Agency Business Cycle*

Mikhail Golosov
Princeton University and NBER

Guido Menzio
University of Pennsylvania and NBER

March 2015

Abstract

A novel theory of labor market fluctuations is advanced. Firms need to fire non-performing workers with some positive probability in order to provide them with an ex-ante incentive to exert effort. In order to provide this incentive at the lowest cost, firms load the firing probability on the states of the world where the worker's cost of losing a job relative to the firm's cost of losing a worker is highest. When there are aggregate decreasing returns to scale to the value of unemployment, the states of the world where the worker's cost of losing a job is highest in relative terms are the states of the world with the highest unemployment. Hence, an individual firm finds it optimal to fire its non-performing workers in exactly the same states where other firms fire their non-performing workers workers. The strategic complementarity between the resolution to the agency problem faced by different firms leads to endogenous, stochastic fluctuations in unemployment, job-destruction and job-finding rates. The magnitude and the morphology of these fluctuations closely remembes those observed in the US economy.

JEL Codes: D86, E24, E32.

Keywords: Unemployment, Moral Hazard, Endogenous Business Cycles.

*Golosov: Department of Economics, Princeton University, 111 Fisher Hall, Princeton, NJ 08544 (email: Golosov@princeton.edu); Menzio: Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104 (email: gmenzio@sas.upenn.edu). We are grateful for comments by Igor Livschits, Nicola Pavoni, David Berger, Luigi Guiso, John Kennan, Rasmus Lentz, Lones Smith, Thijs van Rens, Randy Wright, and seminar participants at the University of Wisconsin Madison, the Search and Matching Workshop at the Philadelphia FED, and the Macro Dynamics Workshop at EIEF.

1 Introduction

The paper proposes a novel theory of cyclical fluctuations in the labor market, which identifies the source of cyclical fluctuations not in exogenous shocks to productivity or other fundamentals, but in a strategic complementarity in the optimal solution of the agency problem faced by different pairs of firms and workers. According to our theory, firms need to fire non-performing workers with some positive probability in order to provide them with an ex-ante incentive to exert effort. In order to provide this incentive at the lowest cost, firms load up the firing probability on the states of the world where the worker's cost of losing a job relative to the firm's cost of losing a worker is highest. When there are aggregate decreasing returns to scale to the value of unemployment, the states of the world where the worker's cost of losing a job is highest are the states with highest unemployment. Therefore, an individual firm finds it optimal to fire its non-performing workers exactly at the time when other firms fire their non-performing workers. The strategic complementarity between the optimal solution to the agency problem of different firms leads to endogenous and stochastic fluctuations in unemployment, job-destruction and job-finding rates.

We consider an economy populated by risk averse workers and risk neutral firms trading in a frictional labor market. Unemployed workers spend time searching for vacant jobs, while firms spend resources to create and maintain vacancies. Unemployed workers and vacant jobs come together through an aggregate matching function with decreasing returns to scale. Once matched, firms and workers face an agency problem. In particular, the output created by a firm-worker pair depends on the effort put in by the worker, but the effort is not observed by the firm. Foreseeing the agency problem, firms and workers bargain over the terms of short-term employment contracts. In particular, an employment contract specifies the wage that the firm pays to the worker in the current period (which is paid out before current output is observed) and the probability that the firm fires the worker at the beginning of next period, conditional on the realization of current period's output and next period's state of the world.

In the first part of the paper, we characterize the optimal employment contract. We find that the optimal contract prescribes that the worker should be fired with probability one if the current period's output is low and next period's state of the world is such that the cost to the worker from losing the job relative to the cost to the firm from losing the worker is above some endogenous threshold. In contrast, if the next period's state of the world is such that the cost to the worker from losing the job relative to the cost to the firm from losing the worker is below the threshold, the worker should be fired with probability zero.

The finding is intuitive. Firing is costly—as it destroys a match that is valuable to both the worker and the firm—but also necessary—as it gives the worker an ex-ante incentive to exert effort. However, only the value of the destroyed match that would have accrued to the worker serves the purpose of providing incentives. The value of the destroyed match that would have accrued to the firm is "collateral damage." The optimal contract minimizes the collateral damage by concentrating firing in states of the economy where the value of the match to the worker is highest relative to the value of the match to the firm.

In the second part of the paper, we characterize the general equilibrium. Our main finding is that there exists an equilibrium in which all firms fire their non-performing workers with probability one in states of the economy when the realization of an inherently meaningless sunspot falls in some region, and with probability zero in states of the economy when the realization of the sunspot falls outside of that region. Furthermore, we argue that this equilibrium is the only one robust to small perturbations in the environment. These findings imply that, in our model, cyclical fluctuations in the labor market emerge endogenously.

The intuition behind the existence of endogenous cycles is easy to explain. First, notice that the cost to the worker from losing a job relative to the cost to the firm from losing a worker is higher in states of the world with higher unemployment. Indeed, since there are decreasing returns to scale in matching, the worker's job-finding probability and the worker's value of unemployment are lower in states with higher unemployment. Since the value of unemployment is the worker's outside option when bargaining over an employment contract, wages are lower in states with higher unemployment. And, since the employment contract is such that the worker's value of a match is proportional to the worker's marginal utility and workers are risk averse, the states of the world with lower wages are those where the worker's cost from losing a job is higher relative to the firm's cost from losing a worker.

Next notice that, since cost to a worker from losing a job is higher in states of the world with higher unemployment, arbitrarily small differences in fundamentals across states of the world are amplified. Indeed, if the worker's cost from losing a job is relatively higher in one state than in another, firms might fire their non-performing workers with probability one in the first state and with probability zero in the second state. This leads to a significantly higher unemployment rate in the first than in the second state, which amplifies the difference in the relative cost of job loss to the worker between the two states of the world and reinforces the firms' incentives to fire non-performing workers in the first state but not in the second. Clearly, the amplification mechanism is so strong that it does not even require any initial difference between the fundamentals in the two states. If other firms fire their non-performing workers in the first state but not in the second state of the world, an individual firm finds it optimal to behave exactly like the others.

In the last part of the paper, we calibrate the model to match some basic features of the US labor market, such as the average rate at which unemployed workers become employed (UE rate), the average rate at which employed workers become unemployed (EU rate). We use the calibrated model to measure the magnitude of "agency business cycles", i.e. the endogenous labor market fluctuations caused by the firms' coordinated firing of non-performing workers. We find that agency business cycles can account for more than 50% of the cyclical volatility of the unemployment rate that is observed in the data, for approximately 40% of the cyclical volatility of the UE rate, and for all of the volatility of the EU rate. Moreover, we find that the fluctuations in the unemployment, UE and EU rates are uncorrelated from fluctuations in labor productivity. Thus, agency business cycles can account for a large fraction of the volatility of the US labor market and they offer a natural explanation for the low empirical correlation between unemployment and labor productivity.

We then show that agency business cycles and cyclical fluctuations in the US labor market have the same structure. According to our theory, a recession starts when the realization of the sunspot induces firms to coordinate on firing their non-performing workers. The increase in the EU rate drives the unemployment rate up. In turn, the increase in the unemployment rate leads to a decline in the UE rate, because of decreasing returns to matching. Once the firms stop firing non-performing workers, the lower UE rate causes a slow decline in the unemployment rate. We find that these predictions of the model are consistent with the data. In the data as in the model, the EU rate leads the unemployment rate by a quarter, while the UE rate and the unemployment rate coincide. In the data as in the model, the unemployment rate moves faster during recessions than during recoveries.

The main contribution of the paper is to advance a novel theory of cyclical fluctuations in the labor market. According to our theory, labor market fluctuations are not driven by exogenous shocks to current productivity (as in the real business cycle theory of Kydland and Prescott, 1982, or in Mortensen and Pissarides, 1994), by shocks to future productivity (as in the Pigovian business cycle models of Beaudry and Portier 2004 and Jaimovich and Rebelo 2009), by monetary shocks (as in New Keynesian models), or by shocks to any other fundamental. Instead, in our model, labor market fluctuations emerge from the firms' incentive to coordinate on firing non-performing workers in the same states of the world. Our theory is also different from existing theories of non-fundamental fluctuations. For one thing, the source of non-fundamental fluctuations in our theory is different than in the existing literature, which has stressed increasing returns in production (e.g., Benhabib and Farmer 1994, and Mortensen 1999), increasing returns to matching (e.g., Diamond 1982), demand externalities (e.g., Heller 1986 and Cooper and John 1988), and shopping externalities (e.g., Kaplan and Menzio 2014). Moreover, the nature of non-fundamental fluctuations in our theory is different than in the existing literature. Indeed, according to our theory, labor market fluctuations are not the result of exogenous changes in equilibrium selection (as in the multiple equilibria models of Benhabib and Farmer 1994 and Kaplan and Menzio 2014), and they are not deterministic (as in the limit cycle models of Diamond and Fudenberg 1989 and Mortensen 1999). Instead, labor market fluctuations emerge in our model because the equilibrium is inherently stochastic.

Finally, our theory proposes a novel view of recessions. In the Real Business Cycle theory, as well as in other fundamental and non-fundamental theories of cyclical fluctuations, a recession is a "rainy day," in the sense that it is a time when the value of market activities relative to the value of home activities is low. For this reason, during a recession, non-employment increases. In our theory, a recession is a time when the value of being employed relative to the value of being unemployed is high. And, according to our theory, it is at exactly such a time that firms find it optimal to coordinate on firing non-performing workers, which leads to a decline in economic activity. Consistently with our theory, the data seems to suggest that recessions are times when the value added from being employed is particularly high. Indeed, we construct time series for the value of employment and unemployment using data on the UE rate, EU rate and on wages and we find that the value added from being employed is clearly countercyclical.

2 Environment and Definition of Equilibrium

2.1 Environment

Time is discrete and continues forever. The economy is populated by a measure 1 of identical workers. Every worker has preferences described by $\sum \beta^t \left[\upsilon(w_t) - ce_t \right]$, where $\beta \in (0,1)$ is the discount factor, $\upsilon(w_t)$ is the utility of consuming w_t units of output in period t, and $-ce_t$ is the disutility of exerting e_t units of effort on the job in period t. We assume that the utility function $\upsilon(w)$ is strictly increasing and strictly concave, with a first derivative $\upsilon'(w)$ such that $\upsilon'(w) \in [\underline{\upsilon}', \overline{\upsilon}']$, and a second derivative $\upsilon''(w)$ such that $-\upsilon''(w) \in [\underline{\upsilon}'', \overline{\upsilon}'']$, with $\overline{\upsilon}' > \underline{\upsilon}' > 0$ and $\overline{\upsilon}'' > \underline{\upsilon}'' > 0$.. We assume that the worker's effort on the job, e_t, can only take on the values 0 and 1. Every worker is endowed with one, indivisible unit of labor.

The economy is also populated by a positive measure of identical firms. Every firm has preferences described by $\sum \beta^t \pi_t$, where β is the discount factor and π_t are the firm's profits in period t. Every firm operates a constant returns to scale production technology that turns one unit of labor (i.e. one employee) into y_t units of output, where y_t is a random variable that depends on the employee's effort e_t. In particular, y_t takes the high value y_h with probability $p_h(e)$ and the low value y_ℓ with probability $p_\ell(e) = 1 - p_h(e)$, with $y_h > y_\ell \geq 0$ and $0 < p_h(0) < p_h(1) < 1$ and $0 < p_\ell(1) < p_\ell(0) < 1$. Moreover, every firm operates a constant return to scale recruitment technology that turns $k > 0$ units of output into a job vacancy.

Every period $t = 1, 2, \ldots.$ is divided into five stages: sunspot, separation, matching, bargaining and production. In the first stage, a random variable, z_t, is independently drawn over time from an identical uniform distribution with support $[0, 1]$. The random variable is aggregate, in the sense that it is publicly observed by all market participants. The random variable is a sunspot, in the sense that it does not directly affect technology, preferences or any other fundamentals, although it may serve to coordinate the behavior of market participants.

In the second stage, some of the employed workers become unemployed. An employed worker leaves his job for exogenous reasons and becomes unemployed with some probability $\delta \in (0, 1)$. In addition, an employed worker is fired from his job and becomes unemployed with some probability $s(y_{t-1}, z_t)$, where $s(y_{t-1}, z_t)$ is determined by the worker's employment contract and is allowed to depend on the worker's performance on the job in the previous period, y_{t-1}, and on the realization of the sunspot in the current period, z_t. For the sake of simplicity, we assume that a worker who becomes unemployed in period t can search for a new job only starting in period $t + 1$.

In the third stage, firms create v_t vacant jobs at the unit cost k. Then, the workers who were unemployed at the beginning of the period, u_{t-1}, and the vacant jobs, v_t, search for each other. The outcome of the search process is described by a decreasing return to scale matching function, $M(u_{t-1}, v_t)$, which gives the measure of bilateral matches formed between unemployed workers and vacant firms. We denote as tht the vacancy-to-unemployment ratio, v_t/u_{t-1}, and we refer to it as the tightness of the labor market. We denote as $\lambda(\theta_t, u_{t-1})$ the probability that an unemployed worker meets a vacancy, i.e. $\lambda(\theta_t, u_{t-1}) = M(u_{t-1}, \theta_t u_{t-1})/u_{t-1}$. Similarly, we denote as $\eta(\theta_t, u_{t-1})$ the probability that a vacancy meets an unemployed worker, i.e. $\eta(\theta_t, u_{t-1}) = M(u_{t-1}, \theta_t u_{t-1})/\theta_t u_{t-1}$. We assume that the job-finding probability $\lambda(\theta_t, u_{t-1})$ is strictly increasing in θ_t and strictly decreasing in u_{t-1}. That is, the higher is the vacancy-to-unemployment ratio, the more likely it is for an unemployed worker to find a match. However, for a given vacancy-to-unemployment ratio, the higher is unemployment, the less likely it is for an unemployed worker to find a match. Similarly, we assume that the job-filling probability $\eta(\theta_t, u_{t-1})$ is strictly decreasing in both θ_t and u_{t-1}. As it is well-known, when the matching function features constant returns to scale, the job-finding and the job-filling probabilities are only functions of the vacancy-to-unemployment ratio. In contrast, when the matching function has decreasing returns to scale, the job-finding and the job-filling probabilities are also (decreasing) functions of unemployment.

In the fourth stage, every matched pair of firm and worker bargains over a one-period contract. Specifically, the contract specifies the effort, e_t, that the worker should exert in the current period, over the wage, w_t, that the worker will be paid in the current period, and over the probability, $s(y_t, z_{t+1})$, with which the worker will be fired at the separation stage of next period conditional on a realization of current period's output y_t and of next period's sunspot z_{t+1}. We assume that the outcome of the contractual bargain between the firm and the worker is given by the Axiomatic Nash Bargaining Solution.

In the last stage, production and consumption take place. An unemployed worker home-produces and consumes b units of output. An employed worker privately chooses an effort level, e_t, and consumes w_t units of output. Then, the output of the employed worker, y_t, is realized.

A few comments about the environment are in order. We assume that the employment contract specifies a wage that is independent of the current realization of output (formally because it is paid out before output is observed) and that the employment contract is renegotiated in every period. The first assumption rules out the current wage as an instrument to reward/punish the worker's performance. The second assumption rules out future wages as an instrument to reward/punish the worker's performance. As a result, firing is the only tool available to the firm to provide the worker with an incentive to exert effort. While these assumptions are very strong, they provide the simplest example of a contractual environment in which firing occurs along the equilibrium path. As we know from the literature, there are more complicated and less restrictive contractual environments that deliver equilibrium firing (see, e.g., Clementi and Hopenhayn, 2006).

We assume that the matching function has decreasing returns to scale. From the theoretical point of view, there are several way to justify the assumption. For example, a simple urn-ball matching process with finite balls (i.e., unemployed workers) and finite urns (i.e., vacant jobs) displays decreasing returns to scale (see, e.g., Burdett, Shi and Wright, 2001). From the empirical point of view, there is no consensus on whether the matching function has increasing, constant or decreasing returns to scale. In their survey of the empirical literature, Petrongolo and Pissarides (2001) report of studies that find the returns to scale to be significantly increasing, decreasing or approximately constant. Furthermore, estimates of the matching function are likely to be biased, as they typically assume that the number of employed workers and the number of vacant jobs are the only input into search, while it is well-know that a significant fraction of vacancies are filled by workers who are either already employed or out of the labor force. For these reasons, we consider the assumption of decreasing returns to matching not unreasonable.

We assume that workers have no access to capital markets. The assumption is made for the sake of simplicity. Indeed, as long as workers are not perfectly insured, there is some variation in their marginal utility of consumption across different states of the world and the theoretical predictions of the model would remain unchanged.

2.2 Definition of equilibrium

Let u denote the unemployment rate at the bargaining stage in the current period. Let $W(x,u)$ and $U(u)$ denote, respectively, the lifetime utility of a worker who, at the production stage, is employed under the contract x and the lifetime utility of a worker who, at the production stage, is unemployed. Denote as $F(x,u)$ the difference between $W(x,u)$ and $U(u)$. Let $G(x,u)$ denote the present value of profits for a firm that, at the production stage, employs a worker under the contract x. Let $x(u) = (e(u), w(u), s(y, \hat{z}, u))$ denote the contract that a firm and a worker agree upon during the bargaining stage. Finally, let $\hat{\theta}(\hat{z})$ denote the tightness of the labor market at the next matching stage given that the realization of next period's sunspot is $\hat{z}$, and let $\hat{u}(\hat{z})$ denote the unemployment rate at the beginning of the next bargaining stage given $\hat{z}$.

