Solving and estimating indeterminate DSGE models

Roger E.A. Farmer [a,*], Vadim Khramov [b], Giovanni Nicolò [a]

[a] Department of Economics, UCLA, United States
[b] International Monetary Fund, United States

ARTICLE INFO

ABSTRACT

We propose a method for solving and estimating linear rational expectations models that exhibit indeterminacy and we provide step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys. Our method redefines a subset of expectational errors as new fundamentals. This redefinition allows us to treat indeterminate models as determinate and to apply standard solution algorithms. We prove that our method is equivalent to the solution method proposed by Lubik and Schorfheide (2003, 2004), and using the New-Keynesian model described in Lubik and Schorfheide (2004), we demonstrate how to apply our theoretical results with a practical exercise.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

It is well known that linear rational expectations (LRE) models can have an indeterminate set of equilibria under realistic parameter choices. Lubik and Schorfheide (2003) provided an algorithm that computes the complete set of indeterminate equilibrium, but their approach has not yet been implemented in standard software packages and has not been widely applied in practice. In this paper, we propose an alternative methodology based on the idea that a model with an indeterminate set of equilibria is an incomplete model. We propose to close a model of this kind by treating a subset of the non-fundamental errors as newly defined fundamentals.

Our method builds on the approach of Sims (2001) who provided a widely used computer code, Gensys, implemented in Matlab, to solve for the reduced form of a general class of linear rational expectations (LRE) models. Sims's code classifies models into three groups: those with a unique rational expectations equilibrium, those with an indeterminate set of rational expectations equilibria, and those for which no bounded rational expectations equilibrium exists. By moving non-fundamental errors to the set of fundamental shocks, we select a unique equilibrium, thus allowing the modeler to apply standard solution algorithms. We provide step-by-step guidelines for implementing our method in the Matlab-based software programs Dynare (Adjemian et al., 2011) and Gensys (Sims, 2001).

Our paper is organized as follows. In Section 2, we provide a brief literature survey and in Section 3 we review solution methods for indeterminate models. In Section 4, we discuss the choice of which expectational errors to redefine as

---

* Corresponding author at: Department of Economics, UCLA, Box 951477 Los Angeles, CA 90095-1477, USA.
E-mail addresses: rfarmer@econ.ucla.edu (R.E.A. Farmer), vkhramov@imf.org (V. Khramov), gnicolo@ucla.edu (G. Nicolò).

fundamental and we prove that all possible alternative selections have the same likelihood. Section 5 compares our method to the work of Lubik and Schorfheide (2003) and establishes an equivalence result between the two approaches. In Section 6, we apply our method to the New-Keynesian model described in Lubik and Schorfheide (2004) and we show how to apply our method using Gensys to simulated data. Section 7 provides step-by-step guidelines for implementing our method in the popular software package, Dynare,[1] and Section 8 provides a brief conclusion.

2. Related literature

Blanchard and Kahn (1980) showed that a LRE model can be written as a linear combination of backward-looking and forward-looking solutions. Since then, a number of alternative approaches for solving linear rational expectations models have emerged (King and Watson, 1998; Klein, 2000; Uhlig, 1999; Sims, 2001). These methods provide a solution if the equilibrium is unique, but there is considerable confusion about how to handle the indeterminate case. Some methods fail in the case of a non-unique solution, for example, Klein (2000), while others, e.g. Sims (2001), generate one solution with a warning message.

All of these solution algorithms are based on the idea that, when there is a unique determinate rational expectations equilibrium, the model's forecast errors are uniquely defined by the fundamental shocks. These errors must be chosen in a way that eliminates potentially explosive dynamics of the state variables of the model.

McCallum (1983) has argued that a model with an indeterminate set of equilibria is incompletely specified and he recommends a procedure, the minimal state variable solution, for selecting one of the many possible equilibria in the indeterminate case. Farmer (1999) has argued instead that we should exploit the properties of indeterminate models to help understand data. Farmer and Guo (1995) took up that challenge by studying a model where indeterminacy arises from a technology with increasing returns-to-scale, and Lubik and Schorfheide (2004) developed methods for distinguishing determinate from indeterminate models which they applied to a New-Keynesian monetary model. There is a growing body of the literature (see, for example, Belaygorod and Dueker, 2009; Bhattarai et al., 2012; Fanelli, 2012; Castelnuovo and Fanelli, 2015; Hirose, 2011; Zheng and Guo, 2013; Bilbiie and Straub, 2013), that directly tackles the econometric challenges posed by indeterminacy. This literature offers the possibility for the theoretical work, surveyed in Benhabib and Farmer (1999), to be directly compared with conventional classical and new-Keynesian approaches in which equilibria are assumed to be locally unique.

The empirical importance of indeterminacy began with the work of Benhabib and Farmer (1994) who established that a standard one-sector growth model with increasing returns displays an indeterminate steady state and Farmer and Guo (1994) who exploited that property to generate business cycle models driven by self-fulfilling beliefs. More recent New-Keynesian models have been shown to exhibit indeterminacy if the monetary authority does not increase the nominal interest rate enough in response to higher inflation (see, for example, Clarida et al., 2000; Kerr and King, 1996). Our estimation method should be of interest to researchers in both literatures.

3. Solving LRE models

Consider the following k-equation LRE model. We assume that $X_t \in R^k$ is a vector of deviations from means of some underlying economic variables. These may include predetermined state variables, for example, the stock of capital, non-predetermined control variables, for example, consumption; and expectations at date t of both types of variables.

We assume that z_t is an $l \times 1$ vector of exogenous, mean-zero shocks and η_t is a $p \times 1$ vector of endogenous shocks.[2] The matrices Γ_0 and Γ_1 are of dimension $k \times k$, possibly singular, Ψ and Π are, respectively, $k \times l$ and $k \times p$ known matrices.

Using the above definitions, we will study the class of linear rational expectations models described by

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t. \tag{1}$$

Sims (2001) shows that this way of representing a LRE is very general and most LRE models that are studied in practice by economists can be written in this form. We assume that

$$E_{t-1}(z_t) = 0 \quad \text{and} \quad E_{t-1}(\eta_t) = 0. \tag{2}$$

We define the $l \times l$ matrix Ω_{zz}:

$$E_{t-1}(z_t z_t^T) = \Omega_{zz}, \tag{3}$$

which represents the covariance matrix of the exogenous shocks. We refer to these shocks as predetermined errors, or equivalently, predetermined shocks. The second set of shocks, η_t, has dimension p. Unlike the z_t, these shocks are endogenous and are determined by the solution algorithm in a way that eliminates the influence of the unstable roots of the

---

[1] Dynare is a Matlab-based software platform for handling a wide class of economic models, in particular dynamic stochastic general equilibrium (DSGE). Visit www.dynare.org for details.

[2] Sims (2001) allows z_t to be autoregressive with non zero conditional expectation. We assume, instead, that z_t always has zero conditional mean. That assumption is unrestrictive since an autoregressive error can be written in our form by defining a new state variable, $\tilde{z}_t$ and letting the innovation of the original variable, z_t, be the new fundamental shock.

system. In many important examples, the $\eta_{i,t}$ have the interpretation of expectational errors and, in those examples,

$$\eta_{i,t} = X_{i,t} - E_{t-1}(X_{i,t}).\tag{4}$$

3.1. The QZ decomposition

Sims (2001) shows how to write Eq. (1) in the form

$$\overbrace{\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}}^{S} \overbrace{\begin{bmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{bmatrix}}^{\tilde{X}_t} = \overbrace{\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}}^{T} \overbrace{\begin{bmatrix} \tilde{X}_{1,t-1} \\ \tilde{X}_{2,t-1} \end{bmatrix}}^{\tilde{X}_{t-1}} + \overbrace{\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix}}^{\tilde{\Psi}} z_t + \overbrace{\begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix}}^{\tilde{\Pi}} \eta_t \tag{5}$$

where the matrices $S, T, \tilde{\Psi}$ and $\tilde{\Pi}$ and the transformed variables $\tilde{X}_t$ are defined as follows: Let

$$\Gamma_0 = QSZ^T, \quad \text{and} \quad \Gamma_1 = QTZ^T,\tag{6}$$

be the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ where Q and Z are $k \times k$ orthonormal matrices and S and T are upper triangular and possibly complex.

The QZ decomposition is not unique. The diagonal elements of S and T are called the *generalized eigenvalues* of $\{\Gamma_0, \Gamma_1\}$ and Sims's algorithm chooses one specific decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in increasing order that is

$$|t_{jj}|/|s_{jj}| \geq |t_{ii}|/|s_{ii}| \quad \text{for } j > i.\tag{7}$$

Sims proceeds by partitioning S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},\tag{8}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},\tag{9}$$

where the first block contains all the equations for which $|t_{jj}|/|s_{jj}| < 1$ and the second block, all those for which $|t_{jj}|/|s_{jj}| \geq 1$. The transformed variables $\tilde{X}_t$ are defined as

$$\tilde{X}_t = Z^T X_t,\tag{10}$$

and the transformed parameters as

$$\tilde{\Psi} = Q^T \Psi \quad \text{and} \quad \tilde{\Pi} = Q^T \Pi.\tag{11}$$

3.2. Using the QZ decomposition to solve the model

The model is said to be determinate if Eq. (5) has a unique bounded solution. To establish existence of at least one bounded solution we must eliminate the influence of all of the unstable roots; by construction, these are contained in the second block:

$$\tilde{X}_{2,t} = S_{22}^{-1} T_{22} \tilde{X}_{2,t-1} + S_{22}^{-1}\left(\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t\right),\tag{12}$$

since the eigenvalues of $S_{22}^{-1} T_{22}$ are all greater than or equal to one in absolute value. Hence a bounded solution, if it exists, will set

$$\tilde{X}_{2,0} = 0,\tag{13}$$

and

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0.\tag{14}$$

Since the elements of $\tilde{X}_{2,t}$ are linear combinations of $X_{2,t}$, a necessary condition for the existence of a solution to Eq. (14) is that there are at least as many non-predetermined variables as unstable generalized eigenvalues. A sufficient condition is that the columns of $\tilde{\Pi}_2$ in the matrix,

$$\left[\tilde{\Psi}_2 \; \tilde{\Pi}_2\right],\tag{15}$$

are linearly independent so that there is at least one solution to Eq. (14) for the endogenous shocks, η_t, as a function of the fundamental shocks, z_t. In the case that $\tilde{\Pi}_2$ is square and non-singular, we can write the solution for η_t as

$$\eta_t = -\tilde{\Pi}_2^{-1}\tilde{\Psi}_2 z_t. \tag{16}$$

More generally, Sims' code checks for existence using the singular value decomposition of (15).

