Labor supply in the past, present, and future: a balanced-growth perspective

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Abstract

We argue that a stable utility function of consumption and hours worked for which income effects are slightly stronger than substitution effects can rationalize the long-run data for the main macroeconomic quantities. In these long-run data, in the U.S. as well as in other countries, as productivity grows at a steady rate, hours worked fall slowly and at an approximately constant rate. We narrow down the set of preferences consistent with balanced growth under constant (negative) hours growth. The resulting class amounts to a slight enlargement of the well-known "balanced-growth preferences" that dominate the macro literature and are based on requiring constant hours worked. Thus, hours falling at a constant rate is not inconsistent with the remaining balanced-growth facts but merely requires a slight broadening of the preference class considered. The broadening of the preference class introduces some well-known cases not previously thought to be consistent with balanced growth. From our perspective, we interpret the recent decades of stationary hours worked in the U.S. as a temporary departure from a long-run pattern, and to the extent productivity will keep growing, we predict that hours will fall further.

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1 Introduction

The purpose of this paper is to propose a choice- and technology-based theory for the long-run behavior of the main macroeconomic aggregates. Such a theory—standard balanced-growth theory, specifying preferences and production possibilities along with a market mechanism to be consistent with the data—already exists, but what we argue here is that it needs to be changed. A change is required because of data on hours worked that we document at some length: over a longer perspective—going back a hundred years and more—and looking across many countries, hours worked are falling at a remarkably steady rate: at a little less than half a percentage point per year. Figure 1 illustrates this for a collection of countries. This finding turns

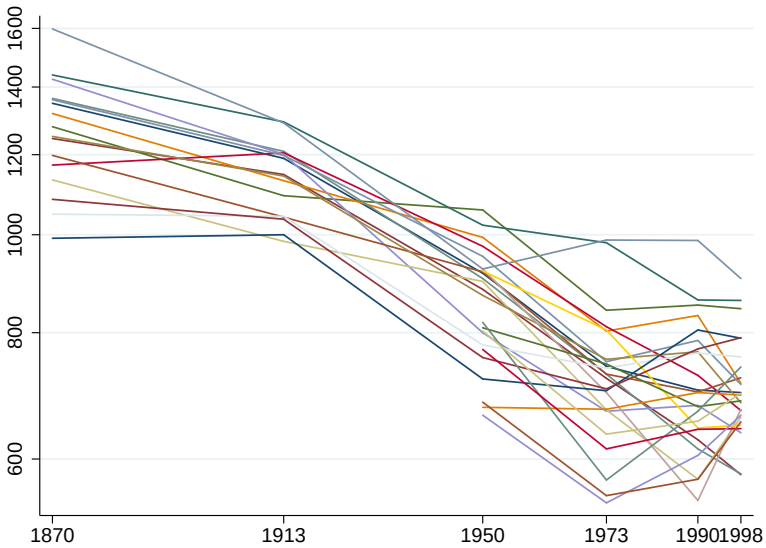


Figure 1: **Average yearly hours worked per capita 1870–1998**

Source: Maddison (2001). The sample includes the following 25 countries: Austria, Belgium, Denmark, Finland, France, Germany, Italy, Netherlands, Norway, Sweden, Switzerland, United Kingdom, Ireland, Spain, Australia, Canada, United States, Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela, Japan. Regressing the log of hours worked on a country fixed effect and year gives a slope coefficient of -0.00462 in the full sample (and -0.00398 for the period 1950–1998). Huberman and Minns (2007) provide similar data.

out to contrast the data in the postwar U.S., where hours are overall well described as stationary, but going back further in time and looking across countries leads one

to view the recent U.S. data rather as an exception.

Since the persistent fall in hours worked is not consistent with the preferences and technology used in the standard framework, we alter this framework. Our alteration is very simple and, on a general level, obvious: to rationalize decreasing hours worked we point to steadily increased productivity over very long periods and preferences over consumption and leisure with the feature that income effects on hours exceed substitution effects. As in the case of the standard setting, we however also impose additional structure by summarizing the long-run data as (roughly, at least) having been characterized by *balanced growth*. So on a balanced growth path, our main economic aggregates—hours worked, output, consumption, investment, and the stock of capital—all grow at constant rates. Characterizing the data as fluctuations around such a path may be viewed as a poor approximation, but here we nevertheless do maintain the position that such a characterization is roughly accurate, at least for the last 150 years of data for many developed countries. Hence, we ask: is there a stable utility function such that consumers choose a balanced growth path, with constant growth for consumption, and constant (negative) growth for hours, given that labor productivity grows at a constant rate? We restrict ourselves to time additivity and constant discounting, in line with the assumptions used to derive the standard preference framework. We find that there indeed are preferences that do deliver the desired properties and our main result is a complete characterization of the class of such preferences.

The modern macroeconomic literature is based on versions of a framework featuring balanced growth with constant hours worked, to a large extent motivated with reference to U.S. data on postwar hours worked; see, e.g., Cooley and Prescott (1995). Our main point here is not to take fundamental issue with this practice; in fact, our proposed utility specification in some ways is quantitatively very similar to the preferences normally used. However, for some issues the distinction may be important. As for discussion of hours historically, there is significant recognition in

the macroeconomic literature that from a longer-run perspective, hours worked have indeed fallen. For example, several broadly used textbooks actually do point to significant decreases over the longer horizon, often with concrete examples of how hard our grand-grandfathers (and -mothers) worked; see, e.g., Barro's (1984) book and Mankiw's (2010) latest intermediate text. In a discussion of some significant length, Mankiw actually reminds us of a very well-known short text wherein John Maynard Keynes speculated that hours worked would fall dramatically in the future—from the perspective he had back then (see Keynes, 1930). Keynes thus imagined a 15-hour work week for his grandchildren, in particular, supported by steadily rising productivity. As it turned out, Keynes was wildly off quantitatively, but we would argue that he was right qualitatively (on this issue...). Finally, in his forthcoming chapter on growth facts, Jones (2015) also points to the tension between the typical description of hours as stationary and the actual historical data.[1]

From Keynes's U.K. perspective, over the postwar period, and in contrast to the U.S. experience, hours worked actually fell steadily until as recently as circa 1980, at which point they appear to have stabilized; we review the data in some detail in Section 2 below. But perhaps more importantly, the picture that arises from looking at a broader set of countries strengthens the case for falling, rather than constant, hours, and going further back in time reinforces this conclusion. With our eye-balling, at least, a reasonable approximation is actually even more stringent: hours worked are falling at a rate that appears roughly constant over longer periods (though, of course, with swings over business cycles, etc.). This rate is slow—somewhere between 0.3% and 0.5% per year—so shorter-run data will not suffice for detecting this trend, to the extent we are right; to halve the number of

[1] Jones writes "A standard stylized fact in macroeconomics is that the fraction of the time spent working shows no trend despite the large upward trend in wages. The next two figures show that this stylized fact is not really true over the longer term, although the evidence is somewhat nuanced".

hours worked at this rate requires around 200 years.

Turning back to the case of the U.S., over the last more than 150 years, thus, as hours have fallen, output has grown at a remarkably steady rate, mainly interrupted only by the Great Depression and World War II. Moreover, over this rather long period, all the other macroeconomic balanced-growth facts also hold up very well; we review these data briefly in Section 3. Thus, as output is growing at a steady rate, hours are falling slowly at a steady rate. The interpretation of these facts that we adopt here is that preferences for consumption and hours belong to the class we define. This preference class is, in fact, very similar to that used ubiquitously in the macroeconomic literature: that defined in King, Plosser, and Rebelo (1988). King, Plosser, and Rebelo showed that the preferences they put forth, often referred to as KPR or, perhaps more descriptively, *balanced-growth* preferences, were the only ones consistent with an exact balanced growth path for all the macroeconomic variables with the restriction to constant hours worked. The class of preferences that we consider in the present paper is thus strictly larger in that it also allows hours worked to shrink over time at a constant rate along a balanced path.

In compact terms, one can describe the period utility function under KPR as a power function of $cv(h)$, where c is consumption and h hours worked and v is an arbitrary (decreasing) function. What we show in our main Theorem 1 is that the broader class has the same form: period utility is a power function of $cv(hc^{\frac{\nu}{1-\nu}})$, where $\nu < 1$ is the preference parameter that guides how fast hours shrink relative to productivity. In terms of gross rates, if productivity grows at rate γ, then hours grow at rate $\gamma^{-\nu}$, whereas consumption grows at $\gamma^{1-\nu}$. For $\nu > 0$, the factor $c^{\frac{\nu}{1-\nu}}$ captures the stronger income effect: as consumption grows, there is an added "penalty" to working (since v is decreasing). Our preference class obviously nests KPR: KPR corresponds to $\nu = 0$.

Having argued that preferences in our class—with a ν that is slightly larger than zero—provide a good account of the longer-run data, what do we then make of the

postwar U.S. experience with stationary hours? The purpose here is not to propose a full account of the shorter-run data, but it seems relevant in this context to also revisit the papers by Prescott (2004), Rogerson (2006, 2008) and Ohanian, Raffo, and Rogerson (2008), who argue that relative tax-rate changes account for the fact that Europeans work less hard now than Americans compared to the early postwar period. If one looks at France and Germany, it is clear that hours decline at a rather fast rate—faster than the rate at which hours decline in the broad cross-section of countries. Our perspective here, then, is that perhaps these countries and the U.S. are temporarily departing from a long-run trend, given that productivity has been growing at a rather constant rate throughout this period. One explanation may well be relative tax-rate changes. Another possible explanation is sharply increased wage inequality, with median wage growth close to zero for many decades; for any worker who does not experience wage growth, the optimal response is of course not to decrease hours. Finally, female labor-force participation took off sooner in the U.S. and has reached a level above that in most European economies, and to the extent this phenomenon represents the loosening of constraints (such as discrimination) it could indeed affect hours in a way that is not fully offset by the reduction of male hours.

Interestingly, our class encompasses some utility functions that are often used in the literature (both in macroeconomics and in other fields). One is another famous functional form of the same vintage as KPR: Greenwood, Hercowitz, and Huffman's (1988) proposed utility function, often referred to as GHH preferences. The GHH class assumes a quasi-linear utility function where utility can be written as a function of c minus an increasing (and convex) function of h. This formulation implies that there is no income effect at all on hours worked. With a judicious choice of v and a $\nu < 0$ we obtain a frequently used case within the GHH class in which the convex function of hours is restricted to be a power function (and the Frisch elasticity is constant). Clearly, without an income effect, hours grow under this formulation (so

long as productivity grows). GHH preferences are often used in applied contexts (see, e.g., Chetty et al., 2011) because they allow simple comparative statics.

Another well-known case is the utility function proposed in MaCurdy (1981) displaying a constant Frisch elasticity of labor supply and a constant intertemporal elasticity of substitution, where the period utility function is additive in a power function of c and a power function of h. However, unless the function of consumption is logarithmic (a special case of the power function), these preferences are well-known not to be consistent with constant hours worked. We show, again by a judicious choice of v, that our preference class actually includes this case. That is, this class of utility functions is consistent with balanced growth—if one admits that hours can change over time along a balanced path. For shrinking hours, one needs the curvature to be high enough (higher than log curvature), since otherwise the marginal utility value of working an hour will grow: if productivity doubles, the marginal utility of consumption must more than halve, because otherwise it will not be optimal to lower hours.

A case that is new relative to the literature is one where the "relative risk aversion" to consumption (RRA) or formally $u_{cc}(c,h)c/u_c(c,h)$ is an increasing (or decreasing) function of the $hc^{\frac{\nu}{1-\nu}}$ composite. Under KPR preferences, the RRA must equal a constant: a preference parameter (usually labeled σ). So, in particular, it is possible that the RRA under our preference specification moves countercyclically, thus displaying higher risk aversion in recessions than in booms. In the cross-section, by the same token, richer households would then be less averse to risk (in the relative sense) and choose riskier portfolios. We briefly discuss this and other possible applications (to growth and business cycles) in Section 6 of our paper.

The paper begins with two data sections. In Section 2 we look at hours worked over different time horizons and in different countries. In Section 3, we then motivate our balanced-growth perspective on longer-run data by revisiting the long-run facts for aggregates, with a focus on the United States. The theory section of the paper

is contained in Section 4 where we lay out the precise balanced-growth restrictions. Then we go on to state our Theorem 1 about what utility function is needed in order for consumers to *choose* balanced-growth consumption and labor sequences. The proof of the theorem is in the Appendix. However, the proof relies heavily on two lemmata—one characterizing the implications of balanced-growth choices for the consumption-hours indifference curves and one for consumption curvature—and we discuss those results in some detail in the main text. The theory section also has a Theorem 2, which is straightforward, showing sufficiency of the stated preference class for balanced-growth choices. The theory section finally contains a sequence of illustrations with examples of utility functions in this preference class. Section 5 comments on consumer heterogeneity, a relevant issue since our theory relies on representative-consumer analysis. This section also briefly discusses the cross-sectional wage-hours-wealth data. Section 6 looks at the Prescott-Rogerson Europe vs. the U.S. postwar comparison of hours worked from the perspective of our theory, and Section 7 concludes.

2 Hours worked over time and across countries

We now go over the hours data from various perspectives: across time and space.

2.1 Hours over time

Figure 2 is the main justification for the assumption of constant hours worked maintained in the macro literature. At least in postwar U.S. data this seems to be a good approximation.

What if we look at some other developed countries? Figure 3 shows hours worked for other selected countries on a logarithmic scale. Now we see that a horizontal line is no longer the best approximation of the data. A country-fixed effect regression suggests that hours fall at roughly 0.45% per year. To be sure, however, there is

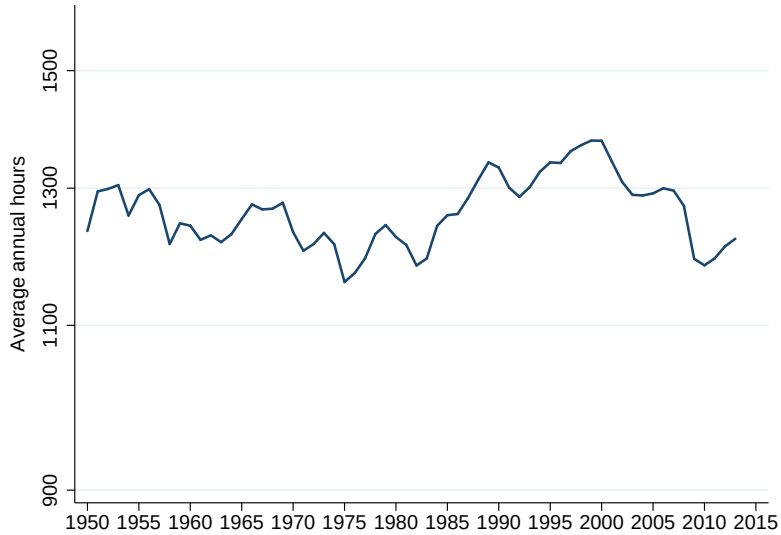


Figure 2: **U.S. average annual hours per capita aged 15–64, 1950–2013**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). Regressing the logarithm of hours worked on time gives an insignificant slope coefficient.

significant heterogeneity; Canada, for example, has stationary hours quite like those in the United States.