The lifetime utility of an unemployed worker, $U(u)$, is given by

$$U(u) = v(b) + \beta E_{\hat{z}} \left[U(\hat{u}(\hat{z})) + \lambda(\hat{\theta}(\hat{z}), u) \left(W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z})) - U(\hat{u}(\hat{z})) \right) \right] \tag{1}$$

In the current period, the worker home produces and consumes b units of output. In the next period, the worker does finds a job with probability $\lambda(\hat{\theta}(\hat{z}), u)$ and does not with the complementary probability $1 - \lambda(\hat{\theta}(\hat{z}), u)$. In the first case, the worker's continuation lifetime utility is $W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z}))$. In the second case, the worker's continuation lifetime utility is $U(\hat{u}(\hat{z}))$.

The lifetime utility of a worker employed at an arbitrary contract $x = (e, w, s)$, $W(x, u)$, is given by

$$\begin{aligned} W(x,u) &= v(w) - ce + \\ &+\beta E_{y,\hat{z}} \left[U(\hat{u}(\hat{z})) + (1-\delta)(1-s(y,\hat{z})) \left(W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z})) - U(\hat{u}(\hat{z})) \right) | e \right]. \end{aligned} \tag{2}$$

In the current period, the worker consumes w units of output and exerts effort e. With probability $p_h(e)$, the worker's output in the current period is y_h. In this case, the worker keeps his job in the next period with probability $(1-\delta)(1-s(y_h, \hat{z}))$ and his continuation lifetime utility is $W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z}))$. With probability $p_\ell(e)$, the worker's output in the current period is y_ℓ. In this case, the worker keeps his job with probability $(1-\delta)(1-s(y_\ell, \hat{z}))$ and his continuation lifetime utility is $W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z}))$. If the worker loses his job, his continuation lifetime utility is $U(\hat{u}(\hat{z}))$.

The difference between the lifetime utility of a worker employed under the contract x and the lifetime utility of an unemployed worker, $F(x, u)$, is given by

$$\begin{aligned} F(x,u) &= v(w) - v(b) - ce + \\ &+\beta E_{y,\hat{z}} \left\{ \left[(1-\delta)(1-s(y,\hat{z})) - \lambda(\hat{\theta}(\hat{z}), u) \right] \left(W(x(\hat{u}(\hat{z})), \hat{u}(\hat{z})) - U(\hat{u}(\hat{z})) \right) | e \right\}. \end{aligned} \tag{3}$$

We denote as $V(u)$ the difference between the lifetime utility of a worker employed at the equilibrium contract $x(u)$ and the lifetime utility of an unemployed worker, i.e. $V(u) = W(x(u), u) - U(u)$. We refer to $V(u)$ as the equilibrium gains from trade accruing to the worker.

The present value of profits for a firm employing a worker under the contract x, $G(x, u)$, is given by

$$\begin{aligned} G(x,u) &= p_h(e)y_h + p_\ell(e)y_\ell - w \\ &+\beta E_{y,\hat{z}} \left[(1-\delta)(1-s(y,\hat{z})) G(x(\hat{u}(\hat{z})), \hat{u}(\hat{z})) | e \right]. \end{aligned} \tag{4}$$

In the current period, the expected output produced by the worker is $p_h(e)y_h + p_\ell(e)y_\ell$ and the wage paid to the worker is w. With probability $p_h(e)$, the worker's current output is y_h. In this case, the firm retains the worker in the next period with probability $(1-\delta)(1-s(y_h, \hat{z}))$ and its continuation profits are $G(x(\hat{u}(\hat{z})), \hat{u}(\hat{z}))$. With probability $p_\ell(e)$, the worker's current output is y_ℓ. In this case, the firm retains the worker in the next period with probability $(1-\delta)(1-s(y_\ell, \hat{z}))$ and its continuation profits are $G(x(\hat{u}(\hat{z})), \hat{u}(\hat{z}))$. If the firm does not retain the worker, its continuation profits are zero. We denote as $J(u)$ the firm's present value of profits from employing a worker at the equilibrium contract $x(u)$, i.e. $J(u) = G(x(u), u)$. We refer to $J(u)$ as the equilibrium gains from trade accruing to the firm.

The equilibrium contract $x(u)$ is given by the Axiomatic Nash Bargaining Solution of the bilateral monopoly problem facing a firm and a worker. A contract x is feasible if and only if it is incentive compatible, in the sense that it gives the worker the incentive to exert the prescribed level of effort e. A contract x is the Axiomatic Nash Bargaining Solution if it maximizes, among all contracts that are feasible, the product of the gains from trade accruing to the worker, $F(x,u)$, and the gains from trade accruing to the firm, $G(x,u)$. Hence, the contract $x(u)$ solves

$$\max_{x=(e,w,s)} F(x,u)G(x,u), \tag{5}$$

subject to

$$e \in \{0,1\} \text{ and } s(y,\hat{z}) \in [0,1].$$

and the worker's incentive compatibility constraint

$$\begin{aligned} c &\leq \beta(p_h(1)-p_h(0))E_{\hat{z}}\left[(1-\delta)(s(y_\ell,\hat{z})-s(y_h,\hat{z}))V(\hat{u}(\hat{z}))\right], \quad \text{if } e=1, \\ c &\geq \beta(p_h(1)-p_h(0))E_{\hat{z}}\left[(1-\delta)(s(y_\ell,\hat{z})-s(y_h,\hat{z}))V(\hat{u}(\hat{z}))\right], \quad \text{if } e=0. \end{aligned}$$

In the matching stage of next period, the firm faces a cost of k to create an additional vacant job. The benefit to the firm to create an additional vacancy is given by $\eta(\hat{\theta}(\hat{z}),u)J(\hat{u}(\hat{z}))$, i.e., the probability of filling the vacancy, $\eta(\hat{\theta}(\hat{z}),u)$, times the lifetime profit from employing an additional worker, $J(\hat{u}(\hat{z}))$. A firm creates infinitely many vacancies if $\eta(\hat{\theta}(\hat{z}),u)J(\hat{u}(\hat{z})) > k$, it creates no vacancies if $\eta(\hat{\theta}(\hat{z}),u)J(\hat{u}(\hat{z})) < k$ and it is indifferent about the number of vacancies it creates if $\eta(\hat{\theta}(\hat{z}),u)J(\hat{u}(\hat{z})) = k$. Thus, the equilibrium vacancy-to-unemployment ratio, $\hat{\theta}(\hat{z})$, is consistent with the firm's optimal vacancy creation strategy if and only if

$$\eta(\hat{\theta}(\hat{z}),u)J(\hat{u}(\hat{z})) \leq k, \tag{6}$$

and $\hat{\theta}(\hat{z}) \geq 0$ with complementary slackness. This conditions imply that the equilibrium vacancy-to-unemployment ratio is equal to

$$\hat{\theta}(\hat{u}(\hat{z})) = \eta^{-1}\left(\min\left\{\frac{k}{J(\hat{u}(\hat{z}))},1\right\},u\right), \tag{7}$$

where $\eta^{-1}(.,u)$ denotes the inverse of the job-filling probability $\eta(\theta,u)$ with respect to θ.

The unemployment rate at the bargaining stage of next period, $\hat{u}(\hat{z})$, is given by a $g(u,\hat{z})$ such that

$$g(u,\hat{z}) = u - u\mu(J(g(u,\hat{z}),u) + (1-u)E_y\left[\delta + (1-\delta)s(y,\hat{z},u)\right], \tag{8}$$

where $\mu(J(g(u,\hat{z}),u)$ is defined as $\lambda(\hat{\theta}(\hat{u}(\hat{z})),u)$. The first term on the right-hand side of (8) is unemployment rate at the bargaining state of the current period, u. The second term on the right-hand side of (8) is the flow of workers out of unemployment. That is, the unemployment rate at the bargaining stage of the current period, u, times the probability that an unemployed worker finds a job during the matching stage of next period, $\mu(J(g(u,\hat{z}),u)$. The last term on the right-hand side of (8) is the flow of workers into unemployment. That is, the employment rate at the bargaining stage of the current period, $1-u$, times the probability that an employed worker becomes unemployed, $E_y\left[\delta+(1-\delta)s(y,\hat{z},u)\right]$.

We are now in the position to define a recursive equilibrium for our model economy.

Definition 1: A Recursive Equilibrium is a tuple (V,J,x,g) such that:

(i) The gains from trade accruing to the worker, V, and the gains from trade accruing to the firm, J, satisfy

$$\begin{aligned}
V(u) &= \upsilon(w(u))-\upsilon(b)-ce(u)+\\
&\quad+\beta E_{y,\hat{z}}\left\{\left[(1-\delta)(1-s(y,\hat{z},u))-\mu(J(g(u,\hat{z}),u))\right]V(g(u,\hat{z}))|e(u)\right\},\\
J(u) &= E_y[y|e(u)]-w+\beta E_{y,\hat{z}}\left[(1-\delta)(1-s(y,\hat{z},u))J(g(u,\hat{z}))|e(u)\right];
\end{aligned}$$

(ii) The employment contract, x, satisfies

$$\begin{aligned}
&\max\nolimits_x F(x,u)G(x,u),\\
\text{s.t.}\quad & e\in\{0,1\} \text{ and } s(y,\hat{z})\in[0,1],\\
& c\leq\beta(p_h(1)-p_h(0))E_{\hat{z}}\left[(1-\delta)(s(y_\ell,\hat{z})-s(y_h,\hat{z}))V(g(u,\hat{z}))\right], \text{ if } e=1,\\
& c\geq\beta(p_h(1)-p_h(0))E_{\hat{z}}\left[(1-\delta)(s(y_\ell,\hat{z})-s(y_h,\hat{z}))V(g(u,\hat{z}))\right], \text{ if } e=0;
\end{aligned}$$

(iii) The law of motion for unemployment, g, satisfies

$$g(u,\hat{z})=u-u\mu(J(g(u,\hat{z}),u)+(1-u)E_y\left[\delta+(1-\delta)s(y,\hat{z},u)\right].$$

In the next two sections, we are going to characterize the properties of the equilibrium of our model economy. For the sake of exposition, we shall assume that the employment contract, $x(u)$, always prescribes the worker to exert effort, i.e. $e(u)=1$. We shall also assume that the gains from trade are always positive, i.e. $J(u)>0$ and $V(u)>0$. It is straightforward to verify that the first assumption is satisfied as long as y_ℓ is sufficiently low relative to y_h, and that the second assumption is satisfied as long as $p_h(1)y_h+p_\ell(1)y_\ell$ is sufficienly high relative to b.

3 Optimal Contract

In this section, we characterize the solution to the bargaining problem (5) between the firm and the worker. For the sake of brevity, we will omit the dependence of F and G from u and we will use $\hat{V}(\hat{z})$ as shorthand for the worker's continuation gains from trade $V(g(u,\hat{z}))$ and $J(\hat{z})$ as shorthand for the firm's continuation gains from trade $J(g(u,\hat{z}))$. We shall denote as $x^* = (e^*, w^*, s^*)$ a solution to (5) and refer to it as an optimal employment contract. We will characterize the features of the optimal contract in a series of lemmas, which we will then summarize in Theorem 1. As mentioned above, we will carry out the analysis under the maintained assumption that the optimal contract requires the worker to exert effort, i.e. $e^* = 1$, and that the worker's and the firm's gains from trade are always positive, i.e. $\hat{V}(\hat{z}) > 0$ and $J(\hat{z}) > 0$.

The first lemma states that any optimal employment contract x^* is such that the worker's incentive compatibility constraint holds with equality. The result is intuitive. Providing the worker with the incentive to exert effort is costly, as it requires severing an employment relationship that has positive value to both the worker and the firm. For this reason, the optimal contract provides the worker with the minimum amount of incentives to induce him to exert effort.

Lemma 1: Any optimal contract x^* is such that the worker's incentive compatibility holds with equality, i.e.

$$c = \beta(p_h(1) - p_h(0))E_{\hat{z}}\left[(1-\delta)(s^*(y_\ell,\hat{z}) - s^*(y_h,\hat{z}))\hat{V}(\hat{z})\right]. \tag{9}$$

Proof: Let $\rho \geq 0$ denote the Lagrange multiplier on the worker's incentive compatibility constraint, let $\overline{\nu}(y,\hat{z}) \geq 0$ denote the multiplier on the constraint $1 - s(y,\hat{z}) \geq 0$ and let $\underline{\nu}(y,\hat{z})$ denote the multiplier on the constraint $s(y,\hat{z}) \geq 0$.

The first order condition with respect to the firing probability $s(y_\ell,\hat{z})$ is given by

$$(1-\delta)\left[G(x)\hat{V}(\hat{z}) + F(x)\hat{J}(\hat{z})\right] = \rho\beta(1-\delta)(p_h(1) - p_h(0))\hat{V}(\hat{z}) + \underline{\nu}(y_\ell,\hat{z}) - \overline{\nu}(y_\ell,\hat{z}), \tag{10}$$

together with the complementary slackness conditions $\overline{\nu}(y_\ell,\hat{z})\cdot(1-s(y_\ell,\hat{z})) = 0$ and $\underline{\nu}(y_\ell,\hat{z})\cdot s(y_\ell,\hat{z}) = 0$. The left-hand side of (10) is the marginal cost of increasing $s(y_\ell,\hat{z})$. This cost is given by the decline in the product of the worker's and firm's gains from trade caused by a marginal increase in the firing probability $s(y_\ell,\hat{z})$. The right-hand side of (10) is the marginal benefit of increasing $s(y_\ell,\hat{z})$. This benefit is given by the value of relaxing the worker's incentive compatibility and the $s(y_\ell, \hat{z}) \geq 0$ constraints net of the cost of tightening the $s(y_\ell, \hat{z}) \leq 1$ constraint by marginally increasing the firing probability $s(y_\ell, \hat{z})$.

Similarly, the first order condition with respect to the firing probability $s(y_h, \hat{z})$ is given by

$$(1-\delta)\left[G(x)\hat{V}(\hat{z}) + F(x)\hat{J}(\hat{z}) + \rho\beta(p_h(1) - p_h(0))\hat{V}(\hat{z})\right] = \underline{\nu}(y_h, \hat{z}) - \overline{\nu}(y_h, \hat{z}), \tag{11}$$

together with the complementary slackness conditions $\overline{\nu}(y_h, \hat{z}) \cdot (1 - s(y_h, \hat{z})) = 0$ and $\underline{\nu}(y_h, \hat{z}) \cdot s(y_h, \hat{z}) = 0$. The left-hand side of (11) represents the marginal cost of increasing $s(y_h, \hat{z})$. The rigth-hand side of (11) represents the marginal benefit of increasing $s(y_h, \hat{z})$. Notice that increasing the firing probability $s(y_h, \hat{z})$ tightens the worker's incentive compatibility constraint and, hence, the term in ρ is now on the left-hand side of (11).

Suppose $\rho = 0$. First, notice that the left-hand side of (10) is strictly positive as $\hat{V}(\hat{z}) > 0$, $\hat{J}(\hat{z}) > 0$ by assumption, and $F(x) > 0$, $G(x) > 0$ at the optimum x^*. The right-hand side of (10) is strictly positive only if $\underline{\nu}(y_\ell, \hat{z}) > 0$. Hence, if $\rho = 0$, the only solution to the first order condition with respect to the firing probability $s(y_\ell, \hat{z})$ is 0. Next, notice that the left-hand side of (11) is strictly positive and the right-hand side is strictly positive only if $\underline{\nu}(y_h, \hat{z}) > 0$. Hence, if $\rho = 0$, the only solution to the first order condition with respect to the firing probability $s(y_h, \hat{z})$ is 0. However, if $s(y_\ell, \hat{z}) = s(y_h, \hat{z}) = 0$, the worker's incentive compatibility constraint is violated. Therefore, $\rho > 0$ and the worker's incentive compatibility constraint holds with equality. ■

The second lemma states that any optimal employment contract x^* is such that the worker is never fired when production is successful. This result is also easy to understand. If the worker is fired with positive probability when production is successful, the value of the contract to both the worker and the firm is reduced as a valuable relationship is destroyed. Moreover, if the worker is fired with positive probability when production is successful, the incentive compatibility constraint becomes tighter as exerting effort increases the probability that production is successful and reduces the probability that production is unsuccessful. Overall, firing the worker with positive probability when production is successful has only costs and no benefits.

Lemma 2: Any optimal contract x^* is such that the worker is fired with probability zero if production is successful. That is, for all $\hat{z} \in [0, 1]$,

$$s^*(y_h, \hat{z}) = 0. \tag{12}$$

Proof: The first order condition with respect to $s(y_h, \hat{z})$ is given by (11) together with the complementary slackness conditions $\overline{\nu}(y_h, \hat{z})(1 - s(y_h, \hat{z})) = 0$ and $\underline{\nu}(y_h, \hat{z})s(y_h, \hat{z}) = 0$. The left-hand side of (11) is strictly positive. The right-hand side of (11) is strictly positive only if $\underline{\nu}(y_h, \hat{z}) > 0$. Therefore, the first order condition is satisfied only if $\underline{\nu}(y_h, \hat{z}) > 0$ and, hence, only if $s(y_h, \hat{z}) = 0$. ■

The next lemma is one of the central results of the paper. Let $\phi(\hat{z})$ denote the ratio of the gains from trade accruing to the worker next period, $\hat{V}(\hat{z})$, and the gains from trade accruing to the firm next period, $\hat{J}(\hat{z})$. Then, any optimal contract x^* prescribes that the worker should be fired with probability one if production is unsuccessful and the realization of next period's sunspot, $\hat{z}$, is such that the gains from trade accruing to the worker relative to those accruing to the firm, $\phi(\hat{z})$, are strictly higher than some cutoff ϕ^*. The worker should be fired with zero probability if production is unsuccessful and the realization of next period's sunspot, $\hat{z}$, is such that the gains from trade accruing to the worker relative to those accruing to the firm, $\phi(\hat{z})$, are strictly lower than the cutoff ϕ^*.

