrowing constraint. But the feedback in this model is two-sided, as real activity affects stock prices through dividend payments. The strength of this channel depends on general equilibrium effects. Equation (13) shows that K_t enters the expression for expected dividends twice. The multiplying factor K_t captures a partial equilibrium effect which is internalized by the firm. The internal rate of return on capital is higher than the cost of debt and the firm therefore wants to increase its capital stock until it exhausts the borrowing constraint, *increasing* expected dividends. At the same time though, higher levels of capital lower its marginal return $R^k(K_t, A_t)$ because of decreasing returns to scale at the aggregate level, increasing wages and *decreasing* expected dividends.

When financial frictions are severe enough (ξ is low), the partial equilibrium effect dominates. This case is depicted in Panel (a) of Figure 3. The figure plots the stock pricing equation (13) and the credit constraint (14). When the degree of financial frictions is high, the credit constraint line is steep. Consider the effect of a positive productivity shock at $t = 1$ as before, when the initial equilibrium is at P_1 and $\hat{\mu}_1$. The immediate effect will be a proportionate rise in stock prices and capital, together with a rise in beliefs from $\hat{\mu}_1$ to $\hat{\mu}_2$. This leads to higher stock prices at $t = 2$ and allows the firm to invest more and increase its expected profits—the partial equilibrium effect dominates. This adds to the rise in realized stock prices, further relaxing the borrowing constraint and increasing next period's beliefs. Stock prices, investment, and output all rise more than proportionally to productivity.

In Panel (b), credit constraints are not very severe (ξ is large) and the credit constraint line is relatively flat. A relaxation of the borrowing constraint due to a rise in $\hat{\mu}_2$ still allows the firm to invest and produce more, but in general equilibrium wages rise by so much that dividends fall. This response of dividends dampens rather than amplifies the dynamics of investment and asset prices.

This dampening effect can be so strong as to eliminate the effects of learning altogether. The appendix shows that in the limit of vanishing financial frictions, stock prices become a pure random walk again:

$$\log P_t - \log P_{t-1} \xrightarrow{\xi \to 1} \varepsilon_t.$$

As a consequence, the entire dynamics of the model become identical to those under rational expectations. As financial frictions disappear, the general equilibrium effects offset any dynamics from stock price learning. This shows that sizable amplification arises neither from learning nor from credit frictions alone, but only from their interaction.

3 Full model for quantitative analysis

This section embeds the mechanism discussed so far into a quantitative New-Keynesian model with a financial accelerator. Compared with the simple model in the previous section, I introduce a number of additional elements. First, firms are allowed to finance capital out of retained earnings. Second, the borrowing constraint is generalized and microfounded by a limited commitment problem. Third, I add investment adjustment costs and nominal rigidities. The former improve the quantitative fit, while the latter help to attenuate the general equilibrium effects discussed above.

3.1 Model setup

The economy is closed and operates in discrete time. It is populated by two types of households. *Lending households* consume final goods and supply labor. They trade debt claims on intermediate goods producers and receive interest from them. *Firm owners* only consume final goods. They trade equity claims on intermediate goods producers and receive dividends from them.

The two households own four types of firms. Only the first type is substantial to the model analysis: *Intermediate goods producers* (or simply *firms*) combine capital and differentiated labor to produce a homogeneous intermediate good. They are owned by the firm owners, borrow funds from households, and are financially constrained. The other three types of firms serve only to add nominal rigidities and adjustment costs to the model. They are owned by the households. *Labor agencies* transform homogeneous household labor into differentiated labor services, which they sell to intermediate goods producers. *Final good producers* transform intermediate goods into differentiated final goods. *Capital goods producers* produce new capital goods from final consumption goods subject to an investment adjustment cost.

Finally, there is a *fiscal authority* setting tax rates to offset steady-state distortions from monopolistic competition, and a *monetary authority* setting nominal interest rates. Most elements of the model are standardand their desription is relegated to the appendix.

3.1.1 Households

A representative household with time-separable preferences maximizes utility as follows:

$$\max_{\left(C_t, L_t, B_{jt}, B_t^g\right)_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t \log\left(C_t\right) - \eta \frac{L_t^{1+\phi}}{1+\phi}$$

$$\text{s.t. } C_t = \tilde{w}_t L_t + B_t^g - \left(1+i_{t-1}\right) \frac{p_{t-1}}{p_t} B_{t-1}^g + \int_0^1 \left(B_{jt} - R_{jt-1} B_{jt-1}\right) dj + \Pi_t$$

Here, $\tilde{w}_t$ is the real wage received by the household and L_t is the amount of labor supplied. B_t^g are real quantities of nominal one-period government bonds (in zero net supply) that pay a nominal interest rate i_t and p_t is the price level, defined below. Households also lend funds B_{jt} to intermediate goods producers indexed by $j \in [0,1]$ at the real interest rate R_{jt}. These loans are the outcome of a contracting problem described later on. Π_t represents lump-sum profits and taxes. Consumption C_t is itself a composite CES utility flow from a variety of differentiated goods with elasticity of substitution σ.

The first-order conditions of the household are standard. In what follows I define the stochastic discount factor of the household as $\Lambda_{t+1} = \beta C_t / C_{t+1}$.

3.1.2 Intermediate good producers (firms)

The production of intermediate goods is carried out by a continuum of firms, indexed $j \in [0,1]$. Firm j enters period t with capital K_{jt-1} and a stock of debt B_{jt-1} which needs to be repaid at the gross real interest rate R_{jt-1}. First, capital is combined with a labor index L_{jt} to produce output:

$$Y_{jt} = \left(K_{jt-1}\right)^{\alpha} \left(A_t L_{jt}\right)^{1-\alpha}, \tag{16}$$

where A_t is aggregate productivity. The labor index is a CES combination of differentiated labor services with elasticity of substitution σ_w, but the firm's problem can be treated as if the labor index was acquired in a competitive market at the real wage index w_t. Output is sold competitively to final good producers at price q_t. During production, the capital stock depreciates at rate δ. This depreciated capital can be traded by the firm at the price Q_t.

The firm's net worth is the difference between the value of its assets and its outstanding debt:

$$N_{jt} = q_t Y_{jt} - w_t L_{jt} + Q_t (1-\delta) K_{jt-1} - R_{jt-1} B_{jt-1}. \tag{17}$$

I assume that the firm exits with probability γ. This probability is exogenous and independent across time and firms. As in Bernanke, Gertler and Gilchrist (1999), exit prevents firms from becoming financially unconstrained. If a firm does not exit, it needs to pay out a fraction $\zeta \in (0,1)$ of its earnings as dividends (where earnings E_{jt} are given by $N_{jt} - Q_t K_{jt-1} + B_{jt-1}$). The number ζ therefore represents the dividend payout ratio for continuing firms.[5] If a firm does exit, it must pay out its entire net worth as dividends. It is subsequently replaced by a new firm, which receives the index j. I assume that this new firm gets endowed with a fixed number of shares, normalized to one, and is able to raise an initial amount of net worth. This amount equals $\omega (N_t - \zeta E_t)$, where $\omega \in (0,1)$ and N_t and E_t are aggregate net worth and earnings, respectively.[6]

The net worth of firm j after equity changes, entry and exit is given by

$$\tilde{N}_{jt} = \begin{cases} N_{jt} - \zeta E_{jt} & \text{for continuing firms,} \\ \omega (N_t - \zeta E_t) & \text{for new firms.} \end{cases}$$

This firm then decides on the new stock of debt B_{jt} and the new capital stock K_{jt}, maximizing the present discounted value of dividend payments using the discount factor of its owners. Its balance sheet must satisfy:

$$Q_t K_{jt} = B_t^j + \tilde{N}_{tj}. \tag{18}$$

3.1.3 Firm owners

Firm owners differ from households in their capacity to own intermediate firms. The representative firm owner is risk-neutral. It can buy shares in firms indexed by $j \in$

[5] The optimal dividend payout ratio in this model would be $\zeta = 0$, as firms would always prefer to build up net worth to escape the borrowing constraint over paying out dividends. However, this would imply that aggregate dividends would be proportional to aggregate net worth, which is rather slow-moving. The resulting dividend process would not be nearly as volatile as in the data. Imposing $\zeta > 0$ allows to better match the volatility of dividends and therefore obtain better asset price properties.

[6] The simplified firm problem of Section 2 is nested as the case$\zeta = 1$ and $\gamma = 0$.

$[0,1]$. As described above, when a firm exits, it pays out its net worth N_{jt} as dividends, and is replaced by a new firm, which raises equity $\omega(N_t - \zeta E_t)$. Let the set of exiting firms in each period t be denoted by $\Gamma_t \subset [0,1]$. Then, the firm owner's utility maximization problem is given by:

$$\max_{\left(C_t^f, S_t^j\right)_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t C_t^f$$

$$\text{s.t. } C_t^f + \int_0^1 S_{jt} P_{jt} dj = \int_{j \notin \Gamma_t} S_{jt-1}\left(P_{jt} + D_{jt}\right) dj \tag{19}$$

$$+ \int_{j \in \Gamma_t} \left[S_{jt-1} D_{jt} - \omega\left(N_t - \zeta E_t\right) + P_{jt}\right] dj \tag{20}$$

$$S_t^j \in \left[0, \bar{S}\right] \tag{21}$$

for some $\bar{S} > 1$. Firm owners' consumption C_t^f is the same aggregator of differentiated final goods as for households.

The first term on the right-hand side of the budget constraint deals with continuing firms and is standard: Each share in firm j pays dividends D_{jt} and continues to trade, at price P_{jt}. The second term deals with firm entry and exit. If the household owns a share in the exiting firm j, it receives a terminal dividend. At the same time, a new firm j appears that is able to raise a limited amount of equity $\omega(N_t - \zeta E_t)$ from the firm owner in exchange for a unit amount of shares that can be traded at price P_{jt}. In addition, firm owners face upper and lower bounds on traded stock holdings.[7] The first-order conditions of the firm owner are

$$S_{jt} \begin{cases} = 0 & \text{if } P_{jt} > \beta \mathbb{E}_t^{\mathcal{P}}\left[D_{jt+1} + P_{jt+1} \mathbb{1}_{\{j \notin \Gamma_{t+1}\}}\right] \\ \in \left[0, \bar{S}\right] & \text{if } P_{jt} = \beta \mathbb{E}_t^{\mathcal{P}}\left[D_{jt+1} + P_{jt+1} \mathbb{1}_{\{j \notin \Gamma_{t+1}\}}\right] \\ = \bar{S} & \text{if } P_{jt} < \beta \mathbb{E}_t^{\mathcal{P}}\left[D_{jt+1} + P_{jt+1} \mathbb{1}_{\{j \notin \Gamma_{t+1}\}}\right] \end{cases}. \tag{22}$$

3.1.4 Borrowing constraint

In choosing their debt holdings, firms are subject to a borrowing constraint. The constraint is the solution to a particular limited commitment problem in which the outside option for the lender in the event of default depends on equity valuations.

Each period, lenders (households) and borrowers (firms) meet to decide on the lending of funds. Pairings are anonymous. Contracts are incomplete because the repayment of loans cannot be made contingent. Only the size B_{jt} and the interest rate R_{jt} of the

[7] This renders demand for stocks finite under arbitrary beliefs. In equilibrium, the bounds are never binding.

loan can be contracted in period t. Both the lender (a household) and the firm have to agree on a contract (B_{jt}, R_{jt}). Moreover, there is limited commitment in the sense that at the end of the period, but before the realization of next period's shocks, firm j can always choose to enter a state of default. In this case, the value of the debt repayment must be renegotiated. If the negotiations are successful, then wealth is effectively shifted from creditors to debtors. The outside option of this renegotiation process is bankruptcy of the firm and seizure by the lender.

