

Variable Annuities in Pension Schemes with Risk Sharing: Valuation, Investment and Communication

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Abstract: This paper explores pension schemes that provide (deferred) variable annuities by sharing risks among current participants on the basis of complete contracts. Annuities are adjusted gradually after an unexpected shock. This is consistent with habit formation and leads to life-cycle investment. We show how these variable annuities can be valued in a market-consistent fashion and discuss how investment policy of a pension fund can be determined endogenously on the basis of the desired risk profiles of the (deferred) variable annuities.

Key words: variable annuity, defined-ambition, liability-driven investment.

JEL Code(s): D91, G11, G23

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1 Introduction

This paper explores pension schemes that provide (deferred) variable annuities by sharing financial, inflation and biometric risks among the participants. These so-called defined-ambition schemes share risks on the basis of complete contracts. In particular, if the value of aggregate liabilities deviates from the value of total assets, the pension contract specifies how individual pension rights are adjusted so that the aggregate value of individual pension rights equals again total assets in the fund. Accordingly, mismatch risk is born by the participants on the basis of ex-ante rules rather than by a corporate sponsor (in corporate schemes), future contributors (in defined-benefit schemes in which contribution rates and capital buffers absorb shocks), tax payers (in public schemes), or the shareholders of an insurance company (in insured schemes). These schemes are similar to defined-contribution schemes in the sense that outside risk sponsors are lacking. Participants can thus trade risk with outsiders only through tradable financial instruments. Hence, income streams during retirement are conditional on the investment performance of the scheme, actual mortality rates of the participants and possibly (wage) inflation.

Annuities are adjusted gradually after an unexpected shock that causes a mismatch between assets and liabilities. Hence, retirees can take some time to adjust their standard of living after a shock. This intertemporal smoothing of shocks is consistent with habit formation. Gradual adjustment to shocks leads to life-cycle investment in which risk exposure declines with age also during the pay-out phase because retired agents have less time to absorb shocks when they become older and their remaining expected life time declines.

The defined-ambition schemes we consider differ from traditional defined-contribution schemes in which individuals hold tradable financial assets. In particular, the schemes provide longevity insurance by pooling and sharing longevity risk. Furthermore, the aspired annuities can be indexed to wages. Accordingly, participants can trade wage-linked and longevity-linked

claims, which are not (yet) available on financial markets. In this respect, the schemes resemble non-financial defined-contribution schemes in which participants hold implicit assets that are not traded and valued on financial markets (see Holzmann, Palmer and Robalino (2011)). Moreover, risk management during the accumulation phase is based on the objective of providing a life-long income stream during retirement. Accordingly, inflation and interest-rate risks are actively managed. In addition, contribution levels are set so as to reach a particular goal for retirement income. During the pay-out phase, the pension contracts specify also how payouts respond to unexpected shocks. By determining contributions and pay-outs, the contracts thus specify not only investment but also (dis)saving and consumption decisions. Accordingly, the stochastic pension rights can be communicated in terms of a risk profile of an income stream in retirement.

The variable annuities we consider can be valued in a market-consistent fashion. Market consistent valuation is relevant for determining the price for buying and selling the annuities. Indeed, we show how pension contributions can be derived endogenously from the stochastic pension promises (i.e. the pension ambitions), which are in fact the liabilities of the defined-ambition scheme. This is reminiscent of defined-benefit schemes in which the pension contributions are determined by the costs of the aspired income stream in retirement. The (deferred) stochastic annuities provided by the defined-ambition schemes are priced on the basis of the nominal term structure of the interest rates amended in two directions. First, a horizon-dependent risk premium that rises with the time horizon is added. Second, break-even inflation capturing the aspired indexation to inflation is subtracted. The overall discount curve to be used typically lies in between the expected return on the actual investment portfolio of the fund as a whole, on the one hand, and the real risk-free term structure, on the other hand.

The paper discusses how the investment policy of the fund as a whole can be determined endogenously from the desired risk profiles of the (deferred) variable annuities. The paper thus extends to defined-ambition schemes the

principle of liability-driven investment familiar from defined-benefit schemes. In effect, asset-liability management (ALM) is generalized to stochastic liabilities. Indeed, the contract is complete in terms of not only the allocation of mismatch risk across participants but also investment policy so that participants obtain the exposures that have been communicated to them. We show, however, that in the linear contracts we consider a change in investment policy of the fund does not affect the market value of individual pension rights.

The defined-ambition schemes are based on proposed risk-sharing systems in the Netherlands, and evolved from traditional defined-benefit schemes with (nominally) guaranteed pension rights. Also in public-sector pension schemes in the United States risk sharing is being considered as a way to reduce the costs of these schemes (see e.g. Novy-Marx and Rauh (2012)). Our paper contributes to the emerging literature on the implications of moving from a defined-benefit design towards defined-ambition schemes. The scheme we explore in this paper encompasses both the accumulation and the decumulation phases but the defined-ambition scheme can be limited to the pay-out phase or the accumulation phase only.

The structure of this paper is as follows. Section 2 lays out our simple benchmark modeling framework for explaining the basics of the defined-ambition schemes. Section 3 introduces the specification of the risk profile of the variable annuity. This is the stochastic liability that the pension fund must match in its investment policy. We also investigate how pension funds can calibrate and communicate the risk surrounding the future income stream. Section 4 explores the market-consistent valuation of these stochastic annuities. This allows us to demonstrate how pension contributions can be derived endogenously from the stochastic pension ambitions and how one can trade annuities with different risk profiles and indexation ambitions. Section 5 discusses the aggregate investment policy of the fund. Section 6 explores a subclass of risk profiles in which adjustments in benefit levels are determined by a single state variable. Section 7 turns to some extensions, namely

additional risk factors (such as interest rate, longevity, and wage risk) and individual choice (in terms of contributions (savings), investment, and pay-outs (dissavings)). Finally, the concluding section 8 discusses the role of public supervision.

2 The benchmark model

We assume a simple financial market with a single risk factor, which we interpret as the stock-market index at time t .¹ The stock-market return during period j is \mathbf{R}_j^s , where we apply bold typesetting to indicate that this is a stochastic variable. The distribution of these returns is lognormal with standard deviation σ and expectation $E[\mathbf{R}_j^s] = r + \lambda$, where r is the nominal risk-free return and λ the risk premium on the single risk factor. Returns are assumed independent and identically distributed (i.i.d).

Perfect insurance of individual longevity risk is available. Aggregate longevity risk is absent. Individuals start their working career at age a_s and retire at age a_r . In order to save on notation, we assume that all individuals live up to age a_{\max} .

The numerical illustrations in this paper are based upon the following default parameters. The risk-free nominal interest rate r is 3% per year. Volatility σ is 20% on an annual basis and the annual risk premium λ is 4%. Individuals start their working file at $a_s = 20$, retire at $a_r = 65$ and live up to age $a_{\max} = 85$.

¹Section 7.1 explores extensions to more risk factors involving expected and unexpected (wage) inflation, systematic longevity, and interest rates.

3 Risk profiles

3.1 Specification

Let $B_{a,t}^i$ denote the pension entitlements of an individual i with birth-year a at time t . The pension entitlement is defined in terms of a (deferred) variable annuity that starts to be paid at the retirement age. An individual in the pension fund receives an actual pension payment only if retired and alive. In our simple model, an individual thus collects a pension benefit at time t if the birth year of the individual lies between $t - a_{\max}$ and $t - a_r$. Hence, the actual pension payment $P_{a,t}$ to an individual i with birth-year a at time t is given by:

$$P_{a,t}^i = B_{a,t}^i I_{(t-a_{\max}) \leq a \leq (t-a_r)}. \quad (1)$$

The risk profile of the variable annuity is specified in terms of the following relationship between the current pension entitlement B_{a,t_0}^i and the future entitlement \mathbf{B}_{a,t_0+h}^i at horizon h (where we apply bold typesetting for \mathbf{B}_{a,t_0+h}^i to indicate that the realization of this variable is not yet known at time t_0):²

$$\begin{aligned} \mathbf{B}_{a,t_0+h}^i &= B_{a,t_0}^i \times [1 + wq_1(\mathbf{R}_{t_0+h}^s - r - \lambda)] \\ &\quad \times \dots \times [1 + wq_h(\mathbf{R}_{t_0+1}^s - r - \lambda)] \\ &= B_{a,t_0}^i \left(\prod_{j=1}^h [1 + wq_j(\mathbf{R}_{t_0+h-(j-1)}^s - r - \lambda)] \right) \end{aligned} \quad (2)$$

We impose $q_h \rightarrow 1$ for $h \rightarrow \infty$, which we denote as $q_\infty = 1$. Hence, the parameter w can be interpreted as the exposure to the risk factor at very long investment horizons.