There is a simple intuition behind this result. Firing is costly—as it destroys a relationship that is valuable to both the worker and the firm—but also necessary—as it gives the worker an incentive to perform. However, only the value of the destroyed relationship that would have accrued to the worker serves the purpose of providing incentives. The value of the destroyed relationship that would have accrued to the firm is "collateral damage." The optimal contract minimizes this collateral damage by concentrating firing in states of the world where the value of the relationship to the worker is highest relative to the value of the relationship to the firm.

Lemma 3: Any optimal contract x^* prescribes that the worker is fired with probability one (*zero*) if production is unsuccessful and the realization of the sunspot $\hat{z}$ is such that $\phi(\hat{z})$ is greater (*smaller*) than ϕ^*. That is, for all $\hat{z} \in [0, 1]$,

$$s^*(y_\ell, \hat{z}) = \begin{cases} 1, & \text{if } \phi(\hat{z}) > \phi^*, \\ 0, & \text{if } \phi(\hat{z}) < \phi^*. \end{cases} \tag{13}$$

Proof: The first order condition with respect to the firing probability $s(y_\ell, z)$ can be written as

$$(1 - \delta)\hat{V}(\hat{z}) \left[G(x) + F(x)/\hat{\phi}(\hat{z}) - \mu\beta(p_h(1) - p_h(0)) \right] = \underline{\nu}(y_\ell, \hat{z}) - \overline{\nu}(y_\ell, \hat{z}), \tag{14}$$

together with the complementary slackness conditions $\overline{\nu}(y_\ell, \hat{z}) \cdot (1 - s(y_\ell, \hat{z})) = 0$ and $\underline{\nu}(y_\ell, \hat{z}) \cdot s(y_\ell, \hat{z}) = 0$. The left-hand side of (14) is strictly decreasing in $\hat{\phi}(\hat{z})$. The right-hand side of (14) is strictly positive if $\underline{\nu}(y_\ell, \hat{z})$ is strictly positive and it is strictly negative if $\overline{\nu}(y_\ell, \hat{z})$ is strictly positive. Therefore, there exists a ϕ^* such that if $\phi(\hat{z}) > \phi^*$, the left-hand side is strictly negative and the solution to (14) requires $\overline{\nu}(y_\ell, \hat{z}) > 0$. In this case, the solution to the first order condition for $s(y_\ell, \hat{z})$ is 1. If $\phi(\hat{z}) < \phi^*$, the left-hand side is strictly positive and the solution to (14) requires $\underline{\nu}(y_\ell, \hat{z}) > 0$. In this case, the solution to the first order condition for $s(y_\ell, \hat{z})$ is 0. ■

The cutoff value ϕ^* is pinned down by the worker's incentive compatibility constraint

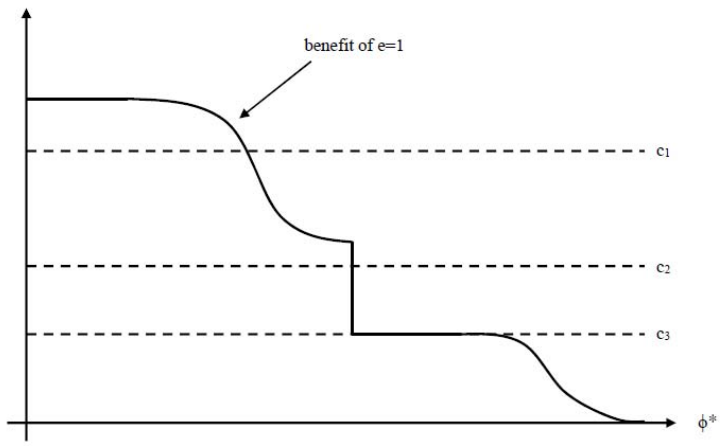
$$c = \beta(p_h(1) - p_h(0))(1-\delta)\left[\int_{\phi(\hat{z})>\phi^*} \hat{V}(\hat{z})d\hat{z} + \int_{\phi(\hat{z})=\phi^*} s^*(y_\ell, \hat{z})\hat{V}(\hat{z})d\hat{z}\right], \tag{15}$$

where the above expression makes use of the optimality condition (12) for $s^*(y_h, \hat{z})$ and of the optimality condition (14) for $s^*(y_\ell, \hat{z})$. Figure 1 plots the right-hand side of (15), i.e. the worker's benefit from extering effort on the job, as a function of the cutoff value ϕ^*. On any interval $[\phi_0^*, \phi_1^*]$ where the distribution of the random variable $\phi(\hat{z})$ has positive density, the right-hand side of (15) is strictly decreasing. On any interval $[\phi_0^*, \phi_1^*]$ where the distribution of $\phi(\hat{z})$ has no density, the right-hand side of (15) is constant. At any value ϕ_0^* where the distribution of $\phi(\hat{z})$ has a mass point, the right-hand side of (15) can take on an interval of values as the firing probability $s^*(y_\ell, \hat{z})$ for $\hat{z}$ such that $\phi(\hat{z}) = \phi_0^*$ varies between zero and one.

The cutoff value ϕ^* is such that right-hand side of (15) equals the worker's cost c from extering effort on the job. If, as in the case of c_1 in Figure 1, the right-hand side of (15) equals c for a ϕ^* in an interval where the distribution of $\phi(\hat{z})$ has positive density, the cutoff value is uniquely determined. If, as in the case of c_2 in Figure 1, the right-hand side of (15) equals c for a ϕ^* in an interval where the distribution of $\phi(\hat{z})$ has a mass point, the cutoff is uniquely determined and the firing probability $s^*(y_\ell, \hat{z})$ for $\hat{z}$ such that $\phi(\hat{z}) = \phi_0^*$ makes the worker's incentive compatibility constraint hold with equality. If, as in the case of c_3 in Figure 1, the right-hand side of (15) equals c over an interval $[\phi_0^*, \phi_1^*]$, the cutoff value ϕ^* can take on any value between ϕ_0^* and ϕ_1^*, but the selection of ϕ^* is immaterial as the random variable $\phi(\hat{z})$ has zero measure over that interval.

The next lemma states that any optimal contract x^* prescribes a wage such that the ratio of the gains from trade accruing to the worker and to the firm is equal to the ratio of the marginal utility of an additional unit of consumption to the worker and to the firm. This is an important result because it tells us when the gains from trade accruing to the worker are high relative to the gains from trade accruing to the firm. Specifically, as the worker's marginal utility is strictly decreasing in the wage w, the result implies that the gains from

Figure 1: Cutoff value ϕ^*



trade accruing to the worker are high relative to those accruing to the firm when the wage is low. Combining this result with Lemma 3, we can conclude that the optimal contract prescribes that the worker should be fired only when production is unsuccessful and the realization of next period's sunspot is such that the worker's wage is sufficiently low.

Lemma 4: Any optimal contract x^* is such that

$$\frac{V}{J} = v'(w^*), \tag{16}$$

where V and J are respectively defined as the worker's and the firm's gains from trade evaluated at the optimal contract x^*, i.e. $V = F(x^*)$ and $J = G(x^*)$.

Proof: The first order condition with respect to the wage w is given by

$$G(x)v'(w) - F(x) = 0. \tag{17}$$

The left-hand side of (17) is the increase in the product of the worker's and firm's gains caused by a marginal increase in the worker's wage w. A marginal increase in w, increases the worker's gains from trade by $v'(w)$ and decreases the firm's gains from trade by 1. Therefore, a marginal increase in w, increases the product of the worker's and firm's gains from trade by $G(x)v'(w) - F(x)$. The first order condition for w states that the effect of a marginal increase in w is zero. Using $V = F(x^*)$ and $J = G(x^*)$ and the first order condition for w, we obtain (16). ■

We are now in the position to summarize the characterization of the optimal contract.

Theorem 1: (*Optimal Contract*) Any optimal contract x^* is such that: (i) the worker is paid the wage w^* given by (16); (ii) if production is successful, the worker is fired with probability $s^*(y_h, \hat{z})$ given by (12); (iii) if production is unsuccessful, the worker is fired with probability $s^*(y_\ell, \hat{z})$ given by (13), where the ϕ^* in (13) is such that the worker's incentive compatibility constraint holds with equality.

4 Sunspots

In the previous section, we characterized the solution x^* to the bargaining problem between the firm and the worker, taking as given the distribution of the worker's relative gains from trade in the next period, $\phi(\hat{z})$. In this section, we characterize the equilibrium distribution of $\phi(\hat{z})$.

To this aim, we first characterize the relationship between the probability $s^*(y_\ell, \hat{z})$ with which firms fire their non-performing workers and the gains from trade accruing to the worker relative to those accruing to the firm $\phi(\hat{z})$. Given $s^*(y_\ell, \hat{z}) = s^*$, the unemployment rate in the next period is given

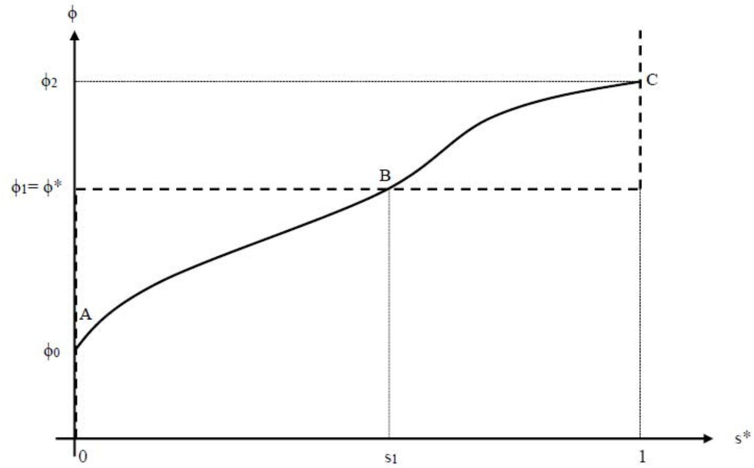
$$\hat{u}(s^*) = u - \mu(J(\hat{u}(s^*)), u) + (1-u)(\delta + (1-\delta)(1-p(1))s^*). \tag{18}$$

Given the unemployment rate in (18), the gains from trade accruing to the worker relative to the gains from trade accruing to the firm are given by

$$\phi(\hat{z}) = \frac{V(\hat{u}(s^*))}{J(\hat{u}(s^*))} = v'(w(\hat{u}(s^*))). \tag{19}$$

We conjecture that the unemployment rate in (18) is strictly increasing in the firing probability s. We also conjecture that the wage function $w(\hat{u})$ on the right hand side of (19) is strictly decreasing in the unemployment rate $\hat{u}$. These conjectures are natural, and we shall verify them in the next section. Intuitively, a higher firing probability tends to increase the unemployment rate. A higher unemployment rate leads to additional congestion in the labor market and, in turn, to a lower value of unemployment and a lower wage. Given these conjectures, the gains from trade accruing to the worker relative to the firm are strictly increasing in the firing probability s^*. The properties of the map from s^* to ϕ are illustrated by the solid line in Figure 2.

Figure 2: Sunspot



Next, we characterize the relationship between the relative gains from trade accruing to the worker and to the firm, $\phi(\hat{z})$, and the probability with which firms fire their non-performing workers, $s^*(y_\ell, \hat{z})$. As established in Lemma 3, for any distribution of the random variable $\phi(\hat{z})$, there exists a ϕ^* such that firms fire their non-performing workers with probability zero if $\phi(\hat{z}) < \phi^*$, with a probability between zero and one if $\phi(\hat{z}) = \phi^*$, and with probability one if $\phi(\hat{z}) > \phi^*$. That is,

$$s^*(y_\ell, \hat{z}) = \begin{cases} 0, & \text{if } \phi(\hat{z}) < \phi^*, \\ \in [0,1], & \text{if } \phi(\hat{z}) = \phi^*, \\ 1, & \text{if } \phi(\hat{z}) > \phi^*, \end{cases} \tag{20}$$

where ϕ^* satisfies the worker's incentive compatibility constraint (15). The properties of the map from $\phi(\hat{z})$ to $s^*(y_\ell, \hat{z})$ are illustrated by the dashed line in Figure 2.

Since in any equilibrium the firing probability $s^*(y_\ell, \hat{z})$ can take at most three values and the relative gains from trade $\phi(\hat{z})$ are a strictly increasing function of $s^*(y_\ell, \hat{z})$, it follows that, in any equilibrium, $s^*(y_\ell, \hat{z})$ can also take at most three values. This is evident from Figure 2. In particular, with some probability π_0, the realization of the sunspot $\hat{z}$ is such that $\phi(\hat{z}) = \phi_0 < \phi^*$ where $\phi_0 = \upsilon'(w(\hat{u}(0)))$.With some probability π_1, the realization of the sunspot $\hat{z}$ is such that $\phi(\hat{z}) = \phi_1 = \phi^*$ where $\phi_1 = \upsilon'(w(\hat{u}(s_1)))$. And with some probability π_2, the realization of the sunspot is such that $\phi(\hat{z}) = \phi_2 > \phi^*$ where $\phi_2 = \upsilon'(w(\hat{u}(1)))$. Moreover, in any equilibrium, the worker's incentive compatibility constraint must hold with equality.

Qualitatively, there are three types of equilibria that can emerge. The first type of equilibrium is such that $\pi_1 = 1$ and $\pi_0 = \pi_2 = 0$. In this type of equilibrium, there are no realizations of the sunspot for which firms fire all of their non-performing workers, nor realizations of the sunspot for which firms do not fire any of their non-performing workers. For all realizations of the sunspot, firms fire their non-performing workers with probability s_1^*. Since firms do not use the sunspot to coordinate on their firing decisions, we refer to this as the no-coordination equilibrium. In a no-coordination equilibrium, the worker's incentive compatibility constraint is

$$c = \beta(p_h(1) - p_h(0))(1-\delta)s_1^* V(\hat{u}(s_1^*)). \tag{21}$$

The incentive compatibility constraint (21) implicitly pins down the probability with which firms fire their non-performing workers. Specifically, the firing probability is given by

$$s_1^* = \frac{c}{\beta(p_h(1) - p_h(0))(1-\delta)V(\hat{u}(s_1^*))}. \tag{22}$$

The second type of equilibrium is such that $\pi_1 \in (0,1)$. In this equilibrium, there is a positive measure of realizations of the sunspot for which firms fire their non-performing workers with probability s_1^*. However, for some particular realizations of the sunspot, firms coordinate on either firing all of their non-performing workers or none of them. Since firms use the sunspot to coordinate only with some probability, we refer to this as a partial coordination equilibrium. In a partial coordination equilibrium, the worker's incentive compatibility constraint is

$$c = \beta(p_h(1) - p_h(0))(1-\delta)\left[\pi_1 s_1^* V(\hat{u}(s_1^*)) + \pi_2 V(\hat{u}(1))\right]. \tag{23}$$

The incentive compatibility constraint (23) can be used to pin down the probability with which firms fire their non-performing workers when they do not use the sunspot as a coordination device. Specifically, this firing probability if given by

$$s_1^* = \frac{c - \beta(p_h(1) - p_h(0))(1-\delta)\pi_2 V(\hat{u}(1))}{\beta(p_h(1) - p_h(0))(1-\delta)\pi_1 V(\hat{u}(s_1^*))}. \tag{24}$$

The third type of equilibrium is such that $\pi_1 = 0$, $\pi_0 \in (0,1)$ and $\pi_2 \in (0,1)$. In this type of equilibrium, firms use every realization of the sunspot to either coordinate on firing their non-performing workers with probability one or on firing them with probability zero. Naturally, we refer to this as a pure coordination equilibrium. In a perfect coordination equilibrium, the worker's incentive compatibility constraint reads

$$c = \beta(p_h(1) - p_h(0))(1-\delta)\pi_2 V(\hat{u}(1)). \tag{25}$$

The incentive compatibility constraint (25) implicitly pins down the probability of a realization of the sunspot for which firms fire all of their non-performing workers, π_2, and the probability of a realization of the sunspot for which firms fire none of their non-performing workers, $\pi_0 = 1 - \pi_2$. In particular, the probability π_2 is given by

$$\pi_2 = \frac{c}{\beta(p_h(1) - p_h(0))(1-\delta)V(\hat{u}(1))}. \tag{26}$$

The above results are summarized in the following theorem.

Theorem 2: (*Sunspots*). Assume that the unemployment rate $\hat{u}(s^*(y_\ell, \hat{z}))$ is strictly increasing in the probability $s^*(y_\ell, \hat{z})$ with which firms fire non-performing workers and that the wage $w(u)$ is strictly decreasing in the unemployment rate u. Then, there are three types of equilibria: (i) no-coordination equilibrium where $\phi(\hat{z}) = \phi^*$ with probability $\pi_1 = 1$; (ii) partial coordination equilibrium where $\phi(\hat{z}) = \phi^*$ with probability $\pi_1 \in (0, 1)$, $\phi(\hat{z}) = \phi_0 < \phi^*$ with probability π_0 and $\phi(\hat{z}) = \phi_2 > \phi^*$ with probability π_2; (iii) perfect coordination equilibrium where $\phi(\hat{z}) = \phi_0 < \phi^*$ with probability $\pi_0 \in (0, 1)$ and $\phi(\hat{z}) = \phi_2 > \phi^*$ with probability $\pi_2 \in (0, 1)$.

