The Marginal Cost of Risk and Capital Allocation in a Multi-Period Model*

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January 2015

Abstract

Capital allocation is used widely within the financial industry for purposes of pricing and performance measurement. In this paper, we consider the theoretical impact of extending the canonical model of a profit maximizing insurer beyond a single period and, in particular, consider the effect of having opportunities to raise external financing in future periods. We show that in this setting, outside of special cases, capital allocation as currently conceived cannot be done in a way to produce prices consistent with marginal cost. We go on to evaluate economically correct capital allocations based on the model in the context of an international catastrophe reinsurer with four lines of business. We find that traditional techniques can be resurrected with 1) an appropriate definition of capital, 2) a broader conception of the cost of underwriting, and 3) an appropriate risk measure connected to the fundamentals of the underlying business.

JEL classification: G20, G22
Keywords: capital allocation; profit maximization; dynamic programming; Return On Risk-Adjusted Capital.

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*We gratefully acknowledge funding from the *Casualty Actuarial Society* (CAS) under the project "Allocation of Costs of Holding Capital," and an anonymous reinsurance company for supplying the data. We are grateful for helpful comments from Richard Derrig, John Gill, Ming Li, Glenn Meyers, Stephen Mildenhall, Kailan Shang, as well as from seminar participants at the 2014 CAS Centennial Meeting, the International Congress of Actuaries 2014, the 2014 Congress on Insurance: Mathematics and Economics, and Georgia State University. The usual disclaimer applies.

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1 Introduction

The *Return on Risk-Adjusted Capital* (RORAC) ratio—where the expected return associated with a risk exposure is divided by an allocation of risk capital and compared to a target rate of return—is widely used by financial institutions for pricing and performance measurement. However, despite ease of implementation, RORAC has been criticized for being inconsistent with value maximization and, in particular, failing to provide an accurate picture of the institution's effective risk aversion. We revisit this issue by extending the canonical one-period model of an insurance company to multiple periods. In particular, we allow for opportunities for raising external financing. We solve for the marginal cost of risk in this dynamic setting and then reconcile the optimality conditions with RORAC calculations.

We confirm earlier findings that RORAC ratios as typically conceived will not, in general, yield results consistent with portfolio optimization in a multiperiod model. However, we show that the practice can be resurrected by generalizing the notions of *return*, *capital*, and the *cost of capital*. The necessary generalizations follow from the multiperiod context and entail 1) adjusting the expected return for the impact of the associated risk on the future value of the firm, 2) broadening the notion of "capital" beyond economic capital to include all financial resources that could conceivably be tapped by the firm, and 3) adjusting the target rate of return to reflect the net cost of raising the marginal unit of capital.

The paper is divided into two longer sections. Section 2 describes the general multi-period insurance model whereas Section 3 presents its implementation and corresponding results using data from a catastrophe reinsurer.

The cost of risk in the model introduced in Section 2 reflects two important influences. First, less risky insurers are able to charge higher prices for insurance coverage due to the risk aversion of their customers. Second, greater risk produces a higher probability of financial distress, which brings the burden of costly external financing and potential default. In the event of default, the owners lose their claim to future profit flows. These two influences create risk aversion at the level of the company and motivate the holding of capital despite its carrying cost.

As detailed in Section 2.1, the company maximizes value by choosing its participation in covering various risks, and its capital raising and shedding (dividend) decisions. The optimization problem yields a Bellman equation, where firm value is a function of the current capital level. At any point in time, the firm may be over- or under-capitalized: Too little capital in the firm leads the company to forego profitable business opportunities, whereas too much capital is too costly relative to (decreasing) profit margins. This is also reflected in the optimal capital raising decision: A meagerly capitalized company will raise funds whereas a high-capital firm will shed by paying dividends. In the case of an undercapitalized firm, the adjustment process is gradual due to convex costs of raising funds.

We go on in Section 2.2 to derive the marginal cost of each risk in the company's portfolio, from which the RORAC of the risk can be generated. Marginal cost in a simple one-period insurance model consists of two parts—a marginal actuarial cost and a risk charge that can be interpreted as a capital allocation times a cost of capital. The RORAC ratio can then be calculated by deducting the marginal actuarial cost from the price and dividing by the allocated capital, with the cost of capital being the hurdle rate for the exposure.[1]

[1] If the portfolio is optimized, the RORACs in all lines will of course be equal to the hurdle rate.

A similar calculation can be recovered in our analysis of the multiperiod model, but adjustments are necessary. In addition to the actuarial component, the marginal cost of an exposure now reflects the influence of risk on the expected costs associated with external financing in the future (in the event such financing is needed to avert default) and the influence of risk on the continuation value of the firm. These new components must be reflected in the numerator of the RORAC ratio when calculating the expected return of the exposure.[2] The denominator of the RORAC ratio also requires adjustment, as the basis of the "capital" allocation that emerges from the optimality conditions no longer corresponds to familiar definitions of capital from the literature. Rather than referring to accounting surplus or economic capital, "capital" here includes all of the company's resources (including untapped ones). Moreover, the relevant capital cost or hurdle rate refers to the *net* marginal cost of raising an additional dollar of equity.

Thus, RORAC techniques may be resurrected *if* 1) an appropriate definition of capital is used, 2) the allocation of that capital flows from the economic context in which the firm is operating (rather than from imposition of an arbitrary risk measure), and 3) a broader conception of the cost of underwriting business—including, for example, penalties for the impact of risk on the continuation value of the firm, as well as an allocation for future costs of external financing—is used when evaluating the profitability of a contract. However, pinpointing the underlying parameters and quantities is difficult (if not impossible) without formulating and solving the firm's optimization problem.

The practical exercise of solving the optimization problem is tackled in Section 3 in the context of a 4-line catastrophe reinsurer under several sets of parameterizations (based on industry data where possible). We find that the value of the firm as a function of capital is concave with an optimal capitalization point that trades off profitability and (re-)capitalization costs. The optimal raising decision essentially pushes capital to the optimal point, although it is rigid in the area around the optimum due to a difference in the cost of shedding (nil) and raising (positive) capital. The optimal risk portfolio increases convexly up to a saturation point after which the portfolio is kept constant and excess capital is shed (down to the saturation point).

We then calculate allocations of the capital to different risks and different components. While capital costs typically present the largest component after expected claims costs, the other parts are significant and when combined may overshadow capital costs. We compare the resulting (correct) RORACs to RORACs calculated using conventional techniques, revealing that failure to make the adjustments discussed above can lead to large errors in performance measurement. In particular, relative errors—in the sense of failing to properly rank the profitability of the lines—occur due to failure to account for all components of marginal cost, as well as failure to allocate capital in a manner consistent with the economic context faced by the firm.

Extensions and adaptations to other financial institutions are possible, and we describe some possible refinements in the final section of the paper. However, these do not change the main message of the paper: As we consider more realistic models of firm value, choosing an optimal portfolio, pricing, and measuring risk-adjusted performance require us to think carefully about defining capital and accounting for the impact of risk on the firm. Standard allocation techniques are not necessarily inconsistent with this guidance, but they can only be correctly implemented

[2]These modifications can also be interpreted as a risk adjustment of the return as within a *Risk-Adjusted Return on Capital* (RAROC). Indeed, since risk adjustments enter the numerator as well as the denominator, it may be more accurate to refer to the ratio as a *Risk-Adjusted Return on Risk-Adjusted Capital* (RARAC).

if the user has a thorough understanding of the economic context in which the firm is operating. While allocation can be deployed thoughtfully, we see no alternative to careful consideration of the firm's objectives, constraints, and institutional context in making optimal decisions.

Relationship to the Literature

The findings of this paper relate most closely to the discussion of RAROC/RORAC found in Froot and Stein (1998) and Stoughton and Zechner (2007). Froot and Stein, in a multi-period model, first observed the pitfalls of reconciling RAROC with correct measures of performance and raised questions about its economic foundation. Stoughton and Zechner (2007), on the other hand, spelled out an economic foundation for RAROC in the context of a single-period model with asymmetric information. Their recommended adjustments to the numerator of RAROC reflect their focus on particular cost sources but are conceptually similar to our adjustments in the sense of illustrating that a broad consideration of the impact of a risk exposure is necessary when calculating expected returns. However, there are at least two important points of clarification offered by our paper's extension of the economic foundation to a multi-period setting. First, we show the basis for capital allocation is no longer necessarily the "economic capital" advocated by Stoughton and Zechner. Second, the hurdle rate is no longer a simple carrying cost of equity or debt, as in their work: Instead, more complex models of external financing make it necessary to consider both the marginal costs *and* benefits of equity raised when calculating a hurdle rate.

The paper also relates to the literature on capital allocation techniques. The prevalence of capital allocation in banking and insurance practice has triggered a large number of contributions in the actuarial science and applied banking literatures on capital allocation methods (Denault, 2001; Tasche, 2004; Tsanakas and Barnett, 2003; Dhaene, Gooverts, and Kaas, 2003; Venter, 2004; Mildenhall, 2004; Albrecht, 2004; Kalkbrener, 2005; Powers, 2007, among others). We refer to the Bauer and Zanjani (2014) for a discussion of the different techniques and illustration in the context of the catastrophe reinsurance data used in this paper.

The practice, however, inspires controversy on several levels. Some question its necessity (Phillips, Cummins, and Allen, 1998; Sherris, 2006). Others argue that it leads to economically suboptimal decisions (Venter, 2002; Gründl and Schmeiser, 2007). Various researchers have picked up on this disparity providing various vantage points why the allocation problem may be misguided or what the debate may be missing (Mango, 2005; Kreps, 2005; Venter, 2010; D'Arcy, 2011). As pointed out by Bauer and Zanjani (2013), there are three key elements for capital allocation to be meaningful: Stakeholders must care about the solvency of the company; markets must be incomplete; and capital must be costly. Allocations differ depending on how these features are embedded into an economic model of the firm. For instance, the focus can be on shareholders worried about future financing costs (Froot and Stein, 1998; Froot, 2007), a regulator assessing the value of the shortfall (Myers and Read, 2001; Erel, Myers, and Read, 2013), or the firm's counterparties concerned with their recoveries (Zanjani, 2002; Ibragimov, Jaffee and Walden, 2010; Bauer and Zanjani, 2015).

The current paper extends the latter strands of literature, particularly by considering multiple periods and various forms of raising capital. Our application compares the ensuing allocation results to conventional allocations as employed in industry—thereby connecting the economic and the applied literatures.

2 Capital Allocation and Profit Maximization: A Multi-Period Model

The canonical model for choosing an optimal insurance portfolio considers the maximization of profits subject to a risk measure constraint. In such a setting, marginal cost will match up with the gradient of the underlying risk measure—which in turn corresponds to the allocated capital within conventional *gradient allocations*. In this section, we reconsider the connection between the marginal cost of risk and the allocation of (risk) capital in a more general multi-period setting. Here, the key question is whether the marginal cost of risk can be reconciled *in theory* with capital allocation in a dynamic model of an insurance company. The next section then considers the *practical* implications by implementing the model in the context of a catastrophe reinsurer.

We start off in Section 2.1 by laying out the model framework, then Section 2.2 derives the marginal cost of risk for the insurance company. Section 2.3 provides a discussion of the results, particularly their implications for capital allocations. Technical derivations and alternative model versions are collected in Appendix A.1.

2.1 Profit Maximization Problem in a Multi-Period Model

Formally we consider an insurance company with N business lines and corresponding loss realizations $L_t^{(i)}$, $i = 1, 2, \ldots, N$, each period $t = 1, 2, \ldots$ These losses could be associated with certain perils, certain portfolios of contracts, or even individual contracts/costumers, depending on context of the question we are addressing: How to choose an optimal exposure among these N lines?

We assume that for fixed i, $L_1^{(i)}$, $L_2^{(i)}, \ldots$ are non-negative, independent, and identically distributed (iid) random variables. We make the iid assumption for convenience of exposition, and since it suits our application in Section 3. However, non-identical distributions arising from, e.g., claims inflation could be easily incorporated, and also extensions to serially correlated (e.g., auto-regressive) loss structures or loss payments developing over several years are feasible at the expense of a larger state space.

We also abstract from risky investments, so that all the uncertainty is captured by the losses; we define the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ that describes the information flow over time via $\mathcal{F}_t = \sigma(L_s^{(i)},\, i \in \{1, 2, \ldots, N\},\, s \leq t)$. However, generalizations with securities markets are possible at the expense of notational complication (Bauer and Zanjani, 2015).

At the beginning of every underwriting period t, the insurer chooses to underwrite certain portions of these risks and charges premiums $p_t^{(i)}$, $1 \leq i \leq N$, in return. More precisely, the underwriting decision corresponds to choosing an indemnity parameter $q_t^{(i)}$, so that the indemnity for loss i in period t is:

$$I_t^{(i)} = I_t^{(i)}(L_t^{(i)}, q_t^{(i)}),$$

where we require $I_t^{(i)}(0, q_t^{(i)}) = 0$, $i = 1, 2, \ldots, N$. For analytical convenience and again because it suits our setting in Section 3, we focus on choosing to underwrite a fraction of the risks, i.e., we assume:

$$I_t^{(i)} = I^{(i)}(L_t^{(i)}, q_t^{(i)}) = q_t^{(i)} \times L_t^{(i)},$$

although, here also, generalizations are possible. We denote the aggregate period-t loss by $I_t = \sum_i I_t^{(i)}$.

The company has the possibility to raise or shed (i.e., pay dividends) capital R_t^b at the beginning of the period at cost $c_1(R_t^b)$, $c_1(x) = 0$ for $x \leq 0$. Moreover, it can raise capital R_t^e, $R_t^e \geq 0$, at the end of the period—after losses have been realized—at a (higher) cost $c_2(R_t^e)$. Here we think of R_t^b as capital raised under normal conditions, whereas R_t^e is emergency capital raised under distressed conditions. In particular, we assume:

$$c_2(x) > c_1(y),\ x, y \geq 0, \tag{1}$$

i.e., raising capital under normal conditions is less costly than in distressed states. We also assume there exists a positive carrying cost for capital a_t within the company as a proportion τ of a_t. One may imagine τ to arise from cost of capital charges by the shareholders (sometimes referred to as the *average cost of capital*) and depreciation, whereas $c_1(\cdot)$ are costs arising from raising capital in the current period (which typically will be higher than τ).

Finally, the (constant) continuously compounded risk-free interest rate is denoted by r. Hence, the law of motion for the company's capital (budget constraint) is:

$$a_t = \left[a_{t-1} \times (1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)} \right] e^r + R_t^e - c_2(R_t^e) - \sum_{j=1}^{N} I_t^{(j)} \tag{2}$$

for $a_{t-1} \geq 0$. We require that:

$$R_t^b \geq -a_{t-1}(1-\tau), \tag{3}$$

i.e., the company cannot pay more in dividends than its capital (after capital costs have been deducted).