Expression (2) in effect specifies the liabilities of the defined-ambition scheme. The marginal risk exposures q_j depend on neither the return $\mathbf{R}_{t+h-(j-1)}^s$

²Agent i is entitled to this future benefit on the basis of current entitlements only. (2) thus assumes that agent i does not accumulate more pension rights by paying contribution rates in the future. This in fact is a so-called discontinuity perspective in which we consider only future pension benefits on account of pension entitlements that have been accumulated in the past.

nor $B_{a,t}^i$. Moreover, these exposures do not depend on time t either. The risk profiles are thus 'sustainable' in the sense that young workers (who have to wait longer for their annuity payment than older workers) do not feature a more risky variable annuity at the same horizon than older workers do.

If $q_h = 1$ for all h , all agents exhibit the same exposure to stock-market risk $\mathbf{R}_{t_0+j}^s$ irrespective of their investment horizon $t_0 + h \geq t_0 + j$. However, (2) allows for the possibility of risk differentiation based on the investment horizon h by setting $q_h < q_\infty = 1$. In particular, compared to smaller horizons, longer investment horizons exhibit a larger exposure to a given risk $\mathbf{R}_{t+h-(j-1)}^s$ if q_h rises with h . In that case, young agents, who feature a longer investment horizon than old agents do, are exposed more to current stock-market risk. We impose that the risk exposures q_h are non-decreasing with the horizon, i.e. $q_h \geq q_{h-1}$ for all $h > 1$. Specification (2) allows risk exposure to depend only on the investment horizon h and thus only indirectly on age. As a direct consequence, life-cycle investment in which stock-market exposure declines with age continues during the pay-out phase. Habit formation can explain this type of horizon differentiation in risk exposure.³

Figure 1 illustrates an example of horizon differentiation q_h as a function of horizon h in the case where $q_h = h/N$ for $h < N$ and $q_h = 1$ with $N = 10$. This is an example in which smaller investment horizons $h < N = 10$ exhibit a smaller risk exposure than longer investment horizons $h \geq N = 10$. In particular, the risk exposure at a one-year horizon ($h = 1$) is only 1/10th of the exposure at long horizons $h \geq N = 10$, the exposure at a two-year horizon ($h = 2$) is 2/10th of the exposure at long horizons, etc.

³If risk differentiation would be based on differential exposures to human-capital risk rather than habit formation, marginal risk exposures would depend not only on horizon j but also on age a (see Bodie, Merton and Samuelson (1992)). The restriction that risk exposures depend only on the investment horizon (and not on age) implies that the adjustment of pension entitlements $\frac{(B_{a,t+1}^i - B_{a,t}^i)}{B_{a,t}^i}$ can be uniform for all age groups (see (6)). This is common practice in Dutch occupation pension schemes. For an extension in which risk exposures also depend on other factors, see section 7.2.

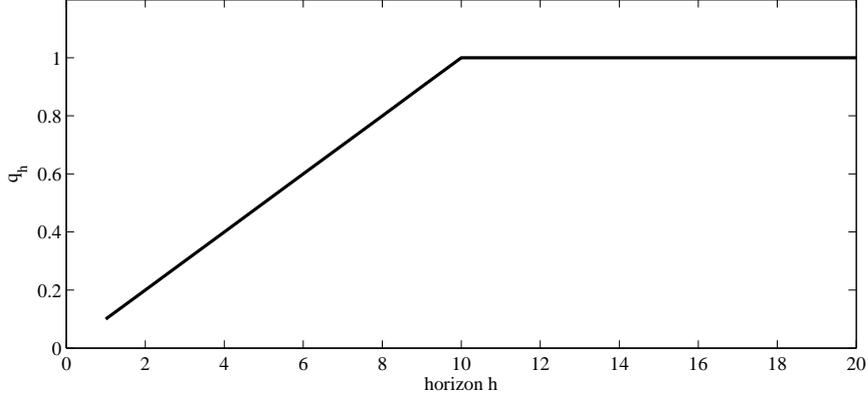


Figure 1: *Illustration of horizon differentiation of risk exposure q_h in the case where $q_h = h/N$ for $h < N$ and $q_h = 1$ for $h \geq N$ with $N=10$.*

3.2 Smoothing: Impact of past shocks

The pension entitlement at time $t_0 + l$ given the information at the current time $t = t_0 + k$ ($l > k$) is given by (see (2), where we define $h = l - k$ so that $t_0 + l = t + h$)

$$\begin{aligned} \mathbf{B}_{a,t+h}^i &= B_{a,t_0}^i \times \prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)] \\ &\quad \times \prod_{j=1}^h [1 + wq_j(\mathbf{R}_{t+h-(j-1)}^s - r - \lambda)], \end{aligned} \quad (3)$$

where the realized shocks in the past R_{t-j}^s for $0 \leq j \leq k$ are separated from the future, uncertain shocks \mathbf{R}_{t+j}^s for $1 \leq j \leq h$.⁴ If the investment policy fits the assumed risk profile, the expectation of the pension entitlement at

⁴Shocks occur at the beginning of a period so that current shocks R_t^s are known and affect pension payments during the current period t .

time $t + h$ given the information at time t is given by (see Appendix)

$$\frac{E_t[\mathbf{B}_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} = F_t^h, \quad (4)$$

where

$$F_t^h \equiv \frac{\prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)]}{\prod_{j=0}^{k-1} [1 + wq_{j+1}(R_{t-j}^s - r - \lambda)]} - 1 \quad (5)$$

captures how future benefits are affected by shocks in the past. In the absence of horizon differentiation, $F_t^h = 0$ (substitute $q_j = 1$ for all j in (5)) because shocks are absorbed immediately in current pay-outs. The adjustment factors (5) are thus the direct consequence of smoothing shocks over time so that shocks are absorbed only gradually. Intuitively, the smoothing of shocks results in funding imbalances that must be absorbed in the future. As a direct consequence, future adjustments in pension entitlements $\frac{E_t[\mathbf{B}_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i}$ become predictable. F_t^h summarizes the predictable changes in annuities at the future horizon h as a result of past shocks that have not been absorbed fully in current annuities yet.

At the begin of each period $t + 1$, pension entitlements (and thus pensions in payment) are adjusted according to (see Appendix)

$$\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} = F_{t+1}^0 = (1 + F_t^1) \times [1 + wq_1(R_{t+1}^s - r - \lambda)] - 1 \quad (6)$$

where R_{t+1}^s is now known. The right-hand side depends neither on a nor i . Hence, pension entitlements (and thus pension benefits) are adjusted uniformly across individuals.

3.3 Calibrating and communicating risk

If the investment policy fits the assumed risk profile, the volatility of the pension payments at time $t+h$ given the information at time t is an increasing

function of the horizon h : (see Appendix)

$$std_t[\mathbf{B}_{a,t+h}] \approx w\sigma \sqrt{\sum_{j=1}^h q_j^2}. \quad (7)$$

Expression (7) results in the following expression for payments at time $t + h$, conditional on information available at time t , for individual i of age a in the 2.5 % quantile $P2.5\%_{a,t}^{i,h}$: (see Appendix)

$$P2.5\%_{a,t}^{i,h} \approx B_{a,t}^i (1 + F_t^h) \left(1 - 1.96w\sigma \sqrt{\sum_{k=1}^h q_k^2} \right), \quad (8)$$

where the 2.5% quantile is defined as $Pr_t[\mathbf{B}_{a,t+h}^i < P2.5\%_{a,t}^{i,h}] \equiv 2.5\%$. The terms $(1+F_t^h)$ and $\left(1 - 1.96w\sigma \sqrt{\sum_{k=1}^h q_k^2}\right)$ represent past and future shocks, respectively.

We can calibrate long-run risk w and horizon-differentiation q_j from the desired risk at each horizon. For example, we can set the desired deviation from the expected pension benefit $E_t[B_{a,t+h}^i]$ in the 2.5 % quantile at each horizon h as follows

$$\frac{P2.5\%_{a,t}^{i,h}}{E_t[\mathbf{B}_{a,t+h}^i]} = -1.96\sigma w \sqrt{\sum_{k=1}^h q_k^2}, \quad (9)$$

and then endogenously determine the risk parameters w and q_j .

Given individual pension entitlement $B_{a,t}^i$, pension funds should communicate to individuals both expected benefits (4) and the 2.5 % quantiles (8) at particular horizons h . The outcomes depend on the stochastic model used. Hence, the supervisory authorities may prescribe the models to be used in order to prevent pension funds from providing too optimistic projections.

Figure 2 illustrates the 2.5 % and 97.5% quantiles as a function of horizon h in the situation with horizon differentiation (with q_h given by $q_h = h/N$ for $h < N$ and $q_h = 1$ for $h \geq 10$, with $N=10$, with $N=10$) and without horizon differentiation ($q_h = 1$ for all $h \geq 1$).