To find a solution for $\tilde{X}_{1,t}$ we take Eq. (16) and plug it back into the first block of (5) to give the expression

$$\tilde{X}_{1,t} = S_{11}^{-1}T_{11}\tilde{X}_{1,t-1} + S_{11}^{-1}\left(\tilde{\Psi}_1 - \tilde{\Pi}_1\tilde{\Pi}_2^{-1}\tilde{\Psi}_2\right)z_t. \tag{17}$$

Even if there is more than one solution to (14) it is possible that they all lead to the same solution for $\tilde{X}_{1,t}$. Sims provides a second use of the singular value decomposition to check that the solution is unique. Eqs. (13) and (17) determine the evolution of $\{\tilde{X}_t\}$ as functions of the fundamental shocks $\{z_t\}$ and, using the definition of $\{\tilde{X}_t\}$ from (10), we can recover the original sequence $\{X_t\}$.

3.3. The indeterminate case

There are many examples of sensible economic models where the number of expectational variables is larger than the number of unstable roots of the system. In that case, Gensys will find a solution but flag the fact that there are many others. We propose to deal with that situation by providing a statistical model for one or more of the endogenous errors.

The rationale for our procedure is based on the notion that agents situated in an environment with multiple rational expectations equilibria must still choose to act. And to act rationally, they must form *some* forecast of the future and, therefore, we can model the process of expectations formation by specifying how the forecast errors covary with the other fundamentals.

If a model has n unstable generalized eigenvalues and p non-fundamental errors then, under some regularity assumptions, there will be $m = p - n$ degrees of indeterminacy. In that situation we propose to redefine m non-fundamental errors as new fundamental shocks. This transformation allows us to treat indeterminate models as determinate and to apply standard solution and estimation methods.

Consider model (1) and suppose that there are m degrees of indeterminacy. We propose to partition the η_t into two pieces, $\eta_{f,t}$ and $\eta_{n,t}$, and to partition Π conformably so that

$$\underset{k \times k}{\Gamma_0} \underset{k \times 1}{X_t} = \underset{k \times k}{\Gamma_1} \underset{k \times 1}{X_{t-1}} + \underset{k \times l}{\Psi} \underset{l \times 1}{z_t} + \underset{k \times m}{\left[\Pi_f} \underset{k \times n}{\Pi_n\right]} \begin{bmatrix} \underset{m \times 1}{\eta_{f,t}} \\ \underset{n \times 1}{\eta_{n,t}} \end{bmatrix}. \tag{18}$$

Here, $\eta_{f,t}$ is an $m \times 1$ vector that contains the newly defined fundamental errors and $\eta_{n,t}$ contains the remaining n non-fundamental errors.

Next, we re-write the system by moving $\eta_{f,t}$ from the vector of expectational shocks to the vector of fundamental shocks:

$$\underset{k \times k}{\Gamma_0} \underset{k \times 1}{X_t} = \underset{k \times k}{\Gamma_1} \underset{k \times 1}{X_{t-1}} + \underset{k \times l}{\left[\Psi} \underset{k \times m}{\Pi_f\right]} \underset{(l+m) \times 1}{\tilde{z}_t} + \underset{k \times n}{\Pi_n}\underset{n \times 1}{\eta_{n,t}}, \tag{19}$$

where we treat

$$\underset{(l+m) \times 1}{\tilde{z}_t} = \begin{bmatrix} \underset{l \times 1}{z_t} \\ \underset{m \times 1}{\eta_{f,t}} \end{bmatrix}, \tag{20}$$

as a new vector of fundamental shocks and $\eta_{n,t}$ as a new vector of non-fundamental shocks. To complete this specification, we define $\tilde{\Omega}$:

$$\underset{(l+m) \times (l+m)}{\tilde{\Omega}} = E_{t-1}\left(\begin{bmatrix} \underset{l \times 1}{z_t} \\ \underset{m \times 1}{\eta_{f,t}} \end{bmatrix} \begin{bmatrix} \underset{l \times 1}{z_t} \\ \underset{m \times 1}{\eta_{f,t}} \end{bmatrix}^T \right) \equiv \begin{pmatrix} \underset{l \times l}{\Omega_{zz}} & \underset{l \times m}{\Omega_{zf}} \\ \underset{m \times l}{\Omega_{fz}} & \underset{m \times m}{\Omega_{ff}} \end{pmatrix}, \tag{21}$$

to be the new covariance matrix of fundamental shocks. This definition requires us to specify $m(m+1+2l)/2$ new variance parameters, these are the $m(m+1)/2$ elements of Ω_{ff}, and $m \times l$ new covariance parameters, these are the elements of Ω_{zf}. By choosing these new parameters and applying Sims' solution algorithm, we select a unique bounded rational expectations equilibrium. The diagonal elements of $\tilde{\Omega}$ that correspond to η_f have the interpretation of a pure 'sunspot' component to the shock and the covariance of these terms with z_t represents the response of beliefs to the original set of fundamentals.

Our approach to indeterminacy is equivalent to defining a new model in which the indeterminacy is resolved by assuming that expectations are formed consistently using the same forecasting method in every period. For example, expectations may be determined by a learning mechanism as in Evans and Honkapohja (2001) or using a belief function as in Farmer (2002). For our approach to be valid, we require that the belief function is time invariant and that shocks to that function can be described by a stationary probability distribution. Our newly transformed model can be written in the form

of Eq. (1), but the fundamental shocks in the transformed model include the original fundamental shocks z_t, as well as the vector of new fundamental shocks, $\eta_{f,t}$.

4. Choice of expectational errors

Our approach raises the practical question of which non-fundamentals should we choose to redefine as fundamental. Here we show that, given a relatively mild regularity condition, there is an equivalence between all possible ways of redefining the model.

Definition 1 (*Regularity*). Let ε be an indeterminate equilibrium of model (1) and use the QZ decomposition to write the following equation connecting fundamental and non-fundamental errors:

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0. \tag{22}$$

Let n be the number of generalized eigenvalues that are greater than or equal to 1 and let $p > n$ be the number of non-fundamental errors. Partition η_t into two mutually exclusive subsets, $\eta_{f,t}$ and $\eta_{n,t}$ such that $\eta_{f,t} \cup \eta_{n,t} = \eta_t$ and partition $\tilde{\Pi}_2$ conformably so that

$$\underset{n\times p}{\tilde{\Pi}_2} \underset{p\times 1}{\eta_t} = \begin{bmatrix} \underset{n\times m}{\tilde{\Pi}_{2f}} & \underset{n\times n}{\tilde{\Pi}_{2n}} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}} \\ \underset{n\times 1}{\eta_{n,t}} \end{bmatrix}. \tag{23}$$

The indeterminate equilibrium, ε, is regular if, for all possible mutually exclusive partitions of η_t, $\tilde{\Pi}_{2n}$ has full rank.

Regularity rules out situations where there is a linear dependence in the non-fundamental errors and all of the indeterminate LRE models that we are aware of, that have been studied in the literature, satisfy this condition.

Theorem 1. *Let ε be an indeterminate equilibrium of model (1) and let $\mathbf{P}$ be an exhaustive set of mutually exclusive partitions of η_t into two non-intersecting subsets, where*

$$\left\{ \mathbf{p} \in \mathbf{P} | \mathbf{p} = \left(\underset{m\times 1}{\eta_{f,t}}, \underset{n\times 1}{\eta_{n,t}} \right)^T \right\}.$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be elements of $\mathbf{P}$ and let $\tilde{\Omega}_1$ be the covariance matrix of the new set of fundamentals, $\left[z_t, \eta_{f,t} \right]$ associated with partition $\mathbf{p}_1$. If ε is regular then there is a covariance matrix $\tilde{\Omega}_2$, associated with partition 2 such that the covariance matrix

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix} \begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix}^T \right), \tag{24}$$

is the same for both partitions. $\mathbf{p}_1$ and $\mathbf{p}_2$, parameterized by $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, are said to be equivalent partitions.

Proof. See Appendix A.

Corollary 1. *The joint probability distribution over sequences $\{X_t\}$ is the same for all equivalent partitions.*

Proof. The proof follows immediately from the fact that the joint probability of sequences $\{X_t\}$, is determined by the joint distribution of the shocks. $\square$

The question of how to choose a partition $\mathbf{p}_i$ is irrelevant since all partitions have the same likelihood. However, the partition *will* matter, if the researcher imposes zero restrictions on the variance covariance matrix of fundamentals.

Why does this matter? Suppose that the researcher choose one of two possible partitions, call this $\mathbf{p}_1$, by specifying one of two expectational errors from the original model as a new fundamental. Under partition $\mathbf{p}_1$, the covariance parameters of the second expectational error with the fundamentals will be complicated functions of all of the parameters of the model.

Suppose instead that the researcher chooses the second expectational error to be fundamental, call this partition $\mathbf{p}_2$. In this case, it is the covariance parameters of the first expectational error that will depend on model parameters. Because the researcher cannot know in advance, which of these specifications is the correct one, we recommend that in practice, the VCV matrix of the augmented shocks, $\tilde{z}$, should be left unrestricted.

Lubik and Schorfheide (2004) refer to 'belief shocks' which they think of as independent causal disturbances that influence all of the endogenous variables at each date. Their belief shocks are isomorphic to what Cass and Shell (1983) refer to as 'sunspots' and what Azariadis (1981) and Farmer and Woodford (1984, 1997) call 'self-fulfilling prophecies'.

In Section 5, we prove that Lubik and Schorfheide's representation of a belief shock can be represented as a probability distribution over the forecast error of a subset of the variables of the model. Farmer (2002) shows how a self-fulfilling belief of this kind can be enforced by a forecasting rule, augmented by a sunspot shock. If agents use this rule in every period, and if their current beliefs about future prices are functions of the current sunspot shock, those beliefs will be validated in a rational expectations equilibrium.

5. Lubik–Schorfheide and Farmer–Khramov–Nicolò compared

The two papers by Lubik and Schorfheide (Lubik and Schorfheide, 2003, 2004), are widely cited in the literature (Belaygorod and Dueker, 2009; Zheng and Guo, 2013; Lubik and Matthes, 2013) and their approach is the one most closely emulated by researchers who wish to estimate models that possess an indeterminate equilibrium. This section compares the Lubik–Schorfheide method to the Farmer–Khramov–Nicolò technique (which we denote by LS and FKN) and proves an equivalence result.

We show in Theorem 2 that every LS equilibrium can be implemented as a FKN equilibrium, and conversely, every FKN equilibrium can be characterized using the LS technique. Because our method can be implemented using standard algorithms, our method provides an easy way for applied researchers to simulate and estimate indeterminate models using widely available computer software. And Theorem 2 shows that the full set of indeterminate equilibria can be modeled using our approach.