The overall falling hours in Figure 3 are not due to the selection of countries. A complementary Figure C.1 in the Appendix C.1 shows the graph for all countries with available data. Average hours are declining clearly in this unrestricted sample, at roughly 0.36% per year. Hence in the cross-country data of the postwar period the United States and Canada overall rather look like outliers. Interestingly, as Figure C.2 in Appendix C.1 shows, a time-use survey shows decreasing hours worked even for the postwar United States.

From a longer-run perspective, the U.S. hours have also clearly been falling (see Figure 4). We also see that once one abstracts from the Great Depression and World War II, hours have been falling at a rather steady rate. Only the period 1980–2000 looks exceptional.

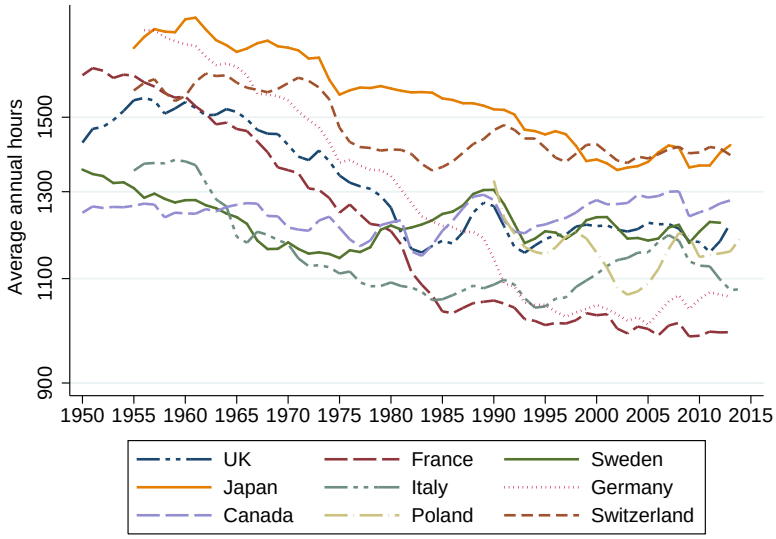


Figure 3: **Selected countries average annual hours per capita aged 15–64, 1950–2015**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). Regressing the logarithm of hours worked on time gives a slope coefficient of -0.00455.

Can the falling trend in hours worked by explained by demographics or the rise in schooling? In Figure C.4 in Appendix C.1 we hold hours worked of different age groups constant at its value of the year 2005 and then check whether the observed changes in the age structure can explain the falling hours. The effect implied by the demographical change is non-monotonic and overall very small. (The baby boomers entering prime working age can however partially explain the observed increase in hours since the 80s.) Furthermore, Ramey and Francis (2009) also provide data on schooling (time attending school and studying at home). As Figure C.5 in Appendix C.1 shows, average weekly hours of schooling increased by less than two hours in total over the period 1900–2005 and cannot, therefore, account for the drop in hours worked (hence: leisure has increased).

The time trend in total hours worked can be split up into trends in participation rates and trends in hours per worker. Figure 5 shows that hours per employed were

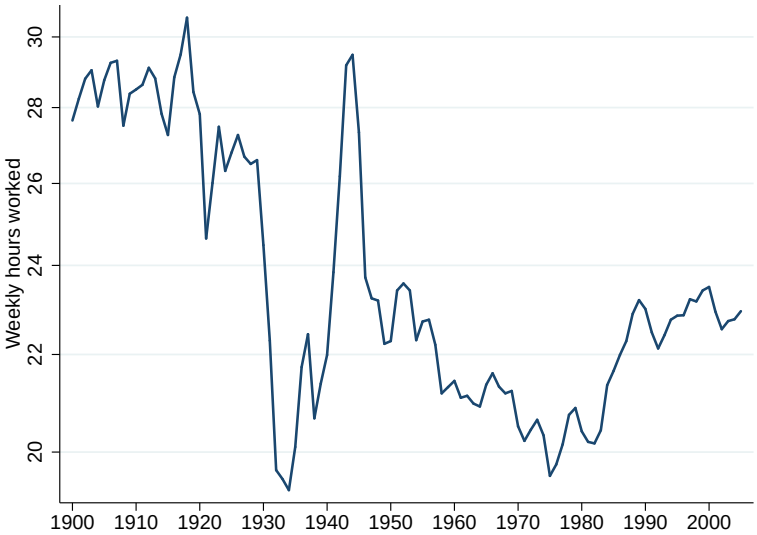


Figure 4: **Weekly U.S. hours worked per population aged 14+, 1900–2005**

Notes: Source: Ramey and Francis (2009). Regressing the logarithm of hours worked on time gives slope coefficient of -0.00285.

in the U.S. declining at a remarkably constant rate, including during the postwar period. This is indeed a remarkably robust fact over time and across countries though the rate of decline differs across countries (see Figure C.3 in Appendix C.1). Figure C.6 and C.7 in Appendix C.1 shows this split in hours per worker and the participation rate again for the U.S. in the postwar period as well as over the last century. In other words: hours in the postwar U.S. are only relatively stable because the participation rate increased steeply.

To sum up: over 100+ years, hours have been falling in all developed countries. In the postwar data hours are still falling in most countries. In countries where they are rather stable, like Canada or the U.S., they are stable only because the participation rate increased quite dramatically. Hours per worker show a clear downward trend in all countries. Participation rates do not show a clear trend over time in developed countries. Hence we conclude that if the participation rate does not increase further in future in the U.S., hours will continue to fall. In fact since the Great Recession, the participation rate fell, as did hours worked per working-age population.

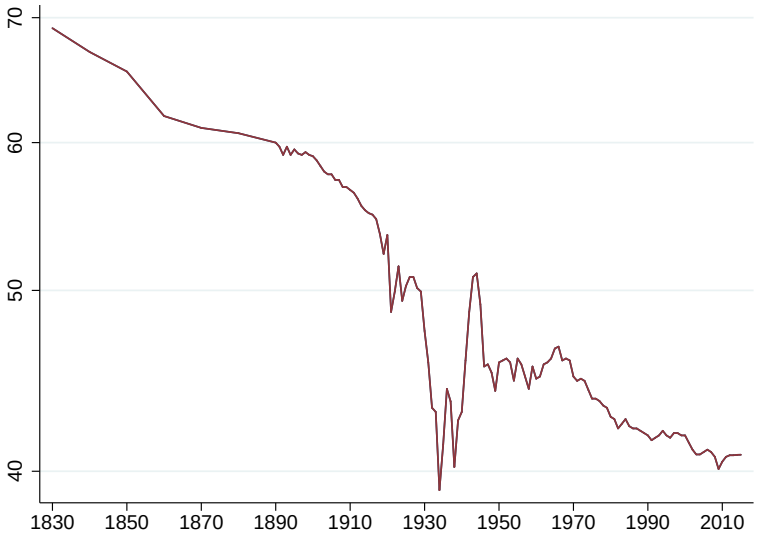


Figure 5: **U.S. weekly hours worked per worker in nonfarm establishments 1830–2015**

Source: Average weekly hours data for 1830–80: Whaples (1990, Table 2.1). 1890-1970: Historical Statistics of the United States: Colonial Times to 1970 (Series D765 and D803). 1970–2015: Statistical Abstract of the United States the number for nonfarm establishments. This graph shows an updates series of the data in Greenwood and Vandenbroucke (2008). Regressing the log of hours on a constant and year gives a slope coefficient of -0.00315 in the full sample (and -0.00208 for the years 1970–2015).

2.2 Hours worked in the cross-section of countries

In the cross-section of countries, our theory predicts that labor productivity (or GDP per capita) should be negatively correlated with hours worked. Winston (1966) establishes such a negative relationship in a sample of 18 countriesand estimates the elasticity of hours worked with respect to GDP per capita to be -0.107. Bick, Fuchs-Schuendeln and Lagakos (2015) document this negative correlation for a larger sample that includes developing countries. Figure C.8 and Figure C.9 in Appendix C.1 shows this negative correlation in the postwar data for the pooled sample and the years 1955 and 2010 separately.

Finally, in Figure C.8 we focus on the 21 countries with data for 1955–2010 and look at the correlation in the growth rates in labor productivity and hours worked over these 55 years. The figure shows again that hours fell for most of the countries.

Moreover, with the exception of South Korea, labor productivity growth is clearly negatively related with growth rate in hours worked.

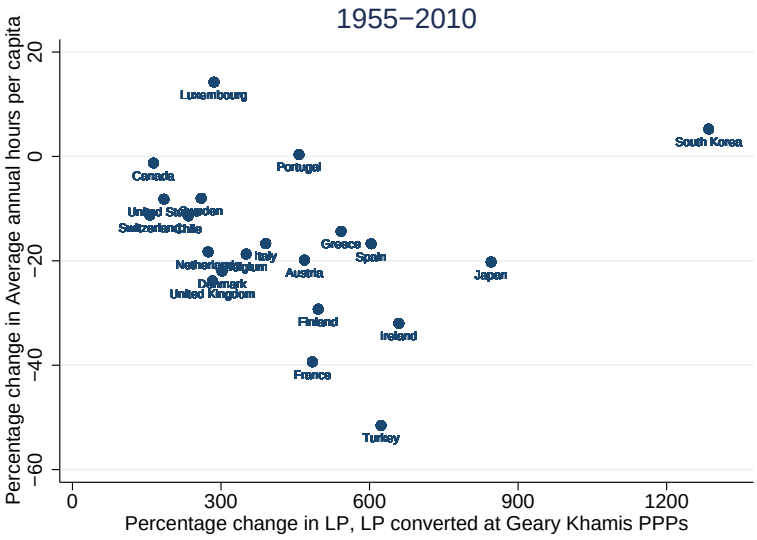


Figure 6: **Changes in hours worked vs. labor productivity**

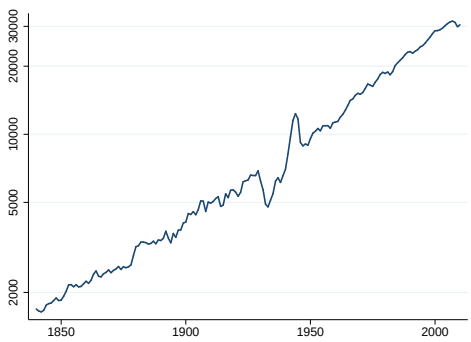
Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64.

3 Balanced-growth facts and theory

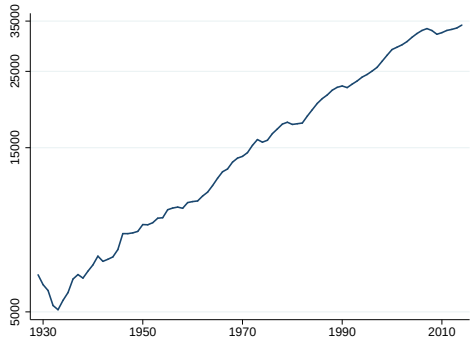
For completeness, we now review the basic "stylized facts of growth" for the United States. These data have been instrumental in guiding the technology and preference specifications in macroeconomic theory.

Figure 7a and 7b show how output and consumption grew over the decades at a very steady rate. Figure 7c and 7d show that the consumption-output ratio and the capital-output ratio remained remarkably stable. (Figure C.10 in Appendix C.1 shows some additional balanced-growth facts often imposed in the macro literature, like constant hours worked or constant factor income shares.)

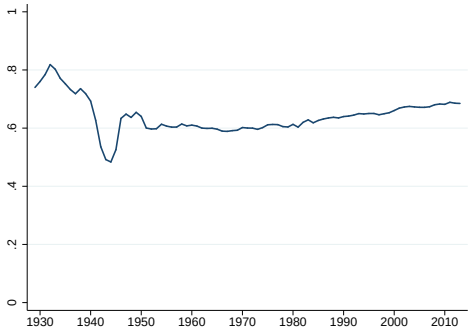
Our main take-away message from Figure 7 is that—in the style of Kaldor



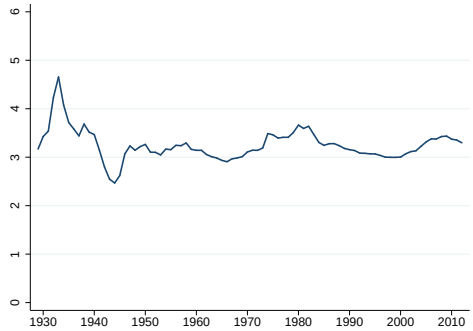
(a) GDP per capita



(b) Consumption per capita



(c) Consumption-output ratio



(d) Capital-output ratio

Figure 7: Balanced growth

Source: BEA and Maddison project.

(1961)—we would like to impose restrictions on our macro framework such that is consistent with these facts. Accordinglly, we define a balanced growth path.

4 Characterization

We now provide our formal analysis. The workhorse macro framework has a resource constraint given by

$$K_{t+1} = F\left(K_t, A_t h_t L_t\right) + (1-\delta)K_t - L_t c_t, \tag{1}$$

where capital letters refer to aggregates and lower-case letters per-capita values, and $F\left(K_t, A_t h_t L_t\right)$ is a neoclassical production function. Here, L is population, h is hours worked per capita and δ the depreciation rate. Growth is of the labor-augmenting kind, because of the Uzawa theorem.[2] We thus assume constant exogenous technology and population growth, i.e.,

$$A_t = A_0\gamma^t, \text{ and } L_t = L_0\eta^t. \tag{2}$$

Turning to preferences, we assume that they are additively separable over time with a constant discount factor β. Quite importantly, and in line with the KPR setting, the instantaneous utility, $u(c_t, h_t)$, is assumed to be stationary. Then households (whether infinitely or finitely lived) maximize

$$\cdots + u(c_t, h_t) + \beta u(c_{t+1}, h_{t+1}) + \ldots \tag{3}$$

subject to a time constraint

$$h_t + l_t = 1, \tag{4}$$

(where l denotes leisure per capita) and a budget constraint

$$a_{t+1} = (1+r_t)a_t + h_t w_t - c_t. \tag{5}$$

A balanced-growth path for this economy is a time path along which K and c grow at constant rates. Such a path thus requires $\frac{K_{t+1}}{K_t} = \frac{A_{t+1}}{A_t}\frac{h_{t+1}}{h_t}\frac{L_{t+1}}{L_t} = \frac{L_{t+1}}{L_t}\frac{c_{t+1}}{c_t}$. This in turn implies

$$\gamma\frac{h_{t+1}}{h_t} = \frac{c_{t+1}}{c_t} \tag{6}$$

[2]Recently, Grossman, Helpman, Oberfield, and Sampson (2015) discusses an interesting exception to the Uzawa theorem.

and

$$\frac{K_{t+1}}{K_t} = \gamma \frac{h_{t+1}}{h_t} \eta. \tag{7}$$

Hence, a balanced-growth path requires $\frac{h_{t+1}}{h_t}$ to be constant.

