Bootstrapping high-frequency jump tests*

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Abstract

In this paper, we consider bootstrap jump tests based on functions of realized volatility and bipower variation, as originally proposed by Barndorff-Nielsen and Shephard (2006). Our aim is to improve the finite sample size of the asymptotic theory-based tests while retaining good power. In order to do so, we generate the bootstrap observations under the null of no jumps, by drawing them randomly from a mean zero Gaussian distribution with a variance given by a local measure of integrated volatility (which we call $\{\hat{v}_i^n\}$).

Our first contribution is to give a set of high level conditions on $\{\hat{v}_i^n\}$ such that any bootstrap method of this form has the asymptotic correct size and is alternative-consistent. We then verify these high level conditions for a specific example of $\{\hat{v}_i^n\}$ based on the product of L multipowers of local realized volatility estimates, each of them computed over M consecutive non-overlapping intraday returns. We show that this choice satisfies our high level conditions under both the null and the alternative hypothesis of jumps when the maximum of the multipowers is strictly less than $1/2$. This is equivalent to letting $L > 2$ when the multipowers are all equal to $1/L$. When $L \leq 2$, the bootstrap is able to mimic the null distribution only under the null of "no jumps". In particular, we cannot guarantee that it is alternative-consistent when $L = 1$ and $M = 1$, which corresponds to the standard wild bootstrap based on a Gaussian external random variable. Our simulations confirm that this choice has very poor finite sample properties. The simulations also show that by appropriately choosing M and L, we can greatly reduce the overrejections that are typically associated with the Barndorff-Nielsen and Shephard (2006) tests without compromising power.

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1 Introduction

A well accepted fact in financial economics is the fact that asset prices do not always evolve continuously over a given time interval, being instead subject to the possible occurrence of jumps (or discontinuous

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movements in prices). The detection of such jumps is crucial for asset pricing and risk management because the presence of jumps has important consequences for the performance of asset pricing models and hedging strategies, often introducing parameters that are hard to estimate (see e.g. Bakshi et al. (1997), Bates (1996), and Johannes (2004)). For this reason, many tests for jumps have been proposed in the literature over the years, most of the recent ones exploiting the rich information contained in high frequency data. These include tests based on bipower variation measures (such as in Barndorff-Nielsen and Shephard (2004, 2006), henceforth BN-S (2004, 2006), Huang and Tauchen (2005), Andersen et al. (2007), Jiang and Oomen (2008), and more recently Mykland, Shephard and Sheppard (2012)); tests based on power variation measures sampled at different frequencies (such as in Aït-Sahalia and Jacod (2009), Aït-Sahalia, Jacod and Li (2012)), and tests based on the maximum of a standardized version of intraday returns (such as in Lee and Mykland (2008, 2012)). In addition, tests based on thresholding or truncation-based estimators of volatility have also been proposed with the objective of disentangling big from small jumps, as in Aït-Sahalia and Jacod (2009) and Cont and Mancini (2011), based on Mancini (2001). See Aït-Sahalia and Jacod (2012, 2014) for a review of the literature on the econometrics of high frequency-based jump tests.

In this paper, we focus on the class of tests based on bipower variation originally proposed by Barndorff-Nielsen and Shephard (2004, 2006). Our main contribution is to propose a bootstrap implementation of these tests with better finite sample properties than the original tests based on the asymptotic normal distribution. In particular, our aim is to improve finite sample size while retaining good power. In order to do so, we generate the bootstrap observations under the null of no jumps, by drawing them randomly from a mean zero Gaussian distribution with a variance given by a local measure of integrated volatility (which we call $\{\hat{v}_i^n\}$).

Our first contribution is to give a set of high level conditions on $\{\hat{v}_i^n\}$ such that any bootstrap method of this form has the asymptotic correct size and is alternative-consistent. We then verify these high level conditions for a specific example of $\{\hat{v}_i^n\}$ based on the product of L multipowers of local realized volatility estimates, each of them computed over M consecutive non-overlapping intraday returns. When $L = 1$, this corresponds to the local Gaussian bootstrap of Hounyo (2013), who proposed this method for inference on integrated volatility under no jumps.

We show that this choice of $\{\hat{v}_i^n\}$ satisfies our high level conditions under both the null *and* the alternative hypothesis of jumps when the maximum of the exponents $\{p_l\}$ used to compute $\hat{v}_i^n$ is strictly less than 1/2. This is equivalent to letting $L > 2$ when $p_l = 1/L$ for $l = 1, \ldots, L$. Thus, under these conditions, the bootstrap is able to mimic the null distribution of the jump test under the null of no jumps as well as under the alternative of jumps. This ensures that the bootstrap test has the correct size asymptotically and is consistent under the alternative. Crucial for this result is the fact that the bootstrap variance of the test is consistent under both the null and the alternative hypothesis. As it turns out, this variance is a function of (efficient) blocked multipower variation measures (cf. Mykland, Shephard and Sheppard (2012), henceforth MSS (2012)) with multipowers given by up to four times the exponents $\{p_l\}$ used to compute $\hat{v}_i^n$. Since these measures are robust to jumps only when we impose the restriction that the multipowers are strictly less than two (here, implied by the condition $4\max(p_l) < 2$), this explains why we obtain the condition $\max(p_l) < 1/2$.

When $\max(p_l) > 1/2$, or equivalently $L < 2$ when all exponents are the same, the bootstrap variance of the test diverges under the alternative of jumps and we can only show that the bootstrap is able to mimic the null distribution in restriction to the null of "no jumps". Although this result implies that the bootstrap has the correct asymptotic size for $L < 2$, its power may be adversely affected. In particular, we show that this is the case when $L = 1$ and $M = 1$, which corresponds to the standard wild bootstrap based on a Gaussian external random variable. For these choices of L and M, the bootstrap test statistic diverges at the same rate as the original test statistic, implying that it might not be alternative-consistent. This is confirmed by our simulations, which show that this choice has very poor finite sample power properties. The simulations also show that by appropriately choosing M and L, we can greatly reduce the overrejections that are typically associated with the Barndorff-Nielsen and Shephard (2006) tests without compromising power.

The rest of the paper is organized as follows. In Section 2, we provide the framework and state our assumptions. In Section 3, we introduce a general bootstrap method for testing for jumps based on the Gaussian wild bootstrap and a general local volatility measure $\{\hat{v}_i^n\}$. After providing examples of $\{\hat{v}_i^n\}$, we give a set of high level conditions on $\{\hat{v}_i^n\}$ such that any such bootstrap method is asymptotically valid when testing for jumps. We end this section with two lemmas that are crucial to proving the remaining results. In Section 4, we verify our high level conditions for the choice of $\{\hat{v}_i^n\}$ introduced by Hounyo (2013) and show that they are verified only under no jumps. For the special case of $M = 1$, we show that this choice of $\{\hat{v}_i^n\}$ implies the divergence of the bootstrap test statistic under the alternative of jumps. For this reason, in Section 5 we discuss a generalization of the local Gaussian bootstrap of Hounyo (2013) that leads to bootstrap tests that have correct asymptotic size and are alternative-consistent. Section 6 gives simulations while Section 7 provides an empirical application. Section 8 concludes. All proofs are relegated to Appendices A and B. Appendix C contains formulas for the log version of our tests.

Finally, a word on notation. As usual in the bootstrap literature, we let P^* describe the probability of bootstrap random variables, conditional on the observed data. Similarly, we write E^* and Var^* to denote the expected value and the variance with respect to P^*, respectively. For any bootstrap statistic $Z_n^* \equiv Z_n^*(\cdot,\omega)$ and any (measurable) set A, we write $P^*(Z_n^* \in A) = P^*(Z_n^*(\cdot,\omega) \in A) = \Pr(Z_n^*(\cdot,\omega) \in A|\mathcal{X}_n)$, where $\mathcal{X}_n$ denotes the observed sample. We say that $Z_n^* \to^{P^*} 0$ in prob-P (or $Z_n^* = o_{P^*}(1)$ in prob-P) if for any $\varepsilon, \delta > 0$, $P(P^*(|Z_n^*| > \varepsilon) > \delta) \to 0$ as $n \to \infty$. Similarly, we say that $Z_n^* = O_{P^*}(1)$ in prob-P if for any $\delta > 0$, there exists $0 < M < \infty$ such that $P(P^*(|Z_n^*| \geq M) > \delta) \to 0$ as $n \to \infty$. For a sequence of random variables (or vectors) Z_n^*, we also need the definition of convergence in distribution in prob-P. In particular, we write $Z_n^* \to^{d^*} Z$, in prob-P (a.s.-P), if $E^*(f(Z_n^*)) \to E(f(Z))$ in prob-P for every bounded and continuous function f (a.s.$-P$).

2 Assumptions and statistics of interest

We assume that the log-price process $(X_t)_{t\geq 0}$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that

$$X_t = Y_t + J_t,\ t \geq 0, \tag{1}$$

where Y_t is a continuous Brownian semimartingale process and J_t is a jump process. Specifically, Y_t is defined by the equation

$$Y_t = Y_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s,\ t \geq 0, \tag{2}$$

where a is a predictable locally bounded drift term, σ is an adapted càdlàg spot volatility process and W is a standard Brownian motion that is not necessarily independent of σ (i.e. we allow for leverage effects).

Although the asymptotic properties of the jump tests studied in this paper have been established under very general conditions on a_t, σ_t and J_t (including leverage effects, possible jumps on σ_t and infinite activity processes; see e.g. Barndorff-Nielsen, Shephard and Winkel (2006)), we require stronger assumptions to prove our bootstrap results. In particular, we rule out jumps in σ_t and assume that J_t is a finite activity jump process. Formally, we make the following additional assumptions.

Assumption 1 σ_t is locally bounded away from zero and is a continuous semimartingale.

Assumption 2 J_t is a finite activity process defined as $J_t = \sum_{j=1}^{N_t} c_j$, $t \geq 0$, where c_j represents the size of the jth jump and N_t is a counting process representing the number of jumps up to time t. We assume that c_j has a continuous distribution at 0 for all j, and N_t is finite for all t.

These assumptions are used by Mykland, Shephard and Sheppard (2012) to derive the asymptotic properties (consistency and CLT) of blocked multipower variation measures. By imposing these assumptions, we are able to build on their results when proving our bootstrap results.

The quadratic variation process of X is given by $[X]_t = IV_t + JV_t$, where $IV_t \equiv \int_0^t \sigma_s^2 ds$ is the quadratic variation of Y_t, also known as the integrated volatility, and $JV_t \equiv \sum_{s\leq t} (\Delta J_s)^2$ is the jump quadratic variation, with $\Delta J_s = J_s - J_{s-}$ denoting the jumps in X. Without loss of generality, we let $t = 1$ and we omit the index t. For instance, we write $IV = IV_1$ and $JV = JV_1$.

We assume that prices are observed within the fixed time interval $[0, 1]$ (which we think of as a given day) and that the log-prices X_t are recorded at regular time points $t_i = i/n$, for $i = 0, \ldots, n$, from which we compute n intraday returns at frequency $1/n$,

$$r_i \equiv X_{i/n} - X_{(i-1)/n},\ i = 1, \ldots, n,$$

where we omit the index n in r_i to simplify the notation.

Our focus is on testing for "no jumps" using the bootstrap. In particular, following Aït-Sahalia and Jacod (2009), we would like to decide on the basis of the observed intraday returns $\{r_i : i = 1, \ldots, n\}$ in which of the two following complementary sets the path we actually observed falls:

$$\begin{aligned} \Omega_0 &= \{\omega : t \longmapsto X_t(\omega) \text{ is continuous on } [0,1]\} \\ \Omega_1 &= \{\omega : t \longmapsto X_t(\omega) \text{ is discontinuous on } [0,1]\}, \end{aligned}$$

where $\Omega = \Omega_0 \cup \Omega_1$ and $\Omega_0 \cap \Omega_1 = \varnothing$. Formally, our null hypothesis can be defined as $H_0 : \omega \in \Omega_0$ whereas the alternative hypothesis is $H_1 : \omega \in \Omega_1$.

Let $RV_n = \sum_{i=1}^n r_i^2$ denote the realized volatility and let

$$BV_n = \frac{1}{k_{1,1}^2} \sum_{i=2}^n |r_{i-1}| \, |r_i|$$

be the bipower variation, where we let $k_{1,1} \equiv E\left(\left|\chi_1^2\right|^{1/2}\right) = E(|Z|) = \sqrt{2}/\sqrt{\pi}$, where $Z \sim N(0,1)$. This is a special case of

$$k_{M,q} \equiv E\left(\left|\chi_M^2\right|^{q/2}\right) = 2^{q/2} \frac{\Gamma\left(\frac{M+q}{2}\right)}{\Gamma\left(\frac{M}{2}\right)},\ q > 0,$$

where χ_M^2 is the standard χ^2 distribution with M degrees of freedom and Γ is the gamma function. Writing $\chi_M \equiv \left(\chi_M^2\right)^{1/2}$ yields $k_{M,q} = E\left(\chi_M^q\right)$. Note that for $M = 1$, $k_{1,q} = E(|Z|^q)$, with $Z \sim N(0,1)$.

The class of statistics we consider is based on the comparison between RV_n and BV_n. It is now well known that under certain regularity conditions including the assumption that X is continuous (BN-S (2006) in conjunction with, e.g., Barndorff-Nielsen, Graversen, Jacod, and Shephard (2006)) that the following joint CLT holds:

$$\sqrt{n} \begin{pmatrix} RV_n - IV \\ BV_n - IV \end{pmatrix} \xrightarrow{st} N(0, \Sigma), \tag{3}$$

where $\xrightarrow{st}$ denotes stable convergence and

$$\Sigma = \begin{pmatrix} 2 & 2 \\ 2 & \theta \end{pmatrix} IQ, \tag{4}$$

with $IQ \equiv \int_0^1 \sigma_u^4 du$ and $\theta = \left(k_{1,1}^{-4} - 1\right) + 2\left(k_{1,1}^{-2} - 1\right) \simeq 2.6090$.

An implication of (3) is that under "no jumps", i.e. in restriction to Ω_0,

$$\frac{\sqrt{n}\left(RV_n - BV_n\right)}{\sqrt{V}} \xrightarrow{st} N\left(0,1\right),$$

where $V \equiv \tau \cdot IQ$ is the asymptotic variance of $\sqrt{n}\left(RV_n - BV_n\right)$ and $\tau = \theta - 2$. Hence, a linear version of the test is given by

$$T_n = \frac{\sqrt{n}\left(RV_n - BV_n\right)}{\sqrt{\hat{V}_n}}, \tag{5}$$

where $\hat{V}_n \equiv \tau \cdot \widehat{IQ}_n$ and $\widehat{IQ}_n$ is a consistent estimator of integrated quarticity IQ.

Among the many estimators of IQ that are robust to jumps, here we focus on the tripower realized quarticity defined as

$$\widehat{IQ}_n = \frac{n}{k_{1,\frac{4}{3}}^3} \sum_{i=3}^{n} \left|r_i\right|^{4/3} \left|r_{i-1}\right|^{4/3} \left|r_{i-2}\right|^{4/3}. \tag{6}$$

We can show that $\widehat{IQ}_n \xrightarrow{P} IQ$ on both Ω_0 and Ω_1. Thus, $T_n \xrightarrow{st} N\left(0,1\right)$, in restriction to Ω_0, and the test that rejects the null of "no jumps" at significance level α whenever $T_n > z_{1-\alpha}$, where $z_{1-\alpha}$ is the $100\left(1-\alpha\right)\%$ percentile of the $N\left(0,1\right)$ distribution has asymptotically correct size. More formally, the critical region $\mathcal{C}_n = \{T_n > z_{1-\alpha}\}$ is such that for any measurable set $S \subset \Omega_0$ such that $P(S) > 0$,

$$\lim_{n\to\infty} P\left(\omega \in \mathcal{C}_n | S\right) = \alpha.$$

Under the alternative hypothesis, we can show that the test T_n is alternative-consistent, i.e. the probability that we make the incorrect decision of "accepting the null" when this is false goes to zero:

$$\lim_{n\to\infty} P\left(\Omega_1 \cap \bar{\mathcal{C}}_n\right) = 0,$$

where $\bar{\mathcal{C}}_n$ is the complement of $\mathcal{C}_n$. Since the above condition implies that $P\left(\bar{\mathcal{C}}_n | \Omega_1\right) \to 0$, as $n \to \infty$, we have that $P\left(\mathcal{C}_n | \Omega_1\right) \to 1$ as $n \to \infty$, which we can interpret as saying that the test has asymptotic power equal to 1.

3 A general bootstrap method

We generate bootstrap intraday returns as

$$r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i, \qquad i = 1, \ldots, n, \tag{7}$$

for some variance measure $\hat{v}_i^n$ based on $\{r_i : i = 1, \ldots, n\}$, and where η_i is generated independently of the data as an i.i.d. $N\left(0,1\right)$ random variable. For simplicity, we again write r_i^* instead of $r_{i,n}^*$.

According to (7), bootstrap intraday returns are conditionally (on the original sample) Gaussian with mean zero and volatility $\hat{v}_i^n$. This bootstrap DGP is motivated by the simplified model $X_t = \int_0^t \sigma_s dW_s$, where σ is independent of W and there is no drift nor jumps. Under these assumptions, conditionally on the path of volatility, $r_i \sim N\left(0, v_i^n\right)$, where $v_i^n = \int_{(i-1)/n}^{i/n} \sigma_u^2 du$ independently across i. Thus, we can think of $\hat{v}_i^n$ as the sample analogue of v_i^n. Although (7) is motivated by this very simple model, as we will prove below, this does not prevent the bootstrap method to be valid more generally. In particular, its validity extends to the case where there is a leverage effect and the drift is non-zero.

The main feature of notice is that (7) generates bootstrap pseudo-returns that are (conditionally) Gaussian and therefore do not contain jumps. Since our goal is to approximate the distribution of jump tests under the null hypothesis of no jumps, this feature is not only natural, but it is important to minimize the probability of a type I error. In particular, Davidson and MacKinnon (1999) (see also MacKinnon (2009)) show that in order to minimize the error in rejection probability under the null (type I error) of a bootstrap test, we should estimate the bootstrap DGP as efficiently as possible. This entails imposing the null hypothesis on the bootstrap DGP.

Our main goal in this section is to discuss a set of high level conditions on $\hat{v}_i^n$ such that any bootstrap method based on (7) is valid when testing for jumps using the BN-S test statistic. Asymptotic validity here means that the bootstrap replicates the null distribution of the test statistic under the null *and* the alternative hypothesis.

The class of bootstrap statistics that we consider can be described as

$$T_n^* = \frac{\sqrt{n}\left(RV_n^* - BV_n^* - E^*\left(RV_n^* - BV_n^*\right)\right)}{\sqrt{\hat{V}_n^*}}, \tag{8}$$

where

$$RV_n^* = \sum_{i=1}^{n} r_i^{*2} \text{ and } BV_n^* = \frac{1}{k_{1,1}^2} \sum_{i=2}^{n} \left|r_{i-1}^*\right| \left|r_i^*\right|$$

denote the bootstrap analogues of RV_n and BV_n. For a given choice of $\hat{v}_i^n$, the bootstrap expectation $E^*\left(RV_n^* - BV_n^*\right)$ is a known function of the data (see Lemma 3.1 below). Similarly, $\hat{V}_n^*$ denotes an estimator of the bootstrap variance $V_n^* = Var^*\left(\sqrt{n}\left(RV_n^* - BV_n^*\right)\right)$, whose form depends on the choice of $\hat{v}_i^n$ (cf. (11) below for the exact definition of $\hat{V}_n^*$).

In this context, a bootstrap test rejects the null of no jumps whenever $T_n > q_{1-\alpha}^*$, where $q_{1-\alpha}^*$ is the $100\left(1-\alpha\right)\%$ quantile of the bootstrap distribution of T_n^*. Next we provide general conditions on $\hat{v}_i^n$ under which $T_n^* \xrightarrow{d^*} N\left(0,1\right)$ in prob-P independently of whether $\omega \in \Omega_0$ or $\omega \in \Omega_1$. This ensures that the bootstrap test controls size and is consistent under the alternative. Before introducing these conditions, we first provide several examples of $\hat{v}_i^n$ which can be used to implement (7).

