Universal Gravity*

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Abstract

The gravity relationship is one of the most robust empirical results in economics. This success has led to a proliferation of general equilibrium models that offer theoretical foundations for the gravity relationship. In this paper we develop a universal framework that nests previous general equilibrium gravity models and show that many of the macro-economic implications these various models depend solely on two key model parameters, which we term the "gravity constants." On the theoretical side, we provide sufficient conditions for the existence and uniqueness of the trade equilibrium and show that the equilibrium can be equivalently considered as the solution to planning problems either maximizing world income or world welfare. On the empirical side, given observed trade flows, we show that gravity models are fundamentally underidentified, yet we can characterize all comparative statics for any change in bilateral trade frictions solely in terms of observed trade flows and the gravity constants. Based on these results, we derive a closed form solution of a new gravity estimator that improves upon standard reduced-form gravity regressions by directly incorporating general equilibrium effects.

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1 Introduction

The gravity relationship – where trade flows increase with the origin and destination countries incomes and decrease with the distance between the two countries – is one of the most robust empirical results in economics.[1] This success has led to a proliferation of general equilibrium models that offer theoretical foundations for the gravity relationship, see e.g. Anderson (1979); Bernard, Eaton, Jensen, and Kortum (2003); Eaton and Kortum (2002); Chaney (2008). However, due to the numerous and varied general equilibrium effects at play in these gravity trade models, little is known about the extent to which the predictions of each model depends on its particular theoretical foundation.

In this paper, we develop a universal framework that nests previous general equilibrium gravity models and show that many macroeconomic implications depend solely on two key model parameters, which we term the "gravity constants."[2] Different micro-economic foundations affect the interpretation of the gravity constants, but do not affect the general equilibrium structure of the model. Simply put, conditional on the value of these constants, all gravity trade models deliver the same macro-economic predictions.

The general equilibrium gravity framework we develop is based on four restrictions: (i) a "modern" version of gravity, whereby bilateral trade flows depend on (endogenous) origin and a destination country shifter and (exogenous) bilateral trade frictions;[3] (ii) aggregate output equals total sales; (iii) trade is balanced; and (iv) (gross) income is a log-linear function of the origin and destination shifters (which practically translates to the condition that gross income is proportional to labor income). The aforementioned gravity constants are simply the coefficients of this log-linear function. It turns out that these assumptions – which are ubiquitous throughout the trade literature – impose sufficient structure on aggregate trade flows to completely characterize all general equilibrium interactions.

We classify our results into two groups: theoretical and empirical. In the first group, we first examine the existence and uniqueness properties common to all general equilibrium gravity trade models. We show that their solution can be represented by a nonlinear op-

[1]The literature on the gravity equation in trade is vast; an excellent starting place are the recent review articles by Baldwin and Taglioni (2006), Anderson (2011) and Head and Mayer (2013).

[2]Examples of models covered under our specification is perfect competition models such as Anderson (1979), Anderson and Van Wincoop (2003), Eaton and Kortum (2002), Caliendo and Parro (2010) monopolistic competition models such as Krugman (1980), Melitz (2003) as specified by Chaney (2008), Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008), Di Giovanni and Levchenko (2008), Dekle, Eaton, and Kortum (2008), and the Bertrand competition model of Bernard, Eaton, Jensen, and Kortum (2003); see Table 1 for details.

[3]This version of the gravity model was first introduced by Eaton and Kortum (2002), Anderson and Van Wincoop (2003), and Redding and Venables (2004). Baldwin and Taglioni (2006) and Head and Mayer (2013) carefully discuss the econometric issues arising from the use of this specification.

erator on a compact set, which allows us to provide sufficient conditions for existence and uniqueness of a trade equilibrium as a function solely of the two gravity constants. Given the simple mapping of different gravity models to gravity constants, these sufficient conditions are straightforward to check and relax the sufficient conditions presented by Alvarez and Lucas (2007). The parameter region where uniqueness applies can be expanded when trade frictions are "quasi-symmetric", as is assumed in much of the empirical gravity literature, e.g. Eaton and Kortum (2002) and Waugh (2010). Our methodology can also be extended to consider multiple sectors of production, as in Chor (2010) and Costinot, Donaldson, and Komunjer (2010).

Second, we show that there exists two "dual" interpretations of the general equilibrium gravity model. In the first interpretation, a planner maximizes world income subject to trade remaining balanced and an aggregate world resource constraint. In the second interpretation, a planner maximizes a weighted average of world welfare subject to only the aggregate world resource constraint, where welfare is assumed to be written as a function of trade openness (as in the class of trade models considered by Arkolakis, Costinot, and Rodríguez-Clare (2012)). Using these dual interpretations, we apply the envelope theorem to derive the elasticity of both world income and world welfare to any bilateral trade costs, which can both be expressed solely as a function of observed trade flows and the gravity constants. While the expression for world income is, to the best of our knowledge, novel, the expression for world welfare has been derived previously for gravity models with CES demand by Atkeson and Burstein (2010), Burstein and Cravino (2012), and Fan, Lai, and Qi (2013); our derivation extends this result to any gravity trade model where welfare can be expressed as a function of trade openness (which Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012) show holds for a large class of homothetic utility functions).

We then turn to the empirical properties of the model by asking what can be said using our framework given observed trade flows. We first characterize the extent to which model fundamentals can be recovered from the trade data. We show that trade models are intrinsically underidentified: the same trade data can be perfectly matched by different combinations of model fundamentals. Notably, the gravity constants cannot be identified from observed trade flows alone. This result provides a general characterization of the non-identification inherent to gravity models which has been discussed for particular models previously by in Waugh (2010), Eaton, Kortum, Neiman, and Romalis (2011), Burstein and Vogel (2012), Ramondo, Rodríguez-Clare, and Saborio-Rodriguez (2012) and Arkolakis, Ramondo, Rodríguez-Clare, and Yeaple (2013).

To examine how changes in bilateral trade frictions affect equilibrium trade flows and incomes we first derive an analytical expression for the (large) matrix of elasticities of all bilateral trade flows and incomes to changes in all bilateral trade frictions. As with the aggregate elasticities, this expression depends only on observed trade flows and the gravity constants, indicating that apart from these two model parameters, all macro-economic implications – i.e. the changes in trade flows and gross incomes – for all gravity models are the same. We then derive a system of equations that show how arbitrary (possibly non-infinitesimal) changes to the trade friction matrix affect macro-economic variables; again this expression only depends on the gravity constants and observed trade flows. While the non-infinitesimal results generalize those developed by Dekle, Eaton, and Kortum (2008) and Arkolakis, Costinot, and Rodríguez-Clare (2012), the closed form solution for the trade elasticities is, the the best of our knowledge, the first in the literature.[4]

Building upon these theoretical results, we develop a new general equilibrium gravity estimator. Unlike the widely used fixed effects gravity estimator made popular by Eaton and Kortum (2002) and Redding and Venables (2004)[5], our estimator, which is in the spirit of Anderson and Van Wincoop (2003), explicitly incorporates the general equilibrium effects that a change in the bilateral trade friction between any two countries has on all other bilateral trade flows.[6] Unlike Anderson and Van Wincoop (2003), however, we derive a closed form solution for the general equilibrium estimator, and show that it can be interpreted as an ordinary least squares regression where the typical gravity regressors have undergone a transformation to account for general equilibrium effects. Using Monte Carlo simulations, we show that the general equilibrium estimator can not only outperform a fixed effects gravity estimator, it can also overcome concerns of omitted variable bias by relying on the general equilibrium effects as a source of identification.

Finally, we put our gravity framework to work in two empirical applications. First, we show how to optimally allocate "infrastructure improvements" that lower bilateral trade frictions to maximize the welfare of any particular country (or total world welfare). Second, we estimate the effect of WTO membership on trade flows, and find that while the WTO substantially increases the welfare of member countries, it does so at a cost to non-members.

Our work is related to a small but growing literature analyzing the structure of general

[4]The analysis of Arkolakis, Costinot, and Rodríguez-Clare (2012) and Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012) applies only to models where the trade elasticity, i.e. the response of trade flows to trade costs, pins down our first gravity constant and the second gravity constant is equal to zero. Formally, models that violate the assumption R2' in Arkolakis, Costinot, and Rodríguez-Clare (2012) are not covered in that class. Models that our framework nests that violate R2' in Arkolakis, Costinot, and Rodríguez-Clare (2012) include Di Giovanni and Levchenko (2009), Arkolakis (2010) when domestic labor is fully or partially used for marketing costs of exporting, and models with intermediate inputs as in Eaton and Kortum (2002).

[5]Earlier applications of the fixed effects estimator include Harrigan (1996) and Hummels (1999).

[6]Other papers that develop general equilibrium estimation procedures for the gravity equation include Balistreri and Hillberry (2007), Anderson and Yotov (2010), and Fally (2012); our paper, however, is the first to derive a closed form least squares estimator that incorporates general equilibrium effects.

equilibrium models of trade. Notably, Arkolakis, Costinot, and Rodríguez-Clare (2012) derive a closed form expression for changes in welfare as a function of openness that holds true across a large set of trade models. This paper follows in their footsteps by deriving closed form expressions for all other outcomes of interest, e.g. changes in bilateral trade flows, incomes, and global welfare that hold universally across gravity models. Our paper is also related to Costinot (2009), who examines the patterns of trade that hold true across many models. His primary focus, however, is on the specialization of countries in particular sectors, whereas we are concerned with the pattern of aggregate trade flows in a gravity framework.

The paper is organized as follows. In the next section, we present the universal framework and discuss how it nests existing general equilibrium gravity models. In Section 3, we present the theoretical results for existence and uniqueness, as well as the dual interpretations of the problem. In Section 4, we present the empirical results for identification, comparative statics and estimation. In Section 5, we present two empirical applications illustrating our results. Section 6 concludes.

2 The general equilibrium gravity framework

Consider a world comprised of a set $S \equiv \{1, ..., N\}$ of locations.[7] Let Y_i denote the gross income of country i, X_{ij} the total value of country j's imports from country i, and $K_{ij} > 0$ the associated bilateral trade frictions. As indicated in the introduction we focus our attention on models satisfying the "modern" version of the gravity equation, first discussed by Eaton and Kortum (2002), Anderson and Van Wincoop (2003), and Redding and Venables (2004). Formally, we define a *gravity trade model* as any model which yields an equation of the following type:

Assumption 1. *For any countries $i \in S$ and $j \in S$, the value of bilateral trade flows is given by $X_{ij} = K_{ij}\gamma_i\delta_j$, where $K_{ij} > 0$ is the exogenous bilateral trade friction and γ_i and δ_i are endogenous model outcomes.*

In this specification, the origin shifter γ_i and the destination shifter δ_j can represent endogenous model outcomes – such as wages or the measure of firms as well as model fundamental parameters – such as productivities or labor endowments. The bilateral trade frictions are exogenous and capture the effects of bilateral trade costs; they could be inverse functions of bilateral distance, various exporting barriers faced by exporting countries, etc. Note that larger values of K_{ij} indicate lower bilateral trade frictions. Whereas we do not

[7] The choice of a finite number of locations is not necessary for the the results that follow, but it saves on notation, avoids several thorny technical issues, and is consistent with the majority of the trade literature.

take a particular stand on the model that yields this gravity specification, we explain how different models map to this specification and to our subsequent results below.

Goods market clearing and trade balance. We proceed by defining two equilibrium conditions that are standard assumptions for modern general equilibrium gravity models: goods market clearing and trade balance. We say that *goods markets clear* if the output for all $i \in S$ is equal to the value of the good sold to all destinations. This condition is practically an accounting identity. Formally:

Assumption 2. *For any country* $i \in S, Y_i = \sum_{j \in S} X_{ij}$.

Furthermore we assume that *trade is balanced,* i.e. that output for all $i \in S$ is equal to the amount spent on good purchased from all other destinations:

Assumption 3. *For any country* $i \in S$, $Y_i = \sum_{j \in S} X_{ji}$.

While balanced trade is a standard equilibrium condition in general equilibrium gravity models, it is important to note that trade is not balanced empirically. This empirical discrepancy is an inherent limitation arising from the use of a static model to explain an empirical phenomenon with dynamic aspects. However, given both its ubiquity in the literature and the necessarily ad hoc nature of any alternative assumption (e.g. exogenously trade deficits), balanced trade seems the natural assumption on which to focus. We relax this assumption in characterization of the empirical properties of the model in Section 4.

Our last assumption postulates a log-linear parametric relationship between gross income and the origin and destination shifter:

Assumption 4. *For any country* $i \in S$, $Y_i = B_i \gamma_i^{\alpha} \delta_i^{\beta}$, *where we define* $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}$ *to be the* ***gravity constants*** *and* $B_i > 0$ *is an (exogenous) country specific shifter.*

Contrasting with A.1 which controls the matrix of bilateral flows, A.4 regulates the extent to which income responds to changes in the two endogenous shifters. The gravity constants determine the importance of the origin and destination shifters in determining a country's income. Let us consider the case that α and β are negative (which turns out to be a useful one). A larger origin shifter represents a higher exporting potential of the country. With a negative α this higher exporting ability is only achieved through a lower income (conditional on the destination shifter). A similar relationship holds between the destination shifter and income with a negative β. These inverse relationships guarantee that there exists a stabilizing force in the gravity network, which will prove important when we discuss the existence and uniqueness of equilibria.

In practice, A.4 is analogous to the standard condition that the income in a location is equal to the income earned by the factors of production in that location but reformulated in terms of the origin and destination shifters of the gravity equation. This formulation is general enough to incorporate a number of seminal gravity trade models, e.g. Armington (1969); Anderson (1979); Krugman (1980); Eaton and Kortum (2002); Melitz (2003).

Finally, to choose the numeraire, we normalize world income equal to one:

$$\sum_i Y_i = 1. \tag{1}$$

In what follows, we define a *general equilibrium gravity model* to be any gravity trade model such that goods market clears, trade is balanced, factor markets clear, (conditions A.1-A.4) and the normalization (1) is satisfied.

Example: the Armington model To make things concrete, we will provide a simple example of a general equilibrium trade model that satisfies our assumptions. In the Armington (1969) model, first formulated in general equilibrium by Anderson (1979), each location produces a differentiated variety (which is sold at marginal cost) and consumers have CES preferences with elasticity of substitution σ and where we denote by P_i the Dixit-Stiglitz CES price index across all varieties. We assume that production combines labor and an intermediate input in a Cobb-Douglas fashion, where the share of labor is given by $\delta \in (0, 1]$, and the intermediate input uses the same CES aggregator of goods from all countries as the final consumption good. Thus, with productivity A_i the unit cost of production in country i is simply $w_i^\delta P_i^{1-\delta}/A_i$.

In this model, the value of bilateral trade between $i \in S$ and $j \in S$ is:

$$X_{ij} = \tau_{ij}^{1-\sigma} \left(\frac{w_i^\delta P_i^{1-\delta}}{A_i} \right)^{1-\sigma} P_j^{\sigma-1} Y_j \tag{2}$$

where w_i is location's i wage, A_i is the location's productivity and the marginal production cost is $\frac{w_i}{A_i}$, τ_{ij} is the iceberg cost of delivering i's good in destination j, and Y_i is again its income. It is also straightforward to show that output is proportional to wage income and is given by

$$Y_i = w_i L_i / \delta \tag{3}$$

where L_i is the population in location i. According to the definition of gravity, A.1, we have

$$\gamma_i \equiv \left(\frac{w_i^\delta P_i^{1-\delta}}{A_i} \right)^{1-\sigma}, \ \delta_i \equiv P_i^{\sigma-1} Y_i,$$

which allows us to write A.4 as

$$Y_i = \gamma_i^{\frac{1}{1-\sigma\delta}} \delta_i^{\frac{1-\delta}{1-\sigma\delta}} A_i^{\frac{\sigma-1}{\sigma\delta-1}} L_i^{\frac{\delta(\sigma-1)}{\sigma\delta-1}},$$

so that $\alpha \equiv \frac{1}{1-\sigma\delta}$, $\beta \equiv \frac{1-\delta}{1-\sigma\delta}$, and $B_i = A_i^{\frac{\sigma-1}{\sigma\delta-1}} L_i^{\frac{\delta(\sigma-1)}{\sigma\delta-1}}$. Note that if $\sigma > 1$ and $\sigma\delta > 1$, then $\alpha, \beta < 0$ and a higher productivity A_i will increase both the exporting ability and the income of the country. At the same time increases in wages increase exports but decrease income as discussed earlier.

Table 1 shows how to express the two gravity constants, parameters α and β, in several models that map to our framework. As we will see below, these two constants can be used to sufficiently characterize whether or not an equilibrium is unique and, along with observed trade flows, fully determine how changes to model parameters will affect trade flows and incomes.

3 Theoretical properties

We first consider the theoretical properties of the general equilibrium gravity framework.

3.1 Existence and Uniqueness

In this section, we provide sufficient conditions for establishing existence and uniqueness in a general equilibrium gravity model. We start by formulating the equilibrium system implied by our assumptions. Using A.2 and A.3 and substituting out X_{ij} and Y_i with the definitions A.1 and A.4, respectively, yields:

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{4}$$

and

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{5}$$

Thus, the solution of a gravity model is given by γ_i and δ_i for all $i \in S$ such that equations (4) and (5) and the normalization from equation (1) are satisfied.

To proceed, we define $x_i \equiv B_i \gamma_i^{\alpha-1} \delta_i^{\beta}$ and $y_i \equiv B_i \gamma_i^{\alpha} \delta_i^{\beta-1}$. By reformulating the system in terms of x_i, y_i (see Appendix A.1 for details), equations (4) and (5) take the form of a standard system of non-linear equations. It turns out that this reformulation of the problem provides a method of solving for the trade equilibrium system using functions that map a compact space onto itself. This has two advantages over the standard formulation given in equations Equations (4) and (5): first, by restricting the potential solution space, it facilitates the calculation of the equilibrium; second, it allows us to generalize results used in the study of integral equations to prove the following theorem regarding the existence and uniqueness of general equilibrium gravity models:

Theorem 1. *Consider any general equilibrium gravity model. Then:*

i) If $\alpha + \beta \neq 1$, the model has a positive solution and all possible solutions are positive;

ii) If $\alpha, \beta \leq 0$ or $\alpha, \beta \geq 1$, then the model has a unique solution.

Proof. See Appendix A.1. □

Note that condition (ii) of Theorem 1 provides sufficient conditions for uniqueness; for certain parameter constellations (e.g. particular geographies of trade costs), equilibria may be unique even if the conditions are not satisfied. In practice, however, we have found that there exist multiple equilibrium for particular geographies when condition (ii) is not satisfied.

Quasi-symmetry

It turns out that we can extend the range in which uniqueness is guaranteed if we constrain our analysis to a particular class of trade frictions which are the focus of a large empirical literature on estimating gravity trade models. We call these trade frictions quasi-symmetric.

Definition 1. Quasi Symmetry: We say the trade frictions matrix K is *quasi-symmetric* if there exists a symmetric $N \times N$ matrix $\tilde{K}$ (i.e. for all $i, j \in S$, $\tilde{K}_{ij} = \tilde{K}_{ji}$) and $N \times 1$ vectors K^A and K^B such that for all $i, j \in S$ we have:

$$K_{ij} = \tilde{K}_{ij} K_i^A K_j^B.$$

Loosely speaking, quasi-symmetric trade frictions are those that are reducible to a symmetric component and an origin- and destination-specific component. While restrictive, it is important to note that the vast majority of papers which estimate gravity equations assume that trade frictions are quasi-symmetric; for example Eaton and Kortum (2002) and Waugh (2010) assume that trade costs are composed by a symmetric component that depends on bilateral distance and on a destination or origin fixed effect.

When trade frictions are quasi-symmetric we can show that the system of equations (28) and (29) can be dramatically simplified, and the uniqueness more sharply characterized.

Theorem 2. *Consider any general equilibrium gravity model with quasi-symmetric trade costs. Then:*

i) The balanced trade condition is equivalent to the origin and destination fixed effects being equal up to scale, i.e.

$$\gamma_i K_i^A = \kappa \delta_i K_i^B \tag{6}$$

for some $\kappa > 0$ that is part of the solution of the equilibrium.

ii) If α and β satisfy

$$\alpha + \beta \leq 0 \text{ or } \alpha + \beta \geq 2 \tag{7}$$

the model has a unique positive solution.

Proof. See Appendix A.2. □

Part i) of the Theorem 2 is particularly useful since it allows to simplify the equilibrium system into a single non-linear equation:

$$\gamma_i^{\alpha+\beta-1} = \kappa^{\beta-1} \sum_j \widetilde{K}_{ij} B_i^{-1} \left(K_i^A\right)^{1-\beta} \left(K_i^B\right)^{\beta} \gamma_j \tag{8}$$

In addition, because the origin and destination shifters in gravity models will (generally) be composites of exogenous and endogenous variables, by showing that the two fixed effects are equal up to scale, Theorem 2 provides a more precise analytical characterization of the equilibrium. We should note that the results of Theorem 2 have already been used in the literature for particular models, albeit implicitly. The most prominent example is Anderson and Van Wincoop (2003), who use the result to show the bilateral resistance is equal to the price index.[8] To our knowledge, Head and Mayer (2013) are the first to recognize the importance of balanced trade and market clearing in generating the result for the Armington model; however, Theorem 2 shows that the result applies more generally to any general equilibrium gravity model with quasi-symmetrical trade costs.