5.1. The singular value decomposition

Determinacy boils down to the following question: Does Eq. (14), which we repeat below as Eq. (25), have a unique solution for the $p \times 1$ vector of endogenous errors, η_t, as functions of the $\ell \times 1$ vector of fundamental errors, z_t?

$$\underset{n \times \ell}{\tilde{\Psi}_2} \underset{\ell \times 1}{z_t} + \underset{n \times p}{\tilde{\Pi}_2} \underset{p \times 1}{\eta_t} = 0. \tag{25}$$

To answer this question, LS apply the singular value decomposition to the matrix $\tilde{\Pi}_2$. The interesting case is when $p > n$, for which $\tilde{\Pi}_2$ has n singular values, equal to the positive square roots of the eigenvalues of $\tilde{\Pi}_2 \tilde{\Pi}_2^T$. The singular values are collected into a diagonal matrix D_{11}. The matrices U_1 and V in the decomposition are orthonormal and $m = p - n$ is the degree of indeterminacy:

$$\underset{n \times p}{\tilde{\Pi}_2} \equiv \underset{n \times n}{U_1} \begin{bmatrix} \underset{n \times n}{D_{11}} & \underset{n \times m}{\mathbf{0}} \end{bmatrix} \underset{p \times p}{V^T}. \tag{26}$$

Replacing $\tilde{\Pi}_2$ in (25) with this expression and premultiplying by U_1^T lead to the equation

$$\underset{n \times n}{U_1^T} \underset{n \times \ell}{\tilde{\Psi}_2} \underset{\ell \times 1}{z_t} + \begin{bmatrix} \underset{n \times n}{D_{11}} & \underset{n \times m}{\mathbf{0}} \end{bmatrix} \underset{p \times p}{V^T} \underset{p \times 1}{\eta_t} = 0. \tag{27}$$

Now partition

$$V = \begin{bmatrix} \underset{p \times n}{V_1} & \underset{p \times m}{V_2} \end{bmatrix},$$

and premultiply (27) by D_{11}^{-1},

$$\underset{n \times n}{D_{11}^{-1}} \underset{n \times n}{U_1^T} \underset{n \times \ell}{\tilde{\Psi}_2} \underset{\ell \times 1}{z_t} + \underset{n \times p}{V_1^T} \underset{p \times 1}{\eta_t} = 0. \tag{28}$$

Because $p > n$ this system has fewer equations than unknowns. LS suggest that we supplement it with the following new $m = p - n$ equations:

$$\underset{m \times \ell}{M_z} \underset{\ell \times 1}{z_t} + \underset{m \times m}{M_\zeta} \underset{m \times 1}{\zeta_t} = \underset{m \times p}{V_2^T} \underset{p \times 1}{\eta_t}. \tag{29}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero and covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamentals, z_t:

$$E[\zeta_t] = 0, \quad E[\zeta_t z_t^T] = 0, \quad E[\zeta_t \zeta_t^T] = \Omega_{\zeta\zeta}. \tag{30}$$

Correlation of the forecast errors, η_t, with fundamentals, z_t, is captured by the matrix M_z. Because the parameters of $\Omega_{\zeta\zeta}$ cannot separately be identified from the parameters of M_ζ, LS choose the normalization

$$M_\zeta = I_m. \tag{31}$$

Appending Eq. (29) as additional rows to Eq. (28), premultiplying by V and rearranging terms lead to the following representation of the expectational errors as functions of the fundamentals, z_t and the sunspot shocks, ζ_t:

$$\underset{p \times 1}{\eta_t} = \left(-\underset{p \times n}{V_1} \underset{n \times n}{D_{11}^{-1}} \underset{n \times n}{U_1^T} \underset{n \times \ell}{\tilde{\Psi}_2} + \underset{p \times m}{V_2} \underset{m \times \ell}{M_z} \right)_{\ell \times 1} z_t + \underset{p \times m}{V_2} \underset{m \times 1}{\zeta_t}. \tag{32}$$

This is Eq. (25) in Lubik and Schorfheide (2004) using our notation for dimensions and where our M_z is what LS call $\tilde{M}$. More compactly

$$\underset{p \times 1}{\eta_t} = \underset{p \times n}{V_1} \underset{n \times \ell}{N} \underset{\ell \times 1}{z_t} + \underset{p \times m}{V_2} \underset{m \times \ell}{M_z} \underset{\ell \times 1}{z_t} + \underset{p \times m}{V_2} \underset{m \times 1}{\zeta_t}, \tag{33}$$

where

$$\underset{n\times\ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}}\, \underset{n\times n}{U_1^T}\, \underset{n\times\ell}{\tilde{\Psi}_2}.$$

is a function of the parameters of the model.

5.2. Equivalent characterizations of indeterminate equilibria

To define a unique sunspot equilibrium when the model is indeterminate, our method partitions η_t into two subsets; $\eta = \{\eta_f, \eta_n\}$. We refer to η_f as new fundamentals. A FKN equilibrium is characterized by a parameter vector $\theta \in \Theta_{FKN}$ which has two parts. $\theta_1 \in \Theta_1$:

$$\theta_1 \equiv \text{vec}(\Gamma_0, \Gamma_1, \Psi, \Omega_z)^T,$$

is a vector of parameters of the structural equations, including the variance covariance matrix of the original fundamentals. And $\theta_2 \in \Theta_2$

$$\theta_2 \equiv \text{vec}(\Omega_{zf}, \Omega_{ff})^T,$$

is a vector of parameters that contains the variance covariance matrix of the new fundamentals and the covariances of these new fundamentals, η_f, with the original fundamentals, z.

A FKN representation of equilibrium is a vector $\theta_{FKN} \in \Theta_{FKN}$ where Θ_{FKN} is defined as

$$\Theta_{FKN} \equiv \{\Theta_1, \Theta_2\}.$$

Theorem 1 establishes that there is an equivalence class of models, all with the same likelihood function, in which the $m \times 1$ vector η_f is selected as a new set of fundamentals and the VCV matrices Ω_{ff} and Ω_{zf} are additional parameters. To complete the model in this way we must add $m(m+1)/2$ new parameters to define the symmetric matrix Ω_{ff} and $m \times \ell$ new parameters to define the elements of Ω_{zf}.

In contrast a LS equilibrium is characterized by a parameter vector

$$\Theta_{LS} \equiv \{\Theta_1, \Theta_3\},$$

where $\theta_3 \in \Theta_3$ is defined as

$$\theta_3 \equiv \text{vec}(\Omega_{\zeta\zeta}, M_z)^T. \tag{34}$$

These parameters characterize the additional equation:

$$\underset{m\times\ell}{M_z}\, \underset{\ell\times1}{z_t} + \underset{m\times1}{\zeta_t} = \underset{m\times p}{V_2^T}\, \underset{p\times1}{\eta_t}, \tag{35}$$

where Eq. (35) adds the normalization (31) to Eq. (29).

The matrix $\Omega_{\zeta\zeta}$ has $m \times (m+1)/2$ new parameters; these are the variance covariances of the sunspot shocks and the matrix M_z has $m \times \ell$ new parameters, these capture the covariances of η with z. To establish the connection between the two characterizations of equilibrium, we establish the following two lemmas.

Lemma 1. *Let ε be a regular indeterminate equilibrium, characterized by $\theta_{FKN} = \{\theta_1, \theta_2\}$ and let $\mathbf{p}_i = \{\eta_{f,t}^i, \eta_{n,t}^i\}$ be an element of the set of partitions, $\mathbf{P}$. Let $\theta_{LS} = \{\theta_1, \theta_3\}$ be the parameters of a Lubik–Schorfheide representation of equilibrium. There are an $m \times m$ matrix G^i and an $m \times \ell$ matrix H^i, where the elements of G^i and H^i are functions of θ_1 and an $m \times \ell$ matrix S^i, respectively:*

$$\underset{m\times\ell}{S^i} = \left(\underset{m\times\ell}{H^i} + \underset{m\times\ell}{M_z} \right), \tag{36}$$

such that the sunspots shocks in the LS representation of equilibrium are related to the fundamentals z_t and the newly defined FKN fundamentals, $\eta_{f,t}^i$ by the equation

$$\underset{m\times1}{\zeta_t} = \underset{m\times m}{G^i}\, \underset{m\times1}{\eta_{f,t}^i} - \underset{m\times\ell}{S^i}\, \underset{\ell\times1}{z_t}. \tag{37}$$

Proof. See Appendix B.

Lemma 1 connects the LS sunspots to the FKN definition of fundamentals. Lemma 2, described below, provides a way of mapping between the original fundamental shocks and the newly defined fundamentals under two alternative partitions $\mathbf{p}_i$ and $\mathbf{p}_j$.

Lemma 2. *Let ε be a regular indeterminate equilibrium, characterized by $\theta_{FKN} = \{\theta_1, \theta_2\}$ and let $\mathbf{p}_i = \{\eta_{f,t}^i, \eta_{n,t}^i\}$ and $\mathbf{p}_j = \{\eta_{f,t}^j, \eta_{n,t}^j\}$ be two elements of the set of partitions, $\mathbf{P}$. There exist an $m \times m$ matrix G^i, an $m \times \ell$ matrix H^i, an $m \times m$ matrix G^j, and an $m \times \ell$ matrix H^j, where the elements of G^i, H^i, G^j and H^j are functions of θ_1. The new FKN fundamentals under*

partition $\mathbf{p}_i$, $\eta^i_{f,t}$, are related to the fundamentals z_t and the new FKN fundamentals under partition $\mathbf{p}_j$, $\eta^j_{f,t}$ by the equation

$$\underset{m\times 1}{\eta^i_{f,t}} = \underset{m\times m}{\left(G^i\right)^{-1}}\left[\underset{m\times m}{G^j}\,\underset{m\times 1}{\eta^j_{f,t}} - \left(\underset{m\times \ell}{H^j} - \underset{m\times \ell}{H^i}\right)\underset{\ell\times 1}{z_t}\right]. \tag{38}$$

Proof. Follows immediately from Eqs. (36) and (37) and the fact that G^i is non-singular for all i. ☐

Eq. (38) defines the equivalence between alternative FKN definitions of the fundamental shocks, without reference to the LS definition. The following theorem, proved in Appendix C, uses Lemma 1 to establish an equivalence between the LS and FKN definitions.

Theorem 2. Let θ_{LS} and θ_{FKN} be two alternative parameterizations of an indeterminate equilibrium in model (1). For every FKN equilibrium, parameterized by θ_{FKN}, there is a unique matrix M_z and a unique VCV matrix $\Omega_{\zeta\zeta}$ such that $\theta_3 = \text{vec}(\Omega_{\zeta\zeta}, M_z)^T$ and $\{\theta_1, \theta_3\} \in \Theta_{LS}$ defines an equivalent LS equilibrium. Conversely, for every LS equilibrium, parameterized by θ_{LS}, and every partition $\mathbf{p}_i \in \mathbf{P}$, there are a unique VCV matrix Ω_{ff} and a unique covariance matrix Ω_{zf} such that $\theta_2 = \text{vec}(\Omega_{ff}, \Omega_{zf})^T$ and $\{\theta_1, \theta_2\} \in \Theta_{FKN}$ defines an equivalent FKN equilibrium.