Both in a partial coordination equilibrium and in a perfect coordination equilibrium, the labor market features aggregate uncertainty, even though there is no uncertainty about technology, preferences, policy or any other fundamental of the economy. The cause of aggregate uncertainty is instead a strategic complementarity in the firing decision of different firms. Specifically, if other firms are more likely to fire their workers in some states of the world, then an individual firm finds it optimal to concentrate the probability with which it fires its non-performing workers exactly in those states of the world. This is because, in the states of the world where other firms are more likely to fire their workers, the unemployment rate is higher, the value to a worker of being unemployed is lower and, hence, the cost to a worker of losing his job is higher. The strategic complementarity in the firing decision of different firms opens the door to coordination, which materializes—in the form of aggregate uncertainty—in both the partial coordination and the perfect coordination equilibria. However, the strategic complementarity in the firing decision of different firms is not sufficient to generate aggregate uncertainty, as exemplified by the no coordination equilibrium.

The emergence of aggregate uncertainty in the absence of any fundamental uncertainty is just an extreme illustration of the amplification power of the strategic complementarity in the firing decisions of different firms. Indeed, in an environment with aggregate fundamental uncertainty, firms could use fundamental fluctuations, rather than sunspots, as a coordination device. Imagine, for instance, an environment in which aggregate labor productivity is uncertain. In such an environment, an individual firm would find it optimal to concentrate the probability of firing its non-performing workers in the states of the world where the realization of labor productivity is lowest, as those are the states of the world where the unemployment rate is highest and the cost to the worker from losing a job is highest. As this is true for each firm, arbitrarily small variations in the realized labor productivity would lead to large changes in firing rates and unemployment. In the language of Cass and Shell (1983), the strategic complementarity would manifest itself as a mechanism of amplification of intrinsic uncertainty, rather than as a source of extrinsic uncertainty.

We now briefly turn to the issue of equilibrium selection. Consider the no-coordination equilibrium. The equilibrium is such that the distribution of the worker's relative gains from trade, $\phi(\hat{z})$, is degenerate at ϕ^*, the value for which firms are indifferent between firing their non-performing workers with any probability between zero and one. Yet, the equilibrium requires firms to break the indifference in a particular way, i.e. by choosing, for any realization of the sunspot $\hat{z}$, the firing probability s_1^* that makes the worker's relative gains from trade equal to ϕ^*. This observation suggests that the no-coordination equilibrium is fragile to small perturbations of the environment. Imagine, for example, adding to the unemployment benefit b a random component drawn from an atomless distribution with an arbitrarily small support. In this case, even if all the firms were to fire workers with the prescribed probability s_1^*, the distribution of the worker's relative gains from trade, $\phi(\hat{z})$, would not be degenerate. Consequently, in all states of the world in which $\phi(\hat{z}) < \phi^*$, an individual firm would want to fire the worker with probability zero rather than s_1^*. In all the other states of the world, those in which $\phi(\hat{z}) > \phi^*$, an individual firm would want to fire the worker with probability one rather than s_1^*. The no-coordination equilibrium would cease to exist. The same is true for a partial coordination equilibrium. In contrast, a perfect coordination equilibrium is robust to small perturbations of the environment. This is because, for every realization of $\hat{z}$, the firm has a strict preference for the firing probability it chooses in equilibrium.

Based on the above argument, it is natural to restrict attention to the perfect coordination equilibrium. In this type of equilibrium, there is a measure $\pi_G = \pi_0$ of realizations of the sunspot for which firms fire their non-performing workers with probability zero and a measure $\pi_B = \pi_2$ of realizations of the sunspot for which firms fire their non-performing workers with probability one, where G and B are mnemonic for Good and Bad. That is, the equilibrium looks like an equilibrium in which there is a sunspot z' that can only take two values, G and B. Moreover, the probability of the two different realization of the sunspot z' is endogenously determined by the worker's incentive compatibility constraint, and it is given by (26).

5 Perfect Coordination Equilibrium

In this section, we prove the existence of a perfect coordination equilibrium. The gist of the proof is showing that a perfect coordination equilibrium is the fixed point of a mapping that satisfies the conditions of Schauder's theorem. In the process of proving the existence of the equilibrium, we will characterize several features of the equilibrium objects, i.e. the properties of the law of motion for unemployment, the wage, and the worker's and firm's gains from trade.

5.1 Preliminaries

Consider the function $\omega(u,i) = (i-1)V^+(u) + iJ^+(u)$, with $u \in [0,1]$ and $i \in \{0,1\}$. The function $V^+(u)$ denotes the worker's expected gains from employment at the end of the production stage given that the unemployment rate is u. We refer to $V^+(u)$ as the worker's continuation gains from trade. Similarly, the function $J^+(u)$ denotes the firm's expected gains from employment at the end of the production stage given that the unemployment rate is u. We refer to $J^+(u)$ as the firm's continuation gains from trade. With a slight abuse of notation we will sometime describe the function ω as (V^+, J^+).

Let Ω denote the set of bounded and continuous functions $\omega : [0,1] \times \{0,1\} \to \mathbb{R}$ such that: (i) for all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the difference $V^+(u_1) - V^+(u_0)$ is greater than $\underline{D}_{V_+}(u_1 - u_0)$ and smaller than $\overline{D}_{V_+}(u_1 - u_0)$; (ii) for all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the difference $J^+(u_1) - J^+(u_0)$ is greater than $\underline{D}_{J_+}(u_1 - u_0)$ and smaller than $\overline{D}_{J_+}(u_1 - u_0)$. In other words, Ω is the set of bounded continuous functions $\omega = (V^+, J^+)$ such that V^+ and J^+ have bounded "derivatives." The bounds on the derivatives are given numbers such that $\overline{D}_{V_+} > \underline{D}_{V_+} > 0$, $\overline{D}_{J_+} > 0 \geq \underline{D}_{J_+}$, and $\underline{D}_{V_+} > \overline{v}'(\overline{D}_{J_+} - \underline{D}_{J_+})$. Following Lemma A.1 in Menzio and Shi (2010), it is straightforward to verify that Ω is a non-empty, bounded, closed and convex subset of the space of bounded continuous functions on $[0,1] \times \{0,1\}$ with the sup norm.

In the remainder of this section, we show that the conditions for a perfect coordination equilibrium implicitly define an operator T that takes an arbitrary pair of continuation gains from trade $\omega = (V^+, J^+)$ in Ω and returns an updated pair of continuation value functions $T\omega = (V^{+\prime}, J^{+\prime})$ in Ω. Then we show that the operator T satisfies the conditions for Schauder's fixed point theorem and, hence, there exists an $\omega^* = (V^{+*}, J^{+*})$ such that $T\omega^* = \omega^*$. Finally, we show that the equilibrium objects associated with the fixed point ω^* constitute a perfect coordination equilibrium. Throughout the analysis, we use a few additional pieces of notation. In particular, we use $\underline{V}$, $\overline{V}$, $\underline{J}$ and $\overline{J}$ to denote lower and upper bounds on the worker's and firm's gains from trade constructed as in (31). Also, we use $\underline{\mu}_u$ and $\overline{\mu}_u$ to denote the minimum and the maximum of the (absolute value) of the partial derivative of the job-finding probability $\mu(J,u)$ with respect to $i = u$. That is, $\underline{\mu}_u$ denotes $\min|\partial\mu(J,u)/\partial u|$ for $(J,u) \in [\underline{J},\overline{J}] \times [0,1]$, and $\overline{\mu}_u$ denotes $\max|\partial\mu(J,u)/\partial u|$ for $(J,u) \in [\underline{J},\overline{J}] \times [0,1]$. Similarly, we use $\underline{\mu}_J$ and $\overline{\mu}_J$ to denote the minimum and the maximum on the (absolute value) of the partial derivative of $\mu(J,u)$ with respect to J.

5.2 Wage

Take an arbitrary pair of value functions $\omega = (V^+, J^+)$ in Ω. Then, for any unemployment rate u, the equilibrium wage function $w(u)$ takes a value w such that

$$v(w) - v(b) - c + V^+(u) = v'(w)\left[E[y|e=1] - w + J^+(u)\right]. \tag{27}$$

For any unemployment rate u, there is a unique wage that satisfies (27). In fact, the right-hand side of (27) is strictly increasing in w, as the worker's gains from trade $v(w) - v(b) - c + V^+(u)$ are strictly increasing in w. The left-hand side of (27) strictly decreasing in w, as the worker's marginal utility of consumption $v'(w)$ and the firm's gains from trade $E[y|e=1] - w + J^+(u)$ are both strictly decreasing in w. Therefore, there exists a unique wage w that solves (27) for any unemployment rate u.

The next lemma proves that the equilibrium wage function $w(u)$ is strictly decreasing in the unemployment rate u. There is a simple explanation behind this result. Given the bounds on the derivatives of the value functions V_+ and J_+, an increase in the unemployment rate leads to a larger increase in the worker's gains from trade (the left-hand side of (27)) than in the product of the firm's gains from trade and the worker's marginal utility (the right-hand side of (27)) for any fixed wage w. Therefore, an increase in the unemployment rate requires the equilibrium wage to fall in order to satisfy equation (27). The next lemma also establishes that the "derivative" of the wage function is bounded.

Lemma 5: For all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the wage function $w(u)$ is such that

$$\underline{D}_w(u_1 - u_0) \leq w(u_0) - w(u_1) \leq \overline{D}_w(u_1 - u_0), \tag{28}$$

where the bounds $\underline{D}_w$ and $\overline{D}_w$ are defined as

$$\underline{D}_w = \frac{\underline{D}_{V_+}/\overline{v}' - \overline{D}_{J_+}}{2 + \overline{J}\overline{v}''/\underline{v}'}, \quad \overline{D}_w = \overline{D}_{V_+}/\underline{v}' - \underline{D}_{J_+}. \tag{29}$$

Proof: In Appendix A.

The next lemma proves that the equilibrium wage function w is continuous with respect to the value functions V^+ and J^+. Specifically, consider two pairs of value functions $\omega_0 = (V_0^+, J_0^+)$ and $\omega_1 = (V_1^+, J_1^+)$ with ω_0, ω_1 in Ω. Denote as w_0 the wage function computed using ω_0 in (27). Similarly, denote as w_1 the wage function computed using ω_1 in (27). Then, if the distance between ω_0 and ω_1 goes to zero, so does the distance between w_0 and w_1.

Lemma 6: For any $\kappa > 0$ and any ω_0, ω_1 in Ω such that $||\omega_0$- $\omega_1|| < \kappa$, we have

$$||w_0 - w_1|| < \alpha_w \kappa, \quad \alpha_w = 1 + 1/\underline{\upsilon}'. \tag{30}$$

Proof: In Appendix A.

5.3 Gains from trade to worker and firm

Given the value functions V^+ and J^+ and the equilibrium wage function w computed using (27), the equilibrium gains from trade accruing to the firm and to the worker, J and V, are respectively given by

$$\begin{aligned} J(u) &= E[y|e=1] - w(u) + J^+(u), \\ V(u) &= \upsilon(w(u)) - \upsilon(b) - c + V^+(u) \end{aligned} \tag{31}$$

The next lemma proves that the equilibrium gains from trade accruing to the firm are strictly increasing in the unemployment rate. This finding is intuitive. An increase in the unemployment rate leads to a larger decline in the equilibrium wage $w(u)$ than in the firm's continuation gains from trade $J^+(u)$. Similarly, the lemma proves that the equilibrium gains from trade accruing to the worker are strictly increasing in the unemployment rate. This finding is also intuitive. An increase in the unemployment rate increases the equilibrium gains from trade accruing to the firm and lowers the equilibrium wage. Since the equilibrium gains from trade accruing to the worker are proportional to the product between the worker's marginal utility and the firm's gains from trade, the result follows. Finally, the lemma establishes that the equilibrium gains from trade accruing to the firm and the worker have bounded derivatives.

Lemma 7: For all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the equilibrium gains from trade accruing to the firm, $J(u)$, and to the worker, $V(u)$, are such that

$$\begin{aligned} \underline{D}_J(u_1 - u_0) \le J(u_1) - J(u_0) \le \overline{D}_J(u_1 - u_0), \\ \underline{D}_V(u_1 - u_0) \le V(u_1) - V(u_0) \le \overline{D}_V(u_1 - u_0), \end{aligned} \tag{32}$$

where the bounds are defined as

$$\begin{aligned}&\underline{D}_J = \underline{D}_w + \underline{D}_{J^+} > 0, \quad \overline{D}_J = \overline{D}_w + \overline{D}_{J^+}, \\ &\underline{D}_V = \underline{v}'(\underline{D}_w + \underline{D}_{J^+}) > 0, \quad \overline{D}_V = \overline{v}'(\overline{D}_w + \overline{D}_{J^+}) + \overline{v}''\overline{D}_w\overline{J}.\end{aligned} \tag{33}$$

Proof: In Appendix B.

The next lemma proves that the firm's and worker's gains from trade are continuous with respect to the choice of the functions V^+ and J^+. Specifically, consider two pairs of value functions $\omega_0 = (V_0^+, J_0^+)$ and $\omega_1 = (V_1^+, J_1^+)$ with ω_0, ω_1 in Ω. Denote as w_0 the equilibrium wage computed using ω_0, and as J_0 and V_0 the equilibrium gains from trade computed using ω_0 and w_0 in (31). Similarly, denote as w_1 the equilibrium wage computed using ω_1, and as J_1 and V_1 the equilibrium gains from trade computed using ω_1 and w_1 in (31). Then, if the distance between ω_0 and ω_1 goes to zero, so does the distance between J_0 and J_1 and between V_0 and V_1.

Lemma 8: For any $\kappa > 0$ and any ω_0, ω_1 in Ω such that $||\omega_0\text{-}\ \omega_1|| < \kappa$, we have

$$\begin{aligned}&||J_0 - J_1|| < \alpha_J\kappa, \quad \alpha_J = 1 + \alpha_w, \\ &||V_0 - V_1|| < \alpha_V\kappa, \quad \alpha_V = 1 + \overline{v}'\alpha_w.\end{aligned} \tag{34}$$

Proof: In Appendix B.

5.4 Law of motion for unemployment

Given the value functions V^+ and J^+, the equilibrium wage function, w, is given by (27) and the equilibrium gains from trade accruing to the firm, J, are given by (31). Given J, the equilibrium law of motion for unemployment, $g(u, \hat{z}')$, is such that—for any current unemployment u and any realization of the sunspot $\hat{z}'$—next period's unemployment takes on a value $\hat{u}$ such that

$$\hat{u} = u - u\mu(J(\hat{u}), u) + (1-u)[\delta + (1-\delta)p_\ell(1)s(y_\ell, \hat{z}')], \tag{35}$$

Since we are looking for a perfect coordination equilibrium, the firing probability $s(y_\ell, \hat{z}')$ equals zero if $\hat{z}' = G$ and one if $\hat{z}' = B$.

Next period's unemployment is uniquely determined by (35). In fact, the left-hand side of (35) equals zero for $\hat{u} = 0$, it is strictly increasing in $\hat{u}$ and it equals one for $\hat{u} = 1$. The right-hand side of (35) is strictly positive for $\hat{u} = 0$ and it is strictly decreasing in $\hat{u}$, as the worker's job finding probability μ is strictly increasing in the firm's gains from trade J, and J is strictly increasing in $\hat{u}$. Therefore, there exists one and only one $\hat{u}$ that satisfies (35).

Next period's unemployment is strictly increasing in the current unemployment u. In fact, the left-hand side of (35) is independent of u. The right-hand side of (35) is strictly increasing in u, as its derivative with respect to u is greater than $(1-\delta)p_h(1)-\mu$, which we assume to be strictly positive. Therefore, the $\hat{u}$ that solves (35) is strictly increasing in u. Moreover, next period's unemployment is strictly higher if the realization of the sunspot $\hat{z}'$ is B rather than G and, more generally, strictly increasing in the probability with which firms fire non-performing workers. To see this, it is sufficient to notice that the left-hand side of (35) is independent of the firing probability, while the right hand side is strictly increasing in it. The properties of the equilibrium law of motion for unemployment, $g(u,\hat{z}')$, are illustrated in Figure 3.

The next lemma proves that the equilibrium law of motion for unemployment $g(u,\hat{z}')$ is continuous in u and that its derivative is bounded.

Lemma 9: For all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the equilibrium law of motion for unemployment $g(u,\hat{z}')$ is such that

$$\underline{D}_g(u_1-u_0) < g(u_1,\hat{z}') - g(u_0,\hat{z}') \leq \overline{D}_g(u_1-u_0), \tag{36}$$

where the bounds $\underline{D}_g$ and $\overline{D}_g$ are defined as

$$\underline{D}_g = 0, \quad \overline{D}_g = 1-\delta+\overline{\mu}_u. \tag{37}$$

Proof: In Appendix C.

Next, we prove that the equilibrium law of motion for unemployment, $g(u,\hat{z}')$, is continuous with respect to V^+ and J^+. Formally, we consider two pairs of value functions $\omega_0 = (V_0^+$, $J_0^+)$ and $\omega_1 = (V_1^+$, $J_1^+)$ with ω_0, ω_1 in Ω. We denote as J_0 the firm's equilibrium gains from trade computed using ω_0 in (31), and as g_0 the equilibrium law of motion for unemployment computed using J_0 in (35). Similarly, we denote as J_1 the firm's equilibrium gains from trade computed using ω_1 in (31), and as g_1 the equilibrium law of motion for unemployment computed using J_1 in (35). Then, we show that, if the distance between ω_0 and ω_1 goes to zero, so does the distance between g_0 and g_1.

