Financial Entanglement: A Theory of Incomplete Integration, Leverage, Crashes, and Contagion

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Abstract

Investors, firms, and intermediaries are located on a circle. Intermediaries facilitate risk sharing by allowing investors at their location to invest in firms at other locations. Access to markets is not frictionless, but involves participation costs that increase with distance. Asset prices, the extent of market integration, the extent of cross-location capital flows, and the resources devoted to the financial industry are jointly determined in equilibrium. Although investors at any location are identical to each other, we find that the financial sector may exhibit diversity, with some financial intermediaries in every location offering high-leverage, high-participation, and high-fee structures and some intermediaries offering unlevered, low-participation, and low-fee structures. The capital attracted by high-leverage strategies is vulnerable to even small changes in market access costs, leading to discontinuous price drops and portfolio-flow reversals. Moreover, an adverse shock to intermediaries at a subset of locations causes contagion, in the sense that it affects prices everywhere.

1 Introduction

The role of financial markets and intermediaries is to allocate capital efficiently and facilitate risk sharing. However, in some situations access to financial markets is subject to frictions, and therefore market integration and the allocation of capital are less than perfect.

These statements are supported by a body of empirical literature (described in detail in the next section) showing that a) capital tends to stay "close" to its origin, b) reward for risk reflects — at least partially — "local" factors that one would expect to be diversifiable, and c) reductions in capital flows and the extent of market integration tend to be associated with sudden and substantial drops in prices and increases in risk premia ("financial crises").

We propose a tractable theoretical framework to capture and investigate the implications of these empirical observations. The model features a continuum of investors and financial markets located on a circle. Investors are endowed with shares of a risky firm domiciled at their location and traded at the same location. The dividends of this security are closely correlated with dividends of securities located in nearby locations and less correlated with securities in distant locations. Even though securities in more distant locations offer greater benefits from the perspective of diversification and risk allocation, participation in such markets involves costs that grow with the distance from the current location. Such costs are reflective of the fact that informational asymmetries, agency costs, etc., are likely to grow as investors participate in progressively more unfamiliar markets; indeed, inside our model distance on the circle should be viewed as a broad measure encompassing the magnitude of these frictions rather than a narrow measure of geographical distance. Alongside the risky markets, there exists a zero-net-supply bond market. Access to that market is assumed costless for everyone, but borrowing must be collateralized with securities.

A competitive intermediation sector in each location offers local investors access to markets in other locations. In an attempt to attract investors, intermediaries offer combinations of portfolios and fee structures that maximize the expected utility of investors — after deducting the fees necessary to cover the operating costs of intermediaries, namely the participation costs in distant financial markets.

Because access to financial markets is subject to frictions, the market equilibrium features limited integration. Investors from nearby locations, who face similar cost structures, choose to participate on arcs of the circle that feature a high degree of overlap. As a result, investors share risks predominantly with investors located near to them, leading to endogenous market segmentation and lower risk sharing than achieved in a frictionless world.

An important feature of this setup is that, even though there may be many assets that share no common investor, neighboring investors overlap on some markets — namely, those located close to both of them. This overlap creates a chain of linkages across markets so that market conditions in one location influence prices in any other location, albeit indirectly. We refer to this interdependence as "financial entanglement".

In such a framework, we address the following set of questions. a) How are asset prices determined in light of limited market integration? b) What structure does the financial industry take, i.e., what combinations of investment choices and cost structures are offered to investors? c) What are the effects of positive or negative shocks to the participation technology? and d) How are prices affected across all markets if a subset of locations experience an adverse shock to their financial sector?

We summarize the answer to each question in turn.

a) Limited market integration implies that investors are over-exposed to the risks of locations in their vicinity. Since these are the investors who primarily invest in securities in that same neighborhood, risk premia are higher than they would be in a frictionless world. An important aspect of the analysis is that the magnitudes of risk premia and of portfolio flows are tightly linked and reflect the extent of market integration.

b) A surprising implication of the analysis is that although investors in any given location are identical, the resulting financial industry may exhibit diversity. Specifically, we show that a symmetric equilibrium, i.e., an equilibrium where financial intermediaries offer the same strategies to all investors, may fail to exist. Instead, there are financial firms offering strategies involving high leverage, high Sharpe ratios, and high fees, along with firms offering unlevered, low-Sharpe ratio, and low-fee strategies. Investors are indifferent between the two types of strategies; however, the amount of capital directed to each type of investment strategy is determinate and dictated by market clearing.

The intuition for this finding is that leverage decisions and participation decisions are complements. For a given value of risk premia, choosing to participate in more markets implies that a portfolio can attain a higher Sharpe ratio, which induces intermediaries to leverage such a portfolio. Since leverage increases the overall variance of the portfolio, it increases the marginal benefit of further market participation and diversification.

This complementarity between participation and leverage decisions introduces multiple local extrema in the problem of determining the optimal participation arc. As we show in the text, in such situations a symmetric equilibrium may fail to exist. Instead, prices adjust so as to leave investors indifferent between unlevered, low-participation, low-Sharpe ratio strategies with low fees versus levered, high-participation, high-Sharpe ratio strategies with high fees. This feature of the model may help explain the co-existence of financial firms such as mutual funds and hedge funds in reality.

c) An interesting situation emerges when the financial industry is diverse and access to the intermediation sector (or financial markets) becomes more costly. In such a situation prices may drop and flows of capital may cease in a discontinuous manner.

The intuition is as follows. An increase in the cost of accessing distant markets makes investors particularly reluctant to continue allocating funds to the strategies that rely on participation in costly and distant markets. The resulting outflows from these levered strategies reduce aggregate market integration and push down the prices of risky securities. The drop in prices has two effects. On one hand, it raises the Sharpe ratio and helps restore the attractiveness of high-leverage, increased-participation strategies. On the other, it reduces the possibility of leverage and increased participation because the collateral value of risky securities declines. When the second effect becomes sufficiently strong, further drops in prices can no longer help in attracting capital for high-leverage, increased-participation strategies. In these situations, even a marginal increase in participation costs eliminates the pre-existing asymmetric equilibrium and triggers a transition to a new type of equilibrium (in the example we discuss in the text it causes a transition from an asymmetric to a symmetric one). The consequence of this transition is that prices of risky securities, capital flows across locations, and the amount of leverage in the economy drop discontinuously, despite the continuous and smooth dependence of investors' objective functions and feasible choices on participation costs.

d) Somewhat surprisingly, the fact that markets are only partially integrated implies stronger correlations between the prices of risky securities in different locations than in fully integrated markets. We present an example where the intermediation sector in a subset of locations — we refer to them as the "affected" locations — ceases to function. We show that such an event pushes prices in almost all locations downward. This is true even for locations that are *not* connected with the affected locations through asset trade and also have negative dividend correlation with them.

The intuition for this finding is as follows. When intermediation breaks down somewhere on the circle, the demand for risky assets becomes lower in the locations these investors used to participate in.To bring the market back to equilibrium, prices in these locations drop so as to attract demand from neighboring locations, where the intermediation sector still operates. This results in a portfolio reallocation in these neighboring locations away from other, farther locations, resulting in weaker demand in these farther locations, necessitating price drops in these locations as well, so as to attract demand from locations neighboring the neighboring locations, etc..

The extent of price drops required to compensate investors for tilting their portfolios to absorb local risks depends on the extent of their overall participation in risky markets. If the extent of their participation is small, so that their portfolio is heavily exposed to risks in their vicinity, then even the smallest tilt towards a nearby location requires a high compensation. By contrast, if investors' portfolios are invested across a broad range of locations, then they are more willing to absorb risks in their vicinity. Hence, somewhat surprisingly, the smaller the degree of integration between risky markets, the higher the interdependence of prices across locations.

This article is related to several strands of theoretical and empirical literature. We discuss connections to the theoretical literature here and postpone a discussion of related empirical literature for the next section. On the technical side, our circle model and especially the assumption on distance-dependent participation costs, is motivated by Salop (1979), who introduced such a model in the industrial-organization literature.[1] We advance the circle model by proposing a tractable dividend specification that helps capture the notion that nearby locations feature stronger dividend correlation. Recent studies featuring a circle include Allen and Gale (2000) on financial contagion and Caballero and Simsek (2012) on financial networks and crises. Our modeling approach is also related to models of overlapping generations. Just as in these models, every subset of investors overlaps with some other subset on some markets, but no market participant participates in all markets. In many ways, the role played by locations in our model is reminiscent of the role played by time in overlapping generations models (except that our locations are aligned on a circle). Finally, our construction of an asymmetric equilibrium is based on Aumann (1966), whose analysis covers the case of non-concave investor-optimization problems in markets with a continuum of agents.

Price crashes in our model are driven by the interaction of a collateral constraint and a non-concavity in investors' optimization problems. There exists a vast literature analyzing the interaction between declining prices and tightening collateral constraints (we do not attempt to summarize this literature and simply refer to Kiyotaki and Moore (1997) for a seminal contribution). A novel aspect of our analysis is that leverage is not caused by differences in preferences or endowments. Instead, leverage and the capital allocated to high-leverage, high-participation strategies arise in response to a non-concave participation problem solved by investors. We show that, following an increase in participation cost, a collateral constraint may prevent falling prices from restoring an equilibrium in which some

[1]In a financial-markets context, the idea that investing in more distant locations involves frictions is also present in Gehrig (1993), Kang and Stulz (1997), and Brennan and Cao (1997).

investors take high leverage — the farther prices fall, the more capital must leave such strategies. In this case, the result is an abrupt change in the nature of the equilibrium accompanied by a downward jump in prices, abrupt reversals in capital flows, and deleveraging. We wish to underscore that the non-concavity we identify in this paper, which is due to the complementarity between participation and leverage, is crucial for the abrupt change in the type of equilibrium.

Modelling price crashes through changes in the type of equilibrium is common in the literature, typically featuring a combination of backward-bending demand curves and multiple equilibria (see, e.g., Gennotte and Leland (1990), Barlevy and Veronesi (2003), Yuan (2005), Basak et al. (2008), and Brunnermeier and Pedersen (2009)). Our economic mechanism is different from that literature. In particular, our model does not feature multiple equilibria, noise traders, or sunspots.[2] Instead, the price crash is obtained when prices can no longer support an asymmetric equilibrium (as in Aumann (1966)), by adjusting so as to attract investors to high-leverage, high-participation strategies. As we explain in greater detail in the next section, our channel is consistent with the typical flight-to-home effect and de-leveraging effect observed during financial crises (e.g., Giannetti and Laeven (2012) and Ahrend and Schwellnus (2012)).

Finally, the domino effects identified in our model are related to the vast literature on contagion. We do not attempt to summarize this literature. Instead we single out a popular mechanism that has been advocated in that literature as an explanation of contagion. (see, e.g., Dumas et al. (2003), Kyle and Xiong (2001), Cochrane et al. (2008), Pavlova and Rigobon (2008)). Specifically, a popular idea is that there exist some agents who price all the assets, and whose marginal utility therefore transmits the shock experienced by one asset to the price of another. In contrast, contagion in our model obtains because of — and, in fact, despite — a fairly strong notion of market segmentation, namely that no agent participate in all markets. Instead, in our paper a shock in one area is transmitted to another area in a sense that is reminiscent of the medical notion of contagion, i.e., by affecting first the immediately neighboring areas, which in turn affect their neighboring areas, etc. A practical implication is that there can be positive interdependence between the prices in locations that have no or negatively correlated dividends, and no common traders in their risky assets.

Overall, we complement the literature by providing a unified, yet tractable, framework that addresses the joint determination of risk premia, the extent of leverage in the financial

[2]Another possibility that does not involve multiple equilibria is evoked by Romer (1993) and Hong and Stein (2003), who argue that small events can reveal substantial information to partially informed investors, leading to large price changes, albeit not a discontinuity, technically speaking.

Table 1: Gravity Equations in International Trade and Finance.

The table reports a survey of the literature on gravity equations in international trade and finance. Portes and Rey (2005) uses bilateral equity flow data among 14 countries from 1989 to 1996. Buch (2005) uses the stock of assets and liabilities of banks from 1983 to 1999. Head and Ries (2008) uses bilateral FDI stocks with 30 OECD reporters and 32 partners in year 2001. Talamo (2007) uses FDI flow data from 1980-2001. Aviat and Coeurdacier (2007) uses bilateral trade data and bank asset holding data in year 2001. Ahrend and Schwellnus (2012) uses IMF's Consolidated Portfolio Investment Survey (CPIS) in 2005-2006, which reports bilateral debt investment for 74 reporting countries and 231 partner countries. Typically the regression performed is

$$\log(X_{i,j}) = \alpha + \beta_1 \log(GDP_i) + \beta_2 \log(GDP_j) + \beta_3 \log(Distance_{i,j}) + controls + \epsilon_{i,j}$$

where $X_{i,j}$ is the equity flow, portfolio holding, FDI flow, FDI stock, trade, or bank asset holdings.

Source	Dependent variable	Distance	t-stat
Portes and Rey (2005)	Equity flows	-0.881	-28.419
Buch (2005)	Bank asset holdings	-0.650	-12.020
Talamo (2007)	FDI flow	-0.643	-9.319
Head and Ries (2008)	FDI stock	-1.250	-17.361
Aviat and Coeurdacier (2007)	Trade	-0.750	-10.000
Aviat and Coeurdacier (2007)	Bank asset holdings	-0.756	-8.043
Ahrend and Schwellnus (2012)	Bond holdings	-0.513	-4.886

system, and the magnitude of capital flows, alongside excess correlation between seemingly unrelated markets and price crashes in response to small shocks to the financial sector. The key channel we highlight is the partial integration of the financial markets.

The paper is organized as follows. Section 2 presents in greater detail the empirical evidence underlying and motivating this paper. Section 3 presents the baseline model. Section 4 presents the solution and the results. Section 7 considers extensions. All proofs are in the appendix.

2 Motivating Facts

As a motivation for the assumptions of the model, we summarize some well-documented facts about the allocation of capital. Table 1 summarizes the evidence on so-called "gravity" equations in international finance. These gravity equations typically specify a linear relation between logs of bilateral flows in various forms of asset trade (equities, bonds, foreign direct investment, etc.) as a linear function of the logs of the sizes of countries and the geographical distance between them.

A striking and robust finding of this literature is that bilateral capital flows and stocks decay substantially with geographical distance. This finding is surprising, since countries that are geographically distant would seem to offer greater diversification benefits; hence one would expect distance to have the opposite sign from the one found in regressions. The literature typically interprets this surprising finding as evidentiary of informational asymmetries that increase with geographical distance — a crude proxy for familiarity and similarity in social, political, legal, cultural, and economic structures.

Supportive of this interpretation is the literature that finds a similar relation between distance and portfolio allocations in domestic portfolio allocations. For instance, Coval and Moskowitz (1999) shows that US mutual fund managers tend to overweight locally headquartered firms. Coval and Moskowitz (2001) shows that mutual fund managers earn higher abnormal returns in nearby investments, suggesting an informational advantage of local investors. Using Finnish data, Grinblatt and Keloharju (2001) shows that investors are more likely to hold, buy, and sell the stocks of Finnish firms that are located close to the investors. Similar evidence is presented in Chan et al. (2005): Using mutual fund data from 26 countries, and using distance as a proxy for familiarity, this paper finds that a version of the gravity equation holds for mutual fund holdings. That is, the bias against foreign stocks is stronger when the foreign country is more distant. In a similar vein, Huberman (2001) documents familiarity-related biases in portfolio holdings.

Further supportive evidence is provided by literature that documents how partial market integration affects the pricing of securities. Bekaert and Harvey (1995) finds that local factors affect the pricing of securities and are not driven out by global factors.

An important additional finding of Bekaert and Harvey (1995) is that the relative importance of global and local factors is time varying, suggesting time-varying integration between markets. Consistent with this evidence, Ahrend and Schwellnus (2012) uses IMF's Consolidated Portfolio Investment Survey (CPIS) and BIS Locational Banking Statistics (ILB) to document a significantly stronger coefficient on distance in cross-border gravity equations during 2008-2009 as compared to previous years. This evidence suggests that capital becomes more concentrated "locally" during times of crisis.

Indeed, crisis periods offer a unique opportunity to visualize the extent of variation in market integration. The left plot of Figure 1 reports the sum of global net purchases of foreign assets by residents (labeled "Gross capital inflows") and the sum of global net purchases of domestic assets by foreigners (labeled "Gross capital outflows"). The figure also reports

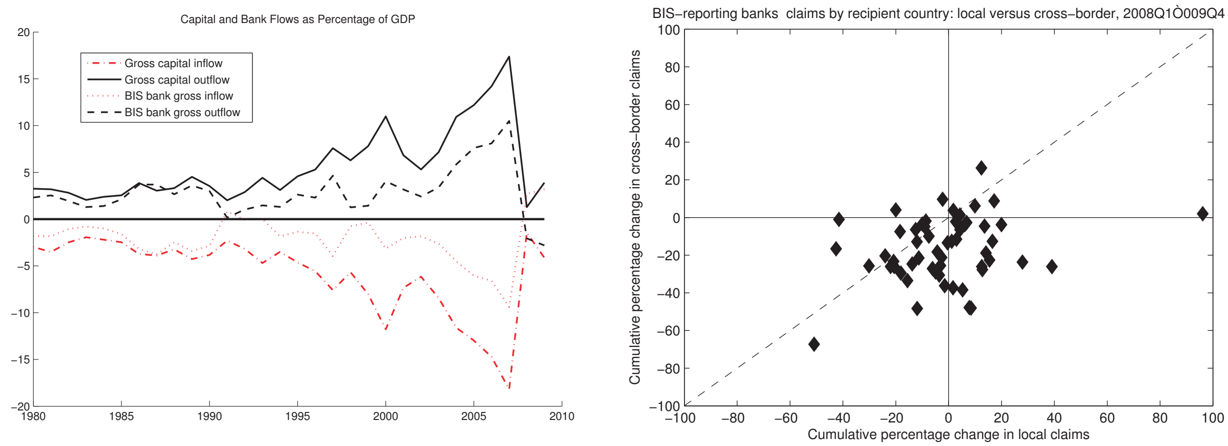


Figure 1: Reversals in global capital flows. Source: Hoggarth et al. (2010). The left plot depicts the sum of global net purchases of foreign assets by residents (labeled "Gross capital inflows") and the sum of global net purchases of domestic assets by foreigners (labeled "Gross capital outflows"). The figure also reports cross-border bank inflows and outflows based on BIS data. Right plot: Cumulative percentage in local claims held by banks against cumulative percentage change in cross-border claims during the 2008Q1 - 2009Q4 period. Source: Hoggarth et al. (2010). based on BIS Data on the 50 largest debtor countries by foreign liabilities. For a detailed description of the sample of reporting countries, see Hoggarth et al. (2010).

cross-border bank inflows and outflows based on BIS data. The picture helps visualize that, in the years preceding the financial crisis of 2008, there was a large increase in gross capital flows. This expansion in capital flows came to a sudden stop in the first quarter of 2008, as the financial crisis took hold.

