Does Incomplete Spanning in International Financial Markets Help to Explain Exchange Rates?

Hanno Lustig
Stanford GSB and NBER

Adrien Verdelhan
MIT Sloan and NBER

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Abstract

Exchange rates are puzzlingly smooth and only weakly correlated with macro-economic fundamentals compared to the predictions of exchange rate models with complete spanning. This paper derives an upper bound on the effects of incomplete spanning in international financial markets. We introduce stochastic wedges between log exchange rate changes and the difference in the countries' log pricing kernels without violating the foreign investors' Euler equations for the domestic risk-free assets. The wedges reconcile the volatility of no-arbitrage exchange rates with the data, provided that the volatility of the wedges is approximately as large as the maximum Sharpe ratio, but these same wedges cannot deliver exchange rates that simultaneously match currency risk premia and the exchange rates' correlation with fundamentals in the data.

Keywords: Exchange rates, Incomplete Markets, Currency Risk Premia.

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*First Version: April 2015. Lustig: Stanford Graduate School of Business, 355 Knight Way, Stanford, Ca 94305 (hlustig@stanford.edu). Verdelhan: MIT Sloan School of Management, 100 Main Street, E62-621, Cambridge, MA 02139 (adrienv@mit.edu). We welcome comments. We are grateful to Jack Favilukis, Bob Hodrick, and Robert Richmond for detailed comments on an early version, and we would like to thank Bob Hodrick for conversations that led to this paper. We are grateful to Bernard Dumas, Francisco Gomes, Leonid Kogan, Sylvain Leduc, Anna Pavlova, Anders Trolle, Raman Uppal and seminar participants at different institutions for their comments.

Over the last thirty years, starting with the seminal work of Lucas (1982), an increasingly common approach to studying exchange rates has been to assume that there is complete spanning in financial markets. Given complete spanning, the percentage appreciation of the foreign currency in units of the domestic numeraire equals the difference between the log of the foreign and domestic stochastic discount factors: $\Delta s = m^* - m$. If, for example, goods markets are not frictionless or domestic and foreign agents consume different goods, then real exchange rates will vary in equilibrium even if financial markets are themselves frictionless. To many in international finance, however, assuming complete spanning is impalatable. Everyone agrees that, in reality, markets certainly do not span all possible states of the world. The real question is: can departures from complete spanning help us understand the puzzling features of exchange rates?

Our paper answers this question by focusing on three key exchange rate puzzles. (i) The volatility puzzle of Brandt, Cochrane, and Santa-Clara (2006) refers to the low volatility of the exchange rate in the data. Hansen and Jagannathan (1991) show that stochastic discount factors have to be highly volatile in order to reproduce equity premia, a robust feature of equity markets across the globe. As Brandt, Cochrane, and Santa-Clara (2006) point out, stochastic discount factors must thus be almost perfectly correlated in order to match the comparatively low exchange rate volatility in the data. This very large correlation seems at odds with the low cross-country correlation of any macroeconomic variable. (ii) The correlation puzzle refers to the disconnect between exchange rates and macroeconomic variables in the data, also known as the Backus-Smith (1993) puzzle. When markets are complete and agents have constant relative risk aversion preferences, changes in exchange rates must be perfectly correlated with relative consumption growth rates in the domestic and foreign economies. As was first pointed out by Kollmann (1991) and Backus and Smith (1993), the low correlation in the data is therefore surprising. (iii) The forward premium puzzle, as documented by Tryon (1979), Hansen and Hodrick (1980) and Fama (1984), refers to the deviations from the uncovered interest rate parity condition: changes in exchange rates are not equal to interest rate differences and therefore large currency carry trade returns exist on average. The size of currency risk premia is a challenge for many models in international economics. These three puzzles highlight the limits of most international economics models, and they are the subject of a very large literature, with hundreds of contributions in the last thirty years.

In this paper, we show that departures from complete spanning in international securities trading cannot improve upon the complete market models' predictions on these puzzles simultaneously. Incomplete spanning can lower the volatility of exchange rates, as well as the correlation between consumption growth and exchange rate changes, or more generally the cyclicality of exchange rates, but it then also restores uncovered interest rate parity and reduces the carry trade risk premium well below its empirical value. There is a wealth of empirical evidence that investors act as if they face an incomplete menu of assets abroad, either because of explicit transactions and capital controls, or because of other frictions (Lewis, 1995). To derive an upper bound on the effects of incomplete spanning in international financial markets on exchange rates, we focus on the case in which investors can only invest abroad in the risk-free asset.

Our starting point is Backus, Foresi, and Telmer's (2001) insight that incomplete spanning introduces a wedge η between the change in exchange rates and the domestic and foreign stochastic discount factors: $\Delta s = \eta + m^* - m$. We take your favorite stochastic discount factor as our starting point and analyze the effect of the wedge η while enforcing the foreign investors' Euler equation only in the domestic risk-free bond market, but not in other domestic securities markets. This is equivalent to assuming that foreign investors have unlimited access to forward currency markets, but perhaps not to other asset markets. Similarly, domestic investors have access to foreign risk-free bonds. In this setup, we find that the volatility of exchange rates can be reduced enough provided that the volatility of the wedges matches the maximum Sharpe ratio in the economy. These wedges act as a reduction of the representative agent's risk aversion in currency markets, thus solving the excess smoothness puzzle but also shrinking currency risk premia and restoring uncovered interest rate parity.

We start by analyzing the log-normal case. Because of the Euler equation restrictions, the wedge η has to be pro-cyclical: it co-varies negatively with m^* and positively with m, thus always reducing the volatility of the exchange rates. Our quantitative assessment relies on two empirical assumptions: (i) the stochastic discount factors m and m^* are volatile, as implied by the Hansen and Jagannathan (1991) bounds; and (ii) the conditional correlation between the domestic and foreign log stochastic discount factors is at most 0.5, a large value considering the low cross-country correlation of any macroeconomic variable in the data. We show that introducing the wedge η between the exchange rate changes and the stochastic discount factors cannot resolve the three major puzzling features of exchange rates simultaneously. For wedges that are as volatile as the complete market stochastic discount factor, the incomplete market model matches the empirical exchange rate volatility and delivers sizable currency risk premia. But the cyclicality of exchange rates is then exacerbated, worsening the Backus-Smith (1993) puzzle. Lowering the correlation between exchange rates and the projection of stochastic discount factors on traded assets goes hand in hand with lowering the currency risk premium. As a result, incomplete market models that match a low cyclicality of exchange rates imply near zero currency risk premia. Considering the three puzzles together dramatically raises the bar for complete and incomplete market models.

These results are derived assuming that all moments of the wedge η are free parameters, the best case scenario for incomplete market models. Yet, in a large class of dynamic asset pricing models, higher moments of the wedge are related to its first moment by no arbitrage restrictions, further restraining the ability of incomplete market models to match the data. We show that incomplete market perturbations always lower currency market risk premia and Sharpe ratios in a large class of dynamic asset pricing models. A given percentage decrease in the volatility of exchange rates relative to the complete markets case implies the same percentage decrease in the currency markets' Sharpe ratio relative to the complete markets benchmark, while the uncovered interest rate parity slope regression coefficient is always pushed towards unity. In a version of the Consumption-CAPM model with heteroskedasticity in consumption growth, we show that the effect of incompleteness is equivalent to lowering the coefficient of risk aversion of the representative agent for the pricing of currency risk by the same percentage.

In this paper, we therefore adopt the perspective of an econometrician who commits to a model for the log domestic and foreign stochastic discount factors, m and m^*, taking allocations (e.g. aggregate consumption growth, the market return) as given. These stochastic discount factors may depend on agents' preferences and may, for example, vary with consumption, real money balances, and work hours. They may also depend on intermediaries' wealth, collateral or liquidity. They may thus reflect the impact of market incompleteness. But the economic variables are what they are in the data, independently from their theoretical interpretation. An econometrician would start from the time-series of these variables in the data and look for the best-fitting model that can reproduce the empirical exchange rate and risk-free rates, testing for example different preference parameters. We do not commit to any class of model sin particular and let the econometrician choose any possible model. Once the model is chosen, we define the projection of the stochastic discount factor in the space of traded assets as our initial m. Then, we investigate how much departures from complete spanning in international financial markets help to explain exchange rate puzzles, taking m and m^* as given. This procedure does not make any assumption about preferences or constraints. Our theoretical results rely instead on three key assumptions: 1) the existence of a stochastic discount factor in the space of traded assets in one country, 2) the existence of a domestic and a foreign risk-free rate in which the other country's investors can invest, and 3) the log normality of stochastic discount factors and wedges.

Our assumptions are reasonable but not trivial. If the law of one price holds and investors can form portfolios freely, then a unique stochastic discount factor exists in the space of traded assets (see Ross, 1978; Cochrane, 2005, for a textbook exposition). From the stochastic discount factor in one country, one can then construct a second one using the change in exchange rate. But there are cases when investors cannot form portfolios freely (e.g., in the presence of short-selling constraints) or when the law of one price fails, and thus the existence of a stochastic discount factor is not guaranteed. Likewise, risk-free assets may not exist. Assuming that a stochastic discount factor exists in the space of traded assets and that risk-free rates exist allows us to derive preference-free results. The log normality assumption allows us to focus on the first two moments of stochastic discount factors and wedges and address the volatility, cyclicality, and risk premium puzzles altogether. Our results, however, are not a simple rejection of lognormality. Relaxing this assumption, we derive restrictions on wedges in terms of their entropy and co-entropy with the pricing kernels. In this general case, increases in the volatility of the exchange rate changes go hand in hand with decreases in currency risk premia. While one cannot rule out the existence of a non-Gaussian model that would match the three exchange rate puzzles simultaneously thanks to incomplete spanning, we do not know of such a model. We show in a classic jump-based model that again the introduction of incomplete spanning cannot address the three puzzles simultaneously.

Our paper relates to a large literature that studies exchange rates in complete markets. In models that feature complete spanning, one can back out the implied changes in exchange rates from the stochastic discount factors at home and abroad. Equivalently, one can start from the domestic (or foreign) stochastic discount factor and the rate of change in exchange rates, and then derive the implicit foreign (domestic) stochastic discount factor. In one form or another, the complete markets assumption underpins many contributions to the literature. To name a few recent examples, Colacito and Croce (2011), Bansal and Shaliastovich (2012), and Farhi and Gabaix (2015) for example address the aforementioned puzzles in models based on long-run risk preferences or disaster risk. Bekaert (1996), Bansal (1997), and Backus, Foresi, and Telmer (2001) make progress on the forward premium puzzle, by notably deriving necessary conditions that term structure models need to satisfy to reproduce the deviations from the uncovered interest rate parity condition.[1]

Our effort is worth entertaining only if no certain solution to the exchange rate puzzle exists. Recent potential solutions include the long-run risk models of Colacito and Croce (2011) and Bansal and Shaliastovich (2012), the disaster risk model of Farhi and Gabaix (2014), or the segmented market models of Gabaix and Maggiori (2014). The long-run risk models assume that the slow moving components of consumption growth are perfectly correlated across countries. The disaster risk model assumes that the expected disaster is common across countries. The segmented market models imply a very large correlation between exchange rate changes and the consumption growth of the market participants. While we find these assumptions very

[1] Other examples of multi-country term structure models that rely on the complete market assumption to address the carry trade and forward premium puzzle include Frachot (1996), Hodrick and Vassalou (2002), Brennan and Xia (2006), Graveline and Joslin (2011), Sarno, Schneider, and Wagner (2012), and Lustig, Roussanov, and Verdelhan (2011, 2014).

plausible, we recognize that more empirical work is needed to validate them. In this paper, we thus pursue a different route, going back to the basic stylized facts of international economics, the low correlations between exchange rates, foreign and domestic macro-economic variables and the large risk premia, and studying the impact of incomplete spanning. Since we find that incomplete spanning can only go so far in explaining exchange rate puzzles, our paper implicitly argue in favor of more empirical work along the lines suggested by the recent contributions to the international finance literature.

Building on Brandt, Cochrane, and Santa-Clara (2006) and Backus, Foresi, and Telmer's (2001), our paper also complements a growing literature in international economics and finance that study exchange rates in incomplete markets. We do not study how market incompleteness changes the projection of the stochastic discount factor on the space of domestically traded assets itself. Such changes are interesting, and our paper therefore does not suggest that models with incomplete markets are uninteresting or not useful. Notable recent contributions include the work by Alvarez, Atkeson, and Kehoe (2002), Chari, Kehoe, and McGrattan (2002), Bacchetta and van Wincoop (2006), Corsetti, Dedola, and Leduc (2008), Alvarez, Atkeson, and Kehoe (2009), Pavlova and Rigobon (2010, 2012), Bruno and Shin (2014), Maggiori (2014), Gabaix and Maggiori (2015), and Favilukis, Garlappi, and Neamati (2015). Instead of specifying a fully-fledged international economics model as these authors do, we seek results that are valid for any stochastic discount factors. These stochastic discount factors could reflect the impact of market incompleteness on the equilibrium allocations. But the exchange rate puzzles remain: macroeconomic variables like consumption growth, whether the response to an optimal behavior in complete or incomplete markets, exhibit a low correlation across countries and a low correlation to the exchange rate changes in the data.

The rest of this paper is organized as follows. Section 1 defines exchange rates in the case of complete spanning, while Section 2 considers the case of incomplete spanning and presents our main result. Section 3 then focuses on the exchange rate volatility, Section 4 on the currency risk premium, and Section 5 on the exchange rate cyclicality. Section 6 gathers all the three puzzles together. Section 7 generalizes the main result to the non-normal case. In Section 8, we consider a large class of log-linear dynamic asset pricing models as examples. Section 9 concludes.

1 Complete Spanning

We start by defining some notation. S_t denotes the nominal exchange rate in domestic currency (e.g., U.S. dollars) per unit of foreign currency. When S_t increases, the foreign currency appreciates and the U.S. dollar depreciates. We use x^* to denote foreign variables expressed in units of f. R_t^* represents the return on a foreign asset expressed in units of foreign currency, while R_t denotes the return on a domestic asset, expressed in units of the domestic currency. M denotes the stochastic discount factor. With this notation, the domestic and foreign investor's Euler equations for any foreign return R_t^* are:

$$E_t \left(M_{t+1} \frac{S_{t+1}}{S_t} R_{t+1}^* \right) = 1, \qquad (1)$$

$$E_t \left(M_{t+1}^* R_{t+1}^* \right) = 1. \qquad (2)$$

When dealing with real variables, the same expressions apply. In a world with multiple goods, we choose one good in each country to be the numéraire. S_t denotes the real exchange rate, expressed in units of the domestic numéraire, per unit of the foreign numéraire. M_t is expressed in the domestic numéraire. Our analysis allows for different consumption baskets at home and abroad.

Markets are complete when investors can invest in any contingent claim, either directly or by synthesizing contingent claims using other securities. In other words, markets are complete when securities' payoffs span all the possible states of nature. The stochastic discount factor is unique when markets are complete.

The spanning condition is different from the law of one price in financial assets. Any Euler equation implies that the law of one price holds: the price of a combination of payoffs X_1 and X_2 is the combination of their prices, i.e $P(aX_1 + bX_2) = aP(X_1) + bP(X_2)$ for any real pair $(a,\, b)$. If investors can form portfolios freely and the law of one price holds, then there exists a unique stochastic discount factor in the space of traded assets (see Cochrane, 2005, Chapter

4). There are, however, potentially many stochastic discount factors outside the space of traded assets.

By comparing the two Euler equations above, one can guess a candidate foreign pricing kernel:

$$M^*_{t+1} = M_{t+1}\frac{S_{t+1}}{S_t},$$

taking the domestic pricing kernel and the exchange rate process as given. If markets are complete (i.e., complete spanning), then this foreign pricing kernel is the only pricing kernel that is consistent with the absence of arbitrage opportunities. If markets are not complete (incomplete spanning), then there are lots of other candidate foreign pricing kernels. In the case of complete markets, the log change in exchange rate is thus:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1}.$$

Throughout the paper, lower case letters denote natural logarithms. This equation implies that the foreign currency appreciates in bad times for foreign investors, while the home currency depreciates in good times for domestic investors. This implication of complete markets is sometimes viewed as undesirable and counterintuitive (e.g., the depreciation of the Argentine peso in 2002), although not always rejected by the data (e.g., the Japanese yen appreciated after the 2014 tsunami in Japan).

In order to break the tight link between the log change in exchange rate and the two log stochastic discount factors, one needs to abandon the assumption of complete spanning . Our paper starts from a simple question: What do we gain by relaxing the complete spanning assumption and introducing a wedge between the exchange rate change and the stochastic discount factors? To address this question, we first derive some necessary conditions on the wedge implied by incomplete spanning.

2 Incomplete Markets

We adopt the approach of an econometrician who commits to a model for the SDF. We start from a pair of pricing kernels (M, M^*) that can be measured by an econometrician with access to sufficiently rich data. For example, if households have access to a rich menu of assets domestically, then we can rely on standard aggregation results to construct a stand-in agent who eats aggregate consumption and price assets off his IMRS. If we adopt the Breeden-Lucas-Rubenstein representative agent model with power utility, then the real pricing kernel $m_{t+1} = \log \delta - \gamma \Delta c_{t+1}$ where γ denotes the coefficient of relative risk aversion and δ denotes the rate of time preference, and Δc_{t+1} denotes log aggregate consumption growth. Similarly, $m_{t+1} = \log \delta^* - \gamma^* \Delta c^*_{t+1}$. The econometrician can test this model by gathering data on aggregate consumption growth at home and abroad.[2] Armed with this pair (M, M^*), we explore the effects of incomplete spanning in financial markets.