Figure 1 illustrates the range of α and β for which uniqueness of the model can be guaranteed. It should be noted that while most of the examination of existence and uniqueness of trade equilibria has proceeded on a model-by-model case, the gross substitute methodology used by Alvarez and Lucas (2007) has proven enormously helpful in establishing conditions for existence and uniqueness. It can be shown (see Online Appendix B.3) that the gross-substitutes methodology works only when $\alpha \leq 0$ and $\beta \leq 0$; hence, the tools used in Theorems 1 and 2 extend the range of trade models for which uniqueness can be proven, including, for example, Armington model with intermediate inputs.

[8] The result is also used in economic geography by Allen and Arkolakis (2013) to simplify a set on non-linear integral equations into a single integral equation.

Example: Armington model with quasi-symmetry Consider again an Armington model with intermediate inputs, but now assume that trade costs are quasi-symmetric. From part (i) of Theorem 2, we have $\gamma_i = \kappa\delta_i$, which implies:

$$\left(\frac{w_i^{\delta} P_i^{1-\delta}}{A_i}\right)^{1-\sigma} K_i^A = \kappa P_i^{\sigma-1} w_i L_i K_i^B,$$

or equivalently:

$$P_i = w_i^{\frac{1+(\sigma-1)\delta}{(1-\sigma)(2-\delta)}} \left(\kappa L_i A_i^{1-\sigma} \frac{K_i^B}{K_i^A}\right)^{\frac{1}{(1-\sigma)(2-\delta)}}. \tag{9}$$

Equation (9) provides some intuition for the uniqueness condition presented in Theorem 2: when $\sigma < \frac{1}{2}$, it is straightforward to show that the elasticity of the price index with respect to the wage is less than one. This implies that the wealth effect may dominate the substitution effect, so that the excess demand function need not be downward sloping.

In addition, combining equation (9) with equation (8), assuming $\delta = 1$, and noting that welfare $W_i = \frac{w_i}{P_i}$ yields the following equation:

$$\kappa W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}} = \sum_j K_{ij} A_i^{(\sigma-1)\tilde{\sigma}} A_j^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}} W_j^{-(\sigma-1)\tilde{\sigma}}, \tag{10}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$. Equation (10) holds for both trade models (where labor is fixed) and economic geography models (where labor is mobile); in the former case, L_i is treated as exogenous parameter and W_i solved for; in the latter case L_i is treated as endogenous and W_i is assumed to be constant across locations. Hence, Theorem 2 highlights the fundamental similarity between trade and economic geography models.[9]

Multiple sectors

Our approach also can be naturally extended to the cases where there are multiple sectors. Suppose there are a set $s \in \left\{1, ..., \tilde{S}\right\}$ of sectors and that the bilateral trade flow between

[9] When there are only two countries (so that trade costs are necessarily quasi-symmetric), we can use equation (10) to derive a single non-linear equation that yields the relative welfare in the two countries

$$K_{22}\left(\frac{W_1}{W_2}\right)^{\sigma\tilde{\sigma}} - K_{11}\left(\frac{W_1}{W_2}\right)^{(1-\sigma)\tilde{\sigma}} + K_{21}\left(\frac{W_1}{W_2}\right)^{\tilde{\sigma}} = K_{12}.$$

Comparative statics for welfare with respect to changes in K_{ij} can be characterized using the implicit function theorem in this case.

country i and country j in sector s is

$$X_{ij}^s = K_{ij}^s \left(\gamma_i\right) \left(\delta_j^s\right).$$

With multi-sector gravity models, we implicitly assume that there are no frictions on labor markets so that the wages in country i is equalized across sectors. That is why we can assume that the origin effect, γ_i, is independent of the sector s. Assumption A.4 becomes:

$$Y_i \quad = \quad B_i \left(\gamma_i\right)^{\alpha} \left(\prod_s \left(\delta_i^s\right)^{\theta t} \right)^{\beta}.$$

The first two terms are the same as before, but the last term is slightly different from what we have in a single-sector economy. In general the income for country i depends on the price index P_i, which can be captured by $\left(\prod_s \left(\delta_i^s\right)^{\theta t}\right)^{\beta}$.

The other two equilibrium conditions are:

$$\begin{aligned} \sum_j X_{j,i}^s &= B_i^s Y_i \\ \sum_s \sum_j X_{i,j}^s &= Y_i. \end{aligned}$$

The first equation assumes that country i's expenditure in each sector is a constant fraction of its total income. The second equation is the extension of the good market clearing condition we have in a single-sector case.

It turns out that the conditions for uniqueness with multiple sectors are the same as with a single sector, which we formalize in the following proposition:

Proposition 1. *(1) There exists a solution to the multi-sector gravity model if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$. (2) That solution is unique if $\alpha, \beta \leq 0$ or $\alpha, \beta > 1$.*

Proof. See Appendix A.3. □

Note that unlike the single sector case, we cannot prove the existence of a solution when it is not unique; this is due to the presence of cross-sectoral linkages.

3.2 Two dual representations

In this section, we show that the solution of the general equilibrium gravity model can be equivalently expressed as the solution to two distinct maximization problems: one for world income and one for world welfare. These dual interpretations allow us to apply the envelope theorem to derive expressions for the elasticity of world income and world welfare, respectively, to any change in bilateral trade frictions.

Consider first the problem of choosing the set of origin and destination fixed effects to maximize world income subject to trade remaining balanced and the aggregate feasibility constraint that world income can be equivalently calculated by summing over trade flows or using assumption A.4:

$$\max_{\{\gamma\},\{\delta\}} \sum_{i\in S}\sum_{j\in S} K_{ij}\gamma_i\delta_j$$

$$\text{s.t.} \sum_j K_{ij}\gamma_i\delta_j = \sum_j K_{ji}\gamma_j\delta_i \;\forall i \in S \text{ and } \sum_{i\in S}\sum_{j\in S} K_{ij}\gamma_i\delta_j = \sum_{i\in S} B_i\gamma_i^{\alpha}\delta_i^{\beta}, \tag{11}$$

where we now choose as a numeraire that $\gamma_1 = 1$ rather than choosing world income as a numeraire (since maximizing the numeraire is not a well defined problem).

Alternatively, consider the problem of maximizing a weighted average of world welfare subject to only the aggregate feasibility constraint. Of course, in the absence of a micro-foundation of the gravity trade model nothing can be directly said about the welfare of the equilibrium (as we have not specified preferences). However, Arkolakis, Costinot, and Rodríguez-Clare (2012) show that for a large class of trade models, the welfare of a country can be written solely as an increasing function of its openness to trade and an exogenous parameter, i.e. for all $i \in S$, welfare in country i, can be written as:

$$W_i = C_i^W \lambda_{ii}^{-\rho} = C_i^W \left(B_i\gamma_i^{\alpha-1}\delta_i^{\beta-1}\right)^{\rho}, \tag{12}$$

where $C_i^W > 0$ is an (exogenous) parameter and $\rho > 0$ is an exogenous scalar equal to negative of the inverse of the elasticity of K_{ij} to the iceberg trade costs. If welfare can be written as in equation (12), we can define world welfare as a weighted average of the welfare in each country:

$$W \equiv \sum_{i\in S}\omega_i W_i = \sum_{i\in S}\omega_i C_i^W\left(B_i\gamma_i^{\alpha-1}\delta_i^{\beta-1}\right)^{\rho},$$

where $\omega_i > 0$ are the weights placed on the welfare in each country. Then the following world welfare maximization problem is well defined:

$$\max_{\{\gamma\},\{\delta\}} W$$

$$\text{s.t.} \sum_{i\in S}\sum_{j\in S} K_{ij}\gamma_i\delta_j = \sum_{i\in S} B_i\gamma_i^{\alpha}\delta_i^{\beta}. \tag{13}$$

It turns out that the solution to both the world income maximization problem (11) and the world welfare maximization problem (13) is the solution to the general equilibrium gravity model, which we prove in the following proposition:

Proposition 2. *Consider any general equilibrium gravity model. If $\alpha+\beta>2$ or $\alpha+\beta<0$ (which by part (ii) of Theorem 2 guarantees uniqueness), Then:*

(i) The solution of the general equilibrium gravity model is equivalent to the solution of the world income maximization problem (11).

(ii) If welfare can be expressed as in equation (12), then there exists a set of weights $\{\omega_i\}$ such that the solution of the general equilibrium trade model is equivalent to the solution of the world welfare maximization problem (13).

Proof. See Appendix A.4. □

An advantage of the dual approach is that it allows us to apply the envelope theorem to derive an expression for how any change in bilateral trade frictions affects world income and world welfare. Using the world income maximization dual interpretation, the elasticity of world income to K_{ij} is:

$$\frac{\partial \ln Y^W}{\partial \ln K_{ij}} = \left[(\kappa_i - \kappa_j) + \frac{\alpha+\beta}{2-\alpha-\beta}\right] \frac{X_{ij}}{Y^W}, \tag{14}$$

where κ_i is the Lagrange multiplier on the balanced trade constraint and can be shown to be the solution to the following linear system:

$$\frac{\beta-\alpha}{2-(\alpha+\beta)} + \kappa_i = \sum_{j\in S} \frac{X_{ij}}{Y_i}\kappa_j.$$

When trade costs are quasi-symmetric, part (i) of Theorem 2 implies that $X_{ij}=X_{ji}$ so that expression (14) becomes even more straightforward:

$$\frac{1}{2}\left(\frac{\partial \ln Y^W}{\partial \ln K_{ij}} + \frac{\partial \ln Y^W}{\partial \ln K_{ji}}\right) = \frac{\alpha+\beta}{2-\alpha-\beta}\frac{X_{ij}}{Y^W}, \tag{15}$$

i.e. a symmetric increase in any pair of K_{ij} (i.e. a symmetric reduction bilateral trade frictions) increases world income by an amount proportional to the importance of those bilateral trade flows, where the proportion is a function of the gravity constants.[10]

Applying the envelope theorem to the world welfare maximization interpretation, the elasticity of world welfare to K_{ij} is even simpler::

$$\frac{\partial \ln W}{\partial \ln K_{ij}} = \rho \frac{X_{ij}}{Y^W}. \tag{16}$$

[10]Note that if $\alpha+\beta>2$ or $\alpha+\beta<0$, then $\frac{\alpha+\beta}{2-\alpha-\beta}>0$.

Since ρ is the inverse of the negative of the trade elasticity, equation (16) says that the elasticity of welfare to the trade cost is simply equal to $-\frac{X_{ij}}{Y^W}$. This expression has been derived for gravity models with CES demand by Atkeson and Burstein (2010), Burstein and Cravino (2012), and Fan, Lai, and Qi (2013); our derivation extends this result to any gravity trade model where welfare can be expressed as in equation (12). Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012) show that this expression holds for a larger class of homothetic demand function that includes the symmetric translog demand function (see also Feenstra (2003b)) and the Kimball demand function (see Kimball (1995)).

4 Empirical implications

Thus far, we have examined the theoretical properties of the general equilibrium gravity framework. In this part of the paper, we ask in what ways can the general equilibrium gravity framework be used in conjunction with an observed set of bilateral trade flows. In particular, given any set of observed trade flows $\{X_{ij}\}$ and gravity constants α and β: we (1) show to what extent model fundamentals such as bilateral trade frictions can be recovered; (2) derive expressions for how the model equilibrium will change with any change in the underlying bilateral trade flows; and (3) use these results to develop a new general equilibrium gravity estimator that can outperform the standard gravity regression.

Before proceeding to these results, however, we must address an issue familiar to trade empiricists: in contrast to assumption A.3, trade data is usually not balanced. It is not obvious how one ought to address unbalanced trade (which we view as a dynamic phenomenon) in the context of a static model. In what follows, we we treat the trade deficits as exogenous. Define $E_i \equiv \sum_{j \in S} X_{ji}$ to be the expenditure in location $i \in S$, $Y_i \equiv \sum_{j \in S} X_{ij}$ to be the output in location $i \in S$ and $\bar{D}_i \equiv E_i - Y_i$ to be the (exogenous) trade deficit. In the derivations that follow, we allow for any set of $\left\{\bar{D}_i\right\}$, which of course includes the case where observed trade flows are balanced (i.e. $\bar{D}_i = 0$ for all $i \in S$). In this case, equation (5) becomes:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta} + \bar{D}_i = \sum_j K_{ji} \gamma_j \delta_i \tag{17}$$

However, there are two disadvantages to allowing for exogenous deficits: first, the theoretical results presented above (in particular, the uniqueness of the equilibrium) do not necessarily hold; second, welfare cannot be expressed as in equation (12). Subject to these caveats, the empirical results below hold with (exogenous) trade deficits.

4.1 Identification

We first examine the extent to which one can recover model parameters given observed trade flows alone. In particular, suppose that we observe trade flows $\{X_{ij}\}$; to what extent can we recover the gravity constants α and β, the income shifter $\{B_i\}$, the trade frictions $\{K_{ij}\}$ and origin and destination fixed effects $\{\gamma_i\}$ and $\{\delta_i\}$?

The following proposition shows the extent to which the remaining model parameters can be identified.

Proposition 3. *For any set of observed trade flows $\{X_{ij}\}$ and gravity constants α and β, there exists a unique set of relative trade frictions $\left\{\frac{K_{ij}}{\left(K_{ii}^{\beta}/K_{jj}^{\alpha}\right)^{\frac{1}{\alpha-\beta}}(B_i/B_j)^{\frac{1}{\alpha-\beta}}}\right\}$ and appropriately-scaled origin and destination fixed effects $\left\{\frac{\gamma_i}{K_{ii}^{\frac{\beta}{\alpha-\beta}}B_i^{\frac{1}{\beta-\alpha}}}\right\}$ and $\left\{\frac{\delta_i}{K_{ii}^{\frac{\alpha}{\beta-\alpha}}B_i^{\frac{1}{\alpha-\beta}}}\right\}$ that are consistent with a trade equilibrium, which can be written solely as a function of observables:*

$$\begin{aligned}
\gamma_i \times \left(\frac{K_{ii}^{\beta}}{B_i}\right)^{\frac{1}{\beta-\alpha}} &= Y_i^{\frac{1}{\alpha-\beta}} X_{ii}^{\frac{\beta}{\beta-\alpha}}, \\
\delta_i \times \left(\frac{B_i}{K_{ii}^{\alpha}}\right)^{\frac{1}{\beta-\alpha}} &= Y_i^{\frac{1}{\beta-\alpha}} X_{ii}^{\frac{\alpha}{\alpha-\beta}}, \text{ and} \\
K_{ij} \times \left(\frac{K_{ii}^{\beta} B_j}{K_{jj}^{\alpha} B_i}\right)^{\frac{1}{\beta-\alpha}} &= X_{ij}\left(\frac{Y_j}{Y_i}\frac{X_{ii}^{\beta}}{X_{jj}^{\alpha}}\right)^{\frac{1}{\alpha-\beta}}.
\end{aligned}$$

Proof. See Appendix A.5. □

Proposition 3 shows that general equilibrium gravity models are fundamentally underidentified in two ways. First, there exists a fundamental inability to determine which model parameter is responsible for the level of trade flows. In particular, the scale of the bilateral trade frictions and the income shifters cannot be separately identified: this is immediately obvious by noting that one could simply divide both sides of equations (4) and (5) by B_i, thereby normalizing $B_i = 1$ in all locations. Intuitively, a larger value of the income shifter can be counteracted with lower bilateral trade frictions without affecting the equilibrium. Similarly, the origin and destination fixed effects cannot be disentangled from either the income shifter or the level of own trade frictions $\{K_{ii}\}$: increasing either fixed effect can be offset by an appropriate decline in either B_i or K_{ii} without affecting the equilibrium. This in turn implies that bilateral trade friction K_{ij} cannot be separately identified from the level of own trade frictions or income shifter in either the origin or destination location. To put it a different way, one can normalize $K_{ii} = B_i = 1$ for all $i \in S$ without affecting the equilibrium of the model.

Second, even with the appropriate normalization, however, the observed trade flows can be rationalized by the model for any chosen value of α and β (as long as $\alpha \neq \beta$). That is, the gravity constants cannot be identified using trade flow data alone. This result underpins why previous attempts to estimate (transformations of) these gravity constants have relied on additional sources of data such as prices (see e.g. Eaton and Kortum (2002), Simonovska and Waugh (2009), and Waugh (2010)).

4.2 Comparative Statics

In this section, we consider how changes in model fundamentals affect trade flows and income. We first consider infinitesimal changes and derive a closed form expression that yields the elasticities of all origin and destination fixed effects to all bilateral trade frictions that depends only on observed trade flows and the gravity constants. We then derive a system of equations that show how arbitrary changes to the trade friction matrix affect trade flows that also depend only on observed trade flows and the gravity constants.

4.2.1 Local Comparative Statics

Consider an infinitesimal change in any bilateral trade friction K_{ij}; how does this affect equilibrium trade flows and incomes? The following proposition provides a analytical expression for the elasticity of all origin or destination fixed effects to all changes in bilateral trade frictions:

Proposition 4. *Consider any general equilibrium gravity model where condition (ii) of Theorem 1 is satisfied. Define the $2N \times 2N$ matrix $A \equiv \begin{pmatrix} (\alpha - 1) Y & \beta Y - X \\ \alpha E - X^T & (\beta - 1) Y \end{pmatrix}^+$, where the "+" denotes the Moore-Penrose pseudo-inverse, Y is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is Y_i, E is the $N \times N$ diagonal income matrix whose i^{th} diagonal element is E_i and X is the $N \times N$ trade flow matrix whose $\langle i, j \rangle^{th}$ element is X_{ij}. Then:*

$$\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{l,i} + A_{N+l,j}) + c \text{ and } \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{N+l,i} + A_{l,j}) + c, \tag{18}$$

where A_{kl} is the $\langle k, l \rangle^{th}$ element of A and c is a scalar[11] *that ensures the normalization $\sum_i B_i \gamma_i^\alpha \delta_i^\beta = Y^W$ holds.*

[11] In particular, $c \equiv \frac{1}{(\alpha+\beta) Y^W} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j}\right) + \beta \left(A_{N+l,i} + A_{l,j}\right)\right)$.

Proof. See Appendix A.6. □

We should note that the choice of the constant c (and hence the elasticities) will depend on the normalization chosen: for example, the alternative normalization that $\gamma_1 = 1$ implies $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, so that $c = X_{ij} \times (A_{1,i} + A_{N+1,j})$. We should also note that while the expression for A will hold even if trade flows are unbalanced, we can only guarantee that the equilibrium is unique (and the elasticities are well-defined) if trade is balanced and condition (ii) of Theorem 1 is satisfied.

Because all model outcomes (e.g. trade flows and country incomes) are functions of the origin and destination fixed effects, Proposition 4 provides a closed form solution for the the complete set of model elasticities. In particular, it is straightforward to determine how changing the trade costs from i to j affects trade flows between any other bilateral trade pair k and l:[12]

$$\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} = \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \propto X_{ij} \times (A_{k,i} + A_{N+k,j} + A_{N+l,i} + A_{l,j}) . \tag{19}$$

Similarly, Proposition 4 can be applied to determine how changing the trade costs from i to j affects income in any country l:

$$\frac{\partial \ln Y_l}{\partial \ln K_{ij}} = \alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \propto X_{ij} \times (\alpha (A_{l,i} + A_{N+l,j}) + \beta (A_{N+l,i} + A_{l,j})) . \tag{20}$$

If trade flows are balanced and welfare can be written as in equation (12), then we can also determine the elasticity of welfare in any country l to any change in trade costs from i to j:

$$\frac{\partial \ln W_l}{\partial \ln K_{ij}} \propto X_{ij} \times \rho ((\alpha - 1) (A_{l,i} + A_{N+l,j}) + (\beta - 1) (A_{N+l,i} + A_{l,j})) \tag{21}$$

Hence, given observed trade flows and the gravity constants α and β (and ρ in the context of welfare), all general equilibrium gravity models deliver identical predictions for all local comparative statics. We use this powerful result in Section 4.3 to derive a new general equilibrium gravity estimator and in Section 5.1 to characterize the optimal set of trade friction reductions.

4.2.2 Global Comparative Statics

Now consider how an arbitrary change in the trade friction matrix K affects bilateral trade flows. The following proposition, which generalizes the results of Dekle, Eaton, and Kortum

[12]If $k = i$ and $l = j$, then $\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} = 1 + \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}}$, where the addition of one accounts for the direct effect on K_{kl}.

(2008) for all general equilibrium gravity trade models, provides an analytical expression relating the change in the origin and destination fixed effects to the change in trade frictions and the initial exporting and importing shares:

Proposition 5. *Consider any given set of observed trade flows X, gravity constants α and β, and change in the trade friction $\hat{K}_{ij}$. Then the percentage change in the fixed effects, $\{\hat{\gamma}_i\}$ and $\left\{\hat{\delta}_i\right\}$, if it exists, will solve the following system of equations:*

$$\hat{\gamma}_i^{\alpha-1}\hat{\delta}_i^{\beta} = \sum_j \left(\frac{X_{ij}}{Y_i}\right)\hat{K}_{ij}\hat{\delta}_j \text{ and } \hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta}\frac{Y_i}{E_i} + \frac{\bar{D}_i}{E_i} = \sum_{j\in S}\left(\frac{X_{ji}}{E_i}\right)\hat{K}_{ji}\hat{\gamma}_j\hat{\delta}_i. \tag{22}$$

Proof. See Appendix A.7. □

Note that equation (22) inherits the same mathematical structure as equations (4) and (17). As a result, if trade is balanced (so that $\bar{D}_i = 0$ and $Y_i = E_i$ for all $i \in S$), then part (i) of Theorem 1 proves that there will exist a solution to equation (22) and part (ii) of Theorem 1 provides conditions for its uniqueness.