Proof. See Appendix C.

Next, we turn to an example that shows how to use our results in practice.

6. Applying our method in practice: the Lubik–Schorfheide example

In this section we generate data from the model described in Lubik and Schorfheide (2004) and we use our method to recover parameter estimates from the simulated data. By using simulated data, rather than actual data, we avoid possible complications that might arise from mis-specification. For the simulated data, we know the true data generation process.

Section 6.1 explains how to implement our method for the case of the New-Keynesian model and in Section 6.2 we establish two results. First, we take Lubik and Schorfheide's (2004) parameter estimates for the pre-Volcker period, and we treat these parameter estimates as truth. Using the LS parameters, we simulate data under two alternative partitions of our model, and we verify that, using the same random seed, the simulated data are identical for both partitions. Second, we estimate the parameters of the model in Dynare, for the two alternative specifications, and we verify that the parameter estimates from two different partitions are the same.

6.1. The LS model with the FKN approach

The model of Lubik and Schorfheide (2004) consists of a dynamic IS curve:

$$x_t = E_t(x_{t+1}) - \tau(R_t - E_t(\pi_{t+1})) + g_t, \tag{39}$$

a New Keynesian Phillips curve:

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa(x_t - z_t), \tag{40}$$

and a Taylor rule:

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)\left[\psi_1 \pi_t + \psi_2(x_t - z_t)\right] + \varepsilon_{R,t}. \tag{41}$$

The variable x_t represents log deviations of GDP from a trend path and π_t and R_t are log deviations from the steady state level of inflation and the nominal interest rate, respectively.

The shocks g_t and z_t follow univariate AR(1) processes:

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{42}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{43}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are defined as σ_g, σ_z and σ_R, respectively. We allow the correlation between shocks, ρ_{gz}, ρ_{gR} and ρ_{zR}, to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{1,t} = x_t - E_{t-1}[x_t], \quad \eta_{2,t} = \pi_t - E_{t-1}[\pi_t]. \tag{44}$$

We define the vector of endogenous variables:

$$X_t = \left[x_t, \pi_t, R_t, E_t(x_{t+1}), E_t(\pi_{t+1}), g_t, z_t\right]^T$$

the vectors of fundamental shocks and non-fundamental errors:

$$\mathbf{z}_t = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right]^T, \quad \boldsymbol{\eta}_t = \left[\eta_{1,t}, \eta_{2,t}\right]^T$$

and the vector of parameters:

$$\theta = \left[\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_R, \rho_{gz}, \rho_{gR}, \rho_{zR}\right]^T.$$

This leads to the following representation of the model:

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\mathbf{z}_t + \Pi(\theta)\boldsymbol{\eta}_t, \tag{45}$$

where Γ_0 and Γ_1 are represented by

$$\Gamma_0(\theta) = \begin{bmatrix}
1 & 0 & \tau & -1 & -\tau & -1 & 0 \\
\kappa & -1 & 0 & 0 & \beta & 0 & -\kappa \\
(1-\rho_R)\psi_2 & (1-\rho_R)\psi_1 & -1 & 0 & 0 & 0 & -(1-\rho_R)\psi_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

and,

$$\Gamma_1(\theta) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\rho_R & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_g & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_z \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix},$$

and the coefficients of the shock matrices Ψ and Π are given by

$$\Psi(\theta) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Pi(\theta) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}.$$

The last two rows of this system define the non-fundamental shocks and it is these rows that we modify when estimating the model with the FKN approach.

6.1.1. The determinate case

When the monetary policy is active, $|\psi_1| > 1$, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, equals the number of unstable roots. The Blanchard–Kahn condition is satisfied and there is a unique sequence of non-fundamental shocks such that the state variables are bounded. In this case the model can be solved using Gensys which delivers the following system of equations:

$$X_t = G_1(\theta)X_{t-1} + G_2(\theta)\mathbf{z}_t \tag{46}$$

where $G_1(\theta)$ represents the coefficients of the policy functions and $G_2(\theta)$ is the matrix which expresses the impact of fundamental errors on the variables of interest, X_t.

6.1.2. Indeterminate models

A necessary condition for indeterminacy is that the monetary policy is passive, which occurs when

$$0 < |\psi_1| < 1. \tag{47}$$

A sufficient condition is that

$$0 < \psi_1 + \frac{(1-\beta)}{\kappa}\psi_2 < 1. \tag{48}$$

This condition is stronger than (47) but the two conditions are close, given our prior, which sets[3]

$$\frac{(1-\beta)}{\kappa}\psi_2 = 0.056.$$

When (48) holds, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, exceeds the number of unstable roots and there is 1 degree of indeterminacy. Using our approach, one can specify two equivalent alternative models depending on choice of the partition $\mathbf{p}_i$, for $i=1,2$.

Fundamental output expectations: Model 1. In our first specification, we choose $\eta_{1,t}$, the forecast error of output, as a new fundamental. We call this partition $\mathbf{p}_1$ and we write the new vector of fundamental shocks

$$\bar{\mathbf{z}}_{1,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{1,t}\right]^T.$$

The model is defined as

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi_x(\theta)\tilde{\mathbf{z}}_{1,t} + \Pi_x(\theta)\eta_{2,t}, \tag{49}$$

where

$$\Psi_x(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi_x(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that the matrices Γ_0 and Γ_1 are unchanged. We have simply redefined $\eta_{1,t}$ as a fundamental shock by moving one of the columns of Π to Ψ. Because the Blanchard–Kahn condition is satisfied under this redefinition, the model can be solved using Gensys to generate policy functions as well as the matrix which describes the impact of the re-defined vector of fundamental shocks on X_t.

Fundamental inflation expectations: Model 2. Following the same logic there is an alternative partition $\mathbf{p}_2$ where the new vector of fundamentals is defined as

$$\tilde{\mathbf{z}}_{2,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{2,t}\right]^T.$$

Here, the state equation is described by

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi_\pi(\theta)\tilde{\mathbf{z}}_{2,t} + \Pi_\pi(\theta)\eta_{1,t}, \tag{50}$$

where now

$$\Psi_\pi(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Pi_\pi(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Gensys, we can find a unique series of non-fundamental shocks $\eta_{1,t}$ such that the state variables are bounded and the state variables X_t are then a function of X_{t-1} and the new vector of fundamental errors $\tilde{\mathbf{z}}_{2,t}$.

6.2. Simulation and estimation using the FKN approach

In this section, we simulate data from the New-Keynesian model using the parameter estimates of Lubik and Schorfheide (2004) for the case when the model is indeterminate. In light of Theorem 2 and Lemma 2, data generated from the two partitions is identical, a result that we verify computationally. In Section 6.2.2, we use our simulated data to estimate model parameters under the two representations and we confirm that the posterior modes from each representation are, in most cases, equal to two decimal places and that all of the estimates lie well within the 90% probability bounds of the alternative specification.[4] These results demonstrate how to apply our theoretical results from Sections 4 and 5 in practice.

---

[3] We thank one of the referees for pointing that the Taylor principle must be modified, when the central bank responds to the output gap as well as to inflation.

[4] The estimates are not identical because of sampling error that arises from the use of a finite number of draws when we approximate posterior distributions with the Metropolis–Hastings algorithm. We did not see an obvious way of setting the same random seed within Dynare and hence we used different draws for each specification.

6.2.1. Simulation

In this section, we generate data for the observables, $\mathbf{y}_t = \{x_{obs,t}, \pi_{obs,t}, R_{obs,t}\}$, in two different ways. These variables are defined as

1. $x_{obs,t}$ the percentage deviations of (log) real GDP per capita from an HP-trend;
2. $\pi_{obs,t}$ the annualized percentage change in the Consumer Price Index for all Urban Consumers;
3. $R_{obs,t}$ the annualized percentage average Federal Funds Rate.

As described in Lubik and Schorfheide (2004), the measurement equation is given by

$$
\mathbf{y}_t = \begin{bmatrix} 0 \\ \pi^* \\ \pi^* + r^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} X_t.
\tag{51}
$$

where π^* and r^* are annualized steady-state inflation and real interest rates expressed in percentages, respectively. The parameter values that we use to run the simulation of the New-Keynesian model in Lubik and Schorfheide (2004) are the posterior estimates that the authors report for the pre-Volcker period and that we reproduce in Table 2. We feed the model with shocks using the FKN method for two alternative partitions.

We take the LS estimates of the standard deviation of the sunspots shock, σ_ζ, and the $m \times \ell$ matrix M_z and we treat these estimates as the truth. By applying Lemma 1 to the LS parameters, we obtain corresponding values[5] for the standard deviation of the newly defined fundamental, $\eta_{f,t}^i$, under two partitions, $\mathbf{p}_i$, $i \in \{1,2\}$:

$$
\underset{m\times m}{\Omega_{ff}^i} = \left(\underset{m\times m}{G^i}\right)^{-1} \left[\underset{m\times m}{\sigma_\zeta^2} + \underset{m\times \ell}{S^i}\ \underset{\ell\times \ell}{\Omega_{zz}}\ \underset{l\times m}{\left(S^i\right)^T} \right] \left(\underset{m\times m}{\left(G^i\right)^T}\right)^{-1},
\tag{52}
$$

and for the covariance of the fundamentals $\mathbf{z}_t$ with the newly defined fundamental $\eta_{f,t}^i$,

$$
\underset{m\times \ell}{\Omega_{fz}^i} = \left(\underset{m\times m}{G^i}\right)^{-1} \underset{m\times \ell}{S^i}\ \underset{\ell\times \ell}{\Omega_{zz}}.
\tag{53}
$$

The details on the construction of the matrices G^i, H^i and S^i are described in Appendix D.

Having defined the new vector of fundamentals $\tilde{\mathbf{z}}_{i,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{i,t} \right]^T$ we construct the following variance–covariance matrix:

$$
\underset{(\ell+m)\times(\ell+m)}{\Omega^i} \equiv E\left(\tilde{\mathbf{z}}_{i,t} \tilde{\mathbf{z}}_{i,t}^T \right).
\tag{54}
$$

Next, we perform of the Cholesky decomposition of the matrix $\Omega^i = L^i \left(L^i\right)^T$, where L^i is a lower triangular $(\ell+m) \times (\ell+m)$ matrix. After defining a $(\ell+m) \times 1$ vector of shocks u_t such that $E(u_t) = \mathbf{0}_{(\ell+m)\times 1}$ and $E(u_t u_t^T) = I_{(\ell+m)}$, we rewrite $\tilde{\mathbf{z}}_{i,t}$ as $\tilde{\mathbf{z}}_{i,t} = L^i u_t$.

