Transparency and Distressed Sales under Asymmetric Information

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Abstract

We analyze a dynamic market with short lived adverse selection and correlated values. Uninformed buyers compete inter- and intra-temporarily for a good that is sold by an informed seller who is suffering a liquidity shock. We contrast a transparent (public price offers) with an opaque (private price offers) information structure. First, we show that with private offers a pure strategy equilibrium is not sustainable if the seller is patient enough. Moreover, we can fully characterize the equilibrium in both information structures in a three period model if the buyers' valuations are a linear function of the seller's costs and costs are uniformly distributed. Finally, we derive that in this setting, any equilibrium with private offers is weakly more efficient than the unique pure strategy equilibrium with public offers.

1 Introduction

We consider a problem of an owner of an indivisible durable asset who suffers a liquidity shock. If information about the value of the asset was symmetric, the owner would sell the asset to a buyer not facing a liquidity shock and who would hence have a higher valuation. The problem is that usually the owner of the asset has better information about its quality. Any potential buyer therefore faces an adverse selection problem. As first stressed by Akerlof [1], if there is only one opportunity to trade, the buyer is only willing to pay his expected valuation of the asset. However, the highest seller type may not want to accept this price, if the adverse selection problem is sufficiently strong. Hence, only low type sellers sell in equilibrium, even though there are positive gains from trade for all types. In a dynamic setting, in which sellers get several chances to sell their good, this logic of a lemons market leads to an inefficient delay in trade.

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In this paper we analyze how the dynamic market (in)efficiency is affected by price disclosure policy in the market. We consider two opposite information structures, one in which buyers observe past prices and one in which they do not (they observe the time on the market in both cases). Following the recent financial crisis, there has been a lot of discussion and even legislation geared towards moving trades that had been taking place in opaque over the counter markets into more transparent exchanges. For example, one of the declared goals of the Dodd–Frank Wall Street Reform and Consumer Protection Act, which came into effect in 2010, is to increase transparency in the financial system. In Europe, the European Commission has recently announced that it is discussing proposals to revise the Markets in Financial Instruments Directive (MiFID) for more efficient, resilient and transparent financial markets in Europe. Oftentimes, transparency is considered to be welfare enhancing because it is a necessary condition for perfect competition, it decreases uncertainty and it increases public trust. Indeed, transparency has many different aspects and it is a complex question whether transparency benefits efficiency. In our setting, we demonstrate that over the counter markets can actually create higher welfare than transparent markets.

We examine a game theoretic model in discrete time with a long-lived, privately-informed seller and a competitive market of buyers in every period (modeled as a number of short-lived buyers competing in prices in every period). In our model, the adverse selection problem is short lived since we assume that at some date T the seller's type is publicly revealed. This could arise because there are some studies needed to determine the quality of the asset but these studies take time to be conducted. Examples for such studies are recent stress tests of financial institutions, as well as land with mining or oil potential which is undergoing a geological survey where the results are revealed at some future date. Another example is of an entrepreneur who can over time prove the value of its idea.

What makes the markets operate differently in these two information regimes? In a transparent market, buyers can observe all previous price offers and thereby learn about the quality of the good through two channels: the number of rejected offers (time on the market) and the price levels that have been rejected by the seller. By rejecting a high offer, the seller can send a strong signal to future buyers that she is of a high type. In contrast, in an opaque over the counter market, in which buyers cannot observe previously rejected prices, the seller signals only via delay. Hence, when offers are made publicly, the seller has a much higher incentive to reject a high offer than with private offers. Put differently, the supply curve faced by the buyer is more elastic with private offers. This difference in seller's responses to price offers drives the differences in equilibrium dynamics that we describe in this paper.

Our main contribution is to understand these driving forces in detail, to characterize the equilibrium outcomes in settings with opaque and transparent information structures and to compare their welfare implications. We first analyze pure strategy PBEs with public offers and show their existence for general discount factors. Then, we show that there are no pure strategy PBEs with private offers if the discounting between two periods is small enough because buyers have an incentive to deviate to higher prices due to the extremely elastic supply curve.[1]

Since there are no pure strategy equilibria with private offers for large discount factors, and in either game that there can be multiple equilibria, welfare analysis in the general setting is hard or impossible. Hence, to provide some intuition about welfare consequences of information disclosure, we consider a benchmark model where buyer value is a linear function of the seller's value, types are distributed uniformly, and there three periods (two opportunities to trade while information is asymmetric). In this setting, there is a unique PBE with public offers. In case of private offers, we characterize all (mixed-strategy) equilibria. This result paves the way to welfare comparisons. We show that any equilibrium with private offers has a higher expected welfare than the unique PBE with public offers.