Bankruptcy carries a cost of a fraction $1-\xi$ of the firm's capital being destroyed. The lender, a household, does not have the ability to operate the firm. It can liquidate the firm's assets, selling the remaining capital in the next period. This results in a recovery value of $\xi Q_{t+1} K_{jt}$. With some probability x (independent across time and firms), the lender receives the opportunity to "restructure" the firm if it wants. Restructuring means that, similar to Chapter 11 bankruptcy proceedings, the firm gets partial debt relief but remains operational. I assume that the lender has to sell the firm to another firm owner, retaining a fraction ξ of the initial debt. In equilibrium, the recovery value in this case will be $\xi\left(P_{jt} + B_{jt}\right)$ and this will always be higher than the recovery value after liquidation. The appendix shows that the debt contract takes the form of a leverage constraint in which total firm value is a weighted average of liquidation and market value:

$$B_{jt} \leq \xi \left((1-x) \underbrace{\mathbb{E}_t^{\mathcal{P}} \Lambda_{t+1} Q_{t+1} \xi K_{jt}}_{\text{liquidation value}} + x \underbrace{\left(P_{jt} + B_{jt}\right)}_{\text{market value}} \right) \tag{23}$$

This borrowing constraint nests the one in the simple model for $x = 1$.

3.1.5 Further model elements and shocks

Investment is subject to quadratic adjustment costs that move the price for capital goods:

$$Q_t \;=\; 1 + \psi \left(\frac{I_t}{I_{t-1}} - 1 \right) \tag{24}$$

Further, the prices for final goods and wages are subject to Calvo rigidities, with price stickiness parameter κ and wage stickiness parameter κ_w. The price for intermediate goods q_t equals the inverse of the gross markup of final goods producers. The monetary authority setting the nominal interest rate according to a Taylor-type interest rate rule:

$$i_t \;=\; \rho_i i_{t-1} + (1-\rho_i)\left(\beta^{-1} + \phi_\pi \pi_t + \varepsilon_{it}\right), \tag{25}$$

where ϕ_π is the reaction coefficient on consumer price inflation π_t, ρ_i is the degree of interest rate smoothing, and ε_{it} is an interest rate shock. Details of the model structure are provided in the appendix.

Finally, the exogenous stochastic processes are productivity and the monetary policy shock:

$$\log A_t = (1-\rho)\log \bar{A} + \rho \log A_{t-1} + \log \varepsilon_{At} \tag{26}$$

$$\varepsilon_{At} \sim iid\mathcal{N}\left(0, \sigma_A^2\right) \tag{27}$$

$$\varepsilon_{it} \sim iid\mathcal{N}\left(0, \sigma_i^2\right) \tag{28}$$

3.2 Rational expectations equilibrium

I first describe the equilibrium under rational expectations. An equilibrium is a set of stochastic processes for prices and allocations, a set of strategies in the limited commitment game, and an expectation measure $\mathcal{P}$ such that the following holds for all states and time periods: Markets clear; allocations solve the optimization programs of all agents given prices and expectations $\mathcal{P}$; the strategies in the limited commitment game are a subgame-perfect Nash equilibrium for all lender-borrower pairs; and the measure $\mathcal{P}$ satisfies rational expectations.

Under a mild restriction on the exit probability γ, there exists a rational expectations equilibrium characterized by the following properties.

1. All firms choose the same capital-labor ratio K_{jt}/L_{jt}. This allows one to define an aggregate production function and an internal rate of return on capital:

$$Y_t = \alpha K_{t-1}^{\alpha}\left(A_t \tilde{L}_t\right)^{1-\alpha} \tag{29}$$

$$R_t^k = q_t \alpha \frac{Y_t}{K_{t-1}} + Q_t (1-\delta) K_{t-1} \tag{30}$$

2. The expected return on capital is higher than the internal return on debt: $\mathbb{E}_t R_{t+1}^k > R_{jt}$.

3. At any time t, the stock market valuation P_{jt} of a firm j is proportional to its net worth after entry and exit $\tilde{N}_{jt}$. This permits one to write an aggregate stock market index as

$$P_t = \int_0^1 P_{jt} = \beta \mathbb{E}_t \left[D_{t+1} + \frac{1-\gamma}{1-\gamma+\gamma\omega} P_{t+1} \right]. \tag{31}$$

4. Borrowers never default on the equilibrium path and borrow at the risk-free rate

$$R_{jt} = R_t = \frac{1}{\mathbb{E}_t \Lambda_{t+1}}. \tag{32}$$

The lender only accepts debt payments up to the limit given by (23), which is proportional to the firm's net worth $\tilde{N}_{jt}$, and the firm always exhausts this limit.

5. As a consequence of the previous properties of the equilibrium, all firms can be aggregated. Aggregate debt, capital, and net worth are sufficient to describe the intermediate goods sector:

$$\begin{aligned} N_t &= R_t^k K_{t-1} - R_{t-1} B_{t-1} && (33) \\ Q_t K_t &= (1-\gamma+\gamma\omega)\left((1-\zeta)\, N_t + \zeta\,(B_{t-1} - Q_t K_{t-1})\right) + B_t && (34) \\ B_t &= x \mathbb{E}_t \Lambda_{t+1} Q_{t+1} \xi K_t + (1-x)\,\xi\,(P_t + B_t)\,. && (35) \end{aligned}$$

Proofs are relegated to the appendix.

3.3 Learning equilibrium

I introduce learning about stock prices as in the simple model of Section 2. Under the subjective belief measure $\mathcal{P}$, agents are assumed to retain the belief that the value of an individual firm is proportional to its net worth, as under rational expectations:

$$P_{jt} = \frac{N_{jt}}{N_t} P_t \, \mathcal{P}\text{-almost surely.} \tag{36}$$

But while investors have the correct belief of the cross-section of prices given the aggregate market value, they are uncertain about the evolution of the aggregate value P_t itself. I construct the belief system as in the simplified model: Under $\mathcal{P}$, i) agents have the correct belief about the exogenous shocks, ii) agents believe that the stock price P_t evolves according to Equations (9)-(11), iii) agents have conditionally model-consistent expectations with respect to stock prices, as defined in Section 2.3.

In practice, I solve the model using a two-stage procedure. The first stage is to solve for the policy functions and beliefs under $\mathcal{P}$. The Kalman filtering equations that describe beliefs about stock prices are as follows:

$$\begin{aligned} \log P_t &= \log P_{t-1} + \hat{\mu}_{t-1} - \frac{\sigma_\nu^2 + \sigma_\eta^2}{2} + z_t && (37) \\ \hat{\mu}_t &= \hat{\mu}_{t-1} - \frac{\sigma_\nu^2}{2} + g z_t, && (38) \end{aligned}$$

where $\hat{\mu}_t$ is the mean belief about the trend in stock price growth, and z_t is the *forecast error.* Under the subjective beliefs $\mathcal{P}$, z_t is normally distributed white noise. I impose that beliefs about any other endogenous variable are consistent with model outcomes conditional on the evolution of stock prices, and so beliefs and policy functions can be calculated using the belief equations about stock prices and the remaining equilibrium equations, taking z_t as an exogenous shock process. The market clearing condition for stocks and consumption goods, however, do not enter this first stage of the problem. Adding either one would effectively impose that beliefs about stock prices, too, be consistent with equilibrium outcomes, thereby falling back to rational expectations. Now, if y_t is the set of model variables and u_t the set of exogenous shocks, solving this first stage leads to the subjective policy function $y_t = h(y_{t-1}, u_t, z_t)$ satisfying conditionally model-consistent expectations.

The second stage of the model consists in finding the value for z_t which leads to market clearing in the stock market and thereby establishes equilibrium. This results in a mapping from the state variables and exogenous shocks to the perceived forecast error $r : (y_{t-1}, u_t) \mapsto z_t$. The resulting process for z_t is clearly not an iid disturbance, and in this respect agents' subjective beliefs are misspecified. The final solution of the model is given by the policy function $y_t = g(y_{t-1}, u_t) = h(y_{t-1}, u_t, r(y_{t-1}, u_t))$. I approximate the policy functions using a second-order perturbation method.

4 Results

I now present the quantitative results of this paper. First, I discuss the choice of parameters. Then, I review standard business cycle statistics. Learning and asset price volatility account for a third of the volatility of output, pointing to the strength of the endogenous amplification mechanism. I then look at asset pricing moments and find that the model with learning closely matches the volatility of stock prices (which is targeted by the estimation), but also the predictability of stock returns, skewness and kurtosis. Next, I present impulse response functions, confirming the strong amplification mechanism. The main channel is the endogenous volatility of asset prices induced by learning. But I also show that this is not the only channel through which learning affects the economy: Expectations about asset prices also cause procyclical movements in aggregate demand, leading to additional amplification in the presence of nominal rigidities. Finally, I compare forecast errors made by agents in the model with those observed in survey data. The patterns of predictability are remarkably similar, lending credibility to the assumed expectation formation process.

4.1 Choice of parameters

I partition the set of parameters into two groups. The first set of parameters is calibrated to first-order moments, and the second set is estimated by simulated method of moments (SMM) on second-order moments of US quarterly data.

4.1.1 Calibration

The capital share in production is set to $\alpha = 0.33$, implying a labor share in output of two thirds. The depreciation rate $\delta = 0.025$ corresponds to 10 percent annual depreciation. The persistence of the temporary component of productivity is set to 0.95.

The discount factor is set such that the steady-state interest rate matches the average annual real return on Treasury bills of 2.5 percent, implying a discount factor $\beta = 0.9938$. The elasticity of substitution between varieties of the final consumption good, as well as that among varieties of labor used in production, is set to $\sigma = \sigma_w = 4$. The Frisch elasticity of labor supply is set to 3, implying $\phi = 0.33$.

The strength of monetary policy reaction to inflation is set to $\phi_\pi = 1.5$, and the degree of nominal rate smoothing is set to $\rho_i = 0.85$.

Four parameters describe the structure of financial constraints: x, the probability of restructuring after default; ξ, the tightness of the borrowing constraint; ω, the equity received by new firms relative to average equity; and γ, the rate of firm exit and entry. I calibrate the restructuring rate to $x = 0.093$. This is the fraction of US business bankruptcy filings in 2006 that filed for Chapter 11 instead of Chapter 7, and that subsequently emerged from bankruptcy with an approved restructuring plan (a sensitivity check is included in Section 5.2).[8] The remaining three parameters are chosen such that the non-stochastic steady state of the model jointly matches the US average investment share in output of 20 percent, average debt-to-equity ratio of 1:1 (as recorded in the Fed flow of funds), and average quarterly P/D ratio of 139 (taken from the S&P500). The parameter values thus are $\gamma = 0.0165$, $\xi = 0.4152$, and $\omega = 0.018$.

4.1.2 Estimation

The remaining parameters are the standard deviations of the technology and monetary shocks (σ_A, σ_i), the degree of nominal price and wage rigidities (κ, κ_w), the size of investment adjustment costs (ψ), the fraction of dividends paid out as earnings by continuing firms (ζ), and the learning gain (g). I estimate these six parameters to minimize the distance to a set of eight moments pertaining to both business cycle and asset price statistics: The standard deviation of output; the standard deviations of consumption, investment hours worked, and stock prices relative to output; and the standard deviations of inflation, the nominal interest rate, and stock returns (see

[8] 2006 is the only year for which this number can be constructed from publicly available data. Data on bankruptcies by chapter are available at http://www.uscourts.gov/Statistics/BankruptcyStatistics.aspx. Data on Chapter 11 outcomes are analyzed in various samples by Flynn and Crewson (2009), Warren and Westbrook (2009), Lawton (2012), and Altman (2014).