The company defaults if $a_t < 0$, which is equivalent to:

$$\left[a_{t-1} \times (1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)} \right] e^r + R_t^e - c_2(R_t^e) < \sum_{j=1}^{N} I_t^{(j)}.$$

Due to limited liability, in this case the company's funds are not sufficient to pay all the claims. We assume that the remaining assets in the firm are paid to claimants at the same rate per dollar of coverage, so that the recovery for policyholder i is:

$$\min \left\{ I_t^{(i)}, \frac{\left[a_{t-1} \times (1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)} \right] e^r + R_t^e - c_2(R_t^e)}{\sum_{j=1}^{N} I_t^{(j)}} \times I_t^{(i)} \right\}.$$

The premium the company is able to charge for providing insurance now depends on the riskiness of the coverage as well as the underwriting decision—that is, price is a function of demand as within an *inverse demand function*. Formally, this means that the total premium for line i, $p_t^{(i)}$, is a quantity known at time $t-1$ (i.e., it is $\mathbf{F}$-predictable) given by a functional relationship:

$$\mathcal{P}_i \left(a_{t-1}, R_t^b, R_t^e, (p_t^{(j)})_{1 \leq j \leq N}, (q_t^{(j)})_{1 \leq j \leq N} \right) = 0,\ 1 \leq i \leq N.$$

One way to specify this functional relationship that is in line with the micro-foundations of insurance is to consider a set of (binding) *participation constraints*:

$$\begin{aligned}\gamma_i \;=\; \mathbb{E}_{t-1}\Bigg[U_i\Bigg(& \left(\mathrm{w}_{t-1}^{(i)} - p_t^{(i)}\right) e^r - L_t^i \\ & + \min\left\{ I_t^{(i)}, \frac{\left[a_{t-1}(1-\tau) + R_t^b - c_1(R_t^b) + \sum_j p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e)}{\sum_j I_t^{(j)}} I_t^{(i)} \right\}\Bigg)\Bigg],\end{aligned} \tag{4}$$

where U_i is increasing and concave, and we call $\mathrm{w}_{t-1}^{(i)}$ *wealth* and γ_i *reservation utility*. This is the approach taken in Bauer and Zanjani (2015) in a simpler setting, and it ultimately relates capitalization and capital allocation to consumer concerns about company solvency. However, of course relating price and demand this way requires a great amount of supporting information for fixing the various inputs—so that relying on it for empirical applications is challenging.

Instead, in the context of our application, we rely on an alternative reduced-form specification that assumes premiums—as markups on discounted expected losses—are a function of the company's default probability. The underlying intuition is that consumers rely on insurance solvency ratings for making their insurance decisions, which in turn are closely related to default probabilities (see Section 3.2 for more details). More precisely, we set:

$$\begin{aligned}p_t^{(i)} \;=\; e^{-r}\,\mathbb{E}_{t-1}\left[I_t^{(i)}\right]\exp\Bigg\{ & \alpha - \gamma \mathbb{E}_{t-1}[I] \\ & -\beta\,\mathbb{P}_{t-1}\left(I_t > \left[a_{t-1}(1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^N p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e)\right)\Bigg\},\end{aligned} \tag{5}$$

that is the premium charged is the expected indemnity multiplied by an exponential function of the default rate and the aggregate expected loss. Obviously, we expect both β and γ to have a positive sign, i.e., the larger the default rate the smaller the premium loading and the more business the company writes the smaller are the profit margins, respectively. Of course, generalizations such as line-specific parameters are straightforward to include in theory, but they will complicate the estimation as well as the (numerical) solution of the optimization problem.

Therefore, all-in-all, the company solves the following profit-maximization problem:

$$\max_{\{p_t^{(j)}\},\{q_t^{(j)}\},\{R_t^b\},\{R_t^e\}} \left\{ \begin{array}{r} \mathbb{E}_{t-1}\Big[\sum_{t=1}^{\infty} 1_{\{a_1\geq 0,\ldots,a_t\geq 0\}}\, e^{-rt}\big[e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} \\ -(\tau\, a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e)\big] \\ -1_{\{a_1\geq 0,\ldots,a_{t-1}\geq 0, a_t<0\}}\, e^{-rt}\left[(a_{t-1}+R_t^b)e^r + R_t^e\right]\Big] \end{array} \right\}, \tag{6}$$

subject to (2); (3); (5); $R_t^e \geq 0$; $\{p_t^{(j)}\}$, $\{q_t^{(j)}\}$, $\{R_t^b\}$ **F**-predictable; and $\{R_t^e\}$ **F**-adapted. We immediately obtain:

Lemma 2.1. *The objective function may be equivalently represented as:*

$$\max_{\{p_t^{(j)}\},\{q_t^{(j)}\},\{R_t^b\},\{R_t^e\}} \left\{ \mathbb{E}\left[\sum_{\{t\leq t^*: a_1\geq 0, a_2\geq 0,\ldots,a_{t^*-1}\geq 0, a_{t^*}<0\}} e^{-rt}\left[-e^r R_t^b - R_t^e\right]\right] - a_0 \right\}. \tag{7}$$

Hence, the problem can be equivalently expressed as a dividend maximization problem.

Denote the optimal value function, i.e., the solution to (6) or (7), by $V(a_0)$.[3] Then, under mild conditions on the loss distributions, the value function is finite and—as the solution to a stationary infinite-horizon dynamic programming problem—satisfies the following Bellman equation:

Proposition 2.1 (Bellman Equation)**.** *Assume $r, \tau > 0$. Then the value function $V(\cdot)$ satisfies the following Bellman equation:*

$$V(a) = \max_{\{p^{(j)}\},\{q^{(j)}\},R^b,R^e} \left\{ \begin{array}{l} \mathbb{E}\Big[1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r+R^e-c_2(R^e)\geq I\}} \\ \times\Big(\sum_j p^{(j)} - e^{-r} I - \tau\, a - c_1(R^b) - e^{-r}\, c_2(R^e) \\ +e^{-r}\, V\left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r + R^e - c_2(R^e) - I \right) \Big) \\ -1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r+R^e-c_2(R^e)< I\}} \left(a + R^b + e^{-r} R^e \right) \Big] \end{array} \right\}. \tag{8}$$

subject to (3)*;* (5)*; $\{p_t^{(j)}\}$, $\{q_t^{(j)}\}$, $R_t^b \in \mathbb{R}$; and $R_t^e \geq 0$ is $\sigma(L^{(j)}, j = 1, \ldots, N)$-measurable.*

Note that since raising capital at the end of the period—which we interpret as raising capital in distressed states—is more costly than raising capital under "normal" conditions (Eq. (1)), it only makes sense to either raise exactly enough capital to save the company or not to raise any capital at all: Raising more will not be optimal since it is possible to raise at the beginning of the next period on better terms; raising less will not be optimal since the company will go bankrupt and the policyholders are the residual claimants. This yields:

Proposition 2.2. *$R^e \in \{0, R^e_*\}$, where R^e_* solves:*

$$\sum_j I^{(j)} - \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r = R^e_* - c_2(R^e_*). \tag{9}$$

More precisely:

- *for $\left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r \geq I$, we have $R^e = 0$;*
- *for $\left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r < I$ and $V(0) \geq R^e_*$, we have $R^e = R^e_*$;*
- *and for $V(0) < R^e_*$, we have $R^e = 0$.*

The latter assertion states that it is optimal to only save the company if the (stochastic) amount of capital to be raised at the end of the period is smaller than the value of the company, i.e., if the investment has a positive net present value. For a linear specification of end-of-period costs, this leads to the following simplification of the optimization problem:

[3]Note that the discounted expected value of dividends less capital raises will equate to $V(a_0) + a_0$, so that a locally decreasing $V(\cdot)$ does not imply that a less capitalized firm is worth less—solely that a slighlty less capitalized firm may be yielding higher profits.

Corollary 2.1. *For linear costs $c_2(x) = \xi\, x,\ x \geq 0$:*

$$R^e_* = \frac{1}{1-\xi}\left[I - \left(a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right) e^r\right].$$

In particular, the Bellman equation becomes:

$$V(a) = \tag{10}$$

$$\max_{\{p^{(j)}\},\{q^{(j)}\},R^b} \left\{ \begin{array}{l} \mathbb{E}\Big[\mathit{1}_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r \geq I\}} \times \Big(\sum_j p^{(j)} - e^{-r} I - \tau\, a - c_1(R^b) \\ \qquad\qquad + e^{-r}\, V\left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r - I \right) \Big) \\ + \mathit{1}_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r < I \leq (a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r + (1-\xi)\,V(0)\}} \times \\ \quad \left(\frac{1}{1-\xi}\left[[\sum_j p^{(j)} + a(1-\tau) + R^b - c_1(R^b)] - e^{-r} I \right] + e^{-r} V(0) - [a + R^b] \right) \\ + \mathit{1}_{\{I > (a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r + (1-\xi)\,V(0)\}} \left(-(a + R^b) \right) \Big] \end{array} \right\}.$$

subject to:

$$\begin{aligned} p^{(i)} \;&=\; e^{-r}\, \mathbb{E}\left[I^{(i)}\right] \times \exp\Bigg\{ \alpha - \gamma \mathbb{E}[I] \\ &\qquad - \beta\, \mathbb{P}\left(I > \left[a(1-\tau) + R^b - c_1(R^b) + \sum_{j=1}^{N} p^{(j)} \right] e^r + (1-\xi)\, V(0) \right) \Bigg\}. \end{aligned} \tag{11}$$

2.2 The Marginal Cost of Risk and RORAC

In what follows, we assume a linear cost for end-of-period capital as in Corollary 2.1, so that we study problem (10) subject to the participation constraint (11). For ease of presentation, we define:

$$S = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right]\, e^r \tag{12}$$

and:

$$D = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right]\, e^r + (1-\xi) V(0) \tag{13}$$

as the thresholds of I for *saving* the company and for letting it *default*, respectively.

As shown in the Appendix, we work with optimality conditions to obtain an expression for the balancing of marginal revenue with marginal cost for the i-th risk:

Proposition 2.3. *We have:*

$$\frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\,\exp\{\alpha-\gamma\mathbb{E}[I]-\beta\mathbb{P}(I\geq D)\}}{(1-c_1')} \tag{14}$$
$$= \underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\,1_{\{I\leq D\}}\right]}_{(i)} + \underbrace{\frac{\gamma}{1-c_1'(R^b)}\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\,\exp\{\alpha-\gamma\,\mathbb{E}[I]-\beta\mathbb{P}(I>D)\}}_{(ii)}$$
$$+\underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\,V'\left(S-I\right)\,1_{\{I\leq S\}}\right]}_{(iii)} + \underbrace{\mathbb{E}\left[\frac{\xi}{1-\xi}\frac{\partial I^{(i)}}{\partial q^{(i)}}\,1_{\{S<I\leq D\}}\right]}_{(iv)}$$
$$+\underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\middle|\, I=D\right]\times\left\{\mathbb{P}(I>D)+\tau^*\right\}}_{(v)},$$

where

$$\tau^* = \frac{c_1'(R^b)}{1-c_1'(R_b)} - \frac{\xi}{1-\xi}\,\mathbb{P}(S\leq I\leq D) - \mathbb{E}\left[V'\left(S-I\right)\,1_{\{S\geq I\}}\right] \tag{15}$$

is the "shadow cost of raising capital."

The *shadow cost of raising capital* can be interpreted as the marginal cost of an additional dollar of capital raised in normal conditions, net of the marginal beneficial impact this dollar has on 1) the expected future cost of "distressed" capital raises (the second term on the right hand side) and 2) the continuation value of the firm (the third term).

We interpret the left-hand side of (14) as the marginal revenue associated with an increase in exposure to the i-th risk. The term represents the first-order impact of the increase in exposure on premium revenue, which increases as actuarial costs $\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]$ increase. Note that this term is amplified by $(1-c_1')$, which reflects the fact that additional premium acts as a substitute for capital raised and thus saves the company the marginal cost of raising capital.

We interpret the right-hand side as the marginal cost of risk, which has five components. First, an additional dollar issued in some line will increase the actuarial value of the liability in solvent states (i). Moreover, the increased supply will yield a decrease in the price of insurance due to our assumption of decreasing returns to scale—i.e., there are "scale costs" (ii). Also, the higher exposure will produce changes in end-of-period outcomes, which will affect the value of the company—that is, there is a (random) cost associated with the continuation value of the company (iii). Since a company with a larger exposure is more expensive to bail out in case of insufficient funds, the increase in exposure leads to an increase in the expected costs associated with saving the company in times of distress (iv). Finally, additional exposure contributes to a marginal impact on the probability of default (v).

It is this last term—the marginal impact on the probability of default (v)—that we interpret as a capital allocation. It represents an allocation of costs associated with D as defined in (13), which is an expanded notion of capital. This broad notiont of capital has two parts: The first is the amount of available resources S (which amounts to previous assets a net of the carrying cost, plus money raised plus premiums, compounded over the period) and the second part is the cost-adjusted *present value of future profits* for a zero-capital firm ($D - S$). When the firm defaults, it loses both components. This capital is allocated by the formula $\mathbb{E}[{}^{\partial I^{(i)}}/_{\partial q^{(i)}}|I = D]$, so that each exposure is allocated capital according to how it influences total losses at the default point threshold. This allocation formula is the one corresponding to a Value-at-Risk measure with the threshold set at D, which in turn comes from the specification of the premium function that assumes policyholders assess company solvency via default probabilities (Basak and Shapiro, 2001).

We can then define the RORAC by rearranging terms, as in:

$$\text{RORAC}_i = \frac{\frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] \exp\{\alpha - \gamma\mathbb{E}[I] - \beta\mathbb{P}(I \geq D)\}}{(1-c_1')} - (i) - (ii) - (iii) - (iv)}{\mathbb{E}[{}^{\partial I^{(i)}}/_{\partial q^{(i)}}|I = D]} = \mathbb{P}(I > D) + \tau^*.$$

Note that the RORAC definition of the numerator resolves concerns of arbitrariness in our assignment of the "scale" component—which reflects the pressure on margins resulting from the expansion of exposure—to marginal cost rather than marginal revenue. The denominator reflects an allocation of (broad) capital per unit of exposure, which is well known to add up to the total capital of the firm when multiplied by the respective exposures and summed over lines of business:

$$\sum_i q_i\, \mathbb{E}[{}^{\partial I^{(i)}}/_{\partial q^{(i)}}|I = D] = D.$$

2.3 Discussion

The multi-period setting, featuring company options to bear external financing costs when raising additional funding, complicates the risk pricing problem considerably. Noteworthy is the fact that this setting no longer features a mapping of capital allocation as currently practiced to marginal cost. That is, allocating the capital of the firm is no longer sufficient to recapture all of the risk components of marginal cost. Risk has the potential now to consume not only the current capital of the firm, but also capital that has yet to be raised. Moreover, unlike the multi-period version of the model in Bauer and Zanjani (2015), the continuation value of the firm is no longer independent of its capital position due to the adjustment costs introduced by frictional costs of raising capital. In summary, the current capital of the firm is no longer the only cushion absorbing risk, so the marginal cost of risk can no longer be recovered solely by allocating the same.

Another difference from the one-period setting is that the firm's marginal costs in this case, when multiplied by the exposure quantity, will not "add up" to the firm's total costs due to potential non-linearities in $V(\cdot)$, $c_1(\cdot)$, and the premium function—several of the components ((i), (iv), and (v) in (14)) can be fully recouped. While it is well-known that capital allocation and risk-adjusted return calculations are only feasible at the margin (Merton and Perold, 1993), assessing the marginal components from accounting statements presents a considerable practical challenge.