The purpose of the Cholesky decomposition is to simplify the estimation procedure in Dynare[6] which we use to estimate the $(\ell+m) \times [(\ell+m)-1]$ parameters of the matrix L^i rather than the variance–covariance terms of the matrix Ω^i. Eq. (55) reports the matrix Ω^i for $i = 1,2$:

$$
\Omega^1 = \begin{bmatrix} 0.05 & - & - & - \\ 0 & 0.07 & - & - \\ 0 & 0.04 & 1.27 & - \\ -0.03 & 0.10 & 0.11 & 0.17 \end{bmatrix}, \quad \Omega^2 = \begin{bmatrix} 0.05 & - & - & - \\ 0 & 0.07 & - & - \\ 0 & 0.04 & 1.27 & - \\ -0.01 & 0.13 & -2.37 & 4.60 \end{bmatrix},
\tag{55}
$$

and Eq. (56) is the corresponding Cholesky decomposition L^i for $i = 1,2$:

$$
L^1 = \begin{bmatrix} 0.23 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 \\ 0 & 0.15 & 1.11 & 0 \\ -0.14 & 0.37 & 0.04 & 0.10 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 0.23 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 \\ 0 & 0.15 & 1.11 & 0 \\ -0.05 & 0.04 & -2.12 & 0.26 \end{bmatrix}.
\tag{56}
$$

Given a draw of u_t, we obtain the new vector of fundamentals $\tilde{\mathbf{z}}_{i,t} = L^i u_t$ for partition $\mathbf{p}_i$ and we construct the corresponding draws of the vector $\tilde{\mathbf{z}}_{j,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{j,t} \right]^T$. Using Lemma 2, Eq. (38), which we reproduce below as Eq. (57),

---

[5] We derive both Eqs. (52) and (53) from the result in Lemma 1 and by recalling that the vector of sunspot shocks ζ_t is now a scalar which, as described in Section 5.1, has the following properties, $E[\zeta_t] = 0$, $E[\zeta_t z_t^T] = 0$ and $E\left[\zeta_t \zeta_t^T\right] = \sigma_\zeta^2$.

[6] In particular, the estimation of the $(\ell+m) \times [(\ell+m)-1]$ elements of the lower triangular matrix L^i substantially reduces issues related to the convergence of the posterior estimates relative to the case of performing the estimation exercise by estimating the elements of the variance-covariance matrix Ω^i.

we derive the non-fundamental shock which is included as fundamental under partition $\mathbf{p}_j$ for $j \neq i$:

$$\eta_{f,t}^j = \left(G^j\right)^{-1} \left[G^i \, \eta_{f,t}^i - \left(H^i - H^j \right) z_t \right]. \tag{57}$$
$$\scriptstyle{m \times 1 \quad\quad m \times m \quad\quad m \times m \quad m \times 1 \quad m \times \ell \quad m \times \ell \quad \ell \times 1}$$

By feeding the two alternative models with the corresponding new vectors of fundamentals $\tilde{\mathbf{z}}_{1,t}$ and $\tilde{\mathbf{z}}_{2,t}$, using the same random seed, we obtain identical simulated data.[7]

6.2.2. Estimation results

Next, we estimate the parameters of the model on the simulated data and we demonstrate that the posterior estimates of the model parameters are equivalent under two alternative model specifications. Table 1 reports the prior distributions of the parameters used in our estimation. With the exception of priors over the elements of L^i, the prior distributions for the other parameters are the same as in Lubik and Schorfheide (2004).[8]

Table 2 compares the posterior estimates of the model parameters. While the first column reports the parameter values used to simulate the data, columns two and three are the estimates for two alternative partitions $\mathbf{p}_1$ and $\mathbf{p}_2$. Partition $\mathbf{p}_1$ treats $\eta_{1,t}$ as fundamental and partition $\mathbf{p}_2$ treats $\eta_{2,t}$ as fundamental. We used a random walk Metropolis–Hastings algorithm to obtain 150,000 draws from the posterior mean and we report 90% probability intervals of the estimated parameters.[9]

Compare the mean parameter estimates across the three columns. Fifteen of these parameters are common to all three specifications; these are the parameters: $\psi_1, \psi_2, \rho_R, \pi^*, r^*, \kappa, \tau^{-1}, \rho_g, \rho_z, L_{11}, L_{22}, L_{33}, L_{21}, L_{31}$ and L_{32}. The remaining four parameters reported in columns 2 and 3, $L_{41}^i, L_{42}^i, L_{43}^i$, and L_{44}^i represent the elements of the L^i matrix that are not comparable across specifications.

Our results show not only that under both models the posterior point estimates are remarkably close to the parameter values which we use to simulate the data, but also that both the posterior point estimates and the probability intervals are statistically indistinguishable when comparing the two alternative models. This correspondence in parameter estimates across specifications is a consequence of Theorems 1 and 2 of our paper.

7. Implementing our procedure in Dynare

This section provides a practical guide to the user who wishes to implement our method in Dynare. Consider the New-Keynesian model described in Section 6, which we repeat below for completeness:

$$x_t = E_t[x_{t+1}] - \tau(R_t - E_t[\pi_{t+1}]) + g_t, \tag{58}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t + z_t, \tag{59}$$

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{60}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{61}$$

The model is determinate when monetary policy is active, $|\psi_1| > 1$. In this case Dynare finds the unique series of non-fundamental errors that keeps the state variables bounded and Table 3 reports the code required to estimate the model in this case.

In the case of the indeterminate models described in Section 6.1.2, running Dynare with the code from Table 3 produces an error with a message "Blanchard–Kahn conditions are not satisfied: indeterminacy." For regions of the parameter space where the code produces that message, we provide two alternative versions of the model that redefine one of the non-fundamental shocks as new fundamental. Following the notation in Section 6.1.2, we refer to these cases as Model 1, where $\eta_{1,t} = x_t - E_{t-1}[x_t]$ is a fundamental shock, and Model 2, where it is $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$ and we present the Dynare code to estimate the two indeterminate cases.

Tables 4 and 5 present the amended code for these cases. In Table 4, we show how to change the model by redefining $\eta_{1,t}$ as fundamental and Table 5 presents an equivalent change to Table 3 in which $\eta_{2,t}$ becomes the new fundamental. We have represented the new variables and new equations in that table using bold typeface.

---

[7] The code is available in the online Appendix and the results are obtained simulating the data by using both Gensys and Dynare.

[8] The only difference with respect to Lubik and Schorfheide (2004) is that we use a flatter prior for the parameter κ. While the authors set a gamma distribution with mean 0.5 and standard deviation 0.2, our prior sets the standard deviation to 0.35, leaving the mean unchanged. Choosing a flatter prior avoids facing an issue in the convergence of the parameter which arises with a relatively tight prior as in Lubik and Schorfheide (2004). Also, Table 1 reports the mean, the standard deviation and the 90% probability interval for each parameter. Note that we were unable to replicate the probability intervals in Lubik and Schorfheide (2004) and we report the 5th and the 95th percentiles of each distribution. However, the differences with Lubik and Schorfheide (2004) in the values for the probability intervals are small.

[9] To run the estimation exercise, we consider a sample of 1000 observations from the simulated data, run 6 chains of 50,000 draws each and we finally discard half of the draws. The acceptance ratio for all the chains is between 25% and 33%.

Table 1
Prior distribution for DSGE model parameters.

Name	Range	Density	Mean	Std. Dev.	90% interval
ψ_1	$\mathbb{R}^+$	Gamma	1.1	0.50	[0.42,2.03]
ψ_2	$\mathbb{R}^+$	Gamma	0.25	0.15	[0.06,0.53]
ρ_R	[0, 1)	Beta	0.50	0.20	[0.17,0.82]
π^*	$\mathbb{R}^+$	Gamma	4.00	2.00	[1.36,7.75]
r^*	$\mathbb{R}^+$	Gamma	2.00	1.00	[0.68,3.87]
κ	$\mathbb{R}^+$	Gamma	0.50	0.35	[0.09,1.17]
τ^{-1}	$\mathbb{R}^+$	Gamma	2.00	0.50	[1.25,2.88]
ρ_g	[0, 1)	Beta	0.70	0.10	[0.54,0.85]
ρ_z	[0, 1)	Beta	0.70	0.10	[0.54,0.85]
L_{11}	$\mathbb{R}^+$	Inverse Gamma	0.2	0.15	[0.07,0.44]
L_{22}	$\mathbb{R}^+$	Inverse Gamma	0.3	0.2	[0.12,0.64]
L_{33}	$\mathbb{R}^+$	Inverse Gamma	1	0.3	[0.61,1.55]
L_{21}		Normal	0	0.1	[−0.16,0.16]
L_{31}		Normal	0	0.1	[−0.16,0.16]
L_{32}		Normal	0.15	0.1	[−0.01,0.31]
L_{41}^1		Normal	0	0.2	[−0.32,0.32]
L_{42}^1		Normal	0.3	0.2	[−0.02,0.62]
L_{43}^1		Normal	0	0.2	[−0.32,0.32]
L_{44}^1		Normal	0.1	0.2	[−0.22,0.42]
L_{41}^2		Normal	0	0.2	[−0.32,0.32]
L_{42}^2		Normal	0	0.2	[−0.32,0.32]
L_{43}^2		Normal	−2	0.5	[−2.82, −1.18]
L_{44}^2		Normal	0.3	0.2	[−0.02,0.62]

Table 2
Posterior means and probability intervals.

L&S (prior 1)	Mean	FKN – Model 1		FKN – Model 2	
		Mean	90% interval	Mean	90% interval
ψ_1	0.77	0.77	[0.73,0.81]	0.77	[0.73,0.81]
ψ_2	0.17	0.21	[0.08,0.33]	0.22	[0.08,0.35]
ρ_R	0.60	0.61	[0.59,0.63]	0.61	[0.59,0.63]
π^*	4.28	4.44	[4.17,4.71]	4.43	[4.16,4.70]
r^*	1.13	1.18	[1.10,1.25]	1.17	[1.10,1.25]
κ	0.77	0.67	[0.47,0.89]	0.71	[0.51,0.91]
τ^{-1}	1.45	1.63	[1.41,1.85]	1.61	[1.39,1.82]
ρ_g	0.68	0.66	[0.62,0.70]	0.66	[0.62,0.70]
ρ_z	0.82	0.83	[0.81,0.84]	0.83	[0.81,0.85]
L_{11}	0.23	0.23	[0.22,0.24]	0.23	[0.22,0.24]
L_{22}	0.27	0.25	[0.21,0.29]	0.25	[0.21,0.29]
L_{33}	1.11	1.14	[0.90,1.37]	1.10	[0.87,1.30]
L_{21}	0	−0.01	[−0.03,0.009]	−0.01	[−0.03,0.009]
L_{31}	0	0.02	[−0.09,0.14]	0.003	[−0.09,0.09]
L_{32}	0.15	0.14	[0.01,0.27]	0.14	[0.04,0.25]
L_{41}^1	−0.14	−0.15	[−0.18, −0.13]	–	–
L_{42}^1	0.37	0.36	[0.34,0.37]	–	–
L_{43}^1	0.04	0.02	[−0.02,0.07]	–	–
L_{44}^1	0.10	0.10	[−0.20,0.42]	–	–
L_{41}^2	−0.05	–	–	−0.07	[−0.25,0.11]
L_{42}^2	0.04	–	–	0.03	[−0.17,0.22]
L_{43}^2	−2.12	–	–	−2.09	[−2.16, −2.01]
L_{44}^2	0.26	–	–	0.30	[−0.02,0.62]

Table 3
Determinate model.