Table 1: Estimated parameters.

param.	σ_a	σ_i	κ	κ_w	ψ	ζ	g
learning	.00884	.000423	.546	.932	13.7	.632	.00563
	(.000967)	(.00195)	(.089)	(.132)	(3.85)	(.0935)	(.000334)
RE	.0114	.000895	.691	.572	.618	.0490	-
	(.00212)	(.00173)	(.168)	(2.73)	(10.7)	(11.3)	
fric.less	.0116	.00121	.671	.847	.558	-	-
	(.00716)	(.000701)	(.261)	(.398)	(.124)		

Parameters as estimated by simulated method of moments. Asymptotic standard errors in parentheses.

also Table 2). The set of estimated parameters θ solves

$$\min_{\vartheta \in \mathcal{A}} \left(m\left(\theta\right) - \hat{m}\right)' W \left(m\left(\theta\right) - \hat{m}\right)',$$

where $m(\theta)$ are moments obtained from model simulation paths with 50,000 periods, $\hat{m}$ are the estimated moments in the data, and W is a weighting matrix.[9] I also impose that θ has to lie in a subset $\mathcal{A}$ of the parameter space which rules out deterministic oscillations of stock prices.[10] Such oscillations are not observed in the data, but can be consistent with equilibrium when asset price volatility is high and subjective beliefs are far away from rational expectations. In a sense, this restriction constrains the degree of departure of subjective beliefs from rational expectations.

Table 1 summarizes the SMM estimates for both the learning and rational expectations version of the model, as well as for a comparison (rational expectations) model in which all financial frictions are eliminated. The first row presents the results under learning. Exogenous shocks come mainly from productivity shocks, since σ_i is estimated to be relatively small. The Calvo price adjustment parameter is set to $\kappa = 0.546$, implying retailers adjust their prices every two quarters. The SMM procedure selects a high degree of nominal wage rigidities κ_w and of adjustment costs ψ. The estimates are substantially larger than what is commonly found in the literature. The fraction of earnings paid out as dividends is fitted to $\zeta = 0.632$, which is in line with the historical average for the S&P500 at about 50 percent. Finally, the learning gain $g = 0.00563$ implies that agents believe the degree of predictability in the stock market to be very small.

The second row contains the parameters estimated under rational expectations. The fit of the asset price moments is worse and the asymptotic standard errors are large,

[9] I choose $W = \operatorname{diag}\left(\hat{\Sigma}\right)^{-1}$ where $\hat{\Sigma}$ is the covariance matrix of the data moments, estimated using a Newey-West kernel with optimal lag order. This choice of W leads to a consistent estimator that places more weight on moments which are more precisely estimated in the data.

[10] $\theta \notin \mathcal{A}$ iff there exists an impulse response of stock prices with positive peak value also having a negative value of more than 20% of the peak value.

implying that the distance of the moments to the data at the point estimate is relatively flat. Nevertheless, at the point estimate, the size of the shocks σ_a and σ_i is substantially larger under learning. This implies that learning about stock prices leads to substantial amplification of shocks: The increased endogenous volatility of asset prices greatly magnifies the financial accelerator effect, just as in the simple model of Section 2. The degree of wage rigidities and investment adjustment costs required to fit the data is smaller than under learning.

The third row contains parameter estimates under rational expectations when additionally, financial frictions are completely shut off. Since the financial structure is eliminated from that model, the dividend payout ratio ζ is not present. The size of the shocks is larger than under the rational expectations model with the financial accelerator present. This points to moderate amplification effects of financial frictions under rational expectations.

4.2 Business cycle and asset price moments

To get a better understanding of the quantitative properties of the model, I review key moments in the data and across model specifications. Table 2 starts with business cycle statistics. The moments in the data are shown in Column (1). Moments for the estimated learning model are shown in Column (2), while Columns (3) and (4) contain the corresponding moments for the model under rational expectations and the frictionless benchmark. Here, the parameters are held constant at the estimated values as for the learning model. By nature of the estimation, the learning model has the best fit across Columns (2) to (4). The comparison serves to single out the contribution of learning and financial frictions to the fit. Column (5) presents the moments under rational expectations when the parameters are re-estimated to fit the data.

The first row reports the standard deviation of detrended output. By construction, this is matched well by the learning model in Column (2). When learning is shut off in Column (2), the standard deviation drops one-third. This shows the great degree of amplification that learning adds to the model. Of course, it is possible to match output volatility with rational expectations, using larger shock sizes, as in Column (5). But the comparison between Columns (2) and (3) singles out the contribution of learning to the internal amplification mechanism. The standard (rational expectations) financial accelerator mechanism is present in the model as well, since the volatility of output drops further in Column (4) when financial frictions are shut off.

The next three rows report the standard deviation of consumption, investment, and hours worked relative to output. Moving from Column (2) to (3), it can be seen that the removal of learning leads to a sharp drop in the relative volatility of both investment and hours worked. This is because the estimated learning model features

Table 2: Business cycle statistics in the data and across model specifications.

	moment	(1) data	(2) learning	(3) RE	(4) fric.less	(5) RE re-estimated
output volatility	$\sigma_{hp}(Y_t)$	1.43% (0.14%)	1.56%*	1.00	.79	1.51%*
volatility rel. to output	$\sigma_{hp}(C_t)/\sigma_{hp}(Y_t)$	.60 (.035)	.60*	.94	1.34	.58*
	$\sigma_{hp}(I_t)/\sigma_{hp}(Y_t)$	2.90 (.12)	2.78*	.48	.31	2.79*
	$\sigma_{hp}(L_t)/\sigma_{hp}(Y_t)$	1.13 (.061)	1.18*	.84	.40	1.10*
correlation with output	$\rho_{hp}(C_t, Y_t)$	.94 (.0087)	.59	.86	1.00	.84
	$\rho_{hp}(I_t, Y_t)$	.95 (.0087)	.87	.89	.25	.90
	$\rho_{hp}(L_t, Y_t)$	.85 (.035)	.88	.65	.24	.76
inflation	$\sigma_{hp}(\pi_t)$	.27% (.047%)	.34%*	.28%	.28%	.25%*
nominal rate	$\sigma_{hp}(i_t)$	.37% (.046%)	.11%*	.11%	.12%	.07%*

Quarterly U.S. data 1962Q1–2012Q4. Standard errors in parentheses. π_t is quarterly CPI inflation. i_t is the federal funds rate. L_t is total non-farm payroll employment. Consumption C_t consists of services and non-durable private consumption. Investment I_t consists of private non-residential fixed investment and durable consumption. Output Y_t is the sum of consumption and investment. $\sigma_{hp}(\cdot)$ is the standard deviation and $\rho_{hp}(\cdot,\cdot)$ is the correlation coefficient of HP-filtered data (smoothing coefficient 1600). Moments used in the SMM estimation are marked with an asterisk.

a high level of investment adjustment costs to match investment volatility. Without large asset price fluctuations generated by learning, investment becomes too smooth, as does the marginal product of capital and hence labor demand. The next rows report the volatility of inflation and the nominal interest rate. Inflation volatility is roughly in line with the data, but the nominal interest rate is less volatile across all model specifications. This might be due to the fact that the data sample includes the volatile '70s and the following Volcker disinflation period.

Next, I present asset price statistics in Table 3. The statistics correspond to some well-known asset price puzzles. The learning model fits them remarkably well, despite being solved only with a second-order perturbation method. Starting with excess volatility in Column (2), the model with learning produces standard deviations of prices, P/D ratio, and returns that are close to the data. By contrast, the model with rational expectations in Column (5) cannot produce a similar amount of volatility, despite the fact that price and return volatility are explicitly targeted by

Table 3: Asset price statistics in the data and across model specifications.

	moment	(1) data	(2) learning	(3) RE	(5) RE re-estimated
excess volatility	$\sigma_{hp}\left(P_t\right)/\sigma_{hp}\left(Y_t\right)$	7.86 (.61)	8.96*	.26	.16*
	$\sigma\left(\frac{P_t}{D_t}\right)$	41.08% (6.11%)	22.62%	4.12%	3.59%
	$\sigma\left(R^e_{t,t+1}\right)$	8.14% (.61%)	7.12%*	.19%	.19%*
return predictability	$\rho\left(\frac{P_t}{D_t}, R^e_{t,t+4}\right)$	-.297 (.092)	-.376	-.040	-.035
	$\rho\left(\frac{P_t}{D_t}, R^e_{t,t+20}\right)$	-.585 (.132)	-.732	-.006	0.011
	$\rho\left(\frac{P_t}{D_t}, \frac{P_{t+4}}{D_{t+4}}\right)$	.904 (.056)	.637	.303	.564
negative skewness	skew $\left(R^e_{t,t+1}\right)$	-.897 (.154)	-.404	.022	.005
heavy tails	kurt $\left(R^e_{t,t+1}\right)$	1.57 (.62)	.92	.04	-.03

Quarterly U.S. data 1962Q1–2012Q4. Standard errors in parentheses. Dividends D_t are four-quarter moving averages of S&P 500 dividends. The stock price index P_t is the S&P 500. Excess returns R^e_t are annualized quarterly excess returns of theS&P 500 over 3-month Treasury yields. $\sigma\left(\cdot\right)$ is the standard deviation; $\sigma_{hp}\left(\cdot\right)$ is the standard deviation of HP-filtered data (smoothing coefficient 1600); $\rho\left(\cdot,\cdot\right)$ is the correlation coefficient; skew$\left(\cdot\right)$ is skewness;kurt$\left(\cdot\right)$ is excess kurtosis. Moments used in the SMM estimation are marked with an asterisk.

the estimation.

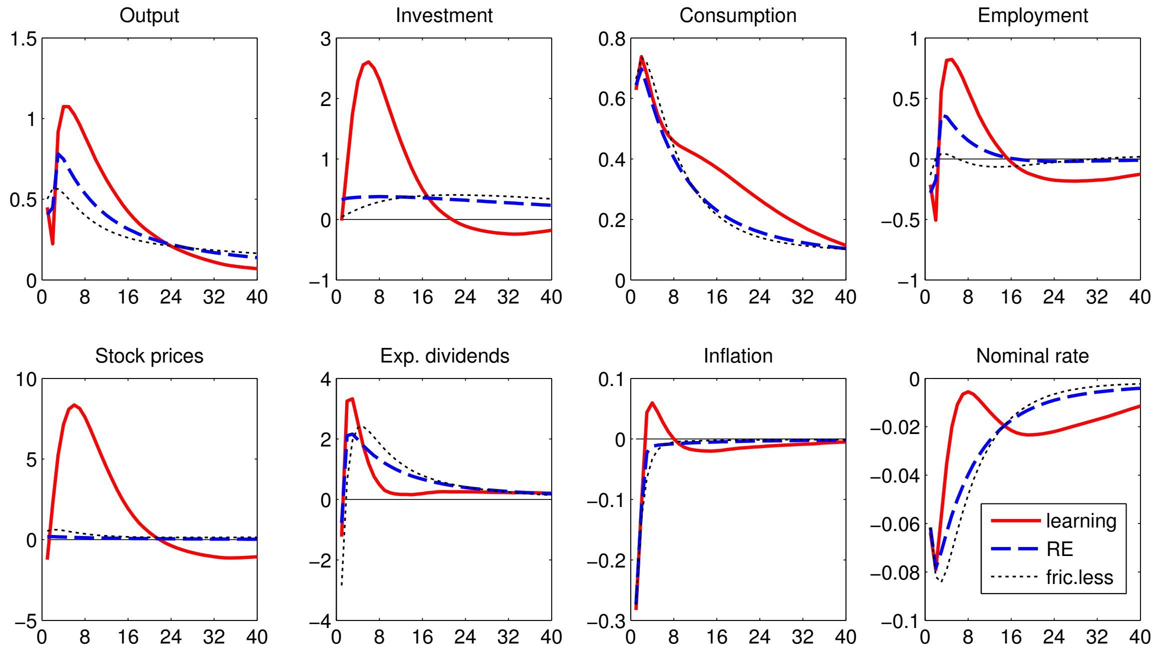
Stock returns also exhibit considerable predictability by the P/D ratio at business-cycle frequency. The same is true in the model with learning. Predictability is not targeted by the estimation, and in fact it is somewhat stronger than in the data, reflected in a persistence of the P/D ratio somewhat lower than in the data. Again, the rational expectations model is not able to produce sizable return predictability.