3.1 Examples of $\hat{v}_i^n$

Throughout, we let $M \in \mathbb{N}$ and assume that n/M is an integer. Although the estimate $\hat{v}_i^n$ depends on M, we do not make this dependence explicit in order to simplify the notation. Note that M is a fixed constant that does not grow with n. Hence, $\hat{v}_i^n$ is not a consistent estimate of v_i^n. Consistency of $\hat{v}_i^n$ is not required for bootstrap validity of jump tests, as our results in the next section make clear. Instead, what is required is consistency of multipower variation measures of $\hat{v}_i^n$. Fixing M rather than letting it grow with n has the advantage that the bootstrap statistics reflect the choice M and this improves finite sample performance of the bootstrap jump tests.

Example 1 (Local RV estimate) Let $M = 1, 2, \ldots$. For $j = 1, \ldots, n/M$, set
$\hat{v}_{i+(j-1)M}^n = \frac{1}{M}\sum_{\ell=(j-1)M+1}^{jM} r_\ell^2 \equiv \bar{R}_j$, for all $i = 1, \ldots, M$.

The main idea is that we split the original sample into non-overlapping blocks of size M and estimate v_i^n within a given block by a local realized volatility measure computed over the M intraday returns. When $M = 1$, $\bar{R}_j = r_j^2$ and we get $\hat{v}_j^n = r_j^2$, for all $j = 1, \ldots, n$, we obtain the wild bootstrap of Gonçalves and Meddahi (2009) based on a Gaussian external random variable. When M is larger than one, we obtain the local Gaussian bootstrap of Hounyo (2013), who related it to the conditional Gaussianity approach of Mykland and Zhang (2009).

As we will see in the next section, Example 1 does not mimic the null distribution of the test when the alternative is true. To understand this, note that the asymptotic normality of T_n^* requires the convergence of V_n^*. This variance depends on

$$Var^*\left(\sqrt{n}RV_n^*\right) = 2n\sum_{i=1}^{n}\left(\hat{v}_i^n\right)^2,$$

which for Example 1 with $M = 1$ is proportional to the standard realized quarticity estimator,

$$2n\sum_{i=1}^{n}\left(\hat{v}_i^n\right)^2 = 2n\sum_{i=1}^{n} r_i^4.$$

As is well known in the literature, this estimator is not robust to jumps. In particular, it diverges to $+\infty$ under H_1. In Section 4 we show that this may have a negative impact on the power of the bootstrap test. To solve this problem, we propose the following choice of $\hat{v}_i^n$.

Example 2 (Multipower local RV estimate) Let $M = 1, 2, \ldots$, $L = 1, 2, \ldots$, and $\{p_l : l = 1, \ldots, L\}$ such that $\sum_{l=1}^{L} p_l = 1$, where $p_l \geq 0$. For $j = L, \ldots, n/M$, set $\hat{v}_{i+(j-1)M}^n = \prod_{l=1}^{L} \bar{R}_{j-l+1}^{p_l}$, for all $i = 1, \ldots, M$.

Example 2 generalizes Example 1 by multiplying together a finite number of local RV estimates raised to some non-negative power. When $L = 1$, Example 2 contains Example 1 as a special case.

To understand why Example 2 ensures the convergence of V_n^* under H_1 consider the simplest case where $M = 1$. For any $L \geq 1$, we obtain

$$\hat{v}_j^n = |r_j|^{2p_1}\,|r_{j-1}|^{2p_2}\ldots|r_{j-L+1}|^{2p_L}\,, \quad \text{for } j = L, \ldots, n.$$

In this case,

$$n\sum_{j=1}^{n}\left(\hat{v}_j^n\right)^2 = n\sum_{j=L}^{n}|r_j|^{4p_1}\,|r_{j-1}|^{4p_2}\ldots|r_{j-L+1}|^{4p_L}$$

is a traditional multipower variation measure. This is a consistent estimator of (a multiple of) IQ under both H_0 and H_1 when we choose p_l such that $\max\left(4p_l\right) < 2$. If we use equal powers $p_l = \frac{1}{L}$, this restriction is satisfied when $L > 2$. The same idea generalizes to $M > 1$, with the difference that the bootstrap variance of the test becomes a function of blocked multipower variation measures. These measures were recently introduced by MSS (2012) as a way of improving the efficiency of the traditional measures. Thus, Example 2 combines the bootstrap with the efficient blocked multipower variation measures as a way of improving the finite sample properties of jump tests under both the null and the alternative hypothesis.

Example 3 (Truncated squared return) We let $\hat{v}_i^n = r_i^2 I_{\{|r_i| \leq \alpha/n^{\varpi}\}}$, $\alpha > 0$, $0 < \varpi < \frac{1}{2}$, where $I_{\{\cdot\}}$ is the usual indicator function.

In Example 3, we exclude all returns containing jumps over a given threshold when computing $\hat{v}_i^n$. See e.g., Mancini (2001) and Aït-Sahalia and Jacod (2009), among others for similar truncated-based statistics. The thresholding ensures the convergence of V_n^* under both H_0 and H_1, as can be verified using results by Jacod and Protter (2012). For brevity, we will not discuss this example in detail and focus instead on Examples 1 and 2. However, we will provide some simulation evidence on the three examples in Section 6.

Next, we provide a set of high level conditions on $\hat{v}_i^n$ such that any such choice is asymptotically valid when estimating the null distribution of the jump tests of BN-S.

3.2 Bootstrap validity under general conditions on $\hat{v}_i^n$

We first provide a set of conditions under which a joint bootstrap CLT holds for $(RV_n^*, BV_n^*)'$. In particular, we would like to establish that

$$\Sigma^{*-1/2}\sqrt{n}\begin{pmatrix} RV_n^* - E^*(RV_n^*) \\ BV_n^* - E^*(BV_n^*) \end{pmatrix} \xrightarrow{d^*} N(0, I_2),$$

in prob-P, where Σ^* is the probability limit of

$$\Sigma_n^* \equiv Var^*\left(\sqrt{n}\begin{pmatrix} RV_n^* \\ BV_n^* \end{pmatrix}\right) = \begin{pmatrix} Var^*(\sqrt{n}RV_n^*) & Cov^*(\sqrt{n}RV_n^*, \sqrt{n}BV_n^*) \\ & Var^*(\sqrt{n}BV_n^*) \end{pmatrix}.$$

The following result gives the first and second order bootstrap moments of $(RV_n^*, BV_n^*)'$. Note that since $r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i$, we can write

$$RV_n^* = \sum_{i=1}^n \hat{v}_i^n \cdot u_i \quad \text{and} \quad BV_n^* = \frac{1}{k_{1,1}^2}\sum_{i=2}^n \left(\hat{v}_{i-1}^n\right)^{1/2}\left(\hat{v}_i^n\right)^{1/2} \cdot w_i$$

where $u_i \equiv \eta_i^2$ and $w_i \equiv |\eta_{i-1}||\eta_i|$, with $\eta_i \sim$ i.i.d. $N(0,1)$. The bootstrap moments of $(RV_n^*, BV_n^*)'$ depend on the moments and dependence properties of (u_i, w_i).

Lemma 3.1 *If* $r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i, \quad i = 1, \ldots, n,$ *where* $\eta_i \sim$ *i.i.d.* $N(0,1)$, *then*

(a1) $E^*(RV_n^*) = \sum_{i=1}^n \hat{v}_i^n$.

(a2) $E^*(BV_n^*) = \sum_{i=2}^n \left(\hat{v}_{i-1}^n\right)^{1/2}\left(\hat{v}_i^n\right)^{1/2}$.

(a3) $Var^*(\sqrt{n}RV_n^*) = 2n\sum_{i=1}^n \left(\hat{v}_i^n\right)^2$.

(a4) $Var^*(\sqrt{n}BV_n^*) = \left(k_{1,1}^{-4} - 1\right) n \sum_{i=2}^n \left(\hat{v}_i^n\right)\left(\hat{v}_{i-1}^n\right) + 2\left(k_{1,1}^{-2} - 1\right) n \sum_{i=3}^n \left(\hat{v}_i^n\right)^{1/2}\left(\hat{v}_{i-1}^n\right)\left(\hat{v}_{i-2}^n\right)^{1/2}$.

(a5) $Cov^*(\sqrt{n}RV_n^*, \sqrt{n}BV_n^*) = n\sum_{i=2}^n \left(\hat{v}_i^n\right)^{3/2}\left(\hat{v}_{i-1}^n\right)^{1/2} + n\sum_{i=2}^n \left(\hat{v}_i^n\right)^{1/2}\left(\hat{v}_{i-1}^n\right)^{3/2}$.

Lemma 3.1 shows that the bootstrap moments of RV_n^* and BV_n^* depend on multipower variation measures of $\{\hat{v}_i^n\}$. In particular, they depend on $n^{-1+q/2}\sum_{i=K}^n \prod_{k=1}^K \left(\hat{v}_{i-k+1}^n\right)^{q_k/2}$, where $K \in \{1,2,3\}$, $q \equiv \sum_{k=1}^K q_k \in \{2,4\}$, and $q_1 \in \{2,4\}$ when $K = 1$, $(q_1, q_2) \in \{(1,1),(2,2),(1,3),(3,1)\}$ when $K = 2$ and $(q_1, q_2, q_3) = (1,2,1)$ when $K = 3$.

The following assumption imposes a convergence condition on these measures as well as other additional high level conditions on $\hat{v}_i^n$ that are sufficient for a bootstrap CLT to hold. Note that this is a high level condition that does not depend on specifying whether we are on Ω_0 or on Ω_1. However, for some examples, such as Example 1, it might hold only in restriction to Ω_0.

Condition A Suppose that $\{\hat{v}_i^n\}$ satisfies the following conditions.

(i) For $K \in \{1,2,3\}$, $q \equiv \sum_{k=1}^{K} q_k \in \{2,4\}$ and $(q_1,\ldots,q_K) \in \{(2),(4),(1,1),(2,2),(1,3),(3,1),(1,2,1)\}$,

$$n^{-1+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}_{i-k+1}^{n}\right)^{q_k/2} \xrightarrow{P} c_{q_1,\ldots,q_K} \cdot \int_0^1 \sigma_u^q du > 0,$$

as $n \to \infty$, where $c_{q_1,\ldots,q_K}$ is a known constant that only depends on $(q_k : k = 1,\ldots,K)$.

(ii) $n^{1+\delta} \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^{2+\delta} = O_P(1)$, for some $\delta > 0$, as $n \to \infty$.

(iii) For the same $\delta > 0$ as in (ii), $n \sum_{j=1}^{[n/(L_n+1)]} \left(\hat{v}_{j(L_n+1)}^{n}\right)^2 = o_P(1)$, where $L_n \propto n^{\alpha}$ with $0 < \alpha < \frac{\delta}{2(1+\delta)}$ and $[x]$ is the largest integer smaller or equal to x.

Condition A.(i) requires the multipower variations of $\hat{v}_i^n$ to converge to a multiple of $\int_0^1 \sigma_u^q du$, where the constants $c_{q_1,\ldots,q_K}$ are known and depend only on the powers $(q_1,\ldots,q_K)$. Under this condition (with $q = 4$), the probability limit of the bootstrap covariance matrix of (RV_n^*, BV_n^*) is a positive definite matrix whose entries are proportional to IQ. As we will see later, it is verified for Example 1 in restriction to Ω_0, but not to Ω_1. For Example 2, Condition A.(i) is verified on both sets, provided we choose $L > 2$. For both examples, $c_{q_1,\ldots,q_K}$ are not equal to 1, unless we let $M \to \infty$ as $n \to \infty$.

Conditions A.(ii) and (iii) are conditions used to show that a CLT holds for $(RV_n^*, BV_n^*)'$ in the bootstrap world. Since the vector $(u_i, w_i)'$ is lag-one dependent, we adopt a large-block-small-block argument, where the large blocks are made of L_n consecutive observations and the small block is made of a single element. Part (ii) is a Lyapunov type condition that drives the asymptotic normality of the average of the large blocks whereas part (iii) ensures that the contribution of the small blocks is asymptotically negligible.

Under this high level condition, we can prove the following results.

Theorem 3.1 *Under Condition A, if $n \to \infty$,*

(a1)

$$\Sigma^{*-1/2} \sqrt{n} \begin{pmatrix} RV_n^* - E^*(RV_n^*) \\ BV_n^* - E^*(BV_n^*) \end{pmatrix} \xrightarrow{d^*} N(0, I_2), \text{ in prob-}P,$$

where

$$\Sigma^* = \begin{pmatrix} \beta & \delta \\ \delta & \alpha \end{pmatrix} IQ$$

and β, δ and α are known constants that depend on $c_{q_1,\ldots,q_K}$. In particular,

$$\begin{aligned} \beta &= 2c_4 \\ \delta &= c_{3,1} + c_{1,3} \\ \alpha &= \left(k_{1,1}^{-4} - 1\right) c_{2,2} + 2\left(k_{1,1}^{-2} - 1\right) c_{1,2,1}. \end{aligned}$$

(a2) *Let $\tau^* = \beta + \alpha - 2\delta$. Then,*

$$V_n^* \equiv Var^*\left(\sqrt{n}\,(RV_n^* - BV_n^*)\right) \xrightarrow{P} V^* \equiv \tau^* \cdot IQ,$$

and

$$S_n^* = \frac{\sqrt{n}\left((RV_n^* - BV_n^*) - E^*(RV_n^* - BV_n^*)\right)}{\sqrt{V^*}} \xrightarrow{d^*} N(0,1), \text{ in prob-}P.$$

Part (a1) of Theorem 3.1 shows that the bootstrap satisfies a joint CLT if we choose $\hat{v}_i^n$ according to Condition A. The bootstrap covariance matrix of $\sqrt{n}\,(RV_n^*, BV_n^*)'$ converges in probability to Σ^*, which in general is not equal to $\Sigma = \lim_{n\to\infty} Var\left(\sqrt{n}\,(RV_n, BV_n)'\right)$ unless the constants $c_{q_1,\ldots,q_K}$ are equal to 1. The implication is that in general V_n^* converges to $V^* \equiv \tau^* \cdot IQ \neq \tau \cdot IQ \equiv V$. However, since we know τ^*, this does not create a problem if we adjust the bootstrap statistic accordingly. In particular, part (a2) implies that the normalized statistic S_n^* is asymptotically normal under Condition A. This result justifies the following bootstrap test. Let $\mathcal{Z}_n^* \equiv \sqrt{n}\left(\left(RV_n^* - BV_n^*\right) - E^*\left(RV_n^* - BV_n^*\right)\right)/\sqrt{\tau^*}$ and $\mathcal{Z}_n \equiv \sqrt{n}\,(RV_n - BV_n)/\sqrt{\tau}$. We reject the null of "no jumps" at level α if $\mathcal{Z}_n > p_{1-\alpha}^*$, where $p_{1-\alpha}^*$ is the $(1-\alpha)$-percentile of the bootstrap distribution of $\mathcal{Z}_n^*$. Under Condition A, the statistic $\mathcal{Z}_n^* \xrightarrow{d^*} N(0, IQ)$, in prob-$P$, implying that this test controls the size under the null and is consistent under the alternative.

In order to obtain asymptotic refinements, we should bootstrap an asymptotically pivotal test statistic. This entails proposing a consistent bootstrap estimator of $V^* = \tau^* \cdot IQ$. Consider the bootstrap analogue of $\widehat{IQ}_n$,

$$\widehat{IQ}_n^* = \frac{n}{k_{1,\frac{4}{3}}^3} \sum_{i=3}^{n} |r_i^*|^{4/3} \left|r_{i-1}^*\right|^{4/3} \left|r_{i-2}^*\right|^{4/3}. \tag{9}$$

It is easy to show that

$$E^*\left(\widehat{IQ}_n^*\right) = n \sum_{i=3}^{n} |\hat{v}_i^n|^{2/3} \left|\hat{v}_{i-1}^n\right|^{2/3} \left|\hat{v}_{i-2}^n\right|^{2/3}.$$

By extending Condition A.(i) to include the power sequence $(4/3, 4/3, 4/3)$, it follows that the probability limit of $E^*\left(\widehat{IQ}_n^*\right)$ is $c_{4/3,4/3,4/3} \cdot IQ$, where the constant $c_{4/3,4/3,4/3}$ is not necessarily equal to one. Thus, we consider the following adjusted bootstrap estimator

$$\widetilde{IQ}_n^* = \frac{1}{c_{4/3,4/3,4/3}} \frac{n}{k_{1,\frac{4}{3}}^3} \sum_{i=3}^{n} |r_i^*|^{4/3} \left|r_{i-1}^*\right|^{4/3} \left|r_{i-2}^*\right|^{4/3}. \tag{10}$$

To show the consistency of $\widetilde{IQ}_n^*$ towards IQ, we impose the following high level condition.

Condition B Suppose $\{\hat{v}_i^n\}$ is such that

(i) $n \sum\limits_{i=3}^{n} \prod\limits_{k=1}^{3} \left(\hat{v}_{i-k+1}^n\right)^{2/3} \xrightarrow{P} c_{4/3,4/3,4/3} \cdot \int_0^1 \sigma_u^4 du.$

(ii) For $K \in \{3, 4, 5\}$ and $q = \sum_{k=1}^{K} q_k = 8$,

$$n^{-2+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}_{i-k+1}^n\right)^{q_k/2} = o_P(1),$$

where $(q_1, \ldots, q_K) \in \{(8/3, 8/3, 8/3), (4/3, 8/3, 8/3, 4/3), (4/3, 4/3, 8/3, 4/3, 4/3)\}$.

Part (i) ensures that $E^*\left(\widetilde{IQ}_n^*\right) \xrightarrow{P} IQ$ whereas part (ii) suffices to show that $Var^*\left(\widetilde{IQ}_n^*\right) = o_P(1)$, thus ensuring that $\widetilde{IQ}_n^* \xrightarrow{P^*} IQ$, in prob-$P$. If $\hat{v}_i^n$ is such that $n^3 \sum\limits_{i=K}^{n} \prod\limits_{k=1}^{K} \left(\hat{v}_{i-k+1}^n\right)^{q_k/2} \xrightarrow{P} c_{q_1,\ldots,q_K} \cdot \int_0^1 \sigma_u^8 du$, then clearly part (ii) is satisfied. As we will see in the next sections, this is true for both Examples 1 and 2 when X is continuous (or more generally, in restriction to Ω_0), but not necessarily when X has jumps (or in restriction to Ω_1).

The next theorem justifies a bootstrap jump test based on the quantiles of the bootstrap studentized statistic T_n^* given in (8) with

$$\hat{V}_n^* = \tau^* \cdot \widetilde{IQ}_n^*. \tag{11}$$

Theorem 3.2 *Suppose Conditions A and B hold. Then, if $n \to \infty$, (a1) $\widetilde{IQ}_n^* \xrightarrow{P^*} IQ$, in prob-$P$ and (a2) $T_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$.*

The first part of Theorem 3.2 implies the convergence of $\hat{V}_n^*$ towards V^* whereas the second part proves the asymptotic normality of T_n^*. Since $T_n \xrightarrow{st} N(0,1)$ on Ω_0, the fact that $T_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, ensures that the test has correct size asymptotically. Under the alternative (i.e. on Ω_1) since T_n diverges at rate $\sqrt{n}$, but we still have that $T_n^* \xrightarrow{d^*} N(0,1)$, the test has power asymptotically.