On a balanced growth path where labor productivity (alternatively, the real wage per hour) changes at constant gross rate $\gamma > 0$, we need to have consumption grow at the same rate as labor income. The derivations above led to $g_c = \gamma g_h$, where g_c is the gross growth rate of consumption and g_h that of hours worked. We thus seek preferences such that g_c and g_h are determined uniquely as a function of the growth rate in (real) wages. Thus, we parameterize preferences with a constant ν so that $g_c = \gamma^{1-\nu}$ and $g_h = \gamma^{-\nu}$.[3] A value of ν greater (smaller) than zero would then correspond to the income effect being stronger (weaker) than the substitution effect. The special case $\nu = 0$ is of interest but we will mainly focus on $\nu \neq 0$; $\nu = 0$ is the standard case, where hours will be constant on a balanced growth path.

Thus, a balanced growth path is one where, for all t, $c_t = c_0 \gamma^{(1-\nu)t}$ and $h_t = h_0 \gamma^{-\nu t}$, for some values c_0 and h_0. One can think of c_0 as a free variable here, determined by the economy's, or the consumer's, overall wealth, with h_0 pinned down by a labor-leisure choice given c_0.

In the following we are interested in an interior solution of the consumption and labor supply decision (i.e., $c_t > 0$, $1 > h_t > 0$) that is consistent with a balanced growth path: we confine attention to the intensive margin of labor supply.[4] Such an interior solution requires that utility to be strictly increasing in consumption and strictly decreasing in hours worked. Two first-order conditions are relevant for the consumer's optimization. The labor-leisure choice is characterized by

$$-\frac{u_2(c_t, h_t)}{u_1(c_t, h_t)} = w_t,$$

[3] With $\nu \geq 1$ the theory would predict decreasing (or constant) consumption as the wage rate increases; we rule this case out.

[4] We comment on the extensive margin in Section 5 below.

where w_t, the return from working one unit of time, grows at rate γ: $w_t = w_0\gamma^t$.[5] On a balanced growth path we thus need this condition to hold for all t. In our theorem below, we will also require that preferences admit a balanced growth path for all $w_0 > 0$. That is, we are looking for preferences that will admit a balanced path for consumption and hours at growth rates $\gamma^{1-\nu}$ and $\gamma^{-\nu}$, respectively, regardless of the (initial) level of the wage rate relative to consumption.

The intertemporal (Euler) equation reads

$$\frac{u_1(c_t, h_t)}{u_1(c_{t+1}, h_{t+1})} = \beta(1 + r_{t+1}),$$

where r is the return on saving and $\beta > 0$ the discount factor. If the economy grows along a balanced path, then we would like this condition to hold for all t, and we need the right-hand side to be equal to an appropriate constant, a constant that moreover depends on the rate of growth of consumption and hours. We will denote this constant R and discuss its dependence on c, h, and γ below. In the subsequent analysis, we will switch from sequence to functional notation. Thus we leave out t subscripts and instead specify the balanced-growth conditions as a requirement that the paths of all the variables start growing from arbitrary positive values (save for those nonlinear restrictions relating the variables to each other that are implied by the equilibrium conditions): they can be scaled arbitrarily.

4.1 Balanced growth using functional language

So note that our balanced-growth path requirements on the utility function can be expressed as follows.

[5] In a decentralized equilibrium, this return denotes the individual wage rate including potential taxes and transfers. Similarly, the return on saving we discuss below should be taken to be net of taxes and transfers.

Assumption 1. *The utility function u has the following properties: for any $w > 0$, $c > 0$, and $\gamma > 0$, there exists an $h > 0$ and an $R > 0$ such that, for any $\lambda > 0$,*

$$-\frac{u_2\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)}{u_1\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)} = w\lambda, \tag{8}$$

and

$$\frac{u_1\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)}{u_1\left(c\lambda^{1-\nu}\gamma^{1-\nu}, h\lambda^{-\nu}\gamma^{-\nu}\right)} = R, \tag{9}$$

where $\nu < 1$.

That is, we must be able to scale variables arbitrarily, but of course consistently with the balanced rates, and still satisfy the two first-order conditions. The scaling is accomplished using λ (for wages/productivity), $\lambda^{1-\nu}$ (for consumption), and $\lambda^{-\nu}$ (for hours) in these conditions. Our main theorem below will thus characterize the class of utility functions u consistent with these conditions. Our theorem will not provide conditions on convexity of the associated maximization problem (of the consumer, or a social planner); obviously, however, conditions must be added such that the first-order conditions indeed characterize the solution. We briefly discuss this issue in the applied contexts below.

4.2 The main theorem

Our main theorem states what restrictions are necessary on the utility function to generate balanced growth.

Theorem 1. *If $u(c, h)$ is twice continuously differentiable, strictly increasing in c and strictly decreasing in h, and satisfies Assumption 1, then (save for additive and multiplicative constants) it must be of the form*

$$u(c, h) = \frac{\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma} - 1}{1-\sigma},$$

for $\sigma \neq 1$, or

$$u(c, h) = \log(c) + \log\left(v(hc^{\frac{\nu}{1-\nu}})\right),$$

where v is an arbitrary, twice continuously differentiable function satisfying, for all $x \equiv hc^{\frac{\nu}{1-\nu}} > 0$, $v'(x) < 0$ and $v(x) + \frac{\nu}{1-\nu}v'(x)x > 0$.

The proof relies crucially on two lemmata, one characterizing the marginal rate of substitution (MRS) function between c and h and one characterizing the curvature with respect to consumption: the relative risk aversion (RRA) function. The proof then uses these lemmata to derive the final characterization. The proofs of the lemmata and of how to use them to complete the proof of the theorem are contained in Appendix A.1. However, we will state and comment on the lemmata, as they are of some independent interest, as well as on the overall method of proof.

4.2.1 The consumption-hours indifference curves

We thus begin with the following lemma:

Lemma 1. *If $u(c,h)$ satisfies (8) for all $\lambda > 0$, and for an arbitrary $c > 0$ and $w > 0$, then its marginal rate of substitution (MRS) function, defined by $u_2(c,h)/u_1(c,h)$, must be of the form*

$$\frac{u_2(c,h)}{u_1(c,h)} = c^{\frac{1}{1-\nu}} v_1(hc^{\frac{\nu}{1-\nu}}), \tag{10}$$

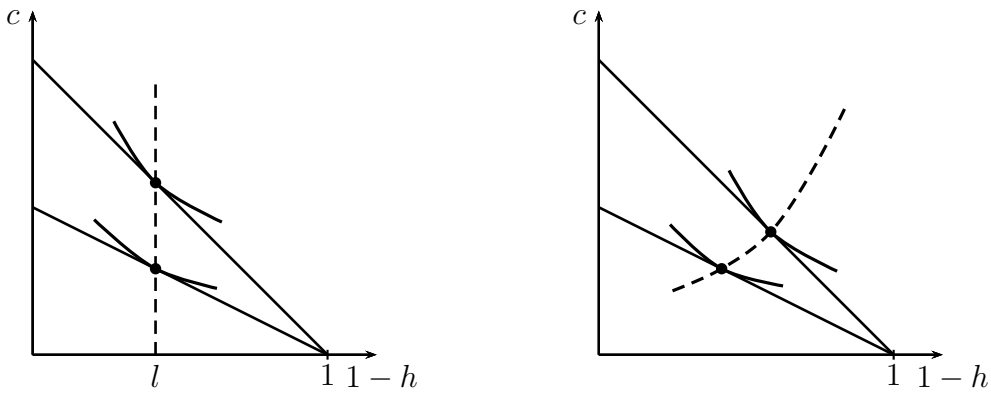
for an arbitrary function v_1.

This lemma characterizes the shape of the within-period indifference curves. Notice here that, in the long run, $hc^{\frac{\nu}{1-\nu}}$ will be constant so that the argument of v_1 will not change over time.

The proof technique for Lemma 1 is very similar to that for Euler's theorem.

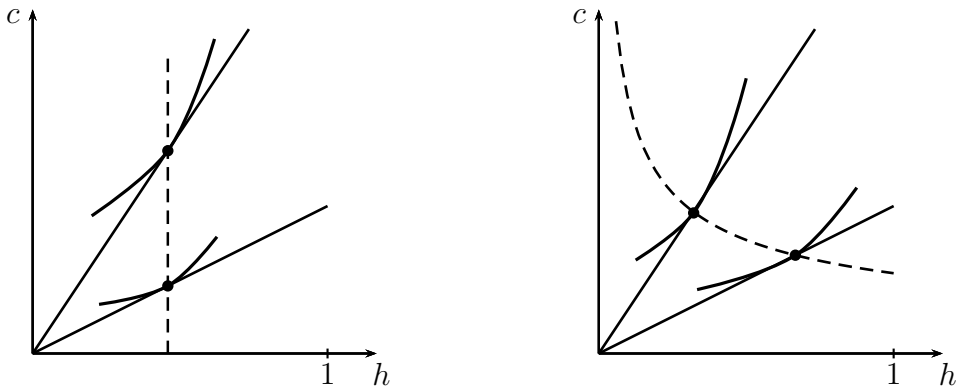
The indifference curves are illustrated with the following sequence of graphs. In Figure 8, we see the KPR indifference curves to the left and a case with $\nu > 0$ to the right, with consumption and leisure on the axes.[6] Clearly, with $\nu > 0$, higher income implies more leisure: the income effect exceeds the substitution effect.

[6]For simplicity we abstract in Figure 8 from non-labor income.



(a) $\nu = 0$ (b) $\nu > 0$

Figure 8: The consumption-leisure trade-off, I

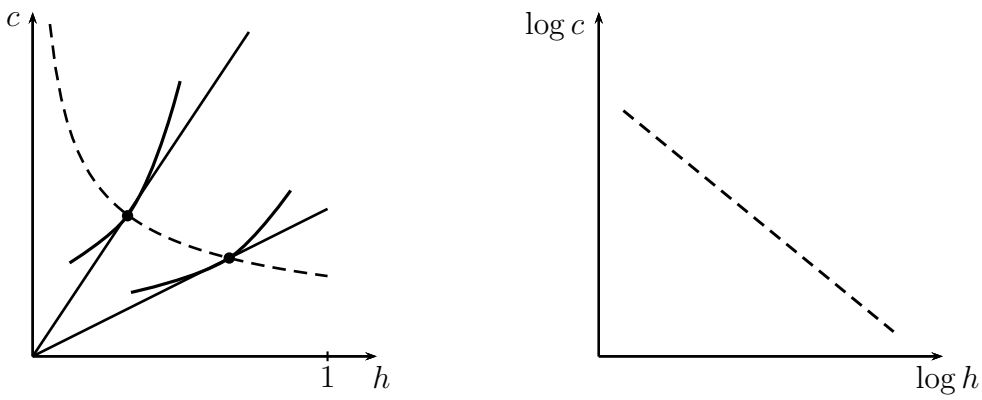


(a) $\nu = 0$ (b) $\nu > 0$

Figure 9: The consumption-leisure trade-off, II

These same preferences can equivalently be depicted with consumption and hours on the axes, as in Figure 9. As in the previous figure, the KPR case is to the left and has constant hours worked, whereas in the right-hand side panel hours decline with higher income.

Finally, Figure 10 takes the right-hand side graph from the previous figure and puts it on the left. On the right, now, we see that same combination of points but on log scales for both the axis. Here, the indifference curves are linear, and that is the defining characteristic of the indifference curves in Lemma 1: that is the precise way in which the income effect has to exceed the substitution effect for the class of



(a) in levels (b) log scale

Figure 10: The consumption-leisure trade-off, III

utility functions delivering balanced growth.[7]

4.2.2 Curvature

Next, let us characterize curvature of u with respect to c with Lemma 2.

Lemma 2. *Under Assumption 1, the relative risk aversion (RRA) function,* $\frac{-cu_{11}(c,h)}{u_1(c,h)}$, *must satisfy*

$$-\frac{cu_{11}(c,h)}{u_1(c,h)} = v_2(hc^{\frac{\nu}{1-\nu}})$$

for an arbitrary function v_2.

As for the previous lemma, let us point out that in the long run, i.e., along a balanced-growth path, $hc^{\frac{\nu}{1-\nu}}$ is constant. Thus, the RRA will be constant. However, its long-run level is endogenous, and over shorter time horizons, in general it will not be constant.

[7]The slope in Figure 10a is $-\frac{1-\nu}{\nu}$. For $\nu < 0$, the substitution effect would be stronger and hours/effort would increase as the wage rises; then the right-hand side panel of Figure 10 would depict a straight line with a positive slope.

The proof of the lemma is straightforward: it involves differentiation of the Euler equation with respect to λ, the use of Lemma 1, and some manipulations.

The term relative risk aversion here is used only for convenience and is not entirely appropriate here: in a dynamic model with leisure, as Swanson (2012) has shown, the appropriate risk aversion concept is based on the value function.

4.2.3 The proof structure and some comments

The structure of the overall proof, based on the lemmata, is as follows. Our description is in two steps that are similar in nature. First, use Lemma 2 to integrate over c to obtain a candidate for u_1; this can be accomplished straightforwardly since the left-hand side of the lemma can be expressed as the derivative of $\log u_c$ with respect to $\log c$. Now note that integration with respect to one variable delivers an unknown function (a "constant") of the other variable. This function can then be restricted by comparison with the characterization in Lemma 1 (a "cross-check").

Second, once the first integration and cross-checking, with its implied restrictions, is completed, integrate again with respect to c, from the obtained u_1, to deliver a candidate for u. Then, as in the previous step, another function of h appears, and it too needs to be cross-checked with Lemma 1 and thus further restricted. This, then, completes the proof.

Notice that, although we were motivated by data displaying increasing productivity growth and falling hours, the proof does not assume $\gamma > 1$ or $\nu \geq 0$. Potentially, the model could thus generate an increasing h at a constant rate, and we shall see an example of this below.

Second, to our surprise, we did not see a full proof of the KPR result in the literature. In particular, in the proofs we have looked at, the fact that the RRA is constant along a balanced path is taken to mean that this constant is exogenous (i.e., given by a preference parameter σ and independent of h). This is a correct presumption but nontrivial to prove, and it is dealt with in our proof in the Appendix

A.1.[8]

4.2.4 Sufficiency

Of course, the v in the theorem has to be such that a characterization based on first-order conditions is valid. Thus, v has to be such that the indifference curves defined by $u_0 = cv(hc^{\frac{\nu}{1-\nu}})$ have the right shape for all u_0. I.e., $v^{-1}(u_0/c)c^{-\frac{\nu}{1-\nu}}$ has to be strictly increasing and concave in c for all u_0.