The right panel of Figure 1 plots the cumulative percentage change in cross-border claims against the respective cumulative change in local claims for various countries. Most points on the graph are below the 45-degree line, suggesting that most banks chose to liquidate foreign holdings disproportionately more than local holdings during the crisis.

Direct evidence of a "flight to home" effect is provided by Giannetti and Laeven (2012), which shows that the home bias of lenders' loan origination increases by approximately 20% if the bank's home country experiences a banking crisis. Giannetti and Laeven (2012) also argues that this flight to home effect is distinct from flight to quality, since borrowers of different quality are equally affected.

Figure 2 provides two further illustrative examples of abrupt reversals of market integration during a crisis by focusing on the recent European debt crisis and the mortgage crisis in the US. The left plot shows that cross-border holdings of European banks in other European

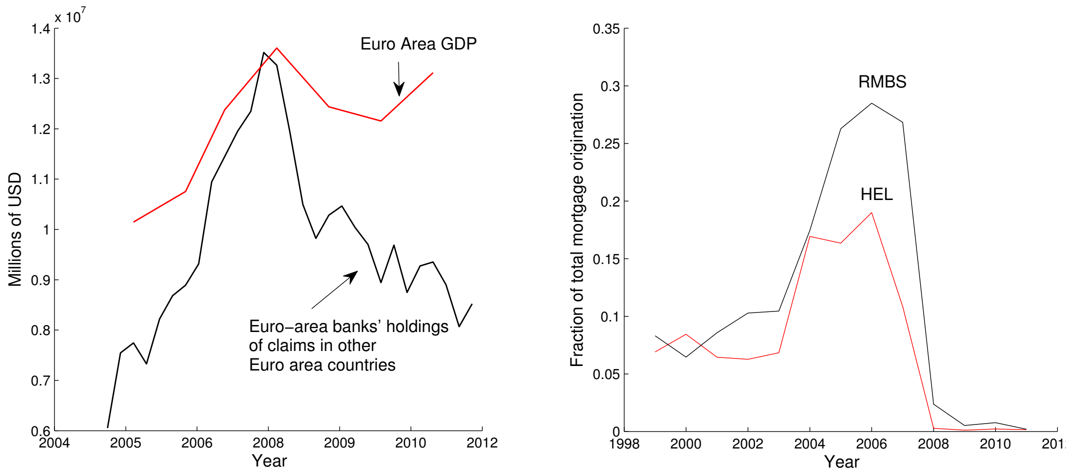


Figure 2: Abrupt reversals in capital allocations. Left plot: European banks' consolidated foreign claims in other European countries - ultimate risk basis. Millions of USD. For comparison we also plot Euro area GDP in millions of USD. Sources: BIS Quarterly Review, 2012 and IMF World Economic Outlook, 2012. Right plot: Issuance of Non-agency residential mortgage backed securities (RMBS) and home equity lines (HEL) as a fraction of new mortgage originations. Sources: Securities Industry and Financial Markets Association (SIFMA) and Mortgage bankers association (MBAA).

countries fell substantially compared to Euro-area GDP. The plot supports the commonly held view that the process of cross-border financial integration within the Euro-area reversed abruptly since the onset of the recent crisis. The right plot documents a similar "sudden stop" in the securitization market in the US. (To avoid issues related to implicit guarantees, we focus on the non-agency market). If one views cross-border financial relationships as a way in which different countries in the European area share local risks, and similarly, if one views the securitization market as allowing regional banks to diversify local real-estate risk, then crisis periods are associated with a collapse in pre-existing mechanisms of risk sharing.

In summary, the empirical evidence supports the following broad conclusions: a) Capital stays "close" to its origin. Despite the potential benefits of diversification from investing in distant locations, the evidence from both domestic and international data is that capital allocations gravitate towards the capital's origin. b) The extent of market integration. More importantly, crises are times when financial integration retreats quite abruptly.

3 Model

3.1 Agents and firms

Time is discrete and there are two dates $t = 0$ and $t = 1$. All trading takes place at time $t = 0$, while at $t = 1$ all payments are made and contracts are settled. Investors are price takers, located at different points on a circle with circumference normalized to one. We index these locations by $i \in [0, 1)$. Investors have exponential utilities and maximize expected utility of time-1 wealth

$$E\left[U\left(W_{1,i}\right)\right] = -E\left[e^{-\gamma W_{1,i}}\right], \tag{1}$$

where $W_{1,i}$ is the time-1 wealth of an investor in location i. The assumptions that investors only care about terminal wealth and have exponential utility are made for tractability and in order to expedite the presentation of the results. In Section 7.1 we allow investors to consume over an infinite horizon.

Besides having identical preferences, investors at any given location are also identical in terms of their endowments and their information sets. Specifically, at time $t = 0$ the investors in location i are equally endowed with the total supply of shares (normalized to one) of a competitive, representative firm, which is domiciled at the same location i. Each firm pays a stochastic dividend equal to D_i in period 1.

We specify the joint distribution of the dividends D_i for $i \in [0, 1]$ so as to obtain several properties. Specifically, we wish that 1) firms be ex-ante symmetric, that is, the marginal distribution of D_i be independent of i; 2) the total dividends paid $\int_0^1 D_i di$ be constant (and normalized to one); 3) firms with indices close to each other (in terms of their shortest distance on the circle) experience a higher dividend correlation than firms farther apart; and 4) dividends at different locations be normally distributed.

To formalize these notions, we let Z_i denote a standard Brownian motion for $i \in [0, 1]$, and we also introduce a Brownian bridge

$$B_i \equiv Z_i - iZ_1 \text{ for } i \in [0, 1].$$

We note that — by construction — the Brownian Bridge satisfies $B_0 = B_1 = 0$. Using these definitions, we let D_i be defined as

$$D_i \equiv 1 + \sigma \left(B_i - \int_0^1 B_j dj \right), \tag{2}$$

where $\sigma > 0$ is a constant controlling the volatility of the dividend process. Since the specification (2) plays a central role in our analysis, we show first that D_i satisfies properties

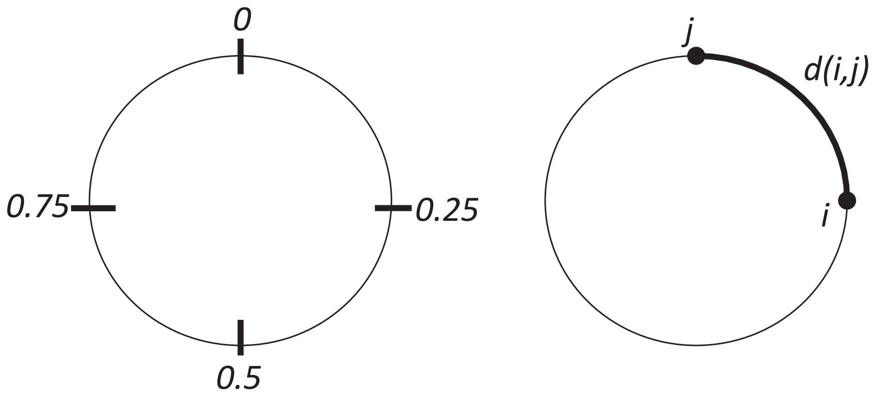


Figure 3: The left plot depicts a circle with circumference 1. The bold arc on the right plot depicts the notion of the shortest distance on the circle $d(i, j)$ between points i and j that we use throughout.

1–4 mentioned above, and then we provide a graphical illustration of these properties to build intuition.

Lemma 1 *D_i satisfies the following properties.*

1. *(Symmetry and univariate normality) The marginal distribution of D_i is the same for all i. Specifically, D_i is normally distributed with mean 1 and variance $\frac{\sigma^2}{12}$.*

2. *(No aggregate risk) $\int_0^1 D_i di = 1$.*

3. *(Continuity on the circle) Let $d(i,j) \equiv \min(|i-j|, 1-|i-j|)$ denote a metric on the interval $[0,1)$. Then D_i is continuous (a.s.) on $[0,1)$ if the interval $[0,1)$ is endowed with the metric d.*

4. *(Joint normality and distance-dependent covariance structure) For any vector of locations $i = (i_1, i_2, \ldots, i_N)$ in $[0,1)$, the dividends $\overrightarrow{D}_i$ are joint normal, with covariances given by*

$$cov\left(D_{i_n}, D_{i_k}\right) = \sigma^2 \left(\frac{1}{12} - \frac{d\left(i_n, i_k\right)\left(1 - d\left(i_n, i_k\right)\right)}{2} \right). \tag{3}$$

It is easiest to undestand the properties of D_i by using a graphical illustration. The left plot of Figure 3 provides an illustration of the interval $[0,1]$ "wrapped" around as a circle with circumference one. The metric $d(i,j)$ can be thought of as the length of the shortest arc connecting i and j. The right plot of Figure 3 provides an illustration. Figure 4 illustrates a path of $Z(i)$ and the associated paths of $B(i)$ and $D(i)$.

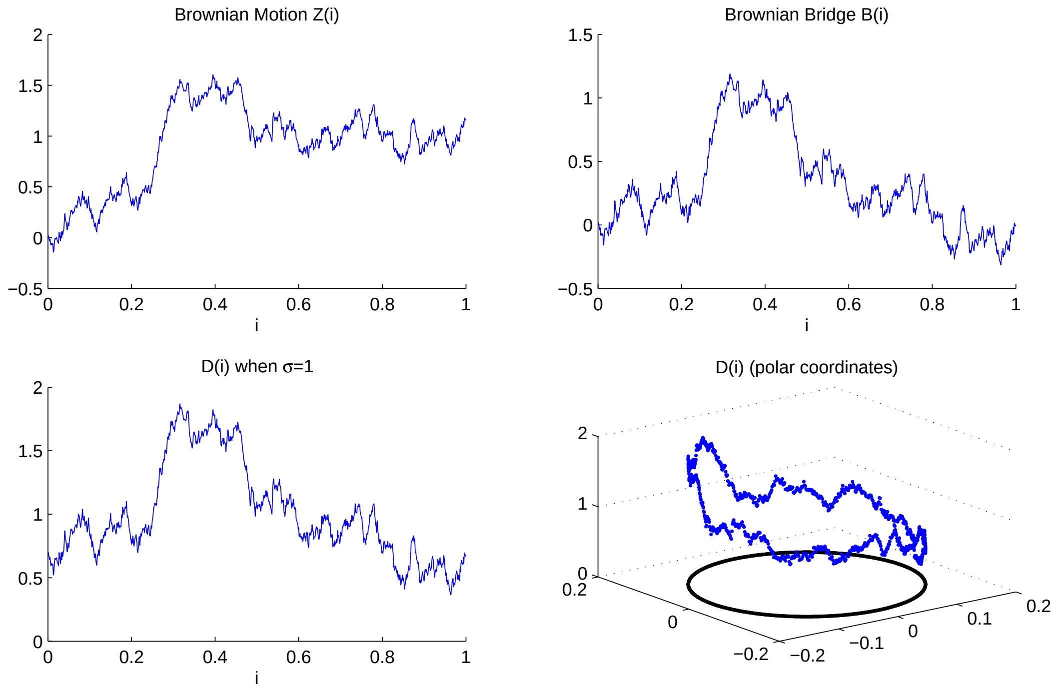


Figure 4: An illustration of the construction of the dividend process D_i. The two plots at the top depict a sample Brownian path Z_i, and the associated path of a Brownian bridge B_i. The two bottom plots depict the associated sample path of D_i, when the indices $i \in [0, 1)$ are aligned on a line and when the same interval is depicted as a circle with circumference one.

A remarkable property of the dividend structure (2) is that the covariance, and therefore correlation, of dividends in any locations i and j depend exclusively on the distance $d(i, j)$ between the two locations, but not the locations themselves. Lemma 1 states the covariances, while the correlations follow immediately as

$$corr\,(D_i, D_j) = 1 - 6d\,(i, j)\,(1 - d\,(i, j))\,. \tag{4}$$

Equation (4) implies that the correlation between D_i and D_j approaches one as the distance $d(i, j)$ approaches zero, and is minimized when $d(i, j) = \frac{1}{2}$, i.e., when the two firms are located diametrically opposite each other on the circle.

3.2 Intermediaries, financial markets, and participation costs

Investors in location i access financial markets in other locations $j \neq i$ through intermediaries domiciled at the location i. These intermediaries receive funds from local investors (obtained from the sale of the local stock the investors are endowed with) and purchase a portfolio of shares of firms domiciled in other and potentially also the local markets along with bonds. Free entry into the intermediation sector implies that intermediaries make zero profits. Moreover, competition between intermediaries in the local market implies that intermediaries create a portfolio of securities so as to maximize the welfare of the local investors.[3]

An important assumption is that participation in financial markets is costly, and the more so the farther away a financial market is from an intermediary's location. This assumption is motivated by the evidence we presented in the previous section, which suggests that investors in one location invest most heavily in nearby locations. The literature has proposed various reasons why participation in more unfamiliar locations is likely to be costlier, with a common reason being the difficulty of overcoming informational frictions[4]. For our results, the assumption of distance-dependent participation costs is meant to encapsulate the frictions associated with leveling the playing field being local and distant investors.

We next propose a mathematical structure for these costs. We assume that the participation decision of intermediaries consists of choosing a subset of all markets $[0, 1)$ in which to invest. To avoid unnecessary complications, we restrict attention to subsets of $[0, 1)$ that can be represented as a finite union of intervals with midpoints $a_{i,n}$ and lengths $\Delta_{i,n}$. If an intermediary in location i chooses to participate in N_i disjoint intervals with midpoints $a_{i,j}$ and total length $\Delta_i \equiv \sum_{n=1}^{N_i} \Delta_{i,n}$, she incurs costs equal to

$$F_i\left(N_i, \{a_{i,1}, \ldots, a_{i,N_i}\}, \Delta_i\right) = \kappa \left[b_{N_i} \sum_{n=1}^{N_i} \times f\left(d(a_{i,n}, i)\right) + g\left(\Delta_i\right) \right] \tag{5}$$

where $\kappa > 0$, $f(x)$ and $g(x)$ are positive, non-decreasing, differentiable, and convex functions for all $x \in (0, 1)$, and b_N is positive and increasing in N. We furthermore assume that f has a discontinuity at zero in the sense that $\lim_{x \to 0} f(x) > 0$, while $f(0) = 0$. Similarly $g(0) = 0$. Finally, we assume that $\lim_{\Delta \to 1} g(\Delta) = \infty$, so that investing in all markets would be prohibitively costly.

We make several remarks on specification (5). First we note that $F_i(1, \{i\}, 0) = 0$, so that participating only in the local market is costless. Second, the fact that f is increasing implies that investing in markets that are farther away (in the sense that the distance $d(a_{i,n}, i)$ from the current location is large) is more costly than participating in markets that are close by. Third, increasing the total mass of markets in which the intermediary participates (Δ_i), while keeping the number (and midpoints) of intervals the same, incurs incremental rather

[3]Otherwise, these local investors would be attracted by other local intermediaries who would charge a sufficiently small $\varepsilon > 0$ in exchange for building a better portfolio for them, an outcome that is inconsistent with the zero-profit condition.

[4]See, e.g., Brennan and Cao (1997).

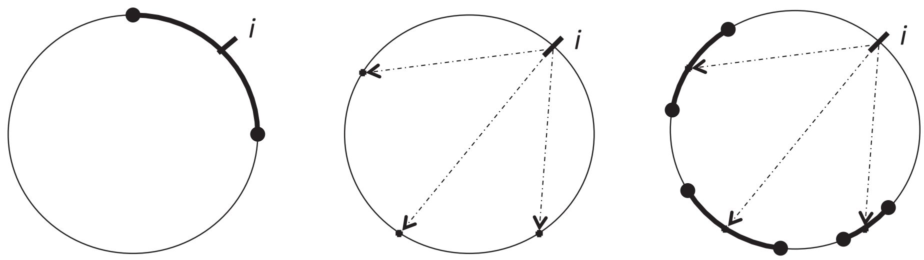


Figure 5: Illustration of participation choices under different assumptions on the participation costs $F_i\left(N_i, \{a_{i,1}; ..; a_{i,N_i}\}, \Delta_i\right)$. The left plot depicts feasible participation choices in the special case $b_N f(x) = \infty$ and $g(y) < \infty$ for any $x, y > 0$ and $n > 1$. The middle plot depicts the respective participation choices in the special case where $g(y) = \infty$ for all $y > 0$ and $b_N f(x)$ is sufficiently small for some $x > 0$ and $n > 1$. The right plot depicts the general case in which both $b_N f(x)$ and $g(y)$ take small enough values for some $x, y > 0$ and $n > 1$.

than fixed costs. This captures the idea that expanding participation to contiguous markets is substantially less costly than participating in a market that is not adjacent to any of the markets where the intermediary has already decided to trade. Fourth, the fact that participation costs depend on the location of the intermediary implies that intermediaries in different locations on the circle face different costs of participating in a given market j, depending on their proximity to that market.