To study the impact of incomplete spanning in international financial markets on exchange rates, we fix the foreign and domestic SDFs, and we follow the approach pioneered by Backus, Foresi, and Telmer (2001) who define a new 'perturbed' SDF denoted $\widehat{M}^*_{t+1}$:

$$\widehat{M}^*_{t+1} = M^*_{t+1} \exp(\eta_{t+1}) = M_{t+1} \frac{S_{t+1}}{S_t}.$$

In essence, we compare two different processes for exchange rates. The first one, highlighted in the previous section, is the complete markets one, the second one allows for incomplete spanning:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1} \qquad vs. \qquad \Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}.$$

Based on this perturbed SDF, the domestic investor's Euler equations for foreign assets, and

[2] However, we could also use an unconstrained individual's IMRS, to be measured with individual consumption data, if we do not wish to rely on aggregation results. Alternatively, we could use a cross-sectional average of the individual IMRS to price assets, in the spirit of Mankiw (1986) and Constantinides and Duffie (1996). This would require data on higher-order moments of individual consumption growth.

the foreign investor's Euler equation for the domestic assets are now given by:

$$E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}R^*_{t+1}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})R^*_{t+1}\right) = 1, \quad (3)$$

$$E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}R_{t+1}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})R_{t+1}\right) = 1. \quad (4)$$

Again, if international financial markets are complete, then of course $\eta = 0$. If international financial markets are incomplete, there are lots of possible η's.[3]

To derive an upper bound on the effects of incompleteness in international financial markets, we start by only considering investments in the foreign risk-free assets. We assume that there exists a risk-free asset at home and abroad that can be bought and sold by domestic and foreign investors. Domestic investors can trade the foreign risk-free asset, and vice-versa, the foreign investor can invest in the domestic risk-free asset. This is equivalent to assuming that investors have access to forward currency markets.

Since the risk-free payoffs are in the space the space of traded assets for all investors, domestic and foreign, they satisfy the Euler equations (1) and (2), but the risk-free payoffs also satisfy the two perturbed Euler equations (3) and (4), in which the incomplete spanning introduces a wedge η in the spot exchange rates. We start from the Euler equations (3) and (4) for the risk-free assets:

$$E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = 1/R^{f,*}_t,$$

$$E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = 1/R^{f,}_t.$$

We can only consider perturbations η that satisfy these Euler equations. If not, we would be violating the assumption of no arbitrage. Below, we explore the restrictions imposed by these conditions on possible η's. To explore these restrictions, we assume conditional joint log normality of the SDFs and the η's. We relax this assumption in the appendix in the Appendix.

[3]Note that incomplete spanning could change M and M^* themselves in interesting ways by changing equilibrium allocations within a country. We do not take these effects into account. The econometrician is committed to a model for M and M^*, and our results are valid for any M and M^*.

Naturally, if foreign investors can invest in other domestic assets, this will give rise to additional restrictions. By ignoring these additional restrictions, we are giving incomplete spanning the best shot at producing promising results that cannot be replicated by a simple reduced form model in complete markets. Our results thus provide an upper bound on the effects of incomplete spanning.

Proposition 1. *We fix the home log stochastic discount factor m and the foreign log stochastic discount factor m^*. Incomplete spanning implies that the exchange rate process S_t satisifies $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$. If the log stochastic discount factors and perturbations η are jointly conditionally normal, then $\mu_{t,\eta} = E_t(\eta_{t+1})$ and $var_t(\eta_{t+1})$ satisfy:*

$$covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - \frac{1}{2} var_t(\eta_{t+1}), \quad (5)$$

$$covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + \frac{1}{2} var_t(\eta_{t+1}), \quad (6)$$

where $\mu_{t,\eta}$ satisfies these additional restrictions for:

$$\begin{aligned}
-\mu_{t,\eta} &\leq std_t(\eta_{t+1})\left(std_t\left(m^*_{t+1}\right) + \frac{1}{2} std_t(\eta_{t+1})\right), \text{ when } \mu_{t,\eta} \leq -\frac{1}{2} var_t(\eta_{t+1}), \\
\mu_{t,\eta} &\leq std_t(\eta_{t+1})\left(std_t\left(m^*_{t+1}\right) - \frac{1}{2} std_t(\eta_{t+1})\right), \text{ when } \mu_{t,\eta} \geq -\frac{1}{2} var_t(\eta_{t+1}), \\
\mu_{t,\eta} &\leq std_t(\eta_{t+1})\left(std_t(m_{t+1}) + \frac{1}{2} std_t(\eta_{t+1})\right), \text{ when } \mu_{t,\eta} \geq \frac{1}{2} var_t(\eta_{t+1}), \\
-\mu_{t,\eta} &\leq std_t(\eta_{t+1})\left(std_t(m_{t+1}) - \frac{1}{2} std_t(\eta_{t+1})\right), \text{ when } \mu_{t,\eta} \leq \frac{1}{2} var_t(\eta_{t+1}), \\
std_t(\eta_{t+1}) &\leq std_t\left(m^*_{t+1} - m_{t+1}\right), \text{ everywhere.}
\end{aligned}$$

Hence, there are limits as to how much incomplete markets noise we can introduce. For example, when the noise has a conditional mean of zero ($\mu_{t,\eta} = 0$), the amount of noise is bounded above by the following two conditions:

$$2min(std_t(m_{t+1}), std_t\left(m^*_{t+1}\right)) \geq std_t(\eta_{t+1}) \text{ and } std_t\left(m^*_{t+1} - m_{t+1}\right) \geq std_t(\eta_{t+1}).$$

The restrictions on the process η highlighted in Proposition 1 have some clear implications for the foreign maximum Sharpe ratio, as shown below.

Corollary 1. *The volatility of the domestic investor's log pricing kernel in the foreign numéraire is given by:*

$$std_t\left(m^*_{t+1} + \eta_{t+1}\right) = \sqrt{var_t(m^*_{t+1}) - 2\mu_{t,\eta}}.$$

The volatility of the foreign investor's log pricing kernel in the domestic numéraire is given by:

$$std_t\left(m_{t+1} - \eta_{t+1}\right) = \sqrt{var_t(m_{t+1}) + 2\mu_{t,\eta}}.$$

At this point, we have made no additional assumptions about the process η that describes the incomplete spanning. In dynamic asset pricing models, the drift term $\mu_{t,\eta}$ imputed to the exchange rate process is not a free parameter but is instead determined by no arbitrage conditions. We analyze a class of dynamic asset pricing models in the last section of this paper, but for now we maintain the full generality of our assumptions. Let us now consider three useful cases.

First, when the disturbance η is mean-zero ($\mu_{t,\eta} = 0$), incomplete spanning does not introduce any non-stationarity in exchange rates. This is a natural case to consider. In this case, the volatility of the domestic investor's log pricing kernel in the foreign numéraire is left unchanged compared to the complete market benchmark. This volatility has an intuitive interpretation: it is equal to the maximal Sharpe ratio when the pricing kernel is lognormal The maximum Sharpe ratio in the foreign country is thus unchanged only when $\mu_{t,\eta} = 0$. In this case, the effects of the η perturbations are completely identical for the home and foreign countries.

Second, when the disturbance η is such that $\mu_{t,\eta} = -var_t(\eta_{t+1})/2$, then the shocks stemming from market incompleteness are unspanned from the domestic investor's perspective. In other words, the disturbance η is orthogonal to the risk-free rate: $E_t\left(R^f_t exp(\eta_{t+1})\right) = 0$. In this case, the maximum Sharpe ratio in the foreign numéraire increases compared to the complete market benchmark: $std_t\left(m^*_{t+1} + \eta_{t+1}\right) = \sqrt{var_t(m^*_{t+1}) + var_t(\eta_{t+1})}$.

Third, when the disturbance η is such that $\mu_{t,\eta} = +var_t(\eta_{t+1})/2$, then the shocks stemming from incomplete spanning are unspanned from the foreign investor's perspective. The maximum Sharpe ratio in the domestic numéraire also increases compared to the complete market benchmark. Additional assets impose additional restrictions on the wedges. If we allow foreign investors to trade risky assets, in addition to the risk-free, this will give rise to additional restrictions. For example, if the foreign investor's can trade an additional risky asset then the wedges cannot covary with these risky returns (r_{t+1}, r^*_{t+1}):

$$covar_t\left(r^*_{t+1}, \eta_{t+1}\right) = 0 = covar_t\left(r_{t+1}, \eta_{t+1}\right). \tag{7}$$

Simple CCAPM Example Consider a model in which aggregate consumption growth Δc consists only of a standard Gaussian component $w \sim N(\mu, \sigma^2)$. The same applies to foreign consumption growth: $w^* \sim N(\mu^*, \sigma^{2,*})$. The representative agents have power utility with risk aversion coefficient γ. Next, we introduce incomplete spanning. Assume that $\eta_{t+1} = \gamma d_{t+1}$, where $d \sim N(\mu_d, \sigma_d^2)$. Domestic and foreign log consumption growth are thus:

$$\begin{aligned} \log \frac{C_{t+1}}{C_t} &= w_{t+1} + d_{t+1}, \\ \log \frac{C^*_{t+1}}{C^*_t} &= w^*_{t+1} + d^*_{t+1}. \end{aligned}$$

In this case, the wedges have to satisfy:

$$\begin{aligned} \mu_d &= \gamma^2\sigma_d^2/2 + \rho_{w,d}\gamma^2\sigma\sigma_d, \\ -\mu_d &= \gamma^2\sigma_d^2/2 - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d, \\ -\gamma^2\sigma_d^2 &= \rho_{w,d}\gamma^2\sigma\sigma_d - \rho_{w^*,d}\gamma^2\sigma\sigma_d. \end{aligned}$$

In the symmetric case in which the countries share all parameters, the drift μ_e has to be zero. In this case, the unspanned risk is always negatively correlated with domestic consumption growth and positively correlated with foreign consumption growth. These results follows directly from Proposition 1. The formal proofs are in Section B of the Appendix.

In this example with only two consumption growth innovations, the domestic investor cannot invest in any foreign risky asset. If we allow the foreign investor to do so, then we need to impose the additional covariance restrictions in Condition (7). These conditions imply that d is orthogonal to w_{t+1} and w^*_{t+1}, because the log return on the domestic (foreign) risky asset is affine in the domestic (foreign) innovation. We are back in the case of complete markets: $\eta_{t+1} = 0$. In this model, incomplete spanning reduces the exchange rate's exposure to the consumption growth innovations. Instead, the exchange rates are now exposed to shocks uncorrelated with aggregate consumption growth in either country. In the following sections, we study the impact of incomplete spanning on each of the key three exchange rate puzzles without restricting ourselves to the Lucas (1982) model. We start with the Brandt, Cochrane, and Santa-Clara (2006) puzzle.

3 Exchange Rate Volatility

Brandt, Cochrane, and Santa-Clara (2006) note that the exchange rate changes predicted by complete market models are much more volatile than in the data, unless stochastic discount factors are almost perfectly correlated across countries, which seems at odds with the evidence on any macroeconomic variable. Colacito and Croce (2011) offer a compelling resolution to the puzzle by assuming that the long-run risk components of the stochastic discount factors are highly correlated across countries.[4] In this section, we pursue another route, adding incomplete spanning to the benchmark model.

Corollary 2 shows that when adding non-zero mean noise to the exchange rate process and thus departing from complete markets, the implied volatility is lowered compared to the complete markets case by the variance of η. The volatility of the exchange rate decreases relative to the complete spanning benchmark one-for-one with the volatility of the wedge.

Corollary 2. *The volatility of exchange rates in incomplete markets is given by:*

$$var_t(\Delta s_{t+1}) \quad = \quad var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - var_t(\eta_{t+1}).$$

[4] In their model, stochastic discount factors are then volatile enough to reproduce the equity premium, but, thanks to their long-run risk components, they are almost perfectly correlated such that exchange rates are as volatile as in the data. The long-run risk components are, however, difficult to measure in the data and most evidence is drawn indirectly from asset prices and not macroeconomic quantities.

This result follows directly from the covariance restrictions in equations (5) and (6). Since the wedges are necessarily pro-cyclical, they offset the effect of m and m^*, and thus reduce the overall volatility of the exchange rate.

Simple CCAPM Example In the simple Gaussian CCAPM, the reduction in exchange rate variance is given by $\gamma^2\sigma_e^2$. The unspanned risk is counter-cyclical, i.e. negatively correlated with domestic consumption growth, and as a result, always reduces the exchange rate's volatility.

To make quantitative progress on the Brandt, Cochrane, and Santa-Clara (2006) puzzle, the wedges need to be volatile. To quantify this statement, we measure the relevant moments in the data. Table 2 reports the annualized volatility of bilateral exchange rates for 15 developed countries (Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S). All exchange rates are defined with respect to the U.S. dollar. Data are quarterly, over the 1973.IV – 2014.IV period. Across all the countries, the average volatility is 11% in this sample. It is precisely estimated, with a standard error (obtained by bootstrapping) of 0.4%, and there are only small variations across countries: the cross-sectional standard deviation is less than 2%.

Suppose therefore that we want to match a 10% per annum volatility of exchange rate changes. Then Corollary 2 implies that:

$$0.10^2 = var_t(\Delta s_{t+1}) \quad = \quad var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - var_t(\eta_{t+1}).$$

As a result, we can simply back out the amount of noise needed to match the volatility of exchange rates in the data:

$$var_t(\eta_{t+1}) \quad = \quad var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - 0.10^2.$$

This equals the difference between the variance of the complete market exchange rates implied by the stochastic discount factors and the target variance. For example, starting from a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$), the wedge must

Table 1: Exchange Rate Puzzles

	Panel A: Volatility			
	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$\sigma_{\Delta s}$	11.21	1.57	6.23	12.70
	(0.44)	(0.22)	(0.56)	(0.61)
$\sigma_{\Delta q}$	11.12	1.64	6.21	12.81
	(0.44)	(0.20)	(0.48)	(0.59)
$corr(\Delta c, \Delta c^*)$	0.17	0.10	0.02	0.35
	(0.05)	(0.02)	(0.07)	(0.08)
Equity S.R.	0.22	0.12	0.00	0.48
	(0.15)	(0.04)	(0.15)	(0.21)
	Panel B: Cyclicality			
	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$corr(\Delta q, \Delta c - \Delta c^*)$	-0.07	0.09	-0.22	0.14
	(0.05)	(0.03)	(0.07)	(0.10)
$\beta_{Backus-Smith}$	-0.01	0.02	-0.03	0.02
	(0.01)	(0.00)	(0.01)	(0.02)
$corr(-\Delta q, \Delta c^*)$	-0.02	0.12	-0.21	0.24
	(0.03)	(0.03)	(0.07)	(0.09)
	Panel C: Risk Premium			
	Time-Series Mean	Time-Series Std	Time-Series Sharpe ratio	
$E_t[rx_{t+1}^{FX}]$	4.42	8.73	0.51	
	(1.36)	(0.97)	(0.19)	
$E_t[rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}]$	4.80	8.73	0.55	
	(1.32)	(0.98)	(0.19)	
$E_t[-rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}]$	-4.04	8.73	-0.46	
	(1.40)	(0.98)	(0.20)	
β_{UIP}	-1.23	0.63	-2.28	0.10
	(0.44)	(0.17)	(0.52)	(0.58)

Notes: The table reports summary statistics on three exchange rate puzzles. Panel A focuses on the exchange rate volatility. It reports the cross-country mean of the bilateral nominal and real exchange rate volatilities, along with the cross-country standard deviation of the bilateral exchange rate volatilities and the corresponding minimum and maximum values across countries. Panel A reports similar moments for the correlation between U.S. and foreign consumption growth rates and equity Sharpe ratios on MSCI country indices. Panel B focuses on the exchange rate cyclicality. It reports similar moments for the correlation between the changes in real exchange rates and the relative consumption growth, the slope coefficient of in a regression of relative consumption growth rates on exchange rate changes and a constant, and for the correlation between the changes in real exchange rates and the foreign consumption growth. Panel C focuses on the exchange rate risk premium. It reports the time-series mean carry trade excess return, its time-series standard deviation and its Sharpe ratio (obtained as the ratio of the mean excess return to its standard deviation). The excess returns are either in logs, or in levels, from the perspective of the U.S. or foreign investor. Finally, Panel C reports the slope coefficient in a regression of log currency excess returns on the domestic minus foreign interest rate difference. Excess returns are annualized (multiplied by 4) and reported in percentages. The standard deviation on the carry trade returns in annualized (multiplied by 2) and reported in percentages. The countries are sorted by the level of their short-term nominal interest rates into four portfolios. The exchange rate risk premium corresponds to the average carry trade excess return obtained by borrowing in low-interest rate currencies (i.e., shorting the first portfolio) and investing in high-interest rate currencies (long the last portfolio). Data are quarterly, over the 1973.IV – 2014.IV period. The panel consists of 15 countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S. The standard errors (reported between brackets) were generated by block-bootstrapping 10,000 samples, each block containing 2 quarters.

have a standard deviation of 48% per annum ($std_t(\eta_{t+1}) = 0.48$). Such a correlation across stochastic discount factors is optimistic. In the data, the correlation of consumption growth varies between 0.02 and 0.35 as shown in Panel A of Table 2. If we decrease the correlation of the SDFs from 0.5 to zero, then we need an even more volatile wedge: $std_t(\eta_{t+1})$ is at least equal to 70%. To stack the deck in favor of the incomplete spanning wedge, we assume throughout the paper a correlation across stochastic discount factors of 0.50.