Finally, the learning model also produces a distribution of returns that is negatively skewed and heavy-tailed to a similar degree as in the data. This points to the importance of non-linearities in the asset price dynamics under learning.

4.3 Impulse response functions

Impulse response functions reveal the amplification mechanism at play. Figure 4 plots the impulse responses to a persistent productivity shock. Red solid lines represent

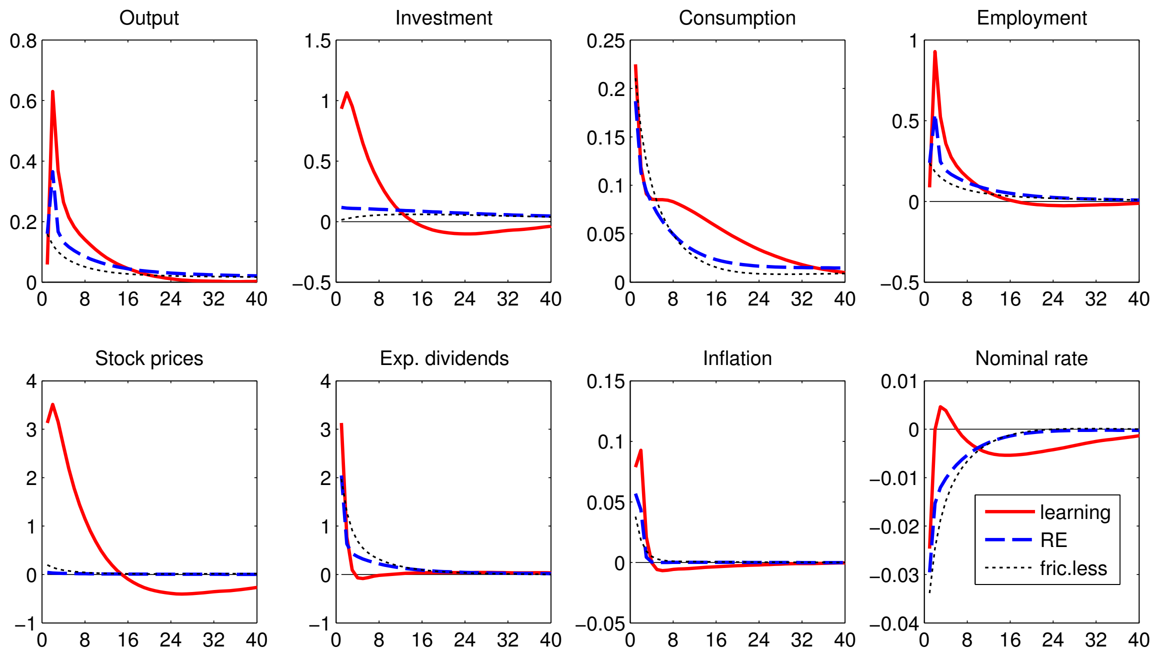
Figure 4: Impulse responses to a persistent productivity shock.



Impulse responses to a one-standard deviation innovation in ε_{At}, averaged over 5,000 random shock paths with a burn-in of 1,000 periods. Stock prices, dividends, output, investment, consumption, and employment are in 100*log deviations. Stock returns and the nominal interest rate are in percentage point deviations.

the learning equilibrium, blue dashed lines represent the rational expectations version, and black thin lines represent the comparison model without financial frictions. The impulse responses are averaged across states and therefore mask the tail dynamics present under learning, but they are nevertheless instructive. Looking at the first row of impulse responses, output rises persistently after the shock due to both the increased productivity and the relaxation of credit constraints from higher asset prices. The increase in output is larger under rational expectations than under the frictionless comparison; this is the standard financial accelerator effect. When learning is introduced, the response to the shock is amplified further. This also translates into amplification of the responses of investment, consumption, and employment. The amplification is due to two channels: First, learning leads to higher stock prices. The increase in firms' market value allows them to borrow more and invest and produce more. Second, agents under learning are not aware of the mean reversion in stock prices and predict the stock price boom to last for a long time. Consequently, they overestimate the availability of credit and therefore production in the future, leading to an aggregate demand effect that increases output today (see also 4.4). The rise in stock prices in the second row of Figure 4 is large under learning and accompanied by an initial spike in dividend payments, although dividends subsequently fall below their counterpart under rational expectations. The nominal interest rate falls less un-

Figure 5: Impulse responses to a monetary shock.



Impulse responses to a innovation in ε_{mt}, averaged over 5,000 random shock paths with a burn-in of 1,000 periods. The size of the innovation is chosen to produce a 10 basis point fall in the equilibrium nominal rate. Stock prices, dividends, output, investment, consumption and employment are in 100*log deviations. Stock returns and the nominal interest rate are in percentage point deviations.

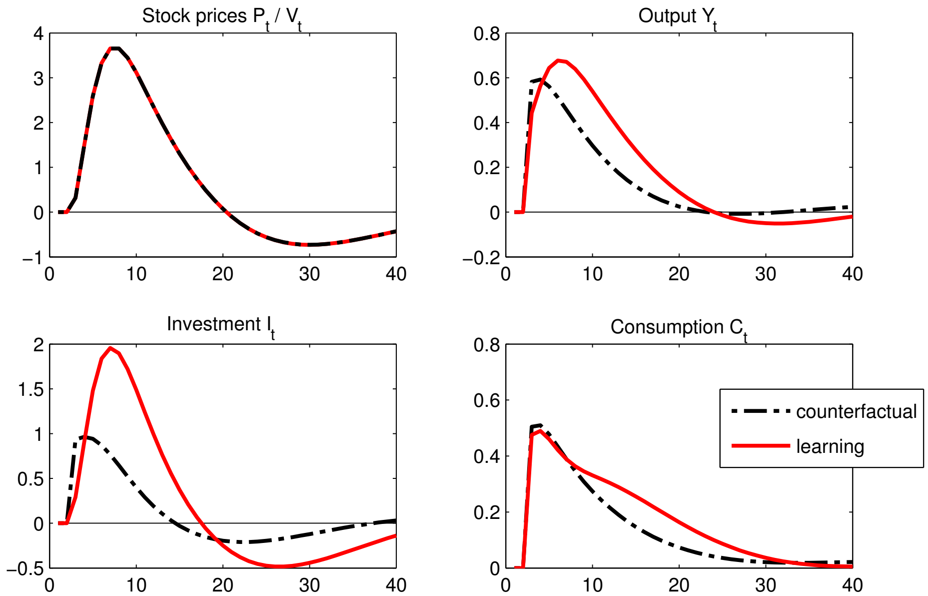
der learning as the monetary authority reacts to the inflationary pressures stemming from the relaxation in credit constraints.

Figure 5 plots the response to a temporary reduction in the nominal interest rate. Again, all macroeconomic aggregates rise substantially more under learning than under both rational expectations and the frictionless benchmark. The monetary stimulus increases stock prices and thus relaxes credit constraints. The consequent increase in aggregate demand raises inflationary pressure, so that the systematic reaction of the interest rate rule raises the interest rate sharply again after the shock.

4.4 Does learning matter?

The discussion so far has mainly focused on how large swings in asset prices lead to large swings in real activity through their effect on credit constraints. But is learning necessary for this story at all? Maybe all that matters for amplification is that asset price volatility has to be increased, by some mechanism or other. In this section, I show that learning has an effect on amplification over and above its effect on asset prices.

Figure 6: Does learning matter?



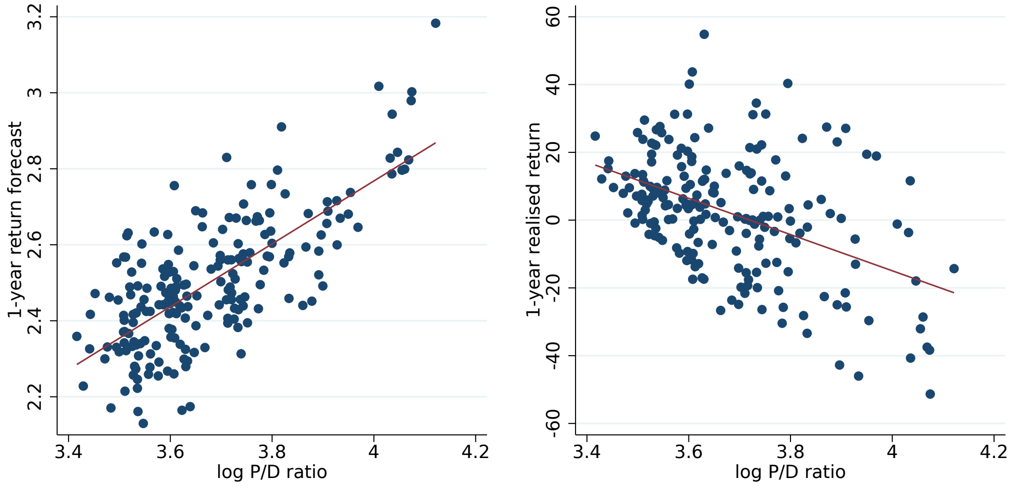
Solid red line: Impulse response to a one standard deviation positive productivity shock under learning. Black dash dotted line: Impulse response to a hypothetical rational expectations model with stock price dynamics identical to those under learning (see text). The impulse responses in the figure are produced using a first-order approximation to the model equations.

I replace the stock market value P_t in the borrowing constraint (35) with an exogenous process V_t that has the same law of motion as the stock price under learning. More precisely, I fit an ARMA(10,5) process for V_t such that its impulse responses are as close as possible to those of P_t under learning (the exogenous shock in the ARMA process are the productivity and monetary shocks). I then solve this model, but with rational expectations. If learning only matters because it affects stock price dynamics, then this hypothetical model should have exactly identical dynamics to the model under learning.[11]

Figure 6 shows that this is not the case. The ARMA process fits stock prices well: The impulse response of P_t under learning and V_t in the counterfactual experiment are indistinguishable. But after a positive productivity shock, output, investment, and consumption rise more under learning, even though the counterfactual model has the same stock price dynamics by construction. The reason is that expectations of future asset prices matter beyond their direct impact on current prices. Under learning, agents do not fully internalize mean reversion in stock prices and therefore predict that credit constraints are loose for longer than they turn out to be. This leads to a wealth effect on households that increases their consumption, raising aggregate demand, and

[11]For this exercise I only compute a first-order approximation to the model equations.

Figure 7: Return expectations and expected returns in a model simulation.



Expected and realized nominal returns along a simulated path of model with learning. Simulation length 200 periods. Theoretical correlation coefficient for subjective expected returns $\rho = .47$, for future realized returns $\rho = -.38$.

it leads to higher future expected prices of capital goods $\mathbb{E}_t Q_{t+1}$, which enters the liquidation value of firms and hence relaxes borrowing constraints, even if stock prices are the same as under rational expectations. These effects are powerful enough to create significant endogenous amplification through the departure of subjective beliefs from rational expectations.