It is helpful to consider a few limiting cases that illustrate the properties of F_i. The first limiting case entails $b_N f(x) = \infty$ and $g(y) < \infty$ for any $x > 0$, $y > 0$, and $N > 1$. In that case, the intermediary finds it optimal to participate only in markets that are contiguous to her local market and her only choice is the length of the interval Δ_i surrounding her local market. This situation is depicted in the left plot of Figure 5. The second limiting case entails $b_N f(x) < \infty$ and $g(y) = \infty$ for any $y > 0$. In that case the intermediary chooses to participate in a finite set of markets around the circle, and the only interesting choice concerns the locations ($a_{i,n}$) and the number of these points (N_i). This situation is depicted in the middle plot of Figure 5. In the general case $b_N f(x) < \infty$ and $g(y) < \infty$, the participation decision involves choosing the number of points N_i, the location of the midpoints $a_{i,n}$, and the length of the intervals $\Delta_{i,n}$ at each location. This situation is depicted in the right plot of Figure 5.

Besides the market for risky shares, there is a zero-net-supply market for riskless bonds that simply pay one unit of wealth at time 1. Participation in the bond market is costless for everyone.

The participation costs act as deadweight costs that are paid out (i.e., reduce consumption) at time 1, but the funds corresponding to the participation costs are deducted from a client's account at time zero and placed in riskless bonds that act as collateral so as to ensure payment of the participation costs at time 1.[5]

Finally, we make the assumption that any borrowing of securities must be collaterlized by other securities. We also allow — but do not require — the possibility of "haircuts". Specifically, if risky securities are used as collateral to borrow bonds, then only a fraction $(1-\chi)$ with $\chi \in [0,1]$ of their value can be used as collateral to borrow riskless securities.

3.3 Maximization problems

To formalize an intermediary's decision problem we let P_i denote the price of a share in a market i, $P_B \equiv \frac{1}{1+r}$ the price of a bond, $dX_j^{(i)}$ the mass of shares in market j bought by an intermediary located in i,[6] and $X_B^{(i)}$ the respective number of bonds. Then, letting

$$W_{0,i} \equiv \int_0^1 P_j dX_j^{(i)} + P_B X_B^{(i)}$$

denote the total financial wealth of an investor in location i, the budget constraint of an investor can be expressed as

$$W_{0,i} = P_i. \tag{6}$$

The assumption that all borrowing of securities must be collateralized, with a haircut χ, is expressed formally as

$$\left[(1-\chi)\int_0^1 P_j dX_j^{(i)}\right] + P_B\left(X_B^{(i)} - F_i\right) \geq 0. \tag{7}$$

Equation (7) stipulates that the maximum amount an investor can borrow (the term inside square brackets) exceeds the one she actually borrows, namely $-P_B(X_B^{(i)} - F_i)$.[7] Combining

[5]If we extend the model to allow for consumption at both times zero and one, then we can adopt the more straightforward assumption that investors need to pay the participation fees directly at time 0 in advance of trading; either assumption delivers the same results.

[6]The function X_j has finite variation. We adopt the natural convention that $X_j^{(i)}$ is continuous from the right and has left limits.

[7]Note that, since the investor must hold F_i in bonds as collateral for the participation fees, lending $X_B^{(i)}$ net requires investing $X_B^{(i)} - F_i$ in bonds.

the definition of $W_{0,i}$ with equation (7) gives

$$\begin{aligned}
W_{0,i} &= \int_0^1 P_j dX_j^{(i)} + P_B X_B^{(i)} \\
&= (1-\chi) \int_0^1 P_j dX_j^{(i)} + P_B \left[X_B^{(i)} - F_i \right] + P_B F_i + \chi \int_0^1 P_j dX_j^{(i)} \\
&\geq P_B F_i + \chi \int_0^1 P_j dX_j^{(i)}. \qquad (8)
\end{aligned}$$

We are now in a position to formulate the intermediary's maximization problem as

$$\max_{w_i^f, G^{(i)}, N_i, \overrightarrow{a}_i, \overrightarrow{\Delta}_i} E\left[U\left(W_{1,i}\right)\right], \qquad (9)$$

subject to (6), (8), and

$$W_{1,i} = W_{0,i} \left(w_i^f (1+r) + \left(1 - w_i^f\right) \int_0^1 R_j dG_j^{(i)} \right) - F_i, \qquad (10)$$

where w_i^f is the fraction of $W_{0,i}$ invested in the risk-free security by an agent in location i, $G_j^{(i)}$ is a bounded-variation function with $\int_0^1 dG_j^{(i)} = 1$, which is constant in locations where the intermediary does not participate (i.e., $dG_j^{(i)}$=0 in these locations), so that $dG_j^{(i)}$ captures the fraction of the risky component of the portfolio $\left(1 - w_i^f\right) W_{0,i}$ invested in the share of stock j by a consumer located in i. Finally, $R_i \equiv \frac{D_i}{P_i}$ is the realized gross return on security i at time 1. We do not restrict G to be continuous, that is, we allow intermediaries to invest mass points of wealth in some locations.

We conclude with a remark on (8). This assumption does not preclude leverage or shorting since bonds can be used as collateral to purchase stocks and vice versa (w_i^f is allowed both to be negative and to exceed one). However, this constraint implies an upper bound on participation costs, namely $P_B F_i \leq W_{0,i} - \chi \int_0^1 P_j dX_j^{(i)} \leq P_i$. Combined with the assumption that participation in all markets would be prohibitively costly ($\lim_{\Delta \to 1} g(\Delta) = \infty$), this constraint implies that investors participate only in a strict subset of all available locations. We note that $\lim_{\Delta \to 1} g(\Delta)$ only needs to be large enough, rather than infinite, to imply that intermediaries do not participate in all markets.[8]

[8] We also note that the constraint (8) and the assumption that $\lim_{\Delta \to 1} g(\Delta)$ is large enough are sufficient but not necessary conditions to ensure that intermediaries don't participate in all locations. Indeed, as we discuss later, if $g'(\Delta)$ becomes arbitrarily large and sufficiently fast as Δ_i increases, then the constraint (8) is not binding, and the investor chooses to participate in a subset of markets.

3.4 Equilibrium

The definition of equilibrium is standard. An equilibrium is a set of prices P_i, a real interest rate r, and participation and portfolio decisions $G^{(i)}$, N_i, and $\{a_{i,1...}a_{i,N_i}, \Delta_{i,1...}\Delta_{i,N_i}\}$ for all $i \in [0,1]$ such that: 1) $G^{(i)}$, N_i, and $\{a_{i,1...}a_{i,N_i}, \Delta_{i,1...}\Delta_{i,N_i}\}$ solve the optimization problem of equation (9), 2) financial markets for all stocks clear: $P_j = \int_{i\in[0,1]} \left(1 - w_i^f\right) W_{0,i} dG_j^{(i)}$, and 3) the bond market clears, i.e., $\int_{i\in[0,1]} W_{0,i} w_i^f di = 0$.

By Walras' law, we need to normalize the price in one market. Since in the baseline model we abstract from consumption at time zero for parsimony, we normalize the price of the bond to be unity (i.e., we choose $r = 0$). We discuss consumption at more dates than time one and an endogenously determined interest rate in Section 7.1.

4 Solution and Its Properties

We solve the model and illustrate its properties in a sequence of steps. First, we discuss a frictionless benchmark, where participation costs are absent. Second, in order to build intuition we discuss in succession the two limiting cases illustrated in the left-most panel and middle plot of Figure 5. We discuss the general case, which allows for both contiguous and non-contiguous participation choices, in a later section.

4.1 A frictionless benchmark

As a benchmark, we consider the case without participation costs: $g(x) = b_N f(x,n) = 0$ for all $x \in [0,1)$ and $n \geq 1$, so that $F_i = 0$. In this case, the solution to the model is trivial. Every intermediary i participates in every market j. The first order condition for portfolio choice is

$$E\left[U'(W_{1,i})\left(R_j - (1+r)\right)\right] = 0. \tag{11}$$

With the above first-order condition in hand, one can verify the validity of the following (symmetric) equilibrium: $w_i^f = 0$, and $G_j^{(i)} = G_j = j$ for all $(i,j) \in [0,1) \times [0,1)$ — i.e., intermediaries in all locations i choose an equally weighted portfolio of every share $j \in [0,1]$ for their clients. Accordingly, $W_{1,i} = \int_0^1 D_j dj = 1$. Since in this equilibrium $W_{1,i} = 1$, the Euler equation (11) implies that

$$E(R_j) = 1 + r. \tag{12}$$

Combining (12) with the definition $R_j = \frac{D_j}{P_j}$ implies

$$P_j = \frac{E\left(D_j\right)}{1+r} = \frac{1}{1+r} = 1, \tag{13}$$

where the last equation follows from the normalization $r = 0$. The equilibrium in the frictionless case is intuitive. Since there are no participation costs, intermediaries maximizing agents' welfare by holding an equally weighted portfolio across all locations. Since — by assumption — there is no risk in the aggregate, the risk of any individual security is not priced. This is reflected in the fact that dividends are discounted at the risk-free rate in equation (13). Indeed, $1 + r$ is the common discount factor for all asset prices in this economy.

4.2 Symmetric equilibria with participation costs

To facilitate the presentation of some of the key results, we focus on the special case depicted on the left panel of Figure 5, i.e., a situation in which the cost of participation in markets that are not contiguous to the ones in which the intermediaries already have an established presence is too large ($b_N f(x) = \infty$ for $x > 0$).

We start by introducing a convention to simplify notation.

Convention 1 *For any real number x, let $\lfloor x \rfloor$ denote the floor of x, i.e., the largest integer weakly smaller than x. We henceforth use the term "location x" (on the circle) to refer to the unique point in $[0, 1)$ given by $x \mod 1 \equiv x - \lfloor x \rfloor$.*

With this convention we can map any real number to a unique location on the circle with circumference one. For example, this convention implies that the real numbers -0.8, 0.2, and 1.2 correspond to the same location on the circle with circumference one, namely 0.2. An implication of this convention is that any function h defined on the circle extends to a function $\hat{h}$ on the real line that is periodic with period one, i.e., $\hat{h}(x) = \hat{h}(x+1) = h(x \mod 1)$. From now on, we adopt the convention that when we refer to a function h on the circle, we also refer to its extension to the real line.

We next introduce two definitions.

Definition 1 *The standardized portfolio associated with $G_j^{(i)}$ is the function L_j defined by $L_j = G_{i+j}^{(i)}$.*

The notion of a standardized portfolio allows us to compare portfolios of investors at different locations on the circle. For example, if all investors choose portfolios with weights that only depend on the distance d between their domicile and the location of investment, then these investors hold the same standardized portfolio.

Definition 2 *A symmetric equilibrium is an equilibrium in which all agents choose the same participation interval Δ, the same leverage w^f, and the same standardized portfolio.*

Due to the symmetry of the problem, it is natural to start by attempting to construct a symmetric equilibrium.

Proposition 1 *For any $\Delta \in (0,1)$ define*

$$L_j^* \equiv \begin{cases} 0 & \text{if } j \in [-\frac{1}{2}, -\frac{\Delta}{2}) \\ j + \frac{1}{2} & \text{if } j \in [-\frac{\Delta}{2}, \frac{\Delta}{2}) \\ 1 & \text{if } j \in [\frac{\Delta}{2}, \frac{1}{2}) \end{cases} \tag{14}$$

and

$$\omega(\Delta) \equiv Var\left(\int_0^1 D_j \, dL_j^*\right) = \frac{\sigma^2}{12}(1-\Delta)^3 . \tag{15}$$

Finally, let Δ^* denote the (unique) solution to the equation

$$\kappa g'(\Delta^*) = -\frac{\gamma}{2}\omega'(\Delta^*) \tag{16}$$

and also consider the set of prices

$$P_i = P = 1 - \gamma\omega(\Delta^*) . \tag{17}$$

Then, assuming that $(1-\gamma\omega(\Delta^*))(1-\chi) > \kappa g(\Delta^*)$ and also that a symmetric equilibrium exists, the choices $\Delta^{(i)} = \Delta^*$, $w_i^f = 0$, and $dG_j^{(i)} = dL_{j-i}^*$ constitue an equilibrium supported by the prices $P_i = P$.

Proposition 1 gives simple, explicit expressions for both the optimal portfolios and participation intervals. To understand how these quantities are derived, we take an individual agent's wealth $W_{1,i}$ from (10) and we assume that prices for risky assets are the same in all locations. Then, using $W_{0,i} = P_i = P$, $r = 0$, $R_j = \frac{D_j}{P}$, and $F_i = \kappa g(\Delta)$, we express $W_{1,i}$ as

$$\begin{aligned} W_{1,i} &= P\left(w_i^f + \left(1 - w_i^f\right)\int_0^1 R_j dG_j^{(i)}\right) - \kappa g(\Delta) \\ &= \left(P w_i^f + \left(1 - w_i^f\right)\int_0^1 D_j dG_j^{(i)}\right) - \kappa g(\Delta) . \end{aligned}$$

Because of exponential utilities and normally distributed returns, maximizing $EU(W_{1,i})$ is equivalent to solving

$$\max_{dG_j^{(i)},\Delta,w_i^f} Pw_i^f + \left(1-w_i^f\right)\int_0^1 ED_j dG_j^{(i)} - \frac{\gamma}{2}\left(1-w_i^f\right)^2 Var\left(\int_0^1 D_j dG_j^{(i)}\right) - \kappa g(\Delta). \quad (18)$$

Noting that $ED_j = 1$, inspection of equation (18) shows that (for any w^f and Δ) the optimal portfolio is the one that minimizes the variance of dividends in the participation interval Δ. Since the covariance matrix of dividends is location invariant, the standardized variance-minimzing portfolio is the same at all locations. Solving for this variance-minimizing portfolio L_j is an infinite-dimensional optimization problem. However, because of the symmetry of the setup we are able to solve it explicitly, and equation (14) provides the solution. To understand the structure of L_j^*, we note that it corresponds to the distribution that minimizes the sum of the vertical distances to the uniform distribution on $\left[-\frac{1}{2},\frac{1}{2}\right)$, subject to the constraints that $L_j = 0$ for $j \in [-\frac{1}{2}, -\frac{\Delta}{2})$ and $L_j = 1$ if $j \in [\frac{\Delta}{2}, \frac{1}{2})$. The resulting optimized variance is given by $\omega(\Delta)$, where $\omega(\Delta)$ is defined in (15).

Accordingly, the agent's problem can be written more compactly as

$$V = \max_{\Delta,w_i^f} Pw_i^f + \left(1-w_i^f\right) - \frac{\gamma}{2}\left(1-w_i^f\right)^2 \omega(\Delta) - \kappa g(\Delta). \quad (19)$$

The first order condition with respect to w_i^f leads to

$$1 - P = \gamma\left(1-w_i^f\right)\omega(\Delta), \quad (20)$$

while the first order condition with respect to Δ leads to

$$\kappa g'(\Delta) = -\frac{\gamma}{2}\left(1-w_i^f\right)^2 \omega'(\Delta). \quad (21)$$

Since in a symmetric equilibrium market clearing requires $w_i^f = 0$, equation (20) becomes identical to (17) and (21) becomes equivalent to (16). Finally, the assumption $(1-\gamma\omega(\Delta^*))(1-\chi) > \kappa g(\Delta^*)$ ensures that the constraint (7) is non-binding.[9]

Equation (16) allows us to view the length Δ as resulting from a tradeoff between spending resources to pay for participation costs in exchange for obtaining a better allocation of risk. In balancing this tradeoff the participation interval is determined as the point where the marginal cost of participation, $\kappa g'(\Delta)$, is equal to the marginal benefit of participation, $-\frac{\gamma}{2}\omega'(\Delta)$.

[9]If the constraint is binding at Δ^*, then Δ is given by the unique solution Δ^{**} to $(1-\gamma\omega(\Delta))(1-\chi) = \kappa g(\Delta)$, while the market clearing price is $P = 1 - \gamma\omega(\Delta^{**})$).

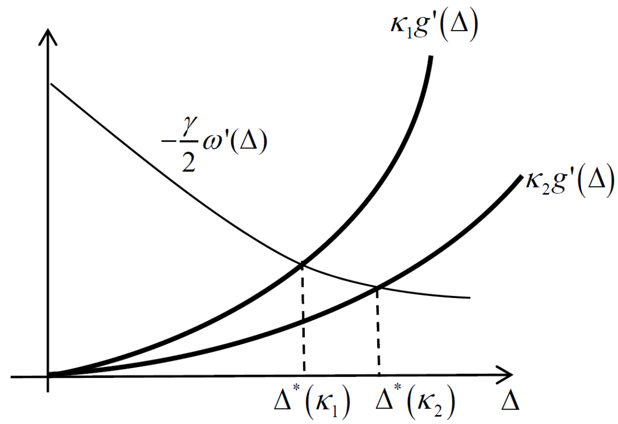


Figure 6: Illustration of the determination of Δ^* when a symmetric equilibrium exists. The figure illustrates that Δ^* increases when $\kappa_2 < \kappa_1$.

Figure 6 illustrates this tradeoff by plotting the marginal cost from increasing Δ, namely $\kappa g'(\Delta)$, against the respective marginal benefit $-\frac{\gamma}{2}\omega'(\Delta)$. Since $g(\Delta)$ is convex, $g'(\Delta)$ is upward sloping. By contrast, the marginal benefit is declining, since $-\omega''(\Delta) = -\frac{\sigma^2}{2}(1-\Delta) < 0$. Since $g'(0) = 0$, $\lim_{\Delta\to 1} g'(\Delta) = \infty$, $-\omega'(0) > 0$, and $\omega'(1) = 0$, the two curves intersect at some point $\Delta^* \in [0,1]$.

Proposition 1 helps capture the economic mechanisms that underlie our model. Consider, for instance, its implications for a reduction in the cost of accessing markets (i.e., a reduction in κ). As Figure 6 illustrates, such a reduction increases the degree of participation Δ and promotes portfolio flows across different locations. In turn, this increased participation improves risk sharing across different locations, which leads to higher prices of risky securities, $P = 1 - \gamma\omega(\Delta)$, and accordingly lower risk premia. By contrast, an increase in the costs of accessing risky markets leads to a lower Δ and a higher degree of concentration of risk. The resulting decline in the extent of risk sharing leads to a drop in the prices of risky assets and an increase in risk premia.

These mechanisms of the model help capture the stylized facts summarized in Section 2. We highlight especially one aspect of our analysis: the extent of market integration and cross-location portfolio flows and the magnitude of risk premia are intimately linked. By contrast, representative-agent approaches to the determination of risk premia are — by their construction — limited in their ability to explain the empirically prevalent joint movements in risk premia and portfolio movements, since the representative agent always holds the market portfolio and prices adjust so as to keep the agent content with her holdings.