Figure 1 plots the implied volatility of the incomplete markets exchange rate against the amount of noise $std_t(\eta_{t+1})$ for various choices of $\mu_{t,\eta}$. We only plot the admissible $(\mu_{t,\eta}, std_t(\eta_{t+1}))$ pairs that satisfy the inequalities in Proposition 1. The figure is drawn assuming a maximum (annualized) Sharpe ratio of 0.50.

In the figure, we gradually increase $\mu_{t,\eta}$, but it has no bearing on the volatility of the exchange rates. For high values of $std_t(\eta_{t+1})$ close to 50% the volatility of exchange rates drops within the plausible range. incomplete spanning helps with the volatility puzzle, but the quantity of perturbation risk needed is of the same order of magnitude as the maximum Sharpe ratio.

4 Currency Risk Premia

Next, we turn to the currency risk premium, assuming that investors have access to domestic and foreign risk-free rates. When markets are incomplete, we do not recover the standard expression for log currency risk premium established in Bekaert (1996), Bansal (1997), Backus, Foresi, and Telmer (2001). But a similar expression emerges, highlighted in the following corollary to Proposition 1.

Corollary 3. *The currency risk premium in logs on a long position in foreign currency is:*

$$\begin{aligned} E_t[rx_{t+1}^{FX}] &= r_t^{f,*} - r_t^f + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m_{t+1}^* + \eta_{t+1}\right)\right] \\ &= \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m_{t+1}^*\right)\right] + \mu_{t,\eta}. \end{aligned}$$

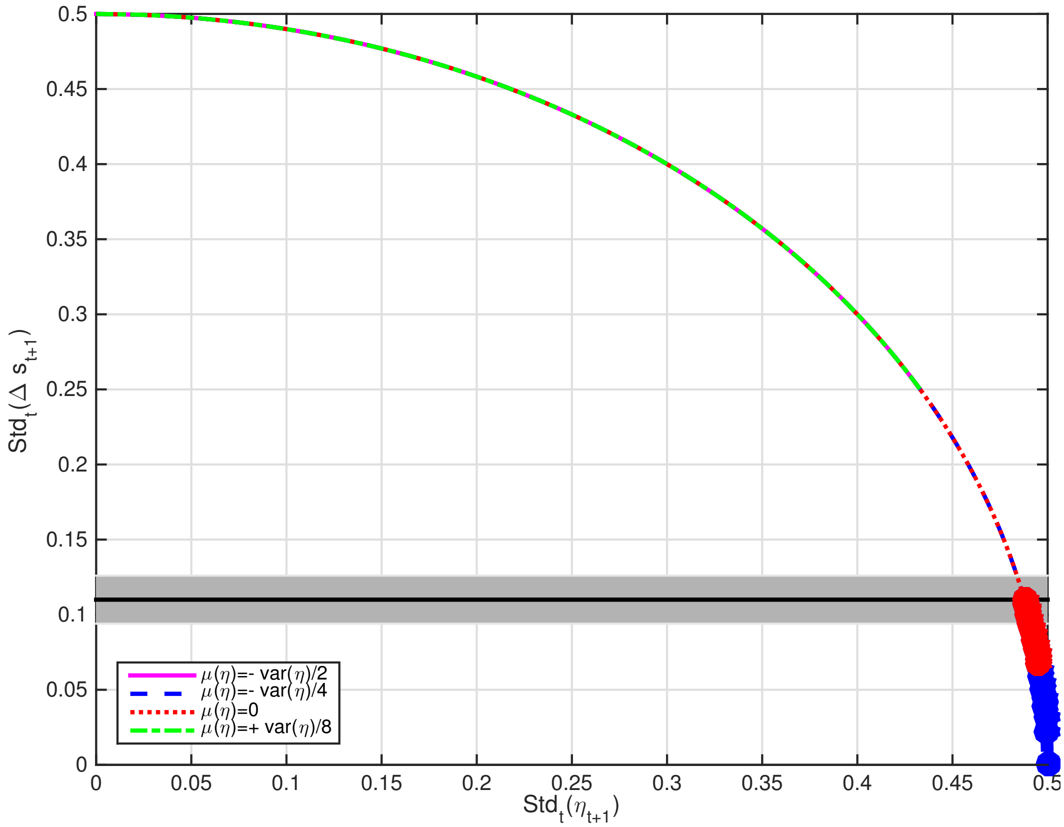

Figure 1: Exchange Rate Volatility: The figure reports the implied volatility of the changes in the log exchange rate, denoted $std_t(\Delta s_{t+1})$, against the volatility of the incomplete market wedge, denoted $std_t(\eta_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$). Large dots indicate parameters that imply exchange rate volatilities less than 11%, the average volatility of exchange rates in our sample ($std_t(\Delta s_{t+1}) \leq 0.11$). The gray area indicates the value of the exchange rate volatility in the data: it is centered around the cross-country mean volatility (11%); the area represents one cross-country standard deviation (1.6%) above and below the cross-country mean volatility.

The currency risk premium in levels on a long position in foreign currency is given by:

$$
\begin{aligned}
E_t[rx^{FX}_{t+1}] + \frac{1}{2} var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1}) \\
&= var_t\left(m_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2} var_t\left(\eta_{t+1}\right) + \mu_{t,\eta}.
\end{aligned}
$$

The currency risk premium in levels, from the perspective of the foreign investor, is given by:

$$\begin{aligned} E_t[-rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m^*_{t+1}, -\Delta s_{t+1}) \\ &= var_t\left(m^*_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - \mu_{t,\eta}. \end{aligned}$$

Simple CCAPM Example Recall that the change in variance from complete to incomplete spanning is given by: $\Delta Var = -(\gamma\sigma_e)^2$. The change in the currency risk premium at home is given by: $\Delta RP = \rho_{w,e}\gamma^2\sigma\sigma_e$. The change in the currency risk premium abroad is: $\Delta RP^* = -\rho_{w^*,e}\gamma^2\sigma\sigma_e$. Hence, the total change in the risk premia has to be negative; the sum of these expressions is negative. That follows from the no no-arbitrage restrictions on wedges in eqns. (8).[5] Furthermore, in the symmetric case in which the countries share all parameters, $\Delta RP = \Delta RP^* = -.5\gamma^2\sigma_e^2$. Recall that $\gamma\sigma$ is of the same order of magnitude as the maximum SR. The SRs decline as well. The SR with incomplete spanning is

$$SR^{FX} = \frac{\gamma}{\sqrt{2}}\sqrt{\sigma^2(1-\rho) - .5\sigma_e^2}.$$

The proofs are in section B of the Appendix.

In the data, as Table 2 reports, the average carry trade excess return is 4.4%, implying a Sharpe ratio of 0.5. To obtain the estimate of the carry trade excess return, the countries are sorted by the level of their short-term nominal interest rates into four portfolios. The exchange rate risk premium corresponds to the average carry trade excess return obtained by borrowing in low-interest rate currencies (i.e., shorting the first portfolio) and investing in high-interest rate currencies (long the last portfolio). Larger average currency risk premia and Sharpe ratio can be obtained on larger sets of countries (Lustig and Verdelhan, 2007).

Figure 2 plots the theoretical currency risk premium in levels against the amount of noise $std_t(\eta_{t+1})$ from the perspective of either the home investor (left panel) or foreign investor (right

[5] A positive drift to mitigate the effect on currency risk premia from the perspective of the domestic investor implies a larger decline for the other investor.

panel). The parameters are identical to those we used in Figure 1. These parameters imply a large complete markets currency risk premium of 12% on average per year, above the actual return on currency carry trades in developed countries. The high value of the currency risk premium in complete markets stacks the deck in favor of incomplete markets, allowing for a potential large decrease in risk premia brought by the incomplete market wedge.

To analyze the results, let us now again go through our three corner cases to study the impact of an increase in the volatility of the incomplete spanning wedge. First, when $\mu_{t,\eta} = -var_t(\eta_{t+1})/2$ (the case of orthogonal η shocks from the perspective of the foreign investor), the risk premium in level from the perspective of the home investor decreases, but it remains equal to its complete market value from the perspective of the foreign investor. Second, when $\mu_{t,\eta} = 0$ (the symmetric case), the log risk premium is the same as in complete markets, but the risk premium in level decreases for both the home and foreign investor: for values of the wedge volatility that matches the exchange rate volatility in the data, these currency risk premia are zero. Third, when $\mu_{t,\eta} = var_t(\eta_{t+1})/2$ (the case of orthogonal η shocks from the perspective of the domestic investor), the risk premium in level from the perspective of the foreign investor decreases, but it remains equal to its complete market value from the perspective of the home investor. Those three corner cases are clearly rejected by the data on currency risk premia. It seems, however, that when the drift term takes on intermediate values ($\mu_{t,\eta} = +var_t(\eta_{t+1})/8$), high values of $std_t(\eta_{t+1})$ around 50% imply currency risk premia in levels that are close to their data counterpart. As we shall see, unfortunately, these parameters imply counterfactual exchange rate cyclicality.

5 Exchange Rate Cyclicality

Complete market models have strong implications for the sign and the magnitude of the exchange rate cyclicality. Complete markets imply, perhaps counterintuitively, that currencies depreciate in relatively good times for home investors, i.e. when they experience low marginal utility growth compared to the foreign investors. Complete markets in fact imply a perfect correlation between the difference in log stochastic discount factors and the log change in exchange rates:

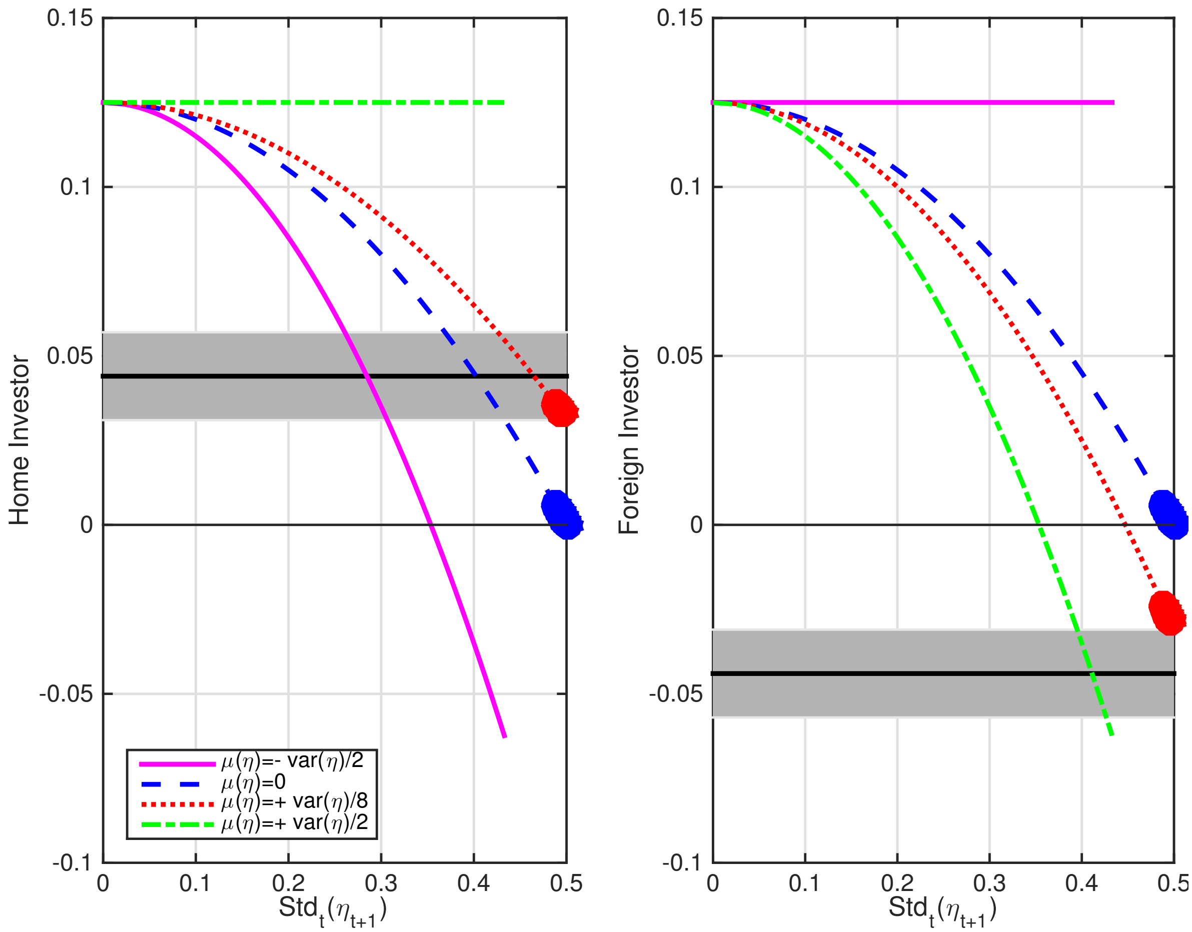

Figure 2: Currency Risk Premia in Level: The figure reports the foreign currency risk premium in level, from the perspective of the home investor (left panel) or foreign investor (right panel), against the volatility of the incomplete market wedge, denoted $std_t(\eta_{t+1})$. For the home investor, the foreign currency risk premium in level is denoted $E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}]$. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$). The gray area indicates the value of the average carry trade excess return in the data: it is centered around the mean excess return (4.4%); the area represents one standard error (1.3%) above and below the mean. Large dots indicate parameters that imply exchange rate volatilities less than 11% ($std_t(\Delta s_{t+1}) \leq 0.11$).

when applied to CRRA preferences, complete markets thus imply a perfect correlation between relative consumption growth rates and exchange rate changes.

In the data, as shown in Panel B of Table 2, the corresponding unconditional correlation is not statistically different from zero. This is the Kollmann (1991) and Backus and Smith (1993) puzzle. Likewise, the unconditional correlation between changes in exchange rates and foreign consumption growth or the slope coefficient in a regression of relative consumption growth rates on exchange rate changes and a constant are also not statistically different from zero. Moving away from complete markets can potentially address the puzzle. Corsetti, Dedola, and Leduc (2008) and Kollmann (1996), for example, offer interesting models in incomplete markets that lower the correlation between exchange rates and consumption growth. Instead of studying one particular model, we consider the potential effect of any wedge introduced by the incomplete spanning on these measures of exchange rate cyclicality.

Corollary 4. *The covariance between the difference in log stochastic discount factors $m^* - m$ and the change in exchange rates in incomplete markets is non-negative:*

$$\begin{aligned} covar_t\left(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}\right) &= covar_t\left(m^*_{t+1} - m_{t+1}, \eta_{t+1}\right) + var_t\left(m^*_{t+1} - m_{t+1}\right), \\ &= var_t\left(m^*_{t+1} - m_{t+1}\right) - var_t\left(\eta_{t+1}\right) = var_t\left(\Delta s_{t+1}\right) \geq 0 \end{aligned}$$

The slope coefficient in a regression of $m^* - m$ on exchange rate changes is equal to 1, its value when markets are complete:

$$\beta_{Backus-Smith} = \frac{covar_t\left(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}\right)}{var_t\left(\Delta s_{t+1}\right)} = 1.$$

Simple CCAPM Example The Backus-Smith correlation coefficient is given by:

$$1 \geq corr_t\left(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta s_{t+1}\right) = \frac{\sqrt{\gamma^2 \sigma^{*,2} + \gamma^2 \sigma^2 - 2\gamma^2 \rho_{w,w^*} \sigma \sigma^* - (\gamma \sigma_e)^2}}{\sqrt{\gamma^2 \sigma^{*,2} + \gamma^2 \sigma^2 - 2\gamma^2 \rho_{w,w^*} \sigma \sigma^*}} \geq 0$$

The correlation is smaller than one, its complete markets value, but always positive. This corollary shows that the impact of incomplete spanning on measures of exchange rate cyclicality is limited for three reasons. First, incomplete spanning does not change the sign of the covariance between exchange rate changes and the difference in log stochastic discount factors spanned by asset markets. Even in complete markets, exchange rates will depreciate when the home investor experience better times than the foreign investor; those times, however, are defined by the marginal utility spanned by asset markets, whereas the total marginal utility of the investor may be high or low. Second, incomplete spanning does not change the slope coefficient in a regression of the difference in log stochastic discount factors on exchange rate changes; it is equal to one, as in complete markets. Third, incomplete spanning decreases the correlation between exchange rates and the stochastic discount factor

$$corr_t\left(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}\right) = \frac{\sqrt{var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) - var_t(\eta_{t+1})}}{\sqrt{var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1})}} \leq 1,$$

only at the cost of a lower Sharpe ratio on the currency risk premium. We now study this tradeoff quantitatively.

Figure 3 plots the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$, in the left panel, and the correlation between consumption growth and the change in exchange rates, $corr_t(\Delta s_{t+1}, \Delta c_{t+1})$, in the right panel, both against the volatility of the incomplete market wedge, denoted $std_t(\eta_{t+1})$. The correlation between the log home stochastic discount factors m and the change in exchange rates is :

$$corr_t\left(m_{t+1}, \Delta s_{t+1}\right) = \frac{covar_t\left(m_{t+1}, \Delta s_{t+1}\right)}{std_t\left(m_{t+1}\right) std_t\left(\Delta s_{t+1}\right)} = -\frac{E_t[rx^{FX}_{t+1}] + \frac{1}{2} var_t[rx^{FX}_{t+1}]}{std_t\left(m_{t+1}\right) std_t\left(\Delta s_{t+1}\right)} = \frac{-SR^{FX}_t}{std_t\left(m_{t+1}\right)},$$

where SR^{FX} denotes the Sharpe ratio on the currency risk premium. As we lower the correlation, we also lower the Sharpe ratios proportionally.