Variable definitions	var x, R, pi, g, z; varexo e_R, e_g, e_z;
Parameter definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1\,psi2$;

Model equations	model(linear);
	$x = x(+1) - tau*(R - pi(+1)) + g$;
	$pi = 0.97*pi(+1) + kappa*(x - z)$;
	$R = rho\_R*R(-1) + (1 - rho\_R)*$
	$(psi1*pi + psi2*(x - z)) + e\_R$;
	$g = rho\_g*g(-1) + e\_g$;
	$z = rho\_z*z(-1) + e\_z$;
	end;

Table 4
Indeterminate model 1. $\eta_{1,t} = x_t - E_{t-1}[x_t]$ is new fundamental.

Variable definitions	var $x, R, pi, g, z, \mathbf{xs}$; varexo $e\_R, e\_g, e\_z, \mathbf{sunspot}$;
Parameter definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1$ $psi2, sigmag, sigmaz, sigmaR$;

Model equations	model(linear);
	$x = \mathbf{xs} - tau*(R - pi(+1)) + g$;
	$pi = 0.97*pi(+1) + kappa*(x - z)$;
	$R = rho\_R*R(-1) + (1 - rho\_R)*$
	$(psi1*pi + psi2*(x - z)) + e\_R$;
	$g = rho\_g*g(-1) + e\_g$;
	$z = rho\_z*z(-1) + e\_z$;
	$x - \mathbf{xs}(-1) = \mathbf{sunspot}$;
	end;

Table 5
Indeterminate model 2. $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$ is new fundamental.

Variable definitions	var $x, R, pi, g, z, \mathbf{pis}$; varexo $e\_R, e\_g, e\_z, \mathbf{sunspot}$;
Parameter definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1$ $psi2, sigmag, sigmaz, sigmaR$;
Model equations	model(linear) ; $x = x(+1) - tau*(R - pi(+1)) + g$; $pi = 0.97*\mathbf{pis} + kappa*(x - z)$; $R = rho\_R*R(-1) + (1 - rho\_R)*$ $(psi1*pi + psi2*(x - z)) + e\_R$; $g = rho\_g*g(-1) + e\_g$; $z = rho\_z*z(-1) + e\_z$; $pi - \mathbf{pis}(-1) = \mathbf{sunspot}$; end;

The following steps explain the changes in more detail. First, we define a new variable, $\mathbf{xs} \equiv E_t[x_{t+1}]$ and include it as one of the endogenous variables in the model. This leads to the declaration

$$\text{var } x, \ R, \ pi, \ \mathbf{xs}; \tag{62}$$

which appears in the first line of Table 4. Next, we add an expectational shock, which we call **sunspot**, to the set of fundamental shocks, e_R, e_g and e_z. This leads to the Dynare statement

$$\text{varexo } e\_R, \ e\_g, \ e\_z, \ \mathbf{sunspot}; \tag{63}$$

which appears in row 2. Then we replace $x(+1)$ by $\mathbf{xs}$ in the consumption-Euler equation, which becomes

$$x = \mathbf{xs} - tau*(R - pi(+1)) + g; \tag{64}$$

and we add a new equation that defines the relationship between $\mathbf{xs}$, x and the new fundamental error:

$$x - \mathbf{xs}(-1) = \mathbf{sunspot}; \tag{65}$$

Similar steps apply in the case of Model 2, but with $\eta_{2,t}$ taking the role of $\eta_{1,t}$. Note that, by substituting expectations of forward-looking variables $x(+1)$ in Model 1, and $pi(+1)$ in Model 2, with **xs** and **pis**, respectively, we decrease the number of forward-looking variables by one. Since these variables are no longer solved forwards, we must add an equation – this appears as Eq. (65) – to describe the dynamics of the new fundamental shock.

How can a researcher know, in advance, if his model is determinate. The answer provided by Lubik and Schorfheide (2004) is that determinate and indeterminate models are alternative representations of data that can be compared either by likelihood ratio tests or by Bayesian model comparison.

The Lubik–Schorfheide approach assumes that the researcher can identify, a priori, determinate and indeterminate regions of the parameter space. For models where that is difficult or impossible, Fanelli (2012) and Castelnuovo and Fanelli (2015) propose an alternative method that may be used to test the null hypothesis of determinacy.

8. Conclusion

Our paper provides a method to solve and estimate indeterminate linear rational expectations models using standard software packages. Our method transforms indeterminate models by redefining a subset of the non-fundamental shocks and classifying them as new fundamentals. Our approach to handling indeterminate equilibria is more easily implementable than that of Lubik and Schorfheide and, one might argue, is also more intuitive. We illustrated our approach using the familiar New-Keynesian monetary model and we showed that, when monetary policy is passive, the New-Keynesian model can be closed in one of the two equivalent ways.

Our procedure raises the question of which non-fundamental shocks to reclassify as fundamental. Our theoretical results demonstrate that the choice of parameterization is irrelevant since all parameterizations have the same likelihood function. We demonstrated that result in practice by estimating a model due to Lubik and Schorfheide (2004) in two different ways and recovering parameter estimates that are statistically indistinguishable between the two. We caution that, in practice, it is important to leave the VCV matrix of errors unrestricted for our results to apply. Our work should be of interest to economists who are interested in estimating models that do not impose a determinacy prior.

Appendix A

Proof of Theorem 1. Let A^1 and A^2 be two orthonormal row operators associated with partitions $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively:

$$\begin{bmatrix} z_t \\ \eta^1_{f,t} \\ \eta^1_{n,t} \end{bmatrix} = A^1 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}, \quad \begin{bmatrix} z_t \\ \eta^2_{f,t} \\ \eta^2_{n,t} \end{bmatrix} = A^2 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}. \tag{A.1}$$

We assume that the operators, A^i have the form

$$A^i = \begin{bmatrix} I_{l\times l} & 0 \\ 0 & \tilde{A}^i_{p\times p} \end{bmatrix}, \tag{A.2}$$

where $\tilde{A}^i$ is a permutation of the columns of an I_p identity matrix. Premultiplying the vector $[z_t, \eta_t]^T$ by the operator A^i permutes the rows of η_t while leaving the rows of z_t unchanged. Define matrices Ω_{ff} and Ω_{zf} for $i \in \{1, 2\}$ to be the new terms in the fundamental covariance matrix:

$$E\left(\begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix} \begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix}^T\right) = \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}.$$

Next, use (22) and (23) to write the non-fundamentals as linear functions of the fundamentals:

$$\eta^i_{n,t} = \Theta^i_z z_t + \Theta^i_f \eta^i_{f,t}, \tag{A.3}$$

where

$$\Theta^i_z \equiv -\left(\tilde{\Pi}^i_{2n}\right)^{-1} \tilde{\Psi}_2, \quad \text{and} \quad \Theta^i_f \equiv -\left(\tilde{\Pi}^i_{2n}\right)^{-1} \tilde{\Pi}^i_{2f}, \tag{A.4}$$

and define the matrix

$$D^i = \begin{bmatrix} \underset{l \times l}{I} & \underset{l \times m}{0} \\ \underset{m \times l}{0} & \underset{m \times m}{I} \\ \underset{(p-m) \times l}{\Theta_z^i} & \underset{(p-m) \times m}{\Theta_f^i} \end{bmatrix}. \tag{A.5}$$

Using this definition, the covariance matrix of all shocks, fundamental and non-fundamental, has the following representation:

$$E\left(\begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix} \begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}^T \right) = D^i \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix} D^{iT}. \tag{A.6}$$

We can also combine the last two row blocks of D^i and write D^i as follows:

$$D^i = \begin{bmatrix} \underset{l \times l}{I} & \underset{l \times m}{0} \\ \underset{p \times l}{D_{21}^i} & \underset{p \times m}{D_{22}^i} \end{bmatrix}, \tag{A.7}$$

where

$$D_{21}^i = \begin{bmatrix} \underset{m \times l}{0} \\ \underset{(p-m) \times l}{\Theta_z^i} \end{bmatrix}, \quad D_{22}^i = \begin{bmatrix} \underset{m \times m}{I} \\ \underset{(p-m) \times m}{\Theta_f^i} \end{bmatrix}. \tag{A.8}$$

Using (A.1) and the fact that A^i is orthonormal, we can write the following expression for the complete set of shocks:

$$\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} = A^{iT} \begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}. \tag{A.9}$$

Using Eqs. (A6) and (A9), it follows that

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T \right) = B^i W^i B^{iT}, \quad \text{for all } \mathbf{p}_i \in \mathbf{P}, \tag{A.10}$$

where

$$W^i \equiv \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}, \tag{A.11}$$

and

$$B^i \equiv A^{iT} D^i = \begin{bmatrix} I & 0 \\ 0 & \tilde{A}^i \end{bmatrix} \begin{bmatrix} I & 0 \\ D_{21}^i & D_{22}^i \end{bmatrix} = \begin{bmatrix} I & 0 \\ B_{21}^i & B_{22}^i \end{bmatrix}. \tag{A.12}$$

Using this expression, we can write out Eq. (A.10) in full to give

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T \right) = \begin{bmatrix} I & 0 \\ B_{21}^i & B_{22}^i \end{bmatrix} \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix} \begin{bmatrix} I & B_{21}^{iT} \\ 0 & B_{22}^{iT} \end{bmatrix}. \tag{A.13}$$

We seek to establish that for any partition $\mathbf{p}_i$, parameterized by matrices Ω_{ff}, and Ω_{zf} that there exist matrices Ω_{ff} and Ω_{zf} for all partitions $\mathbf{p}_j \in \mathbf{P}$, $j \neq i$, such that

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T \right) = B^i W^i B^{iT} = B^j W^j B^{jT}. \tag{A.14}$$

To establish this proposition, we write out the elements of (A.13) explicitly. Since W^i and B^i are symmetric we need to consider only the upper-triangular elements which give three equations in the matrices of Ω_{zf} and Ω_{ff}:

$$\Omega_{11} = \Omega_{zz},$$
$$\Omega_{12} = \Omega_{zz}^i B_{21}^{iT} + \Omega_{zf} B_{22}^{iT},$$
$$\Omega_{22} = B_{21}^i \Omega_{zz}^i B_{21}^{iT} + 2 B_{21}^i \Omega_{zf} B_{22}^{iT} + B_{22}^i \Omega_{ff} B_{22}^{iT}. \tag{A.15}$$