Lemma 10: For any $\kappa > 0$ and any ω_0, ω_1 in Ω such that $||\omega_0$- $\omega_1|| < \kappa$, we have

$$||g_0 - g_1|| < \alpha_g \kappa, \quad \alpha_g = \overline{\mu}_J \alpha_J. \tag{38}$$

Proof: In Appendix C.

5.5 Updated value functions

Given the value functions V^+ and J^+, the equilibrium wage function, w, is given by (27). The equilibrium gains from trade accruing to the firm and to the worker, J and V, are given by (31). The equilibrium law of motion for unemployment, g, is given by (35). In turn, given J, V and g, we can construct an update, $V^{+\prime}$, for the continuation gains from trade accruing to the worker as

$$V^{+\prime}(u) = \beta E_{\hat{z}'} \left\{ [(1-\delta)(1-p_\ell(1)s(y_\ell, \hat{z}')) - \mu(J(g(u,\hat{z}')), u)] V(g(u,\hat{z}')) \right\} \tag{39}$$

We can also construct an update, $J^{+\prime}$, for the ex-wage gains from trade accruing to the firm as

$$J^{+\prime}(u) = \beta E_{\hat{z}'} \left[(1-\delta)(1-p_\ell(1)s(y_\ell, \hat{z}')) J(g(u,\hat{z}')) \right] \tag{40}$$

As we are looking for a perfect coordination equilibrium, the firing probabilities $s(y_\ell, B)$ and $s(y_\ell, G)$ in (39) and (40) are set respectively equal to 1 and 0. Similarly, as we are looking for a perfect coordination equilibrium, the probability π_B that the realization of the sunspot $\hat{z}'$ is B is set equal to

$$\pi_B = \frac{c}{\beta(p_h(1) - p_h(0))(1-\delta)V(g(u,B))}. \tag{41}$$

Notice that the probability that the realization of the sunspot is B is endogenous as it needs to satisfy the worker's incentive compatibility constraint. In particular, the higher is the current unemployment rate u, the higher is next period's unemployment rate $g(u, B)$, the higher are next period's gains from trade accruing to the worker $V(g(u, B))$ and, ultimately, the lower is the probability that the realization of the sunspot is B.

In the next lemma, we prove that the update of worker's continuation value, $V^{+\prime}$, is strictly increasing in the unemployment rate. This finding is intuitive. An increase in the unemployment rate causes congestion in the labor market and, for this reason, leads to a decline in the worker's job finding rate and in the worker's value of unemployment. In turn, the decline in the worker's value of unemployment tends to increase the worker's gains from trade (which is only partly mitigated by a decline in the equilibrium wage).

Lemma 11: For all u_0, $u_1 \in [0, 1]$ with $u_0 < u_1$, the updated continuation gains from trade to the worker, $V^{+\prime}$, and the firm, $J^{+\prime}$, are such that

$$\begin{aligned} \underline{D}'_{V+}(u_1 - u_0) < V^{+\prime}(u_1) - V^{+\prime}(u_0) \leq \overline{D}'_{V+}(u_1 - u_0), \\ \underline{D}'_{J+}(u_1 - u_0) < J^{+\prime}(u_1) - J^{+\prime}(u_0) \leq \overline{D}'_{J+}(u_1 - u_0). \end{aligned} \tag{42}$$

where the bounds $\underline{D}'_{V+}$ and $\overline{D}'_{V+}$ are defined as

$$\begin{aligned}\underline{D}'_{V+} &= \beta \underline{V}(\underline{\mu}_u - \overline{\mu}_J \overline{D}_J \overline{D}_g) - \frac{2c\overline{D}_V \overline{D}_g \overline{V}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2}, \\ \overline{D}'_{V+} &= \beta \left[\overline{V}\overline{\mu}_u + (1-\delta)\overline{D}_V \overline{D}_g\right],\end{aligned} \tag{43}$$

and the bounds $\underline{D}'_{J+}$ and $\overline{D}'_{J+}$ are defined as

$$\begin{aligned}\underline{D}'_{J+} &= -\frac{2c\overline{D}_V \overline{D}_g \overline{J}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2}, \\ \overline{D}'_{J+} &= \beta(1-\delta)\overline{D}_J \overline{D}_g\end{aligned} \tag{44}$$

Proof: In Appendix D.

The next lemma shows that, under some parametric conditions, there exists a fixed point for the derivatives of the continuation gains from trade, i.e. $\underline{D}_{V+}$, $\overline{D}_{V+}$, $\underline{D}_{J+}$, $\overline{D}_{J+}$ such that $\underline{D}'_{V+} = \underline{D}_{V+}$, $\overline{D}'_{V+} = \overline{D}_{V+}$, $\underline{D}'_{J+} = \underline{D}_{J+}$, and $\overline{D}'_{J+} = \overline{D}_{J+}$, such that the conditions $\overline{D}_{V_+} > \underline{D}_{V_+} > 0$, $\overline{D}_{J_+} > 0 \geq \underline{D}_{J_+}$ and $\underline{D}_{V_+} > \overline{v}'(\overline{D}_{J_+} - \underline{D}_{J_+})$ are satisfied. The key parametric condition requires the negative effect of unemployment on the job-finding probability to be sufficiently strong compared to the positive effect of the firm's gains from trade on the job-finding probability. Intuitively, the condition guraantees that the next period's job-finding rate is decreasing in the current period's unemployment rate. In turn, when the discount factor β and the disutility of effort c are small enough, this guarantees that the derivative of the worker's continuation gains from trade is positive and sufficiently larger than the derivative of the firm's continuation gains from trade.

Lemma 12: Assume $\underline{\mu}_u - \overline{\mu}_J(1 - \delta + \overline{\mu}_u) > 0$. Then there exist $\beta^* > 0$ and $c^* > 0$ such that, if $\beta \in (0, \beta^*)$ and $c \in (0, c^*)$, there are bounds $\underline{D}_{V+}$, $\overline{D}_{V+}$, $\underline{D}_{J+}$, $\overline{D}_{J+}$ such that: (i) $\underline{D}'_{V+} = \underline{D}_{V+}$, $\overline{D}'_{V+} = \overline{D}_{V+}$, $\underline{D}'_{J+} = \underline{D}_{J+}$, and $\overline{D}'_{J+} = \overline{D}_{J+}$; (ii) $\overline{D}_{V_+} > \underline{D}_{V_+} > 0$, $\overline{D}_{J_+} > 0 \geq \underline{D}_{J_+}$ and $\underline{D}_{V_+} > \overline{v}'(\overline{D}_{J_+} - \underline{D}_{J_+})$.

Proof: In Appendix D.

In the last lemma, we prove that the updated continuation value functions, $V^{+\prime}$ and $J^{+\prime}$, are continuous with respect to the original value functions V^+ and J^+. Specifically, consider two arbitrary functions $\omega_0 = (V_0^+, J_0^+)$ and $\omega_1 = (V_1^+, J_1^+)$ with ω_0, ω_1 in Ω. Denote as $V_0^{+\prime}$ and $J_0^{+\prime}$ the continuation value functions computed using V_0^+, J_0^+ and g_0 in (39) and (40). Similarly, denote as $V_1^{+\prime}$ and $J_1^{+\prime}$ the continuation value functions computed using V_1^+, J_1^+ and g_1 in (39) and (40). Then, if the distance between ω_0 and ω_1 goes to zero, so does the distance between V_0^+ and V_1^+ and between J_0^+ and J_1^+.

Lemma 13: For any $\kappa > 0$ and any ω_0, ω_1 in Ω such that $||\omega_0 - \omega_1|| < \kappa$, we have $||V_0^{+\prime} - V_1^{+\prime}|| < \alpha_{V_+^\prime}\kappa$ and $||J_0^{+\prime} - J_1^{+\prime}|| < \alpha_{J_+^\prime}\kappa$, where

$$\begin{aligned} \alpha_{V_+^\prime} &= \left[\frac{2c\overline{V}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2} + 1\right]\left(\alpha_V + \overline{D}_V\alpha_g\right) + \overline{V}\overline{\mu}_J\left(\alpha_J + \overline{D}_J\alpha_g\right), \\ \alpha_{J_+^\prime} &= \frac{2c\overline{J}\left(\alpha_V + \overline{D}_V\alpha_g\right)}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2} + \left(\alpha_J + \overline{D}_J\alpha_g\right). \end{aligned} \tag{45}$$

Proof: In Appendix D.

5.6 Existence

In the previous subsections, we have taken a pair of continuation gains from trade V^+ and J^+ and, using the conditions for a perfect coordination equilibrium, we have constructed an updated pair of continuation gains from trade $V^{+\prime}$ and $J^{+\prime}$. We denote as T the operator that takes $\omega(u,i) = (1-i)V^+(u) + iJ^+(u)$ and returns $\omega^\prime(u,i) = (1-i)V^{+\prime}(u) + iJ^{+\prime}(u)$.

The operator T has three key properties. First, the operator T maps functions that belong to the set Ω into functions that also belong to the set Ω. In fact, for any $\omega = (V^+, J^+) \in \Omega$, $\omega^\prime = (V^{+\prime}, J^{+\prime})$ is bounded and continuous and, as established in Lemma 12, it is such that: (i) for all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the difference $V^{+\prime}(u_1) - V^{+\prime}(u_0)$ is greater than $\underline{D}_{V_+}(u_1 - u_0)$ and smaller than $\overline{D}_{V_+}(u_1 - u_0)$; (ii) for all u_0, $u_1 \in [0,1]$ with $u_0 < u_1$, the difference $J^{+\prime}(u_1) - J^{+\prime}(u_0)$ is greater than $\underline{D}_{J_+}(u_1 - u_0)$ and smaller than $\overline{D}_{J_+}(u_1 - u_0)$. Second, the operator T is continuous, as established in Lemma 13. Third, the family of functions $T(\Omega)$ is equicontinuous. To see that this is the case, let $||.||_E$ denote the standard norm on the Euclidean space $[0,1] \times \{0,1\}$. For any $\epsilon > 0$, let $\kappa_\epsilon = \min\{(\max\{\overline{D}_{V+}, \overline{D}_{J+}, |\underline{D}_{J+}|\})^{-1}, 1\}$. Then, for all (u_0, i_0), $(u_1, i_1) \in [0,1] \times \{0,1\}$ such that $||(u_0, i_0) - (u_1, i_1)||_E < \kappa_\epsilon$, we have

$$|(T\omega)(u_0, i_0) - (T\omega)(u_1, i_1)| < \epsilon, \text{ for all } \omega \in \Omega. \tag{46}$$

Since $T : \Omega \longrightarrow \Omega$, T is continuous and $T(\Omega)$ is equicontinuous, the operator T satisfies the conditions of Schauder's fixed point theorem (see Theorem 17.4 in Stokey, Lucas and Prescott 1989). Therefore there exists a $\omega^* = (V^{+*}, J^{+*}) \in \Omega$ such that $T\omega^* = \omega^*$.

Now, we compute the equilibrium wage function w^* using V^{+*} and J^{+*} in (27). We compute the equilibrium gains from trade V^* and J^* using w^*, V^{+*} and J^{+*} in (31). We compute the equilibrium law of motion for unemployment g^* using J^* in (35). Finally, we construct the equilibrium employment contract $x^*(u)$ as the effort $e^* = 1$, the wage function w^* and the firing probabilities $s^*(y_h, \hat{z}') = 0$, $s^*(y_\ell, G) = 0$ and $s^*(y_h, B) = 1$.

The tuple (V^*, J^*, x^*, g^*) constitute an equilibrium. Indeed, if we substitute (39) and (40) into (31), we find that the gains from trade accruing to the worker, V^*, and to the firm, J^*, satisfy condition (i) in the definition of equilibrium. The law of motion for unemployment, g^*, satisfies condition (iii) in the definition of equilibrium. The employment contract, x^*, prescribes a wage w^* that satisfies (16). It prescribes a firing probability $s^*(y_h, \hat{z}')$ that satisfies (12). It prescribes a firing probability $s^*(y_\ell, \hat{z}')$ that satisfies (13). To see why this is the case, notice that the gains from trade accruing to the worker relative to those accruing to the firm are such that $v'(w^*(g^*(u, B))) > v'(w^*(g^*(u, G)))$ since the wage function w^* is strictly decreasing in unemployment and $g^*(u, B)$ is strictly greater than $g^*(u, G)$. Moreover, the probability that the realization of the sunspot is B is such that the worker's incentive compatibility constraint holds with equality. Therefore, by Theorem 1, the employment contract x^* is optimal and satisfies condition (ii) in the definition of equilibrium. Moreover, the tuple (V^*, J^*, x^*, g^*) is a perfect coordination equilibrium by construction.

We have established the following result.

Theorem 3: (*Existence*) Assume $\underline{\mu}_u - \overline{\mu}_J(1-\delta+\overline{\mu}_u) > 0$. A perfect coordination equilibrium exists for all (β, c) such that $\beta \in (0, \beta^*)$ and $c \in (0, c^*)$.

Two comments about Theorem 3 are in order. First, notice that, although the parametric conditions in Theorem 3 are sufficient for the existence of a perfect coordination equilibrium, they are not necessary. Indeed, the version of the model that we calibrate in the next section does not always meet these conditions and, yet, it does admit a perfect coordination equilibrium. Second, notice that Theorem 3 does not rule out the existence of equilibria that do not feature perfect coordination in the firing decision of different firms. Indeed, it is easy to show that under the same conditions as in Theorem 3, there exists a no coordination equilibrium. However, as we argued in Section 4, this type of equilibrium is not robust to the introduction of small shocks to fundamentals.

6 Quantitative Analysis

In this section, we calibrate the model and then use it to measure the magnitude of labor market fluctuations that can be generated by the firms' incentive to coordinate on firing non-performing workers at the same time. We refer to these fluctuations as Agency Business

Cycles. We calibrate the model using US data on the UE, EU and unemployment rates and using, as in identifying assumption, that all of the observed volatility in the EU rate is due to the mechanism highlighted in our theory. First, we find that Agency Business Cycles can account for a substantial fraction of the empirical fluctuations in the labor market, and it can do so without relying on any exogenous shocks to fundamentals such as productivity, unemployment benefits or preferences. Second, we find that Agency Business Cycles generate the same pattern of leads and lags between unemployment, UE and EU rates as in the data. Third, we find that Agency Business Cycles generate the same asymmetries in the velocity at which unemployment rises in recessions and declines in expansions as in the data. Finally, we provide some preliminary evidence suggesting that the difference between the value of employment and the value of unemployment is higher in recessions than in recoveries, in accordance with our theory of recessions and in contrast with the Real Business Cycle view.

6.1 Calibration

The parameters of the model are the following. Preferences are described by the worker's periodical utility function, $v(w) - ce$, and by the discount factor, β. Market production is described by the output distribution, y_h and y_ℓ, by the probability that the output is high given that the worker exerts effort, $p_h(1)$, by the probability that the output is high given that the worker does not exert effort, $p_h(0)$, and by the exogenous match destruction probability δ. Home production is described by the output of an unemployed worker, b. The search and matching technology is described by the vacancy cost, k, and the matching function, $M(u, v)$. We specialize the utility function to be of the form $v(w) - ce = w^{1-\gamma}/(1-\gamma) - ce$, where γ is the coefficient of relative risk aversion. We also specialize the matching function to be of the form $M(u, v) = A(u)m(u, v)$, where $m(u, v) = uv(u^\psi + v^\psi)^{-1/\psi}$ is a constant returns to scale matching function with constant elasticity of substitution ψ between unemployment and vacancies, and $A(u) = \exp(-\rho u)$ is a matching efficiency term with a semi-elasticity with respect to unemployment of negative ρ.

We calibrate the parameters of the model to match features of the US labor market between 1951 and 2014. We measure the unemployment rate as the CPS civilian unemployment rate. We measure the monthly rate at which unemployed workers become employed (henceforth, the UE rate) and the monthly rate at which employed workers become unemployed (henceforth, the EU rate) using the unemployment rate and the short-term unemployment rate from the CPS, following the methodology outlined in Shimer (2005). We measure labor productivity as output per worker in the non-farm sector. We detrend the unemployment, UE and EU rates, and labor productivity using an HP filter with a smoothing parameter of 10^5.

We choose the model period to be one month. We set the discount factor, β, so that the annual real interest rate, $(1/\beta)^{1/12} - 1$, is equal 5 percent. We choose the vacancy cost, k, and the exogenous job destruction rate, δ, so that the model matches the average of the monthly UE rate and of the monthly EU rate that we observe in the data (respectively, 44% and 2.6%). We normalize the expected value of market production, $p_h(1)y_h + (1 - p_h(1))y_\ell$, to be equal to 1. We choose the value of home production, b, to be 70% of the expected value of market production, as suggested by Hall and Milgrom (2010). We choose the disutility of effort, c, so that the model generates the same standard deviation in the time series of log EU rate as in the data (9.85%). In this sense, as discussed above, we calibrate the model under the identifying assumption that all of the fluctuations in the EU rate are caused by the burst of coordinated firings generated by our model. We choose the probability $p_h(1)$ that output is high if the worker exerts effort so that, on average, a coordinated firing episode occurs once every 50 months. Given the normalization of expected output to 1, the distribution of output y_h and y_ℓ does not affect the equilibrium conditions. Similarly, notice that the probability, $p_h(0)$, that output is high if the worker does not exerts effort only enters the equilibrium conditions through the ratio $c/(p_h(1)-p_h(0))$. Therefore, we only need to choose values for $y_h - y_\ell$ and $p_h(1) - p_h(0)$ such that the optimal employment contract prescribes effort.