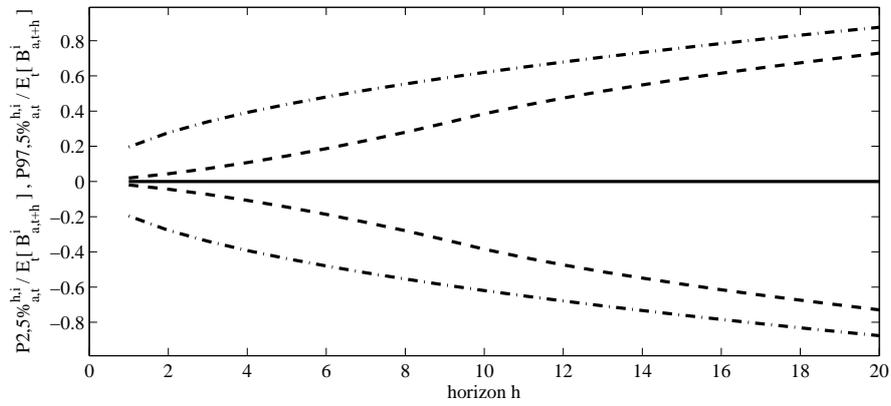


Figure 2: The 2.5 % quantile $\frac{P2.5\%_{a,t}^{i,h}}{E_t[\mathbf{B}_{a,t+h}^i]}$ and the 97.5% quantile $\frac{P97.5\%_{a,t}^{i,h}}{E_t[\mathbf{B}_{a,t+h}^i]}$ as a function of horizon h in the presence of horizon differentiation (with q_h given by $q_h = h/N$ for $h < N$ and $q_h = 1$ for $h \geq 10$, with $N=10$, as in Figure 1) in the dashed lines and in absence of horizon differentiation ($q_h = 1$ for all $h \geq 1$) in the dash-dotted lines. The solid line represents the mean $E_t[\mathbf{B}_{a,t+h}^i]$.

4 Market consistent valuation

Market-consistent valuation of the future stochastic annuity payments determines the resources that are currently needed to honor the pension promises. In a complete market setting, the market consistent value is given by the value of the replicating portfolio, i.e. an investment strategy that exactly matches these promised cashflows.

4.1 Pricing a single cashflow

We employ risk-neutral pricing to derive the market-consistent value at time t of a pension payment paid in period $t + h$. Under risk-neutral pricing, the value of an uncertain cashflow is given by its discounted expectation calculated under the risk-neutral probability measure (see e.g. Cochrane (2001)). Thus, the market-consistent value of the pension payment at horizon h of an individual i with birth date $(t + h - a_{\max}) \leq a \leq (t + h - a_r)$ amounts to: (see Appendix)⁵

$$V_{a,t}^{i,h} = \frac{B_{a,t}^i (1 + F_t^h) \prod_{j=1}^h (1 - w\lambda q_j)}{(1 + r)^h} \approx \frac{B_{a,t}^i (1 + F_t^h)}{(1 + r + w\lambda Q_h)^h}. \quad (10)$$

where Q_t represents the "term structure of risk" Q_h and is defined as

$$Q_h \equiv \frac{1}{h} \sum_{k=1}^h q_k. \quad (11)$$

Q_h and q_h relate to each other in a similar way as the interest rate and the forward rate on horizon h . The non-decreasing nature of the 'marginal' exposures q_h (i.e. $q_h \geq q_{h-1}$ for all $h > 1$) implies that also 'average'

⁵The derivation of (10) in the Appendix uses the no-arbitrage condition in financial markets by imposing that the excess returns on the risky asset (i.e. the return on the risky asset in excess of the risk-free rate) has zero market value. The economic intuition is that, from an ex-ante perspective, the payoff from a one-dollar investment in the risky asset exhibits the same market value as a one-dollar investment in the risk-free asset. Indeed, the same dollar cannot have different market values under the no-arbitrage condition.

exposures Q_h are non-decreasing (i.e. $Q_h \geq Q_{h-1}$ for $h > 1$) and that 'average' exposures do not exceed 'marginal' exposures (i.e. $Q_h \leq q_h$ for all $h \geq 1$).⁶

The linear nature of the pension contract with fixed marginal risk exposures (2) facilitates market-consistent valuation. In particular, the valuation of the cash flows does not depend on actual investment policy. This implies that the market consistent value of the contract does not depend on subjective parameters of a stochastic model.⁷ Indeed, since the value is the same for all investment policies, one can employ a risk-free investment strategy to value the contract. One thus does not have to conduct stochastic simulations.

We can generalize (2) by assuming that

$$B_{a,t_0+h}^i = B_{a,t_0}^i (1 + \pi)^h \left(\prod_{j=1}^h (1 + wq_j(\mathbf{R}_{t_0+h-(j-1)}^s - r - \lambda)) \right), \quad (12)$$

where π is the (constant) desired indexation of pension benefits. In that case (10) becomes

$$V_{a,t}^{i,h} \approx \frac{B_{a,t}^i}{(1 + r - \pi + w\lambda Q_h)^h} (1 + F_t^h). \quad (13)$$

This expression has two parts. The first part at the right-hand side summarizes the future and the second part $(1 + F_t^h)$ represents past shocks that give rise to predictable changes in annuities because of smoothed adjustment to shocks. The solid line in 3 illustrates the discount curve $r - \pi + w\lambda Q_h$ as a function of horizon h in the case where horizon differentiation q_h is given by $q_h = h/N$ for $h < N$ and $q_h = 1$ for $h \geq 10$, with $N=10$, with $N=10$.

The special case of a nominal defined-benefit scheme is given by $w = \pi = 0$. In this case, we have

$$V_{a,t}^{i,h} = \frac{B_{a,t}^i}{(1 + r)^h}. \quad (14)$$

⁶Note that (11) implies that $Q_h < q_h$ if $q_1 < q_h$.

⁷The pension contract (2) depends on the difficult to estimate risk premium λ . Given the selected parameter, however, the valuation of the contract does not depend on a stochastic model and thus does not suffer from model risk.

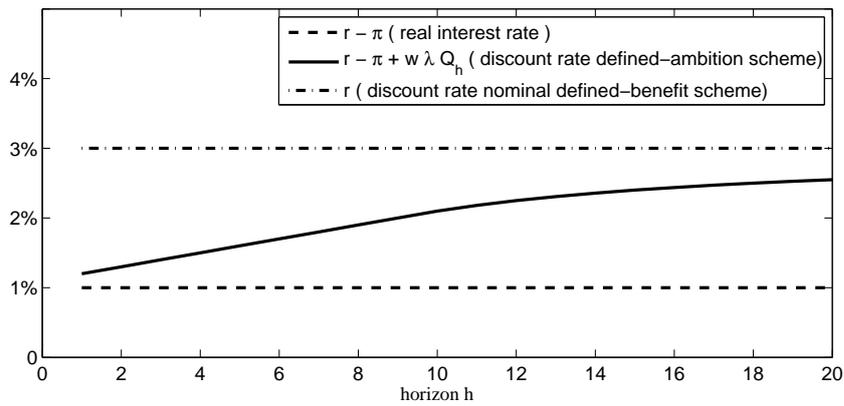


Figure 3: *The real discount rate $r - \pi$ (dashed line), the market-consistent discount rate $r - \pi + w \lambda Q_h$ of the defined-ambition scheme (solid line), and the market-consistent discount rate r of a nominal defined benefit scheme (dash-dotted line). The discount rate of the defined-ambition scheme is based on (11) with horizon differentiation q_h given by $q_h = h/N$ for $h < N$ and $q_h = 1$ for $h \geq 10$, with $N=10$, with $N=10$, as in Figure 1. The figure is based on $\pi=2\%$.*

Figure 3 compares (13) and (14) by showing the three differences between the discount rate of variable annuities and traditional defined-benefit schemes. First of all, a horizon-dependent risk premium $w\lambda Q_h$ that rises with the time horizon is added in (13) compared to (14). Second, aspired indexation π is included in (13). Third, a term $(1 + F_t^h)$ is added representing shocks in the past that have not been absorbed yet in current annuities (if $q_j < 1$).

With a regular variable annuity in which shocks are absorbed immediately (i.e. $q_h = Q_h = 1$ for all $h \geq 1$) and the expected annuity payments are constant in nominal terms, (10) boils down to

$$V_{a,t}^{i,h} = \frac{B_{a,t}^i}{(1 + r + w\lambda)^h}.$$

The term representing shocks in the past is not present in this case because shocks are not smoothed but rather absorbed immediately in current pension rights $B_{a,t}^i$. Hence, predictable future changes in pension payouts as a result of historical shocks that have not been absorbed in current annuities are absent.