This challenge is especially evident when adapting RORAC to the multiperiod environment. Traditional underwriting margins are no longer sufficient statistics for the "return" in the numerator. And an allocation of book equity is no longer appropriate for the "risk-adjusted capital" of the denominator.

We proceed in the next section to numerical analysis of the model, with the aim of comparing the marginal costs and RORACs obtained to those implied by conventional methods. Since we have specified the objectives and constraints facing the firm, we have gone through the "rigors of the pricing project" (Venter, 2010)—with the consequence that, within the context of the model, we have the *right* answer on marginal cost. And while one could of course make different choices in terms of model specification, the model addresses some key areas of interest that affect the marginal cost of risk, such as the continuation value of the firm and costly external financing, that should provide for revealing comparisons with static single period capital allocation models.

3 Implementation in the Context of a Catastrophe Insurer

In this section, we calibrate and numerically solve the model introduced in the previous section using data from a catastrophe reinsurer. We describe the data in Section 3.1, Section 3.2 gives details on the calibration, Section 3.3 presents the results, and we provide marginal cost calculations in Section 3.4. In addition to decomposing the marginal cost of risk into its various components described in Section 2.2, here we compare the model-based results to RORAC calculations for conventional capital allocation techniques to appraise their adequacy. Implementation details and results illustrating the convergence of the algorithm are collected in A.2 whereas Appendix B collects additional figures.

3.1 Data

We are given 50,000 joint loss realizations and premiums for 24 distinct reinsurance lines differing by peril and geographical region. The data have been scaled by the data supplier. Figure 1 provides a histogram of the aggregate loss distribution, and Table 1 lists the lines and provides some descriptive statistics on each line. The largest lines for our reinsurer (by premiums and expected losses) are "US Hurricane," "N American EQ West" (North American Earthquake West), and "Ex-Tropical Cyclone" (Extratropical Cyclone). The expected aggregate loss is 187,819,998 with a standard deviation of 162,901,154, and the aggregate premium income is 346,137,808.

While a practical allocation problem may consider choosing an optimal portfolio among the 24 lines, we aggregate the data to four lines and in what follows focus on the problem of optimally allocating to these (hypothetical) lines. This has the advantage of keeping the numerical analysis tractable and facilitates the presentation of results. Table 1 illustrates the aggregation (column Agg), and Figure 2 shows histograms for each of these four lines.

We notice that the "Earthquake" (Agg 1) distribution is concentrated at low loss levels with only relatively few realizations exceeding 50,000,000 (the 99% VaR slightly exceeds 300,000,000). However, the distribution depicts fat tails with a maximum loss realization of close to one billion. The (aggregated) premium for this line is 46,336,664 with an expected loss of 23,345,695. "Storm & Flood" (Agg 2) is by far the largest line, both in terms of premiums (243,329,704) and expected losses (135,041,756). The distribution is concentrated around loss realizations between 25 and 500 million, although the maximum loss in our 50,000 realizations is almost four times that size. The 99% VaR is approximately 700 million USD. In comparison, the "Fire & Crop" (Agg 3) and "Terror & Casualty" (Agg 4) lines are small with an (aggregated) premiums (expected loss) of about 34 (19) million and 22.5 (11) million, respectively. The maximal realizations are around 500 million for "Fire & Crop" (99% VaR = 163,922,557) and around 190 million for "Terror & Casualty" (99% VaR = 103,308,358).

Line	Statistics			
	Premiums	Expected Loss	Standard Deviation	Agg
N American EQ East	6,824,790.67	4,175,221.76	26,321,685.65	1
N American EQ West	31,222,440.54	13,927,357.33	47,198,747.52	1
S American EQ	471,810.50	215,642.22	915,540.16	1
Australia EQ	1,861,157.54	1,712,765.11	13,637,692.79	1
Europe EQ	2,198,888.30	1,729,224.02	5,947,164.14	1
Israel EQ	642,476.65	270,557.81	3,234,795.57	1
NZ EQ	2,901,010.54	1,111,430.78	9,860,005.28	1
Turkey EQ	214,089.04	203,495.77	1,505,019.84	8
N Amer. Severe Storm	16,988,195.98	13,879,861.84	15,742,997.51	2
US Hurricane	186,124,742.31	94,652,100.36	131,791,737.41	2
US Winterstorm	2,144,034.55	1,967,700.56	2,611,669.54	2
Australia Storm	124,632.81	88,108.80	622,194.10	2
Europe Flood	536,507.77	598,660.08	2,092,739.85	2
ExTropical Cyclone	37,033,667.38	23,602,490.43	65,121,405.35	2
UK Flood	377,922.95	252,833.64	2,221,965.76	2
US Brushfire	12,526,132.95	8,772,497.86	24,016,196.20	3
Australian Terror	2,945,767.58	1,729,874.98	11,829,262.37	4
CBNR Only	1,995,606.55	891,617.77	2,453,327.70	4
Cert. Terrorism xCBNR	3,961,059.67	2,099,602.62	2,975,452.18	4
Domestic Macro TR	648,938.81	374,808.73	1,316,650.55	4
Europe Terror	4,512,221.99	2,431,694.65	8,859,402.41	4
Non Certified Terror	2,669,239.84	624,652.88	1,138,937.44	4
Casualty	5,745,278.75	2,622,161.64	1,651,774.25	4
N American Crop	21,467,194.16	9,885,636.27	18,869,901.33	3

Table 1: Descriptive Statistics (scaled)

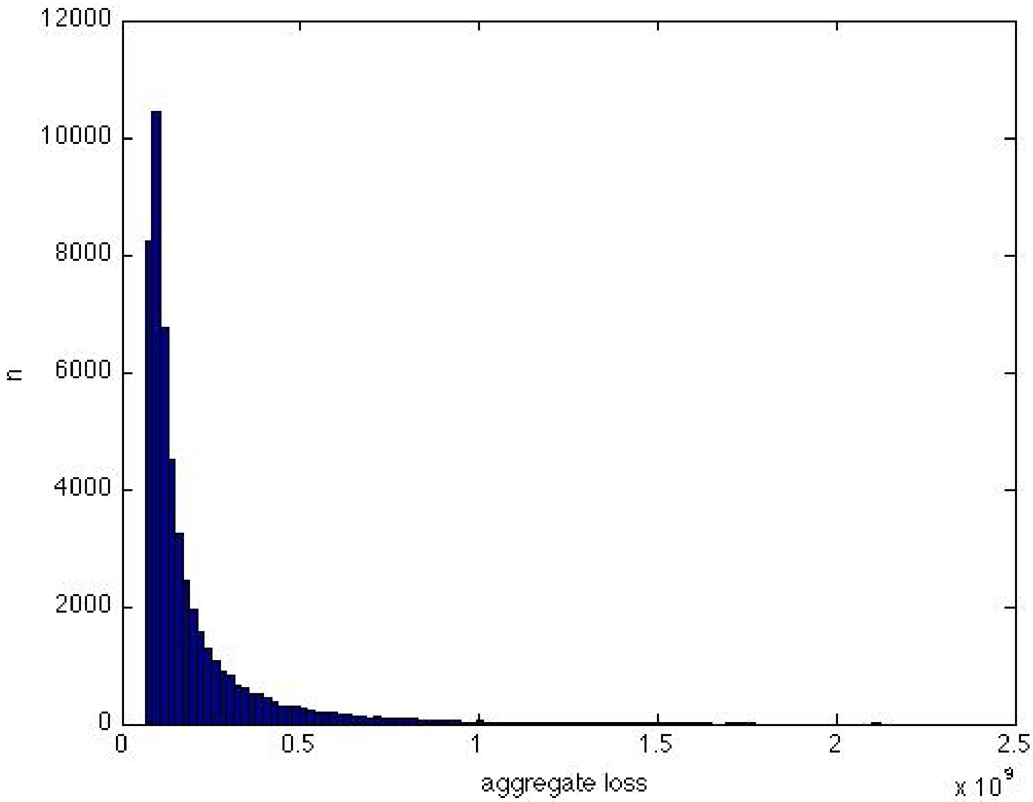
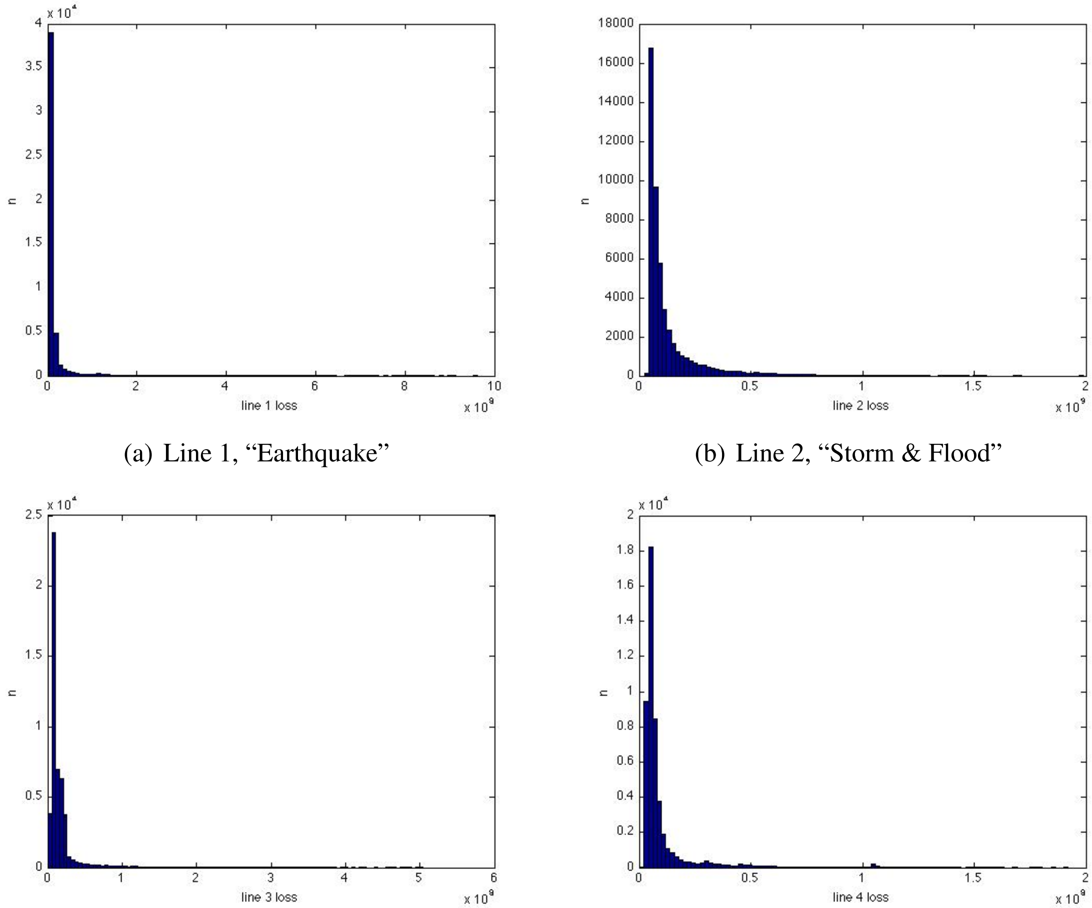


Figure 1: Histograms for the Aggregate Loss (scaled)

3.2 Calibration

The model as developed in Section 2 requires calibration in several areas. It is necessary to specify costs of raising and holding capital. It is also necessary to specify how the insurance company is affected by changes in risk.

As a starting point for the costs of holding capital, Cummins and Phillips (2005) estimate the cost of equity capital for insurance companies using data from the 1997-2000 period. They use several methods to derive a variety of estimates, including a single factor CAPM and Fama-French three-factor cost of capital model (Fama and French, 1993). The estimates for property-casualty insurance fall in the neighborhood of 10% to 20%. Given that the risk-free interest rate used in the analysis was based on the 30-day T-Bill rate, which averaged about 5% over the sample period, the estimates suggest a risk premium for property-casualty insurance ranging from as little as 5% to as much as 15%. However, previous research has found unstable estimates of the cost of capital, suggesting that the risk premium may be considerably smaller; some specifications even suggest that the industry's "beta" may be zero or even negative (Cox and Rudd, 1991; Cummins and Harrington, 1985). Given the range of results, we use τ ranging from 3% to 5% in the model. Calibrating the cost of raising capital is more difficult, as we are not aware of studies specific to the property-casualty industry. Hennessy and Whited (2007), however, analyze the cost of external



(a) Line 1, "Earthquake"

(b) Line 2, "Storm & Flood"

(c) Line 3, "Fire & Crop"

(d) Line 4, "Terror & Casualty"

Figure 2: Histograms for Aggregated (Agg) Lines (scaled)

financing average across industries by using the entire sample of Compustat firms. They find marginal equity flotation costs ranging from 5% for large firms to 11% for small firms, and we base our quadratic specification

$$c_1(x) = c_1^{(1)}\,x + c_1^{(2)}\,x^2,\; x \geq 0,$$

calibration of external financing costs on these figures, with the linear piece ($c_1^{(1)}$) being set at 7.5%.

Changes in risk are known to affect insurance companies. Epermanis and Harrington (2006) focus on the property-casualty insurance industry in particular, documenting significant declines in premium growth following rating downgrades. Sommer (1996) documents a significant connection between default risk and pricing in the property-casualty industry. The foregoing research suggests two possible ways to model the consequences of risk for a property-casualty insurer: Increases in risk could either produce involuntary drops in exposure volume or drops in price, or both. We incorporate both channels in our model setup within the premium function (Eq. (5)).

On the question of how to define "risk," credit ratings are a tempting solution. While credit ratings are widely accepted proxies for market assessments of a company's risk level, using them requires us to map credit ratings to default risk levels, which is a feasible exercise given the validation studies provided by rating companies that document the historical connection between default risk and the various letter ratings.

The question of how to connect risk with pricing is an empirical one, requiring an analysis of the historical relation between default risk inferred from credit ratings and insurance prices. Since the data used for our numerical analysis is drawn from a reinsurance company, we focus on empirical analysis of reinsurers, and specifically those identified in the Reinsurance Association of America's annual review of underwriting and operating results for the years 2008-2012. These reviews yield 30 companies for the analysis, and we collected all available ratings for that set of 30 companies from Moody's, S&P, and A.M. Best.

To calculate the default rate, we use a multi-stage procedure. We start by collecting 1) Moody's, S&P, and A.M. Best ratings from the 2008-2012 period for the sample of insurance companies; 2) the joint distribution of Moody's and S&P ratings for corporate debt as reported in Table 1 of Cantor, Packer, and Cole (1997); and 3) one year default rates by rating as reported in Tables 34 and 35 of Moody's *Annual Default Study: Corporate Default and Recovery Rates, 1920-2012* and Tables 9 and 24 of S&P's *2012 Annual Corporate Default Study and Rating Transitions.* We then fit smoothed default rates for Moody's by choosing default rates for the AA1, AA2, AA3, A1, A2, and A3 categories (AAA, BAA1, and other historical default rates are held at their historic values)[4] and perform a similar procedure for S&P ratings. We calculate an average one-year default rate for A++, A+, A, and A- A.M. Best ratings by calculating an average "Moody's" default rate based on our sample distribution of Moody's ratings for each A.M. Best rating, calculating an average "S&P" default rate in a similar manner, and then averaging the two. This yields one-year default rates for each A.M. Best rating in the 2008-2012 sample as shown in Table 2.[5]

A++	0.006%
A+	0.044%
A	0.072%
A-	0.095%

Table 2: Fitted One-Year Default Rates for A.M. Best Ratings

[4]Fit is assessed by evaluating 8 measures: 1) the weighted average default rate in the Aa category (using the modifier distribution in Cantor, Packer, and Cole (1997) for the weights); 2) the weighted average default rate in the A category, and "fuzzy" default rates for Aa1, Aa2, Aa3, A1, A2, and A3 categories, where the fuzzy rate is calculated by applying the distribution of S&P ratings for each modified category to the default rates (for example, if S&P rated 20% of Moody's Aa1 as AAA, 50% as AA+ and 30% as AA, we would calculate the "fuzzy" default rate for Aa1 as 20%$\times$Aaa default rate + 50%$\times$Aa1 default rate + 30%$\times$ Aa2 default rate). We calculate squared errors between fitted averages and averages using the actual empirical data, and select fitted values to minimize the straight sum of squared errors over the eight measures.