In other words, we show that transparency can hurt efficiency in markets with adverse selection problems. It suggests that governments should take these equilibrium effects for dynamic interactions into account when regulating financial or other markets. In particular, at times such as the recent financial crisis it might be optimal to set up over the counter markets or non-transparent exchanges for the toxic asset classes.

Our result about existence of pure strategy equilibria in the private offers case is related to the result in Kremer and Skrzypacz [11] who study a dynamic version of the education signaling model in a finite horizon model with the type being (partially or fully) revealed in the last period and with all the offers being private. They show that there do not exist fully separating equilibria in a game with a continuum of types or with a finite number of types if the length of periods is short enough. We extend their reasoning to show that in our model with interdependent valuations pure strategy equilibria do not exist if the discount factor is high enough or the periods are short enough.

Moreover, our paper is related to Hörner and Vieille [7] (HV from now on) who also analyze a dynamic lemons market with publicly and privately observable offers. There are two important differences in the setup. First, we consider a market that is liquid and competitive by assuming that every period two or more buyers make offers. In HV there is a monopsonist buyer in every period, which captures thin markets or search frictions. Second, we assume that private information is short-lived while HV assume it is infinitely lived (we study a finite horizon while they study an infinite horizon game). There is also a difference in questions/results. HV show that with private offers there is a positive probability of trade in every period and with probability 1 trade eventually happens. Moreover, if the discount factor is high enough, no buyer plays a pure strategy (other

[1]With the same logic we can show that holding the time horizon T fixed, if we consider a liquid market without seller commitment by taking the length of periods towards zero, there is no pure strategy PBE in the game with private offers.

than possibly the first buyer, see HV Proposition 7). With public offers, trade only happens in the first period and the probability of trade is uniformly bounded away from 1. As a result, for high discount factors, markets with private offers are more efficient. We show a similar result that with private offers and high enough discount factor, the equilibrium is in mixed strategies. However, the equilibrium prices with public offers are quite different than in HV: for example with linear valuations and uniform distribution of types there would be positive probability of trade in every period. In terms of welfare comparisons, instead of an asymptotic result for high discount factors (which would be less interesting in our setup since with a finite horizon all information regimes are asymptotically efficient), we compare welfare in the benchmark model for any discount factor. Finally, we are able to provide a complete characterization of equilibria in this benchmark model.

Similar questions are also addressed by Kaya and Liu [9] who consider a setting with independent valuations (as opposed to interdependent valuations in this paper), and short-lived monopsonistic buyers (as in HV). They as well compare the equilibrium outcomes in transparent and non-transparent setups. In both information structures, they characterize unique equilibrium outcomes and show that price paths are deterministic (pure strategy equilibria in both information regimes). They show that given some regularity assumptions, in the opaque market, trade happens faster and prices are lower.

Kim [10] compares three different information structures in a continuous time setting in which many sellers and buyers, who arrive over time at a constant rate, match randomly. In every match, the buyer makes a price offer that the seller can accept or reject. The type space of the seller is binary. Instead of looking at observability of past offers, steady state equilibria in settings in which buyers do not observe any past histories are compared with settings in which the time on the market or the number of past matches can be observed by buyers. The welfare ordering is not as clear cut as in our paper. It is shown that with small frictions, it is optimal if only the time on the market is observable while with large frictions the welfare ordering can be reversed.

For repeated first-price auctions, Bergemann and Hörner [2] consider three different disclosure regimes and they show that if bidders learn privately about their win, welfare is maximized and information is eventually revealed. An analysis of the Spence signaling model is examined by Nöldeke and Van Damme [13] who assume public offers and Swinkels [16] who focuses on private offers. Welfare is higher with private offers (if offers are sufficiently frequent) because the equilibrium with private offers avoids any costly signaling (there is full pooling at $t = 0$, which is not possible in the lemons model). There are also many papers that explore the role of transparency different from price transparency. These models are mostly in static environments. Pancs [14] illustrates how it can be optimal for stock exchange to allow for iceberg or hidden offers introducing latent buyers who can be attracted by such offers. Likewise, Buti and Rindi [3] show why traders with different preferences choose different levels of information disclosure when they make offers.