We conclude this section by noting that in Section 7.1 we extend the results obtained so far to an intertemporal version of the model with recurrent shocks to participation costs. That extension helps further illustrate how the comparative statics results of this section imply that, in a dynamic setting, the mechanism we identify produces a) a negative correlation between capital flows and excess returns, b) time variation in excess returns that is unrelated to expected dividend growth, aggregate output etc., and c) return correlation across locations that exceeds the respective correlation of dividends.

We postpone the discussion of these issues; instead, we next turn our attention to a different set of issues related to the existence of a symmetric equilibrium.

4.3 Asymmetric, location-invariant equilibria: The role of leverage and the diversity of the financial industry

Proposition 1 contains the premise that a symmetric equilibrium exists. Surprisingly, despite the symmetry of the model setup, a symmetric equilibrium may fail to exist. Instead, the market equilibrium may involve different choices (leverage ratios, portfolios of risky assets, wealth allocations, etc.) for agents in the same location, even though these agents have the same preferences and endowments and are allowed to make the same participation and portfolio choices.

These claims are explained by the observation that the necessary first-order conditions resulting in the prices $P_i = P = 1 - \gamma\omega(\Delta^*)$ are not generally sufficient. We now take a closer look at whether, fixing the prices $P = 1 - \gamma\omega(\Delta^*)$, an investor's optimal participation interval is given by $\Delta = \Delta^*$ along with $w^f = 0$.

To answer this question, we consider again the maximization problem (19). Taking $P = 1 - \gamma\omega(\Delta^*)$ as given, substituting into (20), and re-arranging implies that if investors allocate their wealth over a participation interval Δ (potentially different from Δ^*), then

$$1 - w^f = \frac{\omega(\Delta^*)}{\omega(\Delta)}. \tag{22}$$

Equation (22) contains an intuitive prediction. An investor allocating her wealth over a span $\Delta > \Delta^*$ is facing the same average returns, but a lower variance $\omega(\Delta)$, and hence a higher Sharpe ratio compared to an investor allocating her wealth over an interval of size Δ^*. Accordingly, the former investor finds it optimal to leverage her portfolio. This is reflected in equation (22), which states that $w^f < 0$ $\left(w^f > 0\right)$ whenever $\Delta^* < \Delta$ $(\Delta^* > \Delta)$.

An interesting implication of (21) is that for investors who choose $1 - w^f > 1$, the marginal benefit of an increased participation interval becomes larger. This is intuitive since

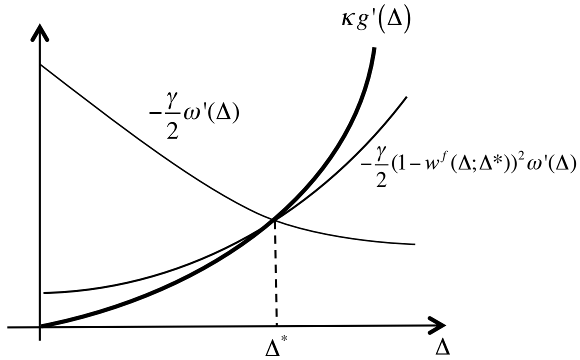


Figure 7: Illustration of the marginal benefit curve $-\frac{\gamma}{2}\left(\frac{\omega(\Delta^*)}{\omega(\Delta)}\right)^2\omega'(\Delta)$, the marginal cost curve $\kappa g'(\Delta)$ and the marginal benefit curve $-\frac{\gamma}{2}\omega'(\Delta)$ assuming that $w^f = 0$. The left plot illustrates a case where a symmetric equilibrium exists whereas the right plot illustrates a case where a symmetric equilibrium fails to exist.

increased leverage implies a more volatile wealth next period and hence a higher marginal benefit of reducing that variance by increasing Δ.

In short, the choice of Δ and the choice of leverage $1 - w^f$ are *complements*.

This complementarity leads to an upward sloping marginal benefit of increased participation. Indeed, substituting (22) into (21) gives

$$\kappa g'(\Delta) = -\frac{\gamma}{2}\left(\frac{\omega(\Delta^*)}{\omega(\Delta)}\right)^2\omega'(\Delta). \tag{23}$$

Using the fact $\omega(\Delta) = \frac{\sigma^2}{12}(1-\Delta)^3$, the right hand side can be expressed as $-\frac{\gamma}{2}\left(\frac{\omega(\Delta^*)}{\omega(\Delta)}\right)^2\omega'(\Delta) = \frac{\gamma\sigma^2}{8}(1-\Delta^*)^6(1-\Delta)^{-4}$, which is increasing in Δ.

Figure 7 helps illustrate these notions. The figure depicts the marginal cost curve $\kappa g'(\Delta)$, the benefit curve $-\frac{\gamma}{2}\left(1 - w^f(\Delta;\Delta^*)\right)^2\omega'(\Delta) = -\frac{\gamma}{2}\left(\frac{\omega(\Delta^*)}{\omega(\Delta)}\right)^2\omega'(\Delta)$, and also the curve $-\frac{\gamma}{2}\omega'(\Delta)$, i.e., the marginal benefit of participation fixing $w^f = 0$. The point where all three curves intersect corresponds to the point $\Delta = \Delta^*$. The left plot of Figure 7 illustrates a case where a symmetric equilibrium exists, whereas the right plot illustrates a case where a symmetric equilibrium fails to exist. The difference between the two plots is the shape of $g'(\Delta)$. In the left plot $g'(\Delta)$ intersects the private marginal benefit curve only once, namely at Δ^*. For values smaller than Δ^*, the private marginal benefit is above the marginal cost and vice versa for values larger than Δ^*. Hence, in this case Δ^* is indeed the optimal choice.

This is no longer the case in the right plot. Here the private marginal benefit curve intersects the marginal cost curve three times (at Δ_1^*, Δ^*, and Δ_2^*). Since the private

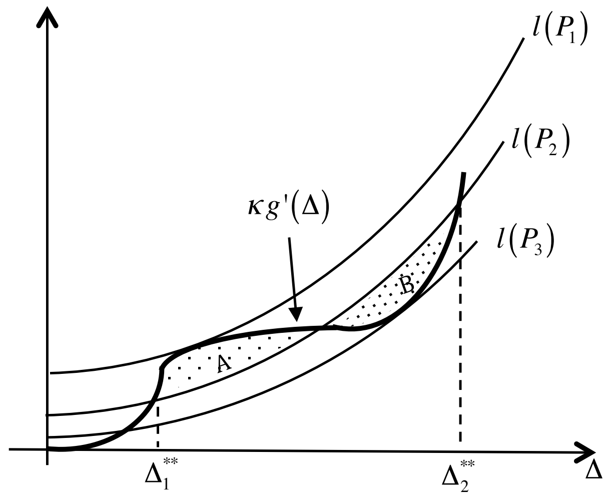


Figure 8: Illustration of an asymmetric equilibrium involving mixed strategies. The price adjusts so that area A is equal to area B. Accordingly, an investor is indifferent between Δ_1^{**} and Δ_2^{**}. Here, $P_3 < P_2 < P_1 < 1$.

marginal benefit is below the marginal cost for values of Δ that are smaller and "close" to Δ^*, while the private marginal benefit is larger than the marginal cost for values of Δ that are larger and "close" to Δ^*, Δ^* is a *local minimum*, and hence a suboptimal choice. By contrast the points Δ_1^* and Δ_2^* correspond to local maxima. (In this particular example the point Δ_1^* is the global maximum, since the area A is larger than the area B.) The fact that Δ^* is not a maximum implies that there does not exist a symmetric equilibrium, since in a symmetric equilibrium it would have to be the case that $w^f = 0$.

The fact that there does not exist a symmetric market equilibrium implies that one should look for equilibria where investors in the same location make different choices, even though they have the same preferences, endowments, and information. Figure 8 presents a simple graphical illustration of such an equilibrium in the context of the example depicted on the right plot of Figure 7.

Specifically we illustrate the construction of an equilibrium that features the same price $P_i = P$ for all markets, but where a fraction π of investors in every location i invest with intermediaries who choose $(w_1^f,\ \Delta_1)$, while the remaining fraction $(1-\pi)$ of agents invest with intermediaries who choose (w_2^f, Δ_2). We introduce a function $l\left(\Delta; P\right)$ that captures the marginal benefit of participation as a function of Δ and P:

$$l\left(\Delta; P\right) \equiv -\frac{\gamma}{2}\left(1 - w^f\left(x; P\right)\right)^2 \omega'\left(x\right) = -\frac{\gamma}{2}\left(\frac{1-P}{\gamma}\right)^2 \frac{\omega'\left(x\right)}{\omega^2\left(x\right)}, \tag{24}$$

where the last equation follows from (20).

Figure 8 depicts the function $l(\Delta; P)$ for three values $P_1 < P_2 < P_3 < 1$. A first observation is that as P declines from P_3 to P_1, the curve $l(\Delta; P)$ shifts up. Moreover there exists a level P_2 and associated values Δ_1^{**} and Δ_2^{**} for which the area "A" equals the area "B", so that investors are indifferent between choosing Δ_1^{**} or Δ_2^{**}. Fixing these values of $P, \Delta_1^{**}, \Delta_2^{**}$, we can determine the values of w_1^f and w_2^f from (20). As part of the proof of Proposition 2 (in the appendix), we show that these values of w_1^f and w_2^f satisfy $1 - w_1^f < 1 < 1 - w_2^f$. In order to make sure that the bond market clears, we need to set π so that $\pi w_1^f + (1-\pi) w_2^f = 0$, which implies a value of $\pi = \frac{w_2^f}{w_2^f - w_1^f} \in (0,1)$.[10] We also show in the appendix that for this value of π all markets for risky assets clear as well. Finally, in order to verify that this is indeed an equilibrium, we additionally need to assume that $\kappa g(\Delta_2) \leq P\left(1 - \chi\left(1 - w_2^f\right)\right)$, so that (7) does not bind. For a non-empty set of parameters, this condition will indeed hold.[11]

The next proposition generalizes the insights of the above illustrative example. It shows the existence of an asymmetric equilibrium, when a symmetric equilibrium fails to exist.

Proposition 2 *When the cost function g is such that a symmetric equilibrium fails to exist, there exists an asymmetric equilibrium. Specifically, there exist (at least) two tuples $\{\Delta_k, w_k^f\}$, $k \geq 2$, and $\pi_k > 0$ with $\sum_k \pi_k = 1$ and $\sum_k \pi_k \left(1 - w_k^f\right) = 1$, such that in every location i a fraction π_k of agents choose the interval, leverage, and portfolio combination $\{\Delta_k, w_k^f, dL^{(k)}\}$, where $dL^{(k)}$ is the measure given in (14) for $\Delta = \Delta_k$.*

In summary, the model predicts that the complementarities between leverage and the incentive to increase participation in risky markets may result in a non-concave objective function for the determination of Δ. In such situations a symmetric equilibrium can fail to exist. Instead, the market equilibrium features a diverse financial industry, with some financial intermediaries pursuing high-cost, high-Sharpe-ratio, high-leverage strategies (such as hedge funds) and some financial intermediaries pursuing low-cost, low-Sharpe-ratio, no-leverage strategies (such as mutual funds). It is useful to underscore that this diversity of the financial industry obtains despite the facts that investors are identical and participation costs are convex. Instead, when markets are partially integrated, the interaction between participation decisions and leverage may actually *necessitate* a diverse financial industry (with some intermediaries pursuing high leverage strategies) for markets to clear.

[10] $\pi \in (0,1)$ since $w_2^f < 0$ and $w_1^f > 0$.

[11] Specifically, it is straightforward to confirm that this condition holds if we set κ proportional to σ and we let σ become sufficiently small.

In the next section we show that the tendency of the model to endogenously require some intermediaries to pursue levered strategies may lead to price crashes in equilibrium in response to small changes in participation costs. In Section 7.2 we examine the robustness of the non-concavities we identified in this section. Specifically, we show that the non-concavities become stronger (in a sense we make precise in that section) when we allow for investment in non-contiguous locations.

5 Collateral Constraints and Price Crashes

In the illustrative example of an asymmetric equilibrium of Section 4.3, we assume that the constraint (7) does not bind on any investor. However, when the equilibrium is asymmetric, a fraction of agents pursues high leverage, high cost strategies. For these strategies it is particularly likely that the constraint (7) may bind. In this section our goal is to show that in this case equilibrium prices may react abruptly (indeed, discontinuously) in response to incremental changes in κ.

In order to keep the exposition uncluttered, we use a stylized specification of participation costs that allows analytical computations. Specifically, we postulate the cost function depicted in the middle graph of Figure 5. In that case, the optimal participation choice amounts to choosing the location of the points a_i and the number of the discrete points N.[12] To keep computations as simple as possible, we assume furthermore that $b_N = \infty$ for $N > 2$, so that only choices involving $N \leq 2$ are feasible.[13] Accordingly, an intermediary's problem comes down to choosing $N = 1$ or $N = 2$, and the distance d from her home location if $N = 2$. At the cost of adding complexity, the results of this section can be extended to multiple disjoint intervals or points; we sketch how in section 7.2.

Conjecture next that in equilibrium $P_j = P$ for all j and let $\pi \in [0, 1]$ denote the fraction of funds invested in the local market. Assuming that a given intermediary chooses $N = 2$ and another location at distance d, equation (3) allows the computation of the minimal portfolio

[12] Recall that in this case there is a minimum "fixed" cost that one needs to pay for every new location that she chooses given by $\kappa b_N \lim_{x \to 0} f(x) > 0$.

[13] The condition $b_N = \infty$ can be substantially weakened. Indeed all that is needed for $N = 1$ or $N = 2$ to be the only feasible choices is that $\kappa b_N \lim_{x \to 0} f(x) > 1$ for $N > 2$. This follows from (8), $P_j \leq 1$, and the fact that f is increasing.

variance:

$$\widehat{\omega}\left(d\right) = \sigma^2 \min_{\pi} \left\{ \left(\pi^2 + \left(1-\pi\right)^2\right) \frac{1}{12} + 2\pi\left(1-\pi\right)\left(\frac{1}{12} - \frac{d\left(1-d\right)}{2}\right) \right\}$$
$$= \sigma^2 \left(\frac{1}{12} - \frac{1}{4} d\left(1-d\right)\right).$$

Using the obtained expression for $\widehat{\omega}\left(d\right)$, the optimal distance d needs to satisfy a first order condition similar to equation (21), namely

$$-\frac{\gamma}{2}\left(1-w^f\right)^2 \frac{\widehat{\omega}'\left(d\right)}{\widehat{\omega}^2\left(d\right)} = \kappa b_2 f'\left(d\right). \tag{25}$$

An interesting special case arises when $f' = 0$, i.e., in the special case where participation in distant markets involves a "fixed" cost $\kappa b_2 f_0 \equiv \kappa b_2 f(x) > 0$ for $x \in (0,1)$, but no distance-dependent, variable costs. In that special case equation (25) implies that the optimal d is given by $d = \frac{1}{2}$ irrespective of w^f, since $\widehat{\omega}'\left(d\right) = 0$ when and only when $d = \frac{1}{2}$. To further simplify calculations we assume for the rest of the section that $f' = 0$. Assuming that the equilibrium is of the asymmetric type, and ignoring momentarily the constraint (7), we can use equation (19) to express the indifference between the choices $N = 1$, respectively $N = 2$ and $d = \frac{1}{2}$, as

$$Pw_1^f + \left(1-w_1^f\right) - \frac{\gamma}{2}\left(1-w_1^f\right)^2 \widehat{\omega}\left(0\right) = Pw_2^f + \left(1-w_2^f\right) - \frac{\gamma}{2}\left(1-w_2^f\right)^2 \widehat{\omega}\left(\frac{1}{2}\right) - \kappa b_2 f_0. \tag{26}$$

Using the first order conditions for leverage

$$1 - P = \gamma\left(1-w_2^f\right)\widehat{\omega}\left(\frac{1}{2}\right) = \gamma\left(1-w_1^f\right)\widehat{\omega}\left(0\right) \tag{27}$$

inside (26) yields — after some simplifications — the equilibrium price:

$$P\left(\kappa\right) = 1 - \sqrt{\frac{2\gamma\kappa b_2 f_0}{\frac{1}{\widehat{\omega}\left(\frac{1}{2}\right)} - \frac{1}{\widehat{\omega}\left(0\right)}}}. \tag{28}$$

For $P\left(\kappa\right)$ to be an equilibrium price, it must also be case that a) $1 - w_1 \leq 1 \leq 1 - w_2$, and b) the constraint (7) is not binding for any agent. In light of (27), the requirement $1 - w_1 \leq 1 \leq 1 - w_2$ is equivalent to $P \in \left[1 - \gamma\widehat{\omega}\left(0\right), 1 - \gamma\widehat{\omega}\left(\frac{1}{2}\right)\right]$. This requirement is satisfied as long as κ lies between κ_1 and κ_2, where

$$\kappa_1 = \frac{\gamma\widehat{\omega}\left(0\right)}{2f_0}\left[\frac{\widehat{\omega}\left(0\right)}{\widehat{\omega}\left(\frac{1}{2}\right)} - 1\right], \quad \kappa_2 = \frac{\gamma\widehat{\omega}\left(\frac{1}{2}\right)}{2f_0}\left[1 - \frac{\widehat{\omega}\left(\frac{1}{2}\right)}{\widehat{\omega}\left(0\right)}\right].$$

For values of κ below κ_1 or above κ_2 the only candidate equilibria are symmetric with everyone choosing $N = 2$ and $d = \frac{1}{2}$ or everyone choosing $N = 1$, respectively. Hence, ignoring momentarily the constraint (7), the left plot of Figure 9 gives a visual depiction of $P(\kappa)$, which is a continuous function of κ. Hence, even though investors make essentially discrete choices, prices are a continuous function of κ.