As with the local comparative statics, equation (22) only depends on trade data and parameters α and β; hence, for any given change in trade frictions, all the gravity trade models with the same α and β must imply the same change in the fixed effects γ_i and δ_i and hence trade flows and incomes. If welfare can be written as in equation (12), the change in country and global welfare will also be the same.

This proposition characterizes the comparative statics for a wide class of gravity trade models. In the case where $\beta = 0$, it can be shown (see Online Appendix B.4) that the comparative statics can be characterized using import shares alone. This special case (and its welfare implications) is discussed in Proposition 2 of Arkolakis, Costinot, and Rodríguez-Clare (2012).

4.3 Estimation

Our final contribution is to develop a new estimator of the gravity equation. For a given set of gravity constants, this estimator directly accounts for the general equilibrium effects that bilateral trade flows between any two locations have on all other trade flows. This "general equilibrium estimator" potentially has two advantages over the standard fixed effects estimator most commonly employed today: first, by using all observed variation in trade flows rather than controlling for origin and destination fixed effects, it can be more efficient;[13] second, it offers a simple way of circumventing the endogeneity issues common

[13]This potential efficiency gain was first noted by Anderson and Van Wincoop (2003).

to standard gravity regressions. Somewhat surprisingly, the general equilibrium estimator is no more difficult to implement than any other gravity regression: using the results from Section 4.2.1, we show that the estimator can be implemented using ordinary least squares once the explanatory variables have been appropriately transformed to incorporate general equilibrium effects.

As we showed in Sections 4.1 and 4.2, while model fundamentals cannot be identified using trade flow levels, observed trade flows (along with the gravity constants) are sufficient to predict counterfactual changes in trade flows. For this reason, in what follows, we consider gravity regressions based on changes in trade flows. Using the "hat" notation from Section 4.2.2 and applying the gravity structure A.1 yields the following gravity equation in differences:

$$\hat{X}_{ij} = \hat{K}_{ij}\hat{\gamma}_i\hat{\delta}_j. \tag{23}$$

Suppose that the (log) change in bilateral trade frictions can be written as a linear function of a vector of observables, i.e. $\ln \hat{K}_{ij} = \hat{T}'_{ij}\mu$, and than an econometrician observes trade flows with measurement error. Then taking logs of equation 23 yields:

$$\ln \hat{X}^o_{ij} = \hat{T}'_{ij}\mu + \ln \hat{\gamma}_i + \ln \hat{\delta}_i + \varepsilon_{ij}, \tag{24}$$

where $\hat{X}^o_{ij}$ are the observed ratio of trade flows between i and j in period 1 to period 0, $\hat{T}_{ij}$ is an $S \times 1$ vector of observables and $\hat{T}'_{ij}$ denotes its transpose, μ is an $S \times 1$ vector of parameters, and ε_{ij} is the measurement error. The goal of the econometrician is to estimate μ, i.e. the effect of the various observables on bilateral trade frictions.

The fixed effects estimator

To provide a point of comparison for our estimator, it is helpful to first describe what has become the standard method of estimating μ, which we refer to as the "fixed effects estimator." The fixed effects estimator estimates μ using equation (24) by including a full set of origin and destination fixed effects in an ordinary least squares regression framework.[14] Formally, the fixed effects estimator μ^* is the one that minimizes the squared error between observed (hatted) trade flows and the gravity regression, conditional on the optimal set of

[14] The fixed effects estimator is discussed in detail in the review articles of Baldwin and Taglioni (2006) and Head and Mayer (2013). The latter review credits Harrigan (1996) as the first to use the fixed effects estimator and Redding and Venables (2004) and Feenstra (2003a) for showing that the fixed effects estimator could be used to control for the endogenous "multilateral resistance" terms present in general equilibrium gravity models. Since then, the fixed effects literature has been used extensively in the empirical trade literature.

fixed effects:

$$\mu_{FE}^* \equiv \arg\min_{\mu\in\mathbb{R}^S}\left(\min_{\hat{\gamma}_i,\hat{\delta}_i\in\mathbb{R}^N}\sum_i\sum_j\left(\ln\hat{X}_{ij}^o-\hat{T}_{ij}'\mu-\ln\hat{\gamma}_i-\ln\hat{\delta}_j\right)^2\right).$$

By taking first order conditions, it is straightforward to derive an analytical solution for μ^* :

$$\mu_{FE}^* = \left(\sum_i\sum_j\hat{T}_{ij}\hat{T}_{ij}'\right)^{-1}\sum_i\sum_j\hat{T}_{ij}\left(\ln\hat{X}_{ij}^o-\ln\hat{\gamma}_i^*-\ln\hat{\delta}_j^*\right), \tag{25}$$

where the estimated fixed effects are identified up to scale:

$$\ln\hat{\gamma}_i^*-\frac{1}{N}\sum_k\ln\hat{\gamma}_k^*=\frac{1}{N}\sum_j\left(\left(\ln\hat{X}_{ij}-\hat{T}_{ij}'\mu^*\right)-\frac{1}{N}\sum_k\left(\ln\hat{X}_{kj}^o-\hat{T}_{kj}'\mu^*\right)\right)$$

and:

$$\ln\hat{\delta}_j^*-\frac{1}{N}\sum_k\ln\hat{\delta}_k^*=\frac{1}{N}\sum_i\left(\left(\ln\hat{X}_{ij}-\hat{T}_{ij}'\mu^*\right)-\frac{1}{N}\sum_k\left(\ln\hat{X}_{ik}-\hat{T}_{ik}'\mu^*\right)\right).$$

We should emphasize that there are a number of attractive properties of the fixed effects estimator, most notably that it is easy to implement, and, as long as the measurement error is uncorrelated with the observables or fixed effects, it is a consistent and unbiased estimator of μ.

The general equilibrium estimator

The major disadvantage of the fixed effects estimator is that it treats the origin and destination fixed effects – which capture the general equilibrium effects of the gravity model – as nuisance parameters to be controlled for. We now develop a "general equilibrium estimator" that directly accounts for these general equilibrium effects, which, as we will see, allows the econometrician to exploit the network structure of trade to overcome some common econometric issues.

As the theoretical portion of the paper demonstrated, the (hatted) origin and destination fixed effects are functions of the entire matrix of (hatted) bilateral trade frictions, i.e. for all $i\in S$ and $j\in S$, we can write $\hat{\gamma}_i\left(\hat{\mathbf{T}}\mu\right)$ and $\hat{\delta}_j\left(\hat{\mathbf{T}}\mu\right)$, where $\hat{\mathbf{T}}\mu$ is an $N\times N$ matrix whose $\langle i,j\rangle$ element is $\hat{T}_{ij}'\mu$. The general equilibrium estimator μ_{GE}^* minimizes the squared deviation from observed (hatted) bilateral trade flows *while accounting for the effect of* μ *on* *the equilibrium (hatted) origin and destination fixed effects*:

$$\mu_{GE}^{*} \equiv \arg\min_{\mu \in \mathbb{R}^S} \left(\sum_i \sum_j \left(\ln \hat{X}_{ij}^o - \hat{T}_{ij}' \mu - \ln \hat{\gamma}_i \left(\hat{\mathbf{T}} \mu \right) - \ln \hat{\delta}_j \left(\hat{\mathbf{T}} \mu \right) \right)^2 \right).$$

By taking first order conditions, it is straightforward to derive an implicit equation for μ_{GE}^{*}. In principal, the general equilibrium estimator could then be calculated through an iterative procedure or through a non-linear least squares routine as in Anderson and Van Wincoop (2003). However, it turns out that we can do better. Consider the following first order approximations of the log change in the origin and destination fixed effects:

$$\ln \hat{\gamma}_i \left(\hat{\mathbf{T}} \mu \right) \approx \sum_k \sum_l \frac{\partial \ln \gamma_i}{\partial \ln K_{kl}} \hat{T}_{kl}' \mu \text{ and } \ln \hat{\delta}_j \left(\hat{\mathbf{T}} \mu \right) \approx \sum_k \sum_l \frac{\partial \ln \delta_j}{\partial \ln K_{kl}} \hat{T}_{kl}' \mu. \tag{26}$$

By taking first order conditions and applying these first order approximations, we can derive a straightforward closed form solution for the general equilibrium estimator (once we turn the $N \times N$ matrices into $N^2 \times 1$ vectors). Let $\hat{\mathbf{T}}$ now denote the $N^2 \times S$ vector whose $\langle i + j(N-1) \rangle$ row is the $1 \times S$ vector $\hat{T}_{ij}'$, $\mathbf{D}$ denote the $N^2 \times N^2$ matrix whose $\langle i + j(N-1), k + l(N-1) \rangle$ element is $\frac{\partial \ln X_{ij}}{\partial \ln K_{kl}}$, and $\hat{\mathbf{y}}$ denote the $N^2 \times 1$ vector whose $\langle i + j(N-1) \rangle$ row is $\ln \hat{X}_{ij}^o$. Then the general equilibrium gravity estimator is:

$$\mu_{GE}^{*} = \left(\left(\mathbf{D} \hat{\mathbf{T}} \right)' \left(\mathbf{D} \hat{\mathbf{T}} \right) \right)^{-1} \left(\mathbf{D} \hat{\mathbf{T}} \right) \hat{\mathbf{y}}. \tag{27}$$

Equation (27) says that, to a first order, the general equilibrium estimator is the coefficient one gets from of an ordinary squares regression of the observed hatted variables on a "general equilibrium transformed" explanatory variable $\hat{T}_{ij}^{GE}$:

$$\ln \hat{X}_{ij}^o = \left(\hat{T}_{ij}^{GE} \right)' \mu + \varepsilon_{ij},$$

where:

$$\hat{T}_{ij}^{GE} \equiv \sum_k \sum_l \frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} \hat{T}_{kl}.$$

Intuitively, the general equilibrium transformed regressors capture the effect of the entire set of explanatory variables on any particular observed bilateral trade flow. To do so, the estimator relies on knowing the elasticities of all bilateral trade flows on all bilateral trade frictions. From Section (4.2.1) and, in particular, equation (19), the complete set of such elasticities can be simultaneously determined by a simple matrix inversion given observed trade flows and a set of gravity constants[15]. As a result, the general equilibrium estimator directly accounts for all (first-order) general equilibrium effects arising from the network structure of trade flows. It is important to emphasize that while the matrix of elasticities $\mathbf{D}$ can be immediately calculated from equation (19), it requires specifying the gravity constants α and β. Intuitively, because the general equilibrium effects depend (only) on the gravity constants and observed trade flows, it is impossible to incorporate the general equilibrium effects into any estimation procedure without specifying the gravity constants.

Comparing the fixed effects and general equilibrium estimators

To assess the relative benefit of the fixed effects and general equilibrium estimators, we conduct a set of Monte Carlo simulations. For each simulation, we draw a random set of initial bilateral frictions $\left\{K_{ij}^0\right\}$ and a random set of (time-invariant) income shifters $\{B_i\}$. We then randomly assign half of the locations to be "existing members" of a "multilateral trade organization" and ten percent of locations to be "new members" of the same trade organization. Next, we suppose that the observed change in trade frictions arises from the new members joining the trade organization; in particular, we assume $\hat{K}_{ij} = \hat{T}_{ij}\mu$, where $\hat{T}_{ij}$ is an indicator variable equal to one if either the origin or destination is a new member of the organization and its trading partner is either an existing or new member. For a given set of gravity constants, we calculate the equilibrium in both periods.[16] We then add idiosyncratic measurement error to the trade flows in both periods and implement the two estimators based on these "observed" trade flows.[17] We calculate the coefficient of variation of the root mean squared deviation "CV(RMSD)" for both estimators over five hundred simulations to assess their relative efficiency.[18] We repeat this procedure for varying numbers of countries, size of measurement error, and magnitudes of the effect of the trade agreement (i.e. μ).

The top panel of Table 2 presents the results. For the sake of readability, we highlight the most efficient estimator under a particular set of simulation parameters in bold. As is evident, which estimator is more efficient depends on the particular set of simulation parameters.

[15] One ought not be concerned that equation (19) provides elasticities for $\frac{\partial \ln X_{ij}}{\partial \ln K_{ij}}$ whereas the elasticities required for the general equilibrium estimator are the "hatted" elasticities $\frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}}$, as it is straightforward to show that $\frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} = \frac{\partial \ln X_{ij}^1}{\partial \ln K_{ij}}$, i.e. the "hatted" elasticities are the same as the period 1 elasticities.

[16] We choose $\alpha = -\frac{2}{3}$ and $\beta = -\frac{1}{3}$; see below.

[17] The results are very similar if we instead add an error term to $\hat{K}_{ij}$, i.e. $\hat{K}_{ij} = \hat{T}_{ij}'\mu + \varepsilon_{ij}$.

[18] The CV(RMSD) is defined as $\frac{\left(\frac{1}{M}\sum_{m=1}^{M}\left(\mu^{true} - \mu_m^*\right)^2\right)^{\frac{1}{2}}}{\mu^{true}}$, where μ^{true} is the true value of μ, and μ_m^* is the estimated value for simulation $m \in \{1, ..., M\}$, i.e. the CV(RSMD) reports the ratio of the standard deviation of an estimator to the true parameter value. Like the root mean squared error, the CV(RMSD) is a statistic that combines both the accuracy and precision of an estimator; unlike the root mean squared error, its value is not dependent on the size of μ^{true}.

When there are a few number of locations, the general equilibrium estimator is more efficient than the fixed effects estimator; this is because fixed effects estimator requires estimating $2N$ nuisance parameters, which reduces the degrees of freedom available for estimating μ. In contrast, with many locations and a large effect size, the fixed effects estimator outperforms the general equilibrium estimator; this is because the first order approximation (26) is less accurate the larger the effect size.

While the general equilibrium estimator often outperforms the fixed effect estimator, its true advantage arises from the ability to exploit the general equilibrium structure of the gravity model to overcome the common econometric concern of omitted variable bias. For example, whether or not a country signs a trade agreement is likely correlated with unobservable variables (e.g. expectations about future trade flows) that are also correlated with observed trade flows. Such omitted variables will result in biased estimates in a typical gravity equation. However, because the identification in the general equilibrium estimator relies on the effect of particular bilateral observables have on all bilateral trade flows, one can estimate μ using only trade flows between locations that did not choose to sign a particular trade agreement. That is, the decision of country i to join a trade agreement will have a general equilibrium effect on trade flows between countries j and k. This general equilibrium effect can be used to infer the effect of a trade agreement without the need to directly consider how the trade flows of country i change.

To illustrate the power of this method, the bottom panel of Table 2 shows the efficiency of the estimators when we include an omitted variable in the error term that increases the observed period 1 trade flows[19] by 5 percent only between countries in which one entered the trade agreement.[20] As is evident, the omitted variable biases both the fixed effects estimator and the baseline general equilibrium estimator upward by an amount equal to the size of the omitted variable. However, when we use the general equilibrium estimator but exclude observations of trade between countries in which one entered the trade agreement (the "GE - switchers" column), the effect of omitted variable on the efficiency of the estimator is small. As a result, the general equilibrium estimator substantially outperforms the fixed effects estimator as long as there are a sufficiently large number of countries to allow for the

[19]We choose to add the omitted variables to period 1 trade flows rather than period 0 trade flows in order to introduce bias into the elasticity calculations used for the general equilibrium estimator.

[20]Because we interpret the error term as measurement error, this procedure should be interpreted as capturing the possibility that countries who sign trade agreements have *observed* trade flows that are on average 5 percent higher than their actual trade flows, while their actual trade flows are affected only by the trade agreement. If we replace the measurement error with an endogeneous error term in the change in bilateral trade frictions (i.e. $\hat{K}_{ij} = \hat{T}_{ij}'\gamma + \varepsilon_{ij}$ where $E\left[\hat{T}_{ij}\varepsilon_{ij}\right] \neq 0$), the general equilibrium estimator excluding the "switchers" still outperforms the fixed effects estimator, although the differences in efficiency are less stark since the endogeneous error term in this case also has a general equilibrium effect on trade flows between all other locations.

indirect identification of the effect of the trade agreement.

5 Empirical applications

In the final part of the paper, we illustrate two potential applications of the tools developed above: first, we determine the optimal reduction in bilateral trade frictions; second, we estimate the effect on trade flows and welfare of joining the WTO.

To illustrate the tools developed above, we use the CEPII gravity data set of Head, Mayer, and Ries (2010). This data set has several advantages: it covers bilateral trade flows between over two hundred countries, allowing us to construct the nearly complete world trade network; it includes both trade flow and GDP data, allowing us to measure own trade flows; and it is widely used, allowing comparability with other empirical studies.

We clean the data in three steps. First, we construct own trade flows. To do so, we rely on the market clearing and balanced trade conditions, which implies that own trade is simply the difference between observed income and total exports or total imports, respectively.[21] Second, to avoid inferring infinitely high trade frictions between bilateral trade flows we replace any missing bilateral trade flow with a small positive value. Finally, we balance the trade flows; while this is not strictly necessary, it guarantees that the equilibrium is unique, and as a result, the elasticities we estimate are well-defined. To do so, we ignore the observed level of trade flows and instead treat the observed import *shares* $\lambda_{ij} \equiv \frac{X_{ij}}{\sum_i X_{ij}}$ as the true data. We then find the unique set of incomes that are consistent with those import shares and balanced trade. In particular, we solve the following linear system of equations:

$$Y_i = \sum_j \lambda_{ij} Y_j.$$

By the Perron-Frobenius theorem, there exists a unique (to-scale) set of Y_i;[22] we pin down the scale with the normalization that $\sum_{i \in S} Y_i = 1$. Given these equilibrium Y_i, we then define the balanced trade flows $X_{ij}^b = \lambda_{ij} Y_j$.[23] The advantage of this procedure is that if the observed X_{ij} are balanced, the resulting re-balanced trade flows will be identical to the

[21] If income exceeds total imports (exports), we define own trade flows as income less total exports (imports); if income exceeds both total imports and exports, we define own trade flows as income less the average of total imports and exports.

[22] The Perron-Frobenius theorem guarantees that there exists a unique (to-scale) strictly positive vector that solves $Y_i = \kappa \sum_j \lambda_{ij} Y_j$ for the largest value of $\kappa > 0$. Since import shares sum to one, it is straightforward to show that $\kappa = 1$ in this case: $\kappa = \frac{\sum_i Y_i}{\sum_i \sum_j \lambda_{ij} Y_j} = \frac{\sum_i Y_i}{\sum_j Y_j \sum_i \lambda_{ij}} = 1$.

[23] It is straightforward to see that these trade flows are balanced: $\sum_j X_{ji}^b = \sum_j \lambda_{ji} Y_i = Y_i \sum_j \frac{X_{ji}}{\sum_j X_{ji}} = Y_i = \sum_j \lambda_{ij} Y_j = \sum_j X_{ij}^b$.

original trade flows, i.e. $X_{ij}^b = X_{ij}$ for all $i \in S$ and $j \in S$. The disadvantage is that it is not the unique way of balancing trade flows; for example, we could have just as easily treated export shares $\pi_{ij} \equiv \frac{X_{ij}}{\sum_j X_{ij}}$ as the true data and found the unique set of incomes consistent with balanced trade and those export shares using the equation $Y_i = \sum_j \pi_{ji} Y_j$.

As Proposition 4 and 5 show, for any given set of gravity constants and observed trade flows, all local and global comparative statics can be calculated without specifying a particular micro-foundation of the gravity model. Hence, it remains to specify our choice of gravity constants. In the main analysis, we choose $\alpha = -\frac{2}{3}$, $\beta = -\frac{1}{3}$, and $\rho = \frac{1}{2}$; in the context of an Armington trade model with intermediate inputs, these constants reflect a trade elasticity of negative four (consistent with Simonovska and Waugh (2009)) and a labor share in the production function of one-half (consistent with Alvarez and Lucas (2007)).

5.1 Optimal trade friction reductions

In this subsection, we demonstrate how the local comparative static results from Proposition 4 can be used to inform the choice of optimal trade policy, as well as estimate the potential welfare gains from such a policy. From equation (21), once the matrix A has been calculated from observed trade flows and the gravity constants, the elasticity of welfare in any country $l \in S$ with respect to the change in trade costs between any two countries $i \in S$ and $j \in S$, i.e. $\frac{\partial \ln W_l}{\partial \ln K_{ij}}$ can be immediately determined from a linear combination of elements of the matrix. These welfare elasticities allow one to address a number of empirically relevant questions, including:

From the perspective of a particular country, which set of world-wide trade friction reductions would benefit it the most and how much would it benefit from these reductions? Suppose that there exists a world trade organization which specifies how much each country in the world ought to reduce its trade frictions subject to two constraints: first, the total amount of trade friction reductions worldwide is fixed (e.g. for political or technological reasons), so that the purpose of the trade organization is to allocate the trade friction reduction across countries; and second, the trade friction reductions cannot discriminate, so that they are applied uniformly to the imports and exports of all other countries.

Formally, let z_i be the percentage change in bilateral trade frictions (both imports and exports) of country $i \in S$ and let $\vec{z}$ denote the $N \times 1$ vector of x_i. To represent the constraint that the total amount of trade friction reductions worldwide is fixed, suppose that $\|\vec{z}\| = 1$.