More formally, let the bootstrap critical region be defined as follows,

$$\mathcal{C}_n^* = \left\{\omega : T_n(\omega) > q_{n,1-\alpha}^*(\omega)\right\},$$

where $q_{n,1-\alpha}^*(\omega)$ is such that

$$P^*\left(T_n^*(\cdot,\omega) \leq q_{n,1-\alpha}^*(\omega)\right) = 1-\alpha.$$

The bootstrap test rejects $H_0 : \omega \in \Omega_0$ against $H_1 : \omega \in \Omega_1$ whenever $\omega \in \mathcal{C}_n^*$. The following theorem follows from Theorem 3.2 and the asymptotic properties of T_n under H_0 and under H_1.

Theorem 3.3 *Suppose $T_n \xrightarrow{st} N(0,1)$, in restriction to Ω_0, and $T_n \xrightarrow{P} +\infty$ on Ω_1. If Conditions A and B hold, then the bootstrap test based on T_n^* controls the strong asymptotic size and is alternative-consistent.*

To control asymptotic size, it suffices that Conditions A and B hold in restriction to Ω_0. Similarly, to ensure consistency, it suffices that the bootstrap statistic $T_n^* = O_{P^*}(1)$, in prob-P. Here, we impose Conditions A and B directly to ensure that $T_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, independently of whether $\omega \in \Omega_0$ or $\omega \in \Omega_1$. This is the ideal situation for the bootstrap test to maximize power and at the same time control size, as our simulation results in Section 6 show.

3.3 Multipower variations of $\hat{v}_i^n$ and blocked multipower variations of r_i

Conditions A and B depend on realized quantities of the form

$$n^{-1+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}_{i-k+1}^n\right)^{q_k/2}, \tag{12}$$

where K is a fixed natural number, $q_k \geq 0$ for all k, and $q = \sum_{k=1}^{K} q_k$. These sums can be interpreted as the multipower variations of $\hat{v}_i^n$. As it turns out, for our Examples 1 and 2, (12) can be written as a linear combination of blocked multipower variation measures of returns r_i as introduced by MSS (2012). For given $q > 0$, $K \in \mathbb{N}$, $q = \sum_{k=1}^{K} q_k$ with $q_k \geq 0$, these are defined as

$$MV_M^{[q,K]}(q_1,\ldots,q_K) = n^{-1+q/2} \frac{M^{q/2}}{\prod_{k=1}^{K} k_{M,q_k}} M \sum_{i=K}^{n/M} \prod_{k=1}^{K} \left(\bar{R}_{i-k+1}\right)^{\frac{q_k}{2}}, \tag{13}$$

where $k_{M,q_k} \equiv E\left(\left|\chi_M^2\right|^{q_k/2}\right)$. Note that $RV_n = MV_M^{[2,1]}(2)$. We will sometimes write $MV_M^{[q,K]}(\{q_k\}) = MV_M^{[q,K]}(q_1, \ldots, q_K)$.

For $j = 1, \ldots, n/M$, recall that Example 2 (which contains Example 1 as a special case when $L = 1$) sets

$$\hat{v}^n_{i+(j-1)M} = \prod_{l=1}^{L} \bar{R}^{p_l}_{j-l+1}, \quad i = 1, \ldots, M,$$

where $\{p_l : l = 1, \ldots, L\}$ is such that $\sum_{l=1}^{L} p_l = 1$, with $p_l \geq 0$. We adopt the convention that $p_l = 0$ whenever $l \leq 0$ or $l > L$.

The following lemma establishes the relation between the multipower variations of $\{\hat{v}_i^n\}$ for Examples 1 and 2 and the blocked multipower variations $MV_M^{[q,K]}(q_1, \ldots, q_K)$.

Lemma 3.2 *Let K be a natural fixed number and let $\{q_k : k = 1, \ldots, K\}$ be such that $q_k \geq 0$. Set $q = \sum_{k=1}^{K} q_k$ and $\bar{q}_k = \sum_{l=1}^{k} q_l$ for $k = 1, \ldots, K$. Then,*

(a1) *For $K = 1$ and $M \geq 1$,*

$$n^{-1+q/2} \sum_{i=1}^{n} (\hat{v}_i^n)^{q/2} = \left(\prod_{l=1}^{L} \frac{k_{M,qp_l}}{M^{q/2}}\right) MV_M^{[q,L]}(qp_1, \ldots, qp_L).$$

(a2) *For $K \geq 2$ and any $M \geq K-1$, we have that*

$$\begin{aligned} n^{-1+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}^n_{i-k+1}\right)^{q_k/2} &= \frac{M-K+1}{M} \left(\prod_{l=1}^{L} \frac{k_{M,qp_l}}{M^{q/2}}\right) MV_M^{[q,L]}(qp_1, \ldots, qp_L) \\ &\quad + \frac{1}{M} \sum_{k=1}^{K-1} \left(\prod_{l=1}^{L+1} \frac{k_{M,\bar{q}_k p_l + (q-\bar{q}_k) p_{l-1}}}{M^{q/2}}\right) MV_M^{[q,L+1]}(\{\bar{q}_k p_l + (q - \bar{q}_k) p_{l-1}\}), \end{aligned}$$

where $MV_M^{[q,L+1]}(\{\bar{q}_k p_l + (q-\bar{q}_k) p_{l-1}\}) \equiv MV_M^{[q,L+1]}(\bar{q}_k p_1, \bar{q}_k p_2 + (q-\bar{q}_k) p_1, \ldots, (q-\bar{q}_k) p_L)$.

(a3) *For $M = 1$, for any $K \geq 1$,*

$$n^{-1+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}^n_{i-k+1}\right)^{q_k/2} = \left(\prod_{l=1}^{L+K-1} k_{1,q_1 p_l + \ldots + q_K p_{l-(K-1)}}\right) MV_1^{[q,L+2]}\left(\left\{q_1 p_l + \ldots + q_K p_{l-(K-1)}\right\}\right).$$

Part (a2) requires that $M \geq K-1$, where $K \geq 2$. For $K = 3$, this restriction excludes the case[1] of $M = 1$, which we include in (a3). We will rely on these results to evaluate the constant τ^* needed to compute T_n^*.

For the special case $L = 1$, part (a1) reads as

$$n^{-1+q/2} \sum_{i=1}^{n} (\hat{v}_i^n)^{q/2} = \frac{k_{M,q}}{M^{q/2}} \cdot MV_M^{[q,1]}(q),$$

[1] Note that Conditions A and B involve the multipower variations of $\hat{v}_i^n$ for $K \leq 5$, implying that the expression in (a2) also does not cover the cases of $M = 2$ when $K = 4$ and of $M \in \{2, 3\}$ when $K = 5$; the reason why we do not cover explicitly these cases here is that for these values of K explicit knowledge of the constants multiplying the $MV_M^{[q.L]}(\{q_k\})$ are not needed; we only need those constants for $K = 3$.

whereas part (a2) is given by

$$n^{-1+q/2}\sum_{i=K}^{n}\prod_{k=1}^{K}\left(\hat{v}_{i-k+1}^{n}\right)^{q_k/2}=\frac{M-K+1}{M}\frac{k_{M,q}}{M^{q/2}}\cdot MV_M^{[q,1]}\left(q\right)+\frac{1}{M}\sum_{k=1}^{K-1}\frac{k_{M,\bar{q}_k}k_{M,q-\bar{q}_k}}{M^{q/2}}\cdot MV_M^{[q,2]}\left(\bar{q}_k,q-\bar{q}_k\right),$$

for any $K\geq 2$ and $M\geq K-1$.

The following lemma gives the asymptotic properties of $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$. Part (a1) is in restriction to Ω_0, whereas (a2) holds for the entire sample space Ω.

Lemma 3.3 *Suppose (1) and (2) and Assumptions 1 and 2 hold. Let $q>0$ such that $q=\sum_{k=1}^{K}q_k$ with $q_k\geq 0$ and $K\in\mathbb{N}$. For any fixed integer $M=1,2,\ldots,$*

(a1) $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)\xrightarrow{P}\int_0^1\sigma_u^q du$ *in restriction to* Ω_0.

(a2) $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)-\int_0^1\sigma_u^q du=O_P\left(n^{-1+\max(q_k)/2}\left|\log n\right|^{\frac{q-\max(q_k)}{2}}\right)$.

Part (a1) shows the convergence in probability of $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$ towards $\int_0^1\sigma_u^q du$ for any $q>0$ and $K\geq 1$ when we restrict $\omega\in\Omega_0$. The proof follows from Theorem 3 of MSS (2012) under Assumptions 1 and 2. Part (a2) gives a bound on the difference between $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$ and $\int_0^1\sigma_u^q du$ for any $\omega\in\Omega$. When $\max\left(q_k\right)<2$, this bound converges to zero at the stated rate, implying that $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$ is a consistent estimator of $\int_0^1\sigma_u^q du$ even under jumps. When $\max\left(q_k\right)\geq 2$, we obtain a bound on $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$. Although this bound is not necessarily sharp, it is sufficient to prove our results (note in particular that the bound diverges to $+\infty$ when $\max\left(q_k\right)\geq 2$, which is certainly not a sharp bound when either $\omega\in\Omega_0$, or ($\omega\in\Omega_1$ and $\max\left(q_k\right)=2$)).

A version of Lemma 3.3 is proven by Barndorff-Nielsen et al. (2006) when $M=1$ under very general conditions on the drift a and the volatility process σ when X is continuous. In particular, they do not rule out jumps in σ, as we do under Assumption 1. By imposing this assumption, we can rely on Theorem 3 of MSS (2012) to obtain the convergence in probability of $MV_M^{[q,K]}\left(q_1,\ldots,q_K\right)$ towards $\int_0^1\sigma_u^q du$ for any fixed value of $M>1$.

4 Example 1: local RV_n estimate

Given the results of Section 3, the asymptotic validity of a bootstrap jump test can be established by verifying Conditions A and B. Here we do so for the choice of $\hat{v}_i^n$ given in Example 1. We first study the asymptotic properties of T_n^* under the null of "no jumps" and then consider what happens under the alternative of jumps.

4.1 Properties under the null of "no jumps"

Recall that T_n^* is given by (8), (10) and (11). Hence, to compute T_n^* we need to know the recentering term $E^*\left(RV_n^*-BV_n^*\right)$; the constant $c_{4/3,4/3,4/3}$ that enters $\widetilde{IQ}_n^*$; and the constants $c_4,c_{2,2},c_{1,3},c_{3,1}$ and $c_{1,2,1}$ that enter the definition of τ^* given in Theorem 3.1.

Given Lemmas 3.1 and 3.2,

$$E^*\left(RV_n^*-BV_n^*\right)=\sum_{i=1}^{n}\hat{v}_i^n-\sum_{i=2}^{n}\left(\hat{v}_{i-1}^n\right)^{1/2}\left(\hat{v}_i^n\right)^{1/2}=RV_n-\frac{M-1}{M}RV_n-\frac{1}{M}\frac{k_{M,1}^2}{M}MV_M^{[2,2]}\left(1,1\right).$$

This expression shows that for finite M, recentering $RV_n^* - BV_n^*$ is important, but if M is sufficiently large this becomes asymptotically negligible.

Our next result shows that Conditions A and B are satisfied for Example 1 under $H_0 : \omega \in \Omega_0$ and identifies the constants needed to compute T_n^*. The proof is in Appendix B. It relies on Lemmas 3.2 and 3.3 for $q \in \{2, 4, 8\}$ and $K \in \{1, 2, 3, 4, 5\}$.

Theorem 4.1 *Suppose (1) and (2) hold and Assumption 1 holds. Then, for any fixed integer $M \geq 1$,*

(a1) *Conditions A and B are satisfied under $H_0 : \omega \in \Omega_0$, where for any $M \geq 1$,*

$$
\begin{aligned}
c_4 &= \frac{k_{M,4}}{M^2}, \\
c_{1,3} &= c_{3,1} = \frac{(M-1)}{M}\frac{k_{M,4}}{M^2} + \frac{1}{M}\frac{k_{M,1}k_{M,3}}{M^2}, \text{ and} \\
c_{2,2} &= \left(\frac{M-1}{M}\right)\frac{k_{M,4}}{M^2} + \frac{1}{M}\frac{k_{M,2}^2}{M^2}.
\end{aligned}
$$

In addition,

$$
c_{1,2,1} = \begin{cases} k_{1,1}^2 k_{1,2} & \text{, for } M = 1 \\ \left(\frac{M-2}{M}\right)\frac{k_{M,4}}{M^2} + 2\frac{1}{M}\frac{k_{M,1}k_{M,3}}{M^2} & \text{, for } M \geq 2, \end{cases}
$$

and

$$
c_{4/3,4/3,4/3} = \begin{cases} k_{1,4/3}^3 & \text{, for } M = 1 \\ \left(\frac{M-2}{M}\right)\frac{k_{M,4}}{M^2} + 2\frac{1}{M}\frac{k_{M,4/3}k_{M,8/3}}{M^2} & \text{, for } M \geq 2. \end{cases}
$$

(a2) *The conclusions of Theorems 3.1, 3.2 and 3.3 hold under $H_0 : \omega \in \Omega_0$.*

Part (a1) identifies the constants needed to compute T_n^*. Part (a2) shows that the results of Theorems 3.1, 3.2 and 3.3 apply to Example 1 under the null of "no jumps" (i.e. in restriction to Ω_0). In particular, under the null of no jumps, Σ_n^*, the local Gaussian bootstrap covariance matrix of $(RV_n^*, BV_n^*)'$, is such that

$$
\Sigma_n^* \xrightarrow{P} \Sigma^* \equiv \begin{pmatrix} \beta_M & \delta_M \\ \delta_M & \alpha_M \end{pmatrix} IQ.
$$

where

$$
\begin{aligned}
\beta_M &= 2c_4 \\
\delta_M &= c_{3,1} + c_{1,3} \\
\alpha_M &= \left(k_{1,1}^{-4} - 1\right) c_{2,2} + 2\left(k_{1,1}^{-2} - 1\right) c_{1,2,1},
\end{aligned}
$$

with $c_{q_1,\ldots,q_K}$ given in part (a1) of Theorem 4.1. This result in turn implies that on Ω_0,

$$
V_n^* \equiv Var^*\left(\sqrt{n}\left(RV_n^* - BV_n^*\right)\right) \xrightarrow{P} V^* \equiv \tau^* \cdot IQ,
$$

where

$$
\tau^* \equiv \beta_M + \alpha_M - 2\delta_M.
$$

When $M = 1$, β_1,δ_1 and α_1 are different from $2, 2$ and θ, respectively, which implies that $\Sigma^* \neq \Sigma$. Also, $\tau^* \neq \tau$, implying that $V^* \neq V$. However, by Remark 2 of MSS (2012), as $M \to \infty$,

$$
\frac{k_{M,q}}{M^{q/2}} \sim 1 + \frac{a_q}{M} + \frac{b_q}{M^2},
$$

for some constants a_q and b_q. Consequently, the constants c_4, $c_{2,2}$, $c_{1,3}$, $c_{3,1}$ and $c_{1,2,1}$ all tend to 1 as $M \to \infty$, implying that

$$\lim_{M\to\infty} \beta_M = \lim_{M\to\infty} \delta_M = 2 \text{ and } \lim_{M\to\infty} \alpha_M = \theta \equiv \left(k_{1,1}^{-4} - 1\right) + 2\left(k_{1,1}^{-2} - 1\right).$$

Hence, by letting $M \to \infty$ we ensure that $V^* = \tau^* \cdot IQ$ approaches $V = \tau \cdot IQ$ (since then $\tau^* \to \tau = \theta - 2$). Nevertheless, in finite samples, fixing M and adjusting the bootstrap statistics accordingly outperforms the approach based on letting $M \to \infty$ and therefore we do not consider this approach here.

4.2 Properties under the alternative of jumps

The results of the previous section imply that the local Gaussian bootstrap controls the size of the test. In this section, we study what happens under the alternative of jumps. As it turns out, Conditions A and B no longer hold. In particular, the bootstrap variances Σ_n^* and V_n^* diverge to infinity. This compromises the asymptotic normality of T_n^* under the alternative hypothesis of jumps.

To ensure that the test has power, the weaker condition that $T_n^* = O_{P^*}(1)$, in prob-P suffices. Nevertheless, as we will see next, this is not guaranteed for the local Gaussian bootstrap. In particular, we show that for the special case of $M = 1$, the fact that V_n^* diverges under the presence of jumps implies that T_n^* diverges at the same rate as T_n. This may imply that the test is not alternative-consistent.

Let us rewrite T_n^* as

$$T_n^* = Z_n^* \sqrt{\frac{n\left(V_n^*/n\right)}{\tau^* \widetilde{IQ}_n^*}},$$

where

$$Z_n^* \equiv \frac{\sqrt{n}\left(RV_n^* - BV_n^* - E^*\left(RV_n^* - BV_n^*\right)\right)}{\sqrt{V_n^*}},$$

and

$$V_n^* = a_1 MV_1^{[4,1]}(4) + a_2 MV_1^{[4,2]}(2,2) + a_3 MV_1^{[4,3]}(1,2,1) + a_4\left(MV_1^{[4,2]}(3,1) + MV_1^{[4,2]}(1,3)\right), \quad (14)$$

where the constants a_1 through a_4 are a function of $c_4, c_{2,2}, c_{1,3}$, and $c_{1,2,1}$ given in Theorem 4.1.(a1) (the exact expression for V_n^* is easily obtained from Lemmas 3.1 and 3.2).

By Lemma 3.3, on Ω_1, V_n^* has an asymptotic order of magnitude $O_P(n)$, the order of its first (and dominant) term. Therefore, V_n^* may diverge at that rate. This is confirmed by Lemma B.1 in Appendix B which establishes that V_n^*/n converges to a random variable that is positive on Ω_1, implying that this rate is sharp so long as $P(\Omega_1) > 0$.

In addition, by Lemma B.2, $\widetilde{IQ}_n^*$ is still convergent towards IQ under the presence of jumps whereas Z_n^* is $O_{P^*}(1)$ in prob-P by construction (since $E^*(Z_n^*) = 0$ and $Var^*(Z_n^*) = 1$). Because we can also show that Z_n^* is *not* $o_{P^*}(1)$ (cf. Lemma B.3), the order of magnitude of T_n^* is equal to $O_P(\sqrt{n})$. This result is summarized in the following theorem.

Theorem 4.2 *Suppose (1), (2) and Assumptions 1 and 2 hold. Then, for $M = 1$, if $P(\Omega_1) > 0$, we have that on Ω_1, $T_n^* = O_{P^*}(\sqrt{n})$, in prob-$P$, where the rate is sharp (i.e. $T_n^* = O_{P^*}(\sqrt{n})$ and $\sqrt{n} = O_{P^*}(T_n^*)$).*

Because the two test statistics T_n and T_n^* diverge at the same rate, we cannot draw any conclusions on the exact asymptotic power of the bootstrap test. However, our simulations suggest that for the models we have simulated the bootstrap test based on $L = M = 1$ has very poor power properties.

5 Example 2: multipower local RV estimate

Here we verify Conditions A and B for Example 2 and identify the constants needed to compute T_n^*. The following result is the analogue of Theorem 4.1 for Example 2.