Under these restrictions, we thus also have the following theorem, guaranteeing sufficiency.

Theorem 2. *Assume that $\nu < 1$. If $u(c,h)$ is given by*

$$u(c,h) = \frac{\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma} - 1}{1-\sigma},$$

for $\sigma \neq 1$, or

$$u(c,h) = \log(c) + \log\left(v(hc^{\frac{\nu}{1-\nu}})\right),$$

where v is an arbitrary, twice continuously differentiable function with $v(x) > -\frac{\nu}{1-\nu}v'(x)x$ and $v'(x) < 0$ for all x and the above-stated concavity requirements, then it satisfies Assumption 1.

Since this proof is much less cumbersome than that for the main theorem, and since it involves the manipulations necessary in applied work based on the preference class we identify here, we include it in the main text.

Proof. Straightforward differentiation delivers

$$u_1(c,h) = \frac{1}{c}\left(1 + \frac{\nu}{1-\nu}\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}\right)\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma}$$

[8] We would be very grateful if someone could point us to a proof somewhere, because we may well have missed it.

and

$$u_2(c,h) = \frac{1}{h}\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}\left(c\cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma}.$$

Dividing the latter by the former we obtain

$$\frac{u_2(c,h)}{u_1(c,h)} = \frac{c}{h}\frac{\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}}{1+\frac{\nu}{1-\nu}\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}}.$$

By multiplying c by $\lambda^{1-\nu}$ and h by $\lambda^{-\nu}$ we obtain that this expression increases by a factor λ. We have thus reproduced the first part of Assumption 1, i.e., the first-order condition for labor on a balanced growth path.

By evaluating $u_1(c,h)/u_1(c\gamma^{1-\nu}, h\gamma^{-\nu})$, we obtain $\gamma^{\sigma(1-\nu)}$, i.e., an expression that is independent of c and h and hence c and h can be scaled arbitrarily. By letting $R = \gamma^{\sigma(1-\nu)}$ we therefore see that also the second condition of Assumption 1 is verified. Finally, it is easy to see that $v(x) > -\frac{\nu}{1-\nu}v'(x)x$ and $v'(x) < 0$ ensure that utility is strictly increasing in c and strictly decreasing in h.

■

4.3 Special cases and relations to the literature

We now look at special cases of interest.

4.3.1 King-Plosser-Rebelo (1988): $\nu = 0$

With $\nu = 0$ we get the following class of preferences

$$u(c,h) = \begin{cases} \frac{(c\cdot v(h))^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log(c) + \log v(h), & \text{if } \sigma = 1. \end{cases} \tag{11}$$

This is the most general preference class that is consistent with a balanced growth path along which h is constant. These preferences were first specified by King, Plosser, and Rebelo (1988). In the KPR class the income and substitution effects

of changes in the wage rate precisely cancel each other and households choose to supply constant hours. Due to this feature the KPR class is dominating the macro literature. Sometimes the KPR class is also referred to as "balanced-growth preferences" and often their use is justified because it seems appealing to restrict attention to a framework that is consistent with a balanced growth path. However, the KPR class is not only characterized by the balanced growth restriction but also by the requirement that labor supply is constant.

Within the KPR class two special cases stand out. One is the Cobb-Douglas case with $v(h) = (1-h)^{\kappa}$ and $\sigma \neq 1$ (or with $\sigma = 1$ and $v(h) = \kappa \log(1-h)$). Thus, $u(c,h) = (c(1-h)^{\kappa})^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$ and otherwise $u(c,h) = \log c + \kappa \log(1-h)$. The Cobb-Douglas case restricts the elasticity of substitution between consumption and leisure to be one. Furthermore, the Cobb-Douglas case is part of the Gorman class, which implies that marginal propensities to consume and work are independent of wealth.

The second often-used case of KPR preferences is

$$u(c,h) = \log c - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}},$$

which is obtained by setting $\sigma = 1$ and $v(h) = \exp\left(-\psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)$. The parameter $\theta > 0$ is then the (constant) Frisch elasticity, i.e., the percentage change in hours when the wage is changed by 1 percent, keeping the marginal utility of consumption (or wealth) constant. We will discuss this elasticity more below.

4.3.2 A case of the Greenwood-Hercowitz-Huffman (1988) preferences

With $v(x) = 1 - x^{-\frac{1-\nu}{\nu}}$ and $\nu < 0$ with $\sigma \neq 1$ (and $v(x) = \log\left(x^{-\frac{1-\nu}{\nu}}\right) + \log\left(x^{\frac{1-\nu}{\nu}} - 1\right)$ with $\sigma = 1$), we obtain the quasi-linear preferences

$$u(c,h) = \begin{cases} \frac{\left(c - h^{1-\frac{1}{\nu}}\right)^{1-\sigma} - 1}{1-\sigma} & \text{if } \sigma \neq 1, \\ \log\left(c - h^{1-\frac{1}{\nu}}\right) & \text{if } \sigma = 1. \end{cases} \tag{12}$$

with $\nu < 0$. This is an often used case of the Greenwood-Hercowitz-Huffman (1988) preferences in which the Frisch elasticity is constant and equal to $-\nu$. These preferences are non-homothetic but they are part of the Gorman class. GHH preferences preclude any income effect on hours worked. Clearly, with a substitution effect alone, GHH preferences imply increasing hours as the wage rate increases. Consequently, we have $\nu < 0$ and there is no overlap with the KPR class. In fact, preferences (12) imply a relative risk aversion which depends on $hc^{\frac{\nu}{1-\nu}}$.

Quasi-linear preferences are widely used in the applied theory and labor literatures, where the household problem is often assumed to be static and σ can be set to zero without loss of generality. However, the quasi-linear formulation does preclude income effects.

4.3.3 MaCurdy (1981)

With $v(x) = \left(1 - \frac{\psi\nu(1-\sigma)}{(1-\nu)(\sigma-1)} x^{\frac{(1-\nu)(\sigma-1)}{\nu}}\right)^{\frac{1}{1-\sigma}}$ for $\sigma \neq 1$, we obtain the case considered by MaCurdy (1981) with

$$u(c,h) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \text{ if } \sigma \neq 1, \tag{13}$$

where $\theta = \frac{\nu}{(1-\nu)(\sigma-1)-\nu}$. The attractiveness of this functional form is that two important elasticities are controlled by two separate parameters: the intertemporal elasticity of substitution (IES) is constant and equal to $1/\sigma$ and the Frisch elasticity is equal to θ; we will discuss these two elasticities more broadly below. As is well known, with $\sigma \neq 1$, preferences of the form (13) are not part of the KPR class. For this reason, as already discussed in subsection 4.3.1, a significant part of the macroeconomic literature restricts itself to the case with a unitary IES by setting $\nu = 0$, $\sigma = 1$ and $v(h) = \exp\left(-\psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)$.[9] Then the preferences become $u(c,h) = \log c - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ and are part of the KPR class.

[9]For instance, Shimer (2010) proposes this preference specification in chapter 1 of his textbook and then writes "This formulation imposes that preferences are additively separable over time

Figure 11 below illustrates how σ and ν have to be restricted on a balanced path with falling hours: $\nu > 0$ requires $\sigma > \frac{1}{1-\nu} > 1$. Thus, any point on the downward-sloping curve is admissible (in the figure ν is set at a quantitatively reasonable value).

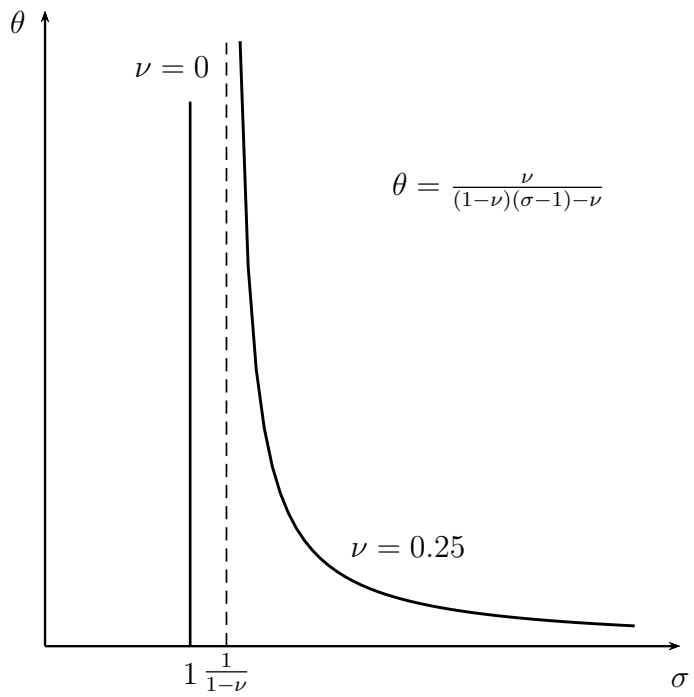


Figure 11: Combinations of elasticities

The figure shows combinations of relative risk aversion in consumption σ and Frisch elasticity θ in the functional form (13) that are consistent with (i) constant hours ($\nu = 0$) and (ii) hours falling at rate $\gamma^{-0.25}$. With two percent productivity growth, i.e., $\gamma = 1.02$, and $\nu = 0.25$ hours worked decline at roughly 0.5 percent per year.

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and across states of the world. It also imposes that preferences are consistent with balanced growth—doubling a household's initial assets and its income in every state of the world doubles its consumption but does not affect its labor supply. [...] I maintain both of these assumptions throughout this book."

4.3.4 Consumption curvature

The IES—the intertemporal elasticity of substitution of consumption—is a key object in some macroeconomic analyses. In a time-additive setting without an hours choice, it is also one divided by the coefficient of relative risk aversion, $u''(c)c/u'(c)$. However, in a context where leisure is valued, it is more difficult to measure these concepts. A natural measure of risk aversion would define a lottery over consumption and hours, or over wealth; Swanson (2015) discusses this question in detail. Thus, what we defined as our RRA function above—$u_{cc}(c,h)c/u_c(c,h)$—is not the most natural measure of risk aversion: it can only be defined if one considers lotteries over consumption only, keeping h fixed. The RRA function is thus merely a measure of curvature in consumption, keeping h fixed. Turning to its characterization, Lemma 2 shows that it is endogenous—it is a function of $hc^{\frac{\nu}{1-\nu}}$—but it is constant along the balanced growth path.

Similarly, the definition of the intertemporal elasticity of substitution of consumption (IES), i.e., $d\log(c_{t+1}/c_t)/d\log(1+r)$, where r is the net interest rate between t and $t+1$, is more complicated when the utility function includes hours worked. Along the lines of the definition above, one can define a restricted IES notion keeping h_t and h_{t+1} constant. When evaluated on the balanced growth path one then obtains the IES $1/(u_{cc}(c,h)c/u_c(c,h))$, which again is constant over time from Lemma 2, but a function of $hc^{\frac{\nu}{1-\nu}}$. Thus, one obtains one divided by the curvature measure used above for the restricted notion of the RRA.

Here we first wish to re-emphasize the point just made: although the consumption curvature remains constant on a balanced growth path, it can be endogenously determined. In contrast, in the standard KPR setting, the IES is not only constant on a balanced growth path but exogenous. We have the following.

Proposition 1. *Given the preferences specified in Theorem 1, with $\nu = 0$, the intertemporal elasticity of substitution is independent of c and h: it equals $1/\sigma$.*

With $\nu \neq 0$, however, the intertemporal elasticity of substitution can, but will not necessarily, depend on $hc^{\frac{\nu}{1-\nu}}$.

Proof. For the KPR class this is verified straightforwardly. For the case $\nu \neq 0$, two cases are dealt with in the text below: one where the IES is decreasing in $hc^{\frac{\nu}{1-\nu}}$ and one where it is constant and exogenous (and equal to $1/\sigma$). ■

For the MaCurdy formulation, it is straightforwardly verified that the RRA is exogenous and equals σ. However, under GHH, it is equally straightforwardly verified, that the RRA/IES is indeed endogenous. For many applications, perhaps particularly in asset pricing, it may be interesting to consider preferences where the RRA in particular is decreasing in the consumption-hours aggregate $ch^{\frac{\nu}{1-\nu}}$: in this case booms involve consumption curvature.

We have not pursued a general investigation into how the RRA may vary under different assumptions on v. It may however be instructive to simply show that a formulation with a decreasing RRA is possible. So let $v(x) = \left(1 - \frac{\psi(1-\sigma)}{\epsilon} x^{\epsilon}\right)^{\frac{1}{1-\sigma}}$ and $\epsilon \equiv \frac{1-\nu}{\nu}$. We then obtain the functional form

$$u(c,h) = \frac{c^{1-\sigma}}{1-\sigma} - \psi \frac{h^{\epsilon} c^{2-\sigma}}{\epsilon}, \tag{14}$$

for $\psi > 0$, $\sigma > 2$ and $\epsilon > \sigma - 1$. In this case, we obtain

$$\text{RRA} = \sigma + \frac{(2-\sigma)\psi \frac{h^{\epsilon} c}{\epsilon}}{1 - (2-\sigma)\psi \frac{h^{\epsilon} c}{\epsilon}}, \tag{15}$$

which is decreasing in $x \equiv hc^{\frac{\nu}{1-\nu}}$.

4.3.5 The Frisch elasticity

The Frisch elasticity of labor supply is defined as the percentage change in hours when the wage rate is changed by 1 percent, keeping the marginal utility of consumption constant. It is a useful concept in the context of intertemporal substitution of hours worked when there is a frictionless market for borrowing and lending: it

relies on the notion that whatever extra labor income is earned by working harder is substituted toward other periods. This is captured by the requirement that the marginal utility of consumption remain unchanged; it would have to change if the income had to be consumed today.

For the KPR formulation, Trabandt and Uhlig (2011) provide a theorem specifying for which subclass of KPR that the Frisch elasticity is constant, i.e., independent of (c, h): it is constant under $u(c, h) = \log c - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ and it is constant under $u(c, h) = \frac{c^{1-\sigma}}{1-\sigma}\left(1 - \kappa(1-\sigma)\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)$ (for $\sigma \neq 1$ and $\kappa > 0$), but it is constant for no other function (other than for affine transformations of the given ones). Our formulation, for $\nu \neq 0$, appears harder to characterize fully in this regard, but it is clear that there are two cases in which the Frisch elasticity is constant. One is our GHH formulation, for which the Frisch elasticity is equal to $-\nu$. Interestingly, although the Frisch elasticity for any given function $u(c, h)$ generally does not feature invariance with respect to monotone transformations—the elasticity for u is different than that for $f(u)$, where f is monotone—it is actually invariant in the GHH case. The second case obtains with the MaCurdy function. So far, we know of no other cases than these two.