Besides our contribution regarding the implications of transparency, our paper also contributes to the literature on dynamic lemons markets in general. One of the most recent works by Deneckere and Liang [5] considers an infinite horizon bargaining situation, i.e. one long-lived buyer and one long-lived seller, with correlated valuations. They show that even in the limit as the discount factor goes to one, there can be an inefficient delay of trade unlike predicted by the Coase conjecture, and they are able to characterize the limiting equilibrium outcome. Janssen and Roy [8] obtain similar results with a dynamic competitive lemons market with discrete time, infinite horizon and a continuum of buyers and sellers. While in their model both market sides compete, we assume that there is only one seller. Unlike most previous papers that consider slightly different market structures, we are able to almost fully characterize equilibrium mixed strategies of buyers with private offers. This makes it possible to understand these kinds of equilibria in more detail. For example, we show that non-offers (i.e. offers rejected by all types) in period one are always part of an equilibrium.

While we focus our discussion on the welfare implications of different information structures, the literature is also concerned with many other interesting questions in similar settings to ours. For example Daley and Green [4] analyze the effect of news on a continuous time lemons market with one risk-neutral seller who faces a competitive crowd of sellers, where offers are assumed to be public and the asset can have only two values. They have information being gradually diffused rather than it being released at some predetermined time as in our paper. The impact of temporary closure of the market in a similar setting as in this paper with public offers, is considered by Fuchs and Skrzypacz [6]. However, none of these works assumes private price offers as we do in this work.

Finally, our paper is also related to the recent literature on the design of optimal government intervention in markets with adverse selection and investment decisions inspired by the classical paper Myers and Majluf [12] on security design. Tirole [17] and Philippon and Skreta [15] both consider models with a static market with adverse selection where the government can intervene ex ante by offering contracts before the firms enter the market. Similar to our model, the rejection of the government offer signals to the market that a firm is a high type. The government can skim away some bad firms (by offering debt buybacks/equity injections and guarantees, respectively) and thereby improve market efficiency.

Our paper is organized as follows. Section 2 formally introduces the model, the equilibrium notion, the Reverse-Skimming Property and its implications. Then, the public offer setting is fully analyzed in section 3. Section 4 first establishes that there are no pure strategy equilibria with private offers if sellers are patient enough, develops some general properties of mixed strategy equilibria in that case and then characterizes the mixed strategy equilibria in the linear example with three periods. Section 5 presents the welfare implications of both information structures. Finally, section 6 discusses the robustness of our results.

2 Model

2.1 General Setup

A seller has an asset that she values at c where $c \in [0,1]$ is distributed according to the cumulative distribution function (cdf) $F(c)$ with a strictly positive density $f(c) > 0$ and no mass points (the domain $[0,1]$ is a normalization). One can think of the asset giving an expected cash flow each period and c being its present value for the seller.[2] The seller's type c is persistent over time and her private information. Every period $t \in \{1, 2, ..., T\}$, for $2 < T < \infty$, two short lived buyers, whom we label by $i = 1, 2$, make simultaneous price offers p_t^i to purchase the asset.[3] The value of the asset for the buyers is given by $v(c)$. Gains from trade $v(c) - c$ are strictly positive for all $c \in [0, 1)$ and $v(1) = 1$. [4] Initially, we only impose $v(c)$ to be differentiable and increasing in c, i.e. $v'(c) > 0$. Later we focus on $v(c) = Ac + B$ and uniformly distributed c.[5] The game ends as soon as the good has been sold. If trade has not taken place by period T, the private information of the seller is revealed before the last buyers make their offers.

The payoff relevant outcome of this game can be described by a quadruple (c, t, i, p), where c is the realized type, t is the time at which agreement is reached, i is the buyer who receives the good and p the price at which the good is traded. All players are risk neutral. The seller discounts payoffs with a discount factor $\delta \in (0, 1)$. Given an outcome (c, t, i, p), the seller's period 1 present value payoff is $(1 - \delta^{t-1})c + \delta^{t-1}p$; a buyer's payoff is $v(c) - p$ if he gets the good and 0 otherwise. Without loss of generality, we restrict prices to be in $[0, v(1)]$, since it is a dominant strategy for the seller to reject any negative price, and for any buyer it is a dominated strategy to offer any price higher than $v(1)$ that has a positive probability of being accepted.