However, when we take into account the constraint (7) a different situation arises, which is depicted on the right plot of Figure 9. The downward-sloping dotted line of that plot depicts the price $P(\kappa)$ that would obtain in the absence of the constraint (7), i.e., it is identical to the solid line of the left plot. The upward-sloping dotted line depicts the locus of κ, P for which

$$\kappa b_2 f_0 = P\left(1 - \chi(1 - w_2^f)\right), \tag{29}$$

where $w_2^f(P)$ is given by (27). For these points the constraint (7) binds with equality.[14]

For values of κ and P that lie above and to the left of the upward-sloping line, the constraint (7) is not binding; hence the equilibrium price (the solid line) coincides with the downward-sloping dotted line for values of κ smaller than κ_c. However, for values of $\kappa \geq \kappa_c$, the downward-sloping dotted line no longer describes the equilibrium price.

To obtain the equilibrium price for $\kappa \geq \kappa_c$ we proceed as follows. We define V_1 and V_2 as

$$V_1 = \max_{w_1^f} P w_1^f + \left(1 - w_1^f\right) - \frac{\gamma}{2}\left(1 - w_1^f\right)^2 \widehat{\omega}(0), \tag{30}$$

$$V_2 = \max_{w_2^f} P w_2^f + \left(1 - w_2^f\right) - \frac{\gamma}{2}\left(1 - w_2^f\right)^2 \widehat{\omega}\left(\frac{1}{2}\right) - \kappa b_2 f_0, \tag{31}$$

where the second maximization is subject to the constraint (7). Indifference requires that $V_1 = V_2$, and hence $\frac{dV_1}{d\kappa} - \frac{dV_2}{d\kappa} = 0$.

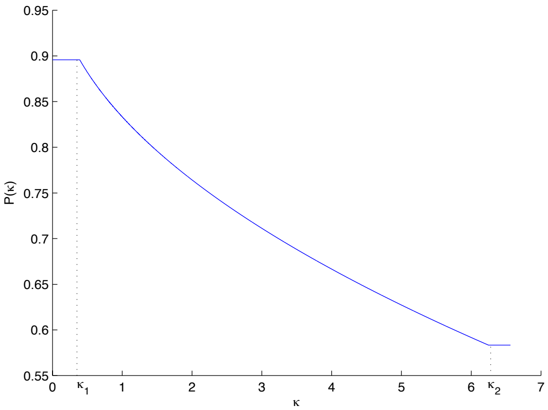
Attaching a Lagrange multiplier $\lambda \geq 0$ to the constraint (29), solving the resulting maximization problem, and utilizing the envelope theorem implies

$$\frac{dV_1}{d\kappa} - \frac{dV_2}{d\kappa} = \left(w_1^f - w_2^f - \lambda\left[1 - \chi\left(1 - w_2^f\right)\right]\right)\frac{dP}{d\kappa} + (1 + \lambda)\, b_2 f_0, \tag{32}$$

with $\lambda = \frac{1 - P - \gamma\left(1 - w_2^f\right)\widehat{\omega}\left(\frac{1}{2}\right)}{\chi P} \geq 0$. Inspection of (32) allows us to draw two conclusions. For values of κ such that the constraint is not binding ($\kappa < \kappa_c$), $\lambda = 0$, and there always exists a value $\frac{dP}{d\kappa}$ such that $\frac{dV_1}{d\kappa} - \frac{dV_2}{d\kappa} = 0$. Indeed, from (32),

$$\frac{dP}{d\kappa} = -\frac{b_2 f_0}{w_1^f - w_2^f} < 0.$$

[14] To see this, evaluate equation (8) with $P_j = P = W_{0,i}$, $P_B = 1$, $F_i = \kappa b_2 f_0$, and $\int_0^1 dX_j = 1 - w^f$, and note that equation (8) is equivalent to (7).



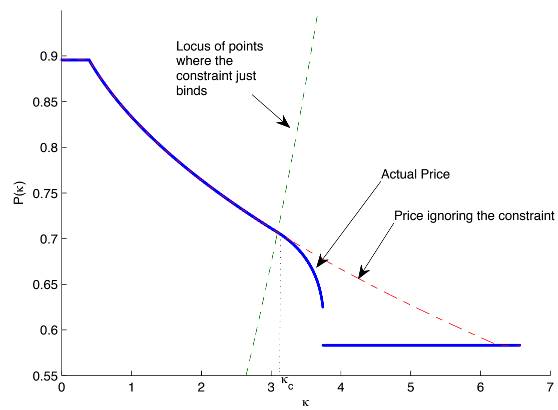


Figure 9: The left plot depicts the price $P(\kappa)$ assuming the leverage constraint is not binding. The downward-sloping dotted line on the right plot plots the same quantity, along with the actual price (solid line) and the locus of points such that the leverage constraint is just binding (upward sloping line). Hence for points above and to the left of the upward sloping line the constraint is not binding, whereas it is binding for points below and to the right. Parameters used in this example: $\gamma = 5$, $b_2 f_0 = 0.1$, $\chi = 0.2$, $\sigma = 1$.

If κ increases sufficiently for the constraint to start binding ($\kappa > \kappa_c$), then $\lambda > 0$. For values of κ larger than, but close to κ_c, the value of λ is positive and small. Therefore

$$w_1^f - w_2^f - \lambda \left[1 - \chi \left(1 - w_2^f\right)\right] > 0$$

and

$$\frac{dP}{d\kappa} = -\frac{(1+\lambda)\, b_2 f_0}{w_1^f - w_2^f - \lambda \left[1 - \chi \left(1 - w_2^f\right)\right]} < -\frac{b_2 f_0}{w_1^f - w_2^f} < 0.$$

Accordingly, the slope of the decline in P as κ increases becomes steeper than it would be absent the constraint. As κ keeps increasing, the constraint becomes progressively more binding, and possibly up to the point where

$$w_1^f - w_2^f - \lambda \left[1 - \chi \left(1 - w_2^f\right)\right] = 0. \tag{33}$$

At that point equation (32) shows that there exists no value $\frac{dP}{d\kappa}$ that can set $\frac{dV_1}{d\kappa} = \frac{dV_2}{d\kappa}$, and hence the price jumps (down) to $P = 1 - \gamma \widehat{\omega}(0)$.

Figure 10 helps illustrate such a discontinuity. For each subplot we fix a value of κ. Given that value of κ, the line denoted "Indifference" in each subplot depicts combinations of $\left(1 - w_2^f\right)$ and P, such that the indifference relation holds:

$$V_1(P) = P w_2^f + \left(1 - w_2^f\right) - \frac{\gamma}{2}\left(1 - w_2^f\right)^2 \widehat{\omega}\left(\frac{1}{2}\right) - \kappa f_0. \tag{34}$$

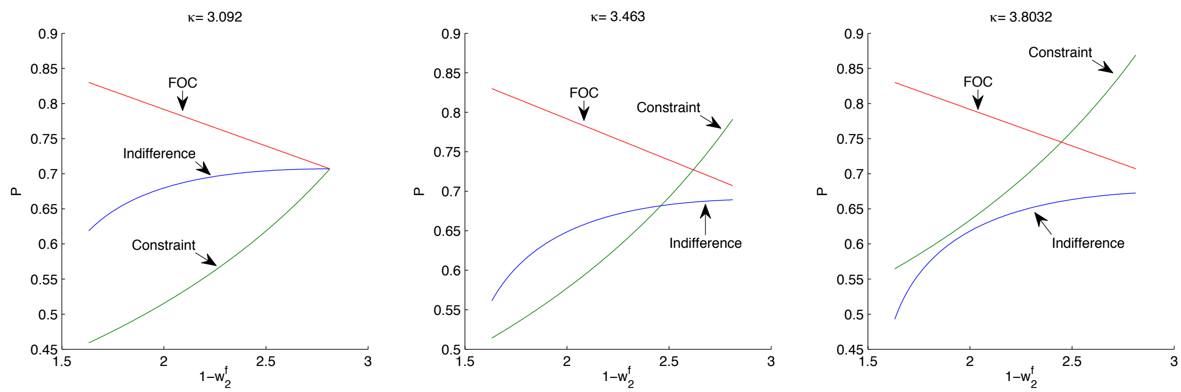


Figure 10: For three different values of the cost parameter κ, the three plots depict combinations of price (P) and leverage ($1 - w_2^f$) so that (i) investors are indifferent between adopting no- or high-leverage strategies (line labeled "Indiference") , (ii) Investors' choices of leverage ($1 - w_2^f$) are (unconstrained) optimal given P (line labeled "FOC"), and (iii) the leverage constraint just binds (line labeled "Constraint"). In each plot, points to the left and above the line labeled "Constraint" are admissible. The intersection of the lines "FOC" and "Indifference" corresponds to the combination of P and $1 - w_2^f$ that would prevail in an equilibrium without the leverage constraint. Similarly, the intersection of the lines "Indifference" and "Constraint" characterize an asymmetric equilibrium with a binding leverage constraint (if one exists). Parameters are identical to Figure 9.

Similarly, the line denoted "Constraint" depicts combinations of P and $1 - w_2^f$ such that the constraint (29) holds as an equality. Accordingly, all the points that lie above that line are admissible combinations of P and $1 - w_2^f$. Finally, the line denoted "FOC" depicts the combination of points that satisfy the first-order condition (27).

The intersection of the lines "Indifference" and "FOC" corresponds to the equilibrium price (and the associated leverage w_2^f) of equation (28). We refer to that intersection as the "unconstrained" point. The left plot depicts the case where κ is set equal to $\kappa = \kappa_c$. In that case the constraint just becomes binding, and all three lines intersect at the same point. As κ increases further, illustrated in the middle panel, the constraint binds actively. Intermediaries reduce leverage and hence the magnitude of their risky positions, and equlibrium prices drop to the point where the constraint is satisfied (i.e., up to the point where the line "Indifference" intersects the line "Constraint").

The right-most plot shows a case where κ drops further and there is no point of intersection between the lines "Indifference" and "Constraint". At that point there can be no asymmetric equilibrium. The equilibrium becomes symmetric, i.e., the price drops to the level that obtains when everyone simply chooses $N = 1$ and $w^f = 0$.

This example helps illustrate that — when the leverage constraint is binding — the price associated with an asymmetric equilibrium may discontinuously drop to the price associated with a symmetric equilibrium even in response to incremental changes in the cost parameter κ.

To understand intuitively why prices react so abruptly, we start by identifying the basic mechanism that helps restore equilibrium when κ increases and the constraint (7) isn't binding: at existing prices, an increase in κ induces a fraction of investors to walk away from high-leverage, increased-market-participation, high-cost strategies; the resulting loss in aggregate participation lowers prices, raises the Sharpe ratio, and helps restore the attractiveness of high-cost, high-leverage strategies. However, when the constraint binds, the decline in prices also lowers the value of collateral that investors are endowed with, thus necessitating further reductions in leverage, which results in even smaller overall participation in risky markets, further reductions in prices, etc. These mutually reinforcing effects can lead to a rapid drop in prices and eventually a discontinuous jump in prices, whereby intermediaries deleverage instantaneously and capital flows drop to zero.

We conclude with a technical remark: even though the example of this section is stylized (to ease computations), it is not special. The rapid decline in prices is driven by the non-concave nature of investors' participation problem: Non-concave objective functions admit multiple optima, and small changes in equilibrium prices — coupled with constraints on leverage — may cause large changes in the optimal leverage and participation choices, which would in turn result in discontinuous changes in prices to clear markets.

6 Reduced Participation in a Subset of Locations and Contagion

In the previous sections we assumed the same cost structure for all intermediaries, regardless of location. This allowed us to focus on (potentially asymmetric but) location-invariant equilibria, which entail the same price in all locations. In this section we relax this assumption in order to study the effect of an adverse shock to the financial sector of a subset of locations. The challenge in analyzing the model with location-specific costs is that an entire price function on $[0, 1)$ has to be computed, rather than a single value. To present the key insights of the model with location-specific costs, we restrict ourselves to a specific setup that is sufficiently simple to analyze, yet rich enough to illustrate how a reduction in market participation in a subset of risky markets (due to a local "breakdown" of intermediation)

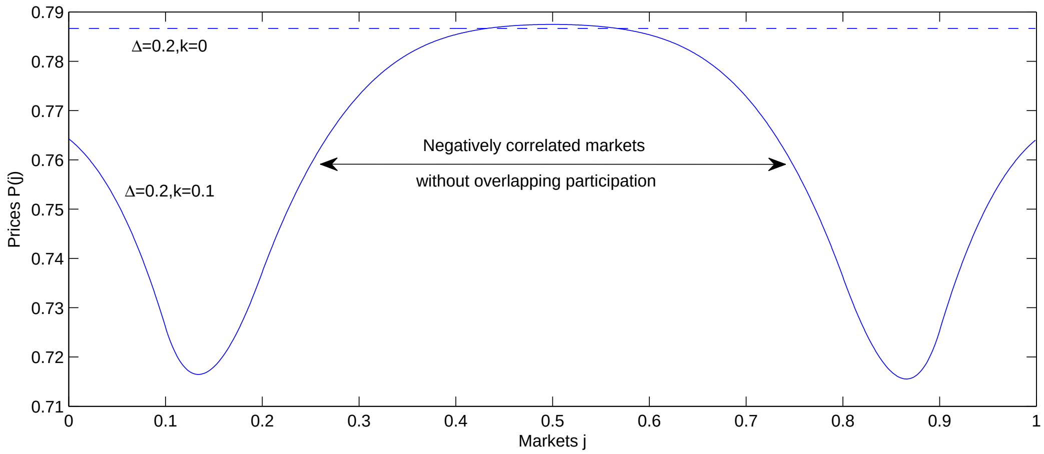


Figure 11: Contagion effects. The price function $P(j)$ depicts the price in different locations when $k = 0$, and $k = 0.1$. We choose $\bar{\Delta} = 0.2$. The locations between the arrows feature both negative correlation of dividends and no overlap between the participants in these markets and the set $A = \left[-\frac{k}{2}, \frac{k}{2}\right]$.

can propagate across all markets, a phenomenon that we refer to as "contagion".

Specifically, we consider the following simple setup: for some positive $k < 1$ intermediation in locations $\left[-\frac{k}{2}; \frac{k}{2}\right]$ "breaks down". Specifically, we assume that in these "affected" locations the cost parameter κ becomes infinite, so that investors choose to participate only in the bond market and the market for the local risky claim. In the rest of the locations intermediaries choose to participate in a single interval of length $\bar{\Delta}$, centered at their "home location" — this would be the outcome of a cost structure in these locations involving, e.g., $\kappa < \infty$, $b_N = \infty$ for $N > 1$, $g'(\bar{\Delta}) = 0$ for $\Delta < \bar{\Delta}$, and $g'(\Delta)$ arbitrarily large for $\Delta > \bar{\Delta}$.

We solve this heterogeneous-participation-costs version of the model in the appendix. First, we calculate analytically the optimal demand of every investor given a price function (Lemma 3), and then aggregate the demands to solve for prices. We are unable to find a closed-form solution for the prices, but we can characterize the solution in terms of a linear integro-differential equation with delay, which can be solved numerically as easily as a matrix inversion problem.

To obtain a visual impression of the solution, Figure 11 depicts the price $P(x)$ for $x \in [0, 1)$ and compares it with the (symmetric) equilibrium price $P^*\left(\bar{\Delta}\right)$ that would prevail if all agents in all locations chose a participation interval with length equal to $\Delta = \bar{\Delta}$. There are several noteworthy facts about Figure 11. As one might expect, prices in the set $A \equiv \left[-\frac{k}{2}, \frac{k}{2}\right]$ are lower than $P^*\left(\bar{\Delta}\right)$. The more important fact is that prices in all other markets are affected, as well. This holds true even in markets in the interval $B \equiv \left[\frac{k}{2} + \bar{\Delta}, 1 - \frac{k}{2} - \bar{\Delta}\right]$, i.e., locations that are in a larger distance than $\bar{\Delta}$ from the set A so that agents in the sets A and B would not trade any risky securities with each other even in a symmetric equilibrium where all agents choose the same participation interval $\bar{\Delta}$. A third observation is that the prices in almost all markets are lower than $P^*\left(\bar{\Delta}\right)$. Indeed, this reduction in price can happen even in markets with dividends that have a zero or negative correlation with the dividends of any risky security in A, as the figure illustrates.

The intuition behind Figure 11 is the following. By assumption, investors in the set A reduce their demand for risky assets to zero in locations other than their home location. Hence compared to the case where everyone participates in an interval of length $\bar{\Delta}$ centered at their location, there is now lower demand for risky securities in locations neighboring the set A. The lower demand for risky securities in these locations leads to lower prices, which attracts demand from locations adjacent to the neighborhood of A, where intermediaries still operate. By tilting their portfolio towards these locations, intermediaries vacate demand for risky securities in locations even farther than A. Accordingly, prices in these farther locations need to drop in order to attract investors from locations adjacent to the neighbourhood of the neighbourhood of A, who also tilt their portfolios and so on.

This chain reaction implies that all locations are affected in equilibrium. In fact, even some prices in locations close to point $\frac{1}{2}$, the farthest from A, drop, as the following result states.[15]

Proposition 3 *Assume $\overline{\Delta} + k < 1$ and that $\frac{1}{2} = \arg\max_{j \in \left[\frac{1}{2} - \bar{\Delta}, \frac{1}{2} + \bar{\Delta}\right]} P_j$. Then there exists a positive-measure subset $D \subset \left[\frac{1}{2} - \bar{\Delta}; \frac{1}{2} + \bar{\Delta}\right]$ such that $P(x) < P^*\left(\bar{\Delta}\right)$ for all $x \in D$.*

Proposition 3 helps formalize the notion that even (some) prices in locations within a radius $\bar{\Delta}$ of the maximal price are lower than the price $P^*(\bar{\Delta})$ that would obtain in a symmetric equilibrium.

The extent of the change in risk premia in the various locations depends on the distance from the arc A, the length of the arc A, and most interestingly, on the length of the participation arc $\bar{\Delta}$. Figure 12 illustrates these statements. If $\bar{\Delta}$ is small, so that their portfolio is heavily exposed to risks in their vicinity, then even the smallest tilt towards a nearby location requires a high compensation. By contrast, if investors' portfolios are invested across a broad range of locations, then they are more willing to absorb risks in their vicinity.

[15] It is easy to show that, in the general case when a subset of investors are limited to a smaller investment set than the original arc of length $\bar{\Delta}$, and therefore in the setting of this section, the average risky-asset price in the economy declines.