Theorem 5.1 *Suppose (1), (2) and Assumptions 1 and 2 hold. Then, for any fixed $M \geq 1$, if $\max(p_l) < \frac{1}{2}$, or equivalently, if $L > 2$ when $p_l = 1/L$ for all $l = 1, \ldots, L$, then*

(a1) *Conditions A and B are verified under both Ω_0 and Ω_1, where for any $M \geq 1$,*

$$\begin{aligned} c_4 &= \prod_{l=1}^{L} \frac{k_{M,4p_l}}{M^2}, \\ c_{1,3} &= c_{3,1} = \frac{(M-1)}{M} \left(\prod_{l=1}^{L} \frac{k_{M,4p_l}}{M^2} \right) + \frac{1}{M} \left(\prod_{l=1}^{L+1} \frac{k_{M,p_l+3p_{l-1}}}{M^2} \right), \\ c_{2,2} &= \left(\frac{M-1}{M} \right) \left(\prod_{l=1}^{L} \frac{k_{M,4p_l}}{M^2} \right) + \frac{1}{M} \left(\prod_{l=1}^{L+1} \frac{k_{M,2p_l+2p_{l-1}}}{M^2} \right), \end{aligned}$$

whereas

$$c_{1,2,1} = \begin{cases} \prod_{l=1}^{L+2} k_{1,p_l+2p_{l-1}+p_{l-2}} & , \text{ for } M = 1 \\ \left(\frac{M-2}{M}\right) \left(\prod_{l=1}^{L} \frac{k_{M,4p_l}}{M^2} \right) + \frac{2}{M} \left(\prod_{l=1}^{L+1} \frac{k_{M,p_l+3p_{l-1}}}{M^2} \right) & , \text{ for } M \geq 2, \end{cases}$$

and

$$c_{4/3,4/3,4/3} = \begin{cases} \prod_{l=1}^{L+2} k_{1,\frac{4}{3}p_l+\frac{4}{3}p_{l-1}+\frac{4}{3}p_{l-2}} & , \text{ for } M = 1 \\ \left(\frac{M-2}{M}\right) \left(\prod_{l=1}^{L} \frac{k_{M,4p_l}}{M^2} \right) + \frac{2}{M} \left(\prod_{l=1}^{L+1} \frac{k_{M,\frac{4}{3}p_l+\frac{8}{3}p_{l-1}}}{M^2} \right) & , \text{ for } M \geq 2. \end{cases}$$

(a2) *The conclusions of Theorems 3.1, 3.2 and 3.3 hold under both Ω_0 and Ω_1.*

Theorem 5.1 proves the asymptotic validity of the bootstrap test based on Example 2. In particular, if we choose $\{p_l\}$ and L appropriately, the bootstrap test based on T_n^* has the correct asymptotic size and is alternative-consistent.

The main difference with respect to the case where $L = 1$ is that we do not need to restrict $\omega \in \Omega_0$ to verify Conditions A and B. Because the multipower variations of $\hat{v}_i^n$ depend on linear combinations of efficient blocked multipower variations of returns whose exponents are a function of $\{p_l : l = 1, \ldots, L\}$, we can choose L and $\{p_l\}$ so as to guarantee that Conditions A and B are verified without restricting ω to belong to Ω_0. In particular, Condition A(i) involves multipower variations of $\hat{v}_i^n$ with $K \in \{1, 2, 3\}$ and $q \in \{2, 4\}$. When $q = 4$, this condition is crucial to show that the bootstrap variance V_n^* converges to a multiple of IQ under both Ω_0 and Ω_1. By Lemma 3.2, for $q = 4$, the multipower variations of $\hat{v}_i^n$ depend on linear combinations of $MV_M^{[4,L]}(4p_1, \ldots, 4p_L)$ and $MV_M^{[4,L+1]}(\bar{q}_k p_1, \bar{q}_k p_2 + (4 - \bar{q}_k) p_1, \ldots, \bar{q}_k p_L + (4 - \bar{q}_k) p_{L-1}, (4 - \bar{q}_k) p_L)$, where $\bar{q}_k = \sum_{j=1}^{k} q_j \in \{1, 2, 3\}$. Thus, by Lemma 3.3, if $\max(4p_l) < 2$ (or equivalently, $L > 2$ when $p_l = 1/L$ for all $l = 1, \ldots, L$), Condition A(i) is satisfied under both the null and the alternative hypothesis, ensuring that V_n^* is robust to jumps.

6 Monte Carlo simulations

In this section, we assess by Monte Carlo simulation the performance of our bootstrap tests. Along with the asymptotic test of BN-S (2006), we report results with $L \in \{1, 5\}$, and $M \in \{1, 2, 3, 4, 6, 12\}$ with $p_l = 1/L$ $(l = 1, \ldots, L)$ for Examples 1 and 2. We also include results for Example 3 and report results for $\varpi = 0.4$ and $\alpha = 2.3\sqrt{BV}$, following Podolskij and Ziggel (2010).

We present results for the SV2F model given by[2]

$$\begin{aligned}
dX_t &= adt + \sigma_{u,t}\sigma_{sv,t}dW_t + dJ_t, \\
\sigma_{u,t} &= C + A \cdot exp\left(-a_1 t\right) + B \cdot exp\left(-a_2\left(1-t\right)\right), \\
\sigma_{sv,t} &= s\text{-}exp\left(\beta_0 + \beta_1\tau_{1,t} + \beta_2\tau_{2,t}\right), \\
d\tau_{1,t} &= \alpha_1\tau_{1,t}dt + dB_{1,t}, \\
d\tau_{2,t} &= \alpha_2\tau_{2,t}dt + \left(1 + \phi\tau_{2,t}\right)dB_{2,t}, \\
corr\left(dW_t, dB_{1,t}\right) &= \rho_1,\ corr\left(dW_t, dB_{2,t}\right) = \rho_2.
\end{aligned}$$

The processes $\sigma_{u,t}$ and $\sigma_{sv,t}$ represent the components of the time-varying volatility in prices. In particular, $\sigma_{sv,t}$ denotes the two factors stochastic volatility model commonly used in this literature. We follow Huang and Tauchen (2005) and set $a = 0.03$, $\beta_0 = -1.2$, $\beta_1 = 0.04$, $\beta_2 = 1.5$, $\alpha_1 = -0.00137$, $\alpha_2 = -1.386$, $\phi = 0.25$, $\rho_1 = \rho_2 = -0.3$. At the start of each interval, we initialize the persistent factor τ_1 by $\tau_{1,0} \sim N\left(0, \frac{-1}{2\alpha_1}\right)$, its unconditional distribution. The strongly mean-reverting factor τ_2 is started at $\tau_{2,0} = 0$. The process $\sigma_{u,t}$ models the diurnal U-shaped pattern in intraday volatility. In particular, we follow Hasbrouck (1999) and Andersen et al. (2012) and set the constants $A = 0.75$, $B = 0.25$, $C = 0.88929198$, and $a_1 = a_2 = 10$. These parameters are calibrated so as to produce a strong asymmetric U-shaped pattern, with variance at the open (close) more than 3 (1.5) times that at the middle of the day. We let $\sigma_{u,t} = 1$ for $t \in [0, 1]$ in the simple case of no diurnality effects. Finally, J_t is a finite activity jump process modeled as a compound Poisson process with constant jump intensity λ and random jump size distributed as $N(0, \sigma^2_{jmp})$. We let $\sigma^2_{jmp} = 0$ under the null hypothesis of no jumps in the return process. Under the alternative, we let $\lambda = 0.058$, and $\sigma^2_{jmp} = 1.7241$. These parameters are motivated by empirical studies by Huang and Tauchen (2005) and Barndorff-Nielsen, Shephard, and Winkel (2006), which suggest that the jump component accounts for 10% of the variation of the price process.

We simulate data for the unit interval $[0, 1]$ and normalize one second to be $1/23,400$, so that $[0, 1]$ is meant to span 6.5 hours. The observed X process is generated using an Euler scheme. We then construct the n-horizon returns $r_i = X_{i/n} - X_{(i-1)/n}$ based on samples of size n. Results are presented for five different samples sizes: $n = 48$, 96, 288, 576, and 1152, corresponding approximately to "8-minute", "4-minute", "1,35-minute", "40-second" and "20-second" frequencies.

Figures 1 through 4 display the results. Figures 1 and 2 contain no diurnal effects, without jumps and with finite activity jumps, respectively. Figures 3 and 4 give the corresponding results under deterministic diurnal effects. In each figure, we present results based on the linear test statistic and its log version[3], with critical values obtained either by the asymptotic theory or by the bootstrap. All tests are carried out at the 5% nominal level. The rejection rates reported in Figures 1 and 3 (under no jumps) are obtained from 10,000 Monte Carlo replications with 999 bootstrap samples for each simulated sample for the bootstrap tests. For finite activity jumps, since J_t is a compound Poisson process, even under the alternative, it is possible that no jump occurs in some sample over the interval [0,1] considered. Thus, to compute the rejection rates under the alternative of jumps (cf. Figures 2

[2]The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: s-$\exp(x) = \exp(x)$ if $x \leq x_0$ and s-$\exp(x) = \frac{\exp(x_0)}{\sqrt{x_0 - x_0^2 + x^2}}$ if $x > x_0$, with $x_0 = \log(1.5)$.

[3]See Appendix C for details on the log-transform of the test statistic T_n and the bootstrap-related formulas.

and 4) we rely on the number n_0 of replications, out of the 10,000, for which at least one jump has occurred. For our parameter configuration, $n_0 = 570$.

Starting with Figure 1, the results show that the linear version of the test based on the asymptotic theory of BN-S (2006) (labeled "AT" in the figures) is substantially distorted for the smaller sample sizes. In particular, the rejection rate is three times larger than the nominal level of the test (at 15.21%) for $n = 48$. Although this rate drops as n increases, it remains significantly larger than the nominal level even when $n = 1152$, with a value equal to 7.08%. As expected, the log version of the test statistic (denoted "AT, log" in the figures) has smaller size distortions: the rejection rates are now 12.54% and 6.25% for $n = 48$ and $n = 1152$, respectively. The rejection rates of the bootstrap tests are always smaller than those of the asymptotic tests and therefore the bootstrap uniformly dominates the latter when controlling size. This is true for both $L = 1$ and $L = 5$ and for both versions of the test, linear and log. However, when $L = 1$ and we rely on the linear version of the test (labeled "$L = 1$"), the bootstrap is very conservative, rejecting the null less than 2% when $n \leq 288$ and we set $M = 1$. Increasing M from 1 to 2 reduces these distortions (for instance, for $M = 2$, this rejection rate increases to 4.14% for $n = 288$) but further increases in M may result in overrejections when n is small (this shows that there is a limit to letting M increase when n is small). Similarly, choosing $L = 5$ (labeled "$L = 5$") may induce slight overrejections under the null for the smaller sample sizes, with rejection rates between 6 and 7% for $n = 48$ and $n = 96$. These rates are nevertheless much smaller than those associated with the asymptotic theory-based tests. When $n \geq 288$, we do not see many differences between the bootstrap tests for the log and the linear version of the statistics (except when $L = M = 1$, where the bootstrap log test does not suffer from the underrejection noted for the linear test). Overall, Figure 1 shows that the bootstrap helps reduce the overrejections associated with the asymptotic theory-based tests, for all values of L and M, and independently of using the linear or the log versions of the test.

Turning now to the analysis of power, Figure 2 shows that the choice of L is very important. In particular, there is a clear separation between $L = 1$ and $L = 5$, especially when M is small. In particular, choosing $L = 1$ and $M = 1$ leads to virtually no power when the bootstrap is applied to the linear test statistic. This confirms our theoretical result. Since T_n^* diverges to $+\infty$ for these choices of L and M, and the divergence rate is the same as that of T_n, the rejection rate under the alternative hypothesis of this bootstrap method is not necessarily equal to 1, even for large n. In the context of the linear version, this test seems to severely underreject under the alternative of jumps. Letting M increase when $L = 1$ helps increase the rejection rates and seems to restore power. We conjecture that the main reason why we see this behavior is that $T_n^* = O_{P^*}(1)$, in prob-P, when $M \geq 2$ and $L = 1$. Thus, even though the local Gaussian bootstrap is not asymptotically normally distributed in this case, it is bounded in probability, which ensures that the bootstrap has power. The log version of the bootstrap test with $L = M = 1$ does not appear to suffer from the almost zero power problem noted for the linear test, but its rejection rate is lower than the rejection rates observed for $L = 5$. Overall, Figure 2 shows that the best choice of L from the power perspective is $L = 5$. This is especially true when using smaller values of M; for $M = 12$, the differences are negligible. However, setting M too large may lead to overrejections under the null. Therefore, our recommendation is to choose $L = 5$.

We also implemented the bootstrap with $\hat{v}_i^n$ given as described by Example 3. To conserve space, we do not present the results (they are available upon request) but provide a brief discussion here. Under the null of no jumps, using truncated squared returns to compute $\hat{v}_i^n$ resulted in rejection rates varying between 2.80% for $n = 48$ and 3.37% for $n = 1152$ for the linear version of the bootstrap jump test. Thus, the thresholding-based bootstrap test was rather conservative and it was dominated by the use of multipower variation measures except when $L = M = 1$, which is characterized by even lower null rejection rates. Using the log version of the test increased the null rejection rates of the thresholding-based bootstrap test to values similar to those of Examples 1 and 2 (in particular, they were equal to 8.5% when $n = 48$ and to 3.99% when $n = 1152$). From the viewpoint of power, the main conclusion that emerged from our simulations was that the thresholding-based bootstrap test had less power than the multipower variation-based bootstrap tests (except when compared to $L = M = 1$ for the linear test). Specifically, the power for Example 3 ranged between 35.14% (for $n = 48$) and 46.28% (for $n = 1152$) for the linear test and between 72.47% (for $n = 48$) and 84.80% (for $n = 1152$) for the log test.

Figures 3 and 4 contain results for the SV2F model with diurnal effects. For brevity, we only present results for $L = 5$. For both the bootstrap and the asymptotic theory methods, two types of tests are considered: tests that do not contain any correction for diurnal effects (labeled with the words "no correction") and tests that contain a nonparametric correction for the diurnal effects. Specifically, we use the nonparametric jump robust estimation of intraday periodicity in volatility suggested by Boudt, Croux and Laurent (2011). This amounts to estimating the intraday volatility pattern $\hat{\sigma}_{u,i}$ using 2,000 days in the simulation and then using this to standardize the intraday returns. The modified data are then used to compute the test statistics, including their bootstrap versions. The results obtained with the transformed data are labeled with "correction" in Figures 3 and 4.

Starting with Figure 3, which presents rejection rates under the null of no jumps, we can see that for the test based on the asymptotic theory of BN-S (2006), a large distortion driven by the difference in volatility across blocks appears even if the sample size is large. For $n = 1152$, the null rejection rate is 10.9% for the linear version of the test, whereas it is 9.9% with the log version. These are twice as large as the desired nominal level of 5%. When n is smaller, the overrejections are much larger. For instance, for $n = 48$ they are equal to 32.8% and 28.7%, respectively. As expected, using the asymptotic tests applied to the modified set of intraday returns helps reduce the distortions. For $n = 48$, the rates are now equal to 15.7% and 12.9%, whereas for $n = 1152$ they are 7.8% and 7.45%. The bootstrap null rejection rates are always smaller than those of the asymptotic theory-based tests, implying that the bootstrap outperforms the latter. This is true even for the bootstrap test applied to the non-transformed intraday returns (labeled "$L = 5$, no correction"), which yields rejection rates that are closer to the nominal level than those obtained with the asymptotic tests based on the correction of the diurnal effect. This is a very interesting finding since it implies that our bootstrap method is robust to the presence of diurnal effects (whereas the asymptotic theory-based test is not). Of course, even better results can be obtained for the bootstrap tests by resampling the transformed intraday returns and this is confirmed by Figure 3, which shows that the results for "$L = 5$, correction" are systematically closer to 5% than those for "$L = 5$, no correction" (and both are closer than the corresponding asymptotic tests). Figure 4 looks at the power properties of these tests under diurnal effects. The main feature of notice is that the bootstrap tests have lower power than their asymptotic counterparts. This is expected given that the asymptotic tests have much larger rejections under the null than the bootstrap tests. In particular, this explains the large discrepancy between the bootstrap and the asymptotic test when both are applied to the non-transformed data. As n increases, we see that this difference decreases. The results also show that power is largest for tests (both asymptotic and bootstrap-based) applied to the transformed returns. For these tests, the difference in power between the bootstrap and the asymptotic tests is very small. Given that the bootstrap essentially eliminates the size distortions of the asymptotic test, these two findings strongly favor the bootstrap over the asymptotic tests.

7 Empirical results

This empirical application uses trade data on the SPDR S&P 500 ETF (SPY), which is an exchange traded fund (ETF) that tracks the S&P 500 index. Data on SPY have been used by MSS (2012) (see also Bollerslev, Law and Tauchen (2008)). Our primary sample comprises 10 years of trade data on SPY starting from June 15, 2004 through June 13, 2014 as available in the New York Stock