The aggregate value of the annuities provided by the pension fund is given by

$$V_t \equiv \sum_{h=1}^{a_{\max} - a_s} V_t^h, \quad (15)$$

where $a_{\max} - a_s$ is the maximum horizon of pension payments and

$$V_t^h \equiv \frac{B_t^h (1 + F_t^h)}{(1 + r - \pi + w\lambda Q_h)^h} \quad (16)$$

represents the market value of actual pension payments at time $t + h$ aggregated over all individuals. B_t^h stands for the sum of all pension entitlements that are in the pay-out phase at time $t + h$ (i.e. at future horizon h at time t)

$$B_t^h \equiv \sum_{a=t+h-a_{\max}}^{t+h-a_r} B_{a,t}, \quad (17)$$

where $B_{a,t} = \sum_i B_{a,t}^i$ denotes aggregate pension entitlements of individuals with birth-year a at time t . Hence, (16) is the aggregate version of (13).

4.2 Liabilities and funding rates

With variable annuities that are adjusted gradually in response to shocks, one can adopt also an alternative definition of 'liabilities'. With this alternative definition of liabilities, the aggregate value of liabilities is no longer necessarily equal to the value of assets so that the funding rate can deviate from unity. The calculation of a funding rate unequal to one makes the pension system reminiscent of defined-benefit systems. The alternative definition of liability is based on the ambition to increase the current pension entitlements B_t in line with aspired indexation π (in expectation and with the desired risk profile). In particular, the value of a defined-ambition liability at horizon h of an individual i with birth date $(t + h - a_{\max}) \leq a \leq (t + h - a_r)$ and pension entitlement $B_{a,t}^i$ amounts to (where the second equality follows from (13))

$$L_{a,t}^{i,h} = \frac{B_{a,t}^i}{(1 + r - \pi + w\lambda Q_h)^h} = \frac{V_{a,t}^{i,h}}{(1 + F_t^h)}, \quad (18)$$

These liabilities are the resources that are currently needed to consistently increase the current pension entitlements $B_{a,t}^i$ in line with aspired indexation π (in expectation and with the desired risk profile (given by w and q_h)). This definition of liabilities thus abstracts from predictable changes in future annuity payments that are the result of past shocks. (18) can be written in aggregate form as (where L_t^h denotes the aggregate value of all liabilities at horizon h)

$$L_t^h \equiv \frac{B_t^h}{(1 + r - \pi + w\lambda Q_h)^h} = \frac{V_t^h}{(1 + F_t^h)}. \quad (19)$$

We can view $(1 + F_t^h)$ as the horizon-specific 'funding rate' because it represents the ratio between the actual value of annuity payments ('assets') and the value of the defined ambition ('liabilities') at a particular horizon. i.e. $(1 + F_t^h) = \frac{V_t^h}{L_t^h}$.

The funding rate for the fund as whole can be computed as the weighted

average of horizon-specific funding rates

$$1 + F_t \equiv \frac{V_t}{L_t} = \sum_{h=1}^{a_{\max} - a_s} \gamma_t^h (1 + F_t^h), \quad (20)$$

where the aggregate value of liabilities L_t is defined by

$$L_t = \sum_{h=1}^{a_{\max} - a_s} L_t^h = \sum_{h=1}^{a_{\max} - a_s} \frac{B_t^h}{(1 + r - \pi + w\lambda Q_h)^h}, \quad (21)$$

and

$$\gamma_t^h \equiv \frac{L_t^h}{L_t}.$$

The budget constraint of a defined-ambition schemes implies (combine (23) and (20) to eliminate V_t)

$$\frac{A_t}{L_t} - 1 = F_t = \sum_{h=1}^{a_{\max} - a_s} \gamma_t^h F_t^h.$$

4.3 Contributions

We can employ (13) to calculate the price $V_{a,t}^i$ of an (deferred) annuity $B_{a,t}^i$ for an individual i with birth-year a :

$$P_{a,t}^i = \frac{V_{a,t}^i}{B_{a,t}^i} \quad (22)$$

The right-hand side of this expression represents the price this individual should pay for each new unit of a pension entitlements $B_{a,t}^i$, where we assume that newly acquired pension entitlements share in current funding gaps F_t^h .⁸ This calculation of the price of the pension entitlement is in line with the

⁸If agents want to acquire pension entitlements that do not share in these funding gaps, they can purchase $\left(\sum_{h=\max(1, a_r + a - t)}^{a_{\max} + a - t} \frac{1}{(1 + r - \pi + w\lambda Q_h)^h} \right) B_{a,t}^i / \left(\sum_{h=\max(1, a_r + a - t)}^{a_{\max} + a - t} \frac{(1 + F_t^h)}{(1 + r - \pi + w\lambda Q_h)^h} \right)$ rather than $B_{a,t}^i$ pension entitlements and buy desired pension entitlements at retirement.

defined-benefit tradition in which the pension premium is determined by the costs of the aspired income stream in retirement. The economically fair price for buying a new annuity ensures that the purchase of new pension entitlements does not impact the value of the existing pension entitlements and that (23).

If annuities that are bought and sold are priced as in (22), the market-value V_t matches the current value of pension fund assets A_t :

$$A_t = V_t. \quad (23)$$

The pension contract thus exhibits a defined-contribution character in the sense that shocks in assets are absorbed immediately in the market-value of individual pension entitlements. Outside sponsors (such as companies, future contributors, insurance companies, tax payers) are lacking. The promises are backed by financial assets so that the system is always fully funded on a so-called discontinuity basis.⁹ Indeed, the funding rate is unity if we measure liabilities in terms of the market value of promised cash flows V_t .

An alternative approach would be that the price $P_{a,t}$ of a newly bought annuities of one euro would be set at the value of an annuity at a funding rate of one

$$P_{a,t}^i = \frac{L_{a,t}^i}{B_{a,t}^i} \quad (24)$$

even though the newly acquired pension entitlements share in current funding gaps F_t^h . This approach (24) would imply intergenerational risk sharing between existing participants and new contributors. In that case, financial assets of the fund A_t would not always be equal to the aggregate value of the pension entitlements of current participants V_t so that (23) does not necessarily holds. Indeed, ex-post transfers between current participants and new contributors¹⁰ would be reflected in buffers $A_t - V_t \neq 0$, which are not

⁹This means that the participants receive their promised benefits even if the pension fund is winded up. In that case, the financial assets of the fund can be transferred to an insurer. Hence, the participants are not exposed to credit risk of the pension fund.

¹⁰Contributors may be able to demand for compensating wage differentials because of

assigned to current participants. Pension promises are not necessarily fully funded by financial assets in this case because pension funds may rely on the inflow of new contributions to finance existing promises. In particular, the purchase of new pension rights raise (decrease) the value of the rights of current participants in case of underfunding, i.e. $F_t^h < 0$ (overfunding, i.e. $F_t^h > 0$) because contributions (24) are larger (smaller) than the actuarially neutral price (22). Contributors who accumulate new pension rights in effect act as risk sponsors for those who have already accumulated pension entitlements in the fund. This paper abstracts from risk trading between current participants and future generations through the transfer of collective buffers. It assumes that (23) holds so that pension promises are fully funded on a discontinuity basis.

5 The investment strategy

This section shows how the current and future investment strategy of the fund as a whole can be determined endogenously from the desired risk profiles of the individual variable annuities parameterized by the long-run risk exposure w and horizon differentiation q_j . In this way, we extend the principle of liability-driven investment to stochastic liabilities. Indeed, with a clearly specified risk profile, the pension contract is complete also in terms of investment policy.

Replication of the risk profiles of the pension entitlements of participants requires a fraction $w\hat{\omega}_t$ of the assets of the pension fund to be invested in stocks at the current time t :

$$w\hat{\omega}_t A_t = w \sum_{h=1}^{a_{\max} - a_s} q_h V_t^h, \quad (25)$$

elastic labor supply to the firm or sector concerned. they . In that case, risk sharing is between shareholders of the firm and present participants.

where the right-hand-side represents the desired exposures of each horizon to current stock market risk. Substitution of (23) into (25) to eliminate A_t yields

$$\hat{\omega}_t = \sum_{h=1}^{a_{\max}-a_s} \alpha_t^h q_h, \quad (26)$$

where α_t^h stands for the value share of horizon h in all pension entitlements:

$$\alpha_t^h \equiv \frac{V_t^h}{V_t}.$$

A popular way to compute pension liabilities is to employ the expected returns from the actual investment portfolio as the discount rate

$$V_t^e = \sum_{h=1}^{a_{\max}-a_s} \frac{B_t^h}{(1+r-\pi+\hat{\omega}_t w \lambda)^h} (1+F_t^h), \quad (27)$$

Linearization of

$$V_t^e(x_t^1, \dots, x_t^{a_{\max}-a_s}) = \sum_{h=1}^{a_{\max}-a_s} \frac{B_t^h}{(1+r-\pi+\hat{\omega}_t w \lambda)^h} (1+F_t^h)$$

around $x_t^h = Q_h$ ($h = 1, \dots, a_{\max} - a_s$) yields

$$\frac{V_t^e - V_t}{V_t} \approx D_t(\theta_t - \hat{\omega}_t)w\lambda, \quad (28)$$

where the duration D_t is defined as $D_t \equiv \sum_{h=1}^{a_{\max}-a_s} \alpha_t^h h$, while $\theta_t \equiv \frac{\sum_{h=1}^{a_{\max}-a_s} \alpha_t^h h Q_h}{D_t} = \sum_{h=1}^{a_{\max}-a_s} \beta_t^h Q_h$ with $\beta_t^h = \frac{\alpha_t^h h}{\sum_{k=1}^{a_{\max}-a_s} \alpha_t^k k}$.