Let us again describe our three corner cases. First, when $\mu_{t,\eta} = -var_t(\eta_{t+1})/2$ (the case of orthogonal η shocks from the perspective of the foreign investor), the correlation decreases in absolute value, and can even change sign. Unfortunately, as we saw in the previous section, this parametrization does not reproduce the exchange rate volatility or risk premia. Second, when $\mu_{t,\eta} = 0$ (the symmetric case), the correlation tends to zero for very volatile incomplete market wedges. Such parameters can reproduce the exchange rate volatility in the data but still imply zero currency risk premia. Third, when $\mu_{t,\eta} = var_t(\eta_{t+1})/2$ (the case of orthogonal η shocks from the perspective of the domestic investor), the correlation actually increases in absolute value. The sweet spot, identified in the previous section, where the drift term takes on interme-

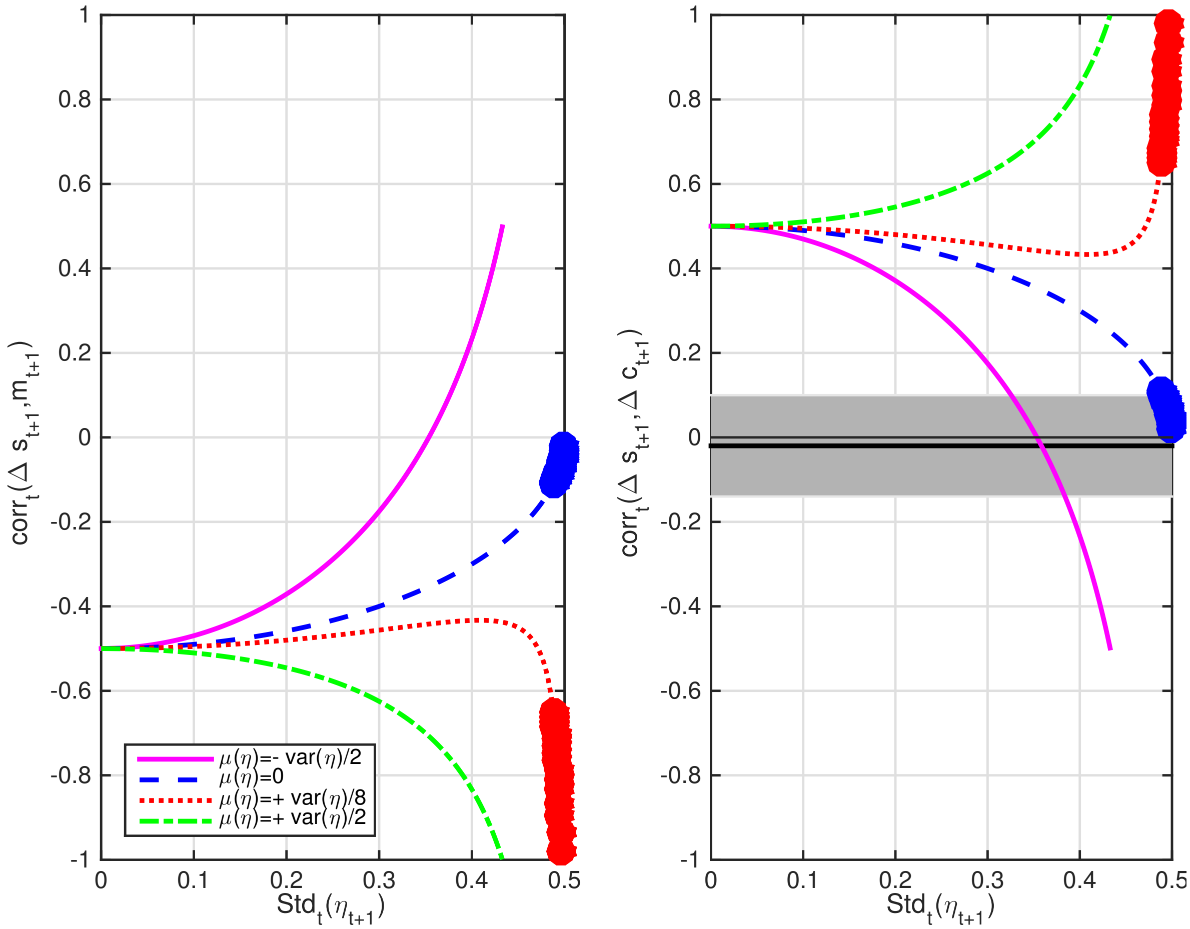

Figure 3: Correlation Between the Changes in Exchange Rates and the SDF or Consumption Growth Rates: The figure reports the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$, in the left panel, and the correlation between consumption growth and the change in exchange rates, $corr_t(\Delta s_{t+1}, \Delta c_{t+1})$, in the right panel, both against the volatility of the incomplete market wedge, denoted $std_t(\eta_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$). Large dots indicate parameters that imply exchange rate volatilities less than 11% ($std_t(\Delta s_{t+1}) \leq 0.11$). In the right panel, the coefficient of constant relative risk aversion is 10 ($\gamma = 10$).

diate values ($\mu_{t,\eta} = +var_t(\eta_{t+1})/8$) and the model implies risk premia close to their empirical values, unfortunately make the correlation larger in absolute values, making the incomplete market model even less attractive than its complete market counterpart. The correlation between exchange rate changes and consumption growth rates exceeds 0.7 in the relevant region of the parameter space.

6 The Intersection of Three Currency Puzzles

In this section, we now bring together the insights gained on exchange rate volatility, cyclicality, and risk premium to show that markets incompleteness cannot improve on the three puzzles at the same time. We summarize our findings in Figure 4.

To summarize our findings, we focus on the parameter space that reproduces the exchange rate volatility. This volatility is precisely measured and fairly similar across developed markets. We assume that it is equal to 11%, the average value we observe in our sample, and derive the implicit relationship between the first and second conditional moments of the wedge η_{t+1}. As in the rest of the paper, the figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$). For those parameters, the standard deviation of the wedge ranges from around 0.4 to 0.5.

The upper left subplot of Figure 4 shows the currency risk premium in complete markets (dotted line) and in incomplete markets. As already noted, our parameters imply that the complete markets risk premium is around 12%. But the wedge introduced by the markets incompleteness reduces the risk premium to less than 5%. The risk premium is in line with its empirical value only for large drift parameters. For zero or negative drifts, the currency risk premium is essentially zero or turn negative. The upper right subplot of Figure 4 shows the same link between the value of the drift and the currency Sharpe ratio. The lower subplots show that large values of the drifts imply exchange rate correlations with log stochastic discount factors (and consumption growth in the case of CRRA preferences) that are in absolute values even larger than their complete markets counterpart. The wedge exacerbates these features of the complete markets models that incomplete spanning is supposed to address.

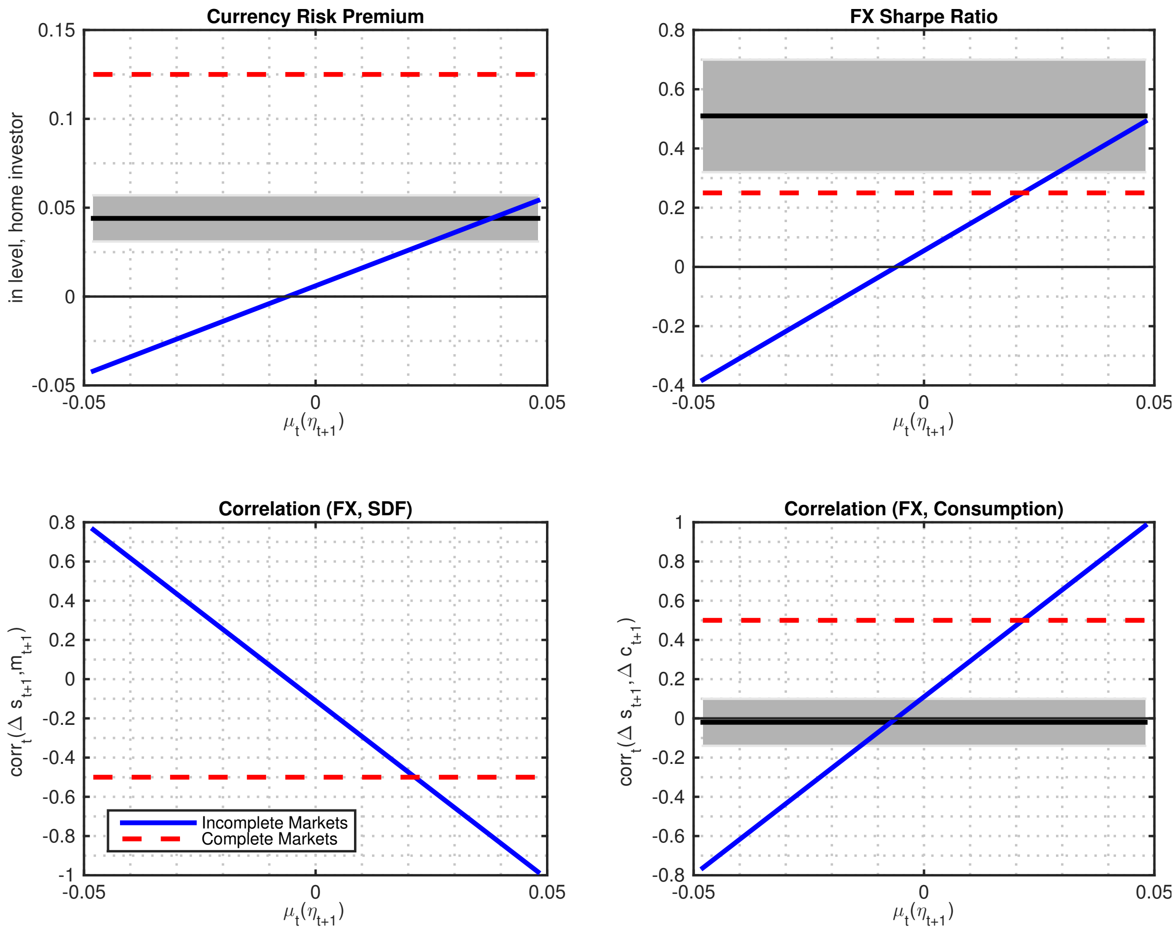

Figure 4: Overview: The figure reports the results for the admissible combinations of $\mu_{t,\eta}$ and $std_t(\eta_{t+1})$ that produce a 11% volatility of exchange rate changes. The figure is drawn assuming a maximum Sharpe ratio of 0.50 in both countries ($std_t(m_{t+1}) = 0.50$ and $std_t(m^*_{t+1}) = 0.50$) and a correlation across stochastic discount factors of 0.50 ($\rho_t(m_{t+1}, m^*_{t+1}) = 0.50$). The first panel reports the required μ_η for each value of $std_t(\eta_{t+1})$. The second panel reports the implied Sharpe ratios on currency risk premia. The third panel reports the correlation of changes in the exchange rate with the domestic SDF, $corr_t(\Delta s_{t+1}, m_{t+1})$. The fourth panel reports the correlation with the consumption growth difference, $corr_t(\Delta s_{t+1}, \Delta c_{t+1})$. The gray area represents the empirical counterpart of each moment.

7 Non-normality

When relaxing the log-normality, we first present preference-free results and then develop a consumption-based example with jumps.

7.1 Preference-free Results

We can show that some general results can be extended to an environment with non-Gaussian shocks. To do so, we use a different, entropy-based measure of risk. The conditional entropy of a random variable X_{t+1} is equal to: $L_t(X_{t+1}) = \log E_t(X_{t+1}) - E_t(\log X_{t+1})$. If the random variable X_{t+1} is log normally distributed, then its entropy is equal to one half of its variance. In general, entropy measures all higher order cumulants κ_i of $\log X$: $L_t(X_{t+1}) = \kappa_{2t}/2! + \kappa_{3t}/3! + \kappa_{4t}/4! + \ldots$ Similarly, the co-entropy is defined as $L_t(X_{t+1}Y_{t+1}) - L_t(X_{t+1}) - L_t(Y_{t+1})$, which is a natural measure of the covariation. This measure is zero if the variables are conditionally independent. Using these measures of risk, we derive an analog to Proposition 1 in the case of non-Gaussian shocks.

Proposition 2. *We fix the home m and foreign m^* log stochastic discount factors. Incomplete spanning implies that the exchange rate process S_t satisfies $\Delta s_{t+1} = \eta_{t+1} + m^*_{t+1} - m_{t+1}$ where η satisfies the following restrictions:*

$$\begin{aligned}
\mu_{t,\eta} - L_t\left(\exp(-\eta_{t+1})\right) &= L_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - L\left(M_{t+1}\right) - L_t\left(\exp(-\eta_{t+1})\right) \\
-\mu_{t,\eta} - L_t\left(\exp(\eta_{t+1})\right) &= L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - L\left(M^*_{t+1}\right) - L_t\left(\exp(\eta_{t+1})\right),
\end{aligned}$$

and where the trend $\mu_{t,\eta}$ satisfies:

$$-\mu_{t\eta} \leq \log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t \log\left(M_{t+1}\right),$$

$$\mu_{t\eta} \leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t \log\left(M^*_{t+1}\right),$$

$$\mu_{t\eta} \leq \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right).$$

These η- drift conditions are the exact equivalent of the covariance conditions in the log-normal case. When the stochastic discount factor and the wedge are jointly lognormal, one recovers the same conditions derived in Proposition 1. We can then compare the entropy of the incomplete markets exchange rates to the entropy of the complete markets version, denoted $L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right)$:

Corollary 5. *The entropy of the changes in exchange rates is:*

$$\begin{aligned} L_t\left(\frac{S_{t+1}}{S_t}\right) &= L_t\left(\frac{M^*_{t+1}\exp(\eta_{t+1})}{M_{t+1}}\right), \\ &= L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) - \mu_{t,\eta} + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right). \end{aligned}$$

Hence, the difference between the entropy of the exchange rate change in incomplete versus complete markets is equal to:

$$\Delta L_t = L_t^{IM} - L_t^{CM} = -\mu_{t,\eta} + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right).$$

The change in entropy of exchange rates introduced by incomplete spanning is tightly linked to the change in currency risk premia. To see this point, let us first define the currency risk premium when shocks are non-Gaussian. The following proposition is the counterpart to Proposition 3 in Section 4.

Proposition 3. *The risk premium in logs on a long position in foreign currency is:*

$$E_t[rx^{FX}_{t+1}] = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + \mu_{t,\eta}.$$

The risk premium in levels on a long position in foreign currency is given by:

$$E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}\exp(\eta_{t+1})}{M_{t+1}}\right).$$

Backus, Foresi, and Telmer (2001) show that the complete markets' risk premium in logs is simply $L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right)$. The complete markets' risk premium in levels is thus given by:

$$E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right).$$

The difference between the currency risk premium in incomplete versus complete markets is thus related to the changes in exchange rate entropy introduced by the incomplete spanning:

$$\Delta RP_t = RP^{IM}_t - RP^{CM}_t = \Delta L_t + \mu_{t,\eta}.$$

A decrease in the entropy of the exchange rate leads to a commensurate decrease in the foreign currency risk premium. Note that exchange rate entropy can be lowered without modifying the risk premium through the first moment of the wedge, $\mu_{t,\eta}$, but, as shown below and in Section 8, the trend $\mu_{t,\eta}$ is constrained by additional no arbitrage restrictions in many dynamic models.

In the most general case, however, we are not able to bound the co-entropy of exchange rates and stochastic discount factors. As a result, the trilemma that we highlight in the lognormal case cannot be formally expressed here. The intuition is simple: as entropy depends on an infinite sum of higher moments, it may be possible to pick some higher moments that affect the co-entropy of exchange rates and stochastic discount factors without affecting much the entropy of the exchange rates or the currency risk premium. We do not know of such a model, but we cannot rule its existence out. In the class of non-normal models often used in the option pricing and macro-finance literature, however, we show that the same trilemma applies: introducing incomplete spanning to decrease the volatility of exchange also decreases the currency risk premium and implies counterfactual links between stochastic discount factors and exchange rate changes.

To illustrate these forces, we study deviations from log normality by introducing jumps in a Consumption-CAPM example. All the proofs are in Section B of the Appendix.

7.2 Consumption-Based Example with Disasters

The domestic and foreign representative agents have power utility with identical risk aversion γ. Consumption growth in each country consists of a standard Gaussian component and a jump component. The first component is the same as in the previous consumption-based example; it is normally distributed, denoted w, and distributed as $\sim N(\mu, \sigma^2)$. The second component is a Poisson mixture of normals, denoted z. Foreign variables are denoted with a *. Log consumption growth is the sum of these two components:

$$\begin{aligned}
\log \frac{C_{t+1}}{C_t} &= w_{t+1} + z_{t+1}, \\
\log \frac{C^*_{t+1}}{C^*_t} &= w^*_{t+1} + z^*_{t+1}.
\end{aligned}$$

At each date, the number of jumps j takes on non-negative integer values with probabilities $e^{-\varpi}\varpi^j/j!$. The parameter ϖ, the jump intensity, is the mean of j. Each jump triggers a draw from a normal distribution with mean θ and variance δ^2 for the domestic agent and with mean θ^* and variance δ^{*2} for the foreign agent. The jumps are thus common across countries, but the jump sizes are not. Conditional on the number of jumps j, the domestic jump component is normally distributed as $z_t|j \sim N(j\theta, j\delta^2)$, while the foreign jump component is normally distributed as $z^*_t|j \sim N(j\theta^*, j\delta^{2,*})$. If ϖ is small, the jump model is well approximated by a Bernoulli mixture of normals. If ϖ is large, multiple jumps can occur frequently. This functional form is known as the Merton (1976) model. In the macro-finance literature, it has been applied by Bates (1988), Naik and Lee (1990), Backus, Chernov, and Zin (2011), and Martin (2013).