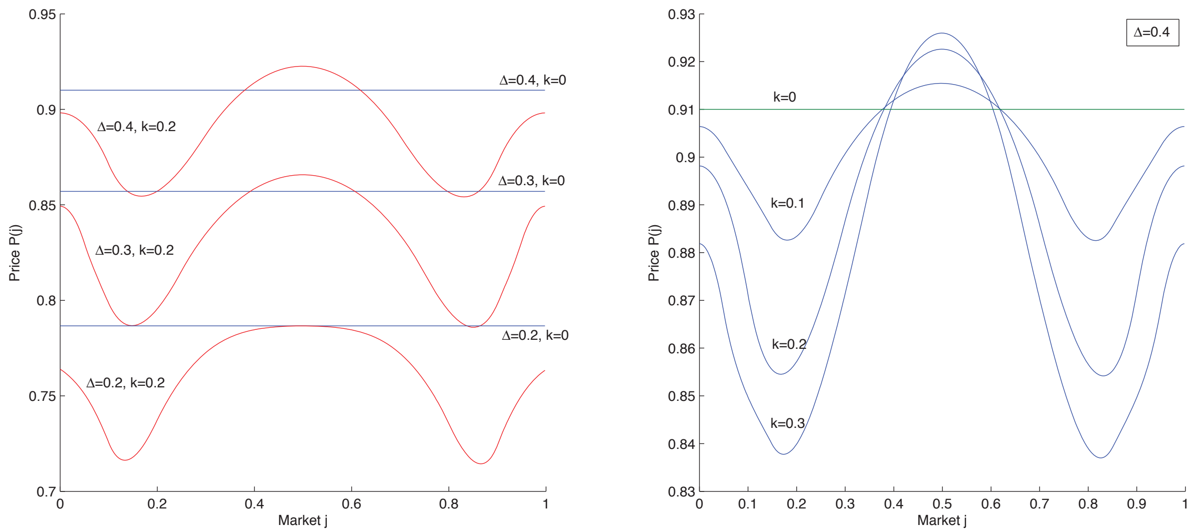


Figure 12: Contagion effects for various choices of $\bar{\Delta}$ and k.

In summary, a surprising result of our analysis is that contagion occurs due to *limited*, rather than excessive, integration of risky markets. Indeed, contagion becomes stronger the weaker the integration of risky markets.

7 Extensions

7.1 An infinite horizon version of the baseline model

In this section we develop an intertemporal version of the model. The intertemporal version allows us to extend the intuitions of our comparative statics exercises to a framework where the shocks to the participation technology are recurrent.

Specifically, we keep the key assumptions of the baseline model (Section 4.2). However, we assume that investors maximize expected discounted utility from consumptiuon

$$-\sum_{t=0}^{\infty}\beta^t E_t\left[e^{-\gamma c_{t,i}}\right],$$

where t denotes (discrete) calendar time. We assume that divdends at location $i \in [0,1)$ are given by

$$D_{t,i} = (1-\rho)\sum_{k=-\infty}^{t}\rho^{t-k}\varepsilon_{k,i} \tag{35}$$

where $\rho < 1$,

$$\varepsilon_{t,i} \equiv 1 + \sigma \left(B_i^{(t)} - \int_0^1 B_j^{(t)} dj \right), \tag{36}$$

and $B_i^{(t)}$ denote a family of Brownian Bridges on $[0, 1]$ drawn independently across times $t = -\infty, ..1, 2, .., +\infty$.

We make a few observations about the above dividend structure. First, we note that equation (36) coincides with equation (2). Accordingly, $\int_0^1 \varepsilon_{t,i} di = 1$, and therefore, equation (35) implies that $\int_0^1 D_{t,i} = 1$ for all t, so that the aggregate dividend is always equal to one. Second, dividends at individual locations follow AR(1) processes, since equation (35) implies that

$$D_{t,i} = (1 - \rho)\, \varepsilon_{t,i} + \rho D_{t-1,i}. \tag{37}$$

Moreover, since (36) coincides with (2), the increments of two dividend processes at two locations i and j have the covariance structure of equation (3).

In terms of participation decisions, we keep the same cost assumption as in the baseline model and further assume that investors participate in a single interval of length $\Delta_t^{(i)}$ centered at their "home" location. Participation costs are paid period by period in advance of trading. Specifically, an investor's intertemporal budget constraint is given by

$$c_{t,i} + F_t \left(\Delta_t^{(i)} \right) + \int_0^1 P_{t,j} dX_{t,j}^{(i)} + P_{B,t} X_{B,t}^{(i)} = \int_0^1 (P_{t,j} + D_{t,j})\, dX_{t-1,j}^{(i)} + X_{B,t-1}^{(i)}. \tag{38}$$

We note that in equation (38) we allow the entire cost function $F_t = \kappa_t g_t(\Delta)$ to be different across different periods in order to capture the effect of repeated shocks to the participation technology.

This intertemporal version of the model presents a challenge that is absent in a static framework: If the interest rate varies over time, then the value function of an agent is not exponential in wealth. In fact, a closed-form expression for the value function most likely does not exist. Furthermore, once the value function is no longer exponential, portfolios are no longer independent of wealth, and hence the entire wealth distribution matters — an infinite-dimensional state variable.

In order to maintain the simple structure of the solution, therefore, we make necessary assumptions to achieve a constant interest rate despite random participation costs. The following proposition states that it is possible to achieve this outcome; the proof is in the appendix. The main thrust of the proposition, however, concerns the form of the risky-asset prices; Proposition 5 below collects the main implications of interest of this form.

Proposition 4 *There exist an interval $[\Delta_l, \Delta_u]$, a (non-trivial) distribution function $\Psi(\cdot)$ on $[\Delta_l, \Delta_u]$, and a cost function $F(\cdot; \Delta_t) : [\Delta_l, \Delta_u] \to \mathbb{R}^+$ such that, if Δ_t is drawn in an i.i.d. fashion from Ψ, then*
(i) intermediaries optimally choose $\Delta = \Delta_t$, thus incurring cost $F(\Delta_t; \Delta_t)$;
(ii) the risk-free rate is constant over time and given as the unique positive solution to

$$1 = \beta (1 + r) E\left[e^{\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta)}\right], \tag{39}$$

where the expectation is taken over the distribution of Δ;
(iii) the risky-asset prices equal

$$P_{t,j}(\Delta_t, D_{t,j}) = \phi (D_{t,j} - 1) + \frac{1}{r} - \Phi_1 \omega (\Delta_t) - \Phi_0 \tag{40}$$

with $\phi \equiv \frac{\rho}{1+r-\rho}$ and

$$\Phi_1 = \gamma \frac{r(1-\rho)^2}{(1+r-\rho)^2} \tag{41}$$

$$\Phi_0 = \frac{\Phi_1}{r} \frac{E\left[e^{\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta)} \omega(\Delta)\right]}{E\left[e^{\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta)}\right]} > 0. \tag{42}$$

Furthermore, investors' optimal portfolios of risky assets are given by (14).

Equation (40) decomposes the price of a security into three components. As in pretty much all CARA models, one of these components equals the expected discounted value of future dividends, $\phi(D_{t,j} - 1) + r^{-1}$. The other two capture the risk premium. The term $\Phi_1 \omega(\Delta_t)$ is the risk premium associated with the realization of time-t+1 dividend uncertainty, to which each investor is exposed according to the breadth Δ_t of her time-t portfolio. Finally, Φ_0 equals the sum of the expected discounted value of risk premia due to future realizations of dividend innovations and Δ_t.

We emphasize that the risk premium decreases with Δ_t and is common for all securities. Alternatively phrased, increases in capital movements across locations are correlated with higher prices for all risky securities (and hence lower expected excess returns). Importantly, these movements in the prices of risky securities are uncorrelated with movements in aggregate output or the interest rate, which are both constant by construction.

A further immediate implication of equation (40) is that the presence of repeated shocks to participation costs introduces comovement in security prices that exceeds the correlation of their dividends. Indeed, taking two securities j and k, and noting that the symmetry of the setup implies $Var\left(D_{t,j}\right) = Var\left(D_{t,k}\right)$, we can use equation (40) to compute

$$corr\left(P_{t,j}, P_{t,i}\right) = \frac{Var(\Phi_1 \omega\left(\Delta_t\right)) + \phi^2 cov\left(D_{t,j}, D_{t,k}\right)}{Var(\Phi_1 \omega\left(\Delta_t\right)) + \phi^2 Var\left(D_{t,j}\right)} > \frac{cov\left(D_{t,j}, D_{t,k}\right)}{Var\left(D_{t,j}\right)} = corr\left(D_{t,j}, D_{t,i}\right).$$

The intuition is that movements in market integration cause common movements in the pricing of risk which make prices more correlated (and volatile) than the underlying dividends.

We collect some basic properties of the price due to the randomness in Δ_t in the following proposition.

Proposition 5 *(i) $P_{t,i}$ increases with Δ_t, and therefore $corr(P_{t,i}, \Delta_t) > 0$;*
(ii) $E_t[P_{i,t+1} - (1+r)P_{i,t}]$ decreases with Δ_t;
(iii) $corr(P_{t,i}, P_{t,j}) > corr(D_{t,i}, D_{t,j})$;
(iv) The portion Φ_0 of the risk premium is higher than the one obtaining for Δ_t constant and equal to $E[\Delta]$. The unconditional expected price is lower for the deterministic Δ_t.

7.2 Multiple arcs on the circle

In our baseline model we have assumed that investors participate in markets spanning a single arc of length Δ around their "home" location. Extending the results to the general case where investors can choose to participate on multiple, disconnected arcs (as illustrated on the right-most graph of Figure 5) is straightforward and involves essentially no new insights. In this section we briefly sketch how to extend the results of the baseline model to this case and we show that allowing for this extra generality introduces an additional source of non-concavity into an investor's optimization problem.

To start, we introduce the function

$$v(I) = \min_{N_i, \overrightarrow{a}_i, \overrightarrow{\Delta}_i, G_j^{(i)}} Var\left(\int_0^1 D_j dG_j^{(i)}\right)$$
$$s.t. \quad I = F\left(N_i, \overrightarrow{a}_i, \sum_{n=1..N_i} \Delta_{i,n}\right).$$

In words, the function $v(I)$ is the minimal variance, per share purchased, of the portfolio payoff that can be obtained by an investor who is willing to spend an amount I on participation costs. Proceeding similarly to Section 4.2 under the assumption that $P_j = P$ for all j, the facts that U is exponential and all D_j are normally distributed imply that maximizing

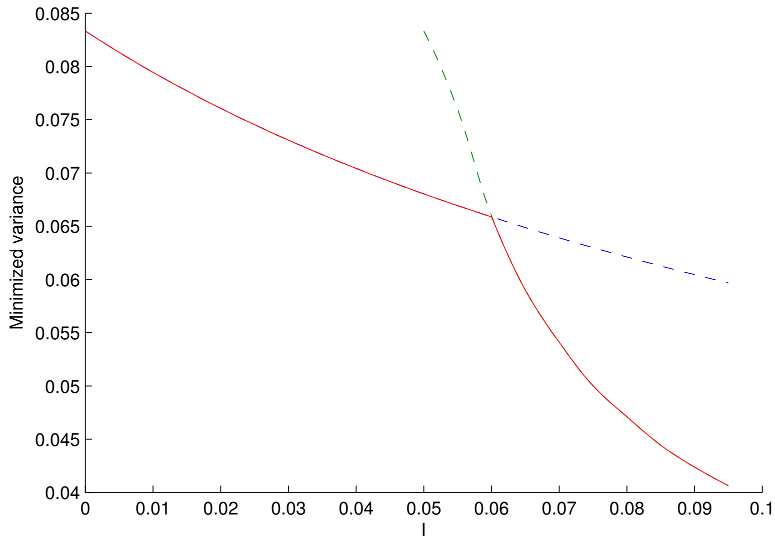


Figure 13: Numerical example to illustrate that $v(I)$ is non-convex. The figure depicts two (dotted) lines and the minimum of the two lines (solid line). The first dotted line starts at $I = 0$ and depicts the minimal variance that can be attained when participation costs are equal to I and the investor chooses to participate only on a single arc centered at her home location. The second dotted line starts at $I = 0.05$, i.e. at the minimum expenditure required to invest in two distinct arcs. This second dotted line depicts the minimal variance that can be attained when participation costs are equal to I and the investor can participate on two separate arcs with locations and lengths chosen so as to minimize variance. The function $v(I)$ (the minimum of the two dotted lines) is given by the solid line. This example helps illustrate that the function $v(I)$ is in general not convex. For this example we chose $\sigma = 1, g(x) = 0.1 \times \left((1-x)^{-6} - 1\right), f(y) = 0.05 + 0.005 \times \left((\frac{1}{2} - y)^{-2} - \frac{1}{0.25}\right)$. (For $I > 0.1$ the function $v(I)$ would in general exhibit further kinks at the critical values I_n, $n = 2, 3..$ where the investor is indifferent between choosing n or $n+1$ distinct arcs.)

utility over the choice of N_i, $\{a_{i,1}; ..; a_{i,N_i}\}$, $\{\Delta_{i,1}; ..; \Delta_{i,N_i}\}$, $G_j^{(i)}$, and w_i^f is equivalent to solving

$$\max_{w_i^f, I} Pw_i^f + \left(1 - w_i^f\right) \int_0^1 E[D_j] dG_j^{(i)} - \frac{\gamma}{2} \left(1 - w_i^f\right)^2 v(I) - I, \tag{43}$$

subject to the constraint $I \leq P\left(1 - \chi\left(1 - w_i^f\right)\right)$. Given that $E[D_j] = 1$, equation (43) can be rewritten as

$$V = \max_{I, w_i^f} Pw_i^f + \left(1 - w_i^f\right) - \frac{\gamma}{2}\left(1 - w_i^f\right)^2 v(I) - I. \tag{44}$$

It is useful to note that (43) is identical to (19) in the special case where investors participate in markets spanning a single arc of length Δ around their "home" location. Indeed, in that special case $v(I) = \omega\left(g^{-1}\left(\frac{I}{\kappa}\right)\right)$. Since $g(\Delta)$ is a monotone function, inspection reveals that (43) is identical to (19). In particular, the first order conditions of the problem (44) become identical to the first order conditions (20) and (21) upon subsituting $I = \kappa g(\Delta)$ and $v'(I) = \frac{\omega'\left(g^{-1}\left(\frac{I}{\kappa}\right)\right)}{\kappa g'\left(g^{-1}\left(\frac{I}{\kappa}\right)\right)} = \frac{\omega'(\Delta)}{\kappa g'(\Delta)}$. Accordingly, all our conclusions of the previous sections remain unaltered: We can simply use the mappings $I = \kappa g(\Delta)$ and $\Delta = g^{-1}\left(\frac{I}{\kappa}\right)$ to translate implications for I into implications for Δ and vice versa.

In the general case where investors' portfolios are invested on disconnected arcs, the function $v(I)$ is different from $\omega\left(g^{-1}\left(\frac{I}{\kappa}\right)\right)$. Interestingly, the function $v(I)$, which is convex in the single-arc case,[16] will in general be non-convex with kinks at the expenditure levels I_n where it becomes optimal to invest in $n+1$ rather than n distinct arcs. Figure 13 provides an illustration. This non-convexity of $v(I)$, which may arise when (and only when) investors participate in markets located on multiple distinct arcs, implies an additional reason for the maximization problem (44) to be non-concave. This reason is distinct from the non-concavity arising from the interaction between leverage and participation decisions that we identified in Section 4.3, and implies an additional reason why a symmetric equilibrium may not exist. Indeed, one can essentially repeat the proof of Proposition 2 to establish the existence of asymmetric equilibria, where strategies involving no leverage, low participation costs I, and low Sharpe ratio coexist with strategies involving high leverage, high participation costs I, and high Sharpe ratio.

[16] To see this note that $v'(I) = \frac{\omega'(\Delta)}{\kappa g'(\Delta)}$ where $\Delta(I) = g^{-1}\left(\frac{I}{\kappa}\right)$. Differentiating again gives $v''(I) = \frac{1}{\kappa}\frac{\omega''(\Delta)g'(\Delta) - \omega'(\Delta)g''(\Delta)}{(g'(\Delta))^2}\Delta'(I) > 0$.

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A Proofs

Proof of Lemma 1. Property 2 follows immediately from integrating (2). To show property 3, note that, for any $i \in (0,1)$, $\lim_{d(i,j)\to 0} D_j = \lim_{j\to i} D_j = D_i$ a.s. by the continuity of the Brownian motion. Continuity at 0 follows from the fact that $B_0 = B_1$.

We turn now to property 1. Since $E(B_i) = 0$ for all $i \in [0,1]$, $E(D_j) = 1$. To compute $cov(D_i, D_j)$ we start by noting that $cov\,(B_s, B_t) = E(B_s B_t) = s\,(1-t)$ for $s \le t$. Therefore, for any $t \in [0,1]$,

$$\begin{aligned}\int_0^1 E\left(B_t B_u\right) du &= \int_0^t u(1-t)du + \int_t^1 t(1-u)du \\ &= \frac{1}{2}(1-t)t^2 + \frac{1}{2}(1-t)^2 t = \frac{t\,(1-t)}{2}.\end{aligned} \tag{45}$$

Accordingly,

$$\begin{aligned}Var\left(\int_0^1 B_u du\right) &= E\left[\left(\int_0^1 B_u du\right)^2\right] = E\left[\left(\int_0^1 B_u du\right)\left(\int_0^1 B_t dt\right)\right] \\ &= \int_0^1 \left(\int_0^1 E(B_u B_t) du\right) dt = \int_0^1 \frac{t\,(1-t)}{2} dt = \frac{1}{12},\end{aligned} \tag{46}$$

where the second line of (46) follows from Fubini's Theorem and (45). Combining (46) and (45) gives

$$\begin{aligned}\frac{1}{\sigma^2} Var\left(D_t\right) &= Var\left(B_t\right) + Var\left(\int_0^1 B_u du\right) - 2cov\left(B_t, \int_0^1 B_u du\right) \\ &= t\,(1-t) + \frac{1}{12} - 2\int_0^1 E(B_t B_u) du = \frac{1}{12}.\end{aligned} \tag{47}$$

This calculation finishes the proof of property 1. For property 4, take any $s \le t$ and use (45) and (46) to obtain

$$\begin{aligned}\frac{cov\left(D_s, D_t\right)}{\sigma^2} &= cov\left(B_s - \int_0^1 B_u du, B_t - \int_0^1 B_u du\right) \\ &= E(B_s B_t) - E\left(B_s \int_0^1 B_u\right) du - E\left(B_t \int_0^1 B_u\right) du + \frac{1}{12} \\ &= s\,(1-t) - \frac{s\,(1-s)}{2} - \frac{t\,(1-t)}{2} + \frac{1}{12} \\ &= \frac{(s-t)(1+s-t)}{2} + \frac{1}{12}.\end{aligned} \tag{48}$$

This establishes property 4. ■

Proof of Proposition 1. We start by establishing the following lemma.