4.3.6 Departing from time invariance or time separability

Our preferences rely on there being a stationary utility function $u(c, h)$ characterizing choice. It is not altogether uncommon in the literature that people use utility functions that are either not stationary or not time-separable. As for non-stationarity, the typical assumption is that some elements of the period utility function shift with labor productivity: preferences change over time, in line with technology. Such functions are often motivated by (but not derived from) some form of home-production structure where the same productivity growth as in final-goods production occurs. As an example, one can make (13) consistent with constant hours in the long run by adding a time-varying term in front of ψ that is growing at the appropriate rate

(see, e.g., Mertens and Ravn, 2011). Such a formulation has been deemed useful when one wants to consider free curvature in consumption and hours separately and yet not violate the balanced-growth conditions.

Reconciling constant hours worked in the long run with a small (or non-existent) income effect has also attracted some attention in the macro literature, since small income effects are sometimes appealing when it comes to fluctuations around a balanced growth path. Hence, the literature has extended the KPR class by giving up the assumption of time separability. A particularly well-known case is Jaimovich and Rebelo (2009). Our analysis shows that even the GHH utility function (12) is part of the general balanced-growth preferences specified in Theorem 1 but of course, as discussed above, they would imply increasing hours worked as wages grow. However, by adding a "habit" term $X_t = c_t^{\rho} X_{t-1}^{1-\rho}$ in front of $h^{1-\frac{1}{\nu}}$ to these preferences, we can also obtain the preferences studied in Jaimovich and Rebelo (2009).

The purpose here is not to take issue with preference formulations that depart from time invariance or time separability. Relatedly, whether the additional terms in such formulations ought to be exogenous to the individual (external) or controlled by the individual (internal) is not a question we address here. It suffices to say that the "tricks" that have been employed in the literature are still possible to employ under our preference class. We will return briefly to the possibility of introducing externalities in the next section.

5 Consumer heterogeneity and cross-sectional facts

We now comment briefly on two important concerns. One is that the analysis so far has exclusively looked at a representative-agent economy, so a natural concern is whether our results are robust to introducing consumer heterogeneity of various forms. Another concern has to do with the cross-sectional implications of our pref-

erence class: if income effects exceed substitution effects, it seems that high-wage workers would work less than low-wage workers. We therefore also discuss this aspect below.

5.1 Models of consumer heterogeneity

Our theory of labor supply in the long run, strictly speaking, holds only for a representative-agent economy. Is it relevant, then, in cases when aggregation does not hold? Whereas it is beyond the scope of the present paper to provide a full answer to this question, let us still conjecture in the affirmative. More precisely, we conjecture that in an environment with a stationary distribution of agents heterogeneous in assets, wages, utility-function parameters, etc., preferences in our class are needed to match the aggregate growth facts (including aggregate hours shrinking at a constant rate).

The reason for this conjecture is perhaps best explained with an example. So consider the modern macro-style models of inequality: the Bewley-Huggett-Aiyagari model. This model by now exists in a vast variety of versions, with the common element being that there are incomplete markets for consumer-specific idiosyncratic shocks of different kinds and implied differences in wealth and consumption. Many of these models also consider substantial additional heterogeneity, such as in preferences (see, e.g., the multiple-discount factor model in Krusell and Smith, 1998), and yet others consider life-cycle versions with and without bequest motives (see, e.g., Huggett, 1996).

These modern-macro models of inequality, then, don't display aggregation (at the very least due to incomplete markets) and they are typically analyzed in steady state. A key question, thus, is: for standard KPR preferences, and more generally for the broader class of preferences considered here, do these models admit balanced growth? The answer is yes. It is straightforward to transform variables and verify this assertion, just like a representative-agent model would be rendered stationary by

variable transformation. Of course, growth makes a difference—the discount rate(s), for example, would need to be transformed—so that some aspects of the aggregate variables (such as the capital-output ratio) will depend on the rate of growth, as will the moments of the stationary distribution of wealth. In Appendix B.1, we formally prove this assertion for a typical Aiyagari model with preference heterogeneity in discounting (and it should be clear from the analysis there that many other sources of heterogeneity could be handled as well). Why does the transformation of variables work? It is straightforward to see that it is precisely because the preferences are in the pre-specified class, and for this reason we conjecture that it would not work outside of this class: balanced growth is, by definition, a set of paths for the economy's different variables that can be rendered constant by standard transformation.

Can life-cycle models with shrinking hours be accommodated? Though the model in Appendix B.1 has infinite-lived agents, it should be clear from the analysis there that the answer is yes. However, in the context of life-cycle models, one is also led to think about participation, and perhaps how participation changes as life expectancy rises. The modeling above focuses entirely on the intensive margin; what can then be said from a theoretical perspective on the extensive margin and how it reacts to productivity growth? How does the extensive margin depend on the form of the utility function? We have begun thinking about these issues, but a full study must be conducted as a separate project. Let us therefore just make a few observations here. So suppose that hours are restricted to be either 0 or a positive, exogenous amount $\bar{h}$ (say, per month or week). Then, first of all, some features of the utility function will not matter. To illustrate, consider the MaCurdy function. Clearly, what matters for choice, then, is the difference between 0 and $\psi\frac{\bar{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$. Hence, the curvature (which contains ν, a key parameter in this paper) is unimportant per se. Of course, if there are two possible choices for positive hours (say, half-time and full-time), then curvature again becomes important. Second, suppose one wants to derive a decreasing participation rate over time: would this be possible for the utility

function class we propose? In general, this may be difficult, but suppose one imagines a planner allocating work to households in the population, 0 or $\bar{h}$, and suppose labor productivity is growing. Then if $u(c, h)$ is not additively separable, the efficient allocation involves different consumption levels across working status. Moreover, it appears difficult to obtain a balanced-growth equilibrium where the participation rate shrinks at a constant rate. However, if $u(c, h)$ is additively separable—the MaCurdy case—it is, in fact possible, so long as $\sigma > 1$. It is thus not possible with KPR preferences, which require $\sigma = 1$. Having said all this, a more realistic model would have decentralized markets with less than full insurance, etc., and the implications of such a model are less obvious. One would ideally construct such a model with both an intensive and an extensive margin.

5.2 Cross-sectional data

Another concern one might have from a perspective of heterogeneity is that perhaps the model with an income effect that is larger than the substitution effect might be inconsistent with what we know from cross-sectional data on households. In particular, there seems to be a view that consumers with higher wages work more, and not less, as would be implied by our theory. We now make a sequence of remarks on this issue. None of these remarks settles the issue entirely and, as in the case with the participation margin, a fuller analysis is needed and such an analysis is outside the scope of the present paper.

First, we are not entirely sure of what the data says. Ideally, one would want a life-time, all-inclusive hours measure and then ceteris-paribus experiments where a permanent wage is changed across households. Arguably, convincing such studies are hard to come by. Interestingly, there is in fact a recently published study that claims that the wage-hours correlation is negative, not positive. In particular, the study of the intensive margin in Heathcote, Storesletten, and Violante (2014) reports such a correlation, after taking out time dummies and age effects. We note in this

context that based on the consumer data they look at, Heathcote, Storesletten and Violante (2014) actually use a MaCurdy preference formulation—thus, one in our class—that implies a strong income effect, even a stronger one that we need to account for the long-run fall in hours: $\nu = 0.184 > 0$.[10] As for what the data says about the extensive margin, it is well documented that the highly educated work longer, but they also start working later. Thus, how life-time participation varies with measures of permanent wage is also not clear.

If, however, the perception that high-skilled people work more, not less, is correct, then we must point out that such a fact would be difficult to explain also with the standard model, i.e., with KPR preferences: our generalization would merely make the challenge slightly more difficult. There are studies in the literature that have attempted to address this issue, using a combination of assumptions. One is that the high wages that are observed—and are observed to be associated with higher working hours—represent a temporary window of opportunity. For such a situation, our preference class is consistent with a positive correlation. Another possibility is a non-convexity of the budget set of consumers in the form of a wage rate that depends on the amount of hours worked (see Erosa, Fuester and Kambourov, 2015). One can easily imagine other channels. Suppose, for example, that people differ in their "utility cost of effort".[11] Then those with high costs will work less and presumably, when effort toward education and learning is factored in, also obtain lower wages. Thus, a positive correlation between wage and hours would be generated in the cross-section. Other elements of heterogeneity could, it seems, also deliver the same

[10] With an annual wage growth of 2 percent, $\nu == 0.184$ implies an annual growth rate of hours work of 0.365 percent.

[11] A formulation of such heterogeneity is entertained in Bick et al. (2015), with a utilitarian planner choosing who works and who does not at any point in time. This formulation would lead, in reduced form, to a utility function of the kind $\log c - v(h)$, where c is per-capita consumption, equalized across all agents, and h is total employment. Bick et al. combine this setting with a Stone-Geary element in order to obtain decreasing hours during the transition.

qualitative result and it is an open question what amount of heterogeneity would be necessary to turn a negative into a positive hours-wage correlation in the cross-section.

Another avenue for explaining the cross-section is to short-circuit the income effect on the individual level but maintain it on the aggregate level, through an externality. This approach would be rather ad hoc but it would work as follows: replace the c in $hc^{\frac{\nu}{1-\nu}}$ in our setting with $c^{\xi}\bar{c}^{1-\xi}$, where $\bar{c}$ is the average consumption level in the economy. Then, if $\xi = 0$, on the individual level, the income effect would equal the substitution effect. One can further weaken the income effect by considering a $\xi < 0$, thus producing a positive hours-wage correlation in the cross-section. Thus, taken together, there are many ways to generate a positive wage-hours correlation, some explored in the literature and some not, and it remains to address the balanced-growth facts with these theories. We believe that a full evaluation of the cross-sectional aspects of the model requires much more work and that it is far to early to dismiss KPR preferences (or those in our proposed class).

6 Hours worked in Europe and the U.S.

Prescott (2004) and Rogerson (2006, 2008) use preferences that imply constant hours worked in the long run. They argue that U.S. hours worked relative to those in Europe have gone up because of upward movements of tax rates in Europe relative to those in the U.S.[12] What does the present theory of long-run labor supply have to say about these comparisons?

The present theory does not present a problem for the Prescott-Rogerson ar-

[12]Interestingly, Ohanian, Raffo, and Rogerson (2008), look at the developments in a number of OECD countries, along with tax-rate data, and also point to falling hours. They interpret this phenomenon as transitional and model it with Stone-Geary preferences similar to Bick et al. (2015).

gument. The perspective offered by our model only calls for a slightly different interpretation of the data. The stable hours in the postwar U.S. may or may not be difficult to explain from the perspective of our theory, but regardless of this, the falling hours in the main European economies may well be because of higher taxes (in relative terms). It is beyond the scope of the present paper to investigate this issue in detail, but let us briefly look at the data and make some remarks.

So Figure 12 shows postwar hours in the U.S., in Germany, and in France. Clearly, hours fall at a fast rate in the European economies, indeed at a much faster rate than in the broader cross-section of countries we looked at in the data section of the paper. Thus, the cross-country average is a rate of decline in hours that lies in between that of the U.S. and those in Germany and France. But to the

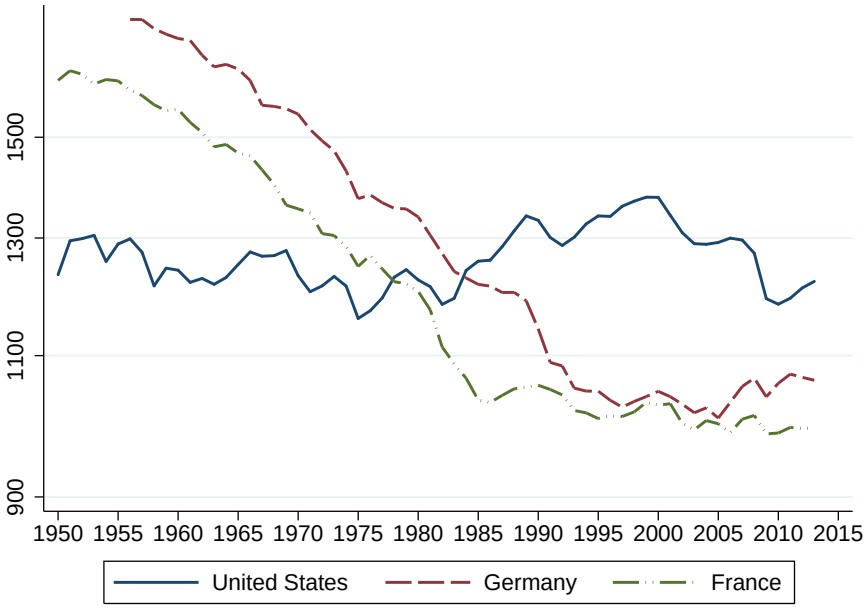

Figure 12: **Hours worked in the U.S., Germany, and France, 1950–2013**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006).

extent one wanted to explain the U.S. experience from the perspective of our model, what could be said? We believe that there are at least three reasons why the U.S.

experience may have represented a temporary departure from a long-run trend of falling hours. One is the tax cuts of the Reagan years, that appear to have been permanent: the Prescott-Rogerson argument. Another is the fact that median wages have not grown much at all in the U.S., and per-capita hours are un-weighted by productivity/wages, so if the vast majority of the population does not experience wage growth, constant hours is of course consistent with our theory. Thus, the well-documented increase in wage inequality from the late 1970s—which is also when hours were rising within the postwar period—is factor two. Factor three is women's increased labor-force participation, which clearly is a transitional phenomenon, and the point here is that women may formerly have been constrained and not able to work (at appropriate wages) so that one cannot simply take a unitary perspective on the household and say that male hours should decline at a very high rate to compensate for women's higher hours. It would be interesting to try to account for these factors together, along with other drivers, but again this would be a project in itself and the comments here just mean to signal how our preference formulation would fit into such an investigation.

What about the developments of hours worked currently in the U.S., and what about their future path? Very recently, participation rates have fallen, and from the perspective of our theory, at least if the bulk of households have begun to experience higher wages, such a fall in the participation rates would be consistent with our theory and would also imply that one would not expect participation to bounce back up.[13] Similarly, for the future, if productivity keeps growing, our theory of course predicts that hours worked will keep falling.

[13]This is of course not to say that there are no cyclical reasons for the participation movements.

7 Conclusions

We have presented an extension to the standard preference framework used to account for the balanced-growth facts. The new preference class admit that hours worked fall at a constant rate when labor productivity grows at a constant rate, as we have also documented the data to show across history and space. The new preference class intuitively involves an income effect that exceeds the substitution effect.

We believe that our new preference class has potentially interesting implications in a range of contexts. As for growth theory and growth empirics, note that on our balanced path, the main macro aggregates (output, investment, consumption) grow at the rate $\gamma^{1-\nu} > 1$ (ignoring population growth), i.e., at a rate lower than productivity and in a way that is determined by the preference parameter ν. Notice also that from a development perspective, falling hours worked is not a sign of economic malfunctioning but rather the opposite: it is the natural outcome given preferences and productivity growth, and it rather instead illustrates clearly how output is an incomplete measure of welfare (see Jones and Klenow, 2015): leisure grows. Interestingly, our theory says that growth theory probably should not abstract from labor supply (which is typically set to "1" in models); rather, it seems an important variable to model in conjunction with the growth process.