For regular variable annuities, the traditional method of using expected returns on the current returns yields the correct result (since in (28) $\theta_t = \hat{\omega}_t = 1$ if $q_h = Q_h = 1$ for all horizons $h \geq 1$ (see (26) for $\hat{\omega}_t = 1$ and $\theta_t = \beta_t^h Q_h$ for $\theta_t = 1$). With horizon differentiation, in contrast, the traditional method of using current expected returns tends to understate actual liabilities since $\theta_t < \hat{\omega}_t$.¹¹ Hence, the overall discount curve for valuing variable annuities is

¹¹This is always the case if liabilities are concentrated around a certain horizon. Horizon differentiation (i.e. $q_h > q_1$) implies $q_h > Q_h$ and thus $\hat{\omega}_t > \theta_t$ if $\beta_t^h \approx \alpha_t^h$. If liabilities are dispersed over various horizons and $\beta_t^h > \alpha_t^h$ for long horizons h , we may theoretically have $\hat{\omega}_t < \theta_t$ because longer horizons with larger Q_h and q_h receive a larger weight in the calculation of θ_t than in the calculation of $\hat{\omega}_t$.

typically in between the expected return on the actual investment portfolio of the fund as a whole, on the one hand, and the risk-free term structure, on the other hand. Intuitively, current expected returns exceed future returns because with life-cycle investment risk is taken back when people age.

6 Exponential decay

6.1 Specification

Expression (5) implies that for each horizon, we need a separate state variable to summarize shocks in the past. For a specific specification of horizon differentiation, however, we can summarize the entire history in one state variable. Consider the following 'exponential decay' subclass in which a fixed fraction $1 - \rho$ of the remaining gap with the long-run exposure ($1 - q_{h-1}$) is annually absorbed in pension payments (i.e. $q_h - q_{h-1} = (1 - \rho)(1 - q_{h-1})$ ¹²):

$$q_h = 1 - \rho^h. \quad (29)$$

The coefficient ρ in (29) governs horizon differentiation in risk exposure since

$$\frac{q_h}{q_1} = \frac{1 - \rho^h}{1 - \rho}.$$

With $\rho = 0$, horizon differentiation is absent and $\frac{q_h}{q_1} = 1$. In this case, shocks are absorbed immediately so that $E_t[\mathbf{B}_{a,t+h}^i] = B_{a,t}^i$ (see (4) with $q_j = 1$ for $j > 0$). With $\rho \uparrow 1$, in contrast, horizon differentiation is maximal and $\frac{q_h}{q_1} \Rightarrow h$. Specification (29) thus implies that the risk profile is parameterized by the long-term risk exposure w and horizon differentiation $0 \leq \rho < 1$.

Figure 4 illustrates horizon differentiation q_t in (29) as a function of the horizon h for three alternative choices of ρ . Figure 5 illustrates the implied term-structure of risk Q_h .

We can write the horizon-specific adjustment factors in terms of one specific fund-specific state variable, namely the aggregate funding rate $1 + F_t$:

¹²This implies $1 - q_h = \rho(1 - q_{h-1})$ and thus $1 - q_h = \rho^h$ since $q_0 = 0$.

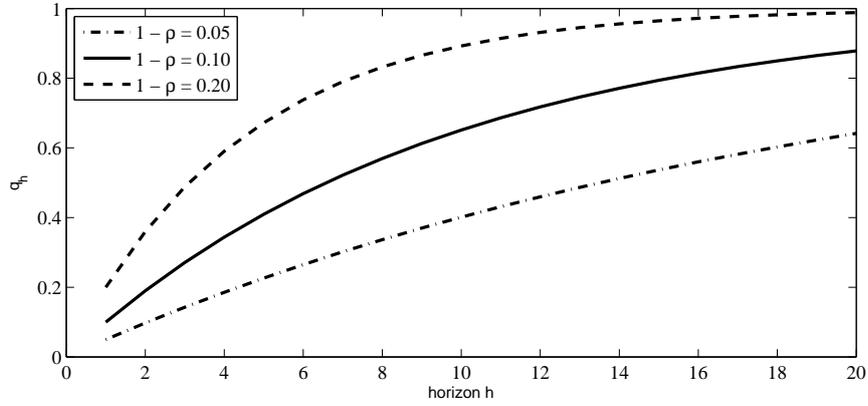


Figure 4: *Illustration of horizon differentiation of risk exposure q_h in the 'exponential decay' subclass defined in (29) for three alternative choices of ρ .*

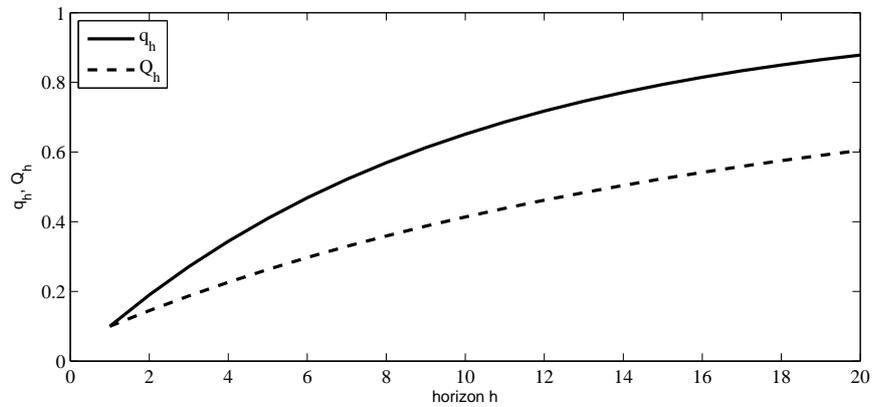


Figure 5: *Illustration of horizon differentiation of risk exposure q_h and the implied term-structure of risk Q_h in the 'exponential decay' subclass defined in (29) for $1 - \rho = 0.1$.*

(see Appendix)

$$F_t^h \approx \frac{q_h}{\omega_t} F_t = \frac{q_h}{\omega_t} \left(\frac{A_t}{L_t} - 1 \right), \quad (30)$$

with

$$\omega_t \equiv \sum_{h=1}^{a_{\max} - a_s} \gamma_t^h q_h.$$

We can write (4) and (19) as (the first equality in (31) follows from the generalization (12) and the second equality in (31) and (32) from substitution of (30) to eliminate F_t^h)

$$\frac{E_t[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} = \pi + F_t^h = \pi + \frac{q_h}{\omega_t} \left(\frac{A_t}{L_t} - 1 \right) = \pi + \frac{q_h}{\omega_t} F_t. \quad (31)$$

$$\frac{V_t^h - L_t^h}{L_t^h} = F_t^h = \frac{q_h}{\omega_t} \left(\frac{A_t}{L_t} - 1 \right) = \frac{q_h}{\omega_t} F_t. \quad (32)$$

The funding rate $1 + F_t = \frac{A_t}{L_t}$ is computed by using the observed actual assets A_t and calculating the liabilities from (21) and the actual pension entitlements (see (17)). Expressions (31) and (32) show that a given funding shortfall $F_t = \left(\frac{A_t}{L_t} - 1 \right)$ leads to a larger adjustments of rights in a greyer fund (with a smaller ω_t). In this way, we guarantee that the aggregate value of claims $V_t \equiv \sum_{h=1}^{a_{\max} - a_s} V_t^h$ matches the assets A_t so that (23) holds.

(32) show how the assets A_t are distributed across the various horizons. It in fact translates pension entitlements $B_{a,t}^i$ of an individual with birth year a into the value of the implicit assets held by that individual:

$$\begin{aligned} V_{a,t}^i &= B_{a,t}^i \sum_{h=\max(1, a_r + a - t)}^{a_{\max} + a - t} \frac{(1 + F_t^h)}{(1 + r - \pi + w\lambda Q_h)^h} \\ &= B_{a,t}^i \sum_{h=\max(1, a_r + a - t)}^{a_{\max} + a - t} \frac{(1 + \frac{q_h}{\omega_t} F_t)}{(1 + r - \pi + w\lambda Q_h)^h}. \end{aligned}$$

Defined 'benefits' $B_{a,t}^i$ (an individual's entitlements to a variable annuity) are translated into the value of an individual retirement account.