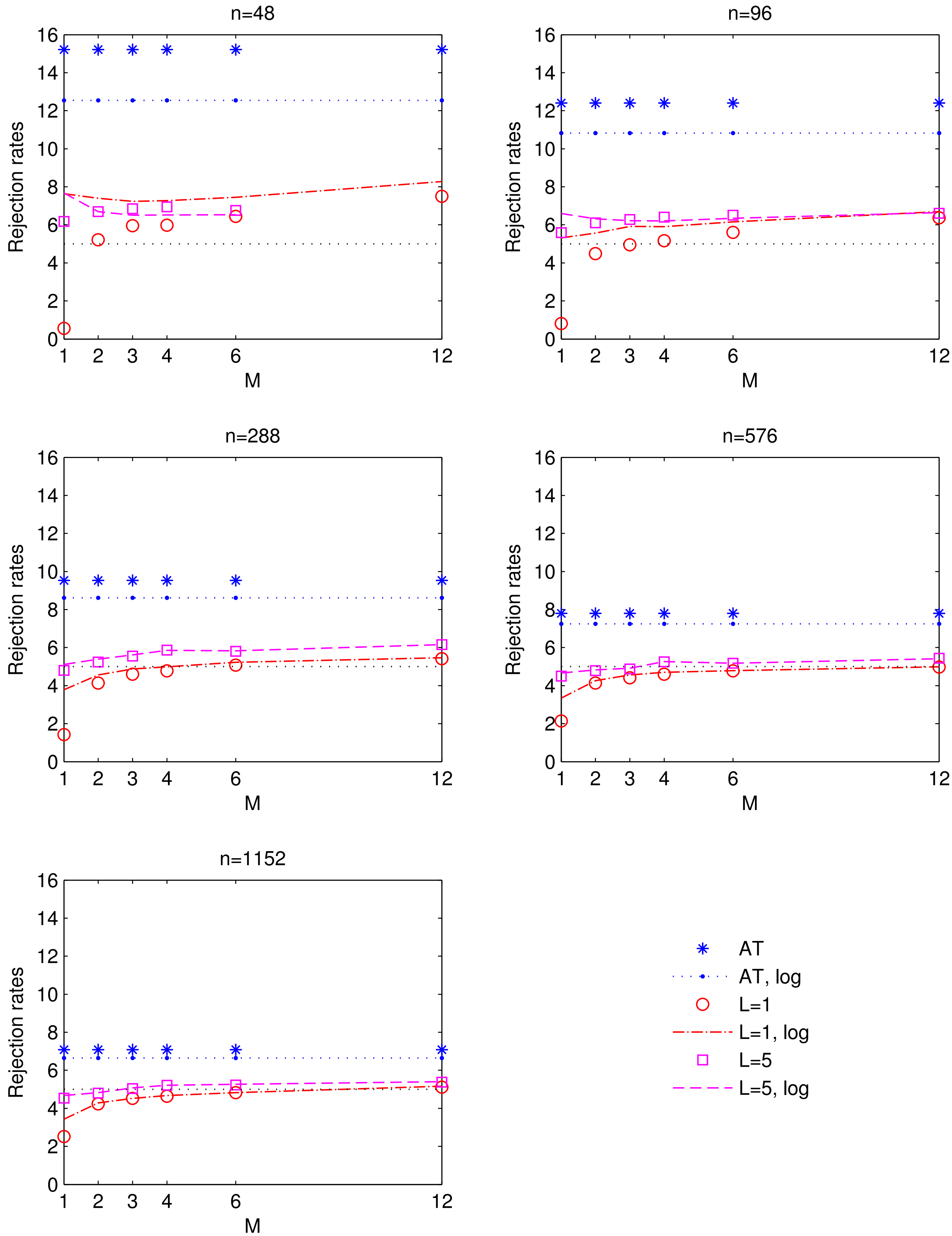

Figure 1: SV2F model without diurnal effects, no jumps

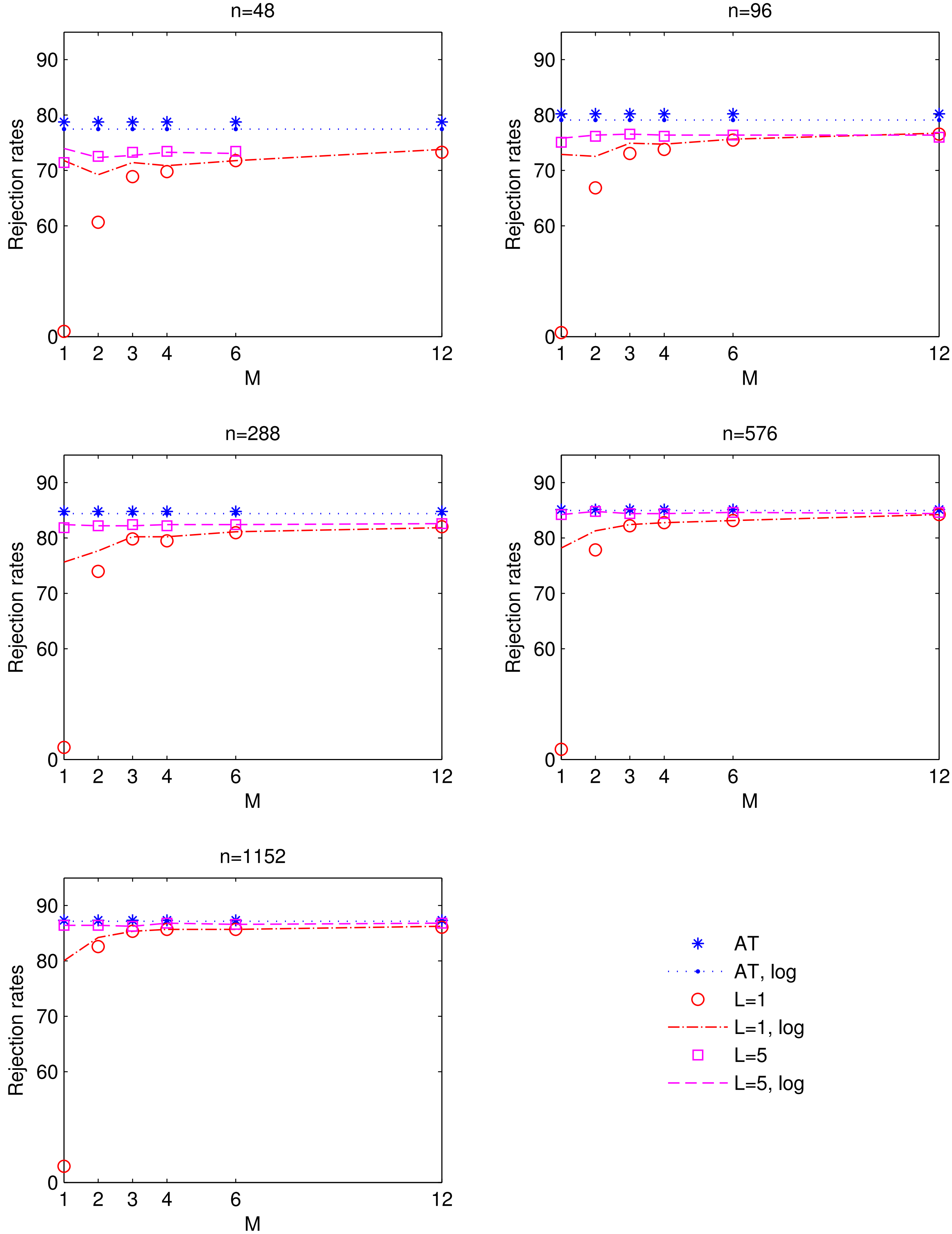

Figure 2: SV2F model without diurnal effects, finite activity jumps

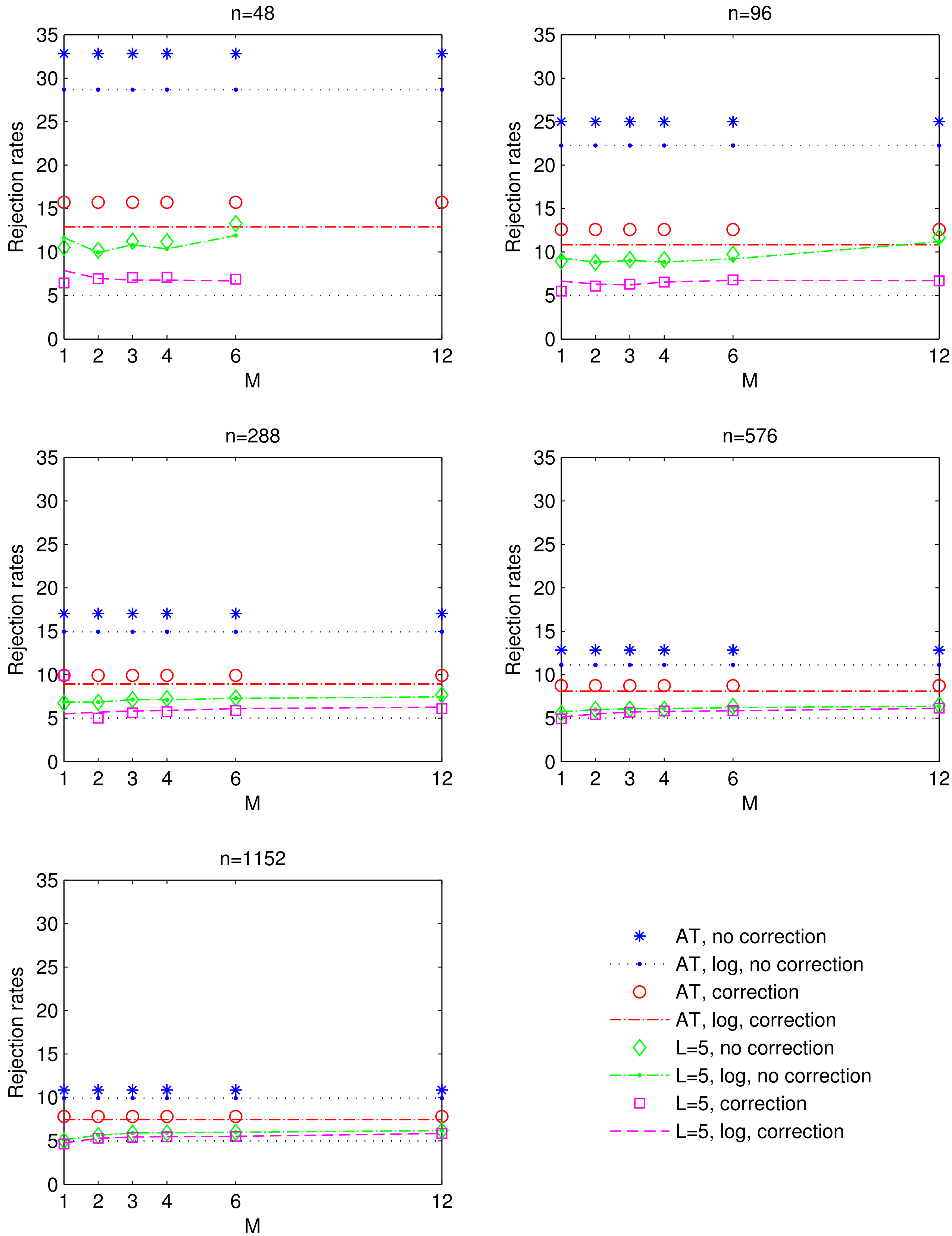

Figure 3: SV2F model with diurnal effects, no jumps

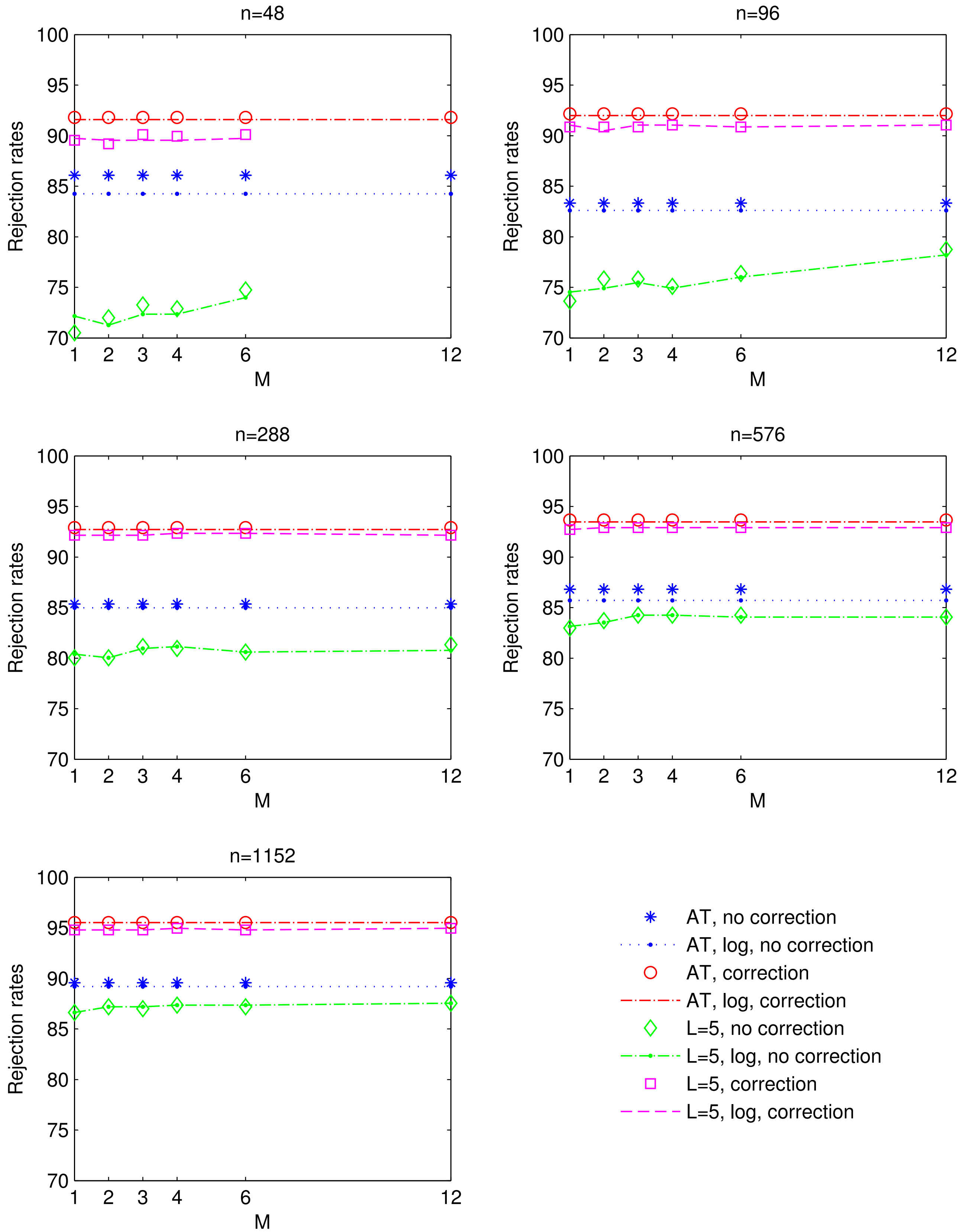

Figure 4: SV2F model with diurnal effects, finite activity jumps

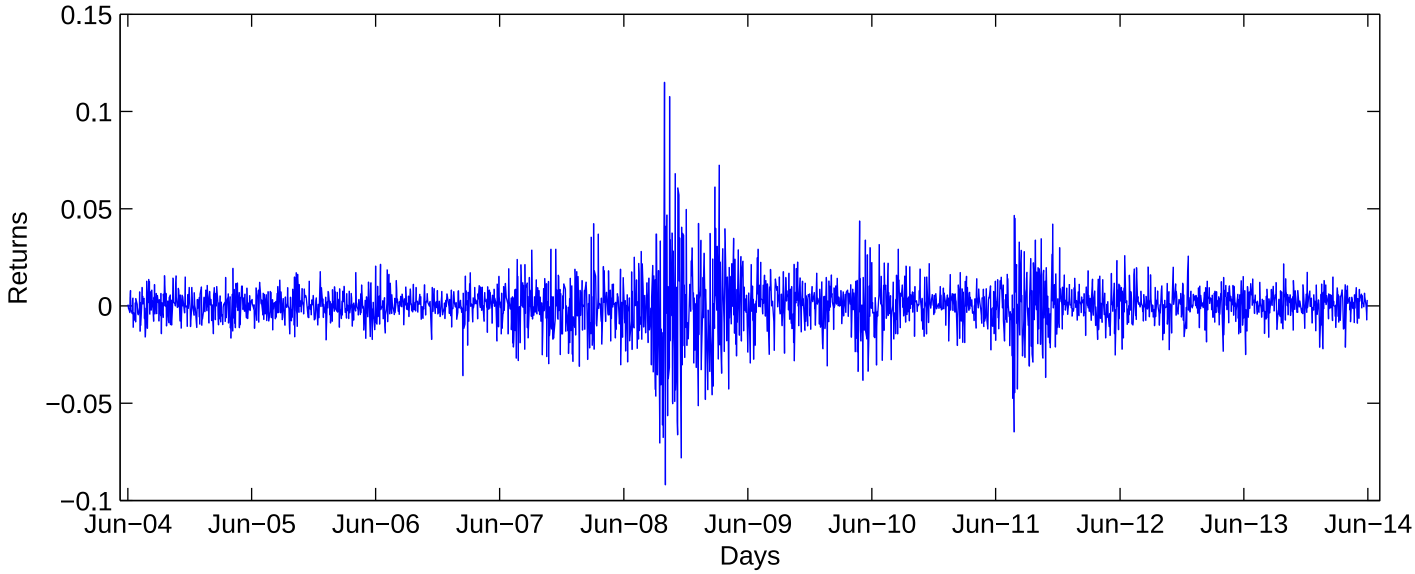

Figure 5: Daily returns on SPY from June 15, 2004 through June 13, 2014.

Figure 5 shows the series of daily returns on SPY over the 2497 trading days considered. The 2008 financial crisis is noticeable with large returns appearing in the third quarter of 2008 and the first quarter of 2009. We can actually distinguish three subperiods for SPY: '*Before crisis*', from the beginning of the sample (June 15, 2004) through August 29 2008 (1053 trading days); '*Crisis*', from September 2, 2008 through May 29, 2009 (185 trading days), and '*After crisis*', from June 2, 2009 through June 13, 2014 (1253 trading days).

Table 1 gives some summary statistics on daily returns and 5-min-return-based realized volatility (RV) and realized bipower variation (BV) over the mentioned periods. The average daily returns before and after the crisis are positive (1.53 and 7.2 basis points, respectively) whereas the average return over the crisis is -12.9 basis points. Daily averages of RV and BV are also quite high during the crisis with both culminating to 6 times their respective level across the whole sample. The average contribution of jumps to realized volatility as measured by $RJ = (RV - BV)/RV$ also deepens during the crisis period to 5%, whereas the 7% found for the full sample and in pre- and post-crisis periods seems in line with the findings of Huang and Tauchen (2005) for S&P 500 future index.

Table 2 shows the percentage of days identified with a jump (jump days) by the asymptotic and the bootstrap tests. Both the linear and the log versions of the test statistic are considered. In line with the simulation findings, the asymptotic tests tend to over detect jumps compared to the bootstrap tests. The asymptotic test based on the linear test statistic detects 26% of jump days in the full sample while the bootstrap tests detect only up to about 16% of jump days. Our simulation results with $n = 96$ (the closest to 78, the number of 5-min returns in a trading day) suggest that the bootstrap test with $L = 1$ performs best at $M = 3, 4$ while the bootstrap with $L = 5$ is quite stable through M, but may lead to overrejections under the null. Under the alternative of jumps, $L = 5$ yields larger rejections than $L = 1$ independently of M. The empirical results in Table 2 confirm these patterns, with the choice of $L = 5$ detecting more jump days than $L = 1$. The lack of power of the bootstrap test for $L = 1$, $M = 1$ (and its conservativeness under the null) means that the 2.8% of jump days detected by this test should not be trusted. Similar observations apply to the periods before and after crisis. During the crisis period, the gap between asymptotic and bootstrap tests narrows from 10 to 7 percentage points. The percentage of jump days detected by the asymptotic test also reduces to 21% and the bootstrap tests to 14%. As expected, the asymptotic test based on the log statistic

Table 1: This table gives the average daily return, realized volatility (RV), realized bipower variation (BV) and the contribution of jumps to realized volatility (RJ) of SPY over each period along with their standard deviations. RV and BV are based on 5-min intra-day returns. These statistics are also given over days identified with and without jumps using the version of the test based on $\log(RV/BV)$ ($\alpha = 0.05$)

	Returns $\times 10^4$	$RV \times 10^4$	$BV \times 10^4$	RJ
		Full sample: June 15, 2004-June 13, 2014 (2497 days)		
Mean	2.93	0.99	0.95	0.07
SD	126.00	2.60	2.52	0.11
		Before crisis: June 15, 2004-August 29, 2008 (1053 days)		
Mean	1.53	0.55	0.51	0.07
SD	86.91	0.66	0.64	0.11
		During crisis: September 2, 2008-May 29, 2009 (185 days)		
Mean	-12.90	6.06	5.82	0.05
SD	313.03	7.31	7.03	0.11
		After crisis: June 1, 2009-June 13, 2014 (1259 days)		
Mean	7.20	0.93	0.89	0.07
SD	117.08	1.61	1.77	0.13
		Days identified with jumps by the asymptotic log test (582 days)		
Mean	10.80	0.82	0.64	0.22
SD	129.53	1.96	1.52	0.07
		Days identified without jumps by the asymptotic log test (1915 days)		
Mean	0.54	1.05	1.04	0.02
SD	124.53	2.76	2.75	0.08
		Days identified with jumps by the bootstrap log test ($M = 1, L = 5$, 361 days)		
Mean	12.41	0.83	0.62	0.25
SD	139.96	1.92	1.42	0.07
		Days identified without jumps by the bootstrap log test ($M = 1, L = 5$, 2136 days)		
Mean	1.33	1.02	1.00	0.04
SD	123.45	2.70	2.66	0.09

Table 2: Percentage of days identified as jumps by the daily statistics ($\alpha = 0.05$) based on 5-min returns

	Tests based on $RV - BV$						Tests based on $\log(RV/BV)$					
		Bootstrap tests						Bootstrap tests				
		M						M				
		1	2	3	4	6		1	2	3	4	6
Full sample: June 15, 2004 through June 13, 2014 (2497 days)												
Asymp.	26.2						23.3					
$L = 1$		2.8	12.0	13.3	13.5	15.1		16.0	16.0	16.5	16.1	17.2
$L = 5$		12.3	15.7	16.4	16.4	16.8		14.5	15.8	15.5	15.5	15.9
Before crisis: June 15, 2004 through August 29, 2008 (1053 days)												
Asymp.	25.4						22.4					
$L = 1$		3.1	11.9	13.7	13.4	15.4		15.6	15.8	16.6	16.1	17.4
$L = 5$		12.1	16.1	16.4	16.4	16.5		14.4	16.1	15.8	15.2	15.5
During crisis: September 2, 2008 through May 29, 2009 (185 days)												
Asymp.	21.6						18.9					
$L = 1$		1.6	10.3	11.4	13.0	12.4		14.1	14.1	13.0	14.1	13.5
$L = 5$		10.3	12.4	12.4	13.0	14.1		11.9	12.4	12.4	13.0	13.0
After crisis: June 1, 2009 through June 13, 2014 (1259 days)												
Asymp.	27.6						24.6					
$L = 1$		2.8	12.4	13.3	13.7	15.3		16.7	16.5	17.0	16.5	17.6
$L = 5$		12.8	15.8	17.0	16.8	17.4		14.9	16.1	15.8	16.0	16.8

detects 3 percentage points less of jump days compared to the linear version. The bootstrap tests are rather stable when applied to log or linear version of the statistic, as in our simulations.