Complete Markets We start from the complete market benchmark.

Result 1. *When markets are complete, the foreign currency risk premium in levels is given by:*

$$\begin{aligned}
E_t\left[rx^{FX}_{t+1}\right] + L_t\left[rx^{FX}_{t+1}\right] &= \gamma^2\sigma^2 + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta^*+(\gamma\delta^*)^2/2} - 1\right) \\
&+ \varpi\left(e^{-\gamma\theta^*+\gamma\theta-2\gamma^2\rho_{z,z^*}\delta\delta^*+(\gamma\delta)^2/2+(\gamma\delta^*)^2/2} - 1\right).
\end{aligned}$$

Incomplete markets Next, we introduce incomplete spanning. We assume that the wedge takes the form $\eta_{t+1} = \gamma d_{t+1}$, where d_{t+1} follows the same Poisson mixture as z_{t+1}, but with parameters θ_d and δ_d. The jumps are common for the z_{t+1} and d_{t+1} components. Conditional on the number of jumps j, the jump and wedge components are jointly normal: $z_t|j \sim N(j\theta, j\delta^2)$ and $d_t|j \sim N(j\theta_d, j\delta_d^2)$. We use $\rho_{z,d}$ and $\rho_{z^*,d}$ to denote the correlation of jump sizes between the spanned and unspanned components of exchange rates.

Result 2. *The wedges satisfy the following restrictions:*

$$\begin{aligned} -\gamma\theta_d + \gamma^2\delta\delta_d\rho_{z,d} + \frac{\gamma^2\delta_d^2}{2} &= 0, \\ \gamma\theta_d - \gamma^2\delta^*\delta_d\rho_{z^*,d} + \frac{\gamma^2\delta_d^2}{2} &= 0. \end{aligned}$$

The change in volatility from complete to incomplete spanning is given by:

$$\Delta L_t = L_t^{IM} - L_t^{CM} = -\gamma\varpi\theta_d + \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2}\left(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1\right).$$

The corresponding change in the risk premium is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2}\left(e^{\gamma^2\delta\delta_d\rho_{z,d}} - 1\right).$$

In the symmetric case, when the wedge does not have a drift ($\theta_d = 0$), the correlation is given by $\rho_{z,d} = -\rho_{z^*,d} = -0.5(\frac{\delta_d}{\delta})$. As before, the market incompleteness reduces the exchange rate volatility and the exchange rate risk premium by the same amount:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi e^{-\gamma\theta^* + \gamma\theta - \gamma^2\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2}\left(e^{-\frac{\gamma^2\delta_d^2}{2}} - 1\right) = \Delta L_t < 0.$$

In the symmetric case, when the wedge does not have a drift ($\theta_d = 0$) and the two countries share the same parameters ($\theta = \theta^*$, $\delta = \delta^*$), the decrease in exchange rate volatility and risk premia is simply:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi e^{-\gamma^2 \rho_{z,z^*} \delta^2 + \gamma^2 \delta^2} \left(e^{-\frac{\gamma^2 \delta_d^2}{2}} - 1 \right) = \Delta L_t < 0.$$

Calibration We follow Backus, Chernov, and Martin (2011) and set the risk-aversion parameter to 5.19, the mean (μ) and standard deviation (σ) of the normal consumption growth shocks to 2.3% and 1%, the jump intensity ϖ to 1.7%, the mean jump size θ to -38%, and the jump size volatility δ to 25%. These parameters were chosen to match the international evidence reported in Nakamura, Steinsson, Barro, and Ursua (2010). We assume the jump sizes are uncorrelated across countries, but the jumps are common.

Figure 5 plots the exchange rate volatility $\sqrt{2L}$ and the currency risk premium on a long position in foreign currency from the perspective of the home investor.

8 Dynamic Asset Pricing Models

In this section, we examine the trade-off between volatility reduction and currency risk premia in some specific dynamic asset pricing models. In a large class of dynamic asset pricing models, the conditional variance of the log SDF is an affine function of the state variables z_t. This class includes many term structure as well as structural models. In such models, no arbitrage implies that the conditional covariances are also affine functions of the state variables. As a result, $-\mu_{t,\eta}$ and $var_t(\eta_{t+1})$ will need to be affine functions of the same state variables in order to satisfy the conditions stated in Proposition 1.

Specifying a law of motion for the stochastic discount factor further restrains the ability of the incomplete spanning wedge to address the main currency puzzles because it completely pins down the drift term imputed by the wedge to the exchange rate process. We use the Cox, Ingersoll, and Ross (1985) model (denoted CIR) model to illustrate this finding. Similar results appear naturally in the case of CRRA preferences with heteroskedastic consumption, since that

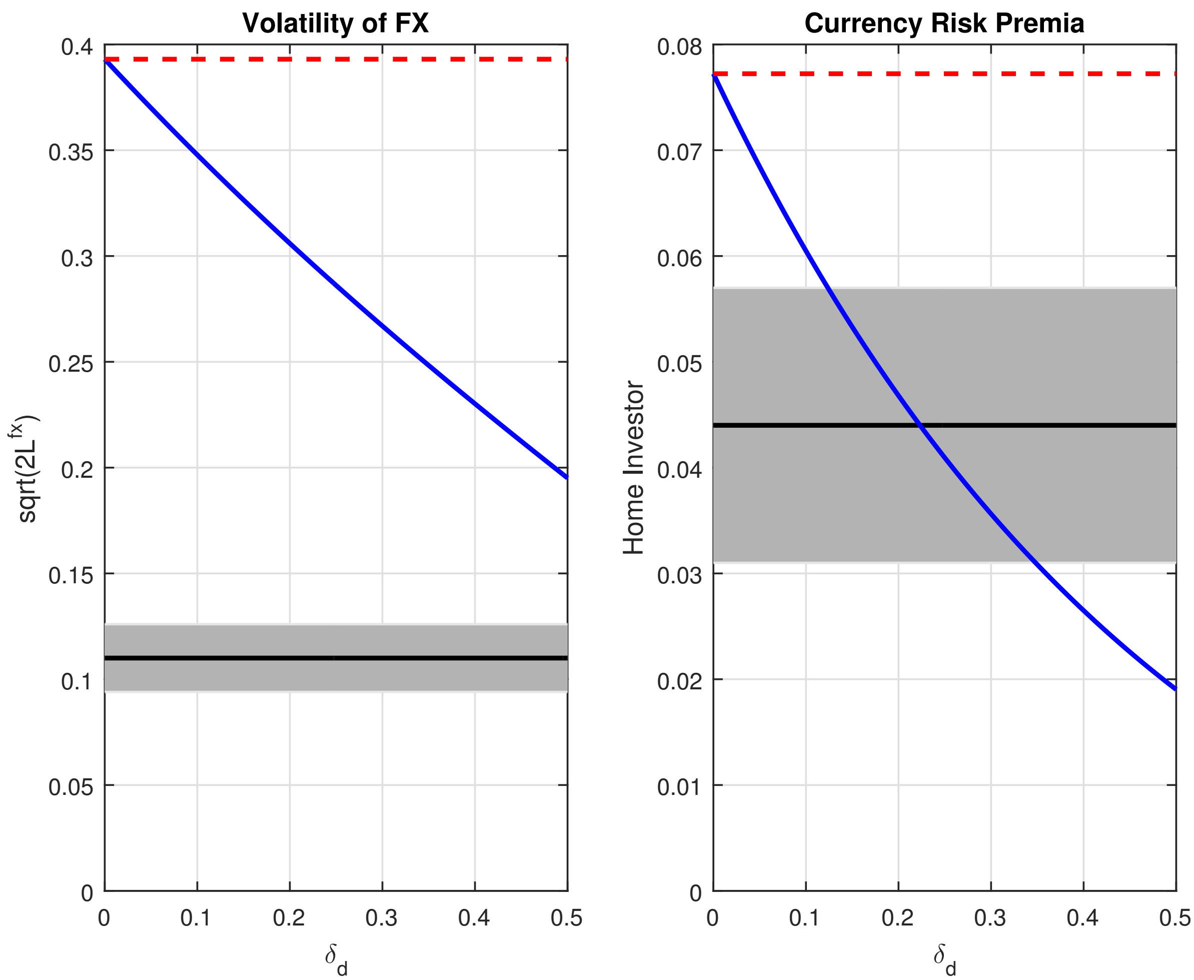

Figure 5: Overview: The figure reports the exchange rate volatility $\sqrt{2L}$ (where L denotes the entropy) and the currency risk premium in levels $E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right]$ for the admissible combinations of the jump parameters $\delta_d \leq 2\delta$. The correlation is given by $\rho_{z,d} = -\rho_{z^*,d} = -\frac{1}{2}\left(\frac{\delta_d}{\delta}\right)$. The parameter θ_d is zero because of symmetry. We follow Backus, Chernov, and Martin (2011) and set the risk aversion parameter to 5.19, the mean (μ) and standard deviation (σ) of the normal consumption growth shocks to 2.3% and 1%, the jump intensity (ϖ) to 1.7%, the mean jump size (θ) to -38%, and the jump size volatility (δ) to 25% The gray area represents the empirical counterpart of each moment.

model is isomorphic to the CIR model.

8.1 CIR Model with Country-specific factors

In discrete time, the simplest version of the CIR model is defined by the following two equations:

$$\begin{aligned} -\log M_{t+1} &= \alpha + \chi z_t + \sqrt{\gamma z_t} u_{t+1}, \\ z_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma \sqrt{z_t} u_{t+1}, \end{aligned}$$

where M denotes the home stochastic discount factor. The disturbances $u_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. over time.[6] The foreign stochastic discount factor follows a similar law of motion but with its own factor z_t^* and shocks u_{t+1}^*.

$$\begin{aligned} -\log M^*_{t+1} &= \alpha + \chi z^*_t + \sqrt{\gamma^* z^*_t} u^*_{t+1}, \\ z^*_{t+1} &= (1-\phi)\theta + \phi z^*_t - \sigma \sqrt{z_t} u^*_{t+1}, \end{aligned}$$

In this model, the maximum Sharpe ratios at home and abroad are $var_t(m_{t+1}) = \gamma z_t$, and $var_t(m^*_{t+1}) = \gamma^* z^*_t$, respectively. The real version of this CIR model with $\chi = 0$ is isomorphic to a model in which the domestic (foreign) representative agent has power utility preferences over consumption with CRRA coefficient $\sqrt{\gamma}$ ($\sqrt{\gamma^*}$) and aggregate consumption growth is heteroskedastic.

We assume that domestic investors can trade at least one risky domestic asset (e.g., a longer maturity bond) and the one-period risk-free bond, but they can only trade the foreign risk-free bond. They cannot trade any foreign risky assets. All domestic shocks are spanned, but not the foreign shocks.

The target volatility of the incomplete spanning exchange rate is: $var_t(\Delta s_{t+1}) = \kappa z_t +$

[6]In this model, log bond prices are affine in the state variable z_t: $p_t^{(n)} = -B_0^n - B_1^n z_t$. The price of a one period-bond is: $P^{(1)} = E_t\left(M_{t+1}\right) = e^{-\alpha-(\chi-\frac{1}{2}\gamma)z_t}$. Bond prices are defined recursively by the Euler equation: $P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)})$. Thus the bond price coefficients evolve according to the following second-order difference equations:

$$\begin{aligned} B_0^n &= \alpha + B_0^{n-1} + B_1^{n-1}(1-\phi)\theta, \\ B_1^n &= \chi - \frac{1}{2}\gamma + B_1^{n-1}\phi - \frac{1}{2}\left(B_1^{n-1}\right)^2\sigma^2 + \sigma\sqrt{\gamma}B_1^{n-1}. \end{aligned}$$

$\kappa^* z_t^*$. The implied volatility of the incomplete spanning exchange rate process is then equal to $var_t(\Delta s_{t+1}) = \gamma z_t + \gamma^* z_t^* - var_t(\eta_{t+1})$, which implies that the volatility of the wedge is $var_t(\Delta \eta_{t+1}) = (\gamma - \kappa) z_t + (\gamma^* - \kappa^*) z_t^*$. The following proposition defines the incomplete markets wedge that matches the desired volatility of the exchange rates while satisfying all the restrictions of Proposition 1.

Proposition 4. *In the CIR model with country-specific factors, we can define an exchange rate process S_t that satisfies $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$ with variance $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. where η_t follows:*

$$\begin{aligned} \eta_{t+1} = \psi z_t + \psi^* z_t^* \quad & - \quad \sqrt{(\gamma - \lambda) z_t} u_{t+1} + \sqrt{(\gamma^* - \lambda^*) z_t^*} u_{t+1}^* \\ & + \quad \sqrt{(\lambda - \kappa) z_t} \epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*) z_t^*} \epsilon_{t+1}^*. \end{aligned}$$

where $\epsilon_{t+1} \sim \mathbb{N}(0,1)$ and $\epsilon_{t+1}^* \sim \mathbb{N}(0,1)$ are i.i.d., and where the parameters λ and λ^* satisfy: $\kappa \leq \lambda \leq \gamma$, $\kappa^* \leq \lambda^* \leq \gamma^*$, as well as

$$\begin{aligned} \kappa^* &= \gamma^* - \sqrt{\gamma^*}\sqrt{\gamma^* - \lambda^*}; \psi^* = \frac{1}{2}(\gamma^* - \kappa^*), \\ \kappa &= \gamma - \sqrt{\gamma}\sqrt{\gamma - \lambda};\ \psi = -\frac{1}{2}(\gamma - \kappa). \end{aligned}$$

The incomplete markets wedges leave the domestic and foreign term structure unchanged. The term $(\gamma - \lambda)$ measures the exchange rate's exposure to spanned noise, while $(\lambda - \kappa)$ measures the exposure to unspanned noise. The drift term in the η process is not a free parameter, but is determined by the other parameters of the model. In the symmetric case, $\psi = -\psi^\star$, and the drift term is zero on average ($E[\mu_{t,\eta}] = 0$). Hence, on average, the wedge has no impact on exchange rates.

If we allow the domestic investor to trade any foreign risky bond, then the wedges are zero again: $\kappa = \gamma = \lambda$ and $\kappa^* = \gamma^* = \lambda^*$, because we need to impose two additional orthogonality conditions given by (7) between log returns and η.

Once we impose these dynamic no-arbitrage restrictions on the drift term, the effect on currency risk premia is unambiguous.

Corollary 6. *The risk premium in logs on a long position in foreign currency is:*

$$E_t[rx_{t+1}^{FX}] = \frac{1}{2}\left[\kappa z_t - \kappa^* z_t^*\right]$$

The risk premium in levels on a long position in foreign currency is always smaller in incomplete markets than in complete markets:

$$E_t[rx_{t+1}^{FX}] + \frac{1}{2} var_t[rx_{t+1}^{FX}] = \kappa z_t \leq \gamma z_t.$$

The Fama slope coefficient in a regression of log currency excess returns on the interest rate difference $r_t - r_t^*$ *is:*

$$\frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)} = \frac{.5\kappa(\chi - \frac{1}{2}\gamma) + .5\kappa^*(\chi - \frac{1}{2}\gamma^*)}{(\chi - \frac{1}{2}\gamma)^2 + (\chi - \frac{1}{2}\gamma^*)^2}$$

Recall that the complete markets log currency risk premium is given by $E_t[rx_{t+1}^{FX}] = \frac{1}{2}(\gamma z_t - \gamma^* z_t^*)$, while the complete markets currency risk premium in levels is given by γz_t. The risk premium in levels is always smaller in incomplete markets than in complete markets because $\kappa \leq \gamma$. If incomplete spanning reduces the standard deviation of exchange rates by 50% ($\sqrt{\kappa/\gamma} = 0.5$), then the currency risk premium is reduced by a factor of 0.25 (or 75%).[7] Since a real version of the CIR model is isomorphic to the Consumption-CAPM with heteroskedastic consumption growth, this result implies that incomplete spanning effectively reduces the representative agent's risk aversion coefficient when pricing currency risk, but not for other risk sources.

To illustrate the trade-off between exchange rate volatility and risk premia, we adopt the following parameters for the two countries: $\lambda_d = -1.07$, $\gamma = \lambda_d^2$, $\theta = 0.004428$, $\phi = 0.976$, $\alpha = 0$, $\chi = -1 + \lambda_d^2/2$, $\sigma = 0.008356$. These parameters are close to those used in Backus, Foresi, and

[7] As a result, currency SR's decrease as well, for all z_t, z_t^*:

$$\frac{E_t[rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}]}{std_t(\Delta s_{t+1})} = \frac{\kappa z_t}{\sqrt{\kappa z_t + \kappa z_t^*}} = \sqrt{\kappa}\frac{z_t}{\sqrt{z_t + z_t^*}} \leq \sqrt{\gamma}\frac{z_t}{\sqrt{z_t + z_t^*}}.$$

Telmer (1998): the only difference is that we defined $\chi = -1 + \lambda_d^2/2$ (instead of $\chi = 1 + \lambda_d^2/2$) in order to obtain counter-cyclical short-term interest rates, a necessary feature to replicate the U.I.P. puzzle in this class of models.

These parameters match the mean short-term interest rate rate, its volatility, and its auto-correlation. Figure 6 reports the annualized volatility of the exchange rate and the UIP slope coefficients for all admissible combinations of the parameters κ and λ. The first panel plots parameters κ and λ against the annualized $std_t(\eta_{t+1})$. The second panel plots the annualized volatility of the exchange rate. The third panel plots the UIP slope coefficient in a regression of log excess returns on the interest rate difference $r_t - r_t^*$. As the volatility of the wedge increases, the exchange rate volatility decreases. It can reach its empirical value, but only at the cost of driving the UIP slope coefficients to zero.