Does our preference class have something to say about business-cycle analysis? We cannot identify any immediate substantive implications, but it is clear that our model can be amended with shocks and transformed to a stationary one that can be analyzed just like in the RBC and NK literatures. The preference class consistent with hours falling at a constant, but low, rate is a bit different than the standard one. From the perspective of a particular case—the MaCurdy constant-Frisch elasticity functional form—one can admit an arbitrarily low elasticity of intertemporal substitution of consumption, though only if the Frisch elasticity is then also very

low.

Other areas where the new preference class may be interesting to entertain include asset pricing and public finance. For asset pricing—as we showed in the paper—it is possible to have attitudes toward risk behave qualitatively differently, and possibly more in line with data, than using standard balanced-growth preferences. These same features would potentially also help explain portfolio-choice patterns across wealth groups. For public finance, the sustainability of government programs, such as social security, and debt service in the future depend greatly on how hours worked will develop (along, of course, with the development of productivity).

We build the explanation into preferences because we do not think institutional factors, for example, can be taken as exogenous over such long horizons and thus must be viewed as responding to preferences. Moreover, the facts—an approximately constant rate of decline in hours worked—are too stark not to propose a "deep", and time- and space-independent, explanation. Of course, we are open to alternatives but our approach seems a reasonable place to start. What are, then, alternative theories that could explain why hours fall? Could an alternative theory explain the past without contradicting the constant-hours presumption of the standard macroeconomic model? Other mechanisms for income effects dominating substitution effects are possible, such as the Stone-Geary formulation proposed in Bick et al. (2015), following Atkeson and Ogaki's estimates (1996). Whether the transition dynamics in such a model are slow enough to generate the long-term, constant percent decline in hours observed in the data we look at here is an open question.

In sum, our present analysis should be viewed as one way to look at the long-run data, and it should carefully be compared to others, especially since their implications for the future differ markedly. For example, the Stone-Geary formulation implies that the future will see flat hours, independently of how future productivity

evolves, whereas the implications of the preferences we propose here suggest a tight hours-productivity link. Our theory also has a number of other implications (over the business cycle, for asset-pricing, growth, and so on) and suggests avenues for follow-up research. We hope to address some of these applications in future work.

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A.1 Appendix A: Proof

We now present the proof of Theorem 1.

Proof. The proof starts by stating and proving two lemmata, one characterizing the marginal rate of substitution (MRS) function between c and h and one characterizing the curvature with respect to consumption: the relative risk aversion (RRA) function. The proof then uses these lemmata to derive the final characterization. Because the proof will involve a large number of auxiliary functions that are either functions of $hc^{\frac{\nu}{1-\nu}}$ or of h, we economize somewhat on notation by sometimes denoting $hc^{\frac{\nu}{1-\nu}}$ by x and by systematically letting f_i be a function of x whereas m_j is a function of h (where i and j are indices for the different functions we will define). A sequence of constants will also appear; they are denoted A_k, accordingly, from $k = 1$ and on.

We now proof the first lemma.

Proof. Because λ is arbitrary, we can set it in (8) so that $c\lambda^{1-\nu} = 1$. This delivers

$$-\frac{u_2(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} = wc^{-\frac{1}{1-\nu}}.$$

Evaluating (8) at $\lambda = 1$ we obtain $-\frac{u_2(c,h)}{u_1(c,h)} = w$. Inserting this expression, we thus obtain

$$\frac{u_2(c,h)}{u_1(c,h)} = c^{\frac{1}{1-\nu}} \frac{u_2(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})}. \tag{A.1}$$

Now identifying $v_1(x)$ as $\frac{u_2(1,x)}{u_1(1,x)}$, where $x = hc^{\frac{\nu}{1-\nu}}$, gives the result in Lemma 1. ■

It follows from Lemma 1 and u being twice continuously differentiable that v_1 is continuously differentiable.

Proof. The second first-order condition, (9), holds for all λ so it can be differentiated with respect to λ and then evaluated at $\lambda = 1$ and divide by (9) again to yield

$$(1-\nu)c\gamma^{1-\nu}\frac{u_{11}\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)}{u_1\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)} - \nu h\gamma^{-\nu}\frac{u_{12}\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)}{u_1\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)} = (1-\nu)c\frac{u_{11}\left(c,h\right)}{u_1\left(c,h\right)} - \nu h\frac{u_{12}\left(c,h\right)}{u_1\left(c,h\right)}. \tag{A.2}$$

This equation has to hold for all γ (and consequently one must adjust R, but R does not appear in the equation). Moreover, it has to hold for all c and h; it has to hold for all h because Assumption 1 allows any w and hence any h (given an arbitrary c). Given this, by setting γ so that $c\gamma^{1-\nu} = 1$ we can state (A.2) as

$$(1-\nu)\frac{u_{11}(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} - \nu h c^{\frac{\nu}{1-\nu}} \frac{u_{12}(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} = (1-\nu)c\frac{u_{11}(c,h)}{u_1(c,h)} - \nu h \frac{u_{12}(c,h)}{u_1(c,h)},$$

which holds for all c and h. We conclude that the right-hand side of equation (A.2) only depends on $hc^{\frac{\nu}{1-\nu}}$, i.e., we can write

$$(1-\nu)c\frac{u_{11}(c,h)}{u_1(c,h)} - \nu h \frac{u_{12}(c,h)}{u_1(c,h)} = f_1(hc^{\frac{\nu}{1-\nu}}), \tag{A.3}$$

where f_1 is then defined by the expression on the left-hand side of equation (A.2). Differentiating (10) with respect to c gives

$$\frac{u_{12}(c,h)u_1(c,h) - u_{11}(c,h)u_2(c,h)}{u_1(c,h)^2} = \frac{c^{\frac{\nu}{1-\nu}} v_1(x)}{1-\nu} + \frac{\nu c^{\frac{1}{1-\nu}} v_1'(x) h c^{\frac{\nu}{1-\nu}-1}}{1-\nu} \equiv c^{\frac{\nu}{1-\nu}} f_2(x),$$

where we used the notation $x = hc^{\frac{\nu}{1-\nu}}$ and the last equality simply defines a new function f_2. Then, again using the characterization of the MRS function to replace $\frac{u_2(c,h)}{u_1(c,h)} = c^{\frac{1}{1-\nu}} v_1(hc^{\frac{\nu}{1-\nu}})$, we obtain

$$\frac{u_{12}(c,h)}{u_1(c,h)} - \frac{u_{11}(c,h)}{u_1(c,h)} c^{\frac{1}{1-\nu}} v_1(x) = c^{\frac{\nu}{1-\nu}} f_2(x),$$

and hence

$$\frac{h u_{12}(c,h)}{u_1(c,h)} = \frac{u_{11}(c,h)}{u_1(c,h)} h c^{\frac{1}{1-\nu}} v_1(x) + h c^{\frac{\nu}{1-\nu}} f_2(x) = \frac{c u_{11}(c,h)}{u_1(c,h)} x v_1(x) + x f_2(x).$$

This expression can be combined with equation (A.3) to conclude that $\frac{-cu_{11}(c,h)}{u_1(c,h)}$ must be a function only of x; we call this function v_2.[14] ■

[14] The function $v_2(x)$ is thus defined by

$$-(1-\nu)v_2(x) + \nu\left(v_2(x)xv_1(x) - xf_2(x)\right) = f_1(x),$$

which straightforwardly offers a solution (that will depend on v_1, f_1, and f_2).

We will now combine the information in Lemmata 1 and 2 to complete our proof. We do this in two steps. First we analyze the case with $\nu \neq 0$ and then the case with $\nu = 0$. Note that the case with $\nu = 0$ is already discussed in King, Plosser and Rebelo (1988).

The strategy of the proof is very similar in the two cases. First, we integrate the RRA function in Lemma 2 with respect to c to obtain a functional form for u_1. As we integrate with respect to c, an unknown function of h appears. Then, by differentiating the obtained function for u_1 with respect to h we arrive at an expression that can be compared to a restriction on $\frac{u_{12}}{u_1}$ found in the proof of Lemma 2. This comparison gives us some additional restrictions on the unknown function of h. Thus, since the proof of Lemma 2 uses Lemma 1, we are in effect making sure that the functional form we arrive at is consistent with both our lemmata. Having arrived at a form for u_1, we again integrate to deliver a candidate for u. Due to the integration a new unknown function of h again appears, but we can again restrict this function by differentiating our candidate u with respect to h and comparing the result to Lemma 1. This, then, delivers our final functional form.

Case with $\nu \neq 0$: note that the characterization of the RRA function in Lemma 2 can be restated as

$$\frac{\partial \log u_1(c,h)}{\partial \log(c)} = -v_2 \left(\exp\left(\log(h) + \frac{\nu}{1-\nu} \log(c) \right) \right).$$

This equation can be integrated straightforwardly with respect to $\log(c)$ to arrive at

$$u_1(c,h) = f_3(hc^{\frac{\nu}{1-\nu}})m_1(h), \tag{A.4}$$

where f_3 is a new function of x and m_1 is an arbitrary function of h.[15]

[15]The integration delivers an expression for $\log u_1(c,h)$ as a function of $\log x$ plus a function of h. The latter function can only be a function of h since c was integrated over. The function of $\log x$ can be rewritten as a function of x. Equation (A.4) is then obtained after raising e to the left- and right-hand sides of this equation and f_3 and m_1 are defined accordingly.

Now observe that it follows from the proof of Lemma 2 that also $\frac{u_{12}(c,h)h}{u_1(c,h)}$ can be written as a function of x alone: it equals $-v_2(x)xv_1(x) + xf_2(x)$. We use this fact to further restrict the function m_1. In particular, by taking derivatives in equation (A.4) with respect to h, multiplying by h, and dividing by u_1, we obtain an expression for $\frac{u_{12}(c,h)h}{u_1(c,h)}$ that can be written as

$$f_4(hc^{\frac{\nu}{1-\nu}}) + \frac{m_1'(h)h}{m_1(h)},$$

where f_4 is defined by $f_4(x) \equiv f_3'(x)x/f_3(x)$. For the consistency of these two expressions for $\frac{u_{12}(c,h)h}{u_1(c,h)}$—the one just stated, and the arbitrary function of x given above $(-v_2(x)xv_1(x) + xf_2(x))$—it must be that $\frac{m_1'(h)h}{m_1(h)}$ is a constant.[16] Hence, $m_1(h) = A_1h^\kappa$ for some constants A_1 and κ, i.e., it is isoelastic. Using this fact in (A.4) gives

$$u_1(c,h) = f_3(hc^{\frac{\nu}{1-\nu}})A_1h^\kappa. \tag{A.5}$$

Since $\nu \neq 0$, the expression on the right-hand side can equivalently be written $f_5(h^{\frac{1-\nu}{\nu}}c)h^\kappa$, by defining $f_5(x) = A_1f_3(x^{\frac{1-\nu}{\nu}})$. Therefore, (A.5) can be easily integrated with respect to c to deliver

$$u(c,h) = f_6(hc^{\frac{\nu}{1-\nu}})h^{\kappa-\frac{1-\nu}{\nu}} + m_2(h), \tag{A.6}$$

where f_6 is the new function that results from the integration of f_5 over c and m_2 is an arbitrary function of h (as the integration was over c). With the aim of further restricting m_2, we can express u_2 as

$$u_2(c,h) = u_1(c,h)c^{\frac{1}{1-\nu}}v_1(x) = f_3(x)A_1h^\kappa c^{\frac{1}{1-\nu}}v_1(x) = f_7(hc^{\frac{\nu}{1-\nu}})h^{\kappa-\frac{1}{\nu}}, \tag{A.7}$$

where we have used the characterization of the MRS function in Lemma 1, (A.5), and finally the definition $f_7(x) \equiv f_3(x)A_1x^{\frac{1}{\nu}}v_1(x)$. Thus, we can now check consistency

[16] If $\frac{m_1'(h)h}{m_1(h)}$ would depend on h, consistency could not be fulfilled for any given combination of c and h.

by taking the derivative of u with respect to h in (A.6) and comparing with (A.7). The derivative becomes

$$\left(\kappa - \frac{1-\nu}{\nu}\right) f_6(x)h^{\kappa-\frac{1}{\nu}} + c^{\frac{\nu}{1-\nu}} f_6'(x)h^{\kappa-\frac{1-\nu}{\nu}} + m_2'(h) \equiv f_8(x)h^{\kappa-\frac{1}{\nu}} + m_2'(h),$$

where the equality comes from collecting terms and defining a new function f_8 accordingly. For consistency, thus, this expression has to equal $f_7(x)h^{\kappa-\frac{1}{\nu}}$ for all x and h. This is possible if and only if $m_2'(h) = A_2 h^{\kappa-\frac{1}{\nu}}$, where A_2 is a constant. Concentrating first on the case where $\kappa - \frac{1}{\nu} \neq -1$, we obtain $m_2(h) = \left(1+\kappa-\frac{1}{\nu}\right)^{-1} A_2 h^{1+\kappa-\frac{1}{\nu}} + A_3 \equiv A_4 h^{1+\kappa-\frac{1}{\nu}} + A_3$. The constant A_3 can be set arbitrarily as it does not affect choice. The second term in (A.6) can thus be merged together with the first term using factorization and we can write $u(c,h)$ as $f_9(x)h^{1+\kappa-\frac{1}{\nu}} + A_3$, with $f_9(x) \equiv f_6(x) + A_4$. Now note that $h^{1+\kappa-\frac{1}{\nu}} = x^{1+\kappa-\frac{1}{\nu}} c^{-\frac{\nu}{1-\nu}\left(1+\kappa-\frac{1}{\nu}\right)}$, so that $u(c,h)$ can be written as $f_9(x)x^{1+\kappa-\frac{1}{\nu}} c^{-\frac{\nu}{1-\nu}\left(1+\kappa-\frac{1}{\nu}\right)} + A_3$. Now define $v(x) \equiv \left((1-\sigma)f_9(x)x^{1+\kappa-\frac{1}{\nu}}\right)^{\frac{1}{1-\sigma}}$ and $\sigma \equiv \kappa\frac{\nu}{1-\nu}$ and we conclude that we can write $u(c,h) = \frac{(cv(x))^{1-\sigma}-1}{1-\sigma}$ (where A_3 has been set to $-1/(1-\sigma)$).