We can now examine what the optimal set of trade friction reductions $\vec{z}$ from the perspective of any country $l \in S$. Country l will want to choose trade friction reductions $\vec{z}$ in order to maximize the (first-order) effect on its welfare:

$$\max_{\{z_i\}} \sum_{i \in S} \sum_{j \in S} \frac{\partial \ln W_l}{\partial \ln K_{ij}} z_i z_l \text{ s.t. } \sum_{i \in S} z_i^2 = 1,$$

or equivalently in matrix notation:

$$\max \vec{z}^T W^l \vec{z} \text{ s.t. } \|\vec{z}\| = 1,$$

where W^l is an $N \times N$ matrix whose $\langle i, j \rangle^{th}$ element is $\frac{\partial \ln W_l}{\partial \ln K_{ij}}$. We set the diagonal elements of W^l equal so that changing bilateral trade frictions does not affect trade frictions with oneself. Let z_i^l denote the optimal trade friction change in country i from the perspective of country l. It is straightforward to show that optimal set of trade frictions from the perspective of country l are simply the eigenvectors of the matrix $\frac{1}{2}\left(W^l + \left(W^l\right)^T\right)$ corresponding to the largest eigenvalue λ^l. Furthermore, it is also straightforward to show the largest eigenvalue λ^l is the total value of welfare to country l under its optimal set of trade frictions:

$$\lambda^l = \left(\vec{z}^l\right)^T W^l \vec{z}^l.$$

These results allow us to immediately determine both the optimal set of trade friction changes from the perspective of any country $l \in S$ and the resulting change in its welfare. Figure 2 depicts the optimal set of trade frictions for all countries from the perspective of the United States. The results are intuitive: to maximize welfare in the U.S., its own trade frictions as well as trade frictions in its major trading partners (e.g. Canada, Mexico, China, Brazil, and Western Europe) ought to fall. In contrast, trade frictions in certain countries like North Korea and Burma actually ought to increase to divert trade to benefit the United States.

How much does the U.S. (or any other country) benefit from such a "selfish" multilateral policy? Figure 3 depicts the welfare gain for the U.S. and for all other countries from their respective optimal set of trade frictions, i.e. it reports the maximum potential gains each country could achieve from multilateral trade friction reduction. The potential benefits of multilateral trade friction reductions are the smallest in countries with sizable domestic production relative to external trade such as the United States, India, and Russia. The potential gains for smaller countries which engage in substantial trade (e.g. Belgium) are larger. However, countries where there exist political constraints that result restrict trade – for example, North Korea, Burma, Somalia, Cuba, and Iraq – face the largest potential gains from freer trade.

We should note that these calculations, while possible without the closed form solution for the complete set of local comparative statics derived in Section 4.2.1, would be onerous, as it would require re-simulating the model N^2 times, each time calculating the welfare effect of a shock to a particular bilateral pair.

What set of world-wide trade friction reductions would increase world welfare the most and what would be the distribution of those gains? Finally, we can use the global welfare elasticity result from Section 3.2 to determine the set of trade friction reductions that would maximize world welfare. Recall from part (ii) of Proposition 2 that the general equilibrium gravity model can be interpreted as maximizing a weighted average of welfare across countries, which implies that the the elasticity of world welfare W to any bilateral trade friction can be written as proportional to the fraction of world income comprised by trade between i and j, i.e. $\frac{\partial \ln W}{\partial \ln K_{ij}} = \rho \frac{X_{ij}}{Y^W}$.

Using this result, we can find the set of trade frictions that maximize world welfare to solve:

$$\max_{\{z_i\}} \sum_{i \in S} \sum_{j \in S} \rho \frac{X_{ij}}{Y^W} z_i z_j \text{ s.t. } \sum_{i \in S} z_i^2 = 1$$

Since ρ does not affect the maximization and Y^W is the numeraire, from above the optimal set of trade friction reductions are simply the eigenvectors corresponding to the largest eigenvalue of the matrix who $\langle i, j \rangle$ element is $X_{ij} + X_{ji}$ (with zeros on the diagonal). That is, *the reduction in bilateral trade frictions which maximizes world welfare is simply the eigenvector of the observed (balanced) trade flows* (corresponding to the largest eigenvalue). Furthermore, that largest eigenvalue represents the elasticity of world welfare to increasing the extent of the trade friction reductions in an optimal way.

Figure 4 depicts the optimal set of trade friction reductions. As is evident, to maximize world welfare, the countries responsible for the most trade in the world (such as the U.S., China, Japan, and Germany) reduce their trade frictions the most, whereas trade frictions in smaller countries fall by less. The largest eigenvalue of the system is 1.006, which implies that increasing the extent of trade friction reductions by one percent yields a world welfare gain of slightly more than one percent. Figure 5 depicts the distribution of these welfare effects; as is evident, North America and South Asia benefit the most from such a policy, while parts of Africa and South America are actually made worse off.

5.2 Estimating the gains from WTO membership

In this subsection, we illustrate the general equilibrium gravity estimator presented in Section 4.3 by estimating the effect of the WTO membership on bilateral trade frictions. We then use this estimate to calculate the welfare gains of WTO membership.

The WTO was founded on January 1, 1995, replacing the General Agreement on Tariffs and Trade (GATT). Of the 201 countries in our trade data, 125 were original WTO members. Between 1995 and 2005, an additional twenty-one countries joined the WTO.[24] In what follows, we will identify the effect of the WTO on trade flows by comparing the observed bilateral trade flows in 1995 to those in 2005 using as identification the twenty-one new countries as members. In particular, we assume that, apart from a common time trend ν, the only change in bilateral frictions between 1995 and 2005 was a (common) reduction in trade costs (i.e. an increase in $\hat{K}_{ij}$) between new WTO members and all other WTO members:

$$\hat{K}_{ij} = \mu \hat{T}_{ij} + \nu,$$

where $\hat{T}_{ij}$ is an indicator variable equal to one if either i or j is a new WTO member and its trading partner is a new or existing WTO member. While this is admittedly a strong assumption, note that by focusing on the change in trade flows rather than their level, we allow for any effect of time-invariant variables (e.g. distance, common language, shared border, etc.) on trade frictions. With this assumption, the parameter of interest μ can be estimated from the following fixed effects gravity regression:

$$\ln \hat{X}_{ij} = \mu \hat{T}_{ij} + \ln \hat{\gamma}_i + \ln \hat{\delta}_j + \varepsilon_{ij},$$

where we interpret ε_{ij} as measurement error. Note that the time trend ν cannot be separately identified from the fixed effects.

Alternatively, parameters μ and ν can be estimated from the following general equilibrium gravity regression:

$$\ln \hat{X}_{ij} = \mu \hat{T}_{ij}^{GE} + \nu I_{ij}^{GE} + \varepsilon_{ij},$$

where:

$$\hat{T}_{ij}^{GE} \equiv \sum_k \sum_l \frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}} \hat{T}_{kl} \text{ and } I_{ij}^{GE} \equiv \sum_k \sum_l \frac{\partial \ln \hat{X}_{ij}}{\partial \ln \hat{K}_{kl}}$$

are the "general equilibrium transformed" variables.

Table 3 presents the results of the two estimators. The first column reports the fixed effects estimator: joining the WTO is associated with a 49.5 percent increase in bilateral trade flows with other WTO members. Interpreting the chosen values of the gravity constants as having a trade elasticity of four, this implies that joining the WTO is associated with a 12.4 percent decline in bilateral iceberg trade costs. Columns two and four present the

[24] The new members were Albania, Armenia, Bulgaria, China, Ecuador, Estonia, Georgia, Croatia, Jordan, Kyrgyzstan, Cambodia, Lithuania, Moldova, Macedonia, Mongolia, Nepal, Oman, Panama, Saudi Arabia, and Taiwan.

general equilibrium estimator using both the raw trade data and the re-balanced trade data; the general estimates are similar to the fixed effects estimates, finding that joining the WTO is associated with a 42.5 and 47.9 percent increase in bilateral trade flows, corresponding to a 10.6 and 12.0 percent reduction in iceberg trade costs, respectively.[25] Finally, we can use the general equilibrium estimator while excluding the observed trade flows of new WTO members in order to mitigate the concerns of endogeneity. Columns three and five present the results using the raw and re-balanced trade data, respectively. While the point estimate for the raw data declines, it increases slightly when trade is balanced, and in both cases is statistically indistinguishable from the point estimates when including the new WTO members in the sample. In fact, we cannot statistically reject that any of the five estimates are different from WTO membership being associated with a 45 percent increase in trade flows (12 percent reduction in iceberg trade costs).

Given these estimates, we can use the methodology of Section 4.2.2 to ask what the welfare effect would be of removing these 21 countries from the WTO. To do so, we take the observed trade flows in 2005 and increase the trade frictions (i.e. reduce $\hat{K}_{ij}$) by 45 percent between these 21 countries and all other WTO members. Figure 6 depicts resulting change in welfare for all countries in the world. For the countries being removed, bilateral trade costs increase substantially with a majority of trading partners. This leads to an average decline in welfare of 9.3 percent. Existing WTO members are also made worse off by an average of 0.9 percent (the effect is smaller in magnitude since trade costs increase for a smaller subset of trading partners). However, non-WTO members actually benefit – their welfare increases by an average of 0.5 percent – since the increase in trade costs between other trading partners results in trade being diverted to non-members. As Figure 6 makes clear, however, these averages mask substantial heterogeneity across countries arising from the network structure of international trade flows.

6 Conclusion

Despite the empirical importance of gravity trade models, little is known about their properties which hold universally, i.e. regardless of the micro-economic foundation of the model. In this paper, we have developed a framework which nests a large set of general equilibrium

[25]While the estimates of joining the WTO agreement are not substantially affected by whether or not we re-balance the observed trade flows, the estimated time trend is substantially different, as the raw data suggesting a slightly negative time trend (i.e. a small increase in trade costs), while the balanced trade data suggest a large and positive time trend. Given that the elasticities used in the calculation of the general equilibrium estimator are not necessarily well-defined when trade is imbalanced, we prefer the balanced trade estimate (which also corresponds to our prior that trade costs in general are falling over time).

gravity models. Using this framework, we have shown that nearly all theoretical and empirical predictions for trade depend only on the value of two "gravity constants." This paper hence contributes to a growing literature emphasizing that the micro-economic foundations are not particularly important for determining a trade model's macro-economic implications.

By providing a universal framework for understanding the general equilibrium forces in gravity trade models, we hope that this paper provides a step toward unifying the quantitative general equilibrium approach with the gravity regression analysis common in the empirical trade literature. Toward this end, we have developed a toolkit that operationalizes all the theoretical results presented in this paper, including the calculation of the equilibrium, identification, calculation of local and global comparative statics, and estimation.[26]

However, in developing a universal framework, several limitations remain. First, since the general equilibrium forces are entirely determined by the value of the two gravity constants, an important future task is finding an effective way of estimating the value of these parameters. Second, there remains the need to address trade imbalances directly rather than relying (as we do) on ad hoc corrections. As trade imbalances are fundamentally dynamic phenomena, we look forward to future research incorporating the gravity structure into dynamic models of trade.

[26] The toolkit is available for download on Allen's webiste.

References

Aliprantis, C. D., and K. Border (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer.

Allen, T., and C. Arkolakis (2013): "Trade and the topography of the spatial economy," *mimeo, Northwestern University.*

Alvarez, F., and R. E. Lucas (2007): "General Equilibrium Analysis of the Eaton-Kortum Model of International Trade," *Journal of Monetary Economics*, 54(6), 1726–1768.

Anderson, J. E. (1979): "A Theoretical Foundation for the Gravity Equation," *American Economic Review*, 69(1), 106–116.

——— (2011): "The Gravity Model," *Annual Review of Economics*, 3(1), 133–160.

Anderson, J. E., and E. Van Wincoop (2003): "Gravity with Gravitas: A Solution to the Border Puzzle," *American Economic Review*, 93(1), 170–192.

Anderson, J. E., and Y. V. Yotov (2010): "The Changing Incidence of Geography," *American Economic Review*, 100(5), 2157–86.

Arkolakis, C. (2010): "Market Penetration Costs and the New Consumers Margin in International Trade," *Journal of Political Economy*, 118(6), 1151–1199.

Arkolakis, C., A. Costinot, D. Donaldson, and A. Rodríguez-Clare (2012): "The Elusive Pro-Competitive Effects of Trade," *mimeo.*

Arkolakis, C., A. Costinot, and A. Rodríguez-Clare (2012): "New Trade Models, Same Old Gains?," *American Economic Review*, 102(1), 94–130.

Arkolakis, C., S. Demidova, P. J. Klenow, and A. Rodríguez-Clare (2008): "Endogenous Variety and the Gains from Trade," *American Economic Review, Papers and Proceedings*, 98(4), 444–450.

Arkolakis, C., N. Ramondo, A. Rodríguez-Clare, and S. Yeaple (2013): "Innovation and production in the global economy," Discussion paper, National Bureau of Economic Research.

Armington, P. S. (1969): "A Theory of Demand for Products Distinguished by Place of Production," *International Monetary Fund Staff Papers*, 16, 159–178.

Atkeson, A., and A. Burstein (2010): "Innovation, Firm Dynamics, and International Trade," *Journal of Political Economy*, 118(3), 433–489.

Baldwin, R., and D. Taglioni (2006): "Gravity for dummies and dummies for gravity equations," Discussion paper, National Bureau of Economic Research.

Balistreri, E. J., and R. H. Hillberry (2007): "Structural estimation and the border puzzle," *Journal of International Economics*, 72(2), 451–463.

Bernard, A. B., J. Eaton, J. B. Jensen, and S. Kortum (2003): "Plants and Productivity in International Trade," *American Economic Review*, 93(4), 1268–1290.

Burstein, A., and J. Cravino (2012): "Measured aggregate gains from international trade," Discussion paper, National Bureau of Economic Research.

Burstein, A., and J. Vogel (2012): "International Trade, Technology, and the Skill Premium," Manuscript, Columbia University and UCLA.

Caliendo, L., and F. Parro (2010): "Estimates of the Trade and Welfare Effects of NAFTA," Manuscript, University of Chicago and Yale University.

Chaney, T. (2008): "Distorted Gravity: The Intensive and Extensive Margins of International Trade," *American Economic Review*, 98(4), 1707–1721.

Chor, D. (2010): "Unpacking Sources of Comparative Advantage: A Quantitative Approach," *forth. Journal of International Economics.*

Costinot, A. (2009): "An elementary theory of comparative advantage," *Econometrica*, 77(4), 1165–1192.

Costinot, A., D. Donaldson, and I. Komunjer (2010): "What Goods Do Countries Trade? A Quantitative Exploration of Ricardo's Ideas," Forthcoming, Review of Economic Studies.

Dekle, R., J. Eaton, and S. Kortum (2008): "Global Rebalancing with Gravity: Measuring the Burden of Adjustment," *IMF Staff Papers*, 55(3), 511–540.

Di Giovanni, J., and A. A. Levchenko (2008): "Putting the Parts Together: Trade, Vertical Linkages, and Business Cycle Comovement," *forthcoming, American Economic Journal: Macroeconomics.*

——— (2009): "Firm Entry, Trade, and Welfare in Zipf's World," Manuscript, University of Michigan.

Eaton, J., and S. Kortum (2002): "Technology, Geography and Trade," *Econometrica*, 70(5), 1741–1779.

Eaton, J., S. Kortum, B. Neiman, and J. Romalis (2011): "Trade and the Global Recession," *NBER Working Paper*, 16666.

Fally, T. (2012): "Structural gravity and fixed effects," *Unpublished Working Paper, University of Colorado-Boulder.*

Fan, H., E. Lai, and H. Qi (2013): "Global Gains from Reduction of Trade Costs," *Unpublished manuscript.*

Feenstra, R. (2003a): *Advanced International Trade.* Princeton University Press, Princeton, New Jersey.

Feenstra, R. C. (2003b): "A Homothetic Utility Function for Monopolistic Competition Models, Without Constant Price Elasticity," *Economics Letters*, 78(1), 79–86.

Harrigan, J. (1996): "Openness to Trade in Manufactures in the OECD," *Journal of international economics*, 40(1), 23–39.

Head, K., and T. Mayer (2013): *Gravity equations: Workhorse, toolkit, and cookbook.* Centre for Economic Policy Research.

Head, K., T. Mayer, and J. Ries (2010): "The erosion of colonial trade linkages after independence," *Journal of International Economics*, 81(1), 1–14.

Hummels, D. (1999): "Toward a geography of trade costs," *GTAP Working Papers*, p. 17.

James, M. (1978): "The generalised inverse," *The Mathematical Gazette*, pp. 109–114.

Karlin, S., and L. Nirenberg (1967): "On a theorem of P. Nowosad," *Journal of Mathematical Analysis and Applications*, 17(1), 61–67.

Kimball, M. S. (1995): "The Quantitative Analytics of the Basic Neomonetarist Model," *Journal of Money, Credit and Banking*, pp. 1241–1277.

Krugman, P. (1980): "Scale Economies, Product Differentiation, and the Pattern of Trade," *American Economic Review*, 70(5), 950–959.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory.* Oxford University Press, Oxford, UK.

Melitz, M. J. (2003): "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 71(6), 1695–1725.

Ramondo, N., A. Rodríguez-Clare, and M. Saborio-Rodriguez (2012): "Scale Effects and Productivity Across Countries: Does Country Size Matter?," *mimeo, University of California, Berkeley.*

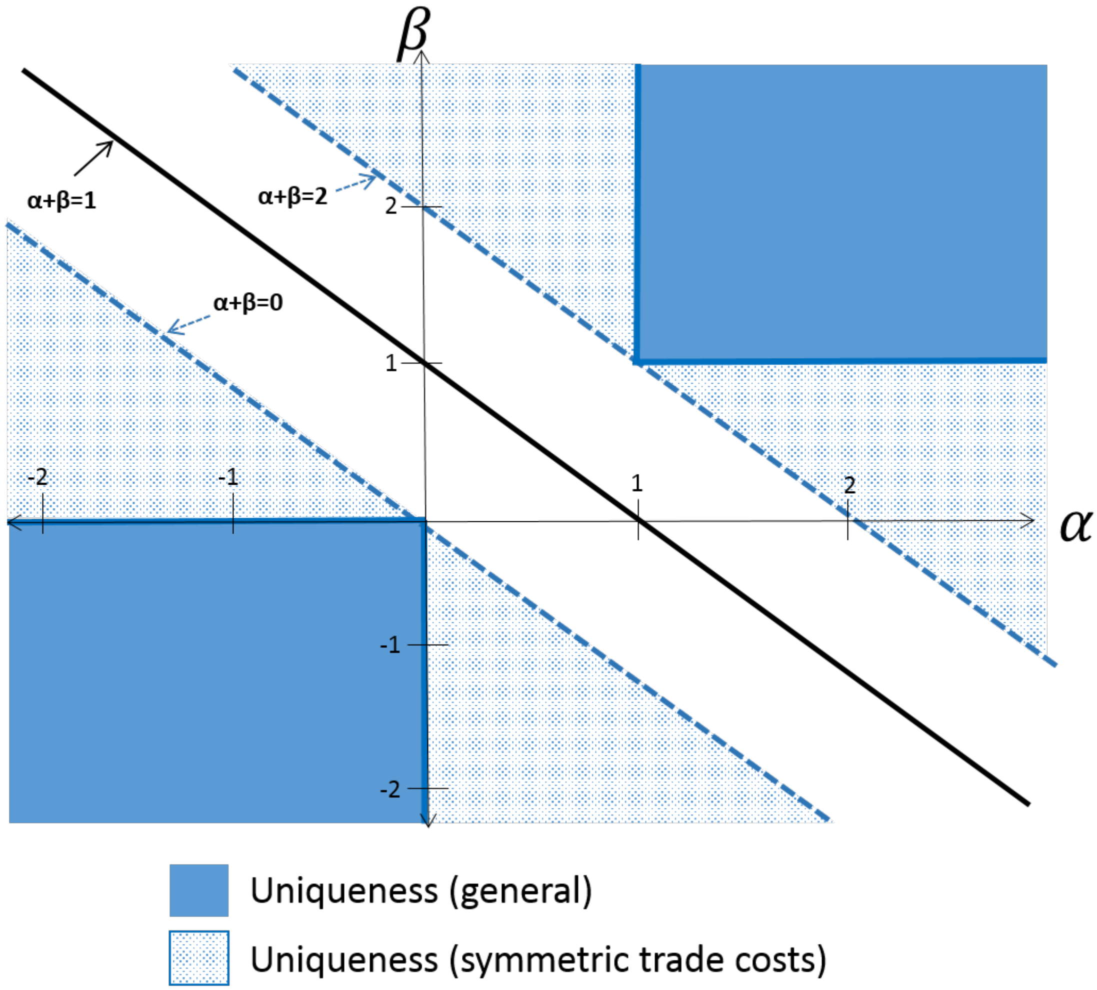
Redding, S., and A. J. Venables (2004): "Economic geography and international inequality," *Journal of international Economics*, 62(1), 53–82.