These conclusions do not depend on the country-specific nature of the factors. When we include common factors, we find that incomplete spanning still lowers the currency risk premium in levels and also forces the return predictability regression coefficient in a regression of log excess returns on the interest rate difference to zero. We analyze this general case with common factors in the Appendix.

8.2 CIR Model with Common Factors

The previous stylized model rules out correlation of interest rates across countries. However, the key insights carry over to a setting with correlated interest rates. To show this result, we use a CIR model with common factors. The Cox, Ingersoll, and Ross (1985) model (denoted CIR) is defined by the following two equations:

$$\begin{aligned} -\log M_{t+1} &= \alpha + \chi z_t + \varphi z_t^* + \sqrt{\gamma z_t} u_{t+1} + \sqrt{\delta z_t^*} u_{t+1}^*, && (8) \\ z_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma \sqrt{z_t} u_{t+1}, \\ z_{t+1}^* &= (1-\phi)\theta + \phi z_t - \sigma \sqrt{z_t^*} u_{t+1}^*, && (9) \end{aligned}$$

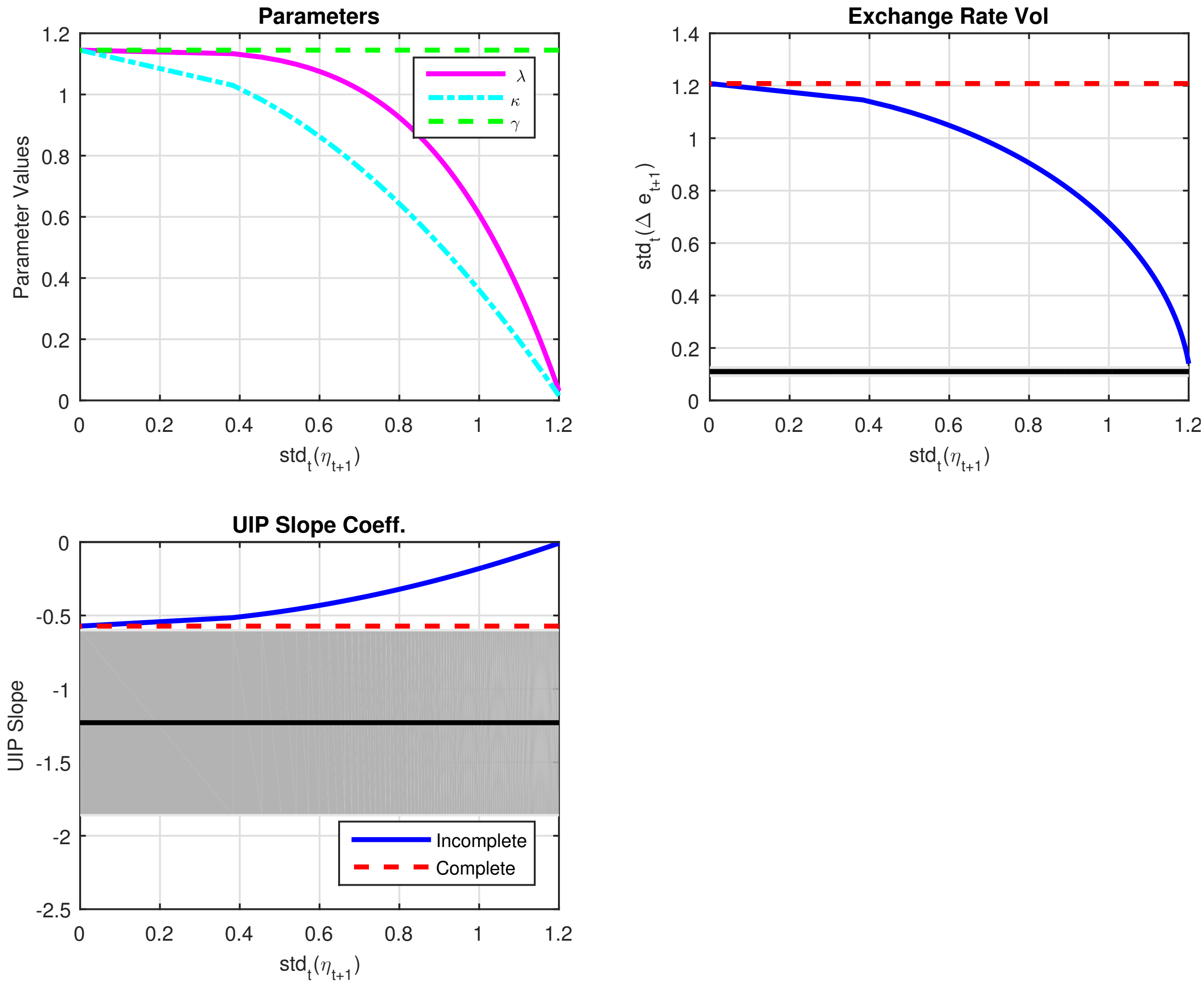

Figure 6: CIR Example: The figure reports the annualized volatility of the exchange rate and the UIP slope coefficients for all admissible combinations of (κ, λ). The first panel plots (κ, λ) against the annualized $std_t(\eta_{t+1})$. The second panel plots the annualized volatility of the exchange rate. The third panel plots the UIP slope coefficient in a regression of log excess returns on $f_t - s_t$. We adopt the following parameters: $\lambda_d = -1.07$, $\gamma = \lambda_d^2$, $\theta = 0.004428$, $\phi = 0.976$, $\alpha = 0$, $\chi = -1 + \lambda_d^2/2$, $\sigma = 0.008356$.

where $u_{t+1} \sim \mathbb{N}(0,1)$ and $u^*_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. The foreign pricing kernel is specified as in Equation (8) with the same parameters. However, the foreign country has different loadings:

$$-\log M_{t+1} = \alpha + \chi^* z_t + \varphi^* z^*_t + \sqrt{\gamma^* z_t} u_{t+1} + \sqrt{\delta^* z^*_t} u^*_{t+1}.$$

To give content to the notion that z_t is a domestic factor and z_t^* is a foreign factor, we assume that $\gamma \geq \gamma^*$ and that $\delta \leq \delta^*$: the domestic (foreign) pricing kernel is more exposed to the domestic (foreign) shock than the foreign (domestic) pricing kernel. We assume that investors can trade the domestic risk-free and at least two risky domestic assets[8], but they can only trade the foreign risk-free asset. The squared maximum SRs at home and abroad are, respectively, $var_t(m_{t+1}) = \gamma z_t + \delta z_t^*$, and $var_t(m_{t+1}^*) = \gamma^* z_t + \delta^* z_t^*$.

We denote the target volatility of the incomplete markets exchange rate can be stated as: $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. We can compute the implied volatility of the incomplete markets exchange rate process using our formula:

$$var_t(\Delta s_{t+1}) = (\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*})z_t + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*})z_t^* - var_t(\eta_{t+1}).$$

Then we simply choose the volatility of the noise to be equal to: $var_t(\eta_{t+1}) = (\gamma + \gamma^* - \kappa)z_t + (\delta + \delta^* - \kappa^*)z_t^*$.

Proposition 5. *In the CIR model with country-specific factors, we can define an exchange rate process S_t that satisfies $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$ with variance $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. where η_t follows:*

$$\begin{aligned} \eta_{t+1} &= \beta + \psi z_t + \psi^* z_t^* - \sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)z_t}u_{t+1} + \sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)z_t^*}u_{t+1}^* \\ &+ \sqrt{(\lambda - \kappa)z_t}\epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*)z_t^*}\epsilon_{t+1}^*, \end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^* \leq \lambda^* \leq \delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}$. The drift imputed to exchange rates is given by $\mu_{t,\eta} = \beta + \psi z_t + \psi^* z_t^*$. where ϵ_{t+1} and ϵ_{t+1}^* are

[8] If they can trade two different longer maturity bonds, then the domestically traded assets span all of the shocks.

$\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ *and* $\kappa^* \leq \lambda^* \leq \gamma^*$ *satisfies:*

$$
\begin{aligned}
\kappa &= -(\sqrt{\gamma}+\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda))}+\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}, \\
\kappa^* &= -(\sqrt{\delta}+\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*))}+\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}, \\
\psi &= -(1/2)(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}, \\
\psi^* &= -(1/2)(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}.
\end{aligned}
$$

If we allowed domestic investors to trade two foreign risky assets, then the wedges disappear. The additional covariance restrictions in (7) imply that $\kappa = \lambda = \gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^* = \lambda^* = \delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}$, because the log returns are affine in the shocks. This in turn implies that the wedges are zero ($\eta = 0$).

Corollary 7. *The risk premium in logs on a long position in foreign currency is:*

$$
\begin{aligned}
E_t[rx_{t+1}^{FX}] &= \frac{1}{2}\left[\gamma-\gamma^*-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z_t^*
\end{aligned}
$$

The risk premium in levels on a long position in foreign currency is given by:

$$
\begin{aligned}
E_t[rx_{t+1}^{FX}]+\frac{1}{2}var_t[rx_{t+1}^{FX}] &= \left[\gamma-\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}-\sqrt{\gamma}\sqrt{\gamma^*}\right]z_t \\
&+ \left[\delta-\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}-\sqrt{\delta}\sqrt{\delta^*}\right]z_t^*.
\end{aligned}
$$

These expressions can readily be compared to the complete markets log currency risk premium, $\frac{1}{2}\left[(\gamma-\gamma^*)z_t+(\delta-\delta^*)z_t^*\right]$, and the complete markets risk premium in levels, $(\gamma-\sqrt{\gamma}\sqrt{\gamma^*})z_t+(\delta-\sqrt{\delta}\sqrt{\delta^*})z_t^*$. Clearly, this establishes that the incomplete markets risk premium in levels is always smaller than the complete markets risk premium. In addition, there is less return predictability as well.

Corollary 8. *The Fama slope coefficient in a regression of log currency excess returns on* $f_t - s_t = r_t - r_t^*$ *is*

$$\frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)}$$
$$= \frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$
$$+ \quad \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

In the relevant region of the parameter space, $(\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*) < 0$ and $(\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*) > 0$. Then the interest rate spread $r_t - r_t^*$ decreases (increases) when z_t increases (z_t^* decreases) –the precautionary motive dominates. This is needed to account for U.I.P. deviations in the data. As a benchmark, we note that the complete markets slope coefficient is given by:

$$= \frac{.5\,(\gamma)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right) + .5\,(\delta)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

Recall that $\gamma \geq \gamma^*$ and $\delta \leq \delta^*$. As a result, the first term now decreases in absolute value relative to the complete markets case. The second term decreases as well in absolute value. Even in the model with common factors, the slope coefficients in the predictability regression are pushed closer to zero by the incomplete spanning and we get closer to U.I.P.

8.3 Dynamic CCAPM

To develop some economic intuition for the dynamics of these wedges, we look at a version of the two-country Lucas (1982) model with heteroskedastic consumption growth. This model produces time-varying risk premia. We use δ to denote the rate of time preference and γ to denote the coefficient of relative risk aversion. The real stochastic discount factor is thus given by:

$$\begin{aligned}
-\log M_{t+1} &= -(\log\delta - \gamma\mu_g) + \gamma\sigma_{g,t}e_{t+1}, \\
\sigma^2_{g,t} &= (1-\phi)\theta + \phi\sigma^2_{g,t} - \sigma_{g,t}e_{t+1}, \\
-\log M^*_{t+1} &= -(\log\delta - \gamma\mu_g) + \gamma\sigma^*_{g,t}e^*_{t+1}, \\
\sigma^{2,*}_{g,t} &= (1-\phi)\theta + \phi\sigma^{2,*}_{g,t} - \sigma^*_{g,t}e^*_{t+1}.
\end{aligned}$$

where $\Delta c_{t+1} = \mu_g + \sigma_{g,t}e_{t+1}$, and $\Delta c_{t+1} = \mu_g + \sigma^*_{g,t}e^*_{t+1}$. The consumption growth innovations $e_{t+1} \sim \mathbb{N}(0,1)$ and $e^*_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. as well as uncorrelated across countries. When markets are complete, the exchange rate variance is thus $var_t(\Delta s_{t+1}) = \gamma^2\sigma^2_{g,t} + \gamma^{2,*}\sigma^{2,*}_{g,t}$. Domestic investors can invest in the domestic risk-free and at least one domestic risky asset (e.g. a longer maturity real zero-coupon bond), and the foreign risk-free, but they cannot invest in foreign risky assets. Hence, only the domestic shocks are spanned.

In this model, we can back out the dynamic process for the wedges that satisfy the necessary conditions of Proposition 1. It turns out that all the wedges take the form:[9]

$$\begin{aligned}
\eta_{t+1} = \psi\sigma_{g,t} + \psi^*\sigma^*_{g,t} &- \sqrt{(\gamma^2-\lambda)}\sigma_{g,t}e_{t+1} + \sqrt{(\gamma^2-\lambda^*)}\sigma^*_{g,t}e^*_{t+1} \\
&+ \sqrt{(\lambda-\kappa)}\sigma_{g,t}\epsilon_{t+1} + \sqrt{(\lambda-\kappa^*)}\sigma^*_{g,t}\epsilon^*_{t+1}.
\end{aligned}$$

where ϵ_{t+1} and ϵ^*_{t+1} are standard i.i.d. Gaussian shocks uncorrelated with the consumption growth innovations e_{t+1} and e^*_{t+1}. These shocks are the unspanned component of the exchange rate changes. The parameters κ and κ^* govern the volatility of the exchange rate when markets are incomplete: $var_t(\Delta s_{t+1}) = \kappa\sigma^2_{g,t} + \kappa^*\sigma^{2,*}_{g,t}$. These wedges only affect exchange rates, and as a result, the returns on foreign investments. The returns on domestic investments remain unchanged.

The parameters κ and κ^* are the only two degrees of freedom in the law of motion of the wedge. The other parameters that describe the wedge are implied. The drift term (denoted $\mu_{t,\eta}$ in Proposition 1 and here equal to $\psi\sigma_{g,t} + \psi^*\sigma^*_{g,t}$) is governed by the consumption growth

[9]This result is a special case of Proposition 4 in Section 8. The proof is in the Appendix.

volatilities; it is determined by the no-arbitrage conditions, which imply that $\psi = -\frac{1}{2}(\gamma^2 - \kappa)$, and $\psi^* = \frac{1}{2}(\gamma^2 - \kappa^*)$. The unexpected component of the wedge depends on the parameters λ and λ^*, which have to satisfy the following restrictions: $\kappa \leq \lambda \leq \gamma^2$ and $\kappa^* \leq \lambda^* \leq \gamma^2$, and are implicitly defined by the following conditions: $\kappa = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2 - \lambda}$, $\kappa^* = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2 - \lambda^*}$.

In this example, the domestic investor cannot invest in any foreign risky asset. If we allow the foreign investor to do so, then we need to impose the additional covariance restrictions in condition (7). These conditions imply that η is orthogonal to e_{t+1} and e^*_{t+1}, because the log return on the domestic (foreign) risky asset is affine in the domestic (foreign) innovation, which in turn implies $\kappa = \lambda = \gamma^2$ and $\kappa^* = \lambda^* = \gamma^{*,2}$. We are back in the case of complete markets: $\eta_{t+1} = 0$.

In the two-country Lucas (1982) model, incomplete spanning reduces the exchange rate's exposure to the consumption growth innovations. Instead, the exchange rates are now exposed to shocks uncorrelated with aggregate consumption growth in either country. In the following sections, we study the impact of incomplete spanning on each of the key three exchange rate puzzles without restricting ourselves to the Lucas (1982) model. We start with the Brandt, Cochrane, and Santa-Clara (2006) puzzle.

The two-country Lucas (1982) model with heteroskedastic consumption growth provides a simple laboratory for understanding the effects of incompleteness. In that model, the complete markets risk premium in logs on a long position in foreign currency is: $E_t[rx^{FX}_{t+1}] = \frac{1}{2}\gamma^2\left[\sigma^2_{g,t} - \sigma^{2,*}_{g,t}\right]$, while the complete markets risk premium in levels is given by: $E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = \gamma^2\sigma^2_{g,t}$. In the incomplete spanning economy, the risk premium in logs on a long position in foreign currency is $E_t[rx^{FX}_{t+1}] = \frac{1}{2}\kappa\left[\sigma^2_{g,t} - \sigma^{2,*}_{g,t}\right]$, while the risk premium in levels on a long position in foreign currency is given by: $E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = \kappa\sigma^2_{g,t}$. The incomplete markets model behaves as if risk aversion γ was effectively reduced to $\sqrt{\kappa}$. There is also less return predictability in the incomplete spanning economy.[10] The Fama slope coefficient in a regression of log currency excess returns on $f_t - s_t = r_t - r^*_t$ is $-2\kappa/\gamma^2$. Hence, the slope coefficient falls below 2, its complete markets value, in absolute value. The percentage reduction in

[10] The proofs are a special case of Corollary 8.

the slope coefficient is twice the percentage reduction in volatility $2\log(\sqrt{\frac{\kappa}{\gamma^2}})$.

9 Conclusion

Our paper investigates whether incomplete spanning in international financial markets can account for the behavior of exchange rates. To attack this question, we allow for a great deal of incompleteness by only enforcing the Euler equations in FX forward and futures markets. To help resolve the currency volatility and correlation puzzles, the quantity of unspanned risk needed in currency markets is of the same size as the maximum Sharpe ratio. Adding this amount of noise to exchange rates shrinks all currency risk premia to zero in violation of large body of empirical evidence from currency markets. This type of incompleteness can only help quantitatively if we ignore the evidence from asset markets that long positions in foreign exchange markets, at least for some currency pairs, are considered risky.