We explore two setups that differ in the information sets of the buyers in terms of what they observe about past offers. In the *public offers* case, we assume that period $t > 1$ buyers observe all past rejected offers $\{p_s\}_{s=1}^{s=t-1}$. A period t buyer's strategy, $\rho_t^{B_i}$, maps a history of prices to a probability measure on $[0, v(1)]$ (the domain of prices). A strategy for the seller is a sequence of acceptance decisions $(\rho_t^S)_{t=1}^T$ that depends on her type as well as on past and current offers. That is, $\rho_t^S : [0,1] \times [0, v(1)]^{2\times t} \to \{(\eta_1, \eta_2) : \eta_1 + \eta_2 \leq 1\}$, where η_i represents the probability with which the seller accepts buyer i's offer in period t.

[2] Alternatively, and mathematically equivalently, c can be thought as the cost of producing the asset.

[3] The analysis is the same if there are more than two buyers since the buyers compete in a Bertrand fashion.

[4] We assume $v(1) = 1$ only to rule out the possibility of trade ending before T. This allows us to avoid making assumptions about off-equilibrium beliefs if the seller does not sell by t even though in equilibrium he is supposed to. If $v(1) > 1$ but T is small enough that not all types trade in equilibrium, our analysis still applies.

[5] In the discounted cashflow interpretation of valuations the proportional valuation $v(c) = \alpha c$ for $\alpha > 1$ follows naturally. In that case we need to assume $c \in [c_{\min}, c_{\max}]$ where $c_{\min} > 0$ in order to ensure that $v(c) > c$ for all $c < c_{\max}$. The role of this assumption is equivalent to the requirement that $B > 0$, which as we show in Section 6 is important in order to guarantee trade with positive probability before period T in equilibrium.

With *private offers*, we assume that period t buyers are aware that the seller has rejected all offers in periods $s < t$ but, crucially, the buyers do not know what these offers actually were. The generation t buyers can condition their offers on their information set. We denote the strategies of generation t buyers by $\sigma_t^{B_i}$ which is a probability measure on $[0, v(1)]$. With a slight abuse of notation we also denote the $[0, v(1)]$-valued random variable that represents buyer i's strategy by $\sigma_t^{B_i}$. A strategy for the seller is of the same form as with public offers, but we will denote them in the private offer setting by σ_t^S. We will assume that the seller responses are buyer identity independent that is, conditional on receiving the same price offer, she will treat both buyers equally. Given the seller's strategy and their information set, the buyers update their beliefs about the current distribution of types and we denote the distribution of seller types, that the period t buyer i ($i \in \{0, 1\}$) believes to be facing at the time of making their offer, by $F_t^i(c)$.

2.2 Equilibrium Notion

We will be interested in characterizing the appropriately extended notion of perfect Bayesian equilibrium (PBE) for our setting. Essentially, this entails a sequence of pricing strategies for the two buyers, $\{(\rho_t^{B_1}, \rho_t^{B_2})\}_t$ and $\{(\sigma_t^{B_1}, \sigma_t^{B_2})\}_t$, for public and private offers, respectively, acceptance rules $\{\rho_t^S\}_t$ and $\{\sigma_t^S\}_t$, respectively, and the buyers' beliefs $\{F_t^1, F_t^2\}_t$ satisfying the following three conditions:

1) Any price offer in the support of $\rho_t^{B_i}$ and $\sigma_t^{B_i}$ must maximize the buyer's payoff conditional on the seller's acceptance rule, the other buyer's strategy $\rho_t^{B_j}$ and $\sigma_t^{B_j}$, respectively, and his belief $F_t^i(c)$.

2) The buyer's beliefs are updated according to Bayes rule taking the seller's and the other buyer's strategies as given.[6] Hence, in particular, $F_1^i(c) = F(c)$ for $i \in \{1, 2\}$. If the seller accepts an offer that no remaining seller type was supposed to accept in equilibrium, we do not need to worry how beliefs evolve off-equilibrium since the game ends in that case. Moreover, by the assumption that $v(1) = 1$, there is a mass of sellers remaining with positive probability in every period in equilibrium. This implies we do not have to consider the off-equilibrium case in which all types should have traded but the seller is still around.[7] Furthermore, note that this implies that in equilibrium, both buyers in any period t must have the same beliefs $F_t \equiv F_t^1 = F_t^2$.

3)The seller's acceptance rule maximizes her profit taking into account the impact of its choices on the agents updating and the future offers she can expect to follow as a result.

[6]If the offer is public the updating is also conditional on the offered amount.

[7]We discuss the case where $v(1) > 1$ in Section 6.