The first of these equations defines the covariance of the fundamental shocks and it holds for all i,j. Now define

$$a = \text{vec}(\Omega_{zz}), \quad x^i = \text{vec}(\Omega_{zf}), \quad y^i = \text{vec}(\Omega_{ff}).$$ (A.16)

Using the fact that

$$\text{vec}(ABC) = \left(C^T \otimes A\right)\text{vec}(B),$$ (A.17)

we can pass the *vec* operator through Eq. (A.15) and write the following system of linear equations in the unknowns x^j and y^j:

$$S^i \begin{bmatrix} x^i \\ y^i \end{bmatrix} + T^i a = S^j \begin{bmatrix} x^j \\ y^j \end{bmatrix} + T^j a,$$ (A.18)

$$S^k = \begin{bmatrix} \left(B_{22}^k \otimes I\right) & 0 \\ \left(B_{22}^k \otimes B_{21}^j\right) & \left(B_{22}^k \otimes B_{22}^k\right) \end{bmatrix}, \quad T^k = \begin{bmatrix} \left(B_{21}^k \otimes I\right) \\ \left(B_{21}^k \otimes B_{21}^k\right) \end{bmatrix}, \quad k \in \{i,j\}.$$ (A.19)

It follows from the assumption that the equilibrium is regular that S^j has full rank for all j hence for any permutation $\mathbf{p}_i$, parameterized by $\{x^i, y^i\}$ we can find an alternative permutation $\mathbf{p}_j$ with associated parameterization $\{x^j, y^j\}$:

$$\begin{bmatrix} x^j \\ y^j \end{bmatrix} = \left(S^j\right)^{-1} \left(S^i \begin{bmatrix} x^i \\ y^i \end{bmatrix} + \left[T^i - T^j\right] a\right),$$ (A.20)

that gives the same covariance matrix $\tilde{\Omega}$ for the fundamental and non-fundamental shocks.   □

Appendix B

Proof of Lemma 1. We seek to characterize the full set of solutions to the equation

$$\underset{n \times \ell}{\tilde{\Psi}_2} \underset{\ell \times 1}{z_t} + \underset{n \times p}{\tilde{\Pi}_2} \underset{p \times 1}{\eta_t} = 0.$$ (B.1)

Let U_1, V and D_{11} characterize the singular value decomposition of $\tilde{\Pi}_2$:

$$\underset{n \times p}{\tilde{\Pi}_2} \equiv \underset{n \times n}{U_1} \begin{bmatrix} \underset{n \times n}{D_{11}} & \underset{n \times m}{\mathbf{0}} \end{bmatrix} \underset{p \times p}{V^T},$$ (B2)

where we partition the matrix V as

$$V = \begin{bmatrix} \underset{p \times n}{V_1} & \underset{p \times m}{V_2} \end{bmatrix},$$

Let θ_{FKN} characterize a regular indeterminate equilibrium for some partition $\mathbf{p}_i$ and we partition η_t into two mutually exclusive subsets, $\eta_{f,t}^i$ and $\eta_{n,t}^i$, such that $\eta_{f,t}^i \cup \eta_{n,t}^i = \eta_t$. From Appendix A, Eq. (A.3), we write the non-fundamentals $\eta_{n,t}^i$ as functions of the fundamentals and where Θ_z^i and Θ_f^i are functions of θ_1:

$$\underset{n \times 1}{\eta_{n,t}^i} = \underset{n \times \ell}{\Theta_z^i} \underset{\ell \times 1}{z_t} + \underset{n \times m}{\Theta_f^i} \underset{m \times 1}{\eta_{f,t}^i}.$$ (B3)

Eq. (B.3) connects the non-fundamental shocks $\eta_{n,t}^i$ to the fundamental shocks $[z_t, \eta_{f,t}^i]$ in the FKN equilibrium. Eq. (33) reproduced below as (B.4), characterizes the additional equations that define an LS equilibrium:

$$\underset{p \times 1}{\eta_t} = \underset{p \times n}{V_1} \underset{n \times \ell}{N} \underset{\ell \times 1}{z_t} + \underset{p \times m}{V_2} \underset{m \times \ell}{M_z} \underset{\ell \times 1}{z_t} + \underset{p \times m}{V_2} \underset{m \times 1}{\zeta_t},$$ (B.4)

where $N \equiv -D_{11}^{-1} U_1^T \tilde{\Psi}_2$. To establish the connection between the LS and FKN representations we split the equations of (B.4) into two blocks:

$$\underset{n \times 1}{\eta_{n,t}^i} = \underset{n \times n}{V_{1,n}^i} \underset{n \times \ell}{N} \underset{\ell \times 1}{z_t} + \underset{n \times m}{V_{2,n}^i} \underset{m \times \ell}{M_z} \underset{\ell \times 1}{z_t} + \underset{n \times m}{V_{2,n}^i} \underset{m \times 1}{\zeta_t}$$ (B5)

$$\underset{m \times 1}{\eta_{f,t}^i} = \underset{m \times n}{V_{1f}^i} \underset{n \times \ell}{N} \underset{\ell \times 1}{z_t} + \underset{m \times m}{V_{2f}^i} \underset{m \times \ell}{M_z} \underset{\ell \times 1}{z_t} + \underset{m \times m}{V_{2f}^i} \underset{m \times 1}{\zeta_t}$$ (B6)

where for $j=1,2$, the matrices $V_{j,f}^i$ and $V_{j,n}^i$ are composed of the row vectors of V_j which, according to partition $\mathbf{p}_i$, correspond to the non-fundamental shocks included as fundamental, $\eta_{f,t}^i$, and those that are still non-fundamental, $\eta_{n,t}^i$.

Using (B.3) to replacing $\eta_{n,t}^i$ in (B.5) and combining with (B.6)

$$\begin{bmatrix} \Theta_f^i \\ {}_{n\times m} \\ I_m \\ {}_{m\times 1} \end{bmatrix} \underset{m\times 1}{\eta_{f,t}^i} = \underset{p\times n}{V_1^i} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} - \begin{bmatrix} \Theta_z^i \\ {}_{n\times \ell} \\ \mathbf{0} \\ {}_{m\times \ell} \end{bmatrix} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2^i} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2^i} \underset{m\times 1}{\zeta_t}, \tag{B.7}$$

where

$$\underset{p\times n}{V_j^i} \equiv \begin{bmatrix} V_{j,n}^i \\ {}_{n\times n} \\ V_{j,f}^i \\ {}_{m\times n} \end{bmatrix}.$$

Premultiplying (B.7) by $\left(V_2^i\right)^T$ and exploiting the fact that V is orthonormal, leads to the equation

$$\underset{m\times m}{G^i} \underset{m\times 1}{\eta_{f,t}^i} = \underset{m\times \ell}{H^i} \underset{\ell\times 1}{z_t} + \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}, \tag{B.8}$$

where

$$\underset{m\times m}{G^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \begin{bmatrix} \Theta_f^i \\ {}_{n\times m} \\ I_m \\ {}_{p\times m} \end{bmatrix}, \quad \text{and} \quad \underset{m\times \ell}{H^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \underset{p\times n}{V_1^i} \underset{n\times \ell}{N} - \underset{m\times p}{\left(V_2^i\right)^T} \begin{bmatrix} \Theta_z^i \\ {}_{n\times \ell} \\ \mathbf{0} \\ {}_{m\times \ell} \\ {}_{p\times l} \end{bmatrix}. \tag{B.9}$$

Rearranging (B.8) and defining

$$\underset{m\times \ell}{S^i} \equiv \underset{m\times \ell}{H^i} + \underset{m\times \ell}{M_z} \tag{B.10}$$

gives

$$\underset{m\times 1}{\zeta_t} = \underset{m\times m}{G^i} \underset{m\times 1}{\eta_{f,t}^i} - \underset{m\times \ell}{S^i} \underset{\ell\times 1}{z_t}, \tag{B.11}$$

which is the expression we seek. □

Appendix C

Proof of Theorem 2. Let $\theta_{FKN} = \{\theta_1, \theta_2\}$ characterize an FKN equilibrium. From (B.8), which we repeat below omitting the superscript i to reduce notation

$$\underset{m\times m}{G} \underset{m\times 1}{\eta_{f,t}} = \underset{m\times \ell}{H} \underset{\ell\times 1}{z_t} + \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}. \tag{C.1}$$

Post-multiplying this equation by z_t^T and taking expectations give

$$\underset{m\times m}{G} \underset{m\times \ell}{\Omega_{fz}} = \underset{m\times \ell}{H} \underset{\ell\times \ell}{\Omega_{zz}} + \underset{m\times \ell}{M_z} \underset{\ell\times \ell}{\Omega_{zz}} = \underset{m\times \ell}{S} \underset{\ell\times \ell}{\Omega_{zz}}, \tag{C.2}$$

which represents $m \times \ell$ linear equations in the $m \times \ell$ elements of vec(M_z) as functions of the elements of H, G and Ω_{zz} (these are functions of θ_1), and Ω_{fz} (these are elements of θ_2). Applying the vec operator to (C.2), using the algebra of Kronecker products, and rearranging terms gives the following solution for the parameters vec(M_z):

$$\underset{(m\times \ell)\times 1}{\text{vec}(M_z)} = \underset{(m\times \ell)\times(\ell\times m)}{(\Omega_{zz} \otimes I_m)^{-1}} \left[\underset{(m\times \ell)\times(\ell\times m)}{(I_\ell \otimes G)} \underset{(m\times \ell)\times 1}{\text{vec}(\Omega_{fz})} - \underset{(m\times \ell)\times \ell^2}{(I_\ell \otimes H)} \underset{\ell^2\times 1}{\text{vec}(\Omega_{zz})} \right]. \tag{C.3}$$

Using Eq. (C.3) we can construct an expression for the elements of S as functions of θ_1 and θ_2. Post-multiplying Eq. (B.11) by itself transposed, and taking expectations, we have

$$\begin{aligned} \underset{m\times m}{\Omega_{\zeta\zeta}} &= \underset{m\times m}{G} \underset{m\times m}{\Omega_{ff}} \underset{m\times m}{G^T} - \underset{m\times m}{G} \underset{m\times \ell}{\Omega_{fz}} \underset{\ell\times m}{S^T} - \underset{m\times \ell}{S} \underset{\ell\times m}{\Omega_{zf}} \underset{m\times m}{G^T} + \underset{m\times \ell}{S} \underset{\ell\times \ell}{\Omega_{zz}} \underset{\ell\times m}{S^T} \\ &= \underset{m\times m}{G} \underset{m\times m}{\Omega_{ff}} \underset{m\times m}{G^T} - \underset{m\times \ell}{S} \underset{\ell\times \ell}{\Omega_{zz}} \underset{\ell\times m}{S^T} \end{aligned} \tag{C.4}$$

where the last equality is obtained using (C.2). The terms on the RHS of (C.4) are all functions of the known elements of θ_1 and θ_2. Since the matrix $\Omega_{\zeta\zeta}$ is symmetric, this gives $m \times (m+1)/2$ equations that determine the parameters of vec($\Omega_{\zeta\zeta}$). This establishes that every $\theta_{FKN} \in \Theta_{FKN}$ defines a unique parameter vector $\theta_{LS} \in \Theta_{LS}$. To prove the converse, solve Eq. (C.3) for vec(Ω_{fz}) as a function of θ_1 and the elements of M_z and apply the vec operator to (C.4) to solve for vec(Ω_{ff}) in terms of θ_1 and vec($\Omega_{\zeta\zeta}$). □