At the beginning of each period $t + 1$, we have the following adjustment in pension entitlements (see Appendix)

$$\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} = \pi + \frac{q_1}{\omega_t} \bar{F}_{t+1}, \quad (33)$$

where

$$\bar{F}_{t+1} = \frac{\bar{A}_{t+1}}{\bar{L}_{t+1}} - 1 = (1 + F_t)(1 + \hat{R}_{t+1}^s - r - \lambda) - 1,$$

and \bar{A}_{t+1} are the aggregate assets in the fund after the return has materialized at the beginning of period $t + 1$ but before pension payments are actually made. Similarly, \bar{L}_{t+1} are the aggregate liabilities at the beginning of period $t + 1$, including the payments that must be made during that period. \hat{R}_{t+1}^s presents the realized return on the aggregate portfolio. (33) ensures that pension rights $B_{a,t}^i$ are adjusted in the wake of the shock ($\hat{R}_{t+1}^s - r - \lambda$) so that the aggregate value of pension rights continues to equal total assets. This expression shows that annuities are not increased in line with the aspired indexation if assets \bar{A}_{t+1} fall short of liabilities \bar{L}_{t+1} .

6.2 Changes in risk profile

An important question is how value is redistributed if discounting (i.e. the desired risk profile w) or the indexation ambition π is changed. We write the value of the liabilities at horizon h from (16) with risk profile (29) (where we have used (30) to eliminate F_t^h)

$$V_t^h \equiv B_t^h \frac{(1 + \frac{q_h}{\omega_t} (\frac{A_t}{L_t} - 1))}{(1 + r - \pi + Q_h w \lambda)^h}. \quad (34)$$

The appendix shows that the value of pension payments at horizon h are changed as follows as a result of changes in the risk profile ($\bar{w} - w$) and indexation ambition ($\bar{\pi} - \pi$) (with constant pension entitlements $B_{a,t}^i$) (here \bar{V}_t^h and V_t^h stand for the values of pension payments at horizon h at, respectively, the new risk profile \bar{w} and indexation ambition $\bar{\pi}$ and the old risk profile w

and indexation ambition π)

$$\frac{\bar{V}_t^h - V_t^h}{V_t^h} \approx \varphi(h)((\bar{\pi} - \pi) - (\bar{w} - w)\lambda). \quad (35)$$

Here $\varphi(h)$ decreases in h and $\sum_{h=1}^{a_{\max}-a_s} \alpha_t^h \varphi(h) = 0$.¹³ More risk (i.e. $\bar{w} - w > 0$) at given pension entitlements $B_{a,t}^i$ thus redistributes value from long horizons (i.e. younger participants) to short horizons (i.e. older participants). Intuitively, a higher discount rate raises the funding rate, thereby increasing the scope to pay out today.

Two alternative ways exist to offset redistribution of value when discounting is changed. These two methods also ensure that the funding rate is not affected. The first method is to reset $B_{a,t}^i$ so that $V_{a,t}^i$ is unchanged:

$$B_{a,t}^{i,\bar{w}} = B_{a,t}^{i,w} \frac{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \frac{(1+F_t^h)}{(1+r-\pi+w\lambda Q_h)^h}}{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \frac{(1+F_t^h)}{(1+r-\pi+\bar{w}\lambda Q_h)^h}} \approx B_{a,t}^{i,w} (1 + (\bar{w} - w)\lambda D_t^a \theta_t^a),$$

where $B_{a,t}^{i,\bar{w}}$ are the pension entitlements with long-term risk \bar{w} . The term D_t^a represents the duration of the annuity for someone of age a

$$D_t^a \equiv \sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \alpha_t^k k,$$

and

$$\theta_t^a \equiv \frac{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \alpha_t^h h Q_h}{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \alpha_t^k k}.$$

A higher discount rate thus must be compensated by higher pension entitlements to keep the market value and thus the funding rate unaffected.

¹³The only exceptions are if all pay outs are at the same horizon (i.e. $\alpha_t^1 = 1$) or if $\rho \uparrow 1$ so that shocks are smoothed over a very long time.

The second method is to compute the relative changes in V_t^h for each horizon and to add these changes $\frac{\Delta V_t^h}{V_t^h}$ to predictable changes at the horizons concerned. The additional annual relative change in pension entitlements $\frac{E_{t+1}[\mathbf{B}_{a,t+h+1}^i] - E_{t+1}[\mathbf{B}_{a,t+h}^i]}{B_{a,t}^i}$ can be approximated by $q_{h+1}(\bar{w} - w)\lambda$.

Both methods involve exchanging two annuities at a fair price. The first method seems more attractive for younger members who are not yet in the pay-out phase because this method guarantees a flat consumption profile (in expectation) if the risk profile changes. The second method avoids discrete changes in current pension entitlements $B_{a,t}^i$ and thus current pay-outs. This may be desirable for those in the pay-out phase if these agents feature habit formation. Whereas the first method reallocates market value across time, the second method leaves the intertemporal allocation of market value constant.

Compared to a more risky investment profile (and a corresponding higher discount rate), a higher indexation ambition $\bar{\pi} - \pi > 0$ exerts the opposite impact on the redistribution of value across horizons at given pension entitlements. If the ambition is to increase annuities over time at a higher rate, more money needs to be reserved and less resources can be paid out today. To prevent these distributional effects, we have to exchange annuities again at fair prices. This can be done by reducing either pension entitlements (i.e. $B_{a,t}^{i,\bar{\pi}} < B_{a,t}^{i,\pi}$) or the predictable increases in payments on the old entitlements $B_{a,t}^{i,\pi}$.

Interestingly enough, (35) shows that a changing a defined-benefit structure (with nominally guaranteed benefits, i.e. $w = \pi = 0$) to a defined-ambition structure with an indexation ambition does not lead to redistribution between time horizons at constant pension entitlements $B_{a,t}^i$ if the long-term risk premium $\bar{w}\lambda$ is equal to the indexation ambition $\bar{\pi}$. This change however leads to a decline in the funding rate in the presence of smoothing (i.e. $q_i < 1$).¹⁴

¹⁴This assumes that the price for buying and selling annuities is based on (22). If the fund instead prices newly bought annuities on the basis of (24), the change in the

7 Extensions

7.1 Additional risk factors

Additional tradable risk factors can be easily included. The pension fund should then form an efficient portfolio that maximizes the Sharpe ratio. In fact, one can reinterpret the single risky asset then as the efficient portfolio.

Interest-rate risk and (expected and unexpected) inflation risk can be included as well. In the valuation equations (13) and (16), the interest rate term r is replaced by the term structure and break-even inflation takes the place of the indexation term π . Interest-rate risks and expected inflation risks give rise to hedging demands as these risks affect also the value of the liabilities (see expression (13)). Indeed, participants can be viewed as holding a combination of a hedge portfolio that hedges the aspired income stream in retirement and a speculative portfolio aimed at capturing the risk premium. The sensitivity of the liabilities to nominal interest rates depends on the correlation between nominal interest rates on the one hand and break-even inflation and the expected returns on risky assets on the other hand. This will determine the degree to which the fund engages in hedging nominal interest-rate risk.

Apart from the sharing of idiosyncratic longevity risk, individual annuities can be matched with tradable assets if all systematic risks are traded on financial markets. However, the risk-sharing contract may include also some non-traded risk factors, such as systematic longevity risk and wage risk (e.g. if π is linked to wage increases). In that case, one can speak of a *non-financial* or *notional* defined contribution scheme in which individuals hold claims on notional assets that are not (yet) traded in financial markets. This complicates valuation since these NDC schemes can not be valued objectively

pension scheme involves distributional effects even though $\bar{w}\lambda = \bar{\pi}$. The same holds true for a system that backloads benefits because it prices newly bought annuities equally irrespective of age. This system currently holds for compulsory sectoral pension funds in the Netherlands.

on the basis of prices in financial markets. This gives rise to a trade-off between complementing financial markets versus objective valuation. On the one hand, NDC schemes allow participants to trade risk factors among themselves that are not yet traded, thereby complementing financial markets and creating value. On the other hand, the prices of these risk factors traded within the mutual insurance vehicle are difficult to determine objectively, which may give rise to additional political risk.

7.2 Individual choice

We can allow individuals set their own risk exposures w_{h-j+1}^i in (2)

$$\mathbf{B}_{a,t_0+h}^i = B_{a,t_0}^i \left(\prod_{j=1}^h \{1 - wq_{h-j+1}\lambda + w_{h-j+1}^i(\mathbf{R}_{t_0+j}^s - r)\} \right),$$

without effecting the market value of the annuity. The expected income stream provided by this annuity is not constant if $w_{h-j+1}^i \neq wq_{h-j+1}$.