In our baseline calibration, we set the coefficient of relative risk aversion, γ, in the utility function to be equal to 1. This is a rather standard value from micro estimates of risk aversion. We set the elasticity of substitution, ψ, between unemployment and vacancy in the matching function to be equal to 1.24. This is the value of the elasticity of substitution estimated by Menzio and Shi (2011) using a model that takes into account for search both off and on the job. We tentatively set the semi-elasticity, ρ, of the matching efficiency with respect to unemployment to be equal to 6. As we shall see, this value implies a relationship between the UE rate and the unemployment rate that is very close to the one observed in the data. In the following pages, we will consider alternative values for these parameters.

6.2 Simulation

The calibrated value of the parameters are reported in Table 1. Given these parameter values, we simulate the model and compute standard business cycle statistics, which are reported and compared to their empirical counterparts in Table 2. The table shows that agency business cycles—i.e. the cycles created by the firms' incentive to coordinate on firing non-performing workers—can account for more than 50% of the overall volatility of the unemployment rate, for 40% of the overall volatility of the UE rate and, by assumption, for all of the volatility of the EU rate. Moreover, the agency business cycles create large fluctuations in the labor market without requiring any exogenous fluctuations in any fundamentals such as technology, preferences and policy.

Table 1: Calibrated Parameters

	Description	Value
β	discount factor	.9967
γ	worker's relative risk aversion	1.000
y_h	high output	1.03
y_ℓ	low output	.000
$p_h(1)$	probability of y_h given $e = 1$	.967
$p_h(0)$	probability of y_h given $e = 0$	.467
b	UI benefit/value of leisure	.700
c	disutility of effort	.007
δ	exogenous job destruction	.025
k	vacancy cost	.257
ψ	elasticity of sub. btw u and v	1.24
ρ	semi-elasticity of A wrt u	6.00

Agency business cycles are a potential explanation for two puzzling features of labor markets. First, they can explain why unemployment fluctuates so much in the face of relatively small movements in labor productivity (see, e.g., Shimer 2005). Second, they can explain why the correlation between unemployment fluctuations and labor productivity fluctuations is low, a phenomenon that has become particularly stark since 1984 (see, e.g., Gali and van Rens 2014). Our theory suggests that the firms' desire to coordinate on firing non-performing workers may be a major source of labor market fluctuations which does not require any contemporaneous shocks to productivity.

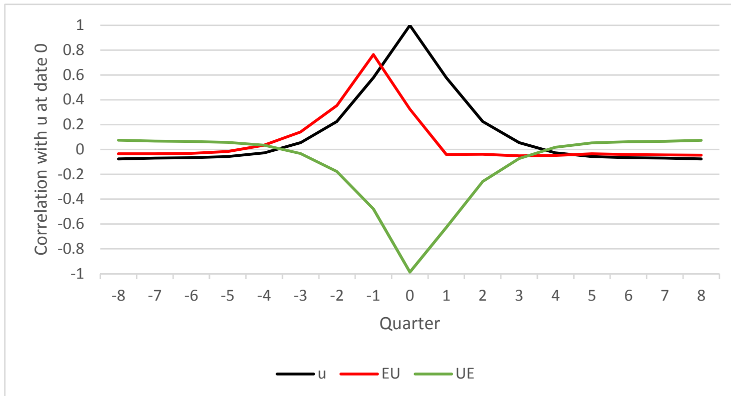
Table 2: Agency Business Cycle

		u rate	UE rate	EU rate	APL
Model	std	9.34	4.09	9.11	0
	cor. wrt u	1	-.98	.32	-
Data: 1951-2014	std	16.9	12.9	9.7	1.98
	cor wrt u	1	-.94	.80	-.37
Data: 1984-2014	std	17.3	13.8	6.91	1.38
	cor wrt u	1	-.96	.70	.09

The literature has advanced several explanations for the high volatility of unemployment (see, e.g., Menzio 2005, Hall 2005, Kennan 2010, Hagedorn and Manovskii 2008, Menzio and Shi 2011) and for the low correlation between unemployment and labor productivity (see, e.g., Kaplan and Menzio 2014, Gali and van Rens 2014). In order to distinguish our theory from others, it is useful to examine in greater detail the pattern of labor market fluctuations generated by our model. Indeed, the agency business cycle follows a rather distinctive pattern. A recession starts with a spike in the EU rate, which leads to a rapid increase in the unemployment rate. In turn, the increase in the unemployment rate decreases the efficiency of matching in the labor market and, hence, depresses the UE rate. This pattern can be seen in Figure 1, which displays the correlation between unemployment in quarter t and the UE rate in quarter $t + i$, for i between -8 and $+8$, as well as the correlation between unemployment in quarter t and the EU rate in quarter $t + i$. The EU rate leads the unemployment rate, in the sense that the peak correlation is between unemployment in quarter t and the EU rate in quarter $t - 1$. In contrast, the UE rate is coincidental with the unemployment rate, in the sense that the peak correlation is between unemployment in quarter t and the UE rate in the same quarter.

The structure of leads and lags generated by the agency business cycle is very similar to the one found in the data and reported in Figure 2. In the data as in the model, the EU rate leads the unemployment rate by a quarter, while fluctuations in the UE rate have the same timing as fluctuations in the EU rate. Moreover, in the data as in the model, the EU rate is correlated with the current unemployment rate further in the past than it is in the future. That is, the past EU rate is more strongly correlated with current unemployment than the future EU rate. In contrast, the correlation between the past UE rate and current unemployment is equally strong as the correlation between the future UE rate and current unemployment. While these features of the data had been already identified by Fujita and Ramey (2010), here we provide an economic theory that rationalizes them. The main

Figure 3: Cross Correlations: Model



shortcoming of our model is that—with i.i.d. sunspots—we generate less persistence in the unemployment, UE and EU rates, as can be seen from the sharper drop of the correlations to the left and right of center.

The mechanics of the agency business cycle imply differences in the velocity at which unemployment increases in recessions and at which it declines in recoveries. Intuitively, unemployment increases quickly in recessions as it is driven by the spurt of firings, while it declines slowly in recoveries as it is bogged down by the congestion created by the large number of unemployed workers. The asymmetry is clearly visible in the top-left panel of Figure 3, which plots a few years of unemployment, UE and EU rates generated by the model. The same asymmetry can be found in the data. The top-right panel of Figure 3 plots the change in the unemployment rate between quarter t and quarter $t+1$ as a function of the unemployment rate in quarter t when unemployment is increasing (black line) and decreasing (red line). The unemployment rate moves significantly faster when it is on the rise than when it is falling.

Our theory also predicts that the EU rate should be higher in recessions, defined as times when the unemployment rate rises, than in recoveries, defined as times when the unemployment rate falls. In contrast, our theory predicts no systematic differences in the UE rate during recessions and recoveries. These features of the model are illustrated in the bottom panels of Figure 3 and compared with the data. One can clearly see that, in the

Figure 4: Cross Correlations: Data

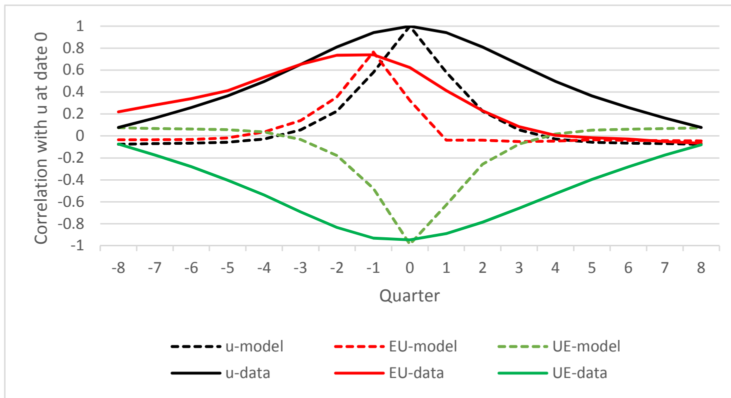
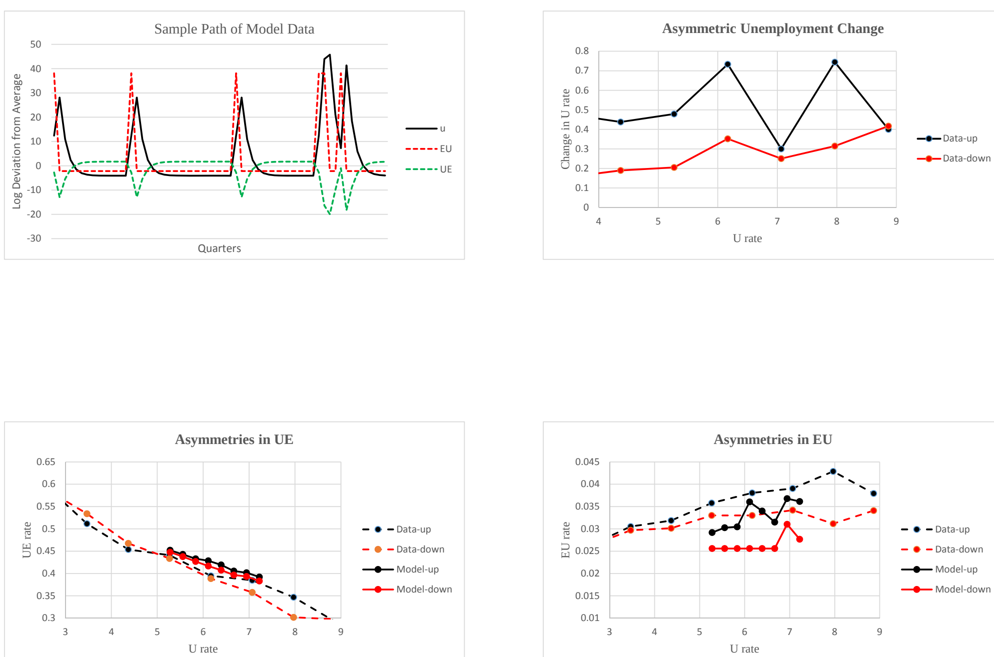


Figure 5: Asymmetries in u, UE and EU rates



data as in the model, the EU rate is systematically higher in recessions than in recoveries, while the UE rate is nearly identical in the two phases of the cycle.

Finally, our theory predicts that the likelihood and magnitude of recessions should depend on the current unemployment rate. In particular, the lower is the unemployment rate, the higher is the probability that firms coordinate on firing non-performing workers and the higher is the increase in the unemployment rate following a firing episode. The first predictions follows immediately from the fact that—when unemployment is lower—the cost to the worker of losing a job is relatively low and the probability that firms fire the non-performing workers must be higher. The second prediction follows from the fact that—when unemployment is lower—the number of employed workers that loses their job in case of a firing episode is larger. The data seems to support both of predictions of the theory. Indeed, given that the unemployment rate is falling between quarters $t-1$ and t, there is a negative correlation ($-.10\%$) between the unemployment rate in t and the probability of observing a turning point in $t+1$—i.e., a point where unemployment turns from decreasing to increasing. Also, conditional on observing a turning point in unemployment in quarter t, there is a negative correlation between the unemployment rate in a quarter t and the increase in unemployment between quarter t and next quarter (-20%), next semester (i.e., -14%) and next year (i.e., -27%).

6.3 Value of a Job

Some of the empirical predictions presented in the previous subsection need not be unique to our model. For instance, the textbook model by Mortensen and Pissarides (1994, henceforth MP94) predicts that, in response to a persistent, negative shock to aggregate productivity, all of the marginal firm-worker matches will be destroyed and firms will create fewer vacancies per unemployed worker. Hence, also MP94 predicts that we should observe recessions starting with an increase in the EU rate followed by a decline in the UE rate. Unlike our theory, though, MP94 implies a perfect correlation between unemployment and labor productivity that is not found in the data. However, there is a much more profound distinction between the prediction of our theory and those in MP94 and all models of Real Business Cycles. If recessions are caused by lower productivity, then recessions should be times when the unemployment rate is high because the gain from finding a job (as well as the cost of losing a job) is relatively low. In contrast, according to our Agency Business Cycle theory, recessions are times when unemployment is high because the cost of losing a job (as well as the value of finding a job) is relatively high.

Measuring the difference between the value of employment and the value of unemployment in the data is clearly a difficult task. However, it is a task that at least is worth attempting. In what follows, we use the empirical time-series of the UE rate, the empirical time-series of the EU rate and the time-series of wages in order to construct a time-series for the value of being employed, the value of being unemployed and for the difference between these two values.

We measure the value of being employed in month t, W_t, as

$$W_t = w_t + \beta \left\{ s_{t+1} U_{t+1} + (1 - s_{t+1}) W_{t+1} \right\}, \tag{47}$$

where w_t is the real wage earned by a worker in month t, s_{t+1} is the EU rate in month $t+1$ and W_{t+1} and U_{t+1} are, respectively, the values of employment and unemployment in quarter $t+1$. Similarly, we measure the value of being unemployed in month t, U_t, as

$$U_t = b_t + \beta \left\{ f_{t+1} W_{t+1} + (1 - f_{t+1}) U_{t+1} \right\}, \tag{48}$$

where b_t is the value of unemployment benefits/leisure for an unemployed worker in month t, f_{t+1} is the UE rate in month $t+1$ and W_{t+1} and U_{t+1} are, as before, the value of employment and unemployment in month $t+1$.

We measure the wage earned by a worker in month t, w_t, using two alternative series constructed by Haefke, Sonntag and van Rens (2013). The first series is the real wage for all employed workers (controlling for changes in the observable characteristics of employed workers). As it is well-know the average wage of employed workers is nearly acyclical. This may be due to fact that wages in continuing relationships are non-allocative and firms and workers do not need to adjust them in response to varying labor market conditions. For this reason, we also consider the average wage of newly hired workers (controlling for changes in the observable characteristics of new hires). As discussed in Haefke, Sonntag and van Rens (2013), the average wage of new hires is much more procyclical than the average wage of all employed workers. When we use the wage of new hires as our measure of w_t, it is natural to assume that the worker is paid the wage at which he was hired throughout the duration of his employment spell. In order to take low-frequency movements out of the data, we HP filter the time series for the wage, compute its cyclical component and add it back to the long-run average value of the wage. We set the value of unemployment benefit/leisure, b_t, as 70% of the long-run average value of the wage. The UE and EU rates, f_t and s_t, are set by HP filtering the time series for these rates, computing their cyclical components and by adding these components back to the long-run averages of the UE and EU rates.

We assume that the value of employment, W_T, and the value of unemployment, U_T, in the last month of observation take the values associated with the long-run averages of w, b, f and s. Given these continuation values, we can use equations (47) and (48) to compute the value of employment, W_t, and the value of unemployment, U_t, for all previous months. Note that these empirical measures of W and U differ from their theoretical counterparts because they are computed using realizations of future wages, job-finding and job-separation rates rather than the expectation of these variables.

Figures 4 and 5 illustrate our findings. Using either time-series for the wage (i.e., the average wage of all employed workers and the average wage of new hires), the difference between the value of employment and the value of unemployment is countercyclical. When the unemployment rate is high, the net value of employment is also high and, conversely, when the unemployment rate is low, the net value of unemployment is also low. The correlation between the net value of a job and the (detrended) unemployment rate is 81% using average wages of all employed workers, and 56% using the average wage of new hires. These results are very robust to different definitions of the value of unemployment benefits/leisure. Indeed, the results follow from the simple fact that the UE rate is very procyclical, the EU rate very countercyclical and wages are not sufficiently procyclical to make the net value of a job go up in expansions. It is perhaps not surprising that this is the case when we use the average wage of all employed workers. It is quite surprising that this is also the case when we use the wage of new hires.

These findings appear to confirm the view of recessions advanced by our theory and cast some shadows on the typical Real Business Cycle view of recessions. Indeed, recessions do not seem to be times when the unemployment rate is high because—due to low productivity of labor—the value of finding a job and the cost of losing a job are relatively low. Recessions seems to be times when the value of becoming employed and the cost of losing a job are relatively large. According to our theory this happens because firms find it optimal to fire non-performing workers exactly when the net value of employment is relatively high. Fluctuations in the net value of employment may be driven by exogenous motives or, as pointed out in our theory, they may be the caused by the self-fulfilling expectations of a firing burst.