4.5 Relation to survey evidence on expectations

The rational expectations hypothesis asserts that "outcomes do not differ systematically [...] from what people expect them to be" (Sargent, 2008). Put differently, a forecast error should not be systematically predictable by information available at the time of the forecast. The absence of predictability is almost always rejected in the data.

Similarly, agents in the model under learning also make systematic, predictable forecast errors. This holds not only for stock prices but also other endogenous model variables, despite the fact that, *conditional* on stock prices, agents' beliefs are model-consistent. A systematic mistake in predicting stock prices will still spill over into a corresponding mistake in predicting the tightness of credit constraints, and hence investment, output, and so forth. Owing to the internal consistency of beliefs, I can compute well-defined forecast errors made by agents in the model at any horizon and for any model variable.

Figure 7 repeats the scatter plot of the introduction, contrasting expected and realized

Table 4: Forecast errors under learning and in the data.

	(1)	(2)	(3)	(4)	(5)	(6)
	$\log PD_t$		$\Delta \log PD_t$		forecast revision	
forecast variable	data	model	data	model	data	model
$R^{stock}_{t,t+4}$	-.44	-.38	.06	.30	-	-.30
	(-3.42)		(.41)			
$Y_{t,t+3}$	-.21	-.16	.22	.16	.29	.28
	(-1.78)		(2.42)		(3.83)	
$I_{t,t+3}$	-.20	-.37	.25	.27	.31	.35
	(-1.74)		(2.88)		(3.79)	
$C_{t,t+3}$	-.19	-.04	.21	.01	.23	.02
	(-1.85)		(2.37)		(2.67)	
$u_{t,t+3}$	.05	.20	-.27	-.20	.43	.32
	(.12)		(-3.07)		(6.07)	

Correlation coefficients for mean forecast errors on one year-ahead nominal stock returns (Graham-Harvey survey) and three quarters-ahead real output growth, investment growth, consumption growth and the unemployment rate (SPF). t-statistics in parentheses. Regressors: Column (1) is the S&P 500 P/D ratio and Column (2) is its first difference. Column (3) is the forecast revision, as in Coibion and Gorodnichenko (2015). Data from Graham-Harvey covers 2000Q3–2012Q4. Data for the SPF covers 1981Q1–2012Q4. For the model, correlations are computed using a simulation of length 50,000, where subjective forecasts are computed using a second-order approximation to the subjective belief system on a path in which no more future shocks occur, starting at the current state in each period. Unemployment in the model is taken to be $u_t = 1 - L_t$. Stock returns in the model $R^{stock}_{t,t+4}$ are quarterly nominal aggregate market returns.

one year-ahead returns in a model simulation. The same pattern as in the data emerges: When the P/D ratio is high, return expectations are most optimistic. In the learning model, this has a causal interpretation: High return expectations drive up stock prices. At the same time, realized future returns are, on average, low when the P/D ratio is high. This is because the P/D ratio is mean-reverting (which agents do not realize, instead extrapolating past price growth into the future): At the peak of investor optimism, realized price growth is already reversing and expectations are due to be revised downward, pushing down prices toward their long-run mean.

Table 4 describes tests using the Federal Reserve's Survey of Professional Forecasters (SPF) as well as the CFO survey data and compares the statistics to those obtained from simulated model data. Each entry corresponds to a correlation of the error of the mean survey forecast with a variable that is observable by respondents at the time of the survey. Under the null of rational expectations, all entries should be zero.

Column (1) shows that the P/D ratio negatively predicts forecast errors. When stock prices are high, people systematically under-predict economic outcomes. This holds in particular for stock returns, as was already shown in the scatter plot above. But it also holds true for macroeconomic aggregates, albeit at lower levels of significance. The same holds true in Column (2), which shows the correlation coefficients obtained from simulated model data.

Column (3) repeats the exercise for the growth rate of the P/D ratio. This measure positively predicts forecast errors, suggesting that agents' expectations are too cautious and under-predict an expansion in its beginning but then overshoot and over-predict it when it is about to end. In the model (Column 4), this pattern also emerges because expectations about asset prices (and hence lending conditions) adjust only slowly. The similarity of the correlations in the data and in the model is striking, with the exception of aggregate consumption. The reason is that consumption forecasts in the model are only biased at longer horizons: A relaxation of borrowing constraints first leads to an increase in investment and only later to an increase in consumption. Agents are aware of this relationship, so that their three-quarter forecasts, as in Table 4 do not become much more optimistic when the P/D ratio increases. At longer forecast horizons, one would observe more predictability for consumption as well.

Column (5) reports the results of a particular test of rational expectations devised by Coibion and Gorodnichenko (2015). Since for any variable x_t, the SPF asks for forecasts at one- through four-quarter horizons, it is possible to construct a measure of agents' revision of the change in x_t as $\hat{\mathbb{E}}_t\left[x_{t+3}-x_t\right]-\hat{\mathbb{E}}_{t-1}\left[x_{t+3}-x_t\right]$. Forecast errors are positively predicted by this revision measure. Coibion and Gorodnichenko take this as evidence for sticky information models in which information sets are gradually updated over time. But it is also consistent with the learning model: The correlation coefficients in Column (6) are very similar to those in the data.[12]

5 Sensitivity checks

5.1 Nominal rigidities

The quantitative model includes price- and wage-setting frictions that complicate the model dynamics. They are nevertheless important for the quantitative fit of the model, as I will argue here. Recall that in the simple model of Section 2, the amplifying effect of asset price learning depended crucially on the behavior of the real wage. After a positive shock, as credit constraints relax and investment picks up, wages rise which work to diminish firms' profits and expected dividend payments. This drives down stock prices and dampens the learning dynamics. The same mechanism is at

[12] The model predicts a negative correlation of forecast errors on stock returns with their forecast revisions. The CFO survey does not allow for the construction of the corresponding statistic in the data, but it is an interesting implication since a negative correlation cannot be produced by rigid information models as in Coibion and Gorodnichenko.

play in the quantitative model. Introducing nominal rigidities greatly helps to obtain amplification.

Specifically, expected dividends in this model are given by:

$$\begin{aligned} \mathbb{E}_t^{\mathcal{P}} D_{t+1} &= (\gamma + (1-\gamma)\zeta)\left(\alpha q_t^{\frac{1}{\alpha}} \mathbb{E}_t^{\mathcal{P}}\left((1-\alpha)\frac{A_{t+1}}{w_{t+1}}\right)^{\frac{1-\alpha}{\alpha}} + (1-\delta)\,\mathbb{E}_t^{\mathcal{P}} Q_{t+1} - R_t \frac{B_t}{K_t}\right) K_t \\ &- (1-\gamma)\zeta\left(1 - \frac{B_t}{K_t}\right) K_t. \end{aligned} \tag{39}$$

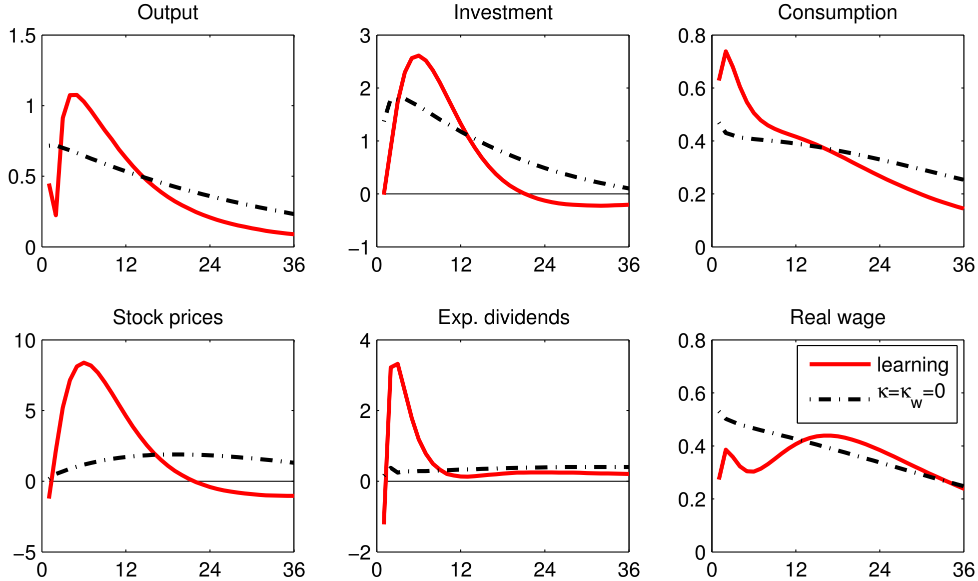
There are four relative prices that enter this equation: The price of intermediates q_t, the real wage w_{t+1}, the price of capital goods Q_{t+1}, and the borrowing rate R_t. Suppose now that asset prices rise because of optimistic investor beliefs, relaxing credit constraints. This directly leads to an increase in the capital stock K_t and also allows for higher leverage B_t/K_t. The rise in investment is expected to persist in the future, so that next period's expected price of capital goods Q_{t+1} increases. This raises expected dividends, helping amplification through higher stock prices. Likewise, increased aggregate demand can raise q_t if prices are sticky. But the increased labor demand, together with a positive wealth effect that households expect from the relaxation of credit constraints, will drive up the real wage w_t. This will tend to reduce D_{t+1} and dampen the dynamics. Also, to the extent that higher investment comes at the expense of lower consumption in the economy, real borrowing rates R_t will rise, also dampening the dynamics. This latter effect is stronger the more leverage there is in the economy.

Nominal rigidities have effects on real wages, the price of intermediates, and real rates. Wage rigidities will counteract the dampening effects of real wage responses to shocks, allowing for greater dividend, and therefore asset price, volatility. They also lead to amplification in the response of employment to movements in financial market sentiment. Price rigidities, together with a relatively loose monetary policy rule, imply that the prices of intermediates q_t are pro-cyclical, and lead to smaller real interest rates movements in response to changes in investor sentiment, also helping amplification. The mirror image of this result, through the consumption Euler equation, is that consumption is not pushed down as much by increases in investment, so its response, too, is amplified.

In sum, nominal price and wage rigidities allow for comovement of all macroeconomic aggregates in response to changes in subjective beliefs. This co-movement property obtains more generally and has been documented in the context of news shocks (Kobayashi and Nutahara, 2010) and financial shocks Ajello (2016).

To illustrate this point, I re-compute impulse responses of the model with learning, but without nominal rigidities (setting $\kappa = \kappa_w = 0$). I also reduce the size of investment adjustment costs to $\psi = 0.125$. With the high degree of adjustment costs in the baseline version, the model would include an explosive two-period oscillation and

Figure 8: Role of nominal rigidities.