2.3 Preliminaries

As in other dynamic games, the seller's equilibrium acceptance rule in each period turns out to be a cutoff rule such that sellers with valuations above a cutoff $k_t(p)$ reject a price offer p in period t while sellers with valuations less than $k_t(p)$ accept it. In the bargaining literature, it is the better types that accept first and this property is known as the skimming property. Since here it is the worse types that trade first, we call it reverse-skimming instead.

Lemma 1 *(Reverse-Skimming Property) In any equilibrium with either type of information structure, for any period t and for any price p, there exists a cutoff type $k_t(p)$ so that a seller of type c accepts the offer p in period t if $c < k_t(p)$ and rejects p in period t if $c > k_t(p)$.*

Note that this lemma holds independently of the information structure in place. The intuition for the lemma is straight forward. If a seller of type c is willing to accept a price that induces a given future price path, then all sellers with $c' < c$ prefer to buy today to buying at a future price. From now on, we will denote the cdf of the cutoff $k_t(p_t)$ by $K_t : [0,1] \to [0,1]$, where p_t is the random variable representing the highest offer in period t, i.e. with public offers $p_t = \max_{j=1,2}\ \rho_t^{B_j}$ and with private offers $p_t = \max_{j=1,2}\ \sigma_t^{B_j}$, respectively.

Thanks to the Reverse-Skimming Property, we can, write a buyer's expected profit *conditional on receiving the good if trade takes place* as

$$\Pi_t(p; f_t) = \int_0^{k_t(p)} (v(c) - p)\, f_t(c)dc, \tag{1}$$

if the belief about the remaining types of the sellers is distributed according to a probability density function f_t. We focus on this conditional expected profit because it is independent of the other buyer's strategy. [8] Note, however, that the actual expected profit of the buyer is the probability that he receives the good times $\Pi_t(p; f_t)$.

Furthermore, in equilibrium, buyers' beliefs about the remaining types are given by mixtures of truncations of the prior distribution F of c. With public offers, all past prices are observable, i.e. $f_t(c) = \frac{f_1(c)}{1-F_1(k_{t-1}(p))}$ where p is the highest price chosen by buyers up to period $t-1$. In contrast, with private offers, it is given by the belief of period t buyers about the strategies of buyers in previous periods. Nevertheless, also in this case, the Reverse-Skimming Property guarantees that equilibrium beliefs about the remaining types at the beginning of period t have a relatively simple structure. Given (possibly mixed) equilibrium strategies of previous buyers $\sigma_1^{B_i}, \ldots, \sigma_{t-1}^{B_i}$, the random variable $k_{t-1}^m \equiv \max\{k_1(\max_i \sigma_1^{B_i}), \ldots, k_{t-1}(\max_i \sigma_{t-1}^{B_i})\}$ represents the cutoff at the

[8] Π_t is also equal to the total expected buyers' surplus, which is collected by one of the agents if he has the higher bid.

beginning of period t. Let us denote its cumulative distribution function by K^m_{t-1}. In equilibrium, it must hold that

$$f_t(c) = f_t[K^m_{t-1}](c) \equiv \int_0^c \frac{f_1(c)}{1-F_1(\tilde{k})} dK^m_{t-1}(\tilde{k}).$$

With private offers, the price history does not affect future buyers' beliefs and hence it does not affect their strategies. This makes it relatively straight forward to show that the mapping from prices to cutoffs in each period is one-to-one, i.e. in a given period, there exists a unique price $p_t(k_t)$ that results in a cutoff k_t. We will call the function p_t inverse supply function. With public offers, the history of prices affects future buyers' strategies, so an out of equilibrium deviation offer of a buyer can change the whole continuation game. Nevertheless, we can show that in every pure strategy equilibrium the price-cutoff mapping is one-to-one in each period. Moreover, for every pure strategy equilibrium outcome, there exists a PBE in which all equilibrium cutoffs are weakly monotone in time. This is summarized in the following lemma.