[5]It is worth noting that these are somewhat lower than suggested by A.M. Best's own review of one-year impairment rates, which indicated 0.06% for the A++/A+ category and 0.17% for the A/A- category (see Exhibit 2 of *Best's Impairment Rate and Rating Transition Study – 1977 to 2011*). In the empirical analysis that followed, it was also necessary to assign a default rate for the B++ rating, which did not occur in the 2008-2012 sample but did surface when performing the analysis over a longer time period. We used 0.20% for this default rate, which is roughly consistent with the default rates for Baa or BBB ratings.

Finally, we identify the relation between price and default risk and volume by fitting the model:

$$\log p_{it} = \alpha + \alpha_t - \beta d_{it} - \gamma E_{it} + e_{it},$$

where p_{it} is calculated as the ratio of net premiums earned to the sum of loss and loss adjustment expenses incurred by company i in year t, d_{it} is the default rate corresponding to the letter rating of company i in year t in percent, E_{it} is the expected loss of company i in year t, and e_{it} is an error term. The expected loss is calculated by applying the average net loss and loss adjustment expense ratio over the sample period for each firm to that year's net premium earned.

We use NAIC data for the period 2002 to 2010 for the sample of companies identified above for the analysis. The results of the regression are presented in Table 3.

Variable	Coefficient	Std. Error	t-value
Intercept (α)	0.6590	0.0614	10.73
Default rate (β)	3.9296	0.5090	-7.72
Expected Loss (γ)	1.48 E-10	2.24 E-11	-6.57

Year dummies are omitted. Observations: 288. Adj. $R^2 = 26\%$

Table 3: Premium Parametrizations

We use three sets of parameters based on the calibrations above. The sets are described in Table 4. We vary the cost of holding capital τ from 3% to 5%; the cost of raising capital in normal circumstances is represented by a quadratic cost function with the linear coefficient $c_1^{(1)}$ fixed at 7.5%; the cost of raising capital in distressed circumstances, ξ, varies from 20% to 75%; the interest rate r varies from 3% to 6%; and the parameters α, β, and γ specify the underwriting margin. For the latter parameters, we use the regression results, with the alpha intercept being adjusted for the average of the unreported year dummy coefficients, and we also use an alternative, more generous specification based on a previous unreported analysis that omits loss adjustment expenses.

Using the loss distributions described in Section 3.1, we solve the optimization problem by value iteration relying on the corresponding Bellman equation (10) on a discretized grid for the capital level a. That is, we commence with an arbitrary value function (constant at zero in our case), and then iteratively solve the one-period optimization problem (10) by using the optimized value function from the previous step on the right hand side. Standard results on dynamic programming guarantee the convergence of this procedure (Bertsekas, 1995). More details on the solution algorithm and its convergence are presented in Appendix A.2.

3.3 Results

The results vary considerably across the parameterizations. While the value function in the base case ranges from approximately 1.8 billion to 2 billion for the considered capital levels, the range for the "profitable company" is in between 21.7 billion to 22.4 billion, and even around 56 to 57

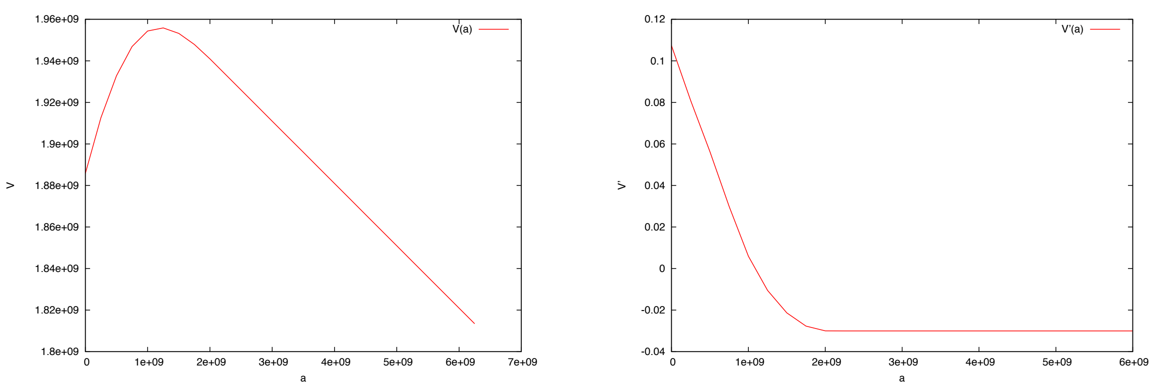
Parameter	1 ("base case")	2 ("profitable company")	3 ("empty company")
τ	3.00%	5.00%	5.00%
$c_1^{(1)}$	7.50%	7.50%	7.50%
$c_1^{(2)}$	1.00E-010	5.00E-011	1.00E-010
ξ	50.00%	75.00%	20.00%
r	3.00%	6.00%	3.00%
α	0.3156	0.9730	0.9730
β	392.96	550.20	550.20
γ	1.48E-010	1.61E-010	1.61E-010

Table 4: Parametrizations

billion for our "empty company." Nonetheless the basic shape of the solution is similar across the first two cases, whereas the "empty company case" yields a qualitatively different form (hence the name).

Base Case Solution

Various aspects of the "base case" solution are depicted in Figures 3, 4, and 5. Table 5 presents detailed results at three key capital levels.

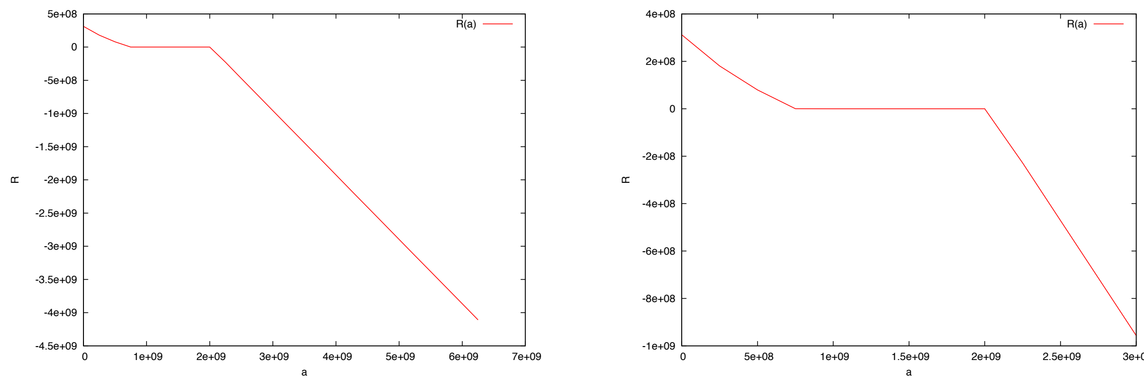


(a) Value Function $V(a)$

(b) Derivative $V'(a)$

Figure 3: Value function V and its derivative V' for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = $ 1.00E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = $ 1.48E-10 (base case).

Figure 3 displays the value function and its derivative. We observe that the value function is "hump-shaped" and concave—i.e., the derivative V' is decreasing in capital. However, for high capital levels, the derivative is approaching a constant level of $-\tau = -3\%$ and the value function is essentially affine.



(a) Raising decisions $R(a)$ (b) Raising decisions $R(a)$ (lim. range)

Figure 4: Optimal raising decision R for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00$E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48$E-10 (base case).

The optimal level of capitalization here is approximately 1 billion. If the company has significantly less than 1 billion in capital, it raises capital as can be seen from Figure 4, where the optimal raising decision for the company is displayed. However, the high and convex cost of raising external financing prevents the company from moving immediately to the optimal level. The adjustment can take time: Since internally generated funds are cheaper than funds raised from investors, the optimal policy trades off the advantages associated with higher levels of capitalization against the costs of getting there. As capitalization increases, there is a rigid region around the optimal level where the company neither raises nor sheds capital. In this region, additional capital may bring a benefit, but it is below the marginal cost associated with raising an additional dollar, which is approximately $c_1^{(1)} = 7.5\%$. The benefit of capital may also be less than its carrying cost of $\tau = 3\%$, but since this cost is sunk in the context of the model, capital may be retained in excess of its optimal level. For extremely high levels of capital, however, the firm optimally sheds capital through dividends to immediately return to a maximal level at which point the marginal benefit of holding an additional unit of capital (aside from the sunk carrying cost) is zero. The transition is immediate, as excess capital incurs an unnecessary carrying cost and shedding capital is costless in the model. This is also the reason that the slope of the value function approaches $-\tau$ in this region.

Figure 5 reveals how the optimal portfolio varies with different levels of capitalization. As capital is expanded, more risk can be supported, and the portfolio exposures grow in each of the lines until capitalization reaches its maximal level. After this point, the optimal portfolio remains constant: Even though larger amounts of risk could in principle be supported by larger amounts of capital, it is, as noted above, preferable to immediately shed any capital beyond a certain point and, concurrently, choose the value maximizing portfolio. Note that the firm here has an optimal

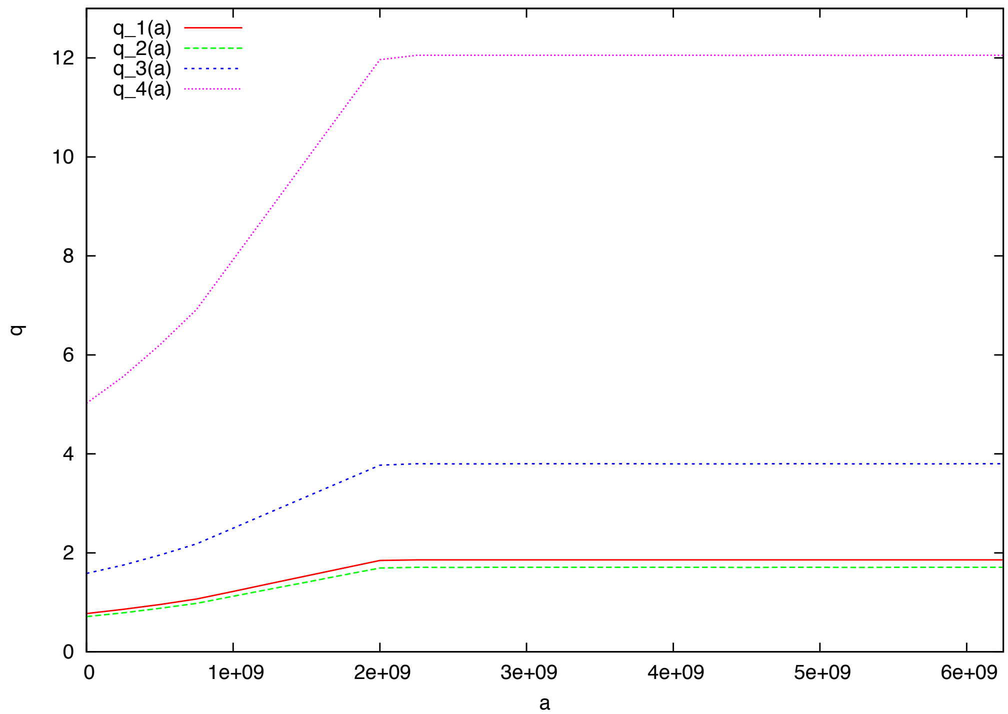


Figure 5: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} =$ 1.00E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma =$ 1.48E-10 (base case).

scale because of the γ parameter in the premium function. As the firm gets larger in scale, margins shrink because of γ.

Table 5 reveals that firm rarely exercises its default option (measured by $\mathbb{P}(I \geq D)$, which is evidently 0.002% even at low levels of capitalization). The firm does experience financial distress more often at low levels of capitalization. For example, the probability of facing claims that exceed immediate financial resources, given by $\mathbb{P}(I > S)$, is 4.54% when initial capital is zero but 0.45% when capital is at the optimal level, and 0.13% when capitalization is at its maximal point. In all of these cases, the firm usually resorts to emergency financing when claims exceed its cash, at a per unit cost of $\xi = 50\%$, to remedy the deficit. Because of the high cost of emergency financing, however, it restrains its risk taking when undercapitalized and also raises capital before underwriting to reduce the probability of having to experience financial distress.

The bottom lines of the table show the various cost parameters at the optimized value. Here, the marginal cost of raising capital is significantly greater than 7.5% for $a = 0$ due to the quadratic adjustment, whereas clearly the marginal cost is zero in the shedding region ($a = 4$ billion). As indicated above, around the optimal capitalization level of 1 billion neither raising nor shedding

	zero capital	optimal capital	high capital
a	0	1,000,000,000	4,000,000,000
$V(a)$	1,885,787,820	1,954,359,481	1,880,954,936
$R(a)$	311,998,061	0	-1,926,420,812
$q_1(a)$	0.78	1.23	1.86
$q_2(a)$	0.72	1.13	1.71
$q_3(a)$	1.60	2.51	3.80
$q_4(a)$	5.06	7.96	12.06
S	550,597,000	1,406,761,416	2,615,202,661
D	1,493,490,910	2,349,655,327	3,558,096,571
$\mathbb{E}[I]$	199,297,482	313,561,933	474,841,815
$\sum p^{(i)}/\mathbb{E}[i]$	1.32	1.30	1.27
$\mathbb{P}(I > a)$	100.00%	2.66%	0.002%
$\mathbb{P}(I > S)$	4.54%	0.45%	0.13%
$\mathbb{P}(I > D)$	0.002%	0.002%	0.002%
$c_1'(R^b)$	13.74%	4.65%	0.00%
$\frac{\xi}{1-\xi}\,\mathbb{P}(S < I < D)$	4.54%	0.45%	0.12%
$\mathbb{E}[V'\, 1_{\{I<S\}}]$	8.03%	1.09%	-2.66%
τ^*	3.36%	3.34%	2.53%

Table 5: Results for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} =$ 1.00E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma =$ 1.48E-10 (base case).

is optimal—so that technically the marginal cost is undefined due to the non-differentiability of the cost function c_1 at zero. To determine the correct "shadow cost" of raising capital, we use an indirect method: We use the aggregated marginal cost condition (14) from Proposition 2.3 to back out the value of $c_1'(0)$ that yields the left- and right-hand side to match up.[6] The cost of emergency raising in this case is exactly the probability of using this option (as $\xi = 50\%$), which—as indicated—decreases in the capital level. Finally, the expected cost in terms of impact on the value function ($-\mathbb{E}[V' \, 1_{\{I<S\}}]$) is negative for low capital levels since the value function is increasing in this region, whereas it is positive and approaching τ for high capital levels. Combining the different cost components, we obtain a "shadow cost" of capital τ^* as defined (15) that is decreasing in a, although the level is not too different across capitalizations. In particular, it is noteworthy that τ^* is considerably below the cost of raising capital. The next subsection provides a more detailed discussion of the marginal cost of risk.