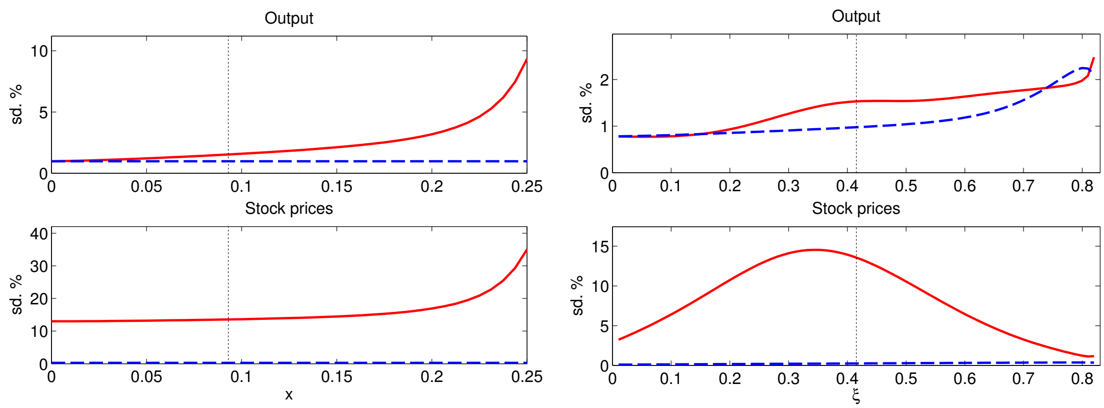
Solid red line: Impulse response to a one-standard deviation positive productivity shock for the model with learning and price and wage rigidities ("nominal" baseline). Black dash-dotted line: Impulse response to a productivity shock for the model with learning but without nominal rigidities, re-estimated as in Section 4.1.2 to fit the data ("real" comparison). The size of the shock shown is the same as in the nominal model.

reducing ψ ensures stability. The low costs of adjusting investment also gives the real version a better chance at delivering strong impulse responses. Even then, the nominal version delivers far greater amplification. Figure 8 plots impulse responses to a positive productivity shock for both the nominal and real version of the model. Owing to lower adjustment costs, the initial response of investment is expectedly stronger in the real version. However, the real wage w_t rises by much more, and also the price of intermediates q_t is fixed, so dividends do not rise as much after the shock. This considerably dampens the learning dynamics and mutes the response of stock prices. By consequence, the response of output, investment, consumption and hours worked (not shown) is overall weaker than in the baseline version of the model.

5.2 Borrowing constraint parameters

I turn to discuss the sensitivity to the two main parameters affecting the borrowing constraint (35): The probability x that a firm can be sold as a going concern after filing for bankruptcy, governing the dependency of the constraint on stock prices; and the fraction of assets ξ preserved in bankruptcy, governing the overall tightness of the constraint. Figure 9 plots the standard deviation of output and stock prices as a function of these two parameters, respectively.

Figure 9: Sensitivity to borrowing constraint parameters.



(a) Market value dependency x.

(b) Recovery rate ξ.

Simulated model data HP-filtered with smoothing parameter 1600, sample length 50,000 periods. The dashed black lines indicate parameter value in the estimated learning model.

Panel (a) shows the role of the x parameter, which crucially affects the amplification mechanism. The value $x = 0$ is a special case. At this point, stock prices do not enter the borrowing constraint and serve no role for allocations in the economy. Allocations under learning and rational expectations coincide perfectly, even though stock price dynamics are still amplified under learning. As x increases, the higher volatility of stock prices under learning translates into higher volatility in real activity as well. Since swings in real activity feed back into asset prices through their effect on dividends, the amplification becomes very strong for high values of x until the dynamics become explosive. Beyond a value of x of about 0.26, no stable learning equilibrium exists. By contrast, the rational expectations equilibrium barely depends on the parameter x.

Panel (b) shows the role of the ξ parameter. Amplification is hump-shaped with respect to ξ. At $\xi = 0$, no collateral is pledgeable and firms have to finance their capital stock entirely out of equity. In this case, fluctuations in stock prices again do not impact the economy and allocations coincide under learning and rational expectations; there is no amplification from learning. However, as pledgeability increases to its maximum value (beyond which a steady state with permanently binding borrowing constraint does not exist), amplification also disappears. This mirrors the analysis of the simplified model.

5.3 Monetary policy rule

Finally, I discuss the sensitivity of the results with respect to the interest rate rule followed by the monetary authority. Consider extending the interest rate rule (25) as follows:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(1/\beta + \phi_\pi \pi_t + \phi_Y \Delta \log Y_t + \phi_P \Delta \log P_t\right) \qquad (40)$$

In addition to raising interest rates when inflation is above its target level (taken to be zero), the monetary authority can raise interest rates $\phi_{\Delta Y}$ percentage points when real GDP growth increases one percentage point and $\phi_{\Delta P}$ percentage points when stock price growth increases one percentage point.[13] I consider several values for the parameters $\phi_\pi, \phi_{\Delta Y}, \phi_{\Delta P}$. Table 5 summarizes how they affect the model under learning. In addition to the standard deviations of output, stock prices, inflation and nominal interest rates, the table also reports a measure of amplification by the learning dynamics and a measure of the welfare cost of business cycles.[14]

Column (1) shows the baseline calibration under learning. Learning amplifies output fluctuations by 33 percent and has a welfare cost of business cycles of 0.0346 percent of average consumption. In Column (2), the coefficient on inflation ϕ_π is doubled. This rule achieves a marked reduction in the volatility of inflation, but at the cost of higher output volatility. The amount of amplification from learning is smaller than in the baseline but still strong at 22 percent. Column (3) augments the baseline rule by a reaction to output growth. This rule reduces the volatility of output and stock prices, while keeping the volatility of inflation unchanged. Again, the amplification effect from learning is dampened but output is still 25 percent more volatile than under rational expectations. Column (4) considers a reaction of interest rates to stock price growth. Such a reaction is highly effective in stabilizing the economy under learning. The volatility of output and stock prices drops markedly. The amplification mechanism is completely eliminated—output volatility under learning is 10 percent lower than under rational expectations, and the welfare cost of business cycles is lower than in Columns (1)–(3). Intuitively, raising rates when asset prices are rising (and vice-versa) acts to stabilize expectations in financial markets. In the model, this

[13] I deliberately exclude levels or gap measures of output or asset prices from the interest rate rule. Doing so would imply that the monetary authority has more knowledge than the private sector under learning, as the perceived equilibrium level of asset prices and the output gap depends on agents' subjective beliefs, and at the same time communicates that knowledge through the rule.

[14] The amplification measure is the ratio of the standard deviation of output under learning over that under rational expectations at the same parameter values. The welfare cost of business cycles χ is defined as the fraction of steady-state consumption the household would need to give up in order to have its period utility at the same level as the average stochastic period utility, in a steady state in which consumption and labor are constant and equal to their average stochastic level, and price and wage dispersion is nil:

$$u\left((1-\chi)\,\mathbb{E}\left[\tilde{C}_t\right], \mathbb{E}\left[\tilde{L}_t\right]\right) = \mathbb{E}\left[u\left(C_t, L_t\right)\right].$$

I simulate a model time series for $(C_t, L_t, \pi_t, \pi_t^w)$ of 10,000 periods using the second-order approximation method above, and I compute series for $\tilde{C}_t$ and $\tilde{L}_t$ using the exact formulae given in the appendix. I then evaluate period utility using its exact formula as well to calculate the welfare loss. The expectation is then computed using averages over time.

Table 5: Alternative monetary policy rules.

	(1)	(2)	(3)	(4)	(5)	(6)
	baseline	alternative rules			optimized rules	
ϕ_π	1.50	3.00	1.50	1.50	1.37	1.20
ϕ_Y			0.50		.61	.95
ϕ_P				0.50		.12
$\sigma(Y)$	3.27%	3.86%	2.77%	2.13%	2.77%	2.04%
$\sigma(P)$	22.8%	26.3%	16.6%	9.39%	15.6%	9.35%
$\sigma(\pi)$	0.35%	0.19%	0.35%	0.33%	0.38%	0.35%
$\sigma(i)$	0.17%	0.17%	0.14%	0.16%	0.17%	0.12%
$\sigma(Y)/\sigma(Y_{RE})$	1.33	1.22	1.25	0.90	1.32	1.07
welfare cost χ	0.0346%	0.0415%	0.0282%	0.0276%	.0275%	.0218%

Standard deviations of output, stock prices, inflation, and interest rates (unfiltered) under learning in percent. The standard deviation of output under rational expectations $\sigma(Y_{RE})$ is calculated at the same parameter values as the learning solution. See Footnote 14 for the definition of the welfare cost χ. The interest rate smoothing coefficient is kept at $\rho_i = 0.85$ for all rules considered.

stabilization works mainly through changes in firms' borrowing costs and dividend payouts, offsetting the self-amplifying learning dynamics in the stock market.

I also compute the coefficient values that minimize the welfare cost of business cycles.[15] Column (5) displays optimized coefficients ϕ_π and ϕ_Y without a reaction to stock prices, $\phi_{\Delta P} = 0$. The coefficients turn out to be quite close to the Taylor-type rule in Column (3). The resulting rule reduces welfare costs to 0.0275 percent, but the amplification through learning dynamics is still strong. Column (6) additionally allows for a reaction to stock prices prices as well. The welfare cost is reduced further to 0.0218 percent, and the amplification effect of learning drops to seven percent.[16] This suggests that a positive interest rate reaction to stock price growth in this model provides stabilizing effects that cannot be achieved by reacting to output and inflation alone. This stands in contrast to what is usually found in the existing literature that uses rational expectations (e.g. Gali, 2014). In line with those findings, the optimization of the rule coefficients in this model under rational expectations (not reported) does not deliver additional stabilization when a reaction to stock prices is included.

[15] Note that this criterion is paternalistic because it minimizes the welfare cost of business cycles over time, rather than the private sector's subjectively expected welfare cost.

[16] It is worth noting that despite the reaction stock prices in Columns (4) and (6), the volatility of the nominal interest rate is lower than in the baseline version. The reaction reduces endogenous asset price volatility so that equilibrium rates do not end being excessively volatile.

6 Conclusion

In this paper, I have analyzed the implications of a learning-based asset pricing theory in a business cycle model with financial frictions. When firms' borrowing constraints depend on their market value, learning in the stock market interacts with credit frictions to form a two-sided feedback loop between stock prices and firm profits that amplifies the learning dynamics encountered in Adam, Beutel and Marcet (2014). At the same time, it makes the financial accelerator mechanism more powerful, amplifying both supply and demand shocks. The model jointly matches standard business cycle and asset pricing moments.

Despite the fact that beliefs are close to rational expectations, agents' forecast errors on stock prices still spill over into their other forecasts. The resulting forecast error predictability was found to closely match survey data on expectations on a range of variables.

An important innovation in developing the model was to introduce a belief system that combines learning about stock prices with a high degree of rationality and internal consistency. Beliefs are restricted in a way such that forecast errors conditional on future prices and fundamentals are zero. This differs from most of the existing adaptive learning literature where every forward-looking equation is parametrized eparately, resulting in a large number of degrees of freedom. The method can be widely applied in other models of the business cycle.

An examination of the sensitivity of the amplification mechanism to the monetary policy rule revealed that a reaction of interest rates to stock price growth is highly beneficial under learning. This is because such a reaction effectively stabilizes expectations in financial markets. The same is generally not true in a rational expectations framework, illustrating that the choice of an asset price theory can have important normative implications. Further research could examine in detail the policy implications of learning-based asset pricing and establish whether a policy of "leaning against the wind" is also desirable in more general settings when agents have to learn about asset prices.

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Online Appendix

A Details on the simplified model

A.1 Formal model description

In the simplified model, the household is risk-neutral and inelastically supplies one unit of labor. Its utility maximization program is as follows:

$$\max_{(C_t,S_t,B_t)_{t=0}^{\infty}} \mathbb{E}^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t C_t$$

$$\begin{aligned} \text{s.t. } C_t + S_t P_t + B_t &= w_t + S_{t-1}\left(P_t + D_t\right) + R_{t-1} B_{t-1} \\ S_t &\in \left[0, \bar{S}\right],\ S_{-1}, B_{-1} \end{aligned}$$

C_t is the amount of nondurable consumption goods purchased by the household in period t. The consumption good serves as the numéraire. w_t is the real wage rate. Moreover, the household can trade two financial assets: one-period bonds, denoted by B_t and paying gross real interest R_t in the next period; and stocks, S_t, which trade at price P_t and entitle their holder to dividend payments D_t. The household cannot short-sell stocks and his maximum stock holdings are capped at some $\bar{S} > 1$. The constraint on S_t is necessary to guarantee existence of the learning equilibrium, but never binds along the equilibrium path. All markets are competitive.