Lemma 2 *(Inverse supply)*[9]
(i) (Private offers) With private offers, in every equilibrium, there exists a unique price $p_t(k)$ for every period t that results in a cutoff type k. $p_t(\cdot) = k_t^{-1}(\cdot)$ is increasing and continuous.
(ii) (Public offers) In any pure strategy equilibrium with public offers, there exists at most one price $p_t(k)$ for each period t that results in each cutoff k. Hence, $p_t(\cdot) = k_t^{-1}(\cdot)$ for all cutoffs k that are attainable.[10] *Moreover, for any pure strategy equilibrium, there exists an outcome-equivalent pure strategy equilibrium that results in cutoffs with $k_1 \leq \cdots \leq k_{T-1}$ and for all t.*

$$p_t(k_t) = \delta p_{t+1}(k_{t+1}) + (1-\delta)k_t.$$

Note that with public offers this inverse supply function does not have to be increasing. The existence of an inverse supply function guarantees that the mapping from prices to cutoffs is one to one, so it allows us to focus on the space of cutoffs only instead of on the space of prices and cutoffs simultaneously. Moreover, it turns out that the slope of the inverse supply function is crucial for the (non-)existence of pure-strategy equilibria with private offers. Abusing notation a little bit, we will represent beliefs about past cutoffs and actual cutoffs expected in future periods as part of the continuation game both by cdfs denoted by K_t.

The most immediate implication is that it allows us to think of the buyers to face supply curves given by the inverse supply function

$$p_t(k_t) = \delta \underbrace{\left[\left(\int_{k_t}^1 p_{t+1}(k_{t+1}) dK_{t+1}(k_{t+1})\right) + K_{t+1}(k_t)p_{t+1}(k_t)\right]}_{\text{continuation payoff}} + \underbrace{(1-\delta)k_t}_{\substack{\text{utility from} \\ \text{keeping the good}}} \tag{2}$$

[9]Note that the function p_t is not an inverse supply function in the classical sense, since it does not have to be increasing, i.e. sellers are not necessarily following a reservation price strategy.

[10]In the construction of the public offer pure strategy PBE (Theorem 1), every k turns out to be attainable for buyers in every period in equilibrium.

in periods $t < T-1$, as derived in the proof of lemma 1 by the indifference condition of the cutoff-seller, and

$$p_{T-1} = (1-\delta)k_{T-1} + \delta v(k_{T-1}) \tag{3}$$

in period $T-1$. Hence, $p_t(k_t)$ is the price that is needed to attract all types up to k_t. The intuition of the right hand side of these expressions is as follows. First, note that $p_t(k)$ is always exactly type k seller's expected payoff of not selling the good today but later because seller k should be indifferent between receiving the price $p_t(k)$ and not selling the good today. Hence, in (2), $(1-\delta)k_t$ represents the payoff of a type k_t buyer if he held on to the good for exactly one more period, $\int_{k_t}^1 p_{t+1}(k_{t+1})dK_{t+1}(k_{t+1})$ is the expected price the seller can get if he sells the asset in period $t+1$ and $p_{t+1}(k_t)$ is the expected payoff that the seller can expect if he does not sell tomorrow either.

One drawback of the representation of a buyer's expected profit given by (1) is that the buyer's belief f_t is defined on the space of cutoffs, whereas buyers choose prices in $[0, v(1)]$. The existence of a unique $p_t(k)$ allows us, on the equilibrium path, to think of buyers essentially choosing cutoffs instead of prices given the seller's optimal cutoff strategy $k_t(\cdot)$. More precisely, we can write a buyer's expected profit conditional on winning the bid on the equilibrium path, if he bids a price $p = k_t^{-1}(k)$ and given belief K_{t-1}^m about the cutoff distribution at the beginning of period t, by

$$\pi_t(k; K_{t-1}^m) \equiv \Pi_t(k_t^{-1}(k); f_t[K_{t-1}^m]) = \int_0^k \int_0^c \frac{1}{1-F(\tilde{k})} dK_{t-1}^m(\tilde{k})\,(v(c) - p_t(k))\,f_1(c)dc. \tag{4}$$

If K_{t-1}^m has a one-point support at l (which is always the case with public offers), then we write $\pi_t(k; l)$ instead of $\pi_t(k; K_{t-1}^m)$, abusing notation slightly.

Finally, it is worth noting that thanks to the Reverse-Skimming Property, the only efficiency relevant outcome of the game is the distribution of the cutoff-types in each period. Hence, if we can determine for any PBE the distribution of cutoffs, we are able to calculate the total surplus and to compare the welfare with private and public offers.