Profitable Company

The results for the profitable company are similar to the "base case" presented above, except that the company is now much more valuable—despite the increases in the carrying cost of capital and in the cost of emergency financing—because of the more attractive premium function. The corresponding results are collected in Appendix B. More precisely, Figure 10 displays the value function and its derivative, Figure 11 displays the optimal raising decision, and Figure 12 displays the optimal exposure to the different lines as a function of capital.

Again, there is an interior optimum for capitalization, and the company optimally adjusts toward that point when undercapitalized. If overcapitalized, it optimally sheds to a point where the net marginal benefit associated with holding a dollar of capital (aside from the current period carrying cost which is a sunk cost) is zero. There is thus a rigid range where the company neither raises nor sheds capital, and the risk portfolio gradually expands with capitalization until it reaches the point where the firm is optimally shedding additional capital on a dollar-for-dollar basis.

As before, Table 6 presents detailed results at three key capital levels. Although parameters have changed, the company again rarely exercises the option to default, which has a probability of occurrence of 0.002% even at low levels of capitalization. In most circumstances, the firm chooses to raise emergency financing when claims exceed cash resources, which happens as much as 3.65% of the time (at zero capitalization).

In contrast to the base case, the "shadow cost" of capital τ^* now is considerably higher than before. To some extent, this originates from the different cost parameters. In particular, the cost of raising emergency capital now is $\xi = 75\%$ and the carrying cost $\tau = 5\%$. However, in addition to higher costs, another aspect is that given the more profitable premium function, it now is optimal to write more business requiring a higher level of capital—which in turn leads to higher capital costs. Essentially, the marginal pricing condition (14) requires marginal cost to equal marginal return/profit—and the point where the two sides align now is at a higher level.

[6] In the differentiable regions ($a = 0$, 4bn, and other values), the aggregated marginal cost condition further validates our results—despite discretization and approximation errors, the deviation between the left- and right-hand side is maximally about 0.025% of the left-hand side.

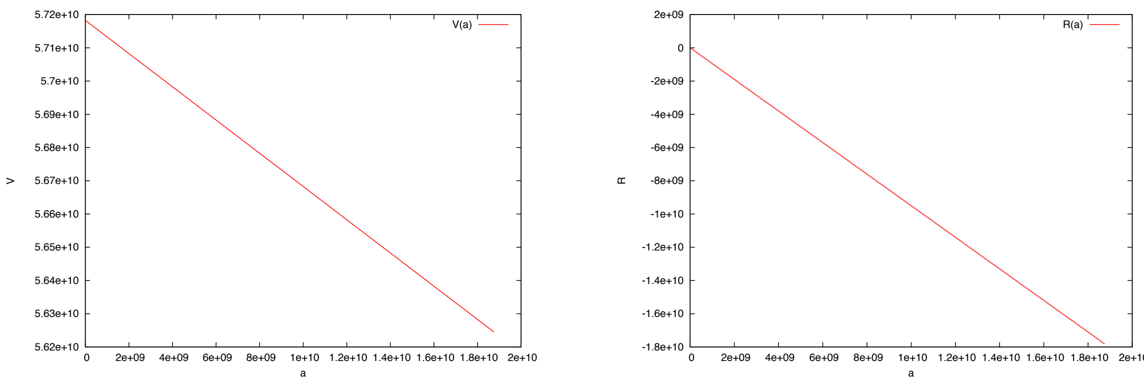
	zero capital	optimal capital	high capital
a	0	3,000,000,000	12,000,000,000
$V(a)$	22,164,966,957	22,404,142,801	22,018,805,587
$R(a)$	1,106,927,845	0	-6,102,498,331
$q_1(a)$	4.81	6.14	7.82
$q_2(a)$	4.42	5.64	7.18
$q_3(a)$	9.83	12.56	15.98
$q_4(a)$	31.19	39.85	50.69
S	3,659,208,135	6,215,949,417	9,412,766,805
D	9,200,449,874	11,757,191,157	14,954,008,545
$\mathbb{E}[I]$	1,227,901,222	1,569,126,466	1,995,776,907
$\sum p^{(i)}/\mathbb{E}[i]$	2.15	2.03	1.90
$\mathbb{P}(I > a)$	1.00%	10.70%	0.07%
$\mathbb{P}(I > S)$	3.65%	0.91%	0.34%
$\mathbb{P}(I > D)$	0.002%	0.002%	0.002%
$c_1'(R^b)$	18.57%	5.97%	0.00%
$\frac{\xi}{1-\xi}\,\mathbb{P}(S < I < D)$	10.94%	2.72%	1.00%
$\mathbb{E}[V'\,1_{\{I<S\}}]$	2.93%	-2.99%	-4.58%
τ^*	8.94%	6.62%	3.58%

Table 6: Results for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} =$ 5.00E-11, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma =$ 1.61E-10 (profitable company).

Empty Company

Figure 6 presents the value function and the optimal raising decision for the "empty company." Figure 7 plots the corresponding optimal exposures to the different business lines.

We call this case the "empty company" because it is optimal to run the company without any capital. This can be seen from Figure 6, which shows that the total continuation value of the company is decreasing in capital and that the optimal policy is to shed any and all accumulated capital through dividends. The optimal portfolio is thus, as can be seen in Figure 7, always the same—corresponding to the portfolio chosen when $a = 0$. Again, note that there is an optimal scale in this case, because greater size is associated with a compression in margins.



(a) Value function $V(a)$ (b) Optimal raising decision R^b

Figure 6: Value function V and optimal raising decision R^b for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 20\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\text{E-}10$ (empty company).

However, even though the company is always empty, it never defaults. This extreme result is produced by two key drivers—the premium function and the cost of emergency financing. As with the "profitable company," the premium function is extremely profitable in expectation. Because of these high margins, staying in business is extremely valuable. Usually, the premiums collected are sufficient to cover losses. When they are not, which happens about 12% of the time, the company resorts to emergency financing. This happens because, in contrast to the "profitable company," emergency financing is relatively cheap at 20% (versus 75% in the "profitable company" case). Thus, it makes sense for the company to forego the certain cost of holding capital—the primary benefit of which is to lessen the probability of having to resort to emergency financing—and instead just endure the emergency cost whenever it has to be incurred. In numbers, the cost of holding capital at $a = 0$ is $\tau \times \mathbb{P}(I \leq S) = 4.38\%$, whereas the cost of raising emergency funds is $\frac{\xi}{1-\xi}\mathbb{P}(I > S) = 3.08\%$.

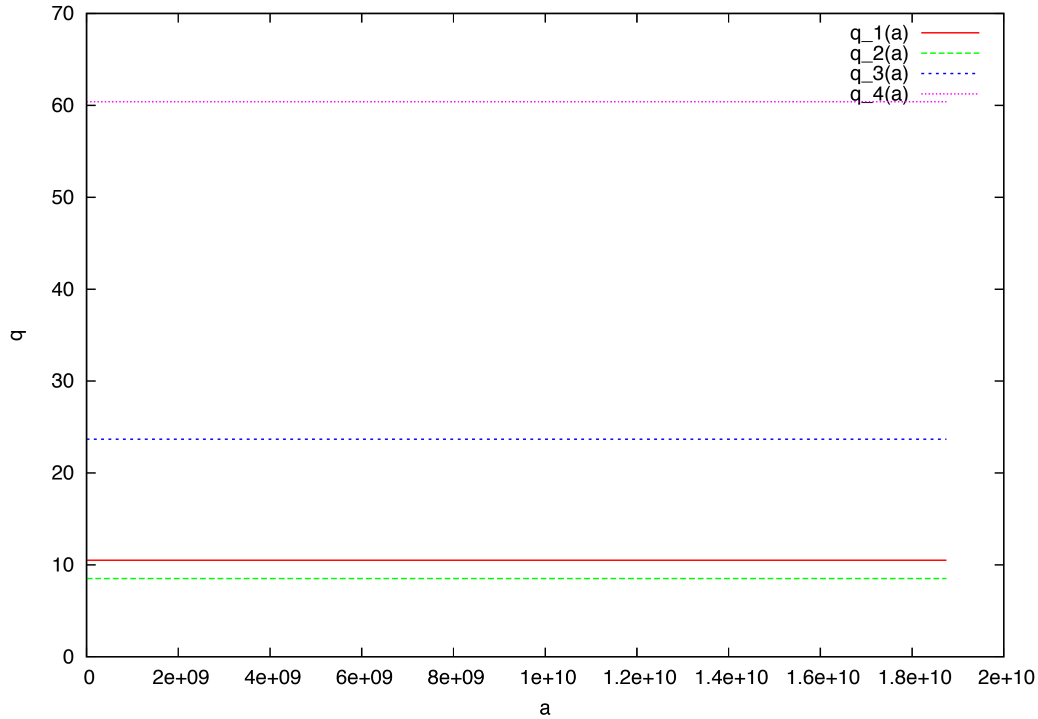


Figure 7: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 20\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\text{E-}10$ (empty company).

3.4 The Marginal Cost of Risk and Capital Allocation

We now contemplate the marginal cost of risk and the allocation of capital within the three example setups. The solutions presented above suggest some immediate difficulties in applying traditional capital allocation methods to price risk. For example, consider the unrealistic (but pedagogically important) example of the "empty company," where it is optimal to hold zero capital. Zero capital would imply zero risk penalties under typical allocation methods. However, risk clearly extends beyond actuarial values, as the insurer, even though it shuns default, may have to raise emergency financing. This example thus raises a key question: If we are to use an risk allocation approach for pricing, *what* exactly should we be allocating? Clearly, the traditional accounting capital, which is equivalent to risk in single period models featuring risk measure constraints, is no longer adequate: After all, in the case of the "empty company," it is zero. Equation (14) suggests that we need a broader conception of what capital is and in fact indicates that the correct quantity to allocate is D, which represents all financial resources currently held by the firm (capital and premiums) *and* the maximum amount of emergency financing that it is willing to raise in case of distress.

However, the problems suggested by Equation (14) go much deeper than relative weightings of

D. Even if we identify the correct quantity to allocate, Equation (14) shows that the marginal cost of risk goes beyond that obtained from a simple allocation of D in two respects. First, calculating the cost of "capital" when allocating D is not straightforward: The theoretical analysis indicates that the key quantity is $\mathbb{P}(I \geq D) + \tau^*$, where τ^* is defined as a marginal cost of raising capital, net of the benefits it provides, as defined by Equation (15). Second, the marginal cost of risk involves terms connected to the scale of the company, the continuation value of the company, and the costs associated with emergency capital raises that lie outside any allocation of capital, whether broadly defined or otherwise: These additional terms are not difficult to allocate to line in theory (as their allocation follows from actuarial values or conditional actuarial values), but 1) not all of them are typically considered in insurance pricing practice, and 2) they are not embedded in the risk penalty emerging from a traditional allocation.

Tables 7, 8, and 9 show the cost allocation decomposition for various capital levels in the three cases.[7]

The component corresponding to the allocation of D—identified as "Capital Cost (v)" in the tables—varies considerably in terms of its importance in the total cost picture. Restricting our attention to parts (ii) through (v), which are the cost components other than claims payments, we see that the capital cost component share of non-payment costs ranges from about 70% (in the "base case" at an initial capitalization of $a = 1,250$ million) to less than a third (in the "profitable" case with $a = 12$ billion).

It is therefore evident that correct risk pricing entails an appreciation of *all* of the components of the marginal cost of risk, one of which can be obtained from an allocation of a broadly defined measure of capital. However, existing allocation methods will misprice risk if the other marginal cost components are overlooked or if capital is defined too narrowly. To get a sense of this, we calculate RORACs for the base case and the profitable company case under Value-at-Risk (VaR) and Expected Shortfall (or Tail-Value-at-Risk, TVaR), the most common risk measures used in practice, where we derive the RORACs as the marginal premium received minus costs divided by allocated capital. We vary both the definition of capital and the types of costs being considered. In particular, in addition to D, we analyze allocations of accounting capital a and surplus S. Moreover, we consider the situations where all non-capital cost components (ii)-(iv) are taken into account in the RORAC calculation, where only actuarial payments are considered ("act. only"), and where also scale effects are considered ("act. and scale"). We compare the results to correct RORACs obtained from the model. The results are presented in Tables 10 and 11.

Since the insurance portfolio has been optimized, the true RORAC is equal across all lines. More precisely, it obviously coincides with the "shadow cost" of capital τ^* (cf. Tables 5 and 6). If all costs are properly accounted for and an appropriately broad definition of capital (D) is used, VaR-based allocation yields RORACs that are quite close to the true RORACs. This is no surprise since VaR is built in our premium function, so that it is the *correct* risk measure on our framework. Indeed, deviations between VaR-RAROCs and the *correct* RAROCs are solely due to inaccuracies in the estimation of VaR. The results based on TVaR, on the other hand, do not tend to give accurate results. In particular, they allocate too little capital to lines 1 and 3, while overpenalizing line 2. The misstatements are significant. For example, although the correct figure never exceeds 10% in our scenarios, RORAC estimated under TVaR for line 1 routinely exceeds 50% and sometimes

[7] In the tables we are taking the product of marginal cost and total exposure quantity, so it is worth noting that the sum total will not add up to total costs because of nonlinearities in the cost and value function.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	199,259,815
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	1,054,415	6,099,197	842,700	486,629	9,001,325
($\frac{\gamma}{1-c_1'(R^b)}\,\mathbb{E}[L^{(i)}]\sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	1,787,494	10,069,585	1,395,397	781,737	14,794,219
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	9.43%	48.77%	15.05%	26.75%	100.00%
Raising cost, (iv)	3,136,921	21,340,216	2,782,890	1,924,195	31,920,536
($\frac{\xi}{1-\xi}\,\mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	7.67%	47.91%	13.91%	30.51%	100.00%
Capital cost, (v)	6,423,322	34,269,301	4,891,903	2,532,602	50,194,848
($\mathbb{E}[L^{(i)}\mid I=D]\times[\mathbb{P}(I>D)+\tau^*]$)	9.99%	48.92%	15.55%	25.54%	100.00%
Cost, (iii)-(v)	11,347,737	65,679,102	9,070,189	5,238,533	96,909,604
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	12,402,152	71,778,299	9,912,889	5,725,162	105,910,928
	9.14%	48.56%	14.93%	27.36%	100.00%
$a = 1,250,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	313,502,671
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	1,475,632	8,535,701	1,179,341	681,028	19,819,580
($\frac{\gamma}{1-c_1'(R^b)}\,\mathbb{E}[L^{(i)}]\sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	557,566	3,411,166	442,838	281,845	7,886,781
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	8.68%	48.76%	14.09%	28.46%	100.00%
Raising cost, (iv)	501,193	3,356,439	489,699	227,776	7,442,867
($\frac{\xi}{1-\xi}\,\mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	8.27%	50.84%	16.52%	24.37%	100.00%
Capital cost, (v)	5,917,572	33,625,361	4,643,976	2,711,437	78,428,268
($\mathbb{E}[L^{(i)}\mid I=D]\times[\mathbb{P}(I>D)+\tau^*]$)	9.27%	48.34%	14.86%	27.53%	100.00%
Cost, (iii)-(v)	6,976,331	40,392,966	5,576,513	3,221,059	93,757,915
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	8,451,962	48,928,667	6,755,854	3,902,086	113,577,496
	9.14%	48.57%	14.93%	27.36%	100.00%
$a = 4,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	474,752,070
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	2,080,451	12,034,241	1,662,719	960,161	42,315,511
($\frac{\gamma}{1-c_1'(R^b)}\,\mathbb{E}[L^{(i)}]\sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-467,409	-2,524,447	-361,943	-198,677	-8,951,208
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	9.71%	48.15%	15.37%	26.77%	100.00%
Raising cost, (iv)	165,605	1,102,308	184,135	65,213	3,676,390
($\frac{\xi}{1-\xi}\,\mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	8.38%	51.19%	19.04%	21.39%	100.00%
Capital cost, (v)	4,479,576	25,627,009	3,517,682	2,062,943	90,335,366
($\mathbb{E}[L^{(i)}\mid I=D]\times[\mathbb{P}(I>D)+\tau^*]$)	9.22%	48.43%	14.80%	27.54%	100.00%
Cost, (iii)-(v)	4,177,771	24,204,869	3,339,874	1,929,479	85,060,548
	9.13%	48.58%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	6,258,222	36,239,111	5,002,593	2,889,640	127,376,059
	9.14%	48.57%	14.93%	27.36%	100.00%