We also investigate the presence of diurnal effect in our return series. The presence of diurnality may further distort the asymptotic test, as shown by our Monte Carlo experiments. Figure 6 plots the diurnal pattern of SPY. The graphs display average absolute 5-min returns over the days in the specified sample. (See Andersen and Bollerslev (1997).) The U-shape of these graphs highlights the presence of diurnality in volatility: higher volatility at the start and the end of the trading sessions. This pattern looks stronger in the crisis period than in the other samples.

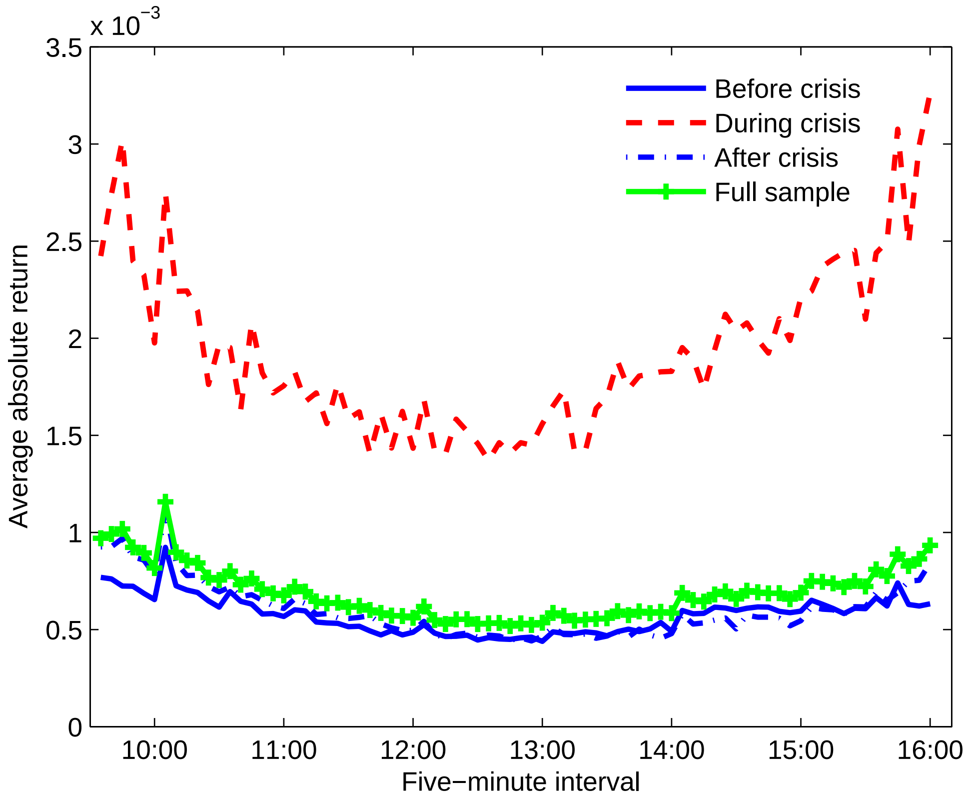

Figure 6: Diurnal pattern of SPY. The graph displays the average (over the specified samples) of absolute 5-min intraday returns of each trading day. ‘Before crisis’ refers to the sample from June 15, 2004 through August 29, 2008 (before the 2008 financial crash). ‘During crisis’ refers to the period from September 2, 2008 through May 29, 2009 and ‘After crisis’ refers to the period from June 1, 2009 through June 13, 2014.

Table 3 is analogue to Table 2 but with tests based on returns corrected for diurnal effect along the procedure described in the section on Monte Carlo experiments. The most noticeable fact is that more jump days are now detected in the period of crisis by the asymptotic test (an increase by 2 percentage points) and less jumps are detected for the other periods. Overall, the jump pattern detected seems uniform through the whole sample studied at about 24% of jump days. The bootstrap tests are rather robust in non crisis periods but detect slightly more jumps in the period of crisis after correction for diurnality. Overall, the bootstrap tests detect about 15% of jump days across the whole sample. Accounting for diurnality also affects the test based on log-statistic. The asymptotic test detects less jumps in non-crisis periods (than without correction for diurnality) and more jumps are detected in the crisis period. The bootstrap tests lead to relatively unchanged conclusions over non crisis periods,

Table 3: Percentage of days identified as jumps by the daily statistics ($\alpha = 0.05$) based on 5-min returns. The tests are applied to returns after correction for diurnal effect.

	Tests based on $RV - BV$						Tests based on $\log(RV/BV)$					
		Bootstrap tests						Bootstrap tests				
		M						M				
		1	2	3	4	6		1	2	3	4	6
Full sample: June 15, 2004 through June 13, 2014 (2497 days)												
Asymp.	24.0						20.5					
$L = 1$		2.8	10.7	13.0	13.4	14.5		14.5	15.6	15.5	15.5	16.0
$L = 5$		11.1	14.8	15.3	15.4	15.4		12.7	14.6	14.5	14.6	13.9
Before crisis: June 15, 2004 through August 29, 2008 (1053 days)												
Asymp.	22.4						19.0					
$L = 1$		2.9	10.4	13.0	12.8	14.8		14.0	14.9	14.9	14.6	15.8
$L = 5$		11.2	14.4	15.4	14.9	15.2		12.3	14.4	14.3	14.2	13.8
During crisis: September 2, 2008 through May 29, 2009 (185 days)												
Asymp.	23.8						21.6					
$L = 1$		2.2	10.3	11.9	13.0	13.5		16.2	17.3	16.8	17.3	15.7
$L = 5$		10.8	15.7	14.6	15.7	15.1		13.5	14.6	15.1	13.5	11.9
After crisis: June 1, 2009 through June 13, 2014 (1259 days)												
Asymp.	25.3						21.7					
$L = 1$		2.9	11.0	13.1	13.9	14.3		14.7	15.9	15.8	16.0	16.2
$L = 5$		11.0	14.9	15.3	15.8	15.6		13.0	14.8	14.5	15.1	14.4

but detect slightly more jumps during the crisis. Overall, the log asymptotic test detects about 20% of jump days whereas the bootstrap tests detect about 15% of jump days.

To conclude, the asymptotic tests over detect jumps compared to the bootstrap, with the log version of the asymptotic test yielding the smallest detection rates among the asymptotic tests. These rates are nevertheless larger than those obtained with the bootstrap by at least 5 percentage points (when $L = 5$), whether the latter is applied to log or linear version of the test statistic and whether the returns are corrected for diurnality or not.

8 Conclusion

The main contribution of this paper is to propose bootstrap methods for testing the null hypothesis of "no jumps". The methods generate bootstrap intraday returns from a Gaussian distribution with variance given by a local realized measure of integrated volatility $\hat{v}_i^n$. We first provide a set of high level conditions on $\{\hat{v}_i^n\}$ such that any bootstrap method of this form is asymptotically valid when testing for jumps using the test statistic proposed by Barndorff-Nielsen and Shephard (2006). This means in particular that the bootstrap is able to control size and is consistent under the alternative of jumps.

We then provide a detailed analysis of two examples. The first example considers $\hat{v}_i^n$ equal to a local realized volatility measure computed over non-overlapping intervals of size M (as suggested by Hounyo (2013) in the context of bootstrap inference for realized volatility). We show that this bootstrap method is able to mimic the null distribution of the bootstrap test of BN-S (2006), for any fixed value of M, thus controlling size. Nevertheless, under the alternative of jumps, this bootstrap method does not replicate the null distribution of the test, which can compromise its ability to reject the null when this hypothesis is false. In particular, for the special case of $M = 1$ (in which case the local Gaussian bootstrap becomes a regular Gaussian wild bootstrap), we show that the bootstrap test statistic diverges to infinity under the alternative of jumps. Given that the two statistics diverge at the same rate, sharp conclusions about the power of the test cannot be obtained. However, our simulations show that this bootstrap method has very low power, even for large sample sizes.

The main reason for the failure of the bootstrap method based on the local realized volatility measure is that this choice leads to a bootstrap test variance that is not robust to jumps. Therefore, we consider a second choice of $\hat{v}_i^n$ that ensures that the bootstrap variance is robust to jumps. More specifically, we let $\hat{v}_i^n$ be equal to multiproducts of powers of local realized volatility measures, where the number of products is L and the multipowers are given by $\{p_l : l = 1, \ldots, L\}$ such that $\sum_{l=1}^{L} p_l = 1$. When $L = 1$, we obtain the choice of $\hat{v}_i^n$ proposed by Hounyo (2013). We show that if we let $\max(p_l) < 1/2$, which is equivalent to letting $L > 2$ when $p_l = 1/L$ for all $l = 1, \ldots, L$, then the bootstrap test statistic is asymptotically $N(0, 1)$ under both the null and the alternative hypothesis. This implies that under these conditions the bootstrap test controls size and is alternative-consistent. In our Monte Carlo experiments, choosing $L = 5$ ensures good size and power properties of the bootstrap across the two different models we simulate.

Although our simulations clearly indicate that the bootstrap provides more accurate inference than the existing asymptotic tests, we do not prove in this paper that the bootstrap provides asymptotic refinements over the asymptotic theory. Because the tests considered here involve multipower variations, computing the necessary higher order cumulants would be extremely cumbersome and would require imposing restrictive conditions such as no leverage effects and no drift. Instead, we have decided to focus on the properties of the tests under the absence *and* the presence of jumps and leave the study of higher order asymptotic refinements for further research.

A Appendix A: proofs of the general bootstrap results in Section 2

Proof of Lemma 3.1. Part (a1) follows from $E^*(u_i) = E^*\left(\eta_i^2\right) = 1$ and (a2) from $E^*(w_i) = E^*(|\eta_{i-1}|\,|\eta_i|) = k_{1,1}^2$. For (a3), note that u_i is i.i.d. χ_1^2, which implies that $Var^*\left(\sqrt{n}\sum_{i=1}^n \hat{v}_i^n \cdot u_i\right) = n\sum_{i=1}^n \left(\hat{v}_i^n\right)^2 \cdot Var^*(u_i) = 2n\sum_{i=1}^n \left(\hat{v}_i^n\right)^2$. For (a4), note that w_i is one lag-dependent, i.e. $Cov^*(w_i, w_j) = 0$ for $|i-j| > 1$. Thus,

$$Var^*\left(\sqrt{n}BV_n^*\right) = \frac{1}{k_{1,1}^4} n\left(\sum_{i=2}^n \left(\hat{v}_i^n\right)\left(\hat{v}_{i-1}^n\right) Var^*(w_i) + 2\sum_{i=3}^n \left(\hat{v}_i^n\right)^{1/2}\left(\hat{v}_{i-1}^n\right)\left(\hat{v}_{i-2}^n\right)^{1/2} Cov^*(w_i, w_{i-1}).\right)$$

The result follows by noting that $Var^*(w_i) = Var^*(|\eta_{i-1}|\,|\eta_i|) = 1 - k_{1,1}^4$ and $Cov^*(w_i, w_{i-1}) = k_{1,1}^2 - k_{1,1}^4$. For part (a5), note that for all $i = 1, \ldots, n$, $Cov^*(u_i, w_i) = k_{1,3}k_{1,1} - k_{1,1}^2$, $Cov^*(u_i, w_{i-j}) = 0$ for $j > 0$, and $Cov^*(u_i, w_{i+j}) = 0$ for all $j > 0$ except when $j = 1$, where $Cov^*(u_i, w_{i+1}) = k_{1,3}k_{1,1} - k_{1,1}^2$. The result follows from standard calculations noting that $k_{1,3} = 2k_{1,1}$.

Proof of Theorem 3.1. Part (a1): Write

$$Z_n^* = \Sigma_n^{*\,-1/2}\sqrt{n}\sum_{i=1}^n D_i e_i^* \equiv \sqrt{n}\sum_{i=1}^n z_i^*,$$

with $z_i^* \equiv \Sigma_n^{*-1/2} D_i e_i^*$, and

$$D_i = \begin{pmatrix} \hat{v}_i^n & 0 \\ 0 & \frac{1}{k_{1,1}^2}\left(\hat{v}_i^n\right)^{1/2}\left(\hat{v}_{i-1}^n\right)^{1/2} \end{pmatrix}, \quad \text{and} \quad e_i^* = \begin{pmatrix} u_i - E^*(u_i) \\ w_i - E^*(w_i) \end{pmatrix},$$

where we set $\hat{v}_0^n = 0$ and where $u_i = \eta_i^2$ and $w_i = |\eta_i|\,|\eta_{i-1}|$ and $\eta_i \sim$ i.i.d. $N(0,1)$. Note that e_i^* is a zero mean vector that is one-dependent. We follow Pauly (2011) and rely on a modified Cramer-Wold device to establish the bootstrap CLT. Let $D = \{\lambda_k : k \in N\}$ be a countable dense subset of the unit circle of R^2. We have to show that for any $\lambda \in D$, $\lambda' Z_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, as $n \to \infty$.

From Lemma 3.1, we have $Var^*(\lambda' Z_n^*) = 1$ for all n. Hence, to conclude, it remains to establish that $\lambda' Z_n^*$ is asymptotically normally distributed, conditionally on the original sample and with probability P approaching one. Since z_i^*'s are lag-one dependent, we adopt the large-block-small-block type of argument to prove this central limit result (see Shao (2010) for an example of this idea). The large blocks are made of L_n successive observations followed by a small bock that is made of a single element.

More precisely, let L_n be an integer such that $L_n \propto n^\alpha$ for $0 < \alpha < \frac{\delta}{2(1+\delta)}$some $\delta > 0$. Let $k_n = \lfloor \frac{n}{L_n+1} \rfloor$. Define the (large) blocks $L_j = \{i \in \mathbb{N} : (j-1)(L_n+1)+1 \le i \le j\,(L_n+1)-1\}$, where $1 \le j \le k_n$ and $L_{k_n+1} = \{i \in N : k_n(L_n+1)+1 \le i \le n\}$. Let $U_j^* = \sum_{i \in \mathcal{L}_j} \lambda' z_i^*$, $j = 1, \ldots, k_n+1$. Clearly,

$$\lambda' Z_n^* = \sqrt{n} \sum_{j=1}^{k_n+1} U_j^* + \sqrt{n} \sum_{j=1}^{k_n} z_{j(L_n+1)}^*.$$

Next, we show that under Condition A,

(i) $\sqrt{n} \sum_{j=1}^{k_n} z_{j(L_n+1)}^* = o_{P^*}(1)$, in prob-$P$; and

(ii) for some $\delta > 0$,

$$\sum_{j=1}^{k_n+1} E^* \left|\sqrt{n} U_j^*\right|^{2+\delta} \xrightarrow{P} 0.$$

Condition (ii) suffices to show that $\sqrt{n}\sum_{j=1}^{k_n+1} U_j^* \to^{d^*} N(0,1)$, in prob-$P$, since $\left\{U_j^*\right\}$ form an independent array, conditionally on the sample. The result then follows given condition (i). Let us establish (i). Since $E^*(z_i^*) = 0$ for all i, it suffices to show that $Var^*(\sqrt{n}\sum_{j=1}^{k_n} z_{j(L_n+1)}^*) = o_P(1)$. For this, since $L_n \ge 1$ for n sufficiently large, $z_{j(L_n+1)}^*$'s are independent along j conditionally on the sample so that

$$Var^* \left(\sqrt{n} \sum_{j=1}^{k_n} z_{j(L_n+1)}^* \right) = \lambda' \Sigma_n^{*-1/2} \Omega_n^* \Sigma_n^{*-1/2} \lambda,$$

where $\Omega_n^* \equiv Var^* \left(\sqrt{n} \sum_{j=1}^{k_n} D_{j(L_{n+1})} e_{j(L_n+1)}^* \right)$. It follows that

$$\left\| Var^* \left(\sqrt{n} \sum_{j=1}^{k_n} z_{j(L_n+1)}^* \right) \right\| \le \|\lambda\|^2 \left\| \Sigma_n^{*-1/2} \right\|^2 \|\Omega_n^*\|.$$

Since $\Sigma_n^* \to^P \Sigma^* > 0$ (by Condition A, and given Lemma 3.1), it follows that $\left\|\Sigma_n^{*-1/2}\right\|^2 = tr\left(\Sigma_n^{*-1}\right) \to^P tr\left(\Sigma^{*-1}\right) = \left\|\Sigma^{*-1}\right\|^2 = 2\lambda_{\max}^2\left(\Sigma^{*-1}\right) = 2\lambda_{\min}^{-2}\left(\Sigma^*\right) = O_P(1)$, given that $IQ > 0$ a.s. Next we analyze $\|\Omega_n^*\|$. We have that

$$\Omega_n^* = n \sum_{j=1}^{k_n} D_{j(L_{n+1})} E^* \left(e_{j(L_n+1)}^* e_{j(L_n+1)}^{*\prime} \right) D_{j(L_{n+1})}',$$

which implies that

$$\left\|\Omega_n^*\right\| \leq n \sum_{j=1}^{k_n} \left\|D_{j(L_{n+1})}\right\|^2 \left\|E^*\left(e^*_{j(L_n+1)} e^{*\prime}_{j(L_n+1)}\right)\right\| \leq Cn \sum_{j=1}^{k_n} \left\|D_{j(L_{n+1})}\right\|^2,$$

for some constant C independent of n (note that the moments of e_i^* do not depend on n). In the following, we use C to denote any constant that is independent of n where the definition may change from line to line. Since for any i,

$$\left\|D_i\right\|^2 = \left(\hat{v}_i^n\right)^2 + \frac{1}{k_{1,1}^4} \left(\hat{v}_i^n\right)\left(\hat{v}_{i-1}^n\right), \tag{15}$$

it follows that

$$\left\|\Omega_n^*\right\| \leq Cn \sum_{j=1}^{k_n} \left\|D_{j(L_{n+1})}\right\|^2 \leq Cn \sum_{j=1}^{k_n} \left(\left(\hat{v}^n_{j(L_n+1)}\right)^2 + \frac{1}{k_{1,1}^4}\left(\hat{v}^n_{j(L_n+1)}\right)\left(\hat{v}^n_{j(L_n+1)-1}\right)\right),$$

where $k_n = \lfloor \frac{n}{L_n+1} \rfloor \leq \frac{n}{L_n} = n^{1-\alpha}$ by letting $L_n = cn^\alpha$. By Condition A.(iii), $x_n \equiv n \sum_{j=1}^{k_n} \left(\hat{v}^n_{j(L_n+1)}\right)^2 = o_P(1)$ and this suffices to prove that $\left\|\Omega_n^*\right\| = o_P(1)$. Next, we verify (ii). For any $1 \leq j \leq k_n$, by the c-r inequality,

$$\left|U_j^*\right|^{2+\delta} \leq \left|\sum_{i \in \mathcal{L}_j} \lambda' z_i^*\right|^{2+\delta} \leq L_n^{2+\delta-1} \left\|\lambda\right\|^{2+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{i \in \mathcal{L}_j} \left\|D_i\right\|^{2+\delta} \left\|e_i^*\right\|^{2+\delta}.$$

It follows that

$$\begin{aligned} E^* \left|U_j^*\right|^{2+\delta} &\leq L_n^{1+\delta} \left\|\lambda\right\|^{2+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{i \in \mathcal{L}_j} \left\|D_i\right\|^{2+\delta} E^* \left\|e_i^*\right\|^{2+\delta} \\ &\leq CL_n^{1+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{i \in \mathcal{L}_j} \left\|D_i\right\|^{2+\delta}, \end{aligned}$$

implying that

$$\begin{aligned} \sum_{j=1}^{k_n+1} E^* \left|\sqrt{n} U_j^*\right|^{2+\delta} &\leq Cn^{1+\delta/2} L_n^{1+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{j=1}^{k_n+1} \sum_{i \in \mathcal{L}_j} \left\|D_i\right\|^{2+\delta} \\ &\leq Cn^{1+\delta/2} L_n^{1+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{j=1}^{k_n+1} \sum_{i \in \mathcal{L}_j} \left(\left(\hat{v}_i^n\right)^{(2+\delta)} + \left(\hat{v}_i^n\right)^{\frac{2+\delta}{2}} \left(\hat{v}_{i-1}^n\right)^{\frac{2+\delta}{2}}\right) \\ &\leq Cn^{1+\delta/2} L_n^{1+\delta} \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} \sum_{i=1}^{n} \left(\left(\hat{v}_i^n\right)^{(2+\delta)} + \left(\hat{v}_i^n\right)^{\frac{2+\delta}{2}} \left(\hat{v}_{i-1}^n\right)^{\frac{2+\delta}{2}}\right) \\ &\leq C \left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} L_n^{1+\delta} n^{-\delta/2} \left(n^{1+\delta} \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^{(2+\delta)}\right), \end{aligned}$$

where the second inequality follows from (15) and the last by the c-r inequality. Given Condition A.(ii), the sum in parenthesis is $O_P(1)$ and therefore the whole term is $O_P\left(\frac{L_n^{1+\delta}}{n^{\delta/2}}\right)$. Setting $L_n = Cn^\alpha$, this term is of order $O\left(\frac{n^{\alpha(1+\delta)}}{n^{\delta/2}}\right) = O\left(n^{\alpha(1+\delta)-\delta/2}\right) = o(1)$ if $\alpha(1+\delta) - \delta/2 < 0$, or equivalently, if

$\alpha < \frac{\delta}{2(1+\delta)}$. This concludes the proof of part (a1). Part (a2) follows from an application of the delta method.