Models that segment international currency markets by only allowing only financial intermediaries (e.g. Gabaix and Maggiori (2015)) or a subset of investors (see, e.g., Chien, Lustig, and Naknoi (2015) and Dou and Verdelhan (2015)) to trade a complete menu of international securities show more promise . These models sever the link between aggregate quantities and real exchange rates by concentrating aggregate risk among a small pool of investors, but these segmented markets models face a challenging measurement test, i.e. to show that changes in exchange rates are highly correlated with marginal utility growth of these market participants.

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Appendix

Section A presents the proof of our main general results. Section B studies a consumption-based example with and without disasters.

A Proofs

Proof of Proposition 1:

Proof. We start from the domestic investor's Euler equation for the foreign risk-free asset, and the foreign investor's Euler equation for the domestic risk-free asset respectively:

$$E_t\left(\widehat{M}^*_{t+1}\right) = E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = 1/R_t^{f,*},$$
$$E_t\left(M_{t+1}\right) = E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = 1/R_t^{f,}.$$

By using conditional joint log normality of the foreign SDF and $exp(\eta)$, the first Euler equation implies that:

$$E_t\left(\log M^*_{t+1}\right) + \frac{1}{2}Var_t\left(\log M^*_{t+1}\right) = E_t\left(\log M^*_{t+1}\right) + \mu_{t,\eta} + \frac{1}{2}Var_t\left(\log M^*_{t+1}\right) + \frac{1}{2}Var_t\left(\eta_{t+1}\right) + covar_t(\eta_{t+1}, \log M^*_{t+1}),$$

where $\mu_{t,\eta} = E_t\left(\eta_{t+1}\right)$. This implies that $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - 0.5var_t\left(\eta_{t+1}\right)$. We move on to the second equation. The second Euler equation for the domestic risk-free asset implies that:

$$E_t\left(\log M_{t+1}\right) + \frac{1}{2}Var_t\left(\log M_{t+1}\right) = E_t\left(\log M_{t+1}\right) - \mu_{t,\eta} + \frac{1}{2}Var_t\left(\log M_{t+1}\right) + (1/2)Var_t\left(\eta_{t+1}\right) - covar_t(\eta_{t+1}, \log M_{t+1}).$$

This implies that $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$.

The inequality restrictions on $\mu_{t,\eta}$ follow directly from the Cauchy-Schwarz inequality for (1) $\left|covar_t\left(m^*_{t+1}, \eta_{t+1}\right)\right| \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right)$ and (2) $\left|covar_t\left(m_{t+1}, \eta_{t+1}\right)\right| \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right)$. Finally, we also impose that (3):

$$\left|covar_t\left(m^*_{t+1} - m_{t+1}, \eta_{t+1}\right)\right| \leq std_t\left(m^*_{t+1} - m_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta} \leq -(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$-(\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right)) \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$-(\mu_{t,\eta}) \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right) + \frac{1}{2}var_t\left(\eta_{t+1}\right)).$$

When $\mu_{t,\eta} \geq -(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right) \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta} \leq std_t\left(m^*_{t+1}\right) std_t\left(\eta_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right).$$

Next, we turn to the second inequality. When $\mu_{t,\eta} \geq (1/2)var_t\left(\eta_{t+1}\right)$, the second inequality implies that:

$$\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right) \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta} \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right) + \frac{1}{2}var_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta} \leq (1/2)var_t\left(\eta_{t+1}\right)$, the second inequality implies that:

$$-(\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right)) \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$-\mu_{t,\eta} \leq std_t\left(m_{t+1}\right) std_t\left(\eta_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right).$$

Finally, the third inequality implies that:

$$std_t\left(\eta_{t+1}\right) \leq std_t\left(m^*_{t+1} - m_{t+1}\right).$$

□

Proof of Corollary 2:

Proof. We start from the definition of log changes in exchange rates: $var_t(\Delta s_{t+1}) = var_t(\eta_{t+1} + m^*_{t+1} - m_{t+1})$. This can be simplified to:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) + var_t(\eta_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- 2cov_t(m_{t+1}, \eta_{t+1}) + 2cov_t(\eta_{t+1}, m^*_{t+1}). \end{aligned}$$

Proposition 1 implies that:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- var_t\left(\eta_{t+1}\right) - var_t\left(\eta_{t+1}\right) + var_t(\eta_{t+1}), \end{aligned}$$

which establishes the result. Finally, we prove the volatility results. The volatility of the log pricing kernel in the foreign country is given by

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2covar_t(m^*_{t+1}, \eta_{t+1}).$$

The result follows directly from the covariance condition. Note that $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right)$.

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2(-\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right)).$$

□

Proof of Corollary 3

Proof. The expression for the log risk premium follows because $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - var_t\left(\eta_{t+1}\right)/2$. The expression for the risk premium in level follows because $var_t[rx^{FX}_{t+1}]/2 = var_t(\Delta s_{t+1})/2$ which is given by:

$$\frac{1}{2}var_t(m_{t+1}) + \frac{1}{2}var_t(m^*_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) - \frac{1}{2}var_t(\eta_{t+1}).$$

The log risk premium is increased by $\mu_{t,\eta}$ relative to the complete markets case. The foreign investor's log risk premium on domestic currency is naturally the opposite of the one above. The symmetry does not hold in levels because of the usual Jensen term. The foreign investor's risk premium in levels on a long position in domestic currency is given by:

$$E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] = cov_t(m^*_{t+1}, \Delta s_{t+1}) = var_t\left(m^*_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - \mu_{t,\eta}.$$

□

Proof of Corollary 4

Proof. This result follows immediately from Proposition 1. We subtract the second $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$ from the first covariance condition $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} + 0.5var_t\left(\eta_{t+1}\right)$. That delivers the results. □

Proof of Proposition 4:

Proof. We need to implement the following conditions:

$$\begin{aligned} covar_t\left(m^*_{t+1},\eta_{t+1}\right) &= -\mu_{t,\eta}-\frac{1}{2}var_t\left(\eta_{t+1}\right), \\ covar_t\left(m_{t+1},\eta_{t+1}\right) &= -\mu_{t,\eta}+\frac{1}{2}var_t\left(\eta_{t+1}\right), \end{aligned}$$

Using the expression for the SDF, we obtain the following conditions:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^*-\lambda^*)}z^*_t &= -(\psi z_t+\psi^* z^*_t)-\frac{1}{2}\left((\gamma-\kappa)z_t+(\gamma^*-\kappa^*)z^*_t\right), \\ +\sqrt{\gamma}\sqrt{(\gamma-\lambda)}z_t &= -(\psi z_t+\psi^* z^*_t)+\frac{1}{2}\left((\gamma-\kappa)z_t+(\gamma^*-\kappa^*)z^*_t\right). \end{aligned}$$

These conditions imply that:

$$\begin{aligned} \psi^* &= \frac{1}{2}(\gamma^*-\kappa^*), \\ \psi &= -\frac{1}{2}(\gamma-\kappa). \end{aligned}$$

as well as:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^*-\lambda^*)} &= -\psi^*-\frac{1}{2}\left(\gamma^*-\kappa^*\right)=-\left(\gamma^*-\kappa^*\right), \\ +\sqrt{\gamma}\sqrt{(\gamma-\lambda)} &= -\psi+\frac{1}{2}\left(\gamma-\kappa\right)=\left(\gamma-\kappa\right), \end{aligned}$$

where we have used the expressions for the ψ's. This delivers the following end result:

$$\begin{aligned} \gamma^*-\sqrt{\gamma^*}\sqrt{(\gamma^*-\lambda^*)} &= \kappa^*, \\ \gamma-\sqrt{\gamma}\sqrt{(\gamma-\lambda)} &= \kappa. \end{aligned}$$

□

Proof of Corollary 6:

Proof. The risk premium in logs on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] &= r^{f,*}_t-r^f_t+E_t(\Delta s_{t+1})=\frac{1}{2}\left[var_t\left(m_{t+1}\right)-var_t\left(m^*_{t+1}+\eta_{t+1}\right)\right] \\ &= \frac{1}{2}\left[(\gamma+2\psi)z_t-(\gamma^*-2\psi^*)z^*_t\right]. \\ &= \frac{1}{2}\left[(\gamma-(\gamma-\kappa))z_t-(\gamma^*-(\gamma^*-\kappa^*))z^*_t\right] \\ &= \frac{1}{2}\left[\kappa z_t-\kappa^* z^*_t\right] \end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}]+\frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1},\Delta s_{t+1}) \\ &= \frac{1}{2}\left[(\kappa+\kappa)z_t-(\kappa^*-\kappa^*)z^*_t\right] \\ &= \kappa z_t \end{aligned}$$

Recall that the short rate is given by: $r_t=\alpha+(\chi-\frac{1}{2}\gamma)z_t$. Hence, the regression slope coefficient on $f_t-s_t=r_t-r^*_t$ is

$$\frac{cov(rx^{FX}_{t+1},f_t-s_t)}{var(f_t-s_t)}=\frac{.5\kappa(\chi-\frac{1}{2}\gamma)+.5\kappa^*(\chi-\frac{1}{2}\gamma^*)}{(\chi-\frac{1}{2}\gamma)^2+(\chi-\frac{1}{2}\gamma^*)^2}$$

Hence, in the symmetric case, we end up with:

$$\frac{.5\kappa}{(\chi - \frac{1}{2}\gamma)}$$

□

Proof of proposition 5:

Proof. Hence, we can write a square root process for η:

$$\begin{aligned}\eta_{t+1} &= \beta + \psi z_t + \psi^* z_t^* - \sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)z_t}u_{t+1} + \sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)z_t^*}u_{t+1}^* \\ &+ \sqrt{(\lambda - \kappa)z_t}\epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*)z_t^*}\epsilon_{t+1}^*,\end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ and $\kappa^* \leq \lambda^* \leq \gamma^*$. The drift imputed to exchange rates is given by $\mu_{t,\eta} = \beta + \psi z_t + \psi^* z_t^*$.

To ensure that the Euler equations for the risk-free are satisfied, we also need to implement the following conditions:

$$\begin{aligned}covar_t\left(m_{t+1}^*, \eta_{t+1}\right) &= -\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right), \\ covar_t\left(m_{t+1}, \eta_{t+1}\right) &= -\mu_{t,\eta} + \frac{1}{2}var_t\left(\eta_{t+1}\right).\end{aligned}$$

Using the expressions for the log SDFs and η, these expressions can be restated as follows:

$$\begin{aligned}- &\sqrt{\gamma^*}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)}z_t - \sqrt{\delta^*}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)}z_t^* \\ = &-(\psi z_t + \psi^* z_t^*) - \frac{1}{2}\left((\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \kappa)z_t + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa^*)z_t^*\right), \\ + &\sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)}z_t + \sqrt{\delta}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda)}z_t^* \\ = &-(\psi z_t + \psi^* z_t^*) + \frac{1}{2}\left((\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \kappa)z_t + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa^*)z_t^*\right).\end{aligned}$$

By collecting the terms in z_t and z_t^*, we obtain the following four equations that need to be solved for 4 unknowns:

$$\begin{aligned}-\sqrt{\gamma^*}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} &= -(\psi) - \frac{1}{2}(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \kappa), \\ -\sqrt{\delta^*}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} &= -(\psi^*) - \frac{1}{2}(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa^*).\end{aligned}$$

$$\begin{aligned}+\sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} &= -(\psi) + \frac{1}{2}(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \kappa) \\ +\sqrt{\delta}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa)} &= -(\psi^*) + \frac{1}{2}(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa^*).\end{aligned}$$

By adding the 1st and 3rd, and the 2nd and 4th equation, we obtain the following expression for the drift terms:

$$\begin{aligned}(\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda} &= -2\psi), \\ (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*} &= -2\psi^*.\end{aligned}$$

By substituting for ψ and ψ^* in the original four equations, we obtain the following conditions:

$$\begin{aligned}+\sqrt{\gamma}\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} &= +\frac{1}{2}(\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda)} + \frac{1}{2}(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \kappa), \\ -\sqrt{\delta^*}\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} &= +\frac{1}{2}(\sqrt{\delta} - \sqrt{\delta^*})\sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*)} + \frac{1}{2}(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \kappa^*).\end{aligned}$$

These conditions can be solved for κ and κ^*:

$$\begin{aligned}
\kappa &= -2\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}),\\
\kappa^* &= +2\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}).
\end{aligned}$$

These conditions imply that:

$$\begin{aligned}
\kappa &= -(\sqrt{\gamma}+\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)})+\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*},\\
\kappa^* &= -(\sqrt{\delta}+\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)})+\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}.
\end{aligned}$$

□

Proof of Corollary 7:

Proof. Note that the risk premium in logs is given by

$$\begin{aligned}
E_t[rx^{FX}_{t+1}] &= r^{f,*}_t - r^f_t + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}+\eta_{t+1}\right)\right]\\
&= \frac{1}{2}\left[(\gamma-\gamma^*+2\psi)z_t + (\delta-\delta^*+2\psi^*)z^*_t)\right]\\
&= \frac{1}{2}\left[\gamma-\gamma^*-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t\\
&+ \frac{1}{2}\left[\delta-\delta^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z^*_t
\end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned}
E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1})\\
&= \frac{1}{2}\left[(\gamma-\gamma^*+2\psi+\kappa)z_t + (\delta-\delta^*+2\psi^*+\kappa^*)z^*_t)\right]\\
&= \frac{1}{2}\left[\gamma-\gamma^*+\kappa-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t\\
&+ \frac{1}{2}\left[\delta-\delta^*+\kappa^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z^*_t,\\
&= \frac{1}{2}\left[\gamma-\gamma^*-2\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*})\right]z_t\\
&+ \frac{1}{2}\left[\delta-\delta^*+2\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*})\right]z^*_t\\
&= \left[\gamma-\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}-\sqrt{\gamma}\sqrt{\gamma^*}\right]z_t\\
&+ \left[\delta-\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}-\sqrt{\delta}\sqrt{\delta^*}\right]z^*_t
\end{aligned}$$

□

Proof of Corollary 8:

Proof. Recall that the short rate is given by: $r_t = \alpha + (\chi - \frac{1}{2}\gamma)z_t + (\phi - \frac{1}{2}\delta)z^*_t$. Hence, the regression slope coefficient on $f_t - s_t = r_t - r_t^*$ is

$$\frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)} =$$

$$\frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

$$+ \quad \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}$$

□

Proof of Proposition 2:

Proof. By definition, the conditional entropy of a random variable X_{t+1} is equal to:

$$L_t(X_{t+1}) = \log E_t(X_{t+1}) - E_t(\log X_{t+1})$$

We assume here that both investors have access to risk-free rates. Let us start again from the Euler equation of the foreign investor:

$$\frac{1}{R_t^{f,*}} = E_t\left(M_{t+1}^* \exp(\eta_{t+1})\right)$$

Taking logs leads to:

$$-r_t^{f,*} = \log E\left(M_{t+1}^* \exp(\eta_{t+1})\right) = L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) + E_t\left(\log M_{t+1}^*\right) + E_t(\eta_{t+1}).$$

But the risk-free rate also satisfies the Euler equation $E\left(M_{t+1}^* R_t^{f,*}\right) = 1$. Taking logs again leads to:

$$\log E\left(M_{t+1}^* R_t^{f,*}\right) = L\left(M_{t+1}^* R_t^{f,*}\right) + E_t\left(\log M_{t+1}^*\right) + r_t^{f,*} = 0$$

Plugging the implied value of the log risk-free rate in the first equation above delivers the result, noting that $L_t(a_t X_{t+1}) = a_t L_t(X_{t+1})$ for any variable a_t known at date t:

$$L\left(M_{t+1}^*\right) + E_t\left(\log M_{t+1}^*\right) = L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) + E_t\left(\log M_{t+1}^*\right) + E_t(\eta_{t+1}),$$

which simplifies to:

$$L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) = L\left(M_{t+1}^*\right) - E_t(\eta_{t+1}).$$

Likewise, one can show that:

$$L_t\left(M_{t+1} \exp(-\eta_{t+1})\right) = L\left(M_{t+1}\right) + E_t(\eta_{t+1}).$$

Finally, we derive restrictions the set of feasible $\mu_{t,\eta}$ from non-negativity of $L_t\left(M_{t+1} \exp(-\eta_{t+1})\right)$, $L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right)$ and $L_t\left(\frac{S_{t+1}}{S_t}\right)$. To start, note that:

$$L_t\left(M_{t+1} \exp(-\eta_{t+1})\right) = \log E_t\left(M_{t+1} \exp(\eta_{t+1})\right) - E_t \log\left(M_{t+1}\right) + E_t(\eta_{t+1}) \geq 0$$

$$L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) = \log E_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) - E_t \log\left(M_{t+1}^*\right) - E_t(\eta_{t+1}) \geq 0$$

This implies that the following restrictions need to be satisfied:

$$-\mu_{t\eta} \leq \log E_t\left(M_{t+1} \exp(-\eta_{t+1})\right) - E_t \log\left(M_{t+1}\right).$$

$$\mu_{t\eta} \leq \log E_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) - E_t \log\left(M_{t+1}^*\right),$$

which in turn implies that:

$$-\left(\log E_t\left(M_{t+1} \exp(-\eta_{t+1})\right) - E_t \log\left(M_{t+1}\right)\right) \leq \mu_{t\eta} \leq \log E_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) - E_t \log\left(M_{t+1}^*\right)$$

Finally, we also know that

$$L_t\left(\frac{S_{t+1}}{S_t}\right) = -E_t(\eta_{t+1}) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq 0$$

This, in turn, implies that:

$$\log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t \log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq \mu_{t\eta}$$

□

Proof of Corollary 5:

Proof. Note that the entropy of the ratio of two random variables is:

$$\begin{aligned} L_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) &= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) - E_t(\log X_{t+1}) + E_t(\log Y_{t+1}) \\ &= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) + L_t(X_{t+1}) - \log E_t(X_{t+1}) - L(Y_{t+1}) + \log E_t(Y_{t+1}). \end{aligned}$$

By applying this fomula to the following expression with $X_{t+1} = M^*_{t+1}/M_{t+1}$ and $Y_{t+1} = M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]$, we obtain

$$\begin{aligned} L_t\left(e^{-\eta_{t+1}}\right) &= L_t\left(\frac{M^*_{t+1}/M_{t+1}}{M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]}\right) \\ &= L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right), \end{aligned}$$

This last step leads to the result in the text as the second term is the entropy of the change in exchange rates.