Table 7: Cost allocation in the base case.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,227,669,151
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	12,171,966	70,408,002	9,727,968	5,617,558	640,202,514
($\frac{\gamma}{1-c_1'(R^b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	1,029,167	5,837,049	782,958	436,750	52,036,839
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	9.51%	49.52%	14.79%	26.18%	100.00%
Raising cost, (iv)	7,837,319	53,467,331	6,897,003	4,951,601	495,968,512
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	7.60%	47.60%	13.67%	31.14%	100.00%
Capital cost, (v)	17,186,342	91,435,424	13,142,696	6,636,818	822,504,916
($\mathbb{E}[L^{(i)} \mid I=D] \times [\mathbb{P}(I>D)+\tau^*]$)	10.05%	49.08%	15.71%	25.17%	100.00%
Cost, (iii)-(v)	26,052,827	150,739,804	20,822,658	12,025,168	1,370,510,267
	9.14%	48.56%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	38,224,794	221,147,806	30,550,626	17,642,726	2,010,712,781
	9.14%	48.56%	14.94%	27.36%	100.00%

$a = 3,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,568,829,904
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	12,749,807	73,750,484	10,189,785	5,884,241	856,948,543
($\frac{\gamma}{1-c_1'(R^b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-235,099	-1,116,997	-202,680	-67,987	-13,002,123
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	11.11%	48.47%	19.58%	20.84%	100.00%
Raising cost, (iv)	2,579,077	18,401,683	2,557,559	1,239,091	201,178,046
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	7.88%	51.61%	15.97%	24.55%	100.00%
Capital cost, (v)	12,029,113	65,894,390	9,133,206	5,463,678	778,163,393
($\mathbb{E}[L^{(i)} \mid I=D] \times [\mathbb{P}(I>D)+\tau^*]$)	9.50%	47.78%	14.74%	27.98%	100.00%
Cost, (iii)-(v)	14,373,091	83,179,077	11,488,085	6,634,782	966,339,316
	9.14%	48.56%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	27,122,898	156,929,561	21,677,870	12,519,023	1,823,287,859
	9.14%	48.56%	14.93%	27.36%	100.00%

$a = 12,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,995,399,708
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	14,236,612	82,350,817	11,378,055	6,570,425	1,217,059,553
($\frac{\gamma}{1-c_1'(R^b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-861,553	-4,764,429	-676,234	-373,605	-70,665,323
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	9.53%	48.38%	15.29%	26.80%	100.00%
Raising cost, (iv)	1,063,664	8,031,264	1,171,071	494,784	109,736,215
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	7.58%	52.52%	17.05%	22.86%	100.00%
Capital cost, (v)	6,522,382	35,669,393	4,880,405	2,983,652	536,155,774
($\mathbb{E}[L^{(i)} \mid I=D] \times [\mathbb{P}(I>D)+\tau^*]$)	9.51%	47.74%	14.54%	28.21%	100.00%
Cost, (iii)-(v)	6,724,493	38,936,228	5,375,242	3,104,832	575,226,667
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	20,961,105	121,287,045	16,753,298	9,675,257	1,792,286,220
	9.14%	48.56%	14.93%	27.36%	100.00%

Table 8: Cost allocation in the profitable company.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,695	135,041,756	18,658,134	10,774,413	2,487,582,817
$(\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}])$	9.86%	46.23%	17.75%	26.16%	100.00%
Scale effect, (ii)	17,919,327	103,653,262	14,321,322	8,270,057	1,909,380,339
$(\frac{\gamma}{1-c_1'(R^b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)})$	9.86%	46.23%	17.75%	26.16%	100.00%
Continuation value, (iii)	-782,453	-4,526,226	-625,396	-361,120	-83,376,677
$(\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V'])$	9.86%	46.23%	17.75%	26.16%	100.00%
Raising cost, (iv)	1,924,160	11,129,311	1,537,555	888,004	205,012,321
$(\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}])$	9.86%	46.23%	17.75%	26.16%	100.00%
Capital cost, (v)	na	na	na	na	na
$(\mathbb{E}[L^{(i)} \vert I = D] \times [\mathbb{P}(I > D) + \tau^*])$	na	na	na	na	na
Cost, (iii)-(v)	1,141,707	6,603,085	912,159	526,885	121,635,644
	9.86%	46.23%	17.75%	26.16%	100.00%
Non payments, (ii)-(v)	19,061,034	110,256,347	15,233,481	8,796,942	2,031,015,983
	9.86%	46.23%	17.75%	26.16%	100.00%

Table 9: Cost allocation in the empty company.

goes beyond 100%.

Things get worse if cost components are overlooked, or if an incorrect notion of capital is used in the allocation process. We consider such errors in the table by reporting what happens if return in the RORAC numerator is calculated only by referencing actuarial (i) or actuarial and scale costs (i) and (ii), or if a more narrow definition of capital—such as a or S—is used when constructing the numerator. In our scenarios, where the additional cost components are positive, both of these errors tend to work in the same direction to inflate estimates of RORAC. In a number of cases, the incorrectly calculated RORAC appears to indicate high levels of profitability across the board. Of course, one may compare these returns to higher hurdle rates, as it is common in practice. For instance, when allocating S according to VaR or TVaR and when only considering actuarial costs, the hurdle rates in the base case are around $7.5\% = c_1^{(1)}$—which is roughly in line with figures used in the industry. However, despite this apparent coherence, of course this approach does not deliver optimal portfolios as it is not founded in a theoretical analysis of the firm's optimization problem.

4 Conclusion

In this paper, we present a multi-period model for a (re-)insurer with multiple sources of financing. We derive capital allocations from the optimality conditions.

The presented model represents a step toward greater sophistication in firm valuation and risk pricing, but only a step. Other nuances—such as regulatory frictions and rating agency requirements—would obviously merit consideration in a richer model. Moreover, calibration of any model would obviously have to be tailored to the unique circumstances of each firm.

For example, different model specifications could favor different risk measures. Our setup was a relatively favorable one for VaR rather than TVaR, and this is rooted in how we specified the premium function. More realistic specifications would undoubtedly point the way to risk measures

	Allocating	Cost considered	Line 1	Line 2	Line 3	Line 4
$a = 0$						
Correct Allocation	D	yes	3.36%	3.36%	3.36%	3.36%
VaR Allocation	D	yes	3.38%	3.73%	3.01%	3.00%
TVaR Allocation	D	yes	98.46%	2.17%	11.37%	4.42%
VaR Allocation	D	act. only	6.53%	7.81%	6.11%	6.79%
VaR Allocation	S	act. only	16.06%	18.23%	20.24%	22.32%
VaR Allocation	a	act. only	na	na	na	na
VaR Allocation	D	act. and scale	5.97%	7.14%	5.59%	6.21%
VaR Allocation	S	act. and scale	14.69%	16.68%	18.52%	20.42%
VaR Allocation	a	act. and scale	na	na	na	na
TVaR Allocation	D	act. only	190.10%	4.55%	23.04%	9.99%
TVaR Allocation	S	act. only	22.95%	19.49%	20.67%	17.26%
TVaR Allocation	a	act. only	na	na	na	na
TVaR Allocation	D	act. and scale	173.94%	4.17%	21.08%	9.14%
TVaR Allocation	S	act. and scale	21.00%	17.83%	18.91%	15.79%
TVaR Allocation	a	act. and scale	na	na	na	na
$a = 1,000,000,000$						
Correct Allocation	D	yes	3.34%	3.34%	3.34%	3.34%
VaR Allocation	D	yes	3.83%	3.38%	3.25%	3.21%
TVaR Allocation	D	yes	90.72%	2.13%	10.79%	4.73%
VaR Allocation	D	act. only	5.46%	4.92%	4.73%	4.62%
VaR Allocation	S	act. only	9.89%	8.00%	8.00%	7.83%
VaR Allocation	a	act. only	16.06%	13.16%	12.23%	8.36%
VaR Allocation	D	act. and scale	4.51%	4.06%	3.90%	3.81%
VaR Allocation	S	act. and scale	8.17%	6.61%	6.60%	6.46%
VaR Allocation	a	act. and scale	13.26%	10.86%	10.09%	6.90%
TVaR Allocation	D	act. only	129.58%	3.10%	15.70%	6.81%
TVaR Allocation	S	act. only	8.99%	7.68%	7.34%	9.08%
TVaR Allocation	a	act. only	12.91%	11.13%	11.98%	11.00%
TVaR Allocation	D	act. and scale	106.96%	2.56%	12.96%	5.62%
TVaR Allocation	S	act. and scale	7.42%	6.34%	6.06%	7.49%
TVaR Allocation	a	act. and scale	10.65%	9.19%	9.89%	9.08%
$a = 4,000,000,000$						
Correct Allocation	D	yes	2.54%	2.54%	2.54%	2.54%
VaR Allocation	D	yes	2.61%	2.70%	2.30%	2.42%
TVaR Allocation	D	yes	68.61%	1.62%	8.17%	3.60%
VaR Allocation	D	act. only	3.64%	3.81%	3.27%	3.39%
VaR Allocation	S	act. only	16.18%	3.65%	4.32%	8.64%
VaR Allocation	a	act. only	3.19%	3.97%	3.04%	3.47%
VaR Allocation	D	act. and scale	2.43%	2.55%	2.18%	2.26%
VaR Allocation	S	act. and scale	10.80%	2.44%	2.89%	5.77%
VaR Allocation	a	act. and scale	2.13%	2.65%	2.03%	2.32%
TVaR Allocation	D	act. only	95.85%	2.30%	11.62%	5.04%
TVaR Allocation	S	act. only	5.44%	4.57%	3.89%	6.24%
TVaR Allocation	a	act. only	85.34%	2.05%	10.34%	4.48%
TVaR Allocation	D	act. and scale	63.99%	1.53%	7.76%	3.36%
TVaR Allocation	S	act. and scale	3.63%	3.05%	2.60%	4.17%
TVaR Allocation	a	act. and scale	56.97%	1.37%	6.90%	2.99%

Table 10: RORAC calculations, base case.

	Allocating	Cost considered	Line 1	Line 2	Line 3	Line 4
$a = 0$						
Correct Allocation	D	yes	8.94%	8.94%	8.94%	8.94%
VaR Allocation	D	yes	9.23%	9.90%	8.19%	7.84%
TVaR Allocation	D	yes	263.48%	5.80%	30.55%	11.58%
VaR Allocation	D	act. only	20.53%	23.93%	19.03%	20.83%
VaR Allocation	S	act. only	83.91%	61.61%	69.60%	38.51%
VaR Allocation	a	act. only	na	na	na	na
VaR Allocation	D	act. and scale	13.99%	16.31%	12.97%	14.20%
VaR Allocation	S	act. and scale	57.19%	41.99%	47.44%	26.25%
VaR Allocation	a	act. and scale	na	na	na	na
TVaR Allocation	D	act. only	586.01%	14.03%	71.00%	30.78%
TVaR Allocation	S	act. only	66.20%	56.01%	60.09%	48.32%
TVaR Allocation	a	act. only	na	na	na	na
TVaR Allocation	D	act. and scale	399.40%	9.56%	48.39%	20.98%
TVaR Allocation	S	act. and scale	45.12%	38.18%	40.96%	32.93%
TVaR Allocation	a	act. and scale	na	na	na	na
$3,000,000,000$						
Correct Allocation	D	yes	6.62%	6.62%	6.62%	6.62%
VaR Allocation	D	yes	7.21%	6.85%	6.08%	6.42%
TVaR Allocation	D	yes	184.42%	4.18%	21.23%	9.53%
VaR Allocation	D	act. only	16.25%	16.32%	14.42%	14.70%
VaR Allocation	S	act. only	33.65%	27.35%	31.83%	30.88%
VaR Allocation	a	act. only	58.55%	59.79%	63.05%	62.84%
VaR Allocation	D	act. and scale	8.61%	8.65%	7.64%	7.79%
VaR Allocation	S	act. and scale	17.83%	14.50%	16.87%	16.37%
VaR Allocation	a	act. and scale	31.03%	31.69%	33.42%	33.31%
TVaR Allocation	D	act. only	415.83%	9.96%	50.38%	21.84%
TVaR Allocation	S	act. only	34.18%	27.54%	27.52%	32.73%
TVaR Allocation	a	act. only	64.22%	59.89%	64.24%	59.52%
TVaR Allocation	D	act. and scale	220.36%	5.28%	26.70%	11.58%
TVaR Allocation	S	act. and scale	18.11%	14.60%	14.58%	17.35%
TVaR Allocation	a	act. and scale	34.03%	31.74%	34.04%	31.55%
$a = 4,000,000,000$						
Correct Allocation	D	yes	3.58%	3.58%	3.58%	3.58%
VaR Allocation	D	yes	3.75%	3.77%	3.16%	3.50%
TVaR Allocation	D	yes	99.82%	2.26%	11.32%	5.20%
VaR Allocation	D	act. only	12.05%	12.82%	10.86%	11.35%
VaR Allocation	S	act. only	19.48%	18.18%	20.07%	20.06%
VaR Allocation	a	act. only	11.41%	15.32%	11.92%	18.85%
VaR Allocation	D	act. and scale	3.87%	4.12%	3.48%	3.64%
VaR Allocation	S	act. and scale	6.25%	5.84%	6.44%	6.44%
VaR Allocation	a	act. and scale	3.66%	4.92%	3.83%	6.05%
TVaR Allocation	D	act. only	320.79%	7.68%	38.87%	16.85%
TVaR Allocation	S	act. only	23.20%	17.53%	16.80%	22.83%
TVaR Allocation	a	act. only	15.96%	14.51%	10.59%	20.03%
TVaR Allocation	D	act. and scale	102.91%	2.47%	12.47%	5.41%
TVaR Allocation	S	act. and scale	7.44%	5.63%	5.39%	7.33%
TVaR Allocation	a	act. and scale	5.12%	4.66%	3.40%	6.43%

Table 11: RORAC calculations, profitable company case.

more complicated than either VaR or TVaR. But a practitioner faces deeper problems: Even if armed with the right risk measure and capital allocation technique, one can get highly distorted results unless the right threshold is chosen and the right costs are considered in setting hurdle rates.