Simonovska, I., and M. Waugh (2009): "The Elasticity of Trade: Estimates and Evidence," *Manuscript, University of California, Davis and New York University.*

Waugh, M. (2010): "International Trade and Income Differences," *forthcoming, American Economic Review.*

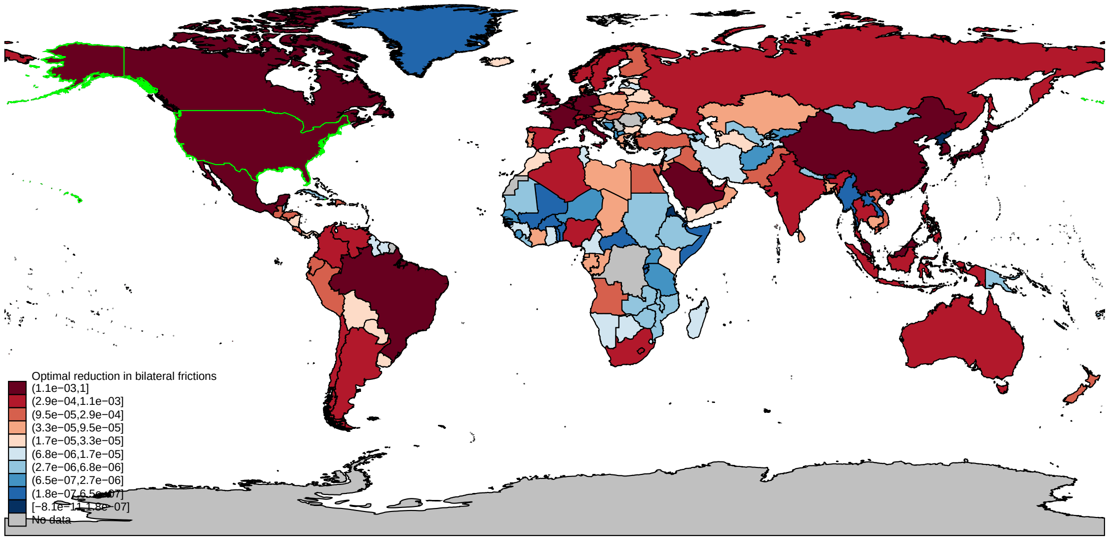
Tables and Figures

Figure 1: Existence and Uniqueness



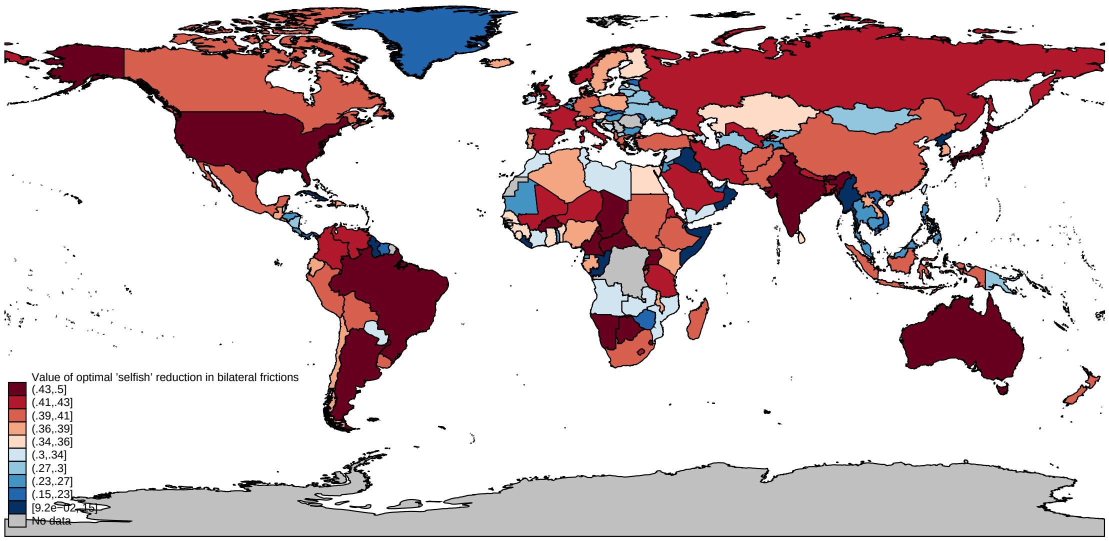
Notes: This figure shows the regions in (α, β) space for which the gravity equilibrium is unique generally and in the special case when trade frictions are quasi-symmetric. Existence can be guaranteed throughout the entire region with the exception of when $\alpha + \beta = 1$.

Figure 2: Optimal multilateral trade friction reduction from the perspective of the U.S.



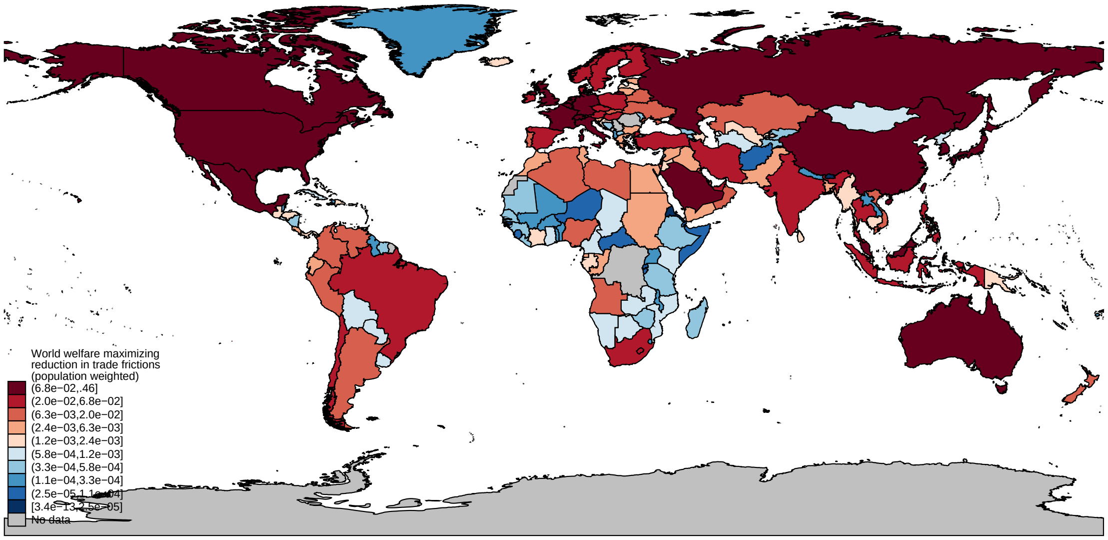
Notes: This figure shows the set of country reductions in trade frictions (subject to the total reduction of bilateral frictions being constant) that maximizes the welfare of the United States. Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction (or even increase) in trade frictions.

Figure 3: Potential welfare gains from multilateral trade friction reductions



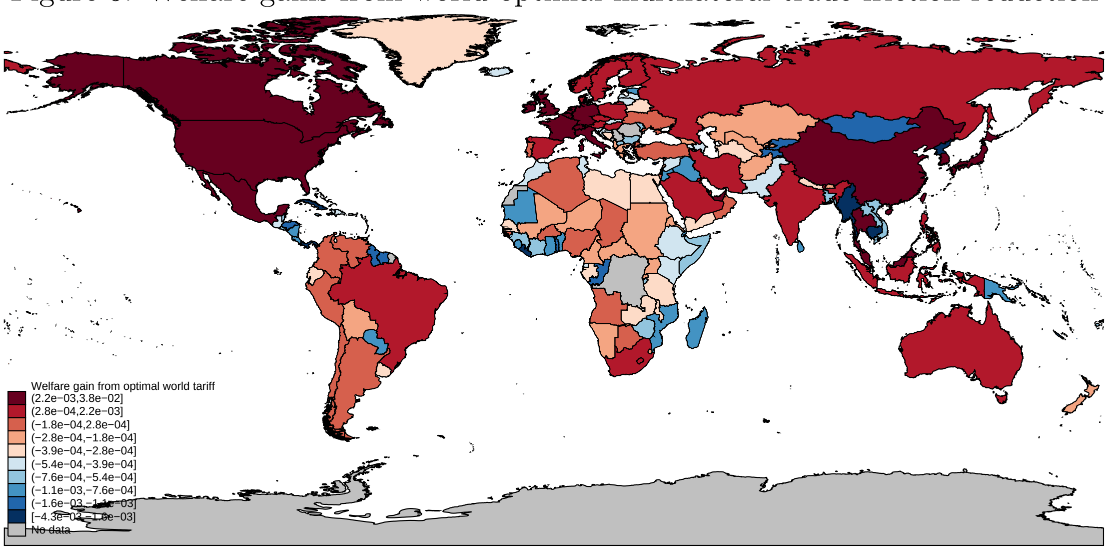
Notes: This figure shows the welfare gain each country would achieve if all countries in the world were to alter their trade frictions in such a way as to maximize the change in welfare of that country, i.e. the figure shows the distribution across countries of the potential gains of reduced trade frictions. Countries are sorted by deciles; red indicates a greater potential increase in welfare while blue indicates a smaller potential increase in welfare.

Figure 4: World optimal multilateral trade friction reduction



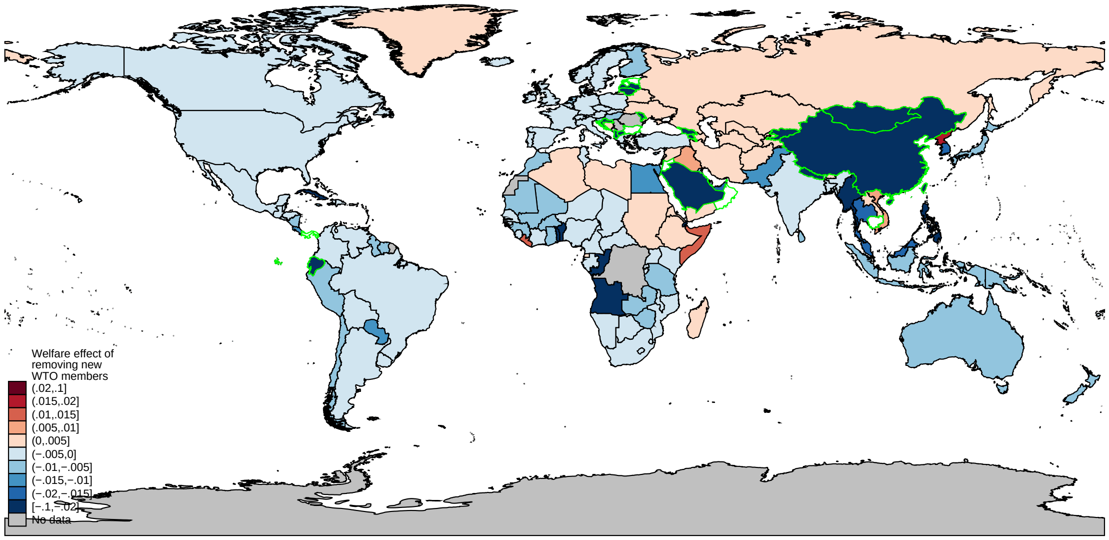
Notes: This figure shows the set of country reductions in trade frictions (subject to the total reduction of bilateral frictions being constant) that maximizes the a population-weighted average of welfare across all countries. Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction (or even increase) in trade frictions.

Figure 5: Welfare gains from world optimal multilateral trade friction reduction



Notes: This figure shows the welfare gain each country would achieve if all countries in the world were to alter their trade frictions in order to maximize a population-weighted average of welfare across the world, i.e. the figure shows the distribution across countries of welfare gains from an optimal multilateral trade friction reduction. Countries are sorted by deciles; red indicates a greater increase in welfare while blue indicates a smaller increase in welfare.

Figure 6: Welfare effect of removing the new members from the WTO



Notes: This figure shows the change in welfare in each country resulting from removing the twenty-one countries (highlighted in green) who joined the WTO between 1995 and 2005. Welfare is measured in percentage changes; the darker the red color, the more positive the increase in welfare; the darker the blue color, the more negative the decrease in welfare.

Table 1: Mapping of trade models to Universal Gravity framework

Model	Citation	Parameter	Symbol	Additional parameters	Symbol	Mapping to α	Mapping to β	Condition for uniqueness (general)	Condition for uniqueness (quasi-symmetry)
Armington	Armington (1969); Anderson (1979); Anderson and Van Wincoop (2003)	Elasticity of substitution	σ	N/A	N/A	$\alpha = \frac{1}{1-\sigma}$	$\beta = 0$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Monopolistic competition, homogeneous firms	Krugman (1980)	Elasticity of substitution	σ	N/A	N/A	$\alpha = \frac{1}{1-\sigma}$	$\beta = 0$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Perfect competition	Eaton and Kortum (2002)	Frechet shape parameter	θ	N/A	N/A	$\alpha = -\frac{1}{\theta}$	$\beta = 0$	$\theta \geq 0$	$\theta \geq 0$
Monopolistic competition, heterogeneous firms, exporting fixed costs in destination	Melitz (2003); Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008); Chaney (2008)	Pareto shape parameter	θ	N/A	N/A	$\alpha = -\frac{1}{\theta}$	$\beta = 0$	$\theta \geq 0$	$\theta \geq 0$
Monopolistic competition, heterogeneous firms, exporting fixed costs in origin	Melitz (2003); Di Giovanni and Levchenko (2009)	Elasticity of substitution	σ	Pareto shape parameter	θ	$\alpha = -\frac{\sigma-1}{\theta\sigma-\sigma+1}$	$\beta = 0$	$\sigma \geq 1$, $\theta \geq \frac{\sigma-1}{\sigma}$	previous region or $\sigma \geq 1$ and $\frac{1}{2}\frac{\sigma-1}{\sigma} \geq \theta$
Monopolistic competition, heterogeneous firms, flexible exporting fixed costs	Arkolakis (2010)	Elasticity of substitution	σ	Pareto shape parameter; share of fixed cost spent locally	θ,δ	$\alpha = -\frac{\sigma-1}{\theta(\sigma-1)-(1-\delta)(\sigma-1-\theta)}$	$\beta = 0$	$\sigma \geq 1; \theta \geq \frac{(\sigma-1)+\delta(\theta-\sigma+1)}{\sigma}$ or $\sigma < 1; \theta < \frac{(\sigma-1)+\delta(\theta-\sigma+1)}{\sigma}$	previous regions or $\theta \leq \frac{\sigma-1}{2\sigma} + \frac{\delta(\theta-\sigma+1)}{\sigma}$
Perfect competition with intermediate inputs	Eaton and Kortum (2002); Dekle, Eaton, and Kortum (2008)	Elasticity of substitution	σ	Share of labor in production	δ	$\alpha = \frac{1}{1-\sigma\delta}$	$\beta = \frac{1-\delta}{1-\sigma\delta}$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Extensions									
Multiple sectors	Chor (2010); Costinot, Donaldson, and Komunjer (2010)	Elasticity of substitution	σ	N/A	N/A	$\alpha = \frac{1}{1-\sigma}$	$\beta = 0$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$
Multiple sectors with intermediate goods	Caliendo and Parro (2010) (with symmetric IO structure and constant σ)	Elasticity of substitution	σ	Share of labor in production	δ	$\alpha = \frac{1}{1-\sigma\delta}$	$\beta = \frac{1-\delta}{1-\sigma\delta}$	$\sigma \geq 1$	$\sigma \geq \frac{1}{2}$

Notes: This table includes a (non-exhaustive) list of trade models that can be examined within the universal gravity framework.

Table 2: Monte Carlo results

		No omitted variable bias								
		10 countries			20 countries			50 countries		
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Measurement error	Size of the effect	FE	GE	GE - no switchers	FE	GE	GE - no switchers	FE	GE	GE - no switchers
0.1	0.05	0.1259	**0.1044**	0.8973	0.0322	**0.0276**	0.1112	**0.0050**	0.0051	0.0235
	0.1	0.0610	**0.0554**	0.3508	0.0163	**0.0147**	0.0603	**0.0026**	0.0064	0.0283
	0.2	0.0307	**0.0279**	0.1453	**0.0079**	0.0124	0.0603	**0.0013**	0.0108	0.0539
0.2	0.05	0.2486	**0.2058**	1.1630	0.0636	**0.0548**	0.2279	0.0104	**0.0095**	0.0392
	0.1	0.1222	**0.1045**	0.6837	0.0318	**0.0276**	0.1050	**0.0051**	0.0072	0.0327
	0.2	0.0614	**0.0548**	0.2739	**0.0153**	0.0161	0.0757	**0.0026**	0.0107	0.0546
0.5	0.05	0.6359	**0.5353**	3.5137	0.1655	**0.1373**	0.5757	0.0251	**0.0207**	0.0930
	0.1	0.3140	**0.2625**	1.5472	0.0824	**0.0695**	0.2950	0.0143	**0.0131**	0.0519
	0.2	0.1594	**0.1269**	1.2624	0.0401	**0.0356**	0.1429	**0.0062**	0.0120	0.0589
		Omitted variable bias								
0.1	0.05	**1.0142**	1.1612	1.1900	1.0025	1.2222	**0.1206**	1.0000	1.2655	**0.0606**
	0.1	0.5034	0.5828	**0.2965**	0.4988	0.6124	**0.0904**	0.5000	0.6371	**0.0714**
	0.2	0.2515	0.2937	**0.1941**	0.2494	0.3116	**0.0965**	0.2500	0.3248	**0.0959**
0.2	0.05	**1.0541**	1.1929	1.1386	1.0042	1.2272	**0.2579**	1.0006	1.2675	**0.0670**
	0.1	**0.5219**	0.5953	0.6438	0.5020	0.6153	**0.1205**	0.4996	0.6376	**0.0725**
	0.2	**0.2593**	0.2976	0.4048	0.2508	0.3111	**0.1087**	0.2500	0.3251	**0.0964**
0.5	0.05	**1.1800**	1.2649	2.5031	1.0198	1.2353	**0.5967**	1.0021	1.2672	**0.1024**
	0.1	**0.6120**	0.6554	1.5613	0.5077	0.6190	**0.2919**	0.4999	0.6363	**0.0848**
	0.2	**0.2918**	0.3171	0.7293	0.2545	0.3156	**0.1625**	0.2497	0.3248	**0.0994**

Notes: This table shows the coefficient of variation of the root mean squared deviation of three estimators of the effect of a multilateral trade agreement using a Monte Carlo procedure. Values in bold indicate which estimator is most precise for a given number of countries, size of measurement error, and size of effect. (We use the coefficient of variation of the RMSE because unlike the RMSE itself, the CV(RMSE) is invariant to the effect size being estimated.) The coefficient of variation of the root mean squared deviation is calculated from 500 simulations of the model, where in each simulation, the set of initial bilateral frictions, the exogenous income shifters B_i, and both the set of countries already in the multilateral trade agreement and the new members are randomly generated. We assign half of the countries to be existing members of the trade agreement and ten percent of countries to be new members. In the top panel, the generated measurement error is independent of whether or not a country is a new member of the multilateral trade agreement. In the bottom panel, the generated measurement error, in addition to an i.i.d. term, includes an omitted variable that increases observed trade flows by 0.05 log points when the trade is between a country who joined the trade agreement and either another new or existing trade agreement member. The size of the effect is the percentage reduction in bilateral frictions (i.e. an increase in K_{ij}); the size of the measurement error is the standard deviation of the measurement error relative to an average bilateral trade flow. The FE estimator is the standard ordinary least squares estimator of the change in trade flows on the change in trade agreement membership with both origin and destination country fixed effects; the GE estimator is the estimator introduced in the text which directly accounts for the general equilibrium effects through the network structure of trade; the GE - no switchers estimator is the GE estimator only identified off of the change in trade flows between countries who did not change their trade agreement membership. To calculate the equilibrium and elasticities, we assume gravity constants $\alpha = -2/3$, $\beta = -1/3$ which in an intermediate goods trade model correspond to a trade elasticity of 4 and a labor share in the production function of 1/2.

Table 3: General equilibrium gravity estimator: Effect of WTO membership

	Fixed effects estimator	General equilibrium estimator			
	(1)	(2)	(3)	(4)	(5)
WTO agreement	0.4948***	0.4246***	0.2738	0.4789***	0.5024
	(0.0975)	(0.0473)	(0.2203)	(0.0495)	(0.2848)
Constant time trend		-0.0823***	-0.0531	0.5992***	0.5768***
		(0.0092)	(0.0427)	(0.0272)	(0.0333)
Sample	Full Sample	Full Sample	Excluding new WTO members	Full Sample	Excluding new WTO members
Data	Raw data or balanced trade	Raw data		Balanced trade	
R-squared (within)	0.0102	0.0683	0.0598	0.0231	0.0148
Observations	17728	17728	14208	17728	14208

Notes: This table shows the estimated effect of joining the WTO on bilateral trade frictions. The dependent variable is the log ratio of bilateral trade flows in 2005 to bilateral trade flows in 1995. Missing and zero trade flows are excluded. WTO agreement is an indicator variable equal to one if the trade is between a country who joined the WTO between 1995 and 2005 and either another new or existing WTO member. The fixed effects estimator includes a full set of country origin and destination fixed effects. The general equilibrium estimator includes only the WTO agreement variable and a time trend; both variables are transformed to account for spillover general equilibrium effects using calculated elasticities. To calculate the elasticities, we assume gravity constants $\alpha = -2/3$, $\beta = -1/3$ which in an intermediate goods trade model correspond to a trade elasticity of 4 and a labor share in the production function of 1/2. The coefficients report the percentage reduction in bilateral trade frictions (i.e. percentage increase in K_{ij}); dividing by 4 yields the percentage reduction in iceberg trade costs. We report the results using both the observed trade flow data and trade flow data that has been balanced using the unique set of balanced trade flows consistent with observed import shares. Robust standard errors are reported in parentheses. Stars indicate statistical significance: * p<.10 ** p<.05 *** p<.01.