Appendix D

To run the simulation of the New-Keynesian model in Lubik and Schorfheide (2004) under indeterminacy, we need to compute the matrices G^i, H^i and S^i. We proceed as follows. First, we apply the QZ decomposition to the representation of the model:

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\mathbf{z}_t + \Pi(\theta)\boldsymbol{\eta}_t, \tag{D.1}$$

where $\Gamma_0(\theta)$, $\Gamma_1(\theta)$, $\Psi(\theta)$ and $\Pi(\theta)$ are described in Section 3. Let

$$\Gamma_0 = QSZ^T, \quad \text{and} \quad \Gamma_1 = QTZ^T, \tag{D.2}$$

be the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ where Q and Z are $k \times k$ orthonormal matrices and S and T are upper triangular and possibly complex. The resulting transformed parameters are

$$\tilde{\Psi} = Q^T\Psi, \quad \text{and} \quad \tilde{\Pi} = Q^T\Pi, \tag{D.3}$$

which then allow us to define the equation connecting fundamental and non-fundamental errors

$$\underset{n\times\ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2} \underset{p\times 1}{\eta_t} = 0, \tag{D.4}$$

where $\tilde{\Psi}_2$ and $\tilde{\Pi}_2$ are described in Section 3. For the New-Keynesian model in Lubik and Schorfheide (2004) the degree of indeterminacy $m = (p-n)$ equals 1 since the number of non-fundamental shocks is $p = 2$, while the number of generalized eigenvalues that are greater than or equal to 1 is $n = 1$.

Second, we follow Lubik and Schorfheide (2004) and apply the singular value decomposition as described in Section 5:

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U_1} \begin{bmatrix} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{bmatrix} \underset{p\times p}{V^T}. \tag{D.5}$$

and we compute

$$\underset{n\times\ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times\ell}{\tilde{\Psi}_2}. \tag{D.6}$$

Third, we partition η_t into two mutually exclusive subsets, $\eta_{f,t}$ and $\eta_{n,t}$ such that $\eta_{f,t} \cup \eta_{n,t} = \eta_t$ and partition $\tilde{\Pi}_2$ conformably so that

$$\underset{n\times p}{\tilde{\Pi}_2} \underset{p\times 1}{\eta_t} = \begin{bmatrix} \underset{n\times m}{\tilde{\Pi}_{2f}^i} & \underset{n\times n}{\tilde{\Pi}_{2n}^i} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}^i} \\ \underset{n\times 1}{\eta_{n,t}^i} \end{bmatrix}. \tag{D.7}$$

For the New-Keynesian model we are considering that there are two possible partitions $i = \{1, 2\}$ for which we include the non-fundamental shock $\eta_{1,t} = x_t - E_{t-1}[x_t]$ or $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$, respectively, as fundamental shock. We then compute the matrices Θ_z^i and Θ_f^i as defined in (A.4) and which we report here:

$$\Theta_z^i \equiv -\left(\tilde{\Pi}_{2n}^i\right)^{-1}\tilde{\Psi}_2, \quad \text{and} \quad \Theta_f^i \equiv -\left(\tilde{\Pi}_{2n}^i\right)^{-1}\tilde{\Pi}_{2f}^i \tag{D.8}$$

Fourth, we partition

$$V = \begin{bmatrix} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{bmatrix}, \tag{D.9}$$

and define the matrices

$$\underset{p\times n}{V_j^i} \equiv \begin{bmatrix} \underset{n\times n}{V_{j,n}^i} \\ \underset{m\times n}{V_{j,f}^i} \end{bmatrix}, \tag{D.10}$$

where the matrices $V_{j,f}^i$ and $V_{j,n}^i$ are composed of the row vectors of V_j which, according to partition $\mathbf{p}_i$, correspond to the non-fundamental shocks included as fundamental, $\eta_{f,t}^i$, and those that are still non-fundamental, $\eta_{n,t}^i$.

Finally, we use the definitions of G^i and H^i:

$$\underset{m\times m}{G^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \begin{bmatrix} \underset{n\times m}{\Theta_f^i} \\ \underset{p\times m}{I_m} \end{bmatrix} \quad \text{and} \quad \underset{m\times\ell}{H^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \underset{p\times n}{V_1^i} \underset{n\times\ell}{N} - \underset{m\times p}{\left(V_2^i\right)^T} \begin{bmatrix} \underset{n\times\ell}{\Theta_z^i} \\ \underset{m\times\ell}{\mathbf{0}} \\ \underset{p\times l}{} \end{bmatrix}. \tag{D.11}$$

for each partition $i = \{1, 2\}$. Therefore, we obtain the matrix

$$\underset{m\times\ell}{S^i} = \underset{m\times\ell}{H^i} + \underset{m\times\ell}{M_z}, \tag{D.12}$$

where the $m \times \ell$ matrix M_z captures the correlation of the forecast errors with the fundamentals in Lubik and Schorfheide (2004) as explained in Section 5.1.

Appendix E. Supplementary data

Supplementary data associated with this paper can be found in the online version at http://dx.doi.org/10.1016/j.jedc.2015.02.012.

References

Adjemian, S., Bastani, H., Juillard, M., Maih, J., Karamé, F., Mihoubi, F., Perendia, G., Pfeifer, J., Ratto, M., Villemot, S., 2011. Dynare: Reference Manual Version 4. Dynare Working Papers, CEPREMAP 1.
Azariadis, C., 1981. Self-fulfilling prophecies. J. Econ. Theory 25 (3), 380–396.
Belaygorod, A., Dueker, M., 2009. Indeterminacy change points and the price puzzle in an estimated DSGE model. J. Econ. Dyn. Control 33 (3), 624–648.
Benhabib, J., Farmer, R.E.A., 1994. Indeterminacy and increasing returns. J. Econ. Theory 63, 19–46.
Benhabib, J., Farmer, R.E.A., 1999. Indeterminacy and sunspots in macroeconomics. In: Taylor, J.B., Woodford, M. (Eds.), Handbook of Macroeconomics, North-Holland, Amsterdam.
Bhattarai, S., Lee, J.W., Park, W.Y., 2012. Policy regimes, policy shifts, and u.s. business cycle. Am. Econ. Rev. 102 (3), 173–178.
Bilbiie, F.O., Straub, R., 2013. Asset market participation, monetary policy rules, and the great inflation. Rev. Econ. Stat. 95 (2), 377–392.
Blanchard, O.J., Kahn, C.M., 1980. The solution of linear difference models under rational expectations. Econometrica 48, 1305–1313.
Cass, D., Shell, K., 1983. Do sunspots matter? J. Polit. Econ. 91, 193–227.
Castelnuovo, E., Fanelli, L., 2015. Monetary policy indeterminacy and identification failures in the U.S.: results from a robust test. J. Appl. Econometr., forthcoming., http://onlinelibrary.wiley.com/doi/10.1002/jae.2423/abstract.
Clarida, R., Galí, J., Gertler, M., 2000. Monetary policy rules and macroeconomic stability: evidence and some theory. Q. J. Econ. 115 (1), 147–180.
Evans, G.W., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press, Princeton.
Fanelli, L., 2012. Determinacy, indeterminacy and dynamic misspecification in linear rational expectations models. J. Econometr. 170, 153–163.
Farmer, R.E.A., 1999. The Macroeconomics of Self-Fulfilling Prophecies, 2nd ed. MIT Press, Cambridge, MA.
Farmer, R.E.A., 2002. Why does the data reject the lucas critique? Ann. Econ. Stat. 67/68, 111–129.
Farmer, R.E.A., Guo, J.T., 1994. Real business cycles and the animal spirits hypothesis. J. Econ. Theory 63, 42–73.
Farmer, R.E.A., Guo, J.-T., 1995. The econometrics of indeterminacy. In: Carnegie Rochester Series on Public Policy, vol. 43, pp. 225–273.
Farmer, R.E.A., Woodford, M., 1984. Self-Fulfilling Prophecies and the Business Cycle. Caress Working Paper 84-12.
Farmer, R.E.A., Woodford, M., 1997. Self-fulfilling prophecies and the business cycle. Macroecon. Dyn. 1 (4), 740–769.
Hirose, Y., 2011. Monetary Policy and Sunspot Fluctuation in the US and Euro Area. Munich Personal RePEc Archive No. 33693, September.
Kerr, W.R., King, R.G., 1996. Limits on interest rate rules in IS-LM models. Fed. Reserve Bank Richmond Econ. Q. 82 (2), 47–75.
Khramov, V., 2013. Essays in Macroeconomics and Finance (Ph.D. thesis). UCLA.
King, R.G., Watson, M., 1998. The solution of singular linear difference systems under rational expectations. Int. Econ. Rev. 39 (4), 1015–1026.
Klein, P., 2000. Using the generalized Schur form to solve a multivariate linear rational expectations model. J. Econ. Dyn. Control 24 (10), 1405–1423.
Lubik, T.A., Matthes, C., 2013. Indeterminacy and learning: an analysis of monetary policy during the geat inflation. Fed. Reserve Bank Richmond, mimeo.
Lubik, T.A., Schorfheide, F., 2003. Computing sunspot equilibria in linear rational expectations models. J. Econ. Dyn. Control 28 (2), 273–285.
Lubik, T.A., Schorfheide, F., 2004. Testing for indeterminacy: an application to U.S. monetary policy. Am. Econ. Rev. 94, 190–219.
McCallum, B.T., 1983. On non-uniqueness in rational expectations models: an attempt at perspective. J. Monet. Econ. 11, 139–168.
Sims, C.A., 2001. Solving linear rational expectations models. J. Comput. Econ. 20 (1–2), 1–20.
Uhlig, H., 1999. A toolkit for analyzing nonlinear dynamic stochastic models easily. In: Marimon, R., Scott, A. (Eds.), Computational Methods for the Study of Dynamic Economies, Oxford University Press, Oxford, England, pp. 30–61.
Zheng, T., Guo, H., 2013. Estimating a small open economy with indeterminacy: evidence from China. Econ. Model. 31, 642–652.