These caveats are addressed only with a deeper understanding of what creates value at the level of the firm. As our models move in this direction, it is evident that greater sophistication is bound to lead to more complication in pricing risk and measuring performance in insurance. Allocation is valid as a pricing guide only if great care is taken in its implementation and interpretation.

References

Albrecht, P. (2004). "Risk Based Capital Allocation." *Encyclopedia of Actuarial Science*, 1459-1466. John Wiley & Sons.

American Academy of Actuaries (2011). "Market Consistent Embedded Values." American Academy of Actuaries.

Basak, S., and Shapiro, A. (2001). "Value-at-risk-based Risk Management: Optimal Policies and Asset Prices." *Review of Financial Studies* 14, 371-405.

Bauer, D. and G. Zanjani (2013). "Capital Allocation and Its Discontents." *Handbook of Insurance, 2nd Ed.*, Dionne, G. (Ed.), 627-645, Springer, New York.

Bauer, D. and G. Zanjani (2014). "Comparison and Sensitivity of Capital Allocation Techniques for a Catastrophe Insurer." Working Paper, Georgia State University.

Bauer, D. and G. Zanjani (2015). "The Marginal Cost of Risk, Risk Measures, and Capital Allocation." *Management Science*, forthcoming.

Bertsekas, D.P. (1995). "Dynamic Programming and Optimal Control, Volume II." Athena Scientific, Belmont, Mass.

Cantor, R., F. Packer, and K. Cole (1997). "Split Ratings and the Pricing of Credit Risk" *Federal Reserve Bank of New York Research Paper* No. 9711.

Cox, L.A., and E.A. Rudd (1991). "Book versus Market Underwriting Betas." *Journal of Risk and Insurance* 58, 312-321.

Cummins, J.D., and S.E. Harrington (1985), "Property-Liability Insurance Regulation: Estimating Underwriting Betas from Quarterly Profit Data." *Journal of Risk and Insurance* 52, 16-43.

Cummins, J.D., and R.D. Phillips (2005). "Estimating the Cost of Equity Capital for Property–Liability Insurers." *Journal of Risk and Insurance* 72, 441-478.

D'Arcy, S.P. 2011. "Capital allocation in the property-liability insurance industry." *Variance* 5, 141-157.

Denault, M. (2001). "Coherent Allocation of Risk Capital." *Journal of Risk* 4, 7-21.

Dhaene, J., M.J. Goovaerts, and R. Kaas (2003). "Economic Capital Allocation Derived from Risk Measures." *North American Actuarial Journal* 7, 44-59.

Epermanis, K. and S.E. Harrington (2006). "Market discipline in property/casualty insurance: Evidence from premium growth surrounding changes in financial strength ratings." *Journal of Money, Credit and Banking* 38, 1515-1544.

Erel, I., Myers, S.C., and Read, J.A. (2013). "Capital Allocation." Working Paper, Ohio State University.

Fama, E.F., and French, K.R. (1993) "Common risk factors in the returns on stocks and bonds." *Journal of Financial Economics* 33, 3-56.

Froot, K.A. (2007). "Risk Management, Capital Budgeting, and Capital Structure Policy for Insurers and Reinsurers." *Journal of Risk and Insurance* 74, 273-299.

Froot, K.A., and Stein, J. (1998). "Risk Management, Capital Budgeting, and Capital Structure Policy for Financial Institutions: An Integrated Approach." *Journal of Financial Economics* 47, 55-82.

Gründl, H. and H. Schmeiser (2007). "Capital Allocation for Insurance Companies – What Good is it?" *Journal of Risk and Insurance* 74(2), 301-17.

Hennessy, C.A., and T.M. Whited (2007). "How costly is external financing? Evidence from a structural estimation." *Journal of Finance* 62, 1705-1745.

Ibragimov, R., D. Jaffee, and J. Walden (2010). "Pricing and Capital Allocation for Multiline Insurance Firms." *Journal of Risk and Insurance* 77(3), 551-78.

Kalkbrener, M. (2005). "An Axiomatic Approach to Capital Allocation." *Mathematical Finance* 15, 425-37.

Kreps, R. (2005). "Riskiness leverage models." *Proceedings of the Casualty Actuarial Society* 92, 31-60.

Mango, D. (2005). "Insurance Capital as a Shared Asset." *ASTIN Bulletin* 35, 471-486.

Merton, R.C., and Perold, A.F. (1993). "Theory of Risk Capital in Financial Firms." *Journal of Applied Corporate Finance* 6, 16-32.

Mildenhall, S.J. (2004). "A Note on Myers and Read Capital Allocation Formula." *North American Actuarial Journal* 8, 32-44.

Myers, S.C. and J.A. Read (2001). "Capital Allocation for Insurance Companies." *Journal of Risk and Insurance* 68, 545-80.

Nelder, J.A. and R. Mead (1965). "A simplex method for function minimization." *Computer Journal* 7, 308-313.

Phillips, R.D., Cummins, J.D., and Allen, F. (1998). "Finanical Pricing of Insurance in the Multiple Line Insurance Company." *Journal of Risk and Insurance* 65, 597-636.

Powers, M.R. (2007). "Using Aumann-Shapley Values to Allocate Insurance Risk: The Case of Inhomogeneous Losses." *North American Actuarial Journal* 11, 113-27.

Sherris, M. (2006). "Solvency, Capital Allocation, and Fair Rate of Return in Insurance." *Journal of Risk and Insurance* 73, 71-96.

Sommer, D.W. (1996). "The impact of firm risk on property-liability insurance prices." *Journal of Risk and Insurance* 63, 501-514.

Stoughton, N.M., and Zechner, J. (2007). "Optimal Capital Allocation Using RAROC and EVA." *Journal of Financial Intermediation* 16, 312-342.

Tasche, D. (2004). "Allocating Portfolio Economic Capital to Sub-portfolios." In: *Economic Capital: A Practitioner Guide*, A. Dev (ed.). Risk Books, 2004, 275-302.

Tsanakas, A., and C. Barnett (2003). "Risk capital allocation and cooperative pricing of insurance liabilities." *Insurance: Mathematics and Economics* 33, 239-254.

Venter, G.G. (2002). "Allocating Surplus–Not!" *Actuarial Review*, February 2002.

Venter, G.G. (2004). "Capital allocation survey with commentary." *North American Actuarial Journal* 8, 96-107.

Venter, G.G. (2010). "Next Steps for ERM: Valuation and Risk Pricing." *Proceedings of 2010 Enterprise Risk Management Symposium* 2010.

Zanjani, G. (2002). "Pricing and Capital Allocation in Catastrophe Insurance." *Journal of Financial Economics* 65, 283-305.

Appendix

A Technical Appendix

A.1 Proofs in Section 2

Proof of Lemma 2.1. With the budget constraint (2):

$$\begin{aligned} & e^{-rt}\, a_t - e^{-r(t-1)}\, a_{t-1} - e^{-rt}[e^r R^b_t + R^e_t] \\ = \;& e^{-rt}\left[e^r \sum_j p^{(j)}_t - \sum_j I^{(j)}_t - (\tau\, a_{t-1} + c_1(R^b_t))\, e^r - c_2(R^e_t)\right]. \end{aligned}$$

Hence, the sum in (6) can be written as:

$$\begin{aligned} & \sum_{t=1}^{\infty} 1_{\{a_1\geq 0,\dots,a_t\geq 0\}}\, e^{-rt}\left[e^r \sum_j p^{(j)}_t - \sum_j I^{(j)}_t - (\tau\, a_{t-1} + c_1(R^b_t))e^r - c_2(R^e_t)\right] \\ & \quad -1_{\{a_1\geq 0,\dots,a_{t-1}\geq 0, a_t<0\}}\, e^{-rt}\left[(a_{t-1} + R^b_t)e^r + R^e_t\right] \\ = & \sum_{\{t<t^*: a_1\geq 0, a_2\geq 0,\dots,a_{t^*-1}\geq 0, a_{t^*}<0\}} \left[e^{-rt}\, a_t - e^{-r(t-1)}\, a_{t-1}\right] - e^{-rt}[e^r R^b_t + R^e_t] \\ & \quad -e^{-rt^*}\left[(a_{t^*-1} + R^b_{t^*})e^r + R^e_{t^*}\right] \\ = & \left[\sum_{t\leq t^*} e^{-rt}[-e^r R^b_t - R^e_t]\right] + e^{-r(t^*-1)}\, a_{t^*-1} - a_0 - e^{-r(t^*-1)}\, a_{t^*-1} \\ = & \left[\sum_{t\leq t^*} e^{-rt}[-e^r R^b_t - R^e_t]\right] - a_0, \end{aligned}$$

which completes the proof. □

Proof of Proposition 2.1. Notice that our per-period profit function in (6) is bounded from above, so the Bellman equation follows from classical infinite-horizon dynamic programming results (see e.g. Proposition 1.1 in Bertsekas (1995, Chap. 3)). □

Proof of Proposition 2.2. Let

$$a' = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r + R^e - c_2(R^e) - \sum_j I^{(j)};$$

then, conditional on $a' < 0$, the objective function is decreasing in R^e so that zero is the optimal choice. Conditional on $a' > 0$, if $R^e > 0$, on the other hand, decreasing R^e by a (small) $\varepsilon > 0$ and increasing R^b in the beginning of the next period will be dominant (since $c_2 > c_1$), so $R_e > 0$ cannot be optimal. Finally, if $a' = 0$ and $R_e > 0$, then $R^e = R^e_*$.

Moreover,

$$
\begin{aligned}
-\left(a+R^b\right) &< \sum_j p^{(j)} - e^{-r}\sum_j I^{(j)} - \tau\, a - c_1(R^b) - e^{-r}\, c_2(R^e_*) \\
&\quad + e^{-r}\, V\left([a(1-\tau)+R^b - c_1(R^b) + \sum_j p^{(j)}]e^r + R^e_* - c_2(R^e_*) - \sum_j I^{(j)}\right) \\
\Leftrightarrow V(0) &> -\left(\sum_j p^{(j)} + (1-\tau)a + R^b - c_1(R^b)\right) e^r + \sum_j I^{(j)} + c_2(R^e_*) \\
\Leftrightarrow V(0) &> R^e_*,
\end{aligned}
$$

which proves the last assertion. □

Proof of Proposition 2.3. We prove an extended version of the proposition in the main text, which additionally includes a regulatory constraint imposed by regulation of the form:

$$
\rho\,(I) \le \underbrace{\left(a(1-\tau) + R^b - c_1(R^b) + \sum_{j=1}^{N} p^{(j)}\right)\, e^r + (1-\xi)\, V(0),}_{\text{Available Capital}} \tag{16}
$$

where ρ is a *monetary risk measure*.[8]

The first order conditions from the Bellman equation (10) are:

$$
\begin{aligned}
[q_i] \qquad & -e^{-r}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\left(1+V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r - I\right)\right)\, 1_{\{S\ge I\}}\right] \\
& -\frac{e^{-r}}{1-\xi}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} 1_{\{S<I\le D\}}\right] \\
& -\sum_{k\neq i}\lambda_k\,\left(\beta\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\big|I=D\right]\, f_I(D) + \gamma\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\right)\, e^{-r}\,\mathbb{E}[I^{(k)}]\,\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\} \\
& +\lambda_i\left(e^{-r}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] - \beta\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\big|I=D\right]\, f_I(D)\,\mathbb{E}[I^{(i)}]\, e^{-r} - \gamma\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\,\mathbb{E}[I^{(i)}]\, e^{-r}\right) \\
& \qquad\qquad \times \exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\} - \zeta\,\frac{\partial\rho}{\partial q^{(i)}}\, e^{-r} = 0, \\
[p_i] \qquad & \mathbb{E}\left[\left(1+V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r - I\right)\right)\, 1_{\{S\ge I\}}\right] + \frac{1}{1-\xi}\mathbb{E}\left[1_{\{S<I\le D\}}\right] \\
& -\lambda_i + \sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\, f_I(D)\,\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\} \\
& +\zeta = 0,
\end{aligned}
$$

[8] This specification assumes that the company can put up the *present value of future profits* (PVFP) as a part if its capital, which is consistent with market-consistent embedded value principles (American Academy of Actuaries, 2011).

$$\begin{aligned}
[R^b] \qquad & \mathbb{E}\left[\left(V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)(1-c_1'(R^b))-c_1'(R^b)\right) I_{\{S\geq I\}}\right] \\
& +\mathbb{E}\left[\left(\frac{1}{1-\xi}-\frac{c_1'(R^b)}{1-\xi}-1\right) I_{\{S<I\leq D\}}\right]-\mathbb{P}(I>D) \\
& +\sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\, f_I(D)\,(1-c_1'(R^b))\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\} \\
& +\zeta(1-c_1'(R^b))=0.
\end{aligned}$$

From $[p_i]$ and $[R^b]$, we obtain:

$$\begin{aligned}
\lambda_i &= \sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\, f_I(D)\,\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}+\frac{1}{1-\xi}\mathbb{E}\left[I_{\{S<I\leq D\}}\right]+\zeta \\
&\quad +\mathbb{E}\left[\left(1+V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)\right) I_{\{S\geq I\}}\right] \\
&= \frac{1}{1-c_1'(R^b)}\left[\zeta(1-c_1'(R^b))+\sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\, f_I(D)\,(1-c_1'(R^b))\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}\right. \\
&\quad +(1-c_1'(R^b))\,\mathbb{E}\left[V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right) I_{\{S\geq I\}}\right] \\
&\quad \left.+\frac{1-c_1'(R^b)}{1-\xi}\mathbb{P}(S<I\leq D)+(1-c_1'(R^b))\,\mathbb{P}(S\geq I)\right] \\
&= \frac{1}{1-c_1'(R^b)}.
\end{aligned}$$

Then, we can write $[R^b]$ as:

$$\begin{aligned}
\zeta &= \frac{\mathbb{P}(I>D)}{1-c_1'(R^b)}+\frac{c_1'(R^b)}{1-c_1'(R^b)}\mathbb{P}(I\leq D)-\frac{\xi}{1-\xi}\,\mathbb{P}(S<I\leq D) \\
&\quad -\mathbb{E}\left[V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right) I_{\{S\geq I\}}\right] \\
&\quad -\frac{1}{1-c_1'(R^b)}\sum_k \mathbb{E}[I^{(k)}]\,\beta\, f_I(D)\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}\,.
\end{aligned} \tag{17}$$

And with $[q_i]$, we obtain:

$$\begin{aligned}
& \frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\,\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}}{1-c_1'(R^b)} \\
&= \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\,I_{\{D\geq I\}}\right]+\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\;V'(S-I)\,I_{\{S\geq I\}}\right]+\frac{\xi}{1-\xi}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}I_{\{S<I\leq D\}}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-c_1'(R^b)}\gamma\,\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\,\mathbb{E}[I]\,\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\} \\
&+\left[\frac{1}{1-c_1'(R^b)}\beta\,f_I(D)\,E[I]\,\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\}\right]\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\big|I=D\right] \\
&+\Bigg[\mathbb{P}(I>D)+\frac{c_1'(R^b)}{1-c_1'(R^b)}-\frac{\xi}{1-\xi}\,\mathbb{P}(S<I\le D)-\mathbb{E}\left[V'\left(S-I\right)\,1_{\{S\ge I\}}\right] \\
&\qquad\qquad \frac{1}{1-c_1'(R^b)}\beta\,f_I(D)\,E[I]\,\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\}\Bigg]\,\frac{\partial\rho}{\partial q^{(i)}}.
\end{aligned}
$$

In the absence of a regulatory constraint, $\zeta=0$ so that Equation (17) yields:

$$
\begin{aligned}
&\frac{1}{1-c_1'(R^b)}\sum_k \mathbb{E}[I^{(k)}]\,\beta\,f_I(D)\exp\left\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\right\} \\
=\;\;&\mathbb{P}(I>D)+\frac{c_1'(R^b)}{1-c_1'(R^b)}-\frac{\xi}{1-\xi}\,\mathbb{P}(S<I\le D)-\mathbb{E}\left[V'\left(S-I\right)\,1_{\{S\ge I\}}\right],
\end{aligned}
$$

which yields the expression in Proposition 2.3. □

A.2 Implementation of the Multi-Period Model from Section 3

For solving the one-period problem in each step, we rely on the following basic algorithm. Details on some of the steps, on the implementation, and evidence on the convergence is provided below.