In the special case where $1+\kappa = 1/\nu$, we obtain from equation (A.6) that $u(c,h) = f_6(hc^{\frac{\nu}{1-\nu}}) + m_2(h)$, but we also see from the arguments above that $m_2(h)$ has to equal $A_2 \log h + A_5$, where A_5 is again an arbitrary constant. Since (given $\nu \neq 0$) we can write $\log h = \log x - \frac{\nu}{1-\nu}\log c$, our candidate u can be rewritten as $u(c,h) = f_6(x) - A_2\frac{\nu}{1-\nu}\log(c) + A_2\log(x) + A_5$. The constant A_5 can be set to zero and we can write $u(c,h) = -A_2\frac{\nu}{1-\nu}\left[\log(c) - \frac{1-\nu}{A_2\nu}f_6(x) - \frac{1-\nu}{\nu}\log(x)\right]$. The factorized constant can be normalized to -1 (as it does not affect choice), and we can then define $\log v(x) \equiv f_6(x) + \frac{1-\nu}{\nu}\log x$, an arbitrary function; this concludes the case $1+\kappa = 1/\nu$. Hence we obtain the utility function

$$u(c,h) = \begin{cases} \frac{\left(c\cdot v(hc^{\frac{\nu}{1-\nu}})\right)^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log(c) + \log v(hc^{\frac{\nu}{1-\nu}}), & \text{if } \sigma = 1. \end{cases}$$

Utility is strictly decreasing in h and strictly increasing in c as long as $v'(x) < 0$ and $v(x) > -\frac{\nu}{1-\nu}v'(x)x$.

<u>Case with $\nu = 0$:</u> in this case we can rewrite the RRA function in Lemma 2 as

$$\frac{\partial \log u_1(c,h)}{\partial \log c} = -v_2(h). \tag{A.8}$$

We can integrate this equation with respect to $\log c$ to obtain

$$\log u_1(c,h) = -v_2(h)\log c + m_3(h), \tag{A.9}$$

where m_3 is an arbitrary function, given that we integrated over c. Differentiating with respect to h then gives

$$\frac{u_{12}(c,h)}{u_1(c,h)} = -v_2'(h)\log c + m_3'(h). \tag{A.10}$$

From the proof of Lemma 2 we know that $\frac{u_{12}(c,h)}{u_1(c,h)}$ must be possible to write as a function of h alone (recall that $\nu = 0$). From this we conclude that we must have $v_2'(h) = 0$, i.e., the only version of equation (A.9) that is possible is $\log u_1(c,h) = -\sigma \log c + m_3(h)$, where σ is a constant. Using this fact and raising e to both sides of (A.9) then delivers

$$u_1(c,h) = c^{-\sigma} m_4(h), \tag{A.11}$$

where $m_4(h) = \exp(m_3(h))$. Integrating (A.11) with respect to c we can write

$$u(c,h) = \begin{cases} \frac{(cv(h))^{1-\sigma}-1}{1-\sigma} + m_5(h) & \text{if } \sigma \neq 1 \\ m_4(h)\log(c) + \log v(h) & \text{if } \sigma = 1; \end{cases} \tag{A.12}$$

here, in the first equation $-1/(1-\sigma) + m_5$ is another function (of h) that appears because of the integration over c and $v(h)$ is defined from $\frac{v(h)^{1-\sigma}-1}{1-\sigma} = m_4(h)$, whereas in the second equation $\log v$ is the function that appears due to the integration.

We will now, along the lines of the case where $\nu \neq 0$, show that m_4 and m_5 will have to have very specific forms. We look at each in turn. So in the case with $\sigma \neq 1$, combine (A.11) with Lemma 1 to write

$$u_2(c,h) = c^{1-\sigma} v_1(h) m_3(h). \tag{A.13}$$

This can be contrasted with the result of differentiating (A.12) with respect to h, an operation that yields

$$u_2(c,h) = c^{1-\sigma} v(h)^{-\sigma} v'(h) + m_5'(h).$$

Since these last two equations both have to hold for all c and h, it must be that $m_5'(h) = 0$, i.e., that $m_5(h)$ is a constant (which can be abstracted from).

Turning to the case where $\sigma = 1$, along the same lines we again derive two expressions for u_2 and check consistency. Combining (A.11) with Lemma 1 one obtains that u_2 cannot depend on c. Differentiating the second line of (A.12) with respect to h, however, delivers a function of c unless $m_4(h)$ is a constant; as it does not affect choice, we set this constant to 1.

This is our final characterization and we have now reproduced the statement in our main theorem. In summary, in the $\sigma \neq 1$ case we obtain $u(c,h) = \frac{(cv(h))^{1-\sigma}-1}{1-\sigma}$ and in the $\sigma = 1$ case we obtain $\log(c) + \log v(h)$. This completes the proof for the case $\nu = 0$. To ensure that $u(c,h)$ is strictly decreasing in h we need $v'(x) < 0$. ■

B.1 Appendix B: A model with consumer heterogeneity

B.1.1 The Aiyagari model without growth

The consumer's problem: for all $(\omega, \epsilon, \beta)$,

$$V(\omega, \epsilon, \beta) = \max_{k', h} u(\omega + h\epsilon w - k', h) + \beta E[V(k'(1 - \delta + r), \epsilon', \beta')|\epsilon, \beta]$$

s.t. $k' \geq \underline{k}, h \in [0, \infty)$. This leads to decision rules $f^k(\omega, \epsilon, \beta)$ and $f^h(\omega, \epsilon, \beta)$.

Labor income is $\epsilon \in \{\epsilon_1, \epsilon_2, \ldots, \epsilon_I\}$ and $\beta \in \{\beta_1, \beta_2, \ldots, \beta_J\}$, with constant and exogenous first-order Markov transition probabilities $\pi(\epsilon', \beta'|\epsilon, \beta)$.

We assume that the economy produces with a neoclassical production function $F(\bar{k}, \bar{h})$ and the production factors ear their marginal products. **Stationary equilibrium:** prices r and w, a value function V, decision rules f^k and f^h, and a stationary distribution Γ such that

1. $f^k(\omega, \epsilon, \beta)$ and $f^h(\omega, \epsilon, \beta)$ attain the maximum in the consumer's problem for all $(\omega, \epsilon, \beta)$.

2. $r = F_1(\bar{k}, \bar{h})$ and $w = F_2(\bar{k}, \bar{h})$, where $\bar{k} \equiv (\sum_{\epsilon,\beta} \int_\omega \omega\Gamma(d\omega, \epsilon, \beta))/(1 - \delta + r)$ and $\bar{h} \equiv \sum_{\epsilon,\beta} \int_\omega \epsilon f^h(\omega, \epsilon, \beta)\Gamma(d\omega, \epsilon, \beta)$.

3. $\Gamma(B, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\omega: f^k(\omega,\hat{\epsilon}\hat{\beta}) \in B} \Gamma(d\omega, \hat{\epsilon}, \hat{\beta})$ for all Borel sets B and for all (ϵ, β).

B.1.2 The Aiyagari model with growth

The consumer's problem: for all $(\omega, \epsilon, \beta)$,

$$V_t(\omega, \epsilon, \beta) = \max_{k', h} u(\omega + h\epsilon w_t - k', h) + \beta E[V_{t+1}(k'(1 - \delta + r), \epsilon', \beta')|\epsilon, \beta]$$

s.t. $k' \geq \underline{k}g^{t+1}, h \in [0, \infty)$. Notice, here, that the borrowing constraint changes over time (unless $\underline{k} = 0$) and gets less and less stringent with $\underline{k} < 0$. This leads to decision rules $f_t^k(\omega, \epsilon, \beta)$ and $f_t^h(\omega, \epsilon, \beta)$.

Labor income and discount factors are as before. Now, however, that $w_t = \gamma^t w$ for all t.

A balanced-growth equilibrium: growth rates g and g_h, prices r and w_t, a value function V_t, decision rules f_t^k and f_t^h, and distributions Γ_t such that, for all t,

1. $g = \gamma g_h$.

2. $f_t^k(\omega, \epsilon, \beta)$ and $f_t^h(\omega, \epsilon, \beta)$ attain the maximum in the consumer's problem for all $(\omega, \epsilon, \beta)$.

3. $r = F_1(\bar{k}_t, \gamma^t \bar{h}_t)$ and $w_t = \gamma^t F_2(\bar{k}_t, \gamma^t \bar{h}_t)$, where $\bar{k}_t \equiv (\sum_{\epsilon,\beta} \int_\omega \omega \Gamma_t(d\omega, \epsilon, \beta))/(1-\delta + r)$ and $\bar{h}_t \equiv \sum_{\epsilon,\beta} \int_\omega \epsilon f_t^h(\omega, \epsilon, \beta) \Gamma_t(d\omega, \epsilon, \beta)$.

4. $\Gamma_{t+1}(B, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\omega: f_t^k(\omega,\hat{\epsilon},\hat{\beta}) \in B} \Gamma_t(d\omega, \hat{\epsilon}, \hat{\beta})$ for all Borel sets B and for all (ϵ, β).

5. $f_t^k(\omega g^t, \epsilon, \beta) = g^t f_0^k(\omega, \epsilon, \beta)$, $f_t^h(\omega g^t, \epsilon, \beta) = g_h^t f_0^h(\omega, \epsilon, \beta)$, and $\Gamma_t(Bg^t, \epsilon, \beta) = \Gamma_0(B, \epsilon, \beta)$ for all ω, B, and (ϵ, β).

Note that due to growth, the distribution ω will not be stationary. However, as we will show below, once ω is detrended by the appropriate growth rate we obtain a stationary distribution.

B.1.3 Transforming the Aiyagari model with growth

Using the last condition of the balanced-growth equilibrium, note that in the third condition we can write $\bar{k}_t = (\sum_{\epsilon,\beta} \int_\omega \omega \Gamma_0(\frac{d\omega}{g^t}, \epsilon, \beta))/(1 - \delta + r)$, which is equivalent to $\tilde{\bar{k}}_t \equiv \frac{\bar{k}_t}{g^t} = (\sum_{\epsilon,\beta} \int_{\tilde{\omega}} \tilde{\omega} \Gamma_0(d\tilde{\omega}, \epsilon, \beta))/(1 - \delta + r)$, where we have defined $\tilde{\omega} = \omega/g^t$. Notice also that $\tilde{\bar{k}}_t = \tilde{\bar{k}}$, i.e., a constant, in a balanced-growth equilibrium.

Similarly, we obtain $\bar{h}_t = \sum_{\epsilon,\beta} \int_\omega \epsilon g_h^t f_0^h(\frac{\omega}{g^t}, \epsilon, \beta)\Gamma_0(\frac{d\omega}{g^t}, \epsilon, \beta)$, implying that $\tilde{\bar{h}}_t \equiv \frac{\bar{h}_t}{g_h^t} = \sum_{\epsilon,\beta} \int_{\tilde{\omega}} \epsilon f_0^h(\tilde{\omega}, \epsilon, \beta)\Gamma_0(d\tilde{\omega}, \epsilon, \beta)$, which also is constant under balanced growth: $\tilde{\bar{h}}_t = \tilde{\bar{h}}$.

Given $g = \gamma g_h$ and that F_1 and F_2 are both homogeneous of degree 0, we now see that the two firm first-order conditions can be expressed as

$$r = F_1(\tilde{\bar{k}}, \tilde{\bar{h}}) \quad \text{and} \quad w_0 = F_2(\tilde{\bar{k}}, \tilde{\bar{h}}). \tag{B.1}$$

Turning to the fourth equilibrium condition, using the (very) last condition stating that the distribution is (in an appropriate sense) constant on the balanced growth path, we obtain

$$\Gamma_0(B/g^{t+1}, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\tilde{\omega}: f_0^k(\tilde{\omega},\hat{\epsilon},\hat{\beta})g^t \in B} \Gamma_0(d\tilde{\omega}, \hat{\epsilon}, \hat{\beta}),$$

where we used the definition of $\tilde{\omega}$. Defining $\tilde{B} = B/g^t$ for any Borel set B, we obtain

$$\Gamma_0(\tilde{B}/g, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\tilde{\omega}: f_0^k(\tilde{\omega},\hat{\epsilon},\hat{\beta}) \in \tilde{B}} \Gamma_0(d\tilde{\omega}, \hat{\epsilon}, \hat{\beta}). \tag{B.2}$$

Looking at consumer optimization under balanced growth, finally, we obtain (after using the same kinds of definitions as above),

$$V_t(\tilde{\omega}g^t, \epsilon, \beta) = \max_{\tilde{k}', \tilde{h}} u(\tilde{\omega}g^t + \tilde{h}g_h^t \epsilon w_0 \gamma^t - \tilde{k}' g^{t+1}, \tilde{h}g_h^t) + \beta E[V_{t+1}(\tilde{k}' g^{t+1}(1-\delta+r), \epsilon', \beta')|\epsilon, \beta]$$

s.t. $\tilde{k}' g^{t+1} \geq \underline{k} g^{t+1}, \tilde{h}g_h^t \in [0, \infty)$.

Now consider our instantaneous utility function for u and let $g_h = \gamma^{-\nu}$ and $g = \gamma^{1-\nu}$. Then $g^{t(1-\sigma)}$ can be factorized out from u. Dividing both sides of the equation by this quantity and defining $V_t(\tilde{\omega}g^t, \epsilon, \beta) \equiv g^{t(1-\sigma)}\tilde{V}(\tilde{\omega}, \epsilon, \beta)$, we can write

$$\tilde{V}(\tilde{\omega}, \epsilon, \beta) = \max_{\tilde{k}', \tilde{h}} u(\tilde{\omega} + \tilde{h}\epsilon w_0 - \tilde{k}' g, \tilde{h}) + \beta g^{1-\sigma} E[\tilde{V}(\tilde{k}'(1-\delta+r), \epsilon', \beta')|\epsilon, \beta] \tag{B.3}$$

s.t. $\tilde{k}' \geq \underline{k}, \tilde{h} \in [0, \infty)$, with associated policy functions $\tilde{f}_t^k(\tilde{\omega}, \epsilon, \beta)$ and $\tilde{f}_t^h(\tilde{\omega}, \epsilon, \beta)$.

Now r, w_0, $\tilde{V}$, $\tilde{f}^k$, $\tilde{f}^h$, and Γ_0, determined by equations (B.3), (B.1), and (B.2), define a stationary equilibrium. Three items differ compared to the formulation above for the stationary equilibrium without growth: the discount factors in the consumer's problem are all multiplied by $g^{1-\sigma}$, an additional gross "cost" of saving, g, appears, and g also appears in the argument on the left-hand side of the equation determining the stationary distribution.