The household maximizes the expectation of discounted future consumption under the probability measure $\mathcal{P}$. This measure is the subjective belief system held by agents in the model economy and might differ from rational expectations.

The firm engages in the production of the consumption good, which can also be used for investment. It is produced using capital K_{t-1}, owned by the firm and depreciating at the rate δ, and labor L_t according to the constant returns to scale technology

$$Y_t = K_{t-1}^{\alpha}\left(A_t L_t\right)^{1-\alpha},$$

where A_t is its productivity. There are two financial claims on the firm: shares and noncontingent bonds. The firm's period budget constraint reads as follows:

$$Y_t + (1-\delta) K_{t-1} + B_t + S_t P_t = w_t L_t + K_t + S_{t-1}\left(P_t + D_t\right) + R B_{t-1} \tag{41}$$

I impose constraints on the issuance of financial instruments. On the equity side, the firm is not allowed to change the quantity of shares outstanding, fixed at $S_t = 1$.

Further, it is not allowed to use retained earnings to finance investment. Instead, all earnings have to be paid out to shareholders:

$$D_t = Y_t - w_t L_t - \delta K_{t-1} - (R-1)\, B_{t-1}.$$

These assumptions imply that the firm's capital stock must be entirely debt-financed: Dividends are paid out until $K_t = B_t$ at the end of every period.[17] The firm's level of debt is limited to a fraction $\xi \in [0,1]$ of its total market value (i.e., the sum of debt and equity):

$$\begin{aligned} B_t &\leq \xi\,(B_t + P_t) \\ \Leftrightarrow K_t &\leq \frac{\xi}{1-\xi} P_t \end{aligned} \tag{42}$$

The firm maximizes the presented discounted sum of future dividends, using the household discount factor:

$$\max_{(K_t, L_t, D_t)_{t=0}^{\infty}} \mathbb{E}^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t D_t \text{ s.t. } (41),\ (42),\ K_{-1}$$

In particular, it makes its decisions under the same belief system $\mathcal{P}$ as the household—expectations are homogenous.

The model is closed by specifying market clearing conditions for the goods, labor and equity markets:

$$\begin{aligned} Y_t + K_t &= C + (1-\delta)\, K_{t-1} \\ L_t &= 1 \\ S_t &= 1. \end{aligned}$$

An equilibrium for an arbitrary subjective probability measure $\mathcal{P}$ is defined as a mapping from realizations of the exogenous variable $(A_t)_{t=0}^{\infty}$ and initial conditions (B_{-1}, K_{-1}, R_{-1}) to the endogenous variables $(B_t, K_t, L_t, D_t, P_t, R_t, w_t, C_t, S_t)_{t=0}^{\infty}$ such that markets clear and agents' choices solve their optimization problem under the probability measure $\mathcal{P}$.

A.2 Limiting case $\xi \to 1$

In the simplified model, the model dynamics under learning approach those under rational expectations in the limiting case $\xi \to 1$. To see this, combine Equation (13)

[17] For very low realizations of the productivity shock, the dividend payment will be negative, which is allowed. The value of the firm will be determined by *expected* dividends, which are always positive.

and (14) to obtain:

$$\begin{aligned}
\frac{1-\xi}{\xi}K_t &= \frac{\left(R^k\left(K_t,A_t\right)-R\right)K_t}{R-\exp\left(\hat{\mu}_t+\frac{1}{2}\sigma_\mu^2\right)} \\
\Leftrightarrow \frac{1-\xi}{\xi}\left(R-\exp\left(\hat{\mu}_t+\frac{1}{2}\sigma_\mu^2\right)\right) &= \alpha\left(\frac{A_t}{K_t}\right)^{1-\alpha}\mathbb{E}_t\left[\varepsilon_{t+1}^{1-\alpha}\right]+1-\delta-R.
\end{aligned}$$

It follows that:

$$\begin{aligned}
\Delta\log P_t &= \Delta\log K_t \\
&= \Delta\log A_t-\frac{1}{1-\alpha}\Delta\log\left(\frac{1-\xi}{\xi}\left(R-\exp\left(\hat{\mu}_t+\frac{1}{2}\sigma_\mu^2\right)\right)-1+\delta+R\right) \\
&\xrightarrow{\xi\to 1} \Delta\log A_t=\varepsilon_t.
\end{aligned}$$

B Details on the full model

B.1 Setup of adjustment costs and nominal rigidities

In the full model, final good producers (indexed by $i\in[0,1]$) transform a homogeneous intermediate good into differentiated final consumption goods using a one-for-one technology. The intermediate good trades in a competitive market at the real price q_t (expressed in units of the composite final good). Each retailer enjoys market power in her output market, and sets a nominal price p_{it} for its production. A standard price adjustment friction à la Calvo means that a retailer cannot adjust her price with probability κ. Hence, the retailer solves the following optimization:

$$\max_{P_{it}}\sum_{s=0}^{\infty}\left(\prod_{\tau=1}^{s}\kappa\Lambda_{t+\tau}\right)\left(\left(1+\tau\right)p_{it}-q_{t+s}p_{t+s}\right)Y_{it+s}$$

$$\text{s.t. } Y_{it+s}=\left(\frac{p_{it}}{p_{t+s}}\right)^{-\sigma}\tilde{Y}_{t+s},$$

where $\tilde{Y}_t$ is aggregate demand for the composite final good. Since all retailers that can re-optimize at t are identical, they all choose the same price $p_{it}=p_t^*$. The derivation of the non-linear aggregate law of motion for the retail sector is standard and the final equations are:

$$\begin{aligned}
\frac{p_t^*}{p_t} &= \frac{1}{1+\tau}\frac{\sigma}{\sigma-1}\frac{\Gamma_{1t}}{\Gamma_{2t}} \\
\Gamma_{1t} &= q_t+\kappa\mathbb{E}_t^{\mathcal{P}}\Lambda_{t+1}\frac{\tilde{Y}_{t+1}}{\tilde{Y}_t}\pi_{t+1}^{\sigma} \\
\Gamma_{2t} &= 1+\kappa\mathbb{E}_t^{\mathcal{P}}\Lambda_{t+1}\frac{\tilde{Y}_{t+1}}{\tilde{Y}_t}\pi_{t+1}^{\sigma-1}
\end{aligned}$$

I assume that the government sets subsidies such that $\tau = 1/(\sigma - 1)$ so that the steady-state markup over marginal cost is zero. Inflation $\pi_t = p_t/p_{t-1}$ and the reset price are linked through the price aggregation equation which can be written as

$$1 = (1-\kappa)\left(\frac{p_t^*}{p_t}\right)^{1-\sigma} + \kappa\pi_t^{\sigma-1}$$

and the Tak-Yun distortion term is

$$\Delta_t = (1-\kappa)\left(\frac{\Gamma_{1t}}{\Gamma_{2t}}\right)^{-\sigma} + \kappa\pi_t^{\sigma}\Delta_{t-1}.$$

This term $\Delta_t \geq 1$ is the wedge due to price distortions between the amount of intermediate goods produced and the amount of the final good consumed. The amount of final goods available for consumption and investment is $\tilde{Y}_t = Y_t/\Delta_t$. Similarly, one can define $\tilde{C}_t = C_t/\Delta_t$ as the level of consumption the household could obtain if price distortions were zero.

Similarly to retailers, labor agencies transform the homogeneous household labor input into differentiated labor goods at the nominal price $\tilde{w}_t p_t$ and sell them to intermediate firms at the price w_{ht}, which cannot be adjusted with probability κ_w. Labor agency h solves the following optimization:

$$\max_{w_{ht}} \mathbb{E}_t^{\mathcal{P}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^{s} \kappa_w \Lambda_{t+\tau}\right) \left(\left(1+\tau_w\right) w_{ht} - \tilde{w}_{t+s}p_{t+s}\right) L_{ht+s}$$

$$\text{s.t. } L_{ht} = \left(\frac{w_{ht}}{\tilde{w}_t}\right)^{-\sigma_w} \tilde{L}_t$$

Since all labor agencies that can re-optimize at t are identical, they all choose the same price $w_{ht} = w_t^*$. The first-order conditions are analogous to those for retailers. Again, I assume that the government sets taxes such that $\tau = 1/(\sigma_w - 1)$ so that the steady-state markup over marginal cost is zero. Wage inflation π_{wt} and the Tak-Yun distortion Δ_{wt} are defined analogously to final good producers.

Capital good producers operate competitively in input and output markets, producing new capital goods using old final consumption goods. For the latter, they have a CES aggregator just like households. Their maximization program is entirely intratemporal:

$$\max_{I_t} Q_t I_t - \left(I_t + \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right)$$

In particular, they take past investment levels I_{t-1} as given when choosing current investment output. Their first-order condition defines the price for capital goods.

All of the profits made by the firms described above accrue to households. Similarly, all subsidies by the government are financed by lump-sum taxes on households. The market clearing conditions are summarized below. Supply stands on the left-hand side, demand on the right-hand side.

$$
\begin{aligned}
Y_t = \int_0^1 Y_{jt} dj &= \int_0^1 Y_{it} di \\
\tilde{Y}_t = \frac{Y_t}{\Delta_t} &= C_t + I_t + -\frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 + C_t^e \\
L_t &= \int_0^1 L_{ht} dh \\
\tilde{L}_t = \frac{L_t}{\Delta_{wt}} &= \int_0^1 L_{jt} dj \\
K_t = \int_0^1 K_{jt} dj &= (1-\delta) K_{t-1} + I_t \\
1 &= S_{jt},\ j \in [0,1] \\
0 &= B_t^g
\end{aligned}
$$

B.2 Properties of the rational expectations equilibrium

The rational expectations equilibrium considered here has the following properties that need to be verified. All statements are local in the sense that for each of them, there exists a neighborhood of the non-stochastic steady-state in which the statement holds.

1. All firms choose the same capital-labor ratio K_{jt}/L_{jt} .
2. The expected return on capital is higher than the internal return on debt: $\mathbb{E}_t R_{t+1}^k > R_t$.
3. At any time t, the stock market valuation P_{jt} of a firm j is proportional to its net worth after entry and exit $\tilde{N}_{jt}$ with a slope that is strictly greater than one.
4. Borrowers never default on the equilibrium path and borrow at the risk-free rate, and the lender only accepts debt payments up to a certain limit.
5. If the firm defaults and the lender seizes the firm, it always prefers restructuring to liquidation.
6. The firm always exhausts the borrowing limit.
7. All firms can be aggregated. Aggregate debt, capital and net worth are sufficient to describe the intermediate goods sector.

I take the following steps to prove the existence of this equilibrium. After setting up the firm value functions, Property 1 just follows from constant returns to scale. I then take Properties 2 and 3 as given and prove 4 to 6. I verify that 3 holds. The aggregation property 7 is then easily verified. I conclude by establishing the parameter restrictions for which 2 holds.