Lemma 2 *The (bounded-variation) function L with $L_{-\frac{\Delta}{2}^-} = 0$ and $L_{\frac{\Delta}{2}} = 1$ that minimizes $Var\left(\int_{-\frac{\Delta}{2}^-}^{\frac{\Delta}{2}} D_j\, dL_j\right)$ is given by (14). Moreover, the minimal variance is equal to $\omega\left(\Delta\right)$.*

Proof of Lemma 2. To simplify notation, we prove a "shifted" version of the lemma, namely finding the minimal-variance portfolio on $[0,\Delta]$ rather than $[-\frac{\Delta}{2},\frac{\Delta}{2}]$. The two versions are clearly equivalent, since covariances depend only on the distances between locations, rather than the locations themselves.

We start by definining $q(d) = \frac{1}{12} - \frac{d(1-d)}{2}$ and therefore $q'(d) = -\frac{1}{2} + d$. In light of (3), $q(d) = \frac{1}{\sigma^2} cov(D_i, D_j)$ whenever $d(i,j) = d$. If $L_u = \int_{0^-}^{u} dL_u$ is a variance-minimizing portfolio of risky assets, it must be the case that the covariance between any gross return $R_s = \frac{D_s}{P}$ for $s \in [0,\Delta]$ and the portfolio $\int_{0^-}^{\Delta} R_u dL_u = \int_{0^-}^{\Delta} \frac{D_u}{P} dL_u$ is independent of s. Thus, the quantity

$$\begin{aligned} \frac{1}{\sigma^2} cov\left(D_s, \int_{0^-}^{\Delta} D_u dL_u\right) &= \frac{1}{\sigma^2}\left[\int_{0^-}^{s} cov(D_s, D_u)\, dL_u + \int_{s}^{\Delta} cov(D_s, D_u)\, dL_u\right] = \\ &= \int_{0^-}^{s} q(s-u)\, dL_u + \int_{s}^{\Delta} q(u-s)\, dL_u \end{aligned} \tag{49}$$

is independent of s. Letting $\tilde{L}(s) = 1 - L(s)$ and integrating by parts we obtain

$$\int_{0^-}^{s} q(s-u)\, dL_u = L(s)\, q(0) - L\left(0^-\right) q(s) + \int_{0^-}^{s} L_u q'(s-u)\, du \tag{50}$$

$$\int_{s}^{\Delta} q(u-s)\, dL_u = \tilde{L}(s)\, q(0) - \tilde{L}(\Delta)\, q(\Delta - s) + \int_{s}^{\Delta} \tilde{L}_u q'(u-s)\, du. \tag{51}$$

Using (50) and (51) inside (49) and recognizing that $q(0) = \frac{1}{12}$, $L(0^-) = 0$, and $\tilde{L}(\Delta) = 0$, we obtain that (49) equals

$$Q(s) \equiv \frac{1}{12} + \int_{0^-}^{s} L_u q'(s-u)\, du + \int_{s}^{\Delta} \tilde{L}_u q'(u-s)\, du. \tag{52}$$

This expression is independent of $s \in [0,\Delta]$ if and only if $Q'(s) = 0$. A straightforward computation yields

$$\begin{aligned} Q'(s) &= \int_{0^-}^{s} L_u q''(s-u)\, du - \int_{s}^{\Delta} \tilde{L}_u q''(u-s)\, du + L_s q'(0) - \tilde{L}_s q'(0) \\ &= \int_{0^-}^{\Delta} L_u du - \Delta + s - L_s + \frac{1}{2} = 0, \end{aligned} \tag{53}$$

where we used $q'(0) = -\frac{1}{2}$, $q'' = 1$, and $\tilde{L}(s) = 1 - L(s)$. Since (53) needs to hold for all $s \in [0,\Delta]$, it must be the case that $L_s = A + s$ for an appropriate constant A. To determine A, we subsitute $L_s = A + s$ into (53) and solve for A to obtain

$$A = \frac{1-\Delta}{2}.$$

It is immediate that the standardized portfolio corresponding to the solution L we computed is L^* of (14).

Using the variance-minimizing portfolio inside (52), implies after several simplifications, that $Q = \frac{1}{12}(1-\Delta)^3$ and hence $cov\left(D_s, \int_{0^-}^{\Delta} D_u dL_u\right) = Q\sigma^2 = \omega(\Delta)$. Accordingly,

$$
\begin{aligned}
Var\left(\int_{0^-}^{\Delta} D_u dL_u\right) &= cov\left(\int_{0^-}^{\Delta} D_s dL_s, \int_{0^-}^{\Delta} D_u dL_u\right) = \int_{0^-}^{\Delta} cov\left(D_s, \int_{0^-}^{\Delta} D_u dL_u\right) dL_s \\
&= \omega(\Delta) \int_{0^-}^{\Delta} dL_s = \omega(\Delta).
\end{aligned}
$$

■

With Lemma 2 in hand it is possible to confirm that the allocations and prices of Proposition 1 constitute a symmetric equilibrium — assuming that one exists. We already argued that the all agents choose the same standardized portfolio (as agent $\frac{\Delta}{2}$). Furthermore, since in a symmetric equilibrium all agents must hold the same allocation of bonds, clearing of the bond market requires $w_i^f = 0$ for all i. By equation (20), $w_i^f = 0$ is supported as an optimal choice for an investor only if $P_i = P$ is given by (17). Similarly, in light of (21), equation (16) is a necessary optimality condition for the interval Δ^*. Since the values of P and Δ^* implied by (17) and (16) are unique, they are necessarily the equilibrium values of P and Δ^* that characterize a symmetric equilirium.

Since the choices Δ^*, $w_i^f = 0$, and $G_{i+j}^{(i)} = L_j^*$ are optimal given prices $P_i = P$, it remains to show that markets clear. We already addressed bond-market clearing. To see that the stock markets clear, we start by noting that, since $P_i = W_{0,i} = P$ for all i, the market clearing condition amounts to $\int_{i\in[0,1)} dG_j^{(i)} = 1$. We have $\int_{i\in[0,1)} dG_j^{(i)} = \int_{i\in[0,1)} dL_{j-i}^* = \int_{j\in[0,1)} dL_j^* = 1$. ■

Proof of Proposition 2. Let $w^*(P)$ denote the set of optimal w_i^f solving the maximization problem (19) when the price in all markets is P. We first note that the assumption that no symmetric equilibrium exists implies that there exists no P such that $0 \in w^*(P)$. (If such a P existed, then we could simply repeat the arguments of Proposition 1 to establish the existence of a symmetric equilibrium with price $P_i = P$, and interval choice $\Delta_i = \Delta^*(P)$).

We next show that since there exists no P such that $0 \in w^*(P)$, it follows that $w^*(P)$ cannot be single-valued for all P. We argue by contradiction. Suppose to the contrary that $w^*(P)$ is single-valued. Since the theorem of the maximum implies that $w^*(P)$ is a upper-hemicontinuous correspondence continuous function, it follows that $w^*(P)$ is actually a continuous function. Inspection of (19) shows that $w^*(1) = 1$. Moreover, as $P \to -\infty$, the optimal solution to (19) subject to the constraint (7) becomes negative: $w^*(-\infty) < 0$. Then an application of the intermediate value theorem gives the existence of P such that $w^*(P) = 0$, a contradiction.

Combining the facts that a) there exists no P such that $0 \in w^*(P)$, b) $w^*(P)$ is multi-valued for at least one value of P, and c) $w^*(P)$ is upper-hemicontinuous, implies that there exists at least one P such that $\{w_1, w_2\} \in w^*(P)$ with $w_1 > 0$ and $w_2 < 0$. An implication of the necessary first-order condition for the optimality of the interval choice $\Delta^*(P)$ is that $\Delta^*(P)$ is also multi-valued with $\Delta_1 < \Delta_2$. Furthermore, since prices in all locations are equal, the (standardized) optimal portfolio of an agent choosing Δ_k is the variance-minimizing portfolio of Proposition 1, denoted $L^{*,k}$.

From this point onwards, an equilibrium can be constructed as follows. By definition, the tuples $\{\Delta_1, w_1, dL^{*,1}\}$ and $\{\Delta_2, w_2, dL^{*,2}\}$ are optimal. Hence it only remains to confirm that asset markets clear. Define $\pi \equiv -\frac{w_2}{w_1 - w_2} \in (0,1)$. By construction, $\pi w_1 + (1-\pi) w_2 = 0$ and, therefore, if in every location π agents choose $\{\Delta_1, w_1, dL^{*,1}\}$ and the remaining fraction $(1-\pi)$ choose $\{\Delta_2, w_2, dL^{*,2}\}$, then the bond market clears by construction. To see that the stock markets clear, we start by noting that, since $P_i = W_{0,i} = P$, the market clearing condition for stock i amounts to

$$\pi \int_{[0,1]} dL^{*,1}_{j-i} + (1-\pi) \int_{[0,1]} dL^{*,2}_{j-i} = 1,$$

which holds because $L^{*,k}_j$ for $k \in \{1,2\}$ is a measure on the circle. ■

Remark 1 *The existence proof of an asymmetric equilibrium (when a symmetric equilibrium fails to exist) obtains whether the leverage constraint binds or not for some subset of agents.*

Lemma 3 *Consider an investor located at $i \notin \left[-\frac{k}{2}; \frac{k}{2}\right]$, and therefore investing in markets $[i - \frac{\bar{\Delta}}{2}, i + \frac{\bar{\Delta}}{2}]$. Suppose that $P(x)$ is continuously differentiable everywhere on $[i - \frac{\bar{\Delta}}{2}, i + \frac{\bar{\Delta}}{2}]$. With $dX_l^{(i)}$ the number of shares purchased on the account of an investor at i in market l and $j \equiv i - \frac{\bar{\Delta}}{2}$,*

$$X^{(i)}_{j+\bar{\Delta}} = \frac{1}{\gamma \omega(\bar{\Delta})} \left[1 - \frac{1-\bar{\Delta}}{2} \left(P_j + P_{j+\bar{\Delta}}\right) - \int_j^{j+\bar{\Delta}} P_u \, du\right]. \tag{54}$$

Furthermore, the function X is given by

$$X^{(i)}_{j+l} = \frac{P'_{j+l}}{\gamma\sigma^2} + X^{(i)}_{j+\overline{\Delta}} \frac{1 - \overline{\Delta} + 2l}{2} + \frac{1}{1-\overline{\Delta}} \frac{P_{j+\overline{\Delta}} - P_j}{\gamma\sigma^2}. \tag{55}$$

If an investor is located at $i \in [-\frac{k}{2}, \frac{k}{2}]$ and only invests in market i then the respective demand for risky asset i is given by

$$\widehat{X}^{(i)}_i = \frac{1}{\gamma\omega(0)} (1 - P_i). \tag{56}$$

Proof of Lemma 3. Notice that optimization problem of agent i is equivalent to

$$\max_X P_i + \int_{j^-}^{j+\bar{\Delta}} (1 - P_u) \, dX_u - \frac{\gamma}{2} Var \left(\int_{j^-}^{j+\bar{\Delta}} D_u dX_u\right) \tag{57}$$

Thus, the first-order condition requires that

$$\gamma \operatorname{cov}\left(D_s, \int_{j^-}^{j+\bar{\Delta}} D_u dX_u\right) = 1 - P_s \tag{58}$$

for all $s \in [j, j+\bar{\Delta}]$. Letting $q(d)$ be defined as in Lemma 2 we can rewrite (58) as

$$\int_{j^-}^{s} q(s-u) \, dX_u + \int_s^{j+\Delta} q(u-s) \, dX_u = \frac{1-P_s}{\gamma\sigma^2}. \tag{59}$$

Let $\widetilde{X}(s) = X\left(j+\bar{\Delta}\right) - X(s)$ and integrating by parts we obtain

$$\int_{j^-}^{s} q(s-u)\,dX_u = X(s)\,q(0) - X\left(j^-\right) q(s) + \int_{j}^{s} X_u q'(s-u)\,du \tag{60}$$

$$\int_{s}^{j+\bar{\Delta}} q(u-s)\,dX_u = \widetilde{X}(s)\,q(0) - \widetilde{X}\left(\bar{\Delta}\right) q(\Delta - s) + \int_{s}^{j+\bar{\Delta}} \widetilde{X}_u q'(u-s)\,du \tag{61}$$

Substituting (60) and (61) into (59), recognizing that $q(0) = \frac{1}{12}$, $X\left(j^-\right) = 0$, and $\widetilde{X}\left(j+\bar{\Delta}\right) = 0$, we obtain

$$\frac{1}{12}X\left(j+\bar{\Delta}\right) + \int_{j}^{s} X_u q'(s-u)\,du + \int_{s}^{j+\bar{\Delta}} \widetilde{X}_u q'(u-s)\,du = \frac{1-P_s}{\gamma\sigma^2}. \tag{62}$$

Since this relation must hold for all s, we may differentiate both sides of (62) to obtain

$$\int_{j}^{s} X_u q''(s-u)\,du - \int_{s}^{j+\bar{\Delta}} \widetilde{X}_u q''(u-s)\,du + X_s q'(0) - \widetilde{X}_s q'(0) = -\frac{P_s'}{\gamma\sigma^2}. \tag{63}$$

This equation holds for all $s \in (j, j+\bar{\Delta})$. Noting that $q'' = 1$, $q'(0) = -\frac{1}{2}$, $\widetilde{X}(s) = X\left(j+\bar{\Delta}\right) - X(s)$, and using (63) to solve for X_s yields

$$X_s = \int_{j}^{j+\bar{\Delta}} X_u du + \left(s - j + \frac{1}{2} - \bar{\Delta}\right) X\left(j+\bar{\Delta}\right) + \frac{P_s'}{\gamma\sigma^2}. \tag{64}$$

Integrating (64) from j to $j+\bar{\Delta}$ and solving for $\int_j^{j+\bar{\Delta}} X_u du$ leads to

$$\int_{j}^{j+\bar{\Delta}} X_u du = \frac{1}{1-\bar{\Delta}}\left[X\left(j+\bar{\Delta}\right)\bar{\Delta}\left(\frac{1-\bar{\Delta}}{2}\right) + \frac{P\left(j+\bar{\Delta}\right) - P(j)}{\gamma\sigma^2}\right], \tag{65}$$

so that

$$\begin{aligned} X_s = {} & \frac{1}{1-\bar{\Delta}}\left[X\left(j+\bar{\Delta}\right)\bar{\Delta}\left(\frac{1-\bar{\Delta}}{2}\right) + \frac{P\left(j+\bar{\Delta}\right) - P(j)}{\gamma\sigma^2}\right] + \\ & \left(s - j + \frac{1}{2} - \bar{\Delta}\right) X\left(j+\bar{\Delta}\right) + \frac{P_s'}{\gamma\sigma^2}. \end{aligned} \tag{66}$$

Evaluating (62) at $s = j+\bar{\Delta}$, and noting that $q'(s) = -\frac{1}{2} + s$ leads to

$$\frac{1}{12}X\left(j+\bar{\Delta}\right) + \int_{j}^{j+\bar{\Delta}} X_u\left[-\frac{1}{2} + \left(\bar{\Delta} - u\right)\right] du = \frac{1-P_{j+\bar{\Delta}}}{\gamma\sigma^2}. \tag{67}$$

An implication of (64) is that $X_u = X_j + \frac{P_u' - P_j'}{\gamma\sigma^2} + X\left(j+\bar{\Delta}\right)u$. Using this expression for X_u inside (67), carrying out the requisite integrations and using integration by parts to express $\int_j^{j+\bar{\Delta}}\left(\frac{P_u'}{\gamma\sigma^2}\right)u\,du = \frac{P_{j+\bar{\Delta}}}{\gamma\sigma^2}(j+\bar{\Delta}) - \frac{P_j}{\gamma\sigma^2}j - \int_j^{j+\bar{\Delta}}\frac{P_u}{\gamma\sigma^2}du$, leads (after some simplifications) to

$$\begin{aligned} & X\left(j+\bar{\Delta}\right)\left(\frac{1}{12} + \frac{\bar{\Delta}^3}{6} - \frac{\bar{\Delta}^2}{4}\right) - \frac{\bar{\Delta}\left(1-\bar{\Delta}\right)}{2}\left(X_j - \frac{P_j'}{\gamma\sigma^2}\right) - \frac{P_{j+\bar{\Delta}} - P_j}{2\gamma\sigma^2} + \int_{j^-}^{j+\bar{\Delta}} \frac{P_u - P_j}{\gamma\sigma^2}du \\ & = \frac{1-P_{\bar{\Delta}}}{\gamma\sigma^2}. \end{aligned} \tag{68}$$

Finally, evaluating (64) at j gives

$$\left(X_j - \frac{P_j'}{\gamma\sigma^2}\right) = \int_j^{j+\bar{\Delta}} X_u du + \left(\frac{1}{2} - \bar{\Delta}\right) X\left(j + \bar{\Delta}\right). \tag{69}$$

Equations (65), (68), and (69) are three linear equations in three unknowns. Solving for $X(j+\bar{\Delta})$ and using the definition of $\omega\left(\bar{\Delta}\right)$ leads to (54). Equation (66) simplifies to (55). Finally, (56) is a direct consequence of (58) when $\bar{\Delta} = 0$. ■