C.1 Appendix C: Additional figures

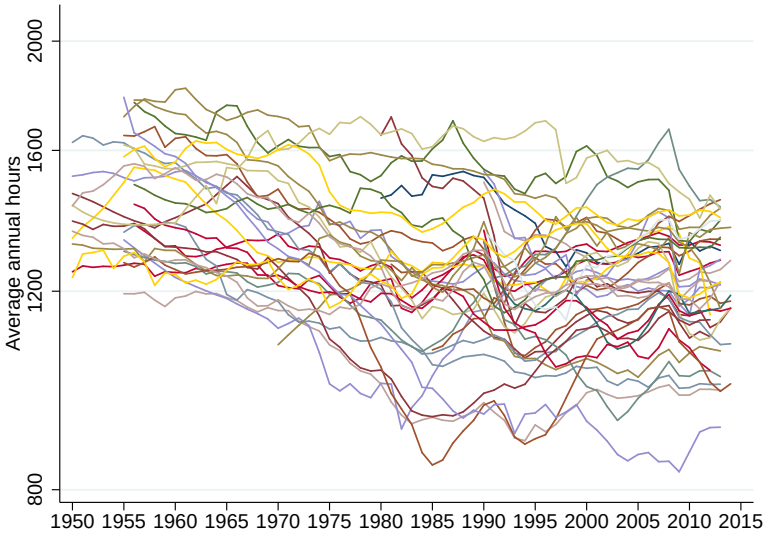


Figure C.1: **Average annual hours per capita aged 15–64, 1950–2015**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). The sample includes 37 countries. Regressing the logarithm of hours worked on time and country fixed effects gives slope coefficient of -0.00336. The R^2 of the regression is 0.64.

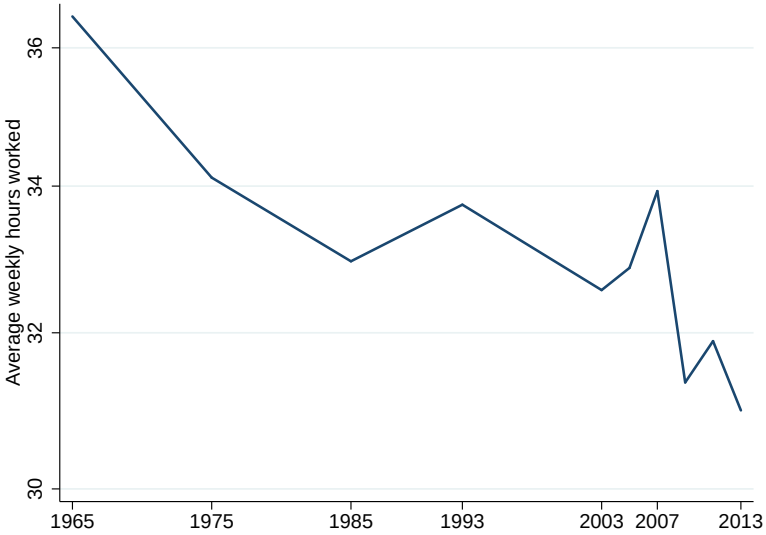


Figure C.2: **U.S. time used survey: Weekly hours worked**

Notes: Source: ATUS, following the methodology in Aguiar and Hurst (2007). The sample contains all non-retired, non-student individuals at age 21–65. For the years 1965–2003 the series is comparable to Aguiar and Hurst (2007) Table II and is updated till 2013 using the same methodology. Regressing the logarithm of hours worked on time gives slope coefficient of -0.0024.

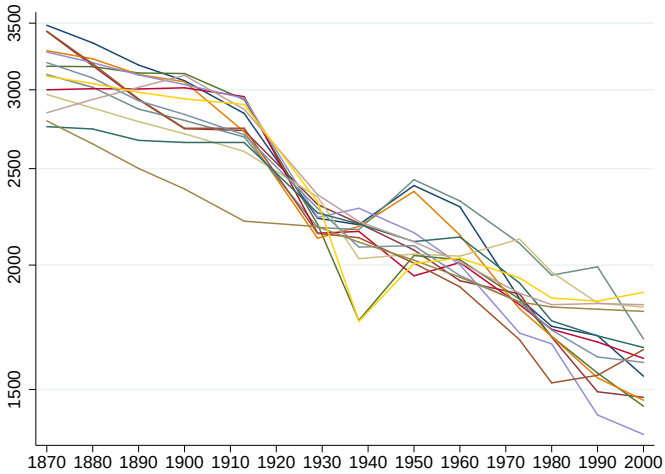


Figure C.3: **Hours worked per worker**

Notes: The figure shows data for the following countries: Belgium, Denmark, France, Germany, Ireland, Italy, Netherlands, Spain, Sweden, Switzerland, U.K., Australia, Canada, and U.S. The scale is logarithmic which suggests that hours fall at roughly 0.57 percent per year. Source: Huberman and Minns (2007).

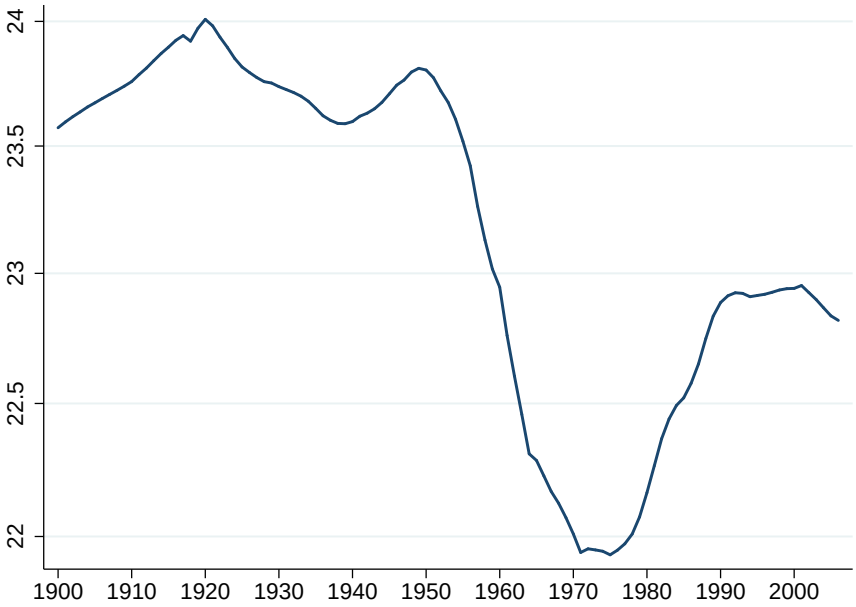
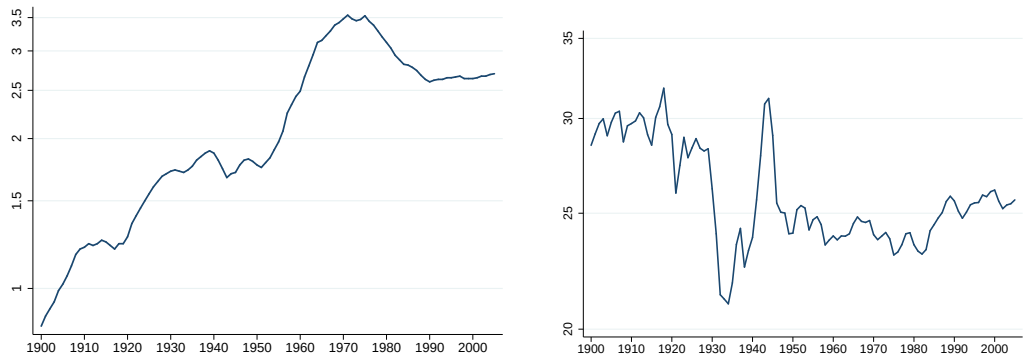


Figure C.4: **Effect of demographics on hours worked, U.S. 1900–2005**

Notes: The figure shows the implied average weekly hours worked per person aged 14+ over time by the variable demographical composition of the society. The scale is logarithmic. Hours worked of each age bracket are held constant at their value in 2005 and only the demographical composition is changing over time. The considered age brackets are 14–17; 18–24; 25–54; 55–64 and 65+. The figure looks very similar for other baselin years than 2005. Source: Ramey and Francis (2009) and U.S. Census.



(a) Weekly hours of school (b) Hours of work plus school

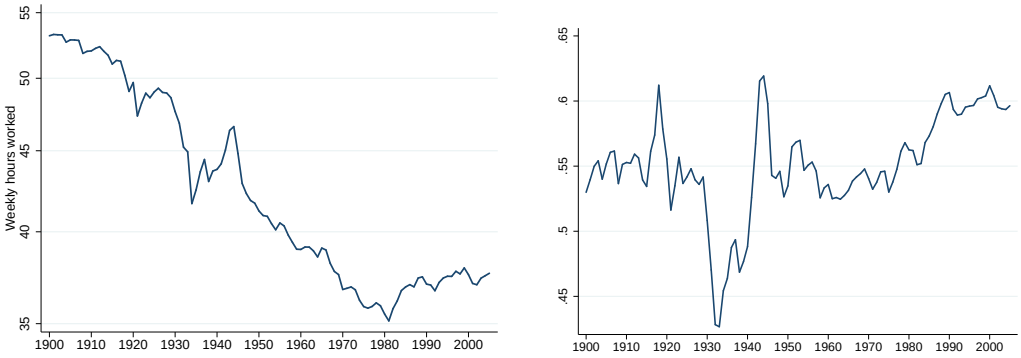
Figure C.5: Hours of schooling

Notes: The figure on the left shows average weekly hours of schooling (time in class and homework) per population aged 14+. The figure on the right shows average hours spent for work plus schooling per population aged 14+. The scale is logarithmic in both figures. Regressing the logarithm of hours worked plus schooling on time gives slope coefficient of -0.0018. Source: Ramey and Francis (2009).

(a) Hours per worker (b) Participation rate

Figure C.6: Hours per worker and participation rate in the postwar U.S.

Notes: The scale is logarithmic in the figure on hours worked per worker. Regressing the logarithm of hours worked per worker on time gives slope coefficient of -0.002. Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64.



(a) Hours per worker (b) Participation rate

Figure C.7: Hours per worker and participation rate in the U.S.

Notes: The scale is logarithmic in the figure on hours worked per worker. Regressing the logarithm of hours worked per worker on time gives slope coefficient of -0.00418. Source: Ramey and Francis (2009).

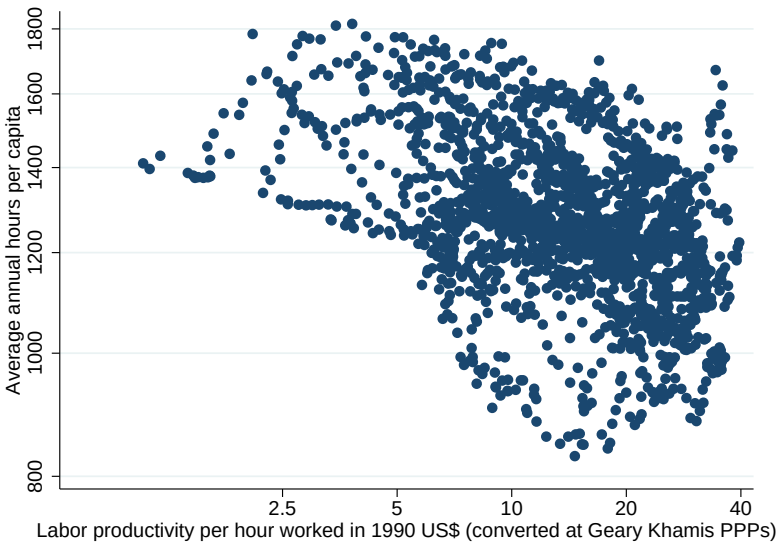


Figure C.8: **Hours worked vs. labor productivity**

Source: Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64. Regression the logarithm of hours worked on the logarithm of labor productivity and a country fixed effect gives a slope coefficient of -0.13 and an R^2 of 0.69.

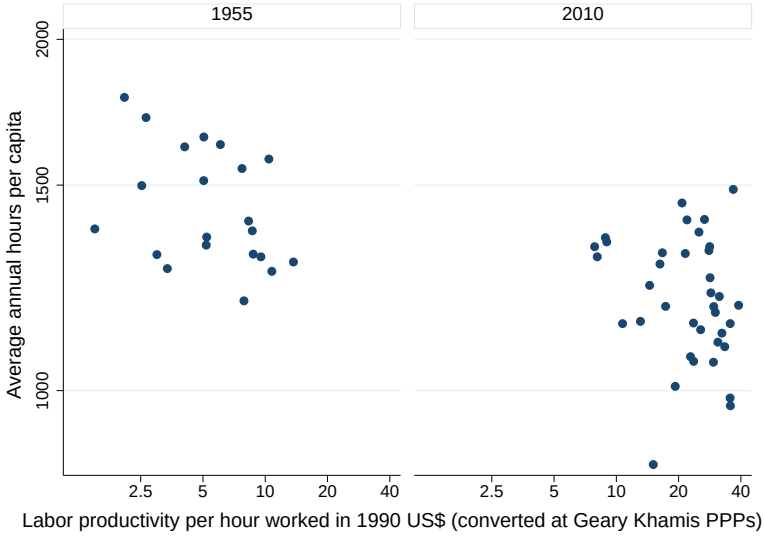
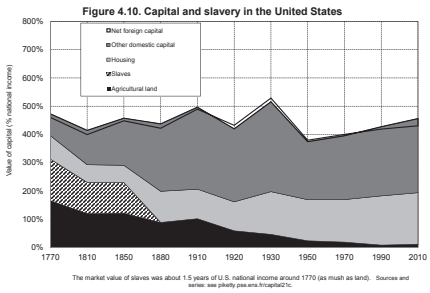
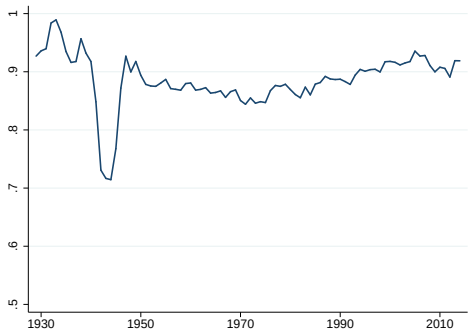


Figure C.9: **Hours worked vs. labor productivity**

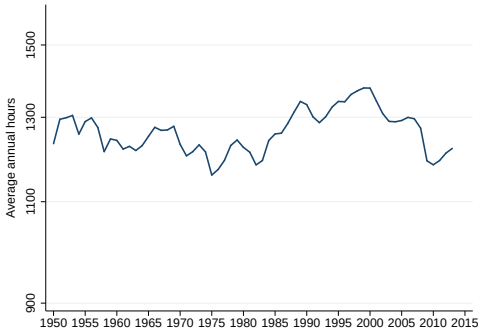
Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64. The figure shows the scatter plot between labor productivity and hours worked for the year 1955 and 2010.



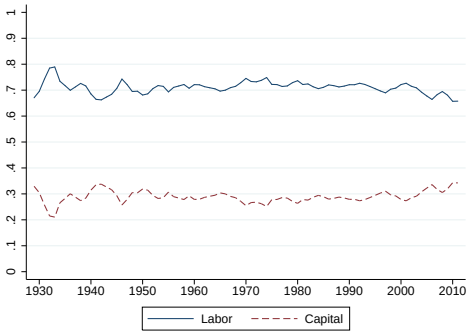
(a) Capital-output ratio



(b) Consumption-income ratio



(c) Hours worked



(d) Factor income shares

Figure C.10: Additional balanced-growth facts

Source: BEA; Piketty and Saez (2006), and Piketty (2014).