The cyclical behavior of layoffs—defined as firm-worker separations initiated by the firm—and quits—defined as firm-worker separations initiated by the worker—provides further support for the above calculations. Figure 6 shows the behavior of the quit rate and of the layoff rate during the last fifteen years. During the Great Recession (December 2007-June

Figure 6: Value of a Job: Wage of all employees

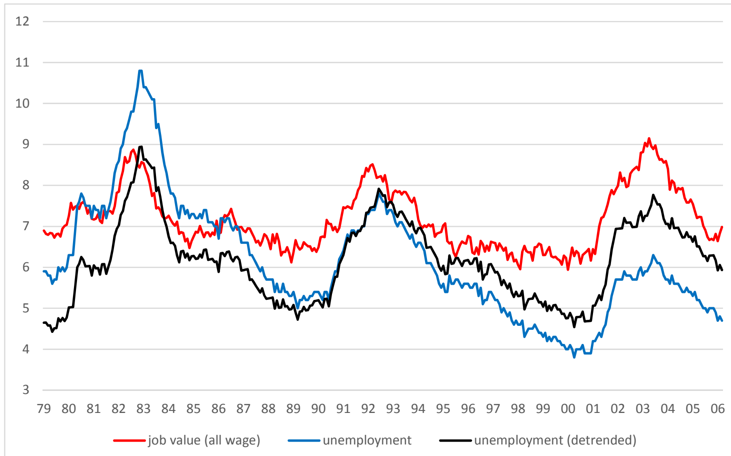


Figure 7: Value of a Job: Wage of new hires

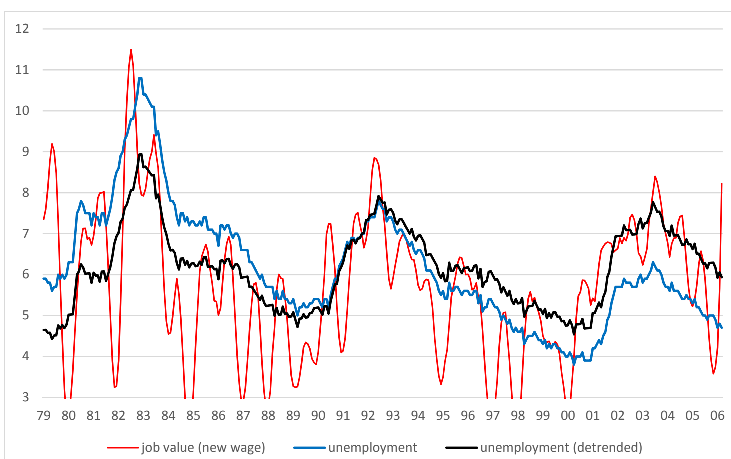
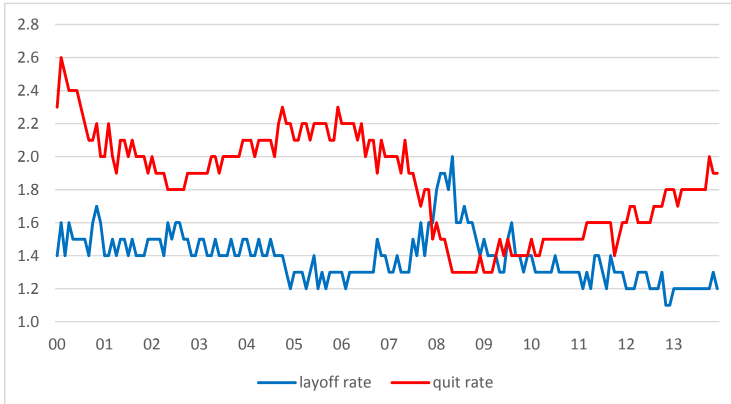


Figure 8: Layoff and Quit Rates



2009), the quit rate fell by 0.7 percentage points (nearly halving). In contrast, the layoff rate increased by 0.7 percentage points (nearly doubling). If the Great Recession was a time when the net value of employment was relatively low, we should have seen the quit rate increase. Instead, the decline in the quit rate suggests that the Great Recession was a time when the net value of employment was relatively high. Moreover, the sharp increase in the layoff rate during the recessions suggests that firms found it optimal to fire workers exactly when the net value of a job was relatively high.

7 Conclusions

In this paper, we advanced a novel theory of cyclical fluctuations in the labor market. The theory is based on two key assumptions. First, firms need to fire workers with positive probability in order to given them an ex-ante incentive to exert effort. Under this assumption, we showed that, in order to provide this incentive at the lowest cost, firms load up the firing probability on the states of the world where the worker's cost of losing a job relative to the firm's cost of losing a worker is highest. Second, there are decreasing returns to scale in the matching process. Under this assumption, we showed that the states of the world where the worker's cost of losing a job is highest are the states with highest unemployment and, hence, an individual firm finds it optimal to fire its non-performing workers exactly at the time when other firms fire their non-performing workers. The strategic complementarity between the optimal solution of the agency problem faced by different firms leads to an equilibrium in which firms use the realization of a sunspot to coordinate on firing their non-performing workers at the same time, leading to aggregate fluctuations in the EU, UE and unemployment rates.

The fluctuations generated by our model are genuinely endogenous, in the sense that they do not require any exogenous shocks to preferences, technology or policy, nor they require any exogenous switch in the selection of the equilibrium, and stochastic, in the sense that the equilibrium features aggregate uncertainty about the firing probability. Quantitatively, we showed that the fluctuations generated by our model can potentially account for a large fraction of the volatility in the unemployment rate and in the UE and EU rates that is observed in the US labor market. Moreover, our model generates fluctuations in labor market variables that are uncorrelated with labor productivity, just as they have been in the US for the past 30 years. We also showed that the fluctuations generated by our model have the same morphology as the labor market fluctuations in the US: the EU rate leads the unemployment rate, while the UE rate and the unemployment rate are contemporaneous, the unemployment rate moves faster in recessions than in recoveries, etc... Finally, we carried out some preliminary calculations that seem to support our view that recessions are times when it is particularly costly to lose a job (or stay unemployed), rather than the standard view that recessions are times when it is particularly cheap to lose a job (or stay unemployed).

Clearly, the theory of labor market fluctuations advanced in this paper is very stark and a lot of work remains to be done. On the modelling side, we want to allow for long-term contracts so that wages and not only firing can be used to provide workers with the incentive to exert effort. On the empirical side, it would be important to find some data to directly calibrate the parameters of the agency problem faced by firms and workers, thus dropping the identifying assumption that all of the volatility in the EU rate observed in the data is due to the theory of cyclical fluctuations proposed in this paper. Finally, we want to explore the welfare properties of the equilibrium and, in particular, address the question of whether the coordination of firms' firings is welfare enhancing or not.

References

[1] Beaudry, P., and F. Portier. 2004. "Exploring Pigou's Theory of Cycles." *Journal of Monetary Economics*, 51: 1183-1216.

[2] Benhabib, J., and R. Farmer. 1994. "Indeterminacy and Increasing Returns." *Journal of Economic Theory* 63: 19–41.

[3] Burdett, K., S. Shi, and R. Wright. 2001. "Pricing and Matching with Frictions." *Journal of Political Economy* 109: 1060-85.

[4] Cass, D., and K. Shell. 1983. "Do Sunspots Matter?" *Journal of Political Economy* 91: 193-227.

[5] Clementi, G., and H. Hopenhayn. 2006. "A Theory of Financing Constraints and Firm Dynamics." *Quarterly Journal of Economics* 121: 229-65.

[6] Cooper, R., and A. John. 1988. "Coordinating Coordination Failures in Keynesian Models." *Quarterly Journal of Economics*, 103: 441–63.

[7] Diamond, P. 1982. "Aggregate Demand Management in Search Equilibrium." *Journal of Political Economy* 90: 881–94.

[8] Diamond, P., and D. Fudenberg. 1989. "Rational Expectations Business Cycles in Search Equilibrium." *Journal of Political Economy* 97: 606–19.

[9] Farmer, R., and J. Guo. 1994. "Real Business Cycles and the Animal Spirits Hypothesis." *Journal of Economic Theory*, 63: 42–72.

[10] Fujita, S., and G. Ramey. 2009. "The Cyclicality of Separation and Job Finding Rates." *International Economic Review* 50: 415–30.

[11] Galí, J., and T. van Rens. 2014. "The Vanishing Procyclicality of Labor Productivity." Manuscript, University of Warwick.

[12] Haefke, C., M. Sonntag, and T. van Rens, 2013. "Wage Rigidity and Job Creation." *Journal of Monetary Economics.*

[13] Hagedorn, M., and I. Manovskii. 2008. "The Cyclical Behavior of Unemployment and Vacancies Revisited." *American Economic Review* 98: 1692–706.

[14] Hall, R., 2005. "Employment Fluctuations with Equilibrium Wage Stickiness." *American Economic Review* 95: 50-65.

[15] Heller, W. 1986. "Coordination Failure Under Complete Markets with Applications to Effective Demand." *Equilibrium analysis: Essays in Honor of Kenneth J. Arrow*, 2: 155–175, 1986.

[16] Jaimovich, N., and S. Rebelo. 2009. "Can News About the Future Drive Business Cycles?" *American Economic Review* 99: 1097-1118.

[17] Kaplan, G., and G. Menzio. 2014. "Shopping Externalities and Self-Fulfilling Unemployment Fluctuations." *Journal of Political Economy*, Forthcoming.

[18] Kennan, J. 2010. "Private Information, Wage Bargaining and Employment Fluctuations." *Review of Economic Studies* 77: 633-64.

[19] Kydland, F., and E. Prescott. 1982. "Time to Build and Aggregate Fluctuations." *Econometrica* 50: 1345-70.

[20] Menzio, G. 2005. "High Frequency Wage Rigidity." Manuscript, Northwestern University.

[21] Menzio, G., and S. Shi. 2010. "Block Recursive Equilibria for Stochastic Models of Search on the Job." *Journal of Economic Theory* 145: 1453-94.

[22] Menzio, G., and S. Shi. 2011. "Efficient Search on the Job and the Business Cycle." *Journal of Political Economy* 119: 468–510.

[23] Mortensen, D. 1999. "Equilibrium Unemployment Dynamics." *International Economic Review*, 40: 889–914.

[24] Mortensen, D., and C. Pissarides. 1994. "Job Creation and Job Destruction in the Theory of Unemployment." *Review of Economic Studies* 61: 397–415.

[25] Petrongolo, B., and C. Pissarides. 2001. "Looking into the Black Box: A Survey of the Matching Function." *Journal of Economic Literature* 39: 390-431.

[26] Pissarides, C. 1985. "Short-Run Equilibrium Dynamics of Unemployment, Vacancies and Real Wages." *American Economic Review* 75: 676–90.

[27] Shimer, R. 2005. "The Cyclical Behavior of Unemployment and Vacancies." *American Economic Review* 95: 25–49.

[28] Stokey, N., R. Lucas, and E. Prescott, "*Recursive Methods in Economic Dynamics.*" Harvard University Press, Cambridge, Massachusetts, 1989.

Appendix

A Proof of Lemma 5 and 6

Proof of Lemma 5: To alleviate notation, let w_0 denote $w(u_0)$ and w_1 denote $w(u_1)$. First, we establish that $w_0 > w_1$. To this aim, notice that

$$\begin{aligned} & \upsilon'(w_0)\left[E[y|e=1]-w_0+J^+(u_1)\right] \\ \leq \;& \upsilon'(w_0)E[y|e=1]-w_0+J^+(u_0)]+\overline{\upsilon}'\overline{D}_{J+}(u_1-u_0). \end{aligned} \tag{A.1}$$

Also, notice that

$$\begin{aligned} & \upsilon(w_0)-\upsilon(b)-c+V^+(u_1) \\ \geq \;& \upsilon(w_0)-\upsilon(b)-c+V^+(u_0)+\underline{D}_{V+}(u_1-u_0) \\ = \;& \upsilon'(w_0)E[y|e=1]-w_0+J^+(u_0)+\underline{D}_{V+}(u_1-u_0) \\ > \;& \upsilon'(w_0)E[y|e=1]-w_0+J^+(u_0)+\overline{\upsilon}'\overline{D}_{J+}(u_1-u_0), \end{aligned} \tag{A.2}$$

where the third line follows from (27) and the last line follows from $\underline{D}_{V_+} > \overline{\upsilon}'\overline{D}_{J_+}$. Taken together (A.1) and (A.2) imply that $\upsilon(w_0)-\upsilon(b)-c+V^+(u_1)$ is strictly greater than $\upsilon'(w_0)\left[E[y|e=1]-w_0+J^+(u_1)\right]$. Therefore, the left-hand side of (27) is strictly smaller than the right-hand side of (27) when evaluated at $w=w_0$ and $u=u_1$. Since the left-hand of (27) is strictly increasing in w and the left-hand side is strictly decreasing in w, w_1 is strictly smaller than w_0.

Next, we derive lower and upper bounds on w_0-w_1. From (27), it follows that

$$\begin{aligned} & \upsilon(w_0)-\upsilon(w_1)+V^+(u_0)-V^+(u_1) \\ = \;& \upsilon'(w_0)\left[E[y|e=1]-w_0+J^+(u_0)\right]-\upsilon'(w_1)\left[E[y|e=1]-w_0+J^+(u_0)\right] \\ + \;& \upsilon'(w_1)\left[E[y|e=1]-w_0+J^+(u_0)\right]-\upsilon'(w_1)\left[E[y|e=1]-w_1+J^+(u_1)\right] \end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned} & w_0-w_1 \\ = \;& \left[\frac{V^+(u_1)-V^+(u_0)}{\upsilon'(w_1)}\right]-\left[J^+(u_1)-J^+(u_0)\right] \\ - \;& \left[\frac{\upsilon(w_0)-\upsilon(w_1)}{\upsilon'(w_1)}\right]-\left[\frac{\upsilon'(w_1)-\upsilon'(w_0)}{\upsilon'(w_1)}\left(E[y|e=1]-w_0+J^+(u_0)\right)\right]. \end{aligned} \tag{A.3}$$

The first term in square brackets on the right-hand side of (A.3) is greater than $(\underline{D}_{V+}/\overline{\upsilon}')(u_1-u_0)$ and smaller than $(\overline{D}_{V+}/\underline{\upsilon}')(u_1-u_0)$. The second term in square brackets is greater than $\underline{D}_{J+}(u_1-u_0)$ and smaller than $\overline{D}_{J+}(u_1-u_0)$. The third term in square brackets is greater than zero and smaller than $w_1 - w_0$. The last term on the right-hand side of (A.3) is greater than zero and smaller than $\overline{\upsilon}'' \overline{D}_w \overline{J} / \underline{\upsilon}'(w_1 - w_0)$. From the above observations, it follows that

$$w_0 - w_1 \leq \left[\frac{\overline{D}_{V+}}{\underline{\upsilon}'} - \underline{D}_{J+} \right] (u_1 - u_0). \tag{A.4}$$

Similarly, we have

$$w_0 - w_1 \geq \left[2 + \frac{\overline{\upsilon}''}{\underline{\upsilon}'} \overline{D}_w \overline{J} \right]^{-1} \left[\frac{\underline{D}_{V+}}{\overline{\upsilon}'} - \overline{D}_{J+} \right] (u_1 - u_0). \tag{A.5}$$

The inequalities (A.4) and (A.5) represent the desired bounds on $w_0 - w_1$. ■

Proof of Lemma 6: Take an arbitrary $u \in [0, 1]$. To alleviate notation, let w_0 denote $w_0(u)$ and w_1 denote $w_1(u)$. From (27), it follows that

$$\begin{aligned}
\upsilon(w_0) - \upsilon(b) - c + V_0^+(u) - \upsilon'(w_0) \left[E[y|e = 1] - w_0 + J_0^+(u) \right] = 0, \\
\upsilon(w_1) - \upsilon(b) - c + V_1^+(u) - \upsilon'(w_1) \left[E[y|e = 1] - w_1 + J_1^+(u) \right] = 0.
\end{aligned}$$

Subtracting the second equation from the first, we obtain

$$\begin{aligned}
& V_0^+(u) - V_1^+(u) + \upsilon(w_0) - \upsilon(w_1) \\
+ \quad & \upsilon'(w_0) \left[J_1^+(u) - J_0^+(u) + w_0 - w_1 \right] \\
+ \quad & \left[\upsilon'(w_1) - \upsilon'(w_0) \right] \left[E[y|e = 1] - w_1 + J_1^+(u) \right] = 0.
\end{aligned}$$

Suppose without loss in generality that $w_0 \geq w_1$. We can rewrite the above equation as

$$\begin{aligned}
w_0 - w_1 \quad &= \left[J_0^+(u) - J_1^+(u) \right] + \left[\frac{V_1^+(u) - V_0^+(u)}{\upsilon'(w_0)} \right] + \left[\frac{\upsilon(w_1) - \upsilon(w_0)}{\upsilon'(w_0)} \right] \\
&+ \left[E[y|e = 1] - w_1 + J_1^+(u) \right] \left[\frac{\upsilon'(w_0) - \upsilon'(w_1)}{\upsilon'(w_0)} \right].
\end{aligned} \tag{A.6}$$

The first term in square brackets on the right-hand side of (A.6) is strictly smaller than κ. The second term in square brackets is strictly smaller than $\kappa / \underline{\upsilon}'$. The third term in square brackets is strictly negative. The last term is strictly negative. Hence, we have

$$0 \leq w_0 - w_1 < (1 + 1/\underline{\upsilon}')\kappa.$$

Since the above inequality holds for any $u \in [0, 1]$, we conclude that $\|w_0 - w_1\| < \alpha_w \kappa$ where $\alpha_w = (1 + 1/\underline{\upsilon}')$. ■

B Proof of Lemma 7 and 8

Proof of Lemma 7: To simplify notation, let w_0 denote $w(u_0)$ and w_1 denote $w(u_1)$. Then the difference $J(u_1) - J(u_0)$ is given by

$$J(u_1) - J(u_0) = w_0 - w_1 + J^+(u_1) - J^+(u_0).$$

From the above expression, it follows that

$$\begin{aligned} J(u_1) - J(u_0) &\geq (\underline{D}_w + \underline{D}_{J^+})(u_1 - u_0), \\ J(u_1) - J(u_0) &\leq (\overline{D}_w + \overline{D}_{J^+})(u_1 - u_0). \end{aligned} \tag{B.1}$$

The difference $V(u_1) - V(u_0)$ is given by

$$\begin{aligned} V(u_1) - V(u_0) &= \upsilon'(w_1)J(u_1) - \upsilon'(w_0)J(u_0) \\ &= [\upsilon'(w_1)J(u_1) - \upsilon'(w_1)J(u_0)] + [\upsilon'(w_1)J(u_0) - \upsilon'(w_0)J(u_0)] \end{aligned}$$