Alternatively, we can provide an annuity with the same risk profile but a constant expected income stream:

$$\hat{\mathbf{B}}_{a,t_0+h}^i = \hat{B}_{a,t_0}^i \prod_{j=1}^h (1 + w_{h-j+1}^i(\mathbf{R}_{t_0+j}^s - r - \lambda)).$$

To ensure that the market values of the two income streams are the same we have to impose

$$B_{a,t_0}^i \sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \left\{ \prod_{j=1}^h (1 - wq_{h-j+1}\lambda) \right\} = \hat{B}_{a,t_0}^i \sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \left\{ \prod_{j=1}^h (1 - w_{h-j+1}^i\lambda) \right\},$$

so that the relative price of the chosen annuity in terms of the original annuity is

$$\frac{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \left\{ \prod_{j=1}^h (1 - w_{h-j+1}^i\lambda) \right\}}{\sum_{h=\max(1,a_r+a-t)}^{a_{\max}+a-t} \left\{ \prod_{j=1}^h (1 - wq_{h-j+1}\lambda) \right\}}.$$

Individuals who enter retirement may not like the predictable changes in annuities as a consequence of shocks in the past. Hence, they can be given the option to exchange their pension entitlement for an entitlement tht in expectations remains constant:

$$\frac{B_{a,t}^{i,t} - B_{a,t}^i}{B_{a,t}^i} = \sum_{h=1}^{a_{\max} - a_r} \alpha_{a,t} F_t^h = F_t \frac{\sum_{h=1}^{a_{\max} - a_r} \alpha_{a,h} q_h}{\omega_t},$$

where $B_{a,t}^{i,t}$ are the new pension entitlements and $\alpha_{a,t}$ represents the weights of the various horizons for generation $t - a_r$.

If an individual does not invest according to the investment profile $q_j w$ but according to w_{j+1}^i , we find

$$E_t[\mathbf{B}_{a,t+h}^i] = B_{a,t}^i (1 + F_t^{h,i}) \prod_{j=1}^h \{1 + (w_{h-j+1}^i - q_{h-j+1} w) \lambda\},$$

where

$$F_t^{h,i} \equiv \frac{\prod_{j=0}^{k-1} [1 - w q_{h+j+1} \lambda + w_{h+j+1}^i (R_{t-j}^s - r)]}{\prod_{j=0}^{k-1} [1 - w q_{j+1} \lambda + w_{j+1}^i (R_{t-j}^s - r)]} - 1,$$

If $w_{h-j+1}^i = \bar{w} q_{h-j+1} q$, we find

$$\frac{E_t[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} \approx F_t^{h,i} + Q_h h (\bar{w} - w) \lambda.$$

The first term at the last right-hand side captures the impact of past shocks whereas the second term represents the effects of a future investment policy that differs from the 'norm' profile w .

The 2.5 % quantile is given by

$$\frac{P2.5\%_{a,t}^{i,h} - B_{a,t}^i}{B_{a,t}^i} \approx F_t^h + Q_h h \{ \bar{w} (\lambda - 1.96\sigma) - w \lambda \}.$$

8 Conclusion

This paper has explored risk-sharing schemes that provide variable annuities. The starting point for the pension contract is a desired risk profile of the income stream in retirement. This risk profile then endogenously determines investment policy, adjustments of benefits to shocks, pricing of the variable annuities and the contribution levels. In line with habit formation, unexpected shocks are transmitted only gradually into pension payments. This smoothing of shocks leads to predictable changes in pension payments. The effects of past shocks on future adjustments in annuities can be captured in one state variable (the funding rate or, alternatively, the current adjustment in pensions in payment) if shocks are smoothed out in an exponentially declining manner.

The governance of the pension scheme can be arranged in alternative ways. The plans for defined-ambition schemes in the Netherlands have evolved from collective schemes in which fiduciaries decide on the risk profile and the associated investment policies and risk-sharing contracts while employers and unions determine contribution levels in collective bargaining. More elements of individual choice (regarding for example risk profiles, pay-out policies, contribution levels, provider or risk pool) could potentially be introduced.

Public supervision plays three important roles. First, it should monitor that the investment policy of the fund is consistent with the promised risk profile. Second, it should ensure that the annuities are priced fairly, especially if participation in the funds is compulsory. To illustrate, if funds change the way they discount benefits, the authorities should check whether the exchange of annuities occurs at fair prices. The third contribution of the supervisory authorities involves the communication of the risk profile. They should induce the funds to communicate the expected income streams and the risks involved (e.g. based on a 'bad weather' scenario) on the basis of standardized stochastic models.

References

Bodie, Zvi, Robert C. Merton, and William F. Samuelson (1992), "Labor Supply Flexibility and Portfolio Choice in a Life-Cycle Model," *Journal of Economic Dynamics and Control*, 16, 427-449.

Cochrane, J., 2001, *Asset Pricing*, Princeton University Press.

Holzmann, R., E. Palmer, and D. Robalino, 2011, *Non-financial Defined Contribution Pension Schemes in a Changing Pension World*, World Bank.

Novy-Marx, R. and J. Rauh, 2012, Linking Benefits to Investment Performance in US Public Pension Systems, NBER Working Paper No. 18491.

Appendix

Derivation of (4) and (6)

Define the pension entitlements at time t in terms of the pension entitlements at time t_0 and the realized unexpected shocks between $t_0 = t - k$ and $t_0 + k = t$ (see (2))

$$B_{a,t}^i \equiv B_{a,t_0}^i \prod_{j=0}^{k-1} [1 + wq_{j+1}(R_{t-j}^s - r - \lambda)]. \quad (\text{A1})$$

Substitute (A1) into (3) to eliminate B_{a,t_0}^i :

$$\begin{aligned} \frac{\mathbf{B}_{a,t+h}^i}{B_{a,t}^i} &= \frac{\prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)]}{\prod_{j=0}^{k-1} [1 + wq_{j+1}(R_{t-j}^s - r - \lambda)]} \\ &\quad \times \prod_{j=1}^h [1 + wq_j(\mathbf{R}_{t+h-(j-1)}^s - r - \lambda)]. \end{aligned} \quad (\text{A2})$$

Take the expectation of (A2) and substitute $E_t[\mathbf{R}_{t+j}^s - r - \lambda] = 0$ for $j = 1, \dots, h$ to obtain (use that returns are i.i.d. so that no cross-terms between future shocks appear in the right-hand-side of the equation when taking

expectations)

$$\frac{E_t[\mathbf{B}_{a,t+h}^i]}{B_{a,t}^i} = \frac{\prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)]}{\prod_{j=0}^{k-1} [1 + wq_{j+1}(R_{t-j}^s - r - \lambda)]}. \quad (\text{A3})$$

Rearrange terms to arrive at equation (4) in the text.

Equation (6) follows from (A2) with $h = 1$ where R_{t+1}^s is now known:

$$\begin{aligned} \frac{B_{a,t+1}^i}{B_{a,t}^i} &= \frac{\prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)]}{\prod_{j=0}^{k-1} [1 + wq_{j+1}(R_{t-j}^s - r - \lambda)]} \\ &\quad \times [1 + wq_1(R_{t+1}^s - r - \lambda)], \end{aligned}$$

so that

$$\begin{aligned} F_{t+1}^0 \equiv \frac{B_{a,t+1}^i}{B_{a,t}^i} - 1 &= \frac{\prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)]}{\prod_{j=0}^{k-1} (1 + wq_{j+1}(R_{t-j}^s - r - \lambda))} \\ &\quad \times [1 + wq_1(R_{t+1}^s - r - \lambda)] - 1 \\ &= (1 + F_t^1) \times [1 + wq_1(R_{t+1}^s - r - \lambda)] - 1, \end{aligned}$$

which leads to equation (6) in the text.

Derivation of (7) and (8)

We specify the following linearization of (3):

$$\begin{aligned} \mathbf{B}_{a,t+h}^i &= B_{a,t_0}^i \times \\ &\quad \prod_{j=0}^{k-1} [1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)] \times \prod_{j=1}^h [1 + wq_j(\mathbf{R}_{t+h-(j-1)}^s - r - \lambda)] \\ &\approx B_{a,t_0}^i \times \\ &\quad \left(1 + w \sum_{j=0}^{k-1} q_{h+j+1}(R_{t-j} - r - \lambda) + w \sum_{j=1}^h q_j(\mathbf{R}_{t+h-(j-1)} - r - \lambda) \right) \quad (\text{A4}) \end{aligned}$$

When we interpret the terms $\mathbf{R}_{t+h-(j-1)} - r - \lambda$ as log returns, then these are i.i.d. normally distributed with mean zero and standard deviation $q_j\sigma$, which leads to the approximations in (7) and (8).