3 Public Offers

Let us first consider the transparent market. With public offers, we can focus our analysis on pure strategy equilibria because if buyers mix between different prices, the realization of prices is observed and hence, the continuation game is the same as if buyers just chose that realized price with probability one in the first place. We have shown in lemma 2 that for any pure strategy PBE, there exists an outcome-equivalent equilibrium with equilibrium cutoffs by $(k_1^*, \dots, k_{T-1}^*)$ and $k_1^* \leq \dots \leq k_{T-1}^*$ where the following seller's indifference conditions must be satisfied:

$$\begin{aligned} p_t(k_t^*) &= (1-\delta)k_t^* + \delta p_{t+1}(k_{t+1}^*) \qquad \forall t < T-1 \quad \text{and} \\ p_{T-1}(k_{T-1}^*) &= (1-\delta)k_{T-1}^* + \delta v(k_{T-1}^*). \end{aligned} \tag{5}$$

Hence, a buyer's expected profit conditional on receiving the good is given by

$$\pi_t(k_t^*;k_{t-1}^*) = \frac{F(k_t^*) - F(k_{t-1}^*)}{1 - F(k_{t-1}^*)} \left[\mathbb{E}^F[v(c)|[k_{t-1}^*, k_t^*]] - p_t(k_t^*)\right] \tag{6}$$

with $\mathbb{E}[v(c)|[k_{t-1}, k_t]] = \frac{\int_{k_{t-1}}^{k_t} v(c)f(c)dc}{F(k_t)-F(k_{t-1})}$. Note that since we have assumed $v(1) = 1$, in any pure strategy equilibrium, there must be some seller types that do not trade in each period. It is immediate that equilibrium prices must increase over time as the deadline approaches.

Lemma 3 *In a pure strategy equilibrium with public offers and with $\delta < 1$, prices are increasing over time, i.e. $p_t(k_t^*) < p_{t+1}(k_{t+1}^*)$ for all t.*

Proof. First, note that $k_1^* \leq k_2^* \leq \cdots \leq k_{T-1}^*$. We argue by backward induction. First, note that $p_{T-1}(k_{T-1}^*) < v(k_{T-1}^*)$. Moreover, if $p_{t+1}(k_{t+1}^*) < p_{t+2}(k_{t+2}^*)$, then

$$p_t(k_t^*) = (1-\delta)k_t^* + \delta p_{t+1}(k_{t+1}^*) < (1-\delta)k_{t+1}^* + \delta p_{t+2}(k_{t+2}^*) = p_{t+1}(k_{t+1}^*).$$

■

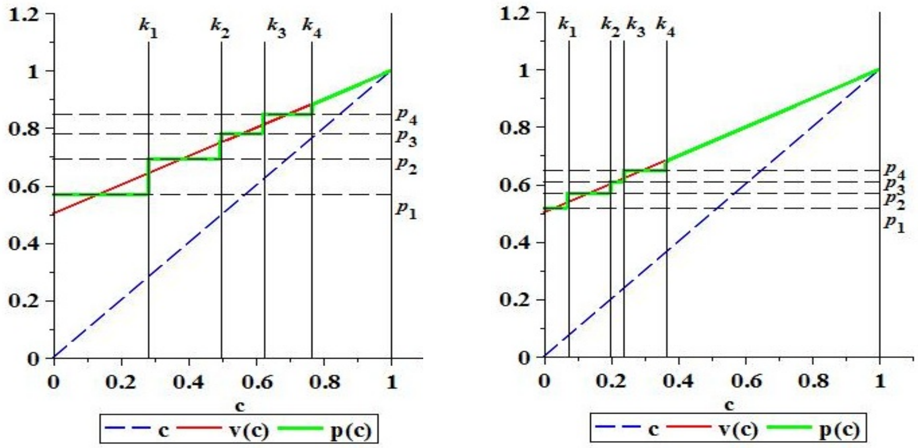
There are two forces driving this result. First, even though buyers never receive any expected surplus due to perfect competition, the seller's reservation price increases over time since she can allocate the good efficiently at T and hence, extract the maximum surplus. Second, by the Reverse-Skimming Property, there are higher types remaining in the market as time goes on, so the willingness to pay of buyers increases over time. Moreover, it is straight forward to show that cutoff types converge to zero as δ goes to 1 since more patient sellers are willing to wait longer until they sell their good at higher prices in later period.[11]

Remark 1 *Note that the above results also hold true for pure strategy PBEs with private offers. We will later even show that any pure strategy PBE with private offers corresponds to a PBE with public offers.*

Figure 1 shows the prices at which the different seller types trade if they are uniformly distributed, $v(c) = 0.5 + 0.5c$, and the time horizon is $T = 5$ for two different discount factors $\delta = 0.7$ and $\delta = 0.9$. p_t denotes the price and k_t denotes the equilibrium cutoff in period t. One can see that for higher δ there is less trade. Moreover, it is interesting to note that the amounts of trade are not monotone over time. For example in period 3 the amount of trade is smaller than in other periods in this example.