□

Proof of Proposition 3:

Proof. The first result just follows from the definition of the log change in the exchange rate and the definition of the risk-free rate at home and abroad. The second result follows immediately because $E_t[rx^{FX}_{t+1}] + L_t(rx^{FX}_{t+1}) = E_t[rx^{FX}_{t+1}] + L_t(S_{t+1}/S_t)$; only S_{t+1}/S_t is random. □

Proof of Equation (**??**):

Proof.

$$\begin{aligned} \Delta RP &= RP^{IM} - RP^{CM} = -L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) \\ &= -L_t\left(e^{-\eta_{t+1}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) + \mu_{t,\eta} \\ &= -\log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) = \Delta L + \mu_{t,\eta}. \end{aligned}$$

The second line uses the entropy of a ratio of two random variables. □

B Consumption-Based Examples With and Without Disasters

We start with a simple lognormal Consumption-CAPM example and then study deviations from log normality by introducing jumps.

B.1 Consumption-Based Example Without Disasters

Complete Markets We start from the complete market benchmark. The model is described in the main text.

Result 3. *The complete markets foreign currency risk premium in levels (defined from the perspective of the home investor) is given by:*

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

The proof of Result 3 is as follows.

Proof. The entropy of the domestic pricing kernel is given by:

$$L_t(M_{t+1}) = L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = \frac{\gamma^2\sigma^2}{2}.$$

As a result, the entropy of the exchange rate is:

$$L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = L_t(e^{-\gamma w_{t+1}^* + \gamma w_{t+1}}) = \frac{\gamma^2\sigma^{*2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

When markets are complete, the log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned} E_t\left[rx_{t+1}^{FX}\right] &= -L_t\left(M_{t+1}^*\right) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c_{t+1}^*}) + L_t(e^{-\gamma\Delta c_{t+1}}) \\ &= -L_t(e^{-\gamma w_{t+1}^*}) + L_t(e^{-\gamma w_{t+1}}) = -\frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2}. \end{aligned}$$

As a result, the currency risk premium in levels (defined from the perspective of the home investor) is given by:

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_{t+1}}{S_t}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

Likewise, the currency risk premium in levels (defined from the perspective of the foreign investor) is given by:

$$-E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_t}{S_{t+1}}\right] = \gamma^2\sigma^{*2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

□

Incomplete markets Next, we introduce incomplete spanning. Assume that the wedge takes the form $\eta_{t+1} = \gamma d_{t+1}$, where $d \sim N(\mu_d, \sigma_d^2)$.

Result 4. *The wedge has to satisfy the following conditions:*

$$\begin{aligned} \mu_d &= \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d, \\ -\mu_d &= \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d. \end{aligned}$$

The change in exchange rate variance from complete to incomplete spanning is given by:

$$\Delta Var_t = Var_t^{IM} - Var_t^{CM} = -\gamma^2\sigma_d^2.$$

The change in the currency risk premium (defined from the perspective of the home investor) from complete to incomplete spanning is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \rho_{w,d}\gamma^2\sigma\sigma_d.$$

The change in the currency risk premium (defined from the perspective of the foreign investor) from complete to incomplete spanning is:

$$\Delta RP_t^* = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Result 4 implies that in the symmetric case (when the drift of the wedge is zero), the change in the currency risk premium in level is $\Delta RP_t = \Delta RP_t^* = -.5\gamma^2\sigma_d^2$. In that case, introducing a wedge decreases the currency risk premium from the perspective of both domestic and foreign agents. The Sharpe ratio declines as well:

$$SR_t^{FX} = \frac{\gamma}{\sqrt{2}}\sqrt{\sigma^2(1-\rho) - \frac{\sigma_d^2}{2}}.$$

The proof of Result 4 is as follows:

Proof. The conditional entropy of the perturbed home pricing kernel is given by:

$$L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) = L_t(e^{-\gamma\Delta c_{t+1}-\gamma d_{t+1}}) = L_t(e^{-\gamma w_{t+1}-\gamma d_{t+1}}) = \frac{\gamma^2\sigma^2}{2} + \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d.$$

Applying Proposition 2, it then implies that the drift of the wedge satisfies:

$$\mu_e = \frac{\gamma^2\sigma_d^2}{2} + \rho_{w,d}\gamma^2\sigma\sigma_d.$$

The conditional entropy of the perturbed foreign pricing kernel is equal to:

$$L_t\left(M_{t+1}^*e^{\eta_{t+1}}\right) = L_t(e^{-\gamma\Delta c_{t+1}^*+\gamma d_{t+1}}) = L_t(e^{-\gamma w_{t+1}^*+\gamma d_{t+1}}) = \frac{\gamma^2\sigma^{2,*}}{2} + \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Proposition 2 then implies that the drift of the wedge satisfies:

$$-\mu_d = \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

When markets are incomplete, the entropy of the 'incomplete spanning' exchange rate is given by:

$$\begin{aligned} L_t\left(\frac{M_{t+1}^*e^{\eta_{t+1}}}{M_{t+1}}\right) &= L_t(e^{-\gamma\Delta c_{t+1}^*+\gamma d_{t+1}+\gamma\Delta c_{t+1}}) = L_t(e^{-\gamma w_{t+1}^*+\gamma w_{t+1}+\gamma d_{t+1}}) \\ &= \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\sigma^*\sigma_d\rho_{w^*,d} + \gamma^2\sigma\sigma_d\rho_{w,d} - \gamma^2\rho_{w,w^*}\sigma\sigma^* + \frac{\gamma^2\sigma_d^2}{2}. \end{aligned}$$

By summing the two conditions that define the drift of the wedge, one obtains that:

$$0 = \gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

The entropy of the 'incomplete spanning' exchange rate is thus simply:

$$L_t\left(\frac{M_{t+1}^*e^{\eta_{t+1}}}{M_{t+1}}\right) = \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^* - \frac{\gamma^2\sigma_d^2}{2}.$$

The entropy gap between the complete and incomplete spanning exchange rate is then:

$$\Delta L_t = L_t\left(\frac{M_{t+1}^*e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = -\frac{\gamma^2\sigma_d^2}{2}.$$

According to Proposition 3, the risk premium in levels on a long position in foreign currency is given by :

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right) - L_t\left(M_{t+1}^*\right) + \mu_{t,\eta} + L_t\left(\frac{M_{t+1}^*e^{\eta_{t+1}}}{M_{t+1}}\right).$$

The change in the risk premium from complete to incomplete spanning is thus given by the change in entropy, $-.5\gamma^2\sigma_d^2$, plus the drift term, $\mu_d = 0.5\gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d$. The difference between the risk premium in incomplete and complete markets is:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) = \rho_{w,d}\gamma^2\sigma\sigma_d.$$

Similarly, the foreign risk premium in levels on a long position in foreign currency is given by :

$$-E_t\left[rx^{FX}_{t+1}\right] + L_t\left(\frac{S_t}{S_{t+1}}\right) = -L_t\left(M_{t+1}\right) + L_t\left(M^*_{t+1}\right) - \mu_{t,\eta} + L_t\left(\frac{M_{t+1}}{M^*_{t+1}e^{\eta_{t+1}}}\right).$$

The change in the foreign risk premium from complete to incomplete spanning is given by:

$$\Delta RP^*_t = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Proposition 2 implies that the restrictions on the wedges are given by:

$$\begin{aligned} \mu_d &= \gamma^2\sigma_d^2/2 + \rho_{w,d}\gamma^2\sigma\sigma_d, \\ -\mu_d &= \gamma^2\sigma_e^2/2 - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d. \end{aligned}$$

□

B.2 Consumption-Based Example with Disasters

Complete Markets We start again from the complete market benchmark. The model is described in the main text. The proof of Result 1 is as follows:

Proof. The conditional entropy of the pricing kernel M_{t+1} is equal to:

$$\begin{aligned} L_t(M_{t+1}) &= L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}}\right) \\ &= \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi\theta. \end{aligned}$$

The entropy of the jump component is presented in Equation (24), page 1981 of Backus, Chernov, and Zin (2011) and derived in their Appendix A. The entropy of the 'complete spanning' exchange rate is given by:

$$\begin{aligned} L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= L_t\left(e^{-\gamma(\Delta c^*_{t+1} - \Delta c_{t+1})}\right) = L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t\left(e^{-\gamma z^*_{t+1} + \gamma z_{t+1}}\right) + L_t\left(e^{\gamma w_{t+1}}\right), \\ &= \frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma\gamma^*\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right) + \gamma^*\varpi\theta^* - \gamma\varpi\theta. \end{aligned}$$

The log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned} E_t\left[rx^{FX}_{t+1}\right] &= -L_t(M^*_{t+1}) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c^*_{t+1}}) + L_t(e^{-\gamma\Delta c_{t+1}}), \\ &= -L_t(e^{-\gamma w^*_{t+1}}) - L_t(e^{-\gamma z^*_{t+1}}) + L_t(e^{-\gamma w_{t+1}}) + L_t(e^{-\gamma z_{t+1}}), \\ &= -\frac{\gamma^{2,*}\sigma^{*,2}}{2} - \varpi\left(e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} - 1\right) \\ &+ \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta + (\gamma\delta)^2/2} - 1\right) - (\gamma^*\varpi\theta^* - \gamma\varpi\theta). \end{aligned}$$

Hence, the foreign currency risk premium in levels is given by:

$$\begin{aligned} E_t\left[rx^{FX}_{t+1}\right] + L_t\left[rx^{FX}_{t+1}\right] &= \gamma^2\sigma^2 + \varpi\left(e^{-\gamma\theta + (\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} - 1\right) \\ &+ \varpi\left(e^{-\gamma\theta^* + \gamma\theta - 2\gamma\gamma^*\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right). \end{aligned}$$

□

Incomplete markets Next, we introduce incomplete spanning as described in the main text. The proof of Result 2 is as follows:

Proof. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned}L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c_{t+1}-\gamma e_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right), \\ &= \gamma^2\sigma^2/2 + \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2}-1\right) + \gamma\varpi(\theta+\theta_d)\end{aligned}$$

The entropy of the sum of two Poisson mixtures ($L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right)$ above) is a generalization of the result presented in Backus, Chernov, and Zin (2011). The co-entropy condition in Proposition 2, $\mu_{t,\eta} = L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right)$, implies here that:

$$\begin{aligned}\gamma\varpi\theta_d &= L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right) \\ &= \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,e}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2}-1\right) - \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2}-1\right) + \gamma\varpi\theta_d.\end{aligned}$$

Simplifying, we obtain:

$$0 = e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - e^{-\gamma\theta+(\gamma\delta)^2/2}.$$

This leads to:

$$-\gamma(\theta+\theta_d) + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 + (\gamma\delta)^2/2 = -\gamma\theta + (\gamma\delta)^2/2.$$

This is equivalent to the following restriction on the wedge:

$$-\gamma\theta_d + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 = 0.$$

Next, we turn to the foreign pricing kernel. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned}L_t\left(M^*_{t+1}e^{\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c^*_{t+1}+\gamma d_{t+1}}\right) = L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t(e^{-\gamma z_{t+1}+\gamma d_{t+1}}) \\ &= \gamma^2\sigma^{2,*}/2 + \varpi^*\left(e^{-\gamma(\theta^*-\theta^*_e)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+(\gamma\delta^*_d)^2/2+(\gamma\delta^*)^2/2}-1\right) + \gamma\varpi(\theta^*) - \gamma\varpi(\theta_d)\end{aligned}$$

The co-entropy condition in Proposition 2, $-\mu_{t,\eta} = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - L\left(M^*_{t+1}\right)$, implies here that:

$$\left[1 - e^{\gamma\theta_d-\gamma^2\delta\delta_d\rho_{z^*,d}+(\gamma\delta_d)^2/2)}\right]\varpi e^{-\gamma\theta^*+(\gamma\delta^*)^2/2} = 0.$$

This is equivalent to the following condition:

$$\gamma\theta_d - \gamma^2\delta^*\delta_d\rho_{z^*,d} + (\gamma\delta_d)^2/2 = 0.$$

Collecting all of the no-arbitrage restrictions, we obtain the conditions first described in Result 2:

$$\begin{aligned}-\gamma\theta_e + \gamma^2\delta\delta_e\rho_{z,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma\theta_e - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma^2\delta\delta_e\rho_{z,e} - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0.\end{aligned}$$

The third condition is implied by the first two conditions.

We turn now to the entropy of the exchange rate. When markets are incomplete, the exchange rate's entropy is given by:

$$L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) = L_t\left(e^{-\gamma\Delta c^*_{t+1}+\gamma d_{t+1}+\gamma\Delta c_{t+1}}\right),$$
$$= L_t\left(e^{-\gamma w^*_{t+1}}\right)+L_t\left(e^{\gamma w_{t+1}}\right)+L_t\left(e^{-\gamma z^*_{t+1}+\gamma z_{t+1}+\gamma d_{t+1}}\right),$$
$$= \frac{\gamma^2\sigma^{*,2}}{2}+\frac{\gamma^2\sigma^2}{2}+\gamma^*\varpi^*\theta^*-\gamma\varpi\theta-\gamma\varpi\theta_d$$
$$+\varpi\left(e^{\gamma(\theta+\theta_d-\theta^*)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta_d)^2}{2}+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right).$$

The entropy gap between the complete and incomplete spanning exchange rate is thus:

$$L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right)-L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) = \varpi\left(e^{\gamma(\theta+\theta_d-\theta^*)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta_d)^2}{2}+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right)$$
$$- \gamma\varpi\theta_d-\varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+(\gamma\delta)^2/2+(\gamma\delta^*)^2/2}-1\right)$$

Using the no-arbitrage condition on the wedges $\gamma\theta_d=\gamma^2\delta^*\delta_d\rho_{z^*,d}-(\gamma\delta_d)^2/2=0$, we obtain the following result:

$$L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right)-L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) = \varpi\left(e^{\gamma(\theta-\theta^*)+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right)$$
$$- \gamma\varpi\theta_d\varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right).$$

This can be restated as :

$$\Delta L_t = L_t^{IM}-L_t^{CM} = L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right)-L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right)$$
$$= -\gamma\varpi\theta_d+\varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}\right)(e^{\gamma^2\delta\delta_d\rho_{z,d}}-1).$$

This is the second part of Result 2. Taking into account the no-arbitrage conditions on the wedge, when the wedge does not have a drift ($\theta_d=0$) and the two countries share the same parameters ($\theta=\theta^*$, $\delta=\delta^*$), we obtain:

$$\Delta L_t=\varpi\left(e^{-\gamma^2\rho_{z,z^*}\delta^2+(\gamma\delta)^2}\right)(e^{(-\gamma^2\delta_e^2}-1)<0.$$

Finally, we turn to the risk premium in levels on a long position in foreign currency, which is given by :

$$E_t\left[rx^{FX}_{t+1}\right]+L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right)-L_t\left(M^*_{t+1}\right)+\mu_{t,\eta}+L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right).$$

Hence, the change in the risk premium from complete to incomplete spanning is given by the change in entropy, $L_t^{IM}-L_t^{CM}$, plus the drift term: $\gamma\varpi\theta_d$. As a result, the change in the risk premium is given by:

$$\Delta RP_t=RP_t^{IM}-RP_t^{CM}=\varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}\right)(e^{\gamma^2\delta\delta_d\rho_{z,d}}-1).$$

This is the third part of Result 2. □

Table 2: Exchange Rate Entropy

	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$L(\Delta s)$	0.64	0.15	0.19	0.80
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma^2_{\Delta s}$	0.64	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$L(\Delta q)$	0.63	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma^2_{\Delta q}$	0.63	0.16	0.19	0.82
	(0.05)	(0.02)	(0.03)	(0.08)

Notes: The table reports summary statistics on exchange rate entropy and volatility. The entropy, denoted $L(\Delta s)$, is measured as the log of the mean change in exchange rate minus the mean of the log change in exchange rate: $L(\Delta s) = logE\left(e^{\Delta s}\right) - E(log\Delta s)$. The volatility is measured as half the variance of the log change in exchange rates. Similar moments are defined for real exchange rates. The table presents the cross-country mean of the bilateral nominal and real exchange rate volatilities, along with the cross-country standard deviation of the bilateral exchange rate volatilities and the corresponding minimum and maximum values across countries. Similar statistics are reported for entropies. Moments are annualized (multiplied by 4) and reported in percentages. Data are quarterly, over the 1973.IV – 2014.IV period. The panel consists of 15 countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S. The standard errors (reported between brackets) were generated by block-bootstrapping 10,000 samples, each block containing 2 quarters.