Proof of Proposition 3. For any $j \in \left(\frac{k}{2}, \frac{1}{2}\right]$ and $l \in \left(-\frac{\bar{\Delta}}{2}, \frac{\bar{\Delta}}{2}\right)$, we have from Lemma 3:

$$\begin{aligned}
X^{(j)}_{j-\frac{\bar{\Delta}}{2}} &= \frac{P'_{j-\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2} + \frac{P_{j-\frac{\bar{\Delta}}{2}} - P_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2(1-\Delta)} + \frac{1-\Delta}{2} X^{(j)}_{j+\frac{\bar{\Delta}}{2}} \\
dX^{(j)}_{j+l} &= \left(\frac{P''_{j+l}}{\gamma\sigma^2} + X^{(j)}_{j+\frac{\bar{\Delta}}{2}}\right) dl \\
X^{(j)}_{j+\frac{\bar{\Delta}}{2}} - X^{(j)}_{\left(j+\frac{\bar{\Delta}}{2}\right)^-} &= -\frac{P'_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2} - \frac{P_{j-\frac{\bar{\Delta}}{2}} - P_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2(1-\Delta)} + \frac{1-\Delta}{2} X^{(j)}_{j+\frac{\bar{\Delta}}{2}}.
\end{aligned}$$

Specialize the first equation to $j = \frac{1}{2} + \frac{\Delta}{2}$, the second to $j = \frac{1}{2} - l$ for all $l \in \left(-\frac{\bar{\Delta}}{2}, \frac{\bar{\Delta}}{2}\right)$, and the third to $j = \frac{1}{2} - \frac{\Delta}{2}$ and aggregate to obtain the total demand for asset $\frac{1}{2}$:

$$1 = \frac{1-\Delta}{2} X^{\left(\frac{1}{2}+\frac{\bar{\Delta}}{2}\right)}_{\frac{1}{2}+\bar{\Delta}} + \frac{1-\Delta}{2} X^{\left(\frac{1}{2}-\frac{\bar{\Delta}}{2}\right)}_{\frac{1}{2}} + \int_{-\frac{\bar{\Delta}}{2}}^{\frac{\bar{\Delta}}{2}} X^{\left(\frac{1}{2}-l\right)}_{\frac{1}{2}-l+\frac{\Delta}{2}} dl + \frac{P''_{\frac{1}{2}}}{\gamma\sigma^2}\Delta + \frac{P_{\frac{1}{2}-\bar{\Delta}} + P_{\frac{1}{2}+\bar{\Delta}} - 2P_{\frac{1}{2}}}{2\gamma\sigma^2(1-\Delta)}.$$

Suppose now that $P_j \geq 1 - \gamma\omega(\bar{\Delta})$ on $\left[\frac{1}{2} - \Delta, \frac{1}{2} + \Delta\right]$, with strict inequality on a positive measure set. It then follows from equation (54) that $X^{(j)}_{j+\frac{\Delta}{2}} \leq 1$, so that

$$0 < \frac{P''_{\frac{1}{2}}}{\gamma\sigma^2}\Delta + \frac{P_{\frac{1}{2}-\bar{\Delta}} + P_{\frac{1}{2}+\bar{\Delta}} - 2P_{\frac{1}{2}}}{2\gamma\sigma^2(1-\Delta)}.$$

This inequality contradicts the assumption that P is maximized at $\frac{1}{2}$. ■

Proof of Proposition 4. We adopt a guess-and-verify approach. We start by noting that the beginning-of-period wealth of investor i at time $t+1$ is $W_{t+1,i} \equiv \int_0^1 \left(P_{t+1,j} + D_{t+1,j}\right) dX^{(i)}_{t,j} + X^{(i)}_{B,t}$. We then conjecture that, as long as

$$F(\Delta) = \frac{M}{\gamma} + \frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega\left(\Delta_t\right) \tag{70}$$

for some $M > -\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega\left(\Delta_u\right)$, (ii) and (iii) obtain. We show at the end that the function F can be chosen to ensure (i).

We also conjecture and verify that investors' holdings of risky assets $X_{t,j}^{(i)}$ coincide with $G_{t,j}^{(i)}$ of Proposition 1, and that their bond holdings equal

$$X_{B,t}^{(i)} = W_{t,i} - (1+r)\,\overline{P}_{t,i} - r\Phi_t, \tag{71}$$

where $\overline{P}_{t,i} \equiv \int_0^1 P_{t,j} dX_{t,j}^{(i)}$ is the average price that investor i pays for her portfolio. Here, to simplify notation, we defined $\Phi_t \equiv \Phi_1\omega(\Delta_t) + \Phi_0$.

We first ensure that with these postulates markets clear. Clearly, all risky markets clear, since the holdings of risky assets are the same as in Proposition 1. To show that bond markets clear, we proceed inductively. First we note that investors are endowed with no bonds at time zero. Hence $\int_0^1 X_{B,-1}^{(i)} di = 0$ and therefore $\int_0^1 W_{0,i} di = \int_0^1 (P_{0,i} + D_{0,i})\, di$. Next we postulate that $\int_0^1 X_{B,t-1}^{(i)} di = 0$, so that $\int_0^1 W_{t,i} di = \int_0^1 P_{t,i} di + \int_0^1 D_{t,j} dj$. Integrating our postulate (71) for $X_{B,t}^{(i)}$ across all investors, we obtain

$$\int_0^1 X_{B,t}^{(i)} di = \int_0^1 W_{t,i} di - (1+r)\int_0^1 \overline{P}_{t,i} di - r\Phi_t. \tag{72}$$

We next note that that (a) $\int_0^1 D_{t,j} dj = 1$, by construction of the dividend process; (b) $\int_0^1 W_{t,i} di = \int_0^1 P_{t,i} di + \int_0^1 D_{t,j} dj = r^{-1} - \Phi_t + 1$, using the induction hypothesis, (40), and (a); and (c) $\int_0^1 \overline{P}_{t,i} di = \int_0^1 P_{t,i} di = r^{-1} - \Phi_t$. Using these three facts, it follows immediately that the right-hand side of (72) is zero, so that the bond market clears.

If investors set their bond holdings according to (71), then their budget constraint implies a consumption of

$$c_{t,i} = W_{t,i} - \frac{1}{1+r} X_{B,t}^{(i)} - \overline{P}_{t,i} - F_t. \tag{73}$$

Using the definition of $W_{t,i}$ and market clearing condition for bond holdings inside (73), and integrating across i implies that the market for consumption goods clears: $\int_0^1 c_{t,i} di = 1 - F_t$.

Having established market clearing given the postulated policies and prices, we next turn to optimality. Equation (73) implies

$$\begin{aligned} c_{t+1,i} - c_{t,i} &= W_{t+1,i} - W_{t,i} - \frac{1}{1+r}\left(X_{B,t+1}^{(i)} - X_{B,t}^{(i)}\right) - \left(\overline{P}_{t+1,i} - \overline{P}_{t,i}\right) - (F_{t+1} - F_t) \\ &= \left(\frac{r}{1+r}\right)(W_{t+1,i} - W_{t,i}) + \frac{r}{1+r}(\Phi_{t+1} - \Phi_t) - (F_{t+1} - F_t)\,, \end{aligned} \tag{74}$$

where the second line follows from (71). We next use the definition of $W_{t,i}$ and (71) to obtain

$$\begin{aligned} W_{t+1,i} - W_{t,i} &= \int_0^1 (P_{t+1,j} + D_{t+1,j})\, dX_{t,j}^{(i)} + X_{B,t}^{(i)} - W_{t,i} \\ &= \int_0^1 (P_{t+1,j} + D_{t+1,j})\, dX_{t,j}^{(i)} - (1+r)\,\overline{P}_{t,i} - r\Phi_t. \end{aligned} \tag{75}$$

Substituting (75) into (74) and using (40) and (71) leads to

$$c_{t+1,i} - c_{t,i} = \left(\frac{r}{1+r}\right)\left[(1+\phi)\int_0^1 D_{t+1,j} dX_{t,j}^{(i)} - (1+r)\,\phi\int_0^1 D_{t,j} dX_{t,j}^{(i)} - 1\right] - (F_{t+1} - F_t)\,. \tag{76}$$

Next use the fact $\overline{D}_{t+1,j} = \rho \overline{D}_{t,j} + (1-\rho)\,\overline{\varepsilon}_{t+1,j}$ along with $\phi = \frac{\rho}{1+r-\rho}$, $(1+\phi)\,\rho = (1+r)\,\phi$, and $(1+\phi)\,(1-\rho) = (1-r\phi)$ inside (76) to arrive at

$$c_{t+1,i} - c_{t,i} = \left(\frac{r}{1+r-\rho}\right)(1-\rho)\int_0^1 (\varepsilon_{t+1,j}-1)\, dX_{t,j}^{(i)} - (F_{t+1}-F_t)\,. \tag{77}$$

Having established (77), the dynamics of agent i's consumption under our postulate, we next turn attention to the Euler equations, starting with the bond Euler equation

$$1 = \beta\,(1+r)\, E_t e^{-\gamma(c_{t+1,i}-c_{t,i})}. \tag{78}$$

Substituting (77) into (78) and noting that $\int_0^1 (\varepsilon_{t+1,j}-1)\, dX_{t,j}^{(i)}$ is normally distributed with mean zero and variance $\omega(\Delta_t)$ gives

$$1 = \beta\,(1+r)\, e^{\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta_t) - \gamma F_t} E_t\left(e^{\gamma F_{t+1}}\right). \tag{79}$$

Now suppose that for any r and a given desired distribution $\Psi(\Delta)$ we set

$$F_t(\Delta_t; r) = \frac{M}{\gamma} + \frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2\,\omega(\Delta_t)\,. \tag{80}$$

Then equation (79) can be written as (39). Since $(1+r)\, E e^{\frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta)}$ is equal to 1 when $r=0$ and increases monotonically to infinity as r increases, it follows that there exists a unique positive r such that equation (39) holds. For that value of r, all investors' bond Euler equations are satisfied.

Finally, we need to determine Φ_t so as to ensure that the Euler equations for risky assets hold, i.e., that

$$P_{t,j} = \beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\,(P_{t+1,j}+D_{t+1,j})\right]. \tag{81}$$

To that end, we use (40) and (37) to express (81) as

$$\frac{1}{r} - \Phi_t + \phi(D_{t,j}-1) = \beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\left(\frac{1}{r} - \Phi_{t+1} + (1+\phi)\,(\rho D_{t,j} + (1-\rho)\,\varepsilon_{t+1,j}) - \phi\right)\right]. \tag{82}$$

We next note that

$$\beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\right](1+\phi)\,\rho D_{t,j} = \frac{(1+\phi)\,\rho}{1+r} D_{t,j} = \phi D_{t,j} \tag{83}$$

using (78). Equation (83) simplifies (82) to

$$\begin{aligned}
\frac{1}{r} - \Phi_t - \phi &= \beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\left(\frac{1}{r} - \Phi_{t+1} + (1+\phi)\,(1-\rho)\,\varepsilon_{t+1,j}\right)\right] \\
&= \frac{1}{r(1+r)} - \beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\Phi_{t+1}\right] + \frac{1}{1+r}\left(-\phi + (1+\phi)\,(1-\rho)\right) \\
&\quad + (1+\phi)\,(1-\rho)\,\beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}(\varepsilon_{t+1,j}-1)\right].
\end{aligned} \tag{84}$$

Using (77), Stein's Lemma, the fact that $cov\left(\int_0^1 (\varepsilon_{t+1,j}-1)\,dX_{t,j}^{(i)}, \varepsilon_{t+1,j}\right) = \omega(\Delta)$ (see Proposition 1), and (78) implies

$$\beta E_t\left[e^{-\gamma(c_{t+1,i}-c_{t,i})}\varepsilon_{t+1,j}\right] = \frac{1-\gamma\frac{r}{1+r-\rho}(1-\rho)\,\omega(\Delta_t)}{1+r}. \tag{85}$$

Substituting (85) into (84) gives linear equations in Φ_0 and Φ_1, solved by (41), respectively (42).

To complete the proof of the claim that Δ_t is chosen optimally, we provide an explicit example of a family of functions for $F_t(\Delta)$ that has the desired properties. To start, we compute the value function of an investor adopting the policies of Proposition 4. Equation (78) along with (77) imply that

$$\begin{aligned} V(W_{t,i},\Delta_t) &= -\frac{1}{\gamma}\sum_{t\geq 0}\beta^t E_t\left[e^{-\gamma c_{t,i}}\right] = -\frac{1}{\gamma}e^{-\gamma c_{0,i}}\sum_{t\geq 1}\beta^t E_t\left[e^{-\gamma\sum_{m=0}^{t}(c_{t+1,i}-c_{t,i})}\right] \\ &= -\frac{1}{\gamma}e^{-\gamma c_{0,i}}\sum_{t\geq 0}(1+r)^{-t} = -\frac{1}{\gamma r}e^{-\gamma c_{0,i}}. \end{aligned}$$

In turn equations (71), (73), and (80) imply that

$$V(W_t,\Delta_t) = -\frac{1}{\gamma r}e^{-\frac{\gamma r}{1+r}W_{t,i}+z(\Delta_t)}, \tag{86}$$

where $z_t(\Delta_t) \equiv \frac{r\Phi_t}{1+r} - M + \frac{\gamma^2}{2}\left(\frac{r}{1+r-\rho}\right)^2(1-\rho)^2\,\omega(\Delta_t)$.

Next we suppose that we no longer impose that the investor choose $\Delta=\Delta_t$, (where Δ_t is the time-t random draw of Δ that we imposed in Proposition 4) . Instead Δ is chosen optimally. However, prices are still given by $P_{t,j}(\Delta_t, D_{t,j})$ from equation (40). We will construct a function $\kappa_t g_t(\Delta)$ that renders the choice $\Delta=\Delta_t$ optimal at the total cost specified in (80).

Throughout we let $\overline{X}_t^{(i)}(\Delta;\Delta_t)$ denote the optimal number of total risky assets chosen by investor i, and assuming that that investor chooses Δ and prices are given by $P_{t,j}(\Delta_t, D_{t,j})$. For future reference, we note that by construction of the price function $P_{t,j}(\Delta_t, D_{t,j})$ it follows that $\overline{X}_t^{(i)}(\Delta_t;\Delta_t)=1$. Using (86) the first order condition characterizing an optimal Δ is

$$F_t'(\Delta) = h(\Delta;\Delta_t),$$

where

$$h(\Delta;\Delta_t) = -\frac{1}{1+r}\frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2(1-\rho)^2\left(\overline{X}_t^{(i)}(\Delta;\Delta_t)\right)^2\omega'(\Delta). \tag{87}$$

Next we fix a value of Δ_t and we simplify notation by writing $h(\Delta)$ rather than $h(\Delta;\Delta_t)$. We also let $q(x)$ denote some continuous function with $q(0)=1$, $q(x)>1$ for $x>0$. Let $\eta\in[0,1]$, take some positive (and small) $\varepsilon<\frac{\Delta_t}{2}$, and consider the following function

$$F_t'(\Delta) = \begin{cases} \frac{\Delta}{\varepsilon}\eta h(\varepsilon) & \text{for } \Delta\leq\varepsilon \\ \eta h(\Delta) & \text{for } \Delta\in(\varepsilon,\Delta_t-\varepsilon] \\ \eta h(\Delta_t-\varepsilon)\frac{\Delta_t-\Delta}{\varepsilon} + h(\Delta_t)\frac{\Delta-\Delta_t+\varepsilon}{\varepsilon} & \text{for } \Delta\in(\Delta_t-\varepsilon,\Delta_t] \\ h(\Delta)\,q(\Delta) & \text{for } \Delta>\Delta_t+\varepsilon. \end{cases}. \tag{88}$$

By construction, $F_t'(0) = 0$, and $F_t'(\Delta)$ is continuous and increasing in Δ. More importantly, $F_t'(\Delta_t) = h(\Delta_t)$, and hence $\Delta = \Delta_t$ satisfies the necessary first order condition (87). Moreover, since $F_t'(\Delta) < (>) h(\Delta)$ for $\Delta < (>) \Delta_t$, it follows that $\Delta = \Delta_t$ is optimal for any $\varepsilon > 0$ and $\eta \in [0,1]$. Finally,

$$\lim_{\varepsilon \to 0} \int_0^{\Delta_t} F_t'(\Delta) = \eta \int_0^{\Delta_t} h(x)\, dx > 0. \tag{89}$$

Now suppose that nature draws $\Delta_t = \Delta_u > 0$. By choosing M that is sufficiently close to $-\frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega\left(\overline{\Delta}\right)$ it follows that

$$0 < \frac{M}{\gamma} + \frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta_u) < \int_0^{\Delta_u} h(x)\, dx. \tag{90}$$

Combining equations (89) and (90) it follows that for sufficiently small $\varepsilon > 0$, there exists some $\eta \in [0,1]$, so that

$$\int_0^{\Delta_u} F_t'(x)\, dx = \frac{M}{\gamma} + \frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta_u) > 0. \tag{91}$$

Hence, when $\Delta_t = \Delta_u$ the cost function $\kappa_t g_t(\Delta)$ renders $\Delta = \Delta_u$, while also satisfying (80). The same argument implies that for any value of Δ_t that satisfies

$$0 < \frac{M}{\gamma} + \frac{\gamma}{2}\left(\frac{r}{1+r-\rho}\right)^2 (1-\rho)^2 \omega(\Delta_t) < \int_0^{\Delta_t} h(x)\, dx, \tag{92}$$

there exists $\eta \in [0,1]$ and sufficiently small $\varepsilon > 0$ such that the optimal Δ coincides with Δ_t, and (80) holds. Continuity of $\omega(\Delta_t)$ and of $\int_0^{\Delta_t} h(x)\, dx$ in Δ_t implies that as long as $\underline{\Delta}$ is sufficiently close to Δ_u, there always exists $\eta \in [0,1]$ and $\varepsilon > 0$ (both depending on the random draw Δ_t) such that $\Delta = \Delta_t$ is optimal and (80) holds. ■

Proof of Proposition 5. Parts (i)–(iii) are proved in the main body of the text. Part (iv) comes down to noticing that

$$cov\left(e^z, z\right) > 0 \tag{93}$$

for any random variable z — in particular, for $z = \omega(\Delta)$. The second statement of (iv) follows from the first and Jensen's inequality applied to the convex function ω. ■