Proof of Theorem 3.2. Given Theorem 3.1, part (a2) follows from (a1). To show (a1), note that

$$E^*\left(\widetilde{IQ}_n^*\right) = \frac{1}{c_{4/3,4/3,4/3}} n \sum_{i=3}^{n} |\hat{v}_i^n|^{2/3} \left|\hat{v}_{i-1}^n\right|^{2/3} \left|\hat{v}_{i-2}^n\right|^{2/3} \to^P IQ,$$

under Condition B.(i) whereas B.(ii) ensures that $Var^*\left(\widetilde{IQ}_n^*\right) \to^P 0$. In particular, let $x_i^* = |r_i^*|^{\frac{4}{3}} |r_{i-1}^*|^{\frac{4}{3}} |r_{i-2}^*|^{\frac{4}{3}}$ and note that x_i^* is lag-2-dependent. Hence,

$$\begin{aligned}
Var^*\left(\widetilde{IQ}_n^*\right) &= \frac{n^2}{k_{1,\frac{4}{3}}^6} Var^*\left(\sum_{i=3}^{n} x_i^*\right) \\
&= \frac{n^2}{k_{1,\frac{4}{3}}^6}\left(\sum_{i=3}^{n} Var^*\left(x_i^*\right) + 2\sum_{i=4}^{n} Cov^*\left(x_{i-1}^*, x_i^*\right) + 2\sum_{i=5}^{n} Cov^*\left(x_{i-2}^*, x_i^*\right)\right) \\
&\leq C\left(n^2 \sum_{i=3}^{n} \left(\hat{v}_i^n\right)^{4/3} \left(\hat{v}_{i-1}^n\right)^{4/3} \left(\hat{v}_{i-2}^n\right)^{4/3} + n^2 \sum_{i=4}^{n} \left(\hat{v}_i^n\right)^{2/3} \left(\hat{v}_{i-1}^n\right)^{4/3} \left(\hat{v}_{i-2}^n\right)^{4/3} \left(\hat{v}_{i-3}^n\right)^{2/3}\right. \\
&\quad \left. + n^2 \sum_{i=5}^{n} \left(\hat{v}_i^n\right)^{2/3} \left(\hat{v}_{i-1}^n\right)^{2/3} \left(\hat{v}_{i-2}^n\right)^{4/3} \left(\hat{v}_{i-3}^n\right)^{2/3} \left(\hat{v}_{i-4}^n\right)^{2/3}\right),
\end{aligned}$$

for some constant C that does not depend on n. By Condition B.(ii), each of the sums inside the parenthesis is $o_P(1)$.

Proof of Theorem 3.3. Strong asymptotic size control: Let $S \subset \Omega_0$ denote any measurable subset of Ω_0 with $P(S) > 0$. Since $T_n \xrightarrow{st} N(0,1)$, in restriction to Ω_0, we have (see Aït-Sahalia and Jacod (2014, Theorem 10.1, p. 339)) that for any $x \in \mathbb{R}$, $P(T_n \leq x|S) \to \Phi(x)$, as $n \to \infty$, where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. Since $\Phi(x)$ is continuous, $\sup_{x\in\mathbb{R}} |P(T_n \leq x|S) - \Phi(x)| \to 0$, *as* $n \to \infty$. By the validity of the bootstrap on Ω_0 under Conditions A and B, we have that $\sup_{x\in\mathbb{R}} |F_n^*(x) - \Phi(x)| \xrightarrow{P} 0$, where $F_n^*(x) \equiv P^*(T_n^* \leq x)$. Thus, $\sup_{x\in\mathbb{R}} |F_n^*(x) - P(T_n \leq x|S)| \xrightarrow{P} 0$. It follows that

$$\left|F_n^*(q_{n,1-\alpha}^*) - P(T_n \leq q_{n,1-\alpha}^*|S)\right| = \left|1 - \alpha - 1 + P(T_n > q_{n,1-\alpha}^*|S)\right| \xrightarrow{P} 0,$$

i.e. $P(T_n > q_{n,1-\alpha}^*|S) \xrightarrow{P} \alpha$ as $n \to \infty$. This establishes the first part of the theorem.

Alternative-consistency: Given Definition 5.19 of Aït-Sahalia and Jacod (2014), this amounts to showing that

$$P(\left\{T_n \leq q_{n,1-\alpha}^*\right\} \cap \Omega_1) \to 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Let $\epsilon > 0$. Since $T_n^* = O_{P^*}(1)$ in prob-P, we have $q_{n,1-\alpha}^* = O_P(1)$. (See proof below.) Hence,

$$\exists N_1 \in \mathbb{N} \quad \text{and} \quad M_0 > 0: \quad P(q_{n,1-\alpha}^* \leq M_0) > 1 - \epsilon, \quad \forall n \geq N_1.$$

Also, since $T_n \xrightarrow{P} +\infty$ on Ω_1, for any $M < \infty$, $P(\{T_n \leq M\} \cap \Omega_1) < \epsilon$ for all n sufficiently large. Thus, in particular,

$$\exists N_2 \in \mathbb{N} : P(\{T_n \leq M_0\} \cap \Omega_1) < \epsilon, \quad \forall n \geq N_2. \tag{17}$$

Hence, for $n \geq \max(N_1, N_2)$,

$$\begin{aligned}
P(\{T_n \leq q^*_{n,1-\alpha}\} \cap \Omega_1) &= P\left(\{T_n \leq q^*_{n,1-\alpha} \cap q^*_{n,1-\alpha} \leq M_0\} \cap \Omega_1\right) + P\left(\{T_n \leq q^*_{n,1-\alpha} \cap q^*_{n,1-\alpha} > M_0\} \cap \Omega_1\right) \\
&\leq P\left(\{T_n \leq M_0\} \cap \Omega_1\right) + P\left(\{T_n \leq q^*_{n,1-\alpha} \cap q^*_{n,1-\alpha} > M_0\} \cap \Omega_1\right) \\
&\leq P\left(\{T_n \leq M_0\} \cap \Omega_1\right) + P\left(\{q^*_{n,1-\alpha} > M_0\} \cap \Omega_1\right) \\
&\leq P\left(\{T_n \leq M_0\} \cap \Omega_1\right) + P\left(q^*_{n,1-\alpha} > M_0\right) < \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, this proves (16). To complete the proof, we prove that $q^*_{n,1-\alpha} = O_P(1)$. Let $\epsilon > 0$. Since $T^*_n = O_{P^*}(1)$, in probability-P, by definition, there exists $M > 0$ such that

$$P\left(P^*\left(T^*_n > M\right) < \alpha\right) > 1 - \epsilon,$$

for any n large enough. By definition of $q^*_{n,1-\alpha}$, $\{P^*(T^*_n > M) < \alpha\}$ implies that $\{q^*_{n,1-\alpha} \leq M\}$. Hence, $P(P^*(T^*_n > M) < \alpha) \leq P(q^*_{n,1-\alpha} \leq M)$. As a result, $P\left(q^*_{n,1-\alpha} \leq M\right) > 1 - \epsilon$ for n large enough, proving that $q^*_{n,1-\alpha} = O_P(1)$.

To prove Lemma 3.2, we rely on the following auxiliary result; the proof of which is omitted since it follows from simple algebra.

Lemma A.1 *Let $\{a_i : i = 1, \ldots, n\}$ be any sequence such that for $j = 1, \ldots, n/M$, $a_{i+(j-1)M} = \bar{a}_j$, $i = 1, \ldots, M$. Then, for any $\{s_1, \ldots, s_K\}$ such that $s = \sum_{k=1}^K s_k$ and $\bar{s}_k = \sum_{l=1}^k s_l$, we have that for $M \geq K - 1$,*

$$\sum_{i=1}^{n} \prod_{k=1}^{K} a_{i-k+1}^{s_k} = (M - K + 1) \sum_{j=1}^{n/M} (\bar{a}_j)^s + \sum_{k=1}^{K-1} \sum_{j=2}^{n/M} (\bar{a}_j)^{\bar{s}_k} (\bar{a}_{j-1})^{s-\bar{s}_k}.$$

Proof of Lemma 3.2. Parts (a1) and (a2) follow from Lemma A.1. Part (a3) follows from replacing $\hat{v}^n_i = |r_i|^{2p_1} |r_{i-1}|^{2p_2} \ldots |r_{i-L+1}|^{2p_L}$ in the multipower variation of $\hat{v}^n_i$, collecting terms and using the definition of $MV_M^{[q,K]}(\{q_k\})$ with $M = 1$.

Proof of Lemma 3.3. Proof of part (a1): Since X_t is continuous on $[0,1]$ for any $\omega \in \Omega_0$, $J_t = \sum_{j=1}^{N_t} c_j$ is constant in $t \in [0,1]$ for any $\omega \in \Omega_0$. Hence $J_t - \sum_{j=1}^{N_0} c_j = 0$ on Ω_0, and for all $t \in [0,1]$. Consider

$$X'_t = Y_t + \sum_{j=1}^{N_0} c_j,$$

where Y_t as defined by Equation (2). Since $\sum_{j=1}^{N_0} c_j$ is constant in t, X'_t is a continuous process as a result of the continuity of Y_t. Let $MV_{M,X'}^{[q,K]}(q_k)$ be associated with X' as $MV_M^{[q,K]}(q_k)$ is associated with X. Since X'_t is continuous, by Theorem 3 of Mykland, Sheppard and Shephard (2012),

$$MV_{M,X'}^{[q,K]}(q_k) \xrightarrow{P} \int_0^1 \sigma_s^q ds.$$

Hence, so long as $P(\Omega_0) > 0$,

$$P\left(\left|MV_{M,X'}^{[q,K]}(q_k) - \int_0^1 \sigma_s^q ds\right| > \epsilon \,\middle|\, \Omega_0\right) \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \epsilon > 0.$$

Also, $X_t = X'_t + J_t - \sum_{j=1}^{N_0} c_j$ and for all $\omega \in \Omega_0$, $X_t = X'_t$, for $t \in [0,1]$. Therefore, on Ω_0, $MV_{M,X'}^{[q,K]}(q_k) = MV_M^{[q,K]}(q_k)$. This completes the proof of (a.1) since

$$P\left(\left|MV_M^{[q,K]}(q_k) - \int_0^1 \sigma_s^q ds\right| > \epsilon \,\middle|\, \Omega_0\right) = P\left(\left|MV_{M,X'}^{[q,K]}(q_k) - \int_0^1 \sigma_s^q ds\right| > \epsilon \,\middle|\, \Omega_0\right).$$

Proof of (a2): Using the decomposition given in equation (2), and letting $MV_{M,Y}^{[q,K]}(q_k)$ denote the blocked multipower variation associated with the process Y, we have that

$$MV_M^{[q,K]}(q_k)-\int_0^1 \sigma_s^q ds = \left(MV_{M,Y}^{[q,K]}(q_k) - \int_0^1 \sigma_s^q ds\right)+\left(MV_M^{[q,K]}(q_k) - MV_{M,Y}^{[q,K]}(q_k)\right) \equiv A_1+A_2. \quad (18)$$

By part (a1) of this lemma, $A_1 = o_P(1)$ given that Y is continuous. Therefore, we only need to establish the order of magnitude of A_2. We use arguments similar to those of Barndorff-Nielsen, Shephard and Winkel (2005, cf. Section 3.1) to do so. For simplicity, we only give the details for $K = 2$. Letting $r_i = y_i + z_i$, where $y_i = Y_{i/n} - Y_{(i-1)/n}$ and $z_i = \sum_{j=N_{(i-1)/n}+1}^{N_{i/n}} c_j$, we can write

$$\begin{aligned}
\bar{R}_j &\equiv \frac{1}{M}\sum_{i=1}^{M} r_{i+(j-1)M}^2 = \frac{1}{M}\sum_{i=1}^{M}\left(y_{i+(j-1)M} + z_{i+(j-1)M}\right)^2 \\
&= \frac{1}{M}\sum_{i=1}^{M} y_{i+(j-1)M}^2 + \frac{1}{M}\sum_{i=1}^{M}\left(2y_{i+(j-1)M}z_{i+(j-1)M} + z_{i+(j-1)M}^2\right) \\
&\equiv \bar{Y}_j + \bar{Z}_j,
\end{aligned}$$

for any $j = 1, \ldots, n/M$. Thus,

$$\begin{aligned}
A_2 &= n^{-1+q/2}\frac{M^{q/2}}{\prod_{k=1}^{2} k_{M,q_k}} M\sum_{j=2}^{n/M}\left|\bar{R}_j\right|^{q_1/2}\left|\bar{R}_{j-1}\right|^{q_2/2} - n^{-1+q/2}\frac{M^{q/2}}{\prod_{k=1}^{2} k_{M,q_k}} M\sum_{j=2}^{n/M}\left|\bar{Y}_j\right|^{q_1/2}\left|\bar{Y}_{j-1}\right|^{q_2/2} \\
&= n^{-1+q/2}\frac{M^{q/2}}{\prod_{k=1}^{2} k_{M,q_k}} M\sum_{j=2}^{n/M}\left|\bar{Y}_j+\bar{Z}_j\right|^{q_1/2}\left|\bar{Y}_{j-1}+\bar{Z}_{j-1}\right|^{q_2/2} - n^{-1+q/2}\frac{M^{q/2}}{\prod_{k=1}^{2} k_{M,q_k}} M\sum_{j=2}^{n/M}\left|\bar{Y}_j\right|^{q_1/2}\left|\bar{Y}_{j-1}\right|^{q_2/2}.
\end{aligned}$$

Suppose $\max(q_k) < 2$, which implies that $0 < q_1/2 < 1$, and $\left|\bar{Y}_j + \bar{Z}_j\right|^{q_1/2} \le \left|\bar{Y}_j\right|^{q_1/2} + \left|\bar{Z}_j\right|^{q_1/2}$ by the C_r inequality (and similarly for the factor whose exponent is $q_2/2$). It follows that

$$|A_2| \le n^{-1+q/2}\sum_{j=2}^{n/M}\left|\bar{Z}_j\right|^{q_1/2}\left|\bar{Z}_{j-1}\right|^{q_2/2} + n^{-1+q/2}\sum_{j=2}^{n/M}\left|\bar{Z}_j\right|^{q_1/2}\left|\bar{Y}_{j-1}\right|^{q_2/2} + n^{-1+q/2}\sum_{j=2}^{n/M}\left|\bar{Y}_j\right|^{q_1/2}\left|\bar{Z}_{j-1}\right|^{q_2/2}$$

where we omit the factors depending on M (this is without loss of generality, since M is fixed). The first term is $o_P(1)$ because the probability that two jumps (or more) occur in two consecutive intervals of length M/n goes to zero as $n \to \infty$ for finite activity processes. Next we show that the same is true for the second and third terms. In particular, the second term can be bounded as follows,

$$n^{-1+q/2}\sum_{j=2}^{n/M}\left|\bar{Z}_j\right|^{q_1/2}\left|\bar{Y}_{j-1}\right|^{q_2/2} \le n^{-1+q/2}\underbrace{\left(\max_{2\le j\le n/M}\left|\bar{Y}_{j-1}\right|^{q_2/2}\right)}_{=O_P\left(n^{-q_2/2}|\log n|^{q_2/2}\right)}\underbrace{\sum_{j=2}^{n/M}\left|\bar{Z}_j\right|^{q_1/2}}_{=O_P(1)} = O_P\left(n^{-1+q_1/2}\left|\log n\right|^{q_2/2}\right), \quad (19)$$

where we use Levy's continuity theorem (see e.g. Proposition 1 of Barndorff-Nielsen, Shephard and Winkel (2005)) to bound the first factor and we use the fact that there are a finite number of jumps to bound the second factor (in particular, $\sum_{j=2}^{n/M}\left|\bar{Z}_j\right|^{q_1/2} \le \sum_{i=1}^{N_1}|c_i|^{q_1} = O_P(1)$). By a similar argument,

$$n^{-1+q/2}\sum_{j=2}^{n/M}\left|\bar{Y}_j\right|^{q_1/2}\left|\bar{Z}_{j-1}\right|^{q_2/2} = O_P\left(n^{-1+q_2/2}\left|\log n\right|^{q_1/2}\right). \quad (20)$$

The dominant term among (19) and (20) is the one associated with $\max(q_1, q_2)$. Thus, $|A_2| = O_P\left(n^{-1+\max(q_k)/2} |\log n|^{\frac{q-\max(q_k)}{2}}\right)$, which is $o_P(1)$ when $\max(q_k) < 2$.

Suppose now that $\max(q_k) \geq 2$. Then, either $q_1 \geq 2$ or $q_2 \geq 2$ (or both). By the C_r-inequality, we have that $\left|\bar{Y}_j + \bar{Z}_j\right|^{q_1/2} \leq 2^{\frac{q_1}{2}-1}\left(\left|\bar{Y}_j\right|^{q_1/2} + \left|\bar{Z}_j\right|^{q_1/2}\right)$, where now the constant in front of the parenthesis is larger than one (since $2^{q_1/2} \geq 2$) (and similarly for the term depending on q_2). It follows that for some constant C_q that depends on q_1, q_2 but not on n, we can bound A_2 as follows:

$$
\begin{aligned}
|A_2| \quad \leq \quad & C_q \left\{ n^{-1+q/2} \sum_{j=2}^{n/M} \left|\bar{Z}_j\right|^{q_1/2} \left|\bar{Z}_{j-1}\right|^{q_2/2} + n^{-1+q/2} \sum_{j=2}^{n/M} \left|\bar{Z}_j\right|^{q_1/2} \left|\bar{Y}_{j-1}\right|^{q_2/2} \right. \\
& \left. + n^{-1+q/2} \sum_{j=2}^{n/M} \left|\bar{Y}_j\right|^{q_1/2} \left|\bar{Z}_{j-1}\right|^{q_2/2} + n^{-1+q/2} \sum_{j=2}^{n/M} \left|\bar{Y}_j\right|^{q_1/2} \left|\bar{Y}_{j-1}\right|^{q_2/2} \right\}.
\end{aligned}
$$

The first three terms can be analyzed as above whereas the last term is $O_P(1)$ (since it depends only on the continuous process Y_t). Since $\max(q_1, q_2) \geq 2$, the dominant term is given by either the second or the third terms, depending on the value of $\max(q_1, q_2)$. If $\max(q_1, q_2) = q_1$, the second term will be dominant, otherwise it will be the third term. We can conclude that $|A_2| \leq O_P\left(n^{-1+\max(q_k)/2} |\log n|^{\frac{q-\max(q_k)}{2}}\right)$, proving the result.