A Proofs

A.1 Proof of Theorem 1

We analyze a transformed system by defining $x_i \equiv B_i \gamma_i^{\alpha-1} \delta_i^{\beta}$ and $y_i \equiv B_i \gamma_i^{\alpha} \delta_i^{\beta-1}$. Then it can be shown that $\delta_i = x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ and $\gamma_i = x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$ so that for any set of $\{B_i\} \in \mathbb{R}^N_{++}$, $\{K_{ij}\} \in \mathbb{R}^{N\times N}_{++}$, $\{\alpha, \beta\} \in \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha + \beta \neq 1\}$, the equilibrium of a general equilibrium gravity model can be written using

$$x_i = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}, \tag{28}$$

and

$$y_i = \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \tag{29}$$

The proof of Theorem 1 proceeds in four parts. In the first part, we consider a general mathematical structure, for which the general equilibrium gravity model (defined by equations (28) and (29)) is a special case. In the second part, we prove a lemma that will allow us to convert the general mathematical result to the particular case of the gravity trade model. In the third and fourth parts, we show how the general mathematical result can be applied to the trade model to prove existence and uniqueness, respectively.

A.1.1 The general case

We start with the result for the general mathematical system, stated as the following lemma. For the proof, we use a version of Schauder's fixed point theorem (FPT for short). The original statement is found in Aliprantis and Border (2006).

Theorem 3. *(Schauder's FPT) Suppose that $D \subset R^N$, where D is a convex and compact set. If a continuous function $f : D \to D$ satisfies the condition that $f(D)$ is a compact subset of D, then there exists $x \in D$ such that $f(x) = x$.*

Lemma 1. *Consider the following system of non-linear equations; for all $i \in S$,*

$$x_i = \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_{i,j} F_{i,j} x_j^a y_j^b} \tag{30}$$

$$y_i = \frac{\sum_j H_{ij} x_j^c y_j^d}{\sum_{i,j} H_{ij} x_j^c y_j^d}, \tag{31}$$

for some $a, b, c, d \in \mathbb{R}$, $C_x, C_y \in \mathbb{R}_{++}$ and matrices F, H with all elements non-negative and *the diagonal strictly positive (i.e. for all $i \in \{1, ..., N\}$, $F_i > 0$ and $H_i > 0$). Then the system has a positive solution $x, y \in \mathbb{R}_+^S$ and all its possible solutions are positive.*

Proof. To apply the Schauder's FPT, we set up a subset D of R^{2S} such that D satisfies the conditions in Schauder's FPT.

Now consider the system (30)-(31). We define the set Γ as

$$\Gamma \equiv \left\{(x, y) \in \Delta\left(R^S\right) \times \Delta\left(R^S\right); m_x \le x_i \le M_x,\ m_y \le y_i \le M_y \quad \text{for all } i\right\},$$

and the following constants

$$\begin{aligned} M_x &\equiv \max_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\ m_x &\equiv \min_{i,j} \frac{F_{i,j}}{\sum_i F_{i,j}} \\ M_y &\equiv \max_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}} \\ m_y &\equiv \min_{i,j} \frac{H_{i,j}}{\sum_i H_{i,j}}. \end{aligned}$$

Γ is convex and compact subset of R^{2S}.

We define the following operator for $d = (x, y) \in \Gamma$.

$$\begin{aligned} Td &= T(x, y) \\ &= \left(\left(T^x(x, y)\right), \left(T^y(x, y)\right)\right), \end{aligned}$$

where

$$\begin{aligned} T_i^x(x, y) &= \frac{\sum_j F_{i,j} x_j^a y_j^b}{\sum_i \sum_j F_{i,j} x_j^a y_j^b} \\ T_i^y(x, y) &= \frac{\sum_j H_{i,j} x_j^c y_j^d}{\sum_i \sum_j H_{i,j} x_j^c y_j^d}. \end{aligned}$$

It is easy to show that

$$m_x \le T_i^x(x, y) \le M_x, m_y \le T_i^x(x, y) \le M_y$$

so that the operator T is from Γ to Γ.

To show that T is continuous, it suffices to show that T_i^x and T_i^y are continuous for all i. Since the range is compact, these functions are trivially continuous.

Since Schauder's FPT is applied for T, then there exists a solution to the system. Also by construction, any fixed points satisfy for all i,

$$
\begin{aligned}
0 &< m_x \leq x_i \\
0 &< m_y \leq y_i.
\end{aligned}
$$

$\square$

A.1.2 Preliminary mathematical result

Second, we prove a result that will allow us to map the general equilibrium gravity model to the general mathematical system.

Lemma 2. *Suppose that (x, y) satisfies*

$$
\begin{aligned}
x_i &= \frac{\sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\
y_i &= \frac{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}.
\end{aligned}
$$

Then we have

$$
\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.
$$

In particular we can take t such that

$$
\begin{aligned}
(tx_i) &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\
y_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.
\end{aligned}
$$

Proof. Note that

$$
x_i = \lambda_x \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}},
$$

where

$$
\lambda_x = \sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}
$$

Multiply both sides by $x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$, which yields:

$$\begin{aligned}
x_i \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) &= \lambda_x \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\
x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} &= \lambda_x \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)
\end{aligned}$$

Now sum over all i and rearrange to solve for λ_x:

$$\begin{aligned}
\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} &= \lambda_x \sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\
\lambda_x &= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)} \\
&= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right) \right] \times \left(x_i^{\frac{1-\beta}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}.
\end{aligned}$$

Now let us consider the second equilibrium condition:

$$y_i = \lambda_y \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}$$

where

$$\lambda_y = \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}.$$

Multiply both sides by $x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}$:

$$\begin{aligned}
y_i \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) &= \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff \\
x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} &= \lambda_y \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)
\end{aligned}$$

Now sum over all i and rearrange to solve for λ_y:

$$\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} = \lambda_y \sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right) \iff$$

$$\lambda_y = \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \sum_j K_{ji} \left(B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \right) \times \left(x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{1-\alpha}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}} \right)}$$

$$= \frac{\sum_i x_i^{\frac{\alpha}{\beta+\alpha-1}} y_i^{\frac{\beta}{\beta+\alpha-1}} B_i^{\frac{1}{1-\alpha-\beta}}}{\sum_i \left[\sum_j K_{ij} \left(x_j^{\frac{\alpha}{\beta+\alpha-1}} y_j^{\frac{1-\alpha}{\beta+\alpha-1}} B_j^{\frac{1}{1-\alpha-\beta}} \right) \right] \times \left(B_i^{\frac{1}{1-\alpha-\beta}} x_i^{\frac{1-\beta}{\alpha+\beta-1}} y_i^{\frac{\beta}{\alpha+\beta-1}} \right)}.$$

Comparing the expressions for λ_x and λ_y, we immediately have $\lambda_x = \lambda_y \equiv \lambda$.

Now take t as

$$t = \left(\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \right)^{\frac{1}{1-\frac{\alpha}{\alpha+\beta-1}}}.$$

To complete the proof of the lemma, we have to show that if (x, y) is a solution to

$$\begin{aligned} (x_i) &= \frac{\sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}} \\ y_i &= \frac{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}{\sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}}, \end{aligned}$$

then (tx, y) is a solution to a general equilibrium trade model. Namely (tx, y) solves

$$\begin{aligned} (tx_i) &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ y_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

To prove this last point, note that

$$\begin{aligned} tx_i &= \underbrace{\frac{t^{1-\frac{\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}}_{=1} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}. \end{aligned}$$

The equality holds by construction of t. Thus first equation that $tx_i = \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (x_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}$ is satisfied. To show the second equation, it suffices to show

$$\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} = 1.$$

This holds since[27]

$$\begin{aligned} \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} &= t^{\frac{1-\beta}{\alpha+\beta-1}} \sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \\ &= \frac{\sum_{i,j} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\sum_{i,j} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}} = 1. \end{aligned}$$

□

A.1.3 Existence for trade models

We next consider the existence of a strictly positive solution to the general equilibrium gravity model defined by equations (28) and (29).

Proof. We apply Lemma 1 with

$$\begin{aligned} a &= \frac{\alpha}{1-\alpha-\beta}, & b &= \frac{1-\alpha}{\alpha+\beta-1} \\ c &= \frac{1-\beta}{\alpha+\beta-1}, & d &= \frac{\beta}{\alpha+\beta-1} \\ F_{i,j} &= K_{ij} B_j^{\frac{1}{1-\alpha-\beta}}, & H_{i,j} &= K_{ji} B_j^{\frac{1}{1-\alpha-\beta}}. \end{aligned}$$

Then there exits a solution, (tx, y), to the system

$$\begin{aligned} (tx_i) &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ y_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}, \end{aligned}$$

proving the result. □

[27] If $\beta = 1$, then this last line is not true, since the equation for y is no longer dependent on x. In this case, however, existence and uniqueness follows immediately from Theorem 1 of Karlin and Nirenberg (1967), as the two integral equations can be treated as distinct from each other.

A.1.4 Uniqueness for trade models

We now consider the uniqueness of the general equilibrium gravity model. We prove uniqueness by contradiction.

Proof. For Part ii), uniqueness, we make use of the same Proposition. Gravity models imply the following restrictions to the coefficients of equations (30) and (31):

$$\begin{aligned} a &= \frac{\alpha}{\alpha+\beta-1}, b = \frac{1-\alpha}{\alpha+\beta-1} \\ c &= a-1 = \frac{1-\beta}{\alpha+\beta-1} \\ d &= b+1 = \frac{\beta}{\alpha+\beta-1}. \end{aligned}$$

Suppose that there are two solutions $(x,y),(\widetilde{x},\widetilde{y})$ for the system. Also assume that there are no constants t such that

$$x = t\widetilde{x}. \tag{32}$$

Without loss of generality, we can assume that for all i,

$$\sum_j F_{i,j} = \sum_j H_{i,j} = 1.$$

Also we can take $(\widetilde{x},\widetilde{y}) = (1,1)$ since

$$\begin{aligned} 1 &= \sum_j F_{i,j} 1^a 1^b \\ 1 &= \sum_j H_{i,j} 1^c 1^d. \end{aligned}$$

Define

$$\begin{aligned} m_x &= \min_i x_i \\ M_x &= \max_i x_i \\ m_y &= \min_i y_i \\ M_y &= \max_i y_i. \end{aligned}$$

From (32), $m_x\,(m_y)$ is strictly less than $M_x\,(M_y)$ respectively.

Under the conditions we make, it is easy to show

$$\begin{aligned} c &< 0 < a \\ b &< 0 < d. \end{aligned}$$

Then we can show that;

$$\begin{aligned} \max x_i &= M_x = \max \sum_j F_{i,j} x_j^a y_j^b \leq M_x^a m_y^b \\ \max y_i &= M_y = \max \sum_j H_{i,j} x_j^c y_j^d \leq m_x^c M_y^d \\ m_x = \min x_i &= \min \sum_j F_{i,j} x_j^a y_j^b \geq m_x^a M_y^b \\ m_y = \min y_i &= \min \sum_j H_{i,j} x_j^c y_j^d \geq M_x^c m_y^d. \end{aligned}$$

It is easy to show[28]

$$\begin{aligned} \left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} &< 1 \\ \left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} &< 1. \end{aligned}$$

[28]To obtain first equation, multiply first and third equation.

$$M_x \left(m_x^b M_y^b\right) \leq m_x \left(M_x^a m_y^b\right),$$

which is equivalent to

$$\left(\frac{M_x}{m_x}\right)^{1-a} \left(\frac{M_y}{m_y}\right)^{b} < 1.$$

For second equation, multiply second and fourth equation.

$$(M_y)\, M_x^c m_y^d \leq \left(m_x^c M_y^d\right) m_y,$$

which implies

$$\left(\frac{M_x}{m_x}\right)^{c} \left(\frac{M_y}{m_y}\right)^{1-d} \leq 1$$

Since $c = a - 1$, and $d = b + 1$,

$$\begin{aligned}\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} &< 1\\ \left(\frac{M_x}{m_x}\right)^{a-1}\left(\frac{M_y}{m_y}\right)^{-b} &< 1.\end{aligned}$$

Therefore the following holds.

$$\left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b} < 1 < \left(\frac{M_x}{m_x}\right)^{1-a}\left(\frac{M_y}{m_y}\right)^{b},$$

which is a contradiction. □

A.2 Proof of Theorem 2

Proof. Part i) This relation comes from assumptions A.2 and A.3 clearing conditions.

$$\sum_i X_{i,j} = \sum_i X_{j,i},$$

which is equivalent to

$$\begin{aligned}\frac{K_i^A\gamma_i}{K_i^B\delta_i} &= \frac{\sum_j \widetilde{K}_{i,j}K_j^A\gamma_j}{\sum_j \widetilde{K}_{i,j}K_j^B\delta_j}\\ &= \sum_j \frac{\widetilde{K}_{i,j}K_j^B\delta_j}{\sum_j\left(\widetilde{K}_{i,j}K_j^B\delta_j\right)} \times \frac{K_j^A\gamma_j}{K_j^B\delta_j}.\end{aligned}$$

It is easy to show that

$$\frac{K_i^A\gamma_i}{K_i^B\delta_i} = 1$$

is a solution to the problem. From the Perron-Frobenius theorem, this solution is unique up to scale. Therefore for some κ, we have

$$\gamma_i K_i^A = \kappa\delta_i K_i^B. \tag{33}$$

Part ii) The relation (33) implies

$$y_i = \frac{\gamma_i}{\delta_i}x_i = \kappa\frac{K_i^B}{K_i^A}x_i.$$

Substituting this expression into (28), we get

$$x_i = \kappa^{\frac{1-\alpha}{\alpha+\beta-1}} \sum_j \widetilde{K}_{i,j} K_i^A K_j^B B_j^{\frac{1}{1-\alpha-\beta}} \left(\frac{K_i^B}{K_i^A}\right)^{\frac{1-\alpha}{\alpha+\beta-1}} x_j^{\frac{1}{\alpha+\beta-1}}. \tag{34}$$

Also if we substitute the same expression into (29), we get the exact same expression. Therefore one of the two equations is trivially satisfied. From Theorem 1 of Karlin and Nirenberg (1967), the system has an unique solution if $\left|\frac{1}{\alpha+\beta-1}\right| \leq 1$, which is equivalent to (7). □

A.3 Proof of Proposition 1

The proof is the same as in a single-sector case, but done with slightly different notation.

A.3.1 Preliminary

The system is rewritten in the following form by redefining $\delta_i^s = \delta_i^s Y_i$.

$$\begin{aligned}
X_{ij}^s &= K_{ij}^s \gamma_i \delta_j^s Y_j \\
Y_i &= \sum_s \sum_j X_{ij}^s \\
B_i^s Y_i &= \sum_j X_{ji}^s \\
Y_i &= B_i \gamma_i^{\frac{\alpha}{1-\beta}} (\delta_i)^{\frac{\beta}{1-\beta}} \\
\delta_i &= \prod_t \left(\delta_i^t\right)^{\theta^t},
\end{aligned}$$

where $\sum_s B_i^s = 1$. The new set of α^* and β^* is $\alpha^* = \frac{\alpha}{1-\beta}$ and $\beta^* = \frac{\beta}{1-\beta}$. Actually it turns out that it is easier to show existence and uniqueness of the system with this notation. However we need to show that it suffices to establish existence and uniqueness for α^* and β^*. The following lemma tells that the mapping between these two is one-to-one so that if the system has an property for (α^*, β^*), then the (original) system has the same property under (α, β).

Lemma 3. *There is an one-to-one mapping between (α, β) and (α^*, β^*)if $\beta \neq 1$.*

Proof. Fix (α, β), then (α^*, β^*) is uniquely pinned down. Fix (α^*, β^*), then there exists an unique β such that

$$\beta = \frac{\beta^*}{1+\beta^*}$$

Then α^* is uniquely pinned down by $\alpha = (1-\beta)\alpha^* = \frac{\alpha^*}{1+\beta^*}$. □

Lemma 4. *Denote the function from (α, β) as f. Then $f(D) = \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$, where $D = \{(\alpha, \beta) \in R^2; \alpha, \beta \leq 0 \text{ or } \alpha, \beta \geq 1\}$.*

Proof. Take $(\alpha, \beta) \in f(D)$. Namely

$$\alpha^* = \frac{\alpha}{1-\beta}$$
$$\beta^* = \frac{\beta}{1-\beta}.$$

Then if α and β are both negative, then, α^* and β^* are both negative. Also

$$\alpha^* - 1 - \beta^* = \frac{\alpha - \beta - 1 + \beta}{1-\beta} = \frac{\alpha - 1}{1-\beta} \leq 0,$$

which implies $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$. Suppose α, β are greater than 1. Then both α^* and β^* are negative, and

$$\alpha^* - 1 - \beta^* = \frac{\alpha - 1}{1-\beta} \leq 0.$$

Again we have $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$.

Fix $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$. Then define (α, β) as follows.

$$\alpha = \frac{\alpha^*}{1+\beta^*}$$
$$\beta = \frac{\beta^*}{1+\beta^*}.$$

Then if $1 + \beta^* < 0$, then

$$\alpha = \frac{\alpha^*}{1+\beta^*} > 1$$
$$\beta = \frac{\beta^*}{1+\beta^*} > 1.$$

If $1 + \beta^* > 0$, then both are negative. Namely $(\alpha^*, \beta^*) \in f(D)$, which completes the proof. □

Lemma 5. *Denote the function from (α, β) as f. Then $f(D) = \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$, where $D = \{(\alpha, \beta) \in R^2; \alpha, \beta \leq 0 \text{ or } \alpha \geq 0, \beta \geq 1\}$.*

Proof. Take $(\alpha, \beta) \in f(D)$. Namely

$$\alpha^* = \frac{\alpha}{1-\beta}$$
$$\beta^* = \frac{\beta}{1-\beta}.$$

Then if α and β are both negative, then, α^* and β^* are both negative,which implies $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$. Suppose $\alpha \geq 0$ and $\beta \geq 1$. Then both α^* and β^* are negative. Again we have $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$.

Fix $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$. Then define (α, β) as follows.

$$\alpha = \frac{\alpha^*}{1+\beta^*}$$
$$\beta = \frac{\beta^*}{1+\beta^*}.$$

Then if $1+\beta^* < 0$, then

$$\alpha = \frac{\alpha^*}{1+\beta^*} \geq 0$$
$$\beta = \frac{\beta^*}{1+\beta^*} \geq 1.$$

If $1+\beta^* > 0$, then both are negative. Namely $(\alpha^*, \beta^*) \in f(D)$, which completes the proof. $\square$

These lemmas imply that if we can establish existence (uniqueness) on (α^*, β^*)-space, then under the associated (α, β), we can show that the system has a solution (unique solution). Strictly speaking, we loose uniqueness result when $\beta = 1$.

From now on, for notational convenience, we omit the star "*". From the previous lemma, it suffices to show that if $(\alpha, \beta) \in f(D)$, the system

$$\begin{aligned}
X_{ij}^s &= K_{ij}^s \gamma_i \delta_j^s Y_j \\
Y_i &= \sum_s \sum_j X_{ij}^s \\
B_i^s Y_i &= \sum_j X_{ji}^s \\
Y_i &= B_i \gamma_i^\alpha (\delta_i)^\beta \\
\delta_i &= \prod_t \left(\delta_i^t\right)^{\theta^t}.
\end{aligned}$$

has an unique solution. Then for $(\alpha, \beta) \in D$, the (original) system has an unique solution.

As we did in a single-sector economy, we can re-define variables as follows.

$$\begin{aligned}
x_i &= B_i \gamma_i^{\alpha-1} (\delta_i)^{\beta} \\
y_i^s &= (\delta_i^s)^{-1} \left(= (P_i^s)^{1-\sigma}\right) \\
z_i &= \prod_s \left((y_i^s)^{\theta^t}\right)^{(\alpha-\beta)} \\
\delta_i &= \prod_t (\delta_i^t)^{\theta^t} = \prod_t (y_i^t)^{-\theta^t} \\
&= \left[\prod_t (y_i^t)^{\theta^t}\right]^{-1} = (z_i)^{-\frac{1}{\alpha-\beta}} .
\end{aligned}$$

Here z_i, loosely speaking, represents the aggregate price index for country i P_i. The power $\alpha - \beta$ is $\frac{1}{1-\sigma}$ for Armington with intermediate case, and $(\delta_i^s)^{-1} = (y_i^s) = (P_i^s)^{1-\sigma}$.