From the above expression, it follows that

$$\begin{aligned} V(u_1) - V(u_0) &\geq \underline{\upsilon}'(\underline{D}_w + \underline{D}_{J^+})(u_1 - u_0), \\ V(u_1) - V(u_0) &\leq \left[\overline{\upsilon}'(\overline{D}_w + \overline{D}_{J^+}) + \overline{\upsilon}''\overline{D}_w\overline{J}\right](u_1 - u_0). \end{aligned} \tag{B.2}$$

Lemma 7 follows directly from the inequalities in (B.1) and (B.2). ■

Proof of Lemma 8: Take an arbitrary $u \in [0, 1]$. The difference $J_0(u) - J_1(u)$ is such that

$$\begin{aligned} |J_0(u) - J_1(u)| &\leq |w_0(u) - w_1(u)| + \left|J_0^+(u) - J_1^+(u)\right| \\ &< (\alpha_w + 1)\kappa. \end{aligned} \tag{B.3}$$

The difference $V_0(u) - V_1(u)$ is such that

$$\begin{aligned} |V_0(u) - V_1(u)| &\leq |\upsilon(w_0(u)) - \upsilon(w_1(u))| + \left|V_0^+(u) - V_1^+(u)\right| \\ &< (\overline{\upsilon}'\alpha_w + 1)\kappa. \end{aligned} \tag{B.4}$$

Since the inequalities (B.3) and (B.4) hold for all $u \in [0, 1]$, we conclude that $\|J_0 - J_1\| < \alpha_{J'}\kappa$ and $\|V_0 - V_1\| < \alpha_{V'}\kappa$, where $\alpha_J = \alpha_w + 1$ and $\alpha_V = \overline{\upsilon}'\alpha_w + 1$. ■

C Proof of Lemma 9 and 10

Proof of Lemma 9: Let $\hat{u}_1$ denote the solution to (35) for $u = u_1$ and let $\hat{u}_0$ denote the solution to (35) for $u = u_0$, i.e.

$$\begin{aligned} \hat{u}_1 &= u_1 - u_1\mu(J(\hat{u}_1), u_1) + (1 - u_1)[\delta + (1 - \delta)p_\ell(1)s(y_\ell, \hat{z}')], \\ \hat{u}_0 &= u_0 - u_0\mu(J(\hat{u}_0), u_0) + (1 - u_0)[\delta + (1 - \delta)p_\ell(1)s(y_\ell, \hat{z}')]. \end{aligned}$$

Subtracting the second equation from the first one, we obtain

$$\begin{aligned}\hat{u}_1 - \hat{u}_0 \quad &= (u_1 - u_0)(1-\delta)[1 - p_\ell(1)s(y_\ell, \hat{z}')] \\ &+ u_0 \left[\mu\left(J(\hat{u}_0), u_0\right) - \mu\left(J(\hat{u}_1), u_0\right)\right] \\ &+ u_0 \left[\mu\left(J(\hat{u}_1), u_0\right) - \mu\left(J(\hat{u}_1), u_1\right)\right] \\ &+ u_0 \mu\left(J(\hat{u}_1), u_1\right) - u_1 \mu\left(J(\hat{u}_1), u_1\right).\end{aligned} \tag{C.1}$$

The first line on the right-hand side of (C.1) is positive. Since $\mu(J, u)$ is increasing in J, J is increasing in u and, as established in the main text, $\hat{u}_1 > \hat{u}_0$, the second line on the right-hand side of (C.1) is negative. Since $\mu(J, u)$ is decreasing in u and $u_1 > u_0$, the third line on the right-hand side of (C.1) is positive. The fourth line on the right-hand side of (C.1) is obviously negative. Hence, an upper bound on $\hat{u}_1 - \hat{u}_0$ is given by

$$\begin{aligned}\hat{u}_1 - \hat{u}_0 \quad &\leq (u_1 - u_0)(1-\delta)[1 - p_\ell(1)s(y_\ell, \hat{z}')] + u_0 \left[\mu\left(J(\hat{u}_1), u_0\right) - \mu\left(J(\hat{u}_1), u_1\right)\right] \\ &\leq (u_1 - u_0)\left[1 - \delta + \overline{\mu}_u\right].\end{aligned}$$

Combining the above inequality with $\hat{u}_1 < \hat{u}_0$, we obtain

$$\underline{D}_g \left(u_1 - u_0\right) < \hat{u}_1 - \hat{u}_0 \leq \overline{D}_g \left(u_1 - u_0\right),$$

where

$$\underline{D}_g = 0, \quad \overline{D}_g = 1 - \delta + \overline{\mu}_u. \quad \blacksquare$$

Proof of Lemma 10: Take an arbitrary $u \in [0, 1]$ and $\hat{z}' \in \{B, G\}$. Let $\hat{u}_0$ denote $g_0(u, \hat{z}')$ and $\hat{u}_1$ denote $g_1(u, \hat{z}')$. From (35), it follows that

$$\begin{aligned}\hat{u}_0 &= u - u\mu(J_0(\hat{u}_0), u) + (1-u)[\delta + (1-\delta)p_\ell(1)s(y_\ell, \hat{z}')], \\ \hat{u}_1 &= u - u\mu(J_1(\hat{u}_1), u) + (1-u)[\delta + (1-\delta)p_\ell(1)s(y_\ell, \hat{z}')].\end{aligned}$$

Without loss in generality suppose that $\hat{u}_0 \geq \hat{u}_1$. In this case,

$$\hat{u}_0 - \hat{u}_1 = u\left\{[\mu(J_1(\hat{u}_1), u) - \mu(J_0(\hat{u}_1), u)] + [\mu(J_0(\hat{u}_1), u) - \mu(J_0(\hat{u}_0), u)]\right\}. \tag{C.2}$$

The term $\mu(J_0(\hat{u}_1), u) - \mu(J_0(\hat{u}_0), u)$ on the right-hand side of (C.2) is negative as $\mu(J, u)$ is increasing in J, J_0 is increasing in u and $\hat{u}_0 \geq \hat{u}_1$. Therefore, we have

$$\hat{u}_0 - \hat{u}_1 \leq u\left[\mu(J_1(\hat{u}_1), u) - \mu(J_0(\hat{u}_1), u)\right] < \overline{\mu}_J \alpha_J \kappa.$$

Since the above inequality holds for any $u \in [0, 1]$ and any $\hat{z}' \in \{B, G\}$, we conclude that $\|g_0 - g_1\| < \alpha_g \kappa$, where $\alpha_g = \overline{\mu}_J \alpha_J$. $\blacksquare$

D Proof of Lemma 11, 12 and 13

Proof of Lemma 11: Take arbitrary $u_0, u_1 \in [0,1]$ with $u_1 > u_0$. To simplify notation, let $\hat{u}_{0,\hat{z}'}$ denote $g(u_0, \hat{z}')$ and let $\hat{u}_{1,\hat{z}'}$ denote $g(u_1, \hat{z}')$. Similarly, let $\pi_{0,\hat{z}'}$ denote the probability that the realization of the sunspot is $\hat{z}' \in \{B, G\}$ defined by using $g(u_0, B)$ in (41), and let $\pi_{1,\hat{z}'}$ denote the probability that the realization of the sunspot is $\hat{z}' \in \{B, G\}$ defined by using $g(u_1, B)$ in (41). Using this notation and (39), we can write the difference $V^{+\prime}(u_1) - V^{+\prime}(u_0)$ as

$$\begin{aligned} & V^{+\prime}(u_1) - V^{+\prime}(u_0) \\ = \;& \beta \textstyle\sum_{\hat{z}'} \{(\pi_{1,\hat{z}} - \pi_{0,\hat{z}}) \left[(1-\delta)(1 - p_\ell(1)s(y_\ell, \hat{z}')) - \mu(J(\hat{u}_{1,\hat{z}}, u_1)\right] V(\hat{u}_{1,\hat{z}}) \\ + \;& \pi_{0,\hat{z}}[(1-\delta)(1 - p_\ell(1)s(y_\ell, \hat{z}')) - \mu(J(\hat{u}_{1,B}, u_1)] \left[V(\hat{u}_{1,\hat{z}}) - V(\hat{u}_{0,\hat{z}})\right] \\ + \;& \pi_{0,\hat{z}} V(\hat{u}_{0,\hat{z}})[\mu(J(\hat{u}_{0,B}, u_0) - \mu(J(\hat{u}_{0,B}, u_1)]. \\ + \;& \pi_{0,\hat{z}} V(\hat{u}_{0,\hat{z}})[\mu(J(\hat{u}_{0,B}, u_1) - \mu(J(\hat{u}_{1,B}, u_1)]\} \end{aligned} \tag{D.1}$$

The first term on the right-hand side of (D.1) is negative. In absolute value, this term is smaller than $c\overline{D}_V \overline{D}_g \overline{V}(u_1 - u_0)/[\beta(p_h(1) - p_h(0))(1-\delta)\underline{V}^2]$. The second term on the right-hand side of (D.1) is geater than zero and smaller than $\pi_{0,\hat{z}}(1-\delta)\overline{D}_V \overline{D}_g(u_1 - u_0)$. The third term on the right-hand side of (D.1) is positive, greater than $\pi_{0,\hat{z}}\underline{V}\underline{\mu}_u(u_1 - u_0)$ and smaller than $\pi_{0,\hat{z}}\overline{V}\overline{\mu}_u(u_1 - u_0)$. The last term on the right-hand side of (D.1) is negative. In absolute value, this term is smaller than $\pi_{0,\hat{z}}\overline{V}\overline{\mu}_J \overline{D}_J \overline{D}_g(u_1 - u_0)$. Overall, we have

$$V^{+\prime}(u_1) - V^{+\prime}(u_0) \geq \left[\beta \underline{V}(\underline{\mu}_u - \overline{\mu}_J \overline{D}_J \overline{D}_g) - \frac{2c\overline{D}_V \overline{D}_g \overline{V}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2}\right](u_1 - u_0), \tag{D.2}$$

and

$$V^{+\prime}(u_1) - V^{+\prime}(u_0) \leq \beta \left[\overline{V}\overline{\mu}_u + (1-\delta)\overline{D}_V \overline{D}_g\right](u_1 - u_0). \tag{D.3}$$

Using (40), we can write the difference $J^{+\prime}(u_1) - J^{+\prime}(u_0)$ as

$$\begin{aligned} & J^{+\prime}(u_1) - J^{+\prime}(u_0) \\ = \;& \beta \textstyle\sum_{\hat{z}'} \{(\pi_{1,\hat{z}} - \pi_{0,\hat{z}}) \left[(1-\delta)(1 - p_\ell(1)s(y_\ell, \hat{z}'))\right] J(\hat{u}_{1,\hat{z}}) \\ + \;& \pi_{0,\hat{z}}[(1-\delta)(1 - p_\ell(1)s(y_\ell, \hat{z}'))] \left[J(\hat{u}_{1,\hat{z}}) - J(\hat{u}_{0,\hat{z}})\right]. \end{aligned} \tag{D.4}$$

The first term on the right-hand side of (D.4) is negative. In absolute value, this term is smaller than $c\overline{D}_V \overline{D}_g \overline{J}(u_1 - u_0)/[\beta(p_h(1) - p_h(0))(1-\delta)\underline{V}^2]$. The second term on the right-hand side of (D.4) is geater than zero and smaller than $\pi_{0,\hat{z}}(1-\delta)\overline{D}_J \overline{D}_g(u_1 - u_0)$. Overall, we have

$$J^{+\prime}(u_1) - J^{+\prime}(u_0) \geq -\frac{2c\overline{D}_V \overline{D}_g \overline{J}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2}(u_1 - u_0), \tag{D.5}$$

and

$$J^{+\prime}(u_1) - J^{+\prime}(u_0) \leq \beta(1-\delta)\overline{D}_J\overline{D}_g(u_1 - u_0). \tag{D.6}$$

Lemma 11 follows directly from the inequalities in (D.2)-(D.3) and (D.5)-(D.6). ■

Proof of Lemma 12: Set $\underline{D}'_{V+}$ and $\overline{D}'_{V+}$ in (43) equal to $\underline{D}_{V+}$ and $\overline{D}_{V+}$, and $\underline{D}'_{J+}$ and $\overline{D}'_{J+}$ in (44) equal to $\underline{D}_{J+}$ and $\overline{D}_{J+}$. Then solve for $\underline{D}_{V+}$, $\overline{D}_{V+}$, $\underline{D}_{J+}$ and $\overline{D}_{J+}$. It is immediate to verify that, since $\underline{\mu}_u - \overline{\mu}_J(1-\delta+\overline{\mu}_u) > 0$, $\overline{D}_{V_+} > \underline{D}_{V_+} > 0$, $\overline{D}_{J_+} > 0 \geq \underline{D}_{J_+}$ and $\underline{D}_{V_+} > \overline{v}'(\overline{D}_{J_+} - \underline{D}_{J_+})$ for β and c small enough. ■

Proof of Lemma 13: Take an arbitrary $u \in [0,1]$. To simplify notation, let $\hat{u}_{0,\hat{z}'}$ denote $g_0(u,\hat{z}')$ and let $\hat{u}_{1,\hat{z}'}$ denote $g_1(u,\hat{z}')$. Similarly, let $\pi_{0,\hat{z}'}$ denote the probability that the realization of the sunspot is $\hat{z}' \in \{B,G\}$ defined by using V_0 and g_0 in (41), and let $\pi_{1,\hat{z}'}$ denote the probability that the realization of the sunspot is $\hat{z}' \in \{B,G\}$ defined by using V_1 and g_1 in (41). Using this notation and (39), we can write the difference $V_0^{+\prime}(u) - V_1^{+\prime}(u)$ as

$$\begin{aligned} & V_0^{+\prime}(u) - V_1^{+\prime}(u) \\ = \; & \beta \textstyle\sum_{\hat{z}'} \{ (\pi_{0,\hat{z}} - \pi_{1,\hat{z}}) \left[(1-\delta)(1 - p_\ell(1)s(y_\ell,\hat{z}')) - \mu(J_0(\hat{u}_{0,\hat{z}},u) \right] V_0(\hat{u}_{0,\hat{z}}) \\ + \; & \pi_{1,\hat{z}}[(1-\delta)(1-p_\ell(1)s(y_\ell,\hat{z}')) - \mu(J_0(\hat{u}_{0,\hat{z}},u)] \left[V_0(\hat{u}_{0,\hat{z}}) - V_1(\hat{u}_{1,\hat{z}}) \right] \\ + \; & \pi_{1,\hat{z}} V_1(\hat{u}_{1,\hat{z}}) [\mu(J_1(\hat{u}_{1,\hat{z}},u) - \mu(J_0(\hat{u}_{0,\hat{z}},u)] \}. \end{aligned} \tag{D.7}$$

From (D.7), it follows that the absolute value of the difference $V_0^{+\prime}(u) - V_1^{+\prime}(u)$ is such that

$$\begin{aligned} & \left| V_0^{+\prime}(u) - V_1^{+}(u) \right| \\ \leq \; & \frac{2c\overline{V}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2} \left(\alpha_V + \overline{D}_V \alpha_g \right) \kappa + \left(\alpha_V + \overline{D}_V \alpha_g \right) \kappa + \overline{V}\overline{\mu}_J \left(\alpha_J + \overline{D}_J \alpha_g \right) \kappa, \end{aligned} \tag{D.8}$$

where the first term on the right-hand side of (D.8) is an upper bound on the absolute value on the first line on the right-hand side of (D.7), the second term on the right-hand side of (D.8) is an upper bound on the absolute value on the second line on the right-hand side of (D.7), and the last term on the right-hand side of (D.8) is an upper bound on the absolute value on the last line on the right-hand side of (D.7).

Using (40) we can write the difference $J_0^{+\prime}(u) - J_1^{+\prime}(u)$ as

$$\begin{aligned} & J_0^{+\prime}(u) - J_1^{+\prime}(u) \\ = \; & \beta \textstyle\sum_{\hat{z}'} \{ (\pi_{0,\hat{z}} - \pi_{1,\hat{z}}) (1-\delta)(1 - p_\ell(1)s(y_\ell,\hat{z}')) J_0(\hat{u}_{0,\hat{z}}) \\ + \; & \pi_{1,\hat{z}}(1-\delta)(1 - p_\ell(1)s(y_\ell,\hat{z}')) \left[J_0(\hat{u}_{0,\hat{z}}) - J_1(\hat{u}_{1,\hat{z}}) \right]. \end{aligned} \tag{D.9}$$

From (D.9), it follows that the absolute value of the difference $J_0^{+\prime}(u) - J_1^{+\prime}(u)$ is such that

$$\begin{aligned} & \left| J_0^{+\prime}(u) - J_1^{+\prime}(u) \right| \\ \leq \; & \frac{2c\overline{J}}{(p_h(1) - p_h(0))(1-\delta)\underline{V}^2} \left(\alpha_V + \overline{D}_V \alpha_g \right) \kappa + \left(\alpha_J + \overline{D}_J \alpha_g \right) \kappa, \end{aligned} \tag{D.10}$$

where the first term on the right-hand side of (D.10) is an upper bound on the absolute value on the first line on the right-hand side of (D.9), and the second term on the right-hand side of (D.10) is an upper bound on the absolute value on the second line on the right-hand side of (D.9).

Since the inequalities (D.8) and (D.10) hold for any $u \in [0,1]$, we conclude that $\left\| V_0^{+\prime} - V_1^{+\prime} \right\| < \alpha_{V+}\kappa$ and $\left\| J_0^{+\prime} - J_1^{+\prime} \right\| < \alpha_{J+}\kappa$, where $\alpha_{V+}\kappa$ denotes the right-hand side of (D.8) and $\alpha_{J+}\kappa$ denotes the right-hand side of (D.10) . ■