B Appendix B: proofs of results for Examples 1 and 2

This Appendix is organized as follows. First, we provide some auxiliary results used in proving Theorem 4.2, followed by their proofs. Then, we provide the proofs of Theorems 4.1, 4.2 and 5.1.

Our first result shows that V_n^* diverges at rate n in restriction to Ω_1.

Lemma B.1 *Suppose (1), (2) and Assumptions 1 and 2 hold. Then, for $M = 1$, on Ω_1,*

$$
\frac{V_n^*}{n} = a_1 \frac{1}{n} MV_1^{[4,1]}(4) + o_P(1),
$$

where $a_1 \frac{1}{n} MV_1^{[4,1]}(4) \xrightarrow{P} v^*$ in restriction to Ω_1, for some r.v. v^*. Hence, $\frac{V_n^*}{n} \xrightarrow{P} v^*$. Furthermore, if $P(\Omega_1) > 0$, then $P(v^* > 0 | \Omega_1) = 1$.

Next we show that $\widetilde{IQ}_n^* \xrightarrow{P^*} IQ$, in prob-$P$ on Ω_1.

Lemma B.2 *Suppose (1), (2) and Assumptions 1 and 2 hold. Then, for $M = 1$, $\widetilde{IQ}_n^* \xrightarrow{P^*} IQ$, in prob-$P$.*

Finally, we show that the order of magnitude of $Z_n^* = O_{P^*}(1)$ defined in Section 4.2 is sharp.

Lemma B.3 *Suppose (1), (2) and Assumptions 1 and 2 hold. Then, for $M = 1$,*

$$
RV_n^* - BV_n^* - E^*\left(RV_n^* - BV_n^*\right) \xrightarrow{d^*} x^*
$$

for some r.v. x^* which, conditionally on Ω_1 is non degenerate at 0. Consequently, $Z_n^* = O_{P^*}(1)$, where this is a sharp order of magnitude.

Proof of Lemma B.1. The order of magnitude of the last three terms in V_n^* given in (14) is obtained from Lemma 3.3(a2) and is equal to $o_p(1)$ when divided by n. Thus, we only need to derive the probability limit of $\frac{1}{n}MV_1^{[4,1]}(4)$. We can write

$$\frac{1}{n}MV_1^{[4,1]}(4) = \frac{1}{n}\left(n\sum_{i=1}^n |r_i|^4\right) = \sum_{i=1}^n |r_i|^4 .$$

Recall that for any $t \geq 0$, $X_t = Y_t + J_t$, where $J_t = \sum_{j=1}^{N_t} c_j$. It follows that $r_i = y_i + z_i$, where $y_i = Y_{i/n} - Y_{(i-1)/n}$ and $z_i = \sum_{j=N_{(i-1)/n+1}}^{N_{i/n}} c_j$. Therefore,

$$\sum_{i=1}^n |r_i|^4 = \sum_{i=1}^n |y_i + z_i|^4 = \sum_{i=1}^n |z_i|^4 + \left(\sum_{i=1}^n |y_i + z_i|^4 - \sum_{i=1}^n |z_i|^4\right) \equiv R_1 + R_2.$$

By the Minkowski inequality,

$$\left|\left(\sum_{i=1}^n |y_i + z_i|^4\right)^{1/4} - \left(\sum_{i=1}^n |z_i|^4\right)^{1/4}\right| \leq \left(\sum_{i=1}^n |y_i|^4\right)^{1/4} = O_P\left(n^{-1/4}\right) = o_P(1),$$

since y_i is the intraday return from the continuous part. This implies that $R_2 = o_P(1)$. Next, under finite activity jumps,

$$R_1 = \sum_{i=1}^n |z_i|^4 \xrightarrow{P} \sum_{j=1}^{N(1)} |c_j|^4 \equiv v^*.$$

Clearly, $v^* > 0$ on Ω_1; which concludes the proof.

Proof of Lemma B.2. First, note that for $M = 1$,

$$E^*\left(\widetilde{IQ}_n^*\right) = \frac{1}{k_{1,4/3}^3} n \sum_{i=1}^n |r_i|^{4/3} |r_{i-1}|^{4/3} |r_{i-2}|^{4/3} .$$

By Lemma 3.3(a1), $E^*\left(\widetilde{IQ}_n^*\right) \xrightarrow{P} IQ$ under Assumptions 1 and 2, since $\max(q_k) = 4/3 < 2$. Next, we analyze the variance of $\widetilde{IQ}_n^*$. This variance is bounded by

$$\begin{aligned} Var^*\left(\widetilde{IQ}_n^*\right) \leq Cn^{-1}\Bigg(& n^3 \sum_{i=3}^n |r_i|^{8/3} |r_{i-1}|^{8/3} |r_{i-2}|^{8/3} \\ & + n^3 \sum_{i=4}^n |r_i|^{4/3} |r_{i-1}|^{8/3} |r_{i-2}|^{8/3} |r_{i-3}|^{4/3} \\ & + n^3 \sum_{i=5}^n |r_i|^{4/3} |r_{i-1}|^{4/3} |r_{i-2}|^{8/3} |r_{i-3}|^{4/3} |r_{i-4}|^{4/3}\Bigg). \end{aligned}$$

By Lemma 3.3(a2), each of the terms above is of order

$$n^{-1} O_P\left(n^{-1+\max(q_k)/2} |\log(n)|^{\frac{q-\max(q_k)}{2}}\right) = O_P\left(n^{-2+\max(q_k)/2} |\log(n)|^{\frac{q-\max(q_k)}{2}}\right),$$

which is $o_P(1)$ provided $\max(q_k) < 4$. Since here $q = 8$ and $\max(q_k) = 8/3$, this condition is satisfied.

Proof of Lemma B.3. We can write

$$RV_n^* - BV_n^* - E^*(RV_n^* - BV_n^*) = (RV_n^* - E^*(RV_n^*)) - (BV_n^* - E^*(BV_n^*)) .$$

We can show that the second term converges to zero under P^*, in prob-P. Indeed, by construction $E^*\left(BV_n^* - E^*\left(BV_n^*\right)\right) = 0$ and

$$Var^*\left(BV_n^*\right) = \frac{1}{n} Var^*\left(\sqrt{n} BV_n^*\right) = O_P\left(\frac{|\log(n)|}{n}\right) = o_P\left(1\right),$$

The order of magnitude of $Var^*\left(BV_n^*\right)$ is explained by the fact that $Var^*\left(\sqrt{n}BV_n^*\right)$ is a function of $n\sum_{i=1}^n |r_i|^2 |r_{i-1}|^2$ and $n\sum_{i=1}^n |r_i| |r_{i-1}|^2 |r_{i-2}|$, which from Lemma 3.3(a2) are $O_P(|\log(n)|)$.

The first term $RV_n^* - E^*\left(RV_n^*\right) = \sum_{i=1}^n r_{i,n}^2\left(\eta_i^2 - 1\right)$ where r_i is replaced by $r_{i,n}$ to stress its dependence on n and $\eta_i^2 \sim$ i.i.d. χ_1^2. Thanks to the assumption of rare jumps, for n large enough, we can assume that each time interval $[\frac{i-1}{n}, \frac{i}{n}]$ has at most one jump and let I_n be the set of i's such that $[\frac{i-1}{n}, \frac{i}{n}]$ contains a jump. Let $\mathbb{I}_{I_n}(\cdot)$ be the usual indicator function. We have that

$$\sum_{i=1}^n r_{i,n}^2\left(\eta_i^2 - 1\right) = \sum_{i=1}^n r_{i,n}^2\left(\eta_i^2 - 1\right)\mathbb{I}_{I_n}(i) + \sum_{i=1}^n r_{i,n}^2\left(\eta_i^2 - 1\right)\left(1 - \mathbb{I}_{I_n}(i)\right) \equiv A_{1n}^* + A_{2n}^*.$$

Clearly, $E^*(A_{2n}^*) = 0$ and

$$Var^*(A_{2n}^*) = \sum_{i=1}^n r_{i,n}^4(1 - \mathbb{I}_{I_n}(i))Var^*(\eta_i^2 - 1) = 2\sum_{i=1}^n r_{i,n}^4(1 - \mathbb{I}_{I_n}(i)) \le 2\sum_{i=1}^n y_{i,n}^4 = o_P(1).$$

As a result, $A_{2n}^* \xrightarrow{P^*} 0$, Prob-$P$. Consider now A_{1n}^* and note that $A_{1n}^* = \sum_{i=1}^n r_{i,n}^2\left(\eta_i^2 - 1\right)\mathbb{I}_{I_n}(i) = \sum_{i=1}^{N_1} r_{i,n}^2\left(\eta_i^2 - 1\right)$, with $r_{i,n} = y_{i,n} + c_i$. Let $x^* = \sum\limits_{i=1}^{N_1} c_i^2(\eta_i^2 - 1)$. We have

$$A_{1n}^* - x^* = \sum_{i=1}^{N_1}(r_{i,n}^2 - c_i^2)(\eta_i^2 - 1).$$

From Proposition 1 of Barndorff-Nielsen, Shephard and Winkel (2006), we claim that, for every $i = 1, \ldots, N_1$,

$$r_{i,n}^2 - c_i^2 = (y_{i,n} + c_i)^2 - c_i^2 = y_{i,n}^2 + 2c_i y_{i,n} = o_P(1).$$

Thus, $(r_{i,n}^2 - c_i^2)(\eta_i^2 - 1) = o_P(1)O_{P^*}(1) = o_{P^*}(1)$, in prob-$P$. Since N_1 is finite P-almost surely, it follows that $A_{1n}^* - x^* = o_{P^*}(1)$, prob-$P$. We deduce that

$$A_{1n}^* \xrightarrow{d^*} x^*,$$

in prob-P. We complete the proof by showing that conditionally on Ω_1, x^* is non degenerate at 0, i.e. $P(x^* = 0|\Omega_1) < 1$. (Note that x^* is not necessarily measurable with respect to the original probability space $(\Omega, \mathcal{F}, P)$. In this expression, P must be seen as the natural extension of the original probability space that makes η_i's measurable. We keep the same notation for simplicity.) We actually show that $P(x^* = 0|\Omega_1) = 0$. Clearly, x^* is function of $(N_1, c_1, \ldots, c_{N_1}, \eta_1, \ldots, \eta_{N_1})$ with $(N_1, c_1, \ldots, c_{N_1})$ independent of $(\eta_1, \ldots, \eta_{N_1})$. Hence,

$$\begin{aligned}
P(x^* = 0|\Omega_1) &= P\left(\left.\sum_{i=1}^{N_1} c_i^2(\eta_i^2 - 1) = 0\right| N_1 \ge 1\right) \\
&= \sum_{m\ge1}\left\{P(N_1 = m|N_1 \ge 1)\int_{z_1,\ldots,z_m} P\left(\left.\sum_{i=1}^{N_1} c_i^2(\eta_i^2 - 1) = 0\right| c = z, N_1 = m\right) dF_c(z|N_1 = m)\right\} \\
&= \sum_{m\ge1}\left\{P(N_1 = m|N_1 \ge 1)\int_{z_1,\ldots,z_m} P\left(\sum_{i=1}^{m} z_i^2(\eta_i^2 - 1) = 0\right) dF_c(z|N_1 = m)\right\}.
\end{aligned}$$

Since $\eta_i^2 - 1$ for $i = 1, 2, \ldots$ are independent with continuous distributions, so are $z_i^2(\eta_i^2 - 1)$ for $i = 1, 2, \ldots$ (so long as $z_i \neq 0$). Hence, if at least one z_i is different from 0, the random variable $\sum_{i=1}^m z_i^2(\eta_i^2 - 1)$ has a continuous distribution and hence, $P\left(\sum_{i=1}^m z_i^2(\eta_i^2 - 1) = 0\right) = 0$. It follows that $P(x^* = 0|\Omega_1) = 0$ since, from Assumption 2, $P(c_i = 0) = 0$ for all $i = 1, 2, \ldots$ which ensures that $P(c = 0|N_1 = m) = 0$ for all $m \geq 1$.

Proof of Theorem 4.1. Part (a1): Condition A(i) is a consequence of Lemmas 3.2 and 3.3. In fact, from Lemma 3.3(a1), RV_n and $MV_M^{[2,2]}(1,1)$ converge in probability to $\int_0^1 \sigma_u^2 du$ in restriction to Ω_0 and $MV_M^{[4,1]}(4)$, $MV_M^{[4,2]}(2,2)$, $MV_M^{[4,2]}(3,1)$, $MV_M^{[4,2]}(1,3)$, $MV_M^{[4,3]}(1,2,1)$ converge in probability to $\int_0^1 \sigma_u^4 du$ in restriction to Ω_0. The constants $c_{q_1,\ldots,q_K}$ are obtained by collecting the coefficients that multiply the integrated quantities. For Condition A(ii), we have

$$n^{1+\delta} \sum_{i=1}^{n} (\hat{v}_i^n)^{2+\delta} = n^{1+\delta} \sum_{j=1}^{n/M} \sum_{i=1}^{M} \left(\hat{v}_{i+(j-1)M}^n\right)^{2+\delta} = n^{1+\delta} \sum_{j=1}^{n/M} M \bar{R}_j^{2+\delta} = \frac{k_{M,4+2\delta}}{M^{2+\delta}} MV_M^{[4+2\delta,1]}(4+2\delta).$$

From Lemma 3.3(a1), $MV_M^{[4+2\delta,1]}(4+2\delta)$ converges in probability to $\int_0^1 \sigma_s^{4+2\delta} ds$ in restriction to Ω_0. This shows that $n^{1+\delta} \sum_{i=1}^n (\hat{v}_i^n)^{2+\delta} = O_P(1)$. For Condition A(iii), let $\delta > 0$, $\alpha \in (0, \delta/(2(1+\delta))$ and $L_n \propto n^\alpha$. We have to show that $n \sum_{j=1}^{k_n} \left(\hat{v}_{j(L_n+1)}^n\right)^2 = o_P(1)$, with $k_n = \left[\frac{n}{L_n+1}\right]$, where $\hat{v}_j^n = \bar{R}_j = \sum_{\ell=(j-1)M+1}^{jM} r_\ell^2/M$. As seen in the proof of Lemma 3.3, on Ω_0, X_t coincides with the continuous process X_t' and, through the same trick as in that proof, we can deal with X_t as though it is a continuous process. From Proposition 1 of Barndorff-Nielsen, Shephard and Winkel (2005),

$$\max_{1 \leq \ell \leq n} |r_\ell| = O_P\left(\sqrt{\frac{\log n}{n}}\right).$$

Hence, $\bar{R}_j = O_P\left(\frac{\log n}{n}\right)$ uniformly over $j = 1, \ldots, n/M$. Therefore

$$n \sum_{j=1}^{k_n} \left(\hat{v}_{j(L_n+1)}^n\right)^2 = n k_n O_P\left(\frac{\log^2(n)}{n^2}\right) = O_P(n^{-\alpha}) = o_P(1). \tag{21}$$

Condition B(i) follows from Lemmas 3.2 and 3.3 similarly to Condition A(i). Finally, for Condition B(ii), let $K \in \{3, 4, 5\}$ and q denote a K-vector of nonnegative numbers with $\sum_{k=1}^K q_k = q$. We show that

$$n^{-2+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}_{i-k+1}^n\right)^{q_k/2} = o_P(1).$$

Note that this quantity is equal to n^{-1} times a linear combination of terms such as $MV_M^{[q,K']}(\{q_\ell'\})$ with coefficients that only depend on M and $\{q_\ell' : \ell = 1, \ldots, K'\}$. See parts (a2) and (a3) of Lemma 3.2. The result now follows from Lemma 3.3(a1) since for any fixed $M \geq 1$, we can deduce that, in restriction to Ω_0,

$$n^{-2+q/2} \sum_{i=K}^{n} \prod_{k=1}^{K} \left(\hat{v}_{i-k+1}^n\right)^{q_k/2} = O_P(n^{-1}) = o_P(1),$$

for any $K \in \{3, 4, 5\}$ and any K-vector of nonnegative numbers q. Part (a2) follows from Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 4.2. The proof follows from Lemmas B.1, B.2 and B.3.

Proof of Theorem 5.1. The proof follows the same arguments as that of Theorem 4.1 with the difference that we now rely on the condition that $\max(p_l) < 1/2$ to apply part (a2) of Lemma 3.3 under both Ω_0 and Ω_1.

C Appendix C: Bootstrap test statistic for the $\log$ version of the jump test

The asymptotic test based on logarithm transformation of the linear version of the jump test as given by (5) has been proposed by Huang and Tauchen (2005). It follows from (3) and 4 that

$$\sqrt{n}\left(\log RV_n - \log BV_n\right) \xrightarrow{st} N\left(0, \tau \frac{IQ}{IV^2}\right), \quad \tau = \theta - 2,$$

and the test statistic of the log version of the jump test is given by

$$T_{\log,n} = \frac{\sqrt{n}\left(\log RV_n - \log BV_n\right)}{\sqrt{\tau \max\left(1, \frac{\widehat{IQ}_n}{BV_n^2}\right)}}.$$

The bootstrap test statistic $T^*_{\log,n}$ for $T_{\log,n}$ derives from Theorem 3.1(a1). By a Taylor expansion, we have

$$\sqrt{n}\left(\log \frac{RV_n^*}{BV_n^*} - \log \frac{E^*(RV_n^*)}{E^*(BV_n^*)}\right) = \begin{pmatrix} \frac{1}{E^*(RV_n^*)} & -\frac{1}{E^*(BV_n^*)} \end{pmatrix} \sqrt{n} \begin{pmatrix} RV_n^* - E^*(RV_n^*) \\ BV_n^* - E^*(BV_n^*) \end{pmatrix} + o_{P^*}(1), \text{ Prob-}P.$$

Conditionally on no jump, $E^*(RV_n^*) \xrightarrow{P} c_2 IV$ and $E^*(BV_n^*) \xrightarrow{P} c_{1,1} IV$. In Example 1, $c_2 = 1$ and $c_{1,1} = 1 - \frac{1}{M} + \frac{k_{M,1}^2}{M^2}$. In Example 2,

$$c_2 = \frac{k^L_{M,\frac{2}{L}}}{M} \text{ and } c_{1,1} = \left(1 - \frac{1}{M}\right) \frac{k^L_{M,\frac{2}{L}}}{M} + \frac{k^2_{M,\frac{1}{L}} k^{L-1}_{M,\frac{2}{L}}}{M^2}; \quad (L = 5),$$

and in Example 3, $c_2 = k_{1,2}$ and $c_{1,1} = k_{1,1}^2$.
From Theorem 3.1(a1), we deduce that

$$\frac{\sqrt{n}\left(\log \frac{RV_n^*}{BV_n^*} - \log \frac{E^*(RV_n^*)}{E^*(BV_n^*)}\right)}{\sqrt{\tau^*_{\log} \frac{IQ}{IV^2}}} \xrightarrow{d^*} N(0,1), \quad \text{in Prob-}P,$$

with $\tau^*_{\log} = \frac{\beta}{c_2^2} - 2\frac{\delta}{c_{1,1}c_2} + \frac{\alpha}{c_{1,1}^2}$. The bootstrap test statistic for $T_{\log,n}$ is given by

$$T^*_{\log,n} = \frac{\sqrt{n}\left(\log \frac{RV_n^*}{BV_n^*} - \log \frac{E^*(RV_n^*)}{E^*(BV_n^*)}\right)}{\sqrt{\tau^*_{\log} \max\left(1, c_{1,1}^2 \frac{\widetilde{IQ}_n^*}{(BV_n^*)^2}\right)}}.$$

Under (1), (2) and Assumption 1, $T_{\log,n}$ satisfies the conditions of Theorem 3.3 and if Conditions (A) and (B) are satisfied, the conclusions of that theorem hold for $T^*_{\log,n}$ and the resulting bootstrap test controls the strong asymptotic size and is alternative-consistent.

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