Then we can express $(\gamma_i, \delta_i, \delta_i^s)$ by (x_i, y_i^s, z_i).

$$\begin{aligned}
\delta_i &= (z_i)^{-\frac{1}{\alpha-\beta}} \left(= (P_i)^{\sigma-1}\right) \\
(\gamma_i)^{\alpha-1} &= (B_i)^{-1} x_i (\delta_i)^{-\beta} \\
&= (B_i)^{-1} x_i \left((z_i)^{-\frac{1}{\alpha-\beta}}\right)^{-\beta} \\
\gamma_i &= (B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
\delta_i^s &= (y_i^s)^{-1} .
\end{aligned}$$

Then substituting these $(\gamma_i, \delta_i, \delta_i^s)$ into the equilibrium conditions, we get

$$
\begin{aligned}
x_i &= \sum_s \sum_j B_j K_{ij}^s \delta_j^s \gamma_j^\alpha \left(\delta_j\right)^\beta \\
&= \sum_s \sum_j B_j K_{ij}^s \left(y_j^s\right)^{-1} \left(B_j\right)^{-\frac{\alpha}{\alpha-1}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(z_j\right)^{\frac{\alpha\beta}{(\alpha-\beta)(\alpha-1)}} \left(z_j\right)^{-\frac{\beta}{\alpha-\beta}} \\
&= \sum_s \sum_j B_j K_{ij}^s \left(y_j^s\right)^{-1} \left(B_j\right)^{-\frac{\alpha}{\alpha-1}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\left(\frac{\alpha}{\alpha-1}-1\right)} \\
&= \sum_s \sum_j K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}} \\
y_i^s &= \left(\delta_i^s\right)^{-1} = \left(B_i^s\right)^{-1} \sum_j K_{ji}^s \gamma_j \\
&= \sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}} \left(x_j\right)^{\frac{1}{\alpha-1}} \left(z_j\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left(\left(y_i^s\right)^{\theta^t}\right)^{(\alpha-\beta)}.
\end{aligned}
$$

The system is finally written in the following form.

$$
\begin{aligned}
x_i &= \sum_s \sum_j K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}} \\
y_i^s &= \sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}} \left(x_j\right)^{\frac{1}{\alpha-1}} \left(z_j\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left(\left(y_i^s\right)^{\theta^t}\right)^{(\alpha-\beta)}.
\end{aligned}
$$

A.3.2 Existence proof

The existence proof consists of two steps. First we consider the following system.

$$
\begin{aligned}
x_i &= \frac{\sum_s \sum_j K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}} \\
y_i^s &= \sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}} \left(x_j\right)^{\frac{1}{\alpha-1}} \left(z_j\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left(\left(y_i^s\right)^{\theta^t}\right)^{(\alpha-\beta)}.
\end{aligned}
$$

Then we know that x_i is bounded since we normalize x_i. The following lemma ensures that we can obtain the bounds for y_i^s and z_i under certain conditions.

Lemma 6. *If $\alpha, \beta \leq 0$ and $\alpha - 1 \leq \beta$, then y_i^s and z_i are bounded.*

Proof. First we construct (candidate) bounds, and show that actually they are bounds. Suppose that

$$m_y \leq y_i^s \leq M_y.$$

Suppose that $\alpha, \beta \leq 0$,$\alpha \geq \beta$, and $\alpha - 1 \leq \beta$. Then z_i is bounded as follows

$$(m_y)^{\alpha-\beta} \leq z_i \leq (M_y)^{\alpha-\beta}.$$

Then

$$\begin{aligned}
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\overbrace{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}^{\geq 0}} \\
&\leq \underbrace{\max_{i,s} \left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq C_y} (m_x)^{\frac{1}{\alpha-1}} \left((M_y)^{\alpha-\beta} \right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
&= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
y_i^s &\geq \underbrace{\min_{i,s} \left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq c_y} (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} \\
&= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}}.
\end{aligned}$$

Set m_y and M_y as follows,

$$\begin{aligned}
M_y &= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
&= \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} = \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} \\
m_y &= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} = \left[c_y (M_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}}.
\end{aligned}$$

It is easy to show $M_y > m_y$ since $\frac{\alpha-1}{\alpha-1-\beta} \geq 0$.

Now suppose that $\alpha \leq \beta$. Then z_i is bounded as follows

$$(M_y)^{\alpha-\beta} \leq z_i \leq (m_y)^{\alpha-\beta}.$$

Then

$$
\begin{aligned}
y_i^s &= \sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}} \left(x_j\right)^{\frac{1}{\alpha-1}} \left(z_j\right)^{\underbrace{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}_{\leq 0}} \\
&\leq \underbrace{\max_{i,s}\left[\sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}}\right]}_{\triangleq C_y} \left(m_x\right)^{\frac{1}{\alpha-1}} \left(\left(M_y\right)^{\alpha-\beta}\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
&= C_y \left(m_x\right)^{\frac{1}{\alpha-1}} \left(M_y\right)^{\frac{\beta}{\alpha-1}} \\
y_i^s &\geq \underbrace{\min_{i,s}\left[\sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{-\frac{1}{\alpha-1}}\right]}_{\triangleq c_y} \left(M_x\right)^{\frac{1}{\alpha-1}} \left(m_y\right)^{\frac{\beta}{\alpha-1}} \\
&= c_y \left(M_x\right)^{\frac{1}{\alpha-1}} \left(m_y\right)^{\frac{\beta}{\alpha-1}}.
\end{aligned}
$$

Set m_y and M_y in the same as before,

$$
\begin{aligned}
M_y &= C_y \left(m_x\right)^{\frac{1}{\alpha-1}} \left(M_y\right)^{\frac{\beta}{\alpha-1}} \\
&= \left[C_y \left(m_x\right)^{\frac{1}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-1-\beta}} = \left[C_y \left(m_x\right)^{\frac{1}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-1-\beta}} \\
m_y &= c_y \left(M_x\right)^{\frac{1}{\alpha-1}} \left(m_y\right)^{\frac{\beta}{\alpha-1}} = \left[c_y \left(M_x\right)^{\frac{1}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-1-\beta}}.
\end{aligned}
$$

Note that $\frac{\alpha-1}{\alpha-1-\beta} \geq 0$. Therefore $M_y > m_y$. $\square$

Since we bound the variables, existence follows immediately from the Schauder's FPT.

Lemma 7. *(Scaling) Suppose that* (x_i, y_i^s, z_i) *solves*

$$
\begin{aligned}
x_i &= \frac{\sum_s \sum_j K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}} \\
y_i^s &= \sum_j K_{ji}^s \left(B_i^s\right)^{-1} \left(B_j\right)^{\frac{1}{1-\alpha}} \left(x_j\right)^{\frac{1}{\alpha-1}} \left(z_j\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left(\left(y_i^s\right)^{\theta^t}\right)^{(\alpha-\beta)}.
\end{aligned}
$$

Then

$$
\sum_{i,s,j} B_j K_{ij}^s \left(B_j\right)^{-\frac{\alpha}{\alpha-1}} \left(x_j\right)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} \left(z_j\right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} = 1.
$$

Proof. Define λ_x for notational convenience.

$$\lambda_x = \sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}.$$

Multiply $\left[(B_j)^{\frac{1}{1-\alpha}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right]$ for the first equation and take a sum w.r.t. i.

$$\begin{aligned}
\lambda_x &= \frac{\sum_i x_i \left[(B_j)^{\frac{1}{1-\alpha}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right]}{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right]} \\
&= \frac{\sum_i \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right]}{\sum_s \sum_j \left(K_{ij}^s\right) \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right] (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}}
\end{aligned}$$

Also multiply $\left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right]$ for the second equation, and sum up w.r.t. i, s.

$$\begin{aligned}
1 &= \frac{\sum_{i,s} \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right] B_i^s y_i^s}{\sum_{i,s} \sum_j K_{ji}^s (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right]} \\
&= \frac{\sum_i \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right] \overbrace{\sum_s B_i^s}^{=1}}{\sum_{i,s} \sum_j K_{i,j}^s (B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right]} \\
&= \frac{\sum_i \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}\right]}{\sum_{i,s} \sum_j \left(K_{i,j}^s\right) \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}\right] (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}} \\
&= \lambda_x.
\end{aligned}$$

$\square$

This lemma tells that it suffices to prove existence for

$$\begin{aligned}
x_i &= \frac{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} \left(y_j^s\right)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta}\frac{1}{\alpha-1}}} \\
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left((y_i^s)^{\theta^t}\right)^{(\alpha-\beta)}.
\end{aligned}$$

A.3.3 Uniqueness proof

Proof. Suppose that there are two solutions. As in a single-sector case, we can take one of the solutions as follows without loss of generality

$$x_i = y_i^s = z_i = 1.$$

Suppose that $\alpha \leq \beta \leq 0$. Then we can bound z_i.

$$(N_y)^{\alpha-\beta} \leq z_i \leq (n_y)^{\alpha-\beta}.$$

Then we can bound the maximums of x_i and y_i^s and the minimums of them as follows. As for x, it is easy to show

$$\left(\frac{N_x}{n_x}\right)^{\frac{1}{1-\alpha}} \leq \left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1}.$$

As for y, we get

$$\left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1} \leq \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}}.$$

Since there are two solutions, one of the inequalities is strict.

$$1 < \left(\frac{N_x}{n_x}\right)^{\frac{1}{1-\alpha}} < \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}} < 1,$$

which is a contradiction. Therefore the system has an unique solution.

Suppose that $\alpha - \beta > 0$, and $\alpha, \beta < 0$, and $\alpha - 1 < \beta$. Then we can bound z_i.

$$(n_y)^{\alpha-\beta} \leq z_i \leq (N_y)^{\alpha-\beta}.$$

Then we can bound the maximums of x_i and y_i^s and the minimums of them as follows. As for x, it is easy to show

$$\left(\frac{N_x}{n_x}\right)^{-\frac{1}{\alpha-1}} \leq \left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1}$$

Then As for y, we get

$$\left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1} \leq \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}}.$$

Since there are two solutions, one of the inequalities is strict.

$$1 < \left(\frac{N_x}{n_x}\right)^{-\frac{1}{\alpha-1}} < \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}} < 1,$$

which is a contradiction. Therefore the system has an unique solution. $\square$

A.4 Proof of Proposition 2

A.4.1 Part (i): The trade equilibrium solves the world income maximization problem.

Proof. To show that the trade equilibrium maximizes the world income, we show that the FONCs for the maximization problem coincide with the equilibrium conditions for the trade model. Mathematically we show that any solutions to the world income maximization satisfy the trade equilibrium conditions.

The associated Lagrangian of the maximization problem is:

$$\mathcal{L}:\sum_{i\in S}\sum_{j\in S}K_{ij}\gamma_i\delta_j-\sum_{i\in S}\kappa_i\left(\sum_j K_{ij}\gamma_i\delta_j-\sum_j K_{ji}\gamma_j\delta_i\right)-\lambda\left(\sum_{i\in S}\sum_{j\in S}K_{ij}\gamma_i\delta_j-\sum_{i\in S}B_i\gamma_i^{\alpha}\delta_i^{\beta}\right)\iff$$

$$\mathcal{L}:(1-\lambda)\sum_{i\in S}\sum_{j\in S}K_{ij}\gamma_i\delta_j-\sum_{i\in S}\kappa_i\left(\sum_j K_{ij}\gamma_i\delta_j-\sum_j K_{ji}\gamma_j\delta_i\right)+\lambda\sum_{i\in S}B_i\gamma_i^{\alpha}\delta_i^{\beta},$$

where $\{\kappa_i\}$ are the Lagrange multipliers on the balanced trade constraint and λ is the Lagrange multiplier on the aggregate factor market clearing.

First order conditions with respect to γ_i are:

$$(1-\lambda-\kappa_i)\sum_j K_{ij}\gamma_i\delta_j+\sum_j K_{ij}\gamma_i\delta_j\kappa_j+\alpha\lambda B_i\gamma_i^{\alpha}\delta_i^{\beta}=0 \tag{35}$$

First order conditions with respect to δ_i are:

$$(1-\lambda+\kappa_i)\sum_j K_{ji}\gamma_j\delta_i-\sum_j K_{ji}\gamma_j\delta_i\kappa_j+\beta\lambda B_i\gamma_i^{\alpha}\delta_i^{\beta}=0 \tag{36}$$

We first solve for the λ. Add the two FOC together and sum over all $i\in S$:

$$2(1-\lambda)\sum_i\sum_j K_{ij}\gamma_i\delta_j+\sum_i\sum_j\left(K_{ij}\gamma_i\delta_j-K_{ji}\gamma_j\delta_i\right)\kappa_j+(\alpha+\beta)\lambda\sum_i B_i\gamma_i^{\alpha}\delta_i^{\beta}=0,$$

which implies

$$\lambda=\frac{2}{2-\alpha-\beta}. \tag{37}$$

The FONCs for γ_i and δ_i become:

$$B_i \gamma_i^\alpha \delta_i^\beta = \left(\frac{\alpha+\beta}{2\alpha} + \frac{2-\alpha-\beta}{2\alpha}\kappa_i \right) \sum_j K_{ij}\gamma_i\delta_j - \frac{2-\alpha-\beta}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j\kappa_j \tag{38}$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \left(\frac{\alpha+\beta}{2\beta} - \frac{2-\alpha-\beta}{2\beta}\kappa_i \right) \sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{2\beta} \sum_j K_{ji}\gamma_j\delta_i\kappa_j \tag{39}$$

We now try to solve for the κ. Equating the two FOC yields:

$$\left(\frac{\alpha+\beta}{2\alpha} + \frac{2-\alpha-\beta}{2\alpha}\kappa_i \right) \sum_j K_{ij}\gamma_i\delta_j - \frac{2-\alpha-\beta}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j\kappa_j = \left(\frac{\alpha+\beta}{2\beta} - \frac{2-\alpha-\beta}{2\beta}\kappa_i \right) \sum_j K_{ji}\gamma_j\delta_i + 2$$

$$\frac{\beta-\alpha}{2-\alpha-\beta} + \kappa_i = \frac{\sum_j \left(\frac{\alpha}{\alpha+\beta} K_{ji}\gamma_j\delta_i + \frac{\beta}{\alpha+\beta} K_{ij}\gamma_i\delta_j \right)\kappa_j}{\sum_j K_{ij}\gamma_i\delta_j}. \tag{40}$$

Substituting (40) back into the FOC for γ_i yields:

$$B_i \gamma_i^\alpha \delta_i^\beta = \left(\frac{\alpha+\beta}{2\alpha} + \frac{2-\alpha-\beta}{2\alpha}\kappa_i \right) \sum_j K_{ij}\gamma_i\delta_j - \frac{2-\alpha-\beta}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j\kappa_j \iff$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \frac{\alpha+\beta}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j + \left(\frac{2-\alpha-\beta}{2\alpha} \left(\frac{\sum_j \left(\frac{\alpha}{\alpha+\beta} K_{ji}\gamma_j\delta_i + \frac{\beta}{\alpha+\beta} K_{ij}\gamma_i\delta_j \right)\kappa_j}{\sum_j K_{ij}\gamma_i\delta_j} - \frac{\beta-\alpha}{2-\alpha-\beta} \right)\right) \sum_j K_{ij}\gamma_i$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \frac{\alpha+\beta}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j + \frac{2-\alpha-\beta}{2\alpha} \left(\sum_j \left(\frac{\alpha}{\alpha+\beta} K_{ji}\gamma_j\delta_i + \frac{\beta}{\alpha+\beta} K_{ij}\gamma_i\delta_j \right)\kappa_j \right) - \frac{\beta-\alpha}{2\alpha} \sum_j K_{ij}\gamma_i\delta_j -$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \sum_j K_{ij}\gamma_i\delta_j + \frac{2-\alpha-\beta}{\alpha+\beta} \sum_j \left(\frac{K_{ji}\gamma_j\delta_i - K_{ij}\gamma_i\delta_j}{2} \right)\kappa_j \tag{41}$$

Substituting (40) back into the FOC for δ_i yields:

$$B_i \gamma_i^\alpha \delta_i^\beta = \left(\frac{\alpha+\beta}{2\beta} - \frac{2-\alpha-\beta}{2\beta}\kappa_i \right) \sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{2\beta} \sum_j K_{ji}\gamma_j\delta_i\kappa_j \iff$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \left(\frac{\alpha+\beta}{2\beta} - \frac{2-\alpha-\beta}{2\beta} \left(\left(\frac{\sum_j \left(\frac{\alpha}{\alpha+\beta} K_{ji}\gamma_j\delta_i + \frac{\beta}{\alpha+\beta} K_{ij}\gamma_i\delta_j \right)\kappa_j}{\sum_j K_{ij}\gamma_i\delta_j} - \frac{\beta-\alpha}{2-\alpha-\beta} \right)\right)\right) \sum_j K_{ji}\gamma_j\delta_i + 2$$

$$B_i \gamma_i^\alpha \delta_i^\beta = \sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{\alpha+\beta} \sum_j \left(\frac{K_{ij}\gamma_i\delta_j - K_{ji}\gamma_j\delta_i}{2} \right)\kappa_j \tag{42}$$

Note that equating the two FOC yields:

$$\begin{aligned}
\sum_j K_{ij}\gamma_i\delta_j + \frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ji}\gamma_j\delta_i - K_{ij}\gamma_i\delta_j}{2}\right)\kappa_j &= \sum_j K_{ji}\gamma_j\delta_i + \frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ij}\gamma_i\delta_j - K_{ji}\gamma_j\delta_i}{2}\right)\kappa_j \\
\frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ji}\gamma_j\delta_i - K_{ij}\gamma_i\delta_j}{2}\right)\kappa_j &= \frac{2-\alpha-\beta}{\alpha+\beta}\sum_j\left(\frac{K_{ij}\gamma_i\delta_j - K_{ji}\gamma_j\delta_i}{2}\right)\kappa_j \iff \\
\sum_j K_{ji}\gamma_j\delta_i\kappa_j &= \sum_j K_{ij}\gamma_i\delta_j\kappa_j,
\end{aligned}$$

where the second to last line imposed balanced trade. Hence the first order conditions become:

$$\begin{aligned}
B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j K_{ij}\gamma_i\delta_j \\
B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j K_{ji}\gamma_j\delta_i.
\end{aligned}$$

Therefore the solution to the problem is unique and coincides with the allocation of the general equilibrium gravity model. □

A.4.2 Part (ii) : The trade equilibrium solves the world welfare maximization problem.

Proof. With the assumption we made that the utility for country i is expressed in the following form

$$u_i = \left(B_i\left(\gamma_i\right)^{\alpha-1}\left(\delta_i\right)^{\beta-1}\right)^{\rho},$$

the welfare maximization problem is to maximize the weighted sum of $\{u_i\}_i$ subject to the same constraints. To show that the competitive allocation is Pareto efficient, we show that under a particular choice of (θ_i), the competitive allocation $\left(\gamma_i^{CE},\delta_i^{CE}\right)_i$ solves the planning problem.

Set the Pareto weights $(\omega_i)_i$ as follows.

$$(\omega_i) = \sum_k \frac{(B_k)^{\rho}\left(\gamma_k^{CE}\right)^{\rho(\alpha-1)}\left(\delta_k^{CE}\right)^{\rho(\beta-1)}}{(B_i)^{\rho}\left(\gamma_i^{CE}\right)^{\rho(\alpha-1)}\left(\delta_i^{CE}\right)^{\rho(\beta-1)}\frac{\sum_j (B_i)^{\rho}\left(\gamma_i^{CE}\right)^{\alpha}\left(\delta_i^{CE}\right)^{\beta}}{\sum_j K_{j,i}\gamma_j^{CE}\delta_i^{CE}}}(\omega_k).$$

From Karlin and Nirenberg (1967), we know there is a solution to the system.