Derivation of (10)

Equation (10) is derived as follows (the first line is the risk neutral pricing formula; the second line follows from substitution of (3) to eliminate $\mathbf{B}_{a,t+h}^i$; the third line follows from the no-arbitrage condition $E_t^{\mathbb{Q}}[\mathbf{R}_{t+j}^s] = r$ for $j \geq 1$ and the i.i.d. shocks; the fourth line uses (A1) to eliminate B_{a,t_0}^i and employs also (5); the approximation on the last line utilizes (11))

$$\begin{aligned}
V_{a,t}^{i,h} &= E_t^{\mathbb{Q}} \left[\frac{\mathbf{B}_{a,t+h}^i}{(1+r)^h} \right] \\
&= \frac{B_{a,t_0}^i \left(\prod_{j=0}^{k-1} (1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)) \right)}{(1+r)^h} \\
&\quad \times E_t^{\mathbb{Q}} \left[\prod_{j=1}^h (1 + wq_j(\mathbf{R}_{t+h-j+1}^s - r - \lambda)) \right] \\
&= \frac{B_{a,t_0}^i \left(\prod_{j=0}^{k-1} (1 + wq_{h+j+1}(R_{t-j}^s - r - \lambda)) \right) \prod_{j=1}^h (1 - w\lambda q_j)}{(1+r)^h} \\
&= \frac{B_{a,t}^i (1 + F_t^h) \prod_{j=1}^h (1 - w\lambda q_j)}{(1+r)^h} \\
&\approx \frac{B_{a,t}^i (1 + F_t^h)}{(1+r + w\lambda Q_h)^h}.
\end{aligned}$$

Derivation of (30)

We specify the following linearization of (5):

$$F_t^h \approx \left(\frac{w \sum_{j=0}^{k-1} \{(q_{h+j+1} - q_{j+1})(R_{t-j} - r - \lambda)\}}{1 + w \sum_{j=0}^{k-1} q_{j+1}(R_{t-j} - r - \lambda)} \right) \quad (\text{A5})$$

With (29), we can write (A5) as

$$F_t^h = q_h \hat{F}_t, \quad (\text{A6})$$

where

$$\begin{aligned}\hat{F}_t &\equiv w \left(\frac{\sum_{j=0}^{k-1} \{\rho^{j+1}(R_{t-j} - r - \lambda)\}}{1 + w \sum_{j=0}^{k-1} \{q_{j+1}(R_{t-j} - r - \lambda)\}} \right) \\ &= \frac{w \sum_{j=0}^{k-1} \{\rho^{j+1}(R_{t-j} - r - \lambda)\}}{1 + w \sum_{j=0}^{k-1} \{(1 - \rho^{j+1})(R_{t-j} - r - \lambda)\}}.\end{aligned}\quad (\text{A7})$$

Relative adjustments in pension entitlements $\frac{E_t[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i}$ are governed by a single state variable summarizing historical shocks (substitute (A6) in (4) (with generalization (A5)) to eliminate F_t^h)

$$\frac{E_t[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} - \pi = q_h \hat{F}_t. \quad (\text{A8})$$

At the beginning of period $t + 1$ when the return R_{t+1} has just been realized, we have the following adjustment in pension entitlements

$$\frac{E_{t+1}[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} - \pi = q_h \check{F}_{t+1}, \quad (\text{A9})$$

where

$$\check{F}_{t+1} \equiv \frac{w \sum_{j=0}^k \{\rho^j (R_{t+1-j} - r - \lambda)\}}{1 + w \sum_{j=0}^{k-1} \{(1 - \rho^{j+1})(R_{t-j} - r - \lambda)\}}.$$

\check{F}_{t+1} differs from \hat{F}_t in that it includes also the information that became available at the beginning of the period, namely the realization of R_{t+1} .

At the beginning of each period $t + 1$, we have the following adjustment in pension entitlements and thus pensions in payment (use (A9) since $E_{t+1}[B_{a,t+1}^i] = B_{a,t+1}^i$)

$$\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} - \pi = q_1 \check{F}_{t+1} = (1 - \rho) \check{F}_{t+1}, \quad (\text{A10})$$

With a single state variable governing the adjustment, we can write the predictable changes in future benefits in terms of the present adjustment of pensions in payment only (combine (A9) and (A10) to eliminate \check{F}_{t+1})

$$\frac{E_{t+1}[B_{a,t+h}^i] - B_{a,t}^i}{B_{a,t}^i} - \pi = \frac{q_h}{q_1} \left(\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} - \pi \right), \quad (\text{A11})$$

so that the predictable change of pension entitlements in period $t + h + 1$ is (the first equality follows from (A11) and the second from (29))

$$\begin{aligned}
& \frac{E_{t+1}[B_{a,t+h+1}^i] - E_{t+1}[B_{a,t+h}^i]}{B_{a,t}^i} - \pi \\
&= \frac{(q_{h+1} - q_h)}{q_1} \left(\left(\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} \right) - \pi \right) \\
&= (1 - q_h) \left(\left(\frac{B_{a,t+1}^i - B_{a,t}^i}{B_{a,t}^i} \right) - \pi \right).
\end{aligned} \tag{A12}$$

With the exponential decay formulation of horizon differentiation (29), we find (substitute (A6) in (20)) the following relationship between aggregate realized risk \hat{F}_t and fund-specific exposure F_t

$$F_t = \omega_t \hat{F}_t, \tag{A13}$$

where

$$\omega_t \equiv \sum_{h=1}^{a_{\max} - a_s} \gamma_t^h q_h.$$

Equation (A13) represents the relationship between the economy-wide state variable \hat{F}_t , which summarizes realized economy-wide risk in the past, and the fund-specific funding rate, which depends on the fund-specific age composition determining the exposure of the fund to this aggregate risk. The older the participants in the fund, the less the fund is exposed to past macroeconomic shocks if horizon differentiation implies smoothing of shocks. With horizon differentiation, q_i rises with i so that larger weights γ_t^h of the shorter horizons (with the smaller risk exposures q_i 's) reduces the exposure ω_t of the fund to macro shocks \hat{F}_t .

Equation (30) is obtained by writing the horizon-specific adjustment factors in terms of one specific fund-specific state variable, namely the aggregate funding rate $1 + F_t$ (by substituting (A13) into (A6) to eliminate \hat{F}_t)

$$F_t^h = \frac{q_h}{\omega_t} F_t = \frac{q_h}{\omega_t} \left(\frac{A_t}{L_t} - 1 \right), \tag{A14}$$

where the second equality follows from substituting (20) to eliminate F_t and subsequently using (23) to eliminate V_t from the resulting expression).

Derivation of (35)

We linearize (34) around $\pi, w\lambda$:

$$\bar{V}_t^h = B_t^h \frac{(1 + \frac{q_h}{\omega_t}(\frac{A_t}{L_t} - 1))}{(1 + r - \pi + Q_h w \lambda)^h} \times \left(1 + \left\{ \frac{q_h}{\bar{\omega}_t} D_t - h \right\} (\bar{\pi} - \pi) - \left\{ \frac{q_h}{\bar{\omega}_t} D_t \theta_t - Q_h h \right\} (\bar{w} - w) \lambda \right). \quad (\text{A15})$$

where the duration D_t is defined as

$$D_t \equiv \sum_{h=1}^{a_{\max} - a_s} \alpha_t^h h, \quad (\text{A16})$$

and

$$\theta_t \equiv \frac{\sum_{h=1}^{a_{\max} - a_s} \alpha_t^h h Q_h}{\sum_{k=1}^{a_{\max} - a_s} \alpha_t^k k} = \sum_{h=1}^{a_{\max} - a_s} \beta_t^h Q_h, \quad (\text{A17})$$

and

$$\beta_t^h = \frac{\alpha_t^h h}{\sum_{k=1}^{a_{\max} - a_s} \alpha_t^k k} \equiv \frac{\alpha_t^h h}{D_t}.$$

(29) implies that

$$Q_i \equiv \frac{1}{i} \sum_{j=1}^i q_j = 1 - \frac{\rho}{1 - \rho} \frac{(1 - \rho^i)}{i} = 1 - \frac{\rho}{1 - \rho} \frac{q_i}{i}. \quad (\text{A18})$$

If we substitute $jQ_j = j - \frac{\rho}{1 - \rho} q_j$ (from (A18)) into (A17) to eliminate jQ_j , we find (where we have used (26) and (A16))

$$\theta_t = 1 - \frac{\rho}{1 - \rho} \frac{\hat{\omega}_t}{D_t}. \quad (\text{A19})$$

Substitution of (A18) for $i = h$ and of (A19) into (A15) to eliminate hQ_h and θ_t , respectively, yields

$$\bar{V}_t^h \approx B_t^h \frac{(1 + \frac{q_h}{\omega_t}(\frac{A_t}{L_t} - 1))}{(1 + r - \pi + Q_h w \lambda)^h} (1 + \varphi(h)(\pi - w\lambda)), \quad (\text{A20})$$

where

$$\varphi(h) \equiv \frac{q_h}{\hat{\omega}_t} D_t - h.$$

Rewriting (A20) (use (34) to eliminate B_t^h), we derive (35) in the text.