Value functions

An operating firm j enters period t with a predetermined stock of capital and debt. It is convenient to decompose its value function into two stages. The first stage is given by:

$$\Upsilon_1(K,B) = \max_{N,L,D} \gamma N + (1-\gamma)(D + \Upsilon_2(N-D))$$

$$\begin{aligned} \text{s.t. } N &= qY - wL + (1-\delta)QK - RB \\ Y &= K^{\alpha}(AL)^{1-\alpha} \\ D &= \zeta(N - QK + B) \end{aligned}$$

I suppress the time and firm indices for the sake of notation. After production, the firm exits with probability γ and pays out all net worth as dividends. The second stage of the value function consists in choosing debt and capital levels as well as a strategy in the default game:

$$\begin{aligned} \Upsilon_2\left(\tilde{N}\right) = \max_{K',B',\text{strategy in default game}} \quad & \beta\mathbb{E}\left[\Upsilon_1(K',B'),\ \text{no default}\right] \\ + \quad & \beta\mathbb{E}\left[\Upsilon_1(K',B^*),\text{debt renegotiated}\right] \\ + \quad & \beta\mathbb{E}\left[0,\ \text{lender seizes firm}\right] \end{aligned}$$

$$\text{s.t. } K' = N + B'$$

A firm that only enters in the current period starts directly with an exogenous net worth endowment and the value function Υ_2.

Characterizing the first stage

The first order conditions for the first stage with respect to L equalizes the wage with the marginal revenue: $w = q(1-\alpha)(K/L)^{\alpha} A^{1-\alpha}$. Since there is no firm heterogeneity apart from capital K and debt B, this already implies Property 1 that all firms choose the same capital-labor ratio. Hence the internal rate of return on capital is common across firms:

$$R^k = \alpha q \left((1-\alpha)\frac{qA}{w}\right)^{\frac{1-\alpha}{\alpha}} + (1-\delta)Q \tag{43}$$

Taking Property 3 as given for now, Υ_2 is a linear function with slope strictly greater than one. Then the following holds for the first-stage value function Υ_1:

$$\begin{aligned}\Upsilon_1(K,B) &= N + (1-\gamma)(D - N + \Upsilon_2(N-D)) \\ &= N + (1-\gamma)(\Upsilon_2' - 1)((1-\zeta)N + \zeta(QK - B)) \\ &> N \\ &= R^k K - RB \end{aligned} \tag{44}$$

This property will be used repeatedly in the next step of the proof.

Characterizing the second stage

The second stage involves solving for the subgame-perfect equilibrium of the default game between borrower and lender. Pairings are anonymous, so repeated interactions are ruled out. Also, only the size B and the interest rate $\tilde{R}$ of the loan can be contracted (in equilibrium $\tilde{R} = R$ but this is to be established first). The game is played sequentially:

1. The firm (F) proposes a borrowing contract $\left(B, \tilde{R}\right)$.
2. The lender (L) can accept or reject the contract.
   - A rejection corresponds to setting the contract $\left(B, \tilde{R}\right) = (0, 0)$. Payoff for L: 0. Payoff for F: $\beta \mathbb{E}\left[\Upsilon_1\left(\tilde{N}, 0\right)\right]$.
3. F acquires capital and can then choose to default or not.
   - If F does not default, it has to repay in the next period. Payoff for L: $\mathbb{E}Q_{t,t+1}\tilde{R}B - B$. Payoff for F: $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B\right)\right]$.
4. If F defaults, the debt needs to be renegotiated. F makes an offer for a new debt level B^*.[18]
5. L can accept or reject the offer.
   - If L accepts, the new debt level replaces the old one. Payoff for L: $\mathbb{E}\Lambda\tilde{R}B^* - B$. Payoff for F: $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B^*\right)\right]$.
6. If L rejects, then she seizes the firm. A fraction $1-\xi$ of the firm's capital is lost in the process. Nature decides randomly whether the firm can be "restructured."

[18] That the interest rate on the repayment is fixed is without loss of generality.

- If the firm cannot be restructured, or it can but the lender chooses not to do so, then the lender has to liquidate the firm.
Payoff for L: $\mathbb{E}\Lambda\xi QK - B$. Payoff for F: 0.
- If the firm can be restructured and the lender chooses to do so, she retains a debt claim of present value ξB and sells the residual equity claim in the firm to another investor.
Payoff for L: $\xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B\right)\right] - B$. Payoff for F: 0.

Backward induction leads to the (unique) subgame-perfect equilibrium of this game. Start with the possibility of restructuring. L prefers this to liquidation if

$$\xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B\right)\right] \geq \mathbb{E}\Lambda\xi QK. \tag{45}$$

This holds true at the steady state because $R^k > R$ (Property 2), $Q = 1$, $\tilde{\beta} = \Lambda$ and

$$\begin{aligned} \xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B\right)\right] &> \xi B + \beta\mathbb{E}\left[R^k \xi K - R\xi B\right] \\ &= \beta\mathbb{E}\left[R^k \xi K\right] \\ &> \xi K \end{aligned} \tag{46}$$

Since the inequality is strict, the statement holds in a neighborhood around the steady-state as well. This establishes Property 5.

Next, L will accept an offer B^* if it gives her a better expected payoff (assuming that lenders can diversify among borrowers so that their discount factor is invariant to the outcome of the game). The probability of restructuring is given by x. The condition for accepting B^* is therefore that

$$\mathbb{E}\Lambda\tilde{R}B^* \geq x\left(\xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B\right)\right]\right) + (1-x)\,\mathbb{E}\Lambda\xi QK. \tag{47}$$

Now turn to the firm F. Among the set of offers B^* that are accepted by L, the firm will prefer the lowest one—i.e., that which satisfies (47) with equality. This follows from Υ_1 being a decreasing function of debt. This lowest offer will be made if it leads to a higher payoff than expropriation: $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B^*\right)\right] \geq 0$. Otherwise, F offers zero and L seizes the firm.

Going one more step backwards, F has to decide whether to declare default or not. It is preferable to do so if the B^* that L will just accept is strictly smaller than B or if expropriation is better than repaying, $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B\right)\right] \geq 0$.

What is then the set of contracts that L accepts in the first place? From the perspective of L, there are two types of contracts: those that will not be defaulted on and those that will. If F does not default ($B^* \geq B$), L will accept the contract simply if it pays at least the risk-free rate, $\tilde{R} \geq R$. If F does default ($B^* < B$), then L accepts if the expected discounted recovery value exceeds the size of the loan—i.e., $\mathbb{E}\Lambda\tilde{R}B^* \geq B$.

Finally, let us consider the contract offer. F can offer a contract on which it will not default. In this case, it is optimal to offer just the risk-free rate $\tilde{R} = R$. Also note that the payoff from this strategy is strictly positive since

$$\begin{aligned}
\tilde{\beta}\mathbb{E}\left[\Upsilon_1(K,B)\right] &> \beta\mathbb{E}\left[R^k K - RB\right] \\
&= \beta\mathbb{E}\left[R^k\tilde{N} + \left(R^k - R\right)B\right] \\
&> 0. \qquad (48)
\end{aligned}$$

The payoff is also increasing in the size of the loan B. So conditional on not defaulting, it is optimal for F to take out the maximum loan size $B = B^*$, and this is preferable to default with expropriation. However, it might also be possible for F to offer a contract that only leads to a default with debt renegotiation. The optimal contract of this type is the solution to the following problem:

$$\max_{\tilde{R},B,B^*} \beta\mathbb{E}\left[\Upsilon_1\left(\tilde{N} + B, \frac{\tilde{R}}{R}B^*\right)\right]$$

$$\begin{aligned}
\text{s.t. } \mathbb{E}\Lambda\tilde{R}B^* &\geq B \\
\mathbb{E}\Lambda\tilde{R}B^* &= x\left(\xi B + \tilde{\beta}\mathbb{E}\left[\Upsilon_1\left(\xi\left(\tilde{N} + B\right), \xi B\right)\right]\right) \\
&\quad + (1-x)\,\mathbb{E}\Lambda Q\xi\left(\tilde{N} + B\right)
\end{aligned}$$

The first thing to note is that only the product $\tilde{R}B^*$ appears, so the choice of the interest rate $\tilde{R}$ is redundant. Further, $B = B^*$ and $\tilde{R} = R$ solve this problem, and this amounts to the same as not declaring default. This choice solves the maximization problem above if the following condition is satisfied at the steady state:

$$\frac{\xi}{R}\left(1 - x + xR + x\Upsilon_1'\left[\frac{R^k}{R} - 1\right]\right) < 1 \qquad (49)$$

For the degree of stock price dependence x sufficiently small, this condition is satisfied. This establishes Properties 4 and 6.

Linearity of firm value

Since firms do not default and exhaust the borrowing limit B^*, the second-stage firm value can be written as follows:

$$\begin{aligned}
\Upsilon_2\left(\tilde{N}\right) &= \beta\mathbb{E}\left[\Upsilon_1\left(\tilde{N} + B, B\right)\right] \qquad (50) \\
\text{where } B &= x\left(\xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi\left(\tilde{N} + B\right), \xi B\right)\right]\right) + (1-x)\,Q\xi\left(\tilde{N} + B\right) \qquad (51)
\end{aligned}$$

We already know that if Υ_2 is a linear function, then Υ_1 is also linear. The converse also holds: The constraint above, together with linearity of Υ_1 imply that B is linear in $\tilde{N}$, and thus Υ_2 is linear, too.

To establish Property 3, it remains to show that the slope of Υ_2 is greater than one. This is easy to see in steady state:

$$\begin{aligned}
\Upsilon_2' &= \beta \frac{\Upsilon_1(K,B)}{\tilde{N}} \\
&= \beta \frac{\gamma\left(R^k K - RB\right) + (1-\gamma)\,\Upsilon_2\left(R^k K - RB\right)}{\tilde{N}} \\
&= \beta\left(\gamma + (1-\gamma)\,\Upsilon_2'\right)\left(R^k \frac{K}{\tilde{N}} - R\frac{B}{\tilde{N}}\right) \\
&= \left(\gamma + (1-\gamma)\,\Upsilon_2'\right)\underbrace{\frac{R^k + \left(R^k - R\right)\frac{B}{\tilde{N}}}{R}}_{=:c_0>1} \\
&= \frac{\gamma c_0}{1-(1-\gamma)\,c_0} \\
&> 1
\end{aligned} \tag{52}$$

Finally, the aggregated law of motion for capital and net worth needs to be established (Property 7). Denoting again by $\Gamma_t \subset [0,1]$ the indices of firms that exit and are replaced in period t, we have

$$\begin{aligned}
K_t = \int_0^1 K_{jt}dj &= \int_{j\notin\Gamma_t}\left(N_{jt} - \zeta E_{jt} + B_{jt}\right)dj + \int_{j\in\Gamma_t}\left(\omega\left(N_t - \zeta E_t\right) + B_{jt}\right)dj \\
&= (1-\gamma+\gamma\omega)\left(N_t - \zeta E_t\right) + B_t
\end{aligned} \tag{53}$$

$$N_t = \int_0^1 N_{jt}dj = R_t^k K_{t-1} - R_{t-1}B_{t-1} \tag{54}$$

$$B_t = \int_0^1 B_{jt}dj = x\xi\left(B_t + P_t\right) + (1-x)\,\xi\mathbb{E}_t\Lambda_{t+1}Q_{t+1}K_t \tag{55}$$

So far, then, all model properties are established except $R^k > R$.

Return on capital

It can now be shown under which conditions the internal rate of return is indeed greater than the return on debt. From the steady-state versions of equations (53) and (54), and the definition of earnings $E = N - K + B$, it follows that

$$R^k = R + \left(\frac{1-\zeta(1-\gamma+\gamma\omega)}{(1-\zeta)(1-\gamma+\gamma\omega)} - R\right)\left(1 - \frac{B}{K}\right). \tag{56}$$

Further, it is straightforward to show that for any combination of parameters, the steady state of the model has $B/K < 1$. A necessary and sufficient condition for $R^k > R$ is therefore that the first term in parentheses above is strictly positive, or equivalently:

$$\gamma > \frac{\zeta(1-\beta)}{1-\zeta(1-\beta)} \frac{1}{1-\omega}. \tag{57}$$