The associated Lagrangian is

$$\mathcal{L} = \sum_i \omega_i B_i^\rho \gamma_i^{\rho(\alpha-1)} \delta_i^{\rho(\beta-1)} - \lambda \left(\sum_i \sum_j K_{ij} \gamma_i \delta_j - \sum_i B_i \gamma_i^\alpha \delta_i^\beta \right).$$

Taking the FONCs w.r.t. γ_i and δ_i, we get

$$\begin{aligned}
\rho\left(\alpha-1\right)\omega_i B_i^\rho \gamma_i^{\rho(\alpha-1)} \delta_i^{\rho(\beta-1)} &= \lambda \sum_j K_{ij}\gamma_i\delta_j - \alpha\lambda B_i \gamma_i^\alpha \delta_i^\beta \\
\rho\left(\beta-1\right)\omega_i B_i^\rho \gamma_i^{\rho(\alpha-1)} \delta_i^{\rho(\beta-1)} &= \lambda \sum_j K_{ji}\gamma_j\delta_i - \lambda\beta B_i \gamma_i^\alpha \delta_i^\beta.
\end{aligned}$$

Adding the two equations, and solving for λ, we have

$$\lambda = \rho \frac{W}{Y}.$$

Substitute this expression into the FONCs.

$$\begin{aligned}
\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\omega_i B_i^\rho \gamma_i^{\rho(\alpha-1)} \delta_i^{\rho(\beta-1)}}{W} Y^W - \sum_j K_{ij}\gamma_i\delta_j\right) + \sum_j K_{ij}\gamma_i\delta_j &= B_i \gamma_i^\alpha \delta_i^\beta \\
\left(\frac{\beta-1}{\beta}\right)\left(\frac{\omega_i B_i^\rho \gamma_i^{\rho(\alpha-1)} \delta_i^{\rho(\beta-1)}}{W} Y^W - \sum_j K_{ji}\gamma_j\delta_i\right) + \sum_j K_{ji}\gamma_j\delta_i &= B_i \gamma_i^\alpha \delta_i^\beta.
\end{aligned}$$

From the construction of ω_i, the bracket term is zero if we evaluate the system at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$.

$$\left(\omega_i B_i^\rho \left(\gamma_i^{CE}\right)^{\rho(\alpha-1)} \left(\delta_i^{CE}\right)^{\rho(\beta-1)} \frac{\sum_j B_j \left(\gamma_j^{CE}\right)^\alpha \left(\delta_j^{CE}\right)^\beta}{\sum_j \omega_i B_i^\rho \left(\gamma_i^{CE}\right)^{\rho(\alpha-1)} \left(\delta_i^{CE}\right)^{\rho(\beta-1)}} - \sum_j K_{ij}\gamma_i^{CE}\delta_j^{CE}\right) = 0.$$

Then the second equation is solved at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$ since

$$\sum_j K_{ji}\gamma_j^{CE}\delta_i^{CE} = B_i \left(\gamma_i^{CE}\right)_i^\alpha \left(\delta_i^{CE}\right)^\beta.$$

$\square$

A.5 Proof of Proposition 3

Proof. From the gravity equation A.1 we have:

$$\begin{aligned} X_{ij} &= K_{ij}\gamma_i\delta_j \iff \\ K_{ij} &= \frac{X_{ij}}{\gamma_i\delta_j} \end{aligned} \tag{43}$$

Combining factor market clearing A.4 with goods market clearing yields:

$$\begin{aligned} B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j X_{ij} \iff \\ \gamma_i^{\alpha}\delta_i^{\beta} &= \frac{\sum_j X_{ij}}{B_i}. \end{aligned} \tag{44}$$

The gravity equation A.1 yields the following relationship between origin and destination fixed effects:

$$\begin{aligned} X_{ii} &= K_{ii}\gamma_i\delta_i \iff \\ \delta_i &= \frac{X_{ii}}{K_{ii}\gamma_i}. \end{aligned} \tag{45}$$

Combining equations (44) and (45) to solve for γ_i and δ_i yields:

$$\gamma_i = \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\alpha-\beta}} \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\beta}{\beta-\alpha}} \text{ and } \delta_i = \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\sum_j X_{ij}}{B_i}\right)^{\frac{1}{\beta-\alpha}},$$

which substituting into equation (43) yields an expression for trade frictions K_{ij} that depends only on observed model parameters and trade flows

$$K_{ij} = X_{ij} \times \left(\frac{\sum_k X_{jk}}{\sum_k X_{ik}} \times \frac{B_i}{B_j} \times \frac{X_{ii}^{\beta}}{X_{jj}^{\alpha}} \times \frac{K_{jj}^{\alpha}}{K_{ii}^{\beta}}\right)^{\frac{1}{\alpha-\beta}},$$

thereby proving the claim. □

A.6 Proof of Proposition 4

Proof. The proof is simply done by implicit function theorem. First some notation is necessary. Define $y_i \equiv \ln\gamma_i$, $z_i \equiv \ln\delta_i$, $k_{ij} \equiv \ln K_{ij}$. Let $\vec{y} \equiv \{y_i\}$ and $\vec{z} \equiv \{z_i\}$ both be $N \times 1$ vectors and let $\vec{x} \equiv \{\vec{y};\vec{z}\}$ be a $2N \times 1$ vector. Let $\vec{k} \equiv \{k_{ij}\}$ be a $N^2 \times 1$ vector. Now consider the function $f\left(\vec{x};\vec{k}\right): R^{2N} \times R^{N^2} \to R^{2N}$ given by:

$$f\left(\vec{x};\vec{k}\right) = \begin{bmatrix} \left[B_i \left(\exp\left\{y_i\right\}\right)^{\alpha} \left(\exp\left\{z_i\right\}\right)^{\beta} - \sum_j \exp\left\{k_{i,j}\right\} \left(\exp\left\{y_i\right\}\right) \left(\exp\left\{z_j\right\}\right)\right]_i \\ \vdots \\ \left[B_i \left(\exp\left\{y_i\right\}\right)^{\alpha} \left(\exp\left\{z_i\right\}\right)^{\beta} - \sum_j \exp\left\{k_{j,i}\right\} \left(\exp\left\{y_j\right\}\right) \left(\exp\left\{z_i\right\}\right)\right]_i \end{bmatrix}.$$

In the general equilibrium trade model, we have:

$$f\left(\vec{x};\vec{k}\right) = 0.$$

Full differentiation of the function hence yields:

$$f_{\vec{x}} D_{\vec{k}} \vec{x} + f_{\vec{k}} = 0, \tag{46}$$

where $f_{\vec{x}}$ is the $2N \times 2N$ matrix:

$$f_{\vec{x}}\left(\vec{x};\vec{k}\right) = \begin{pmatrix} (\alpha-1)\,Y & \beta Y - X \\ \alpha Y - X^T & (\beta-1)\,Y \end{pmatrix},$$

where Y is a $N \times N$ diagonal matrix whose i^{th} diagonal is equal to Y_i and X is the $N \times N$ trade matrix.

Similarly, $f_{\vec{k}}$ is a $2N \times N^2$ matrix that depends only on trade flows:

$$f_{\vec{k}}\left(\vec{x},\vec{k}\right) = -\begin{pmatrix} X_{11} & \cdots & X_{1N} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{21} & \cdots & X_{2N} & \cdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & X_{N1} & \cdots & X_{NN} \\ X_{11} & \cdots & 0 & X_{21} & \cdots & 0 & \cdots & X_{N1} & \cdots & 0 \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots \\ 0 & \cdots & X_{1N} & 0 & \cdots & X_{2N} & \cdots & 0 & \cdots & X_{NN} \end{pmatrix}$$

If $f_{\vec{x}}$ was of full rank, we could immediately invert equation (46) (i.e. apply the implicit function theorem) to immediately yield:

$$D_{\vec{k}} \vec{x} = -\left(f_{\vec{x}}\right)^{-1} f_{\vec{k}}.$$

However, because Walras Law holds and we can without loss of generality apply a normalization to $\{\gamma_i\}$ and $\{\delta_i\}$ (see Online Appendix B.1 for details), we effectively have $N-1$ equations and $N-1$ unknowns, i.e. matrix $f_{\vec{x}}$ is of rank $2N-1$. Hence, there exists an infinite number of solutions to equation (46), each corresponding to a different normalization. To find the solution that corresponds to our choice of world income as the numeraire, note that from equation (1):

$$\sum_l B_l \gamma_l^{\alpha} \delta_l^{\beta} = Y^W \implies$$
$$\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right) = 0. \tag{47}$$

We claim that if $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{l,i} + A_{N+l,j}) - c$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times (A_{N+l,i} + A_{l,j}) - c$, where $c \equiv \frac{1}{Y^W(\alpha+\beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j} \right) + \beta \left(A_{N+l,i} + A_{l,j} \right) \right)$, then $\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}}$ and $\frac{\partial \ln \delta_l}{\partial \ln K_{ij}}$ solve equations (46) and (47). It is straightforward to see that our assumed solution ensures equation (46) holds, as the generalized inverse is a means of choosing from one of the infinitely many solutions; see James (1978). It remains to scale the set of elasticities appropriately to ensure that our normalization holds as well. Given our definition of the scalar c, it is straightforward to verify that equation (47) holds:

$$\begin{aligned}
\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right) &= \sum_l Y_l (\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) - c \right) + \\
&\quad + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) - c \right)) \\
&= X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) \right) \right) \\
&\quad - c \left(\alpha + \beta \right) \sum_l Y_l \\
&= X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A_{l,i} + A_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A_{N+l,i} + A_{l,j} \right) \right) \right) - \\
&\quad \left(\frac{1}{Y^W(\alpha+\beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A_{l,i} + A_{N+l,j} \right) + \beta \left(A_{N+l,i} + A_{l,j} \right) \right) \right) \left(\alpha + \beta \right) Y^W \\
&= 0,
\end{aligned}$$

i.e. equation (47) also holds. More generally, different choices of c correspond to different normalizations. A particularly simple example is if we choose the normalization $\gamma_1 = 1$. Since this implies that $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, $c = X_{ij} \times \left(A_{1,i} + A_{N+1,j} \right)$. In this case, however, an alternative procedure is even simpler: the elasticities for all $i > 1$ can be calculated directly by inverting the $(2N-1) \times (2N-1)$ matrix generated by removing the first row and first column of $f_{\vec{x}}$. $\square$

A.7 Proof of Proposition 5

Proof. We want to rewrite the equilibrium conditions in changes by defining $(\hat{x}_i) = x_i'/x_i$. Starting from (4) we have

$$\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} = \sum_j \frac{K_{ij}'\gamma_i'\delta_j'}{\sum_j K_{ij}\gamma_i\delta_j} \implies$$
$$\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} = \sum_j \pi_{ij}\hat{K}_{ij}\hat{\gamma}_i\hat{\delta}_j \implies$$
$$\hat{\gamma}_i^{\alpha-1}\hat{\delta}_i^{\beta} = \sum_j \pi_{ij}\hat{K}_{ij}\hat{\delta}_j$$

where $\pi_{ij} = X_{ij}/\sum_j X_{ij}$ represents the exporting shares. Similarly we can rewrite the second equilibrium condition, Equation (5), in changes as

$$\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} = \frac{\sum_j K_{ji}'\gamma_j'\delta_i'}{\sum_j K_{ji}\gamma_j\delta_i} \implies$$
$$\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta} = \sum_j \lambda_{ij}\hat{K}_{ji}\hat{\gamma}_j\hat{\delta}_i \implies$$
$$\hat{\gamma}_i^{\alpha}\hat{\delta}_i^{\beta-1} = \sum_j \lambda_{ji}\hat{K}_{ji}\hat{\gamma}_j$$

where $\lambda_{ij} = X_{ij}/\sum_i X_{ij}$ represents the import shares. This system of equations in changes is the same as the system of equations in levels. As long as λ_{ij}, π_{ij} are the same and α,β are the same all the gravity models give the same changes in γ_i, δ_j for a given change in K_{ij}.

□

B Online Appendix (not for publication)

This Online Appendix provides some additional theoretical results referenced in the paper.

B.1 Normalization

Without loss of generality we can normalize the world income.

Proposition 6. *Suppose that* (γ, δ) *solves the non-linear system. Denote the associated* (x, y). *Then* $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ *induces* $\left(t^{-\frac{1-\alpha}{1-\beta}}x, ty\right)$, *which again solves the non-linear equation. The world income* Y^W *under* $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ *is* $t^{\frac{\alpha-\beta}{1-\beta}}Y^w$. *In particular if* $t = (Y^w)^{-\frac{1-\beta}{\alpha-\beta}}$, *then* $Y^W = 1$.

Proof. Take $(t\gamma, s\delta)$, where

$$s = t^{-\frac{1-\alpha}{1-\beta}}.$$

Denote the associated $(x(t,s), y(t,s))$ .Then

$$\begin{aligned}
x(t,s) &= t^{\alpha-1}s^{\beta}x \\
&= t^{\alpha-1}t^{-\beta\frac{1-\alpha}{1-\beta}}x \\
&= t^{\alpha-1-\beta\frac{1-\alpha}{1-\beta}}x \\
&= t^{-\frac{1-\alpha}{1-\beta}}x \\
y(t,s) &= t^{\alpha}s^{\beta-1}y. \\
&= ty.
\end{aligned}$$

It is easy to show

$$\begin{aligned}
x_i(t,s) &= t^{\alpha-1}s^{\beta}x_i = \sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}\left(t^{\alpha-1}s^{\beta}x_j\right)^{\frac{\alpha}{\alpha+\beta-1}}\left(t^{\alpha}s^{\beta-1}y_j\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\
&= \sum_j K_{ij}B_j^{\frac{1}{1-\alpha-\beta}}\left(x_j(t,s)\right)^{\frac{\alpha}{\alpha+\beta-1}}\left(x_j(t,s)\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\
y_i(t,s) &= \sum_j K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}\left(t^{\alpha-1}s^{\beta}x_j\right)^{\frac{1-\beta}{\alpha+\beta-1}}\left(t^{\alpha}s^{\beta-1}y_j\right)^{\frac{\beta}{\alpha+\beta-1}} \\
&= \sum_j K_{ji}B_j^{\frac{1}{1-\alpha-\beta}}\left(x_j(t,s)\right)^{\frac{1-\beta}{\alpha+\beta-1}}\left(y_j(t,s)\right)^{\frac{\beta}{\alpha+\beta-1}}.
\end{aligned}$$

Thus a solution to the The world income induced by $\left(t\gamma, t^{-\frac{1-\alpha}{1-\beta}}\delta\right)$ is

$$\begin{aligned}\sum_i B_i (t\gamma_i)^\alpha \left(t^{-\frac{1-\alpha}{1-\beta}}\delta_i\right)^\beta &= t^{\alpha-\frac{1-\alpha}{1-\beta}\beta}\sum_i B_i\gamma_i^\alpha\delta_i^\beta \\ &= t^{\frac{\alpha-\beta}{1-\beta}}Y^w.\end{aligned}$$

In particular if we take $t^{-\frac{\alpha-\beta}{1-\beta}} = Y^w$, then the world income is normalized to 1. $\square$

B.2 Walras law

In the previous section, we showed that without loss of generality, we can normalize the system of equations so that world income is equal to an arbitrary constant. In this section, we show that Walras law holds, i.e. if all equilibrium equations but one hold with equality, then the remaining one holds with equality as well. The two facts together imply that the equilibrium is really defined by $2N-1$ equations and $2N-1$ unknowns.

To see this, define $\gamma \equiv \{\gamma_i\}$, $\delta \equiv \{\delta_i\}$ and $x \equiv \{\gamma;\delta\}$, where x is a $2N \times 1$ vector. Consider the function $f(x) : R^{2N} \to R^{2N}$ given by:

$$f(x) = \begin{bmatrix} \left[B_i\gamma_i^{\alpha-1}\delta_i^\beta - \sum_j K_{ij}\delta_j\right]_i \\ \vdots \\ \left[\sum_j K_{ji}\gamma_j - B_i\gamma_i^\alpha\delta_i^{\beta-1}\right]_i \end{bmatrix}.$$

Note that the general equilibrium trade model is in equilibrium if $f(x) = 0$. Walras law can be written as:

$$f(x)\cdot x = 0.$$

To see this is the case, note that:

$$\begin{aligned} f(x)\cdot x = 0 &\iff \\ \sum_i \left(B_i\gamma_i^{\alpha-1}\delta_i^\beta - \sum_j K_{ij}\delta_i\right)\times\gamma_i + \sum_i\left(\sum_j K_{ij}\gamma_i - B_i\gamma_i^\alpha\delta_i^{\beta-1}\right)\times\delta_i = 0 &\iff \\ \sum_i B_i\gamma_i^\alpha\delta_i^\beta - \sum_i\sum_j K_{ij}\gamma_i\delta_i + \sum_i\sum_j K_{ji}\gamma_j\delta_i - \sum_i B_i\gamma_i^\alpha\delta_i^\beta = 0 &\iff \\ 0 = 0. \end{aligned}$$

Hence, Walras law holds.

B.3 Existence and Uniqueness using Gross Substitutes Methodology (a la Alvarez and Lucas (2007))

We will illustrate the application of the gross-substitute property to prove uniqueness equilibrium in an excess demand system. This is a necessary step in the proof of Alvarez and Lucas (2007) but it is not sufficient, as a number of other properties need to be proved for an equation to be an excess demand system, as we discuss below.

Because of the complexity of the system that we analyze we cannot apply the gross-substitutes property directly to equations (4) and (5).

$$B_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_j K_{ij} \delta_j \tag{48}$$

Combining gravity A.1 with balanced trade A.(3) and asumption A.4 yields:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \tag{49}$$

In order to find the equation that can be used to prove, we need to eliminate one variable. Use (5) to express δ_i as

$$\delta_i = \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^{\alpha}} \right)^{\frac{1}{\beta-1}} \tag{50}$$

into equation (4), we obtain

$$\begin{aligned} B_i \gamma_i^{\alpha} \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^{\alpha}} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \gamma_i \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij} \iff \\ B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha}{1-\beta}-1} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij} \end{aligned} \tag{51}$$

We define the corresponding excess demand function might be

$$Z_i(\gamma) = \frac{1}{\gamma_i} \left[B_i^{\frac{1}{\beta-1}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} - \sum_{j' \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj'}}{B_{j'} \gamma_{j'}^{\alpha}} \right)^{\frac{1}{\beta-1}} K_{ij'} \right]$$

This system written as such needs to satisfy 5 properties to be an excess demand system and the gross substitute property to establish existence and uniqueness (see Propositions 17.B.2, 17.C.1 and 17.F.3 of Mas-Colell, Whinston, and Green (1995)). The six conditions are:

1. $Z(\gamma)$ is continuous for $\gamma \in \left(\Delta\left(R_+^N\right)\right)^o$
2. $Z(\gamma)$ is homogenous of degree zero.
3. $Z(\gamma)\cdot\gamma = 0$ (Walras' Law).
4. There exists a $k > 0$ such that $Z_j(\gamma) > -k$ for all j.
5. If there exists a sequence $w^m \rightarrow w^0$, where $w^0 \neq 0$ and $w_i^0 = 0$ for some i, then it must be that:

$$max_j\{Z_j(w^m)\}\rightarrow\infty \tag{52}$$

and the gross-substitute property:

6. Gross substitutes property: $\frac{\partial Z(w_j)}{\partial w_k} > 0$ for all $j \neq k$.

Properties 1-3 are trivial by the way we define the system. Properties 4 and 5 are challenging and may require an analysis case-by-case which restrict further the set of parameters that uniqueness applies. We thus only discuss the region where gross-substitutes applies. To consider this system as an excess demand system and apply the tools originally developed in Alvarez and Lucas (2007), we need to differentiate the expression above. We only use the bracketed term without loss of generality. We have:

$$\begin{aligned}
\frac{\partial Z_i(\gamma)}{\partial \gamma_j} = {} & \frac{\beta}{\beta-1}K_{ji}B_i^{\frac{1}{1-\beta}}\gamma_i^{\frac{\alpha+\beta-1}{1-\beta}}\left(\sum_{s\in S}\gamma_s K_{si}\right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1}\sum_{j'\in S, j'\neq j}\left[K_{ij'}\left(\frac{\sum_{s\in S}\gamma_s K_{sj'}}{B_{j'}\gamma_{j'}^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}}K_{jj'}\right] - \\
& -\frac{1}{\beta-1}K_{ij}\left(\frac{\sum_{s\in S}\gamma_s K_{sj}}{B_j\gamma_j^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}}\left[\frac{K_{jj}B_j\gamma_j^{\alpha} - \alpha B_j\gamma_j^{\alpha-1}\sum_{s\in S}\gamma_s K_{sj'}}{B_j\gamma_j^{\alpha}}\right] \\
= {} & \frac{\beta}{\beta-1}K_{ji}B_i^{\frac{1}{1-\beta}}\gamma_i^{\frac{\alpha+\beta-1}{1-\beta}}\left(\sum_{s\in S}\gamma_s K_{si}\right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1}\sum_{j'\in S, j'\neq j}\left[K_{ij'}\left(\frac{\sum_{s\in S}\gamma_s K_{sj'}}{B_{j'}\gamma_{j'}^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}}K_{jj'}\right] - \\
& -\frac{1}{\beta-1}K_{ij}\left(\frac{\sum_{s\in S}\gamma_s K_{sj}}{B_j\gamma_j^{\alpha}}\right)^{\frac{-\beta+2}{\beta-1}}\left[\frac{\gamma_j K_{jj} - \alpha\sum_{s\in S}\gamma_s K_{sj'}}{\gamma_j}\right]
\end{aligned}$$

Let $\beta < 0$ and $\alpha < 0$ then the expression is positive and the gross-substitute property holds. Similar results can be easily established for $\beta = 0$, $\alpha < 0$ and $\beta < 0$, $\alpha = 0$. The same cannot be, in generally, established if $\beta > 1$ or $\alpha > 1$ since the expression cannot be signed in that case, and in particular we have found parametric specifications where the gross-substitutes property may fail.[29] Thus, the region that uniqueness applies with this approach is $\alpha \leq 0, \beta \leq 0$.

[29] In particular, we analyzed the Armington case with intermediate inputs as in Section 2. We can show that this model for $\sigma = 3$ and $\gamma = 1/4$ corresponds to the case $\alpha, \beta > 1$ but the gross-substitute condition does not obtain in the case of many symmetric regions with symmetric trade costs or even two regions with no trade costs.

B.4 Comparative Statics when $\beta = 0$

Let us consider a particularly interesting special case, $\beta = 0$. We have in this case that the equilibrium is characterized by

$$B_i \gamma_i^{\alpha-1} = \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^{\alpha}} \right)^{-1} K_{ij} \implies$$

$$\gamma_i^{\alpha-1} = \sum_{j \in S} \left(\frac{B_j \gamma_j^{\alpha}}{\sum_{s \in S} \gamma_s K_{sj}} \right) B_i K_{ij},$$

which is the standard single-equation gravity model that we find in papers such as Anderson (1979); Eaton and Kortum (2002); Chaney (2008). We can rewrite this system re-written using A.4 as

$$Y_i = \sum_{j \in S} \left(\frac{\gamma_i K_{ij}}{\sum_{s \in S} \gamma_s K_{sj}} \right) Y_j$$

In this last equation the technique developed by Dekle, Eaton, and Kortum (2008) can be applied (see details in Arkolakis, Costinot, and Rodríguez-Clare (2012)) so that computing the changes in γ_i require only knowledge of changes in K_{ij} and initial trade and output levels across all the models that can be captured by this formulation.

Notice that given equation 50 and the above equation we have for $\beta = 0$ that we can express the origin fixed effects as a function of the destination fixed effects and parameters

$$\gamma_i = \left(\frac{\sum_j K_{ij} \delta_j}{B_i} \right)^{\frac{1}{\alpha-1}}. \tag{53}$$