Algorithm A.1. *For a given (discretizted) end-of-period value function V^{end}, and for capital levels $a_k=$ ADEL $\times\, k$, $k=0,1,2,\ldots,$AGRID :*

1. *Given capital level a_k, optimize over $q^{(1)},q^{(2)},q^{(3)},q^{(4)}$.*
2. *Given capital level a_k and a portfolio $(q^{(1)},q^{(2)},q^{(3)},q^{(4)})$, optimize over $R^{(b)}$.*
3. *Given a_k, portfolio $(q^{(1)},q^{(2)},q^{(3)},q^{(4)})$, and raising decision $R^{(b)}$, determine the premium levels by evaluating Equation* (11).
4. *Given a_k, portfolio $(q^{(1)},q^{(2)},q^{(3)},q^{(4)})$, $R^{(b)}$, and premiums $(p^{(1)},p^{(2)},p^{(3)},p^{(4)})$ evaluate $V^{beg}(a_k)$ based on the given end-of-period function V^{end} by interpolating in between the grid and extrapolating off the grid.*

Discretization

We rely on an equidistant grid of size of 26 (AGRID $=25$) with different increments depending on the parameters (ADEL $=250,000,000$ for the base case and ADEL $=750,000,000$ for the "profitable company" and "empty company" cases). We experimented with larger grids with finer intercepts but 26 points proved to be a suitable compromise between accuracy and run time of the program.

Optimization

For carrying out the numerical optimization of the portfolio values $q^{(i)}$, $i = 1, 2, 3, 4$, we rely on the so-called *downhill simplex method* proposed by Nelder and Mead (1965) as available within most numerical software packages. For the starting values, we rely on the optimized values from the previous step, with occasional manual adjustments during the early iteration in order to smooth out the portfolio profiles.

For the optimization of the optimal raising decision R^b, in order to not get stuck in a local maximum, we first calculate the value function based on sixty different values across the range of possible values $[-a \times (1-\tau), \infty)$. We then use the optimum of these as the starting value in the Nelder-Mead method to derive the optimized value.

Calculation of the Premium Levels

The primary difficulty in evaluating the optimal premium level is that premiums enter the constraint (11) on both sides of the equation as the default rate itself depends on the premium, and this dependence is discontinuous (given our discrete loss distributions). We use the following approach: Starting from a zero default rate, we calculate the minimal amount necessary to attain the given default rate; we then check whether this amount is incentive-compatible, i.e., if the policyholders would be willing to pay it given the default probability; if so, we calculate a smoothed version of the premium level using (11) by deriving the (hypothetical) default rate considering how much the policyholders are willing to pay over to the minimal amount at that level relative to the amount necessary to decrease the default rate based on the our discretized distribution; if not, we move to the next possible default rate given our discrete loss distribution and check again.

Interpolation and Extrapolation

For arguments in between grid points, we use linear interpolation. For values off the grid ($a >$ AGRID $\times$ ADEL), supported by the general shape of the value functions across iterations, we use either linear or quadratic extrapolation. More precisely, in case fitting a quadratic regression in a to the five greatest grid values does not yield a significant quadratic coefficient—i.e., if the value function appears linear in this region—we use linear extrapolation starting from the largest grid point. Otherwise, we use a quadratic extrapolation fitted over the entire range starting from the largest grid point.

Convergence

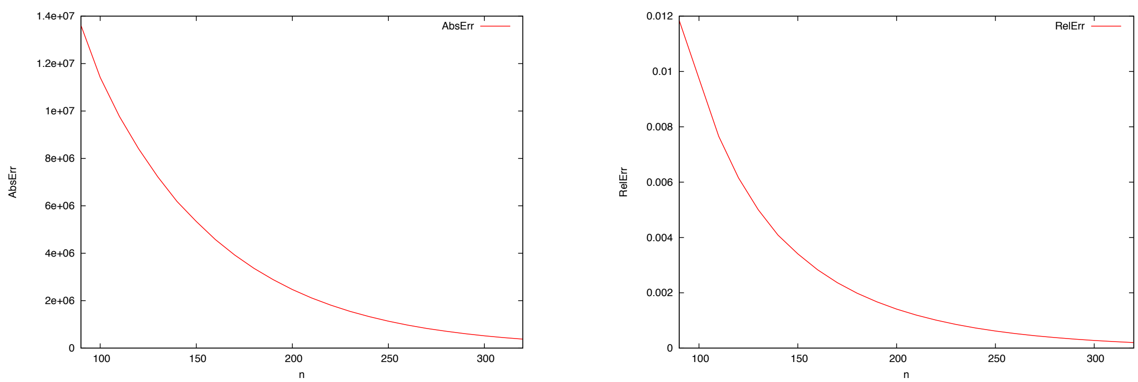
We assess convergence by calculating absolute and relative errors in the value and the policy functions from one iteration to the next. These errors are directly proportional to error bounds for the algorithm, where the proportionality coefficients depend on the interest and the default rate (an upper bound is given by $\bar{c} = \frac{e^{-r}}{1-e^{-r}}$, see e.g. Proposition 3.1 in (Bertsekas, 1995, Chap. 1))). More precisely, we define the absolute and relative errors for the value function by:

$$
\begin{aligned}
\text{AbsErr}_n &= \max_k \left\{ \| V_n(a_k) - V_{n-1}(a_k) \| \right\}, \\
\text{RelErr}_n &= \max_k \left\{ \frac{\| V_n(a_k) - V_{n-1}(a_k) \|}{V_n(a_k)} \right\},
\end{aligned}
$$

where V_n denotes the value function after iteration n. Similarly, we define absolute and relative errors for the policy function by:

$$\begin{aligned}\text{AbsErr}(q^{(i)})_n &= \max_k \left\| q_n^{(i)}(a_k) - q_{n-1}^{(i)}(a_k) \right\|, \ i = 1,2,3,4, \\ \text{RelErr}(q^{(i)})_n &= \max_k \frac{\left\| q_n^{(i)}(a_k) - q_{n-1}^{(i)}(a_k) \right\|}{q_n^{(i)}(a_k)}, \ i = 1,2,3,4,\end{aligned}$$

where $q_n^{(i)}$ denotes the (optimized) exposure to line i after iteration n.



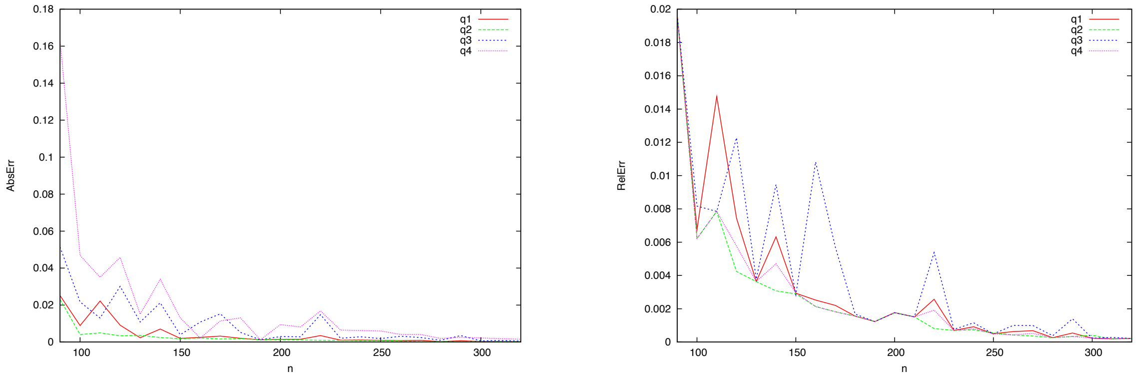
(a) Absolute Error in V

(b) Relative Error in V

Figure 8: Absolute and relative error in the value function for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} =$ 1.00E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma =$ 1.48E-10 (base case).

Figure 8 show the errors for the value function using the base case parameters and different values of n for between 90 and 320 (in increments of 10). Results (and error bounds) for the remaining parametrizations are even smaller. After 320 iterations, the absolute error in the value function is $379,230$, which is only a very small fraction of the value function ranging from $1,813,454,921$ to $1,955,844,603$ (about 0.02%). In particular, considering the rather conservative error bound above, these results imply that the error in V amounts to less than one percent. Similarly, Figure 9 shows the absolute and relative errors for the portfolio functions. Again, we observe that relative changes from one iteration to the next after 320 iterations are maximally around 0.02%.

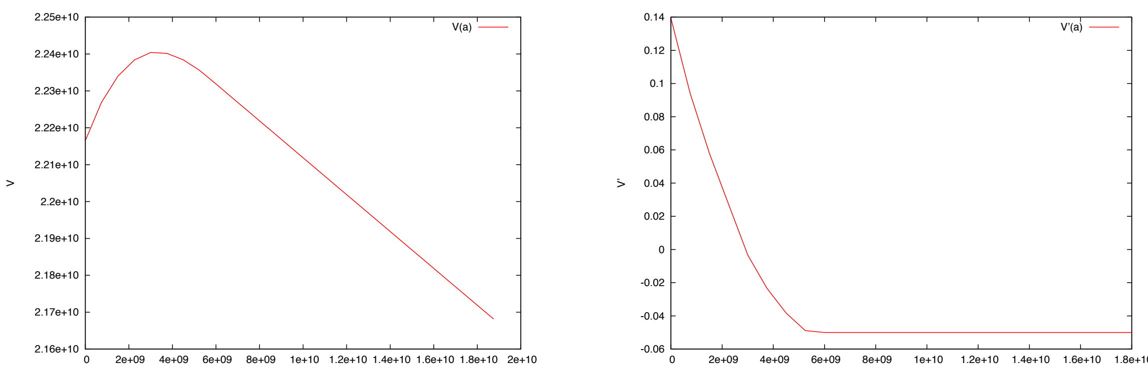
B Additional Figures



(a) Absolute Error in $q^{(i)}$, $i = 1, 2, 3, 4$

(b) Relative Error in $q^{(i)}$, $i = 1, 2, 3, 4$

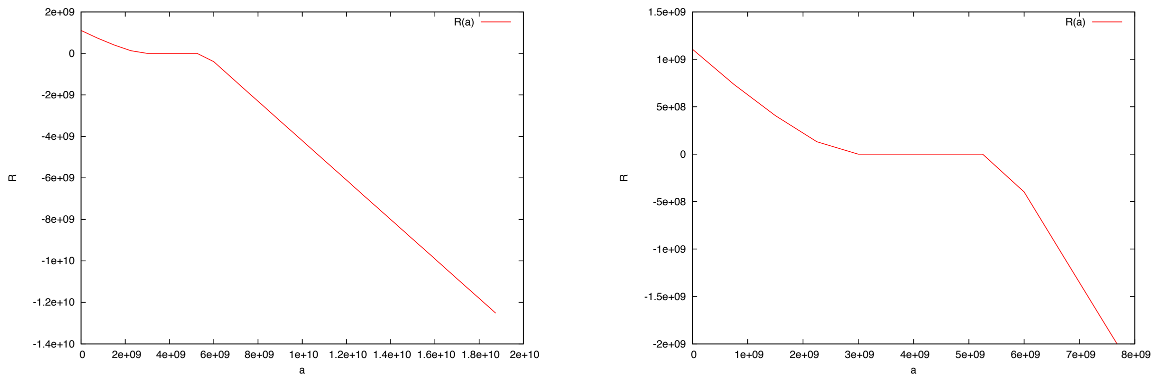
Figure 9: Absolute and relative error in the optimal portfolio weights for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00$E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.3156$, $\beta = 392.96$, and $\gamma = 1.48$E-10 (base case).



(a) Value function $V(a)$

(b) Derivative $V'(a)$

Figure 10: Value function V and its derivative V' for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00$E-11, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61$E-10 (profitable company).



(a) Raising decisions $R(a)$

(b) Raising decisions $R(a)$ (lim. range)

Figure 11: Optimal raising decision R for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\text{E-}10$ (profitable company).

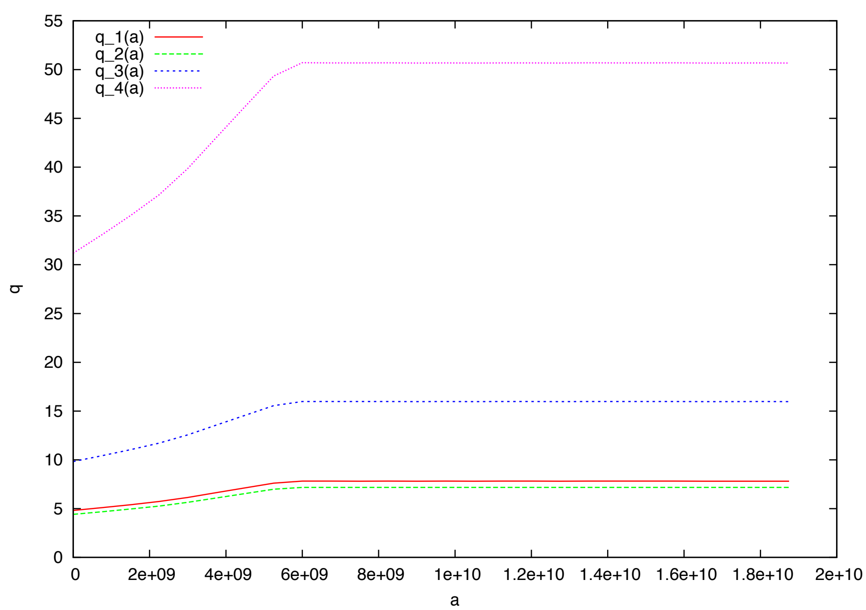


Figure 12: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.9730$, $\beta = 550.20$, and $\gamma = 1.61\text{E-}10$ (profitable company).