Finally, we show the existence of PBEs with public offers in the general setting and uniqueness of equilibrium prices if $v(c)$ is linear and $F(c) = c$.

[11]Formally, this can be seen by noting that because of (5), $\lim_{\delta\to 1} p_t(k_t^*) = \lim_{\delta\to 1} p_T(k_T^*) = v(k_T^*)$ for all $t < T$, that is by (10) $k_t^* = k_{t-1}^* = k_T^*$ for all t.



(a) $\delta = 0.7$

(b) $\delta = 0.9$

Figure 1: Pure strategy equilibria with $T = 5$ and $v(c) = 0.5 + 0.5c$

Theorem 1 *(Public offers)*
(i) With public offers, there exists a pure strategy equilibrium for all $0 < \delta < 1$.
(ii) With linear valuations and uniformly distributed costs, there are unique equilibrium prices (and cutoffs) which result from a PBE in pure strategies.

We show the existence of a pure strategy equilibrium with public offers (i) by construction. The best response function of the seller is given by her indifference condition (5). Buyers form their beliefs about the cutoff type accordingly and if an off equilibrium price greater than $p_{t-1}(1) = 1$ was rejected, then we can assume that buyers believe that the seller is of type 1.[12] We show that, if all buyers choose pricing strategies that result in a cutoff seller $c_t^*(k_{t-1})$ (defined below) given beliefs k_{t-1} about the current cutoff, this constitutes an equilibrium. We define $c_t^*(\cdot)$ inductively for $t = 1, \ldots, T-1$ as follows

$$c_{T-1}^*(k_{T-2}) = \sup\left\{k \in [k_{T-2}, 1] \middle| \frac{1}{1-F(k_{T-2})} \int_{k_{T-2}}^{k} \left(v(c) - p_{T-1}(k)\right) f(c)dc > 0\right\}.$$

with $p_{T-1}(k) = \delta v(k) + (1-\delta)k$ and $\sup \emptyset = k_{T-1}$ and for $t < T-1$

$$c_t^*(k_{t-1}) = \sup\left\{k \in [k_{t-1}, 1] \middle| \frac{1}{1-F(k_{t-1})} \int_{k_{t-1}}^{k} \left(v(c) - p_t(k)\right) f(c)dc > 0\right\}$$

with $p_t(k) = \delta p_{t+1}(c_{t+1}^*(k)) + (1-\delta)k$ and $\sup \emptyset = k_{t-1}$. In order prove this defines an equilibrium, we need to make sure that buyers do not make negative expected profits. This is guaranteed if

$$k \mapsto \frac{1}{1-F(k_{t-1})} \int_{k_{t-1}}^{k} \left(v(c) - (\delta p_{t+1}(c_{t+1}^*(k)) + (1-\delta)k)\right) f(c)dc$$

[12]Any other belief that is a one-point distribution, e.g. that the type is 0, would work, too. This way we only need to characterize strategies given beliefs that are not mixed.

is left-continuous. Therefore, it is sufficient to show that p_{t+1} is left-continuous as a function of c^*_{t+1}. We defer the proof of this technical claim to the appendix. The equilibrium cutoffs $(k^*_1, \dots, k^*_{T-1})$ are then, given by $k^*_1 = c^*_1(0), \dots, k^*_{T-1} = c^*_{T-1}(c^*_{T-2}(\dots c^*_1(0)))$. None of the buyers has an incentive to deviate from this equilibrium, since by increasing the price offer, buyers will either make zero or negative expected profits by definition of $c^*_t(\cdot)$ and by decreasing the price they will not receive the good and make zero expected profits. Note that there are generally multiple equilibria because there can be several prices that result in zero expected profits for the buyers. If $v(c) = Ac + B$, $A, B > 0$, $A + B = 1$ and $F(x) = x$, $x \in [0, 1]$, then for any $k' > k$, $k, k' \in [0, 1]$, $\mathbb{E}[Ac + B|[k, k']] = \frac{A}{2}k' + \frac{A}{2}k + B$ is linear in k' and prices $p_t(k_t)$ are also linear in k_t. Hence, there is only one price at which buyers earn zero expected profits given beliefs that are one-point distributions. Hence, there can only be unique equilibrium cutoffs and prices.

As an example, let us explicitly calculate the unique pure strategy cutoffs in the linear setting for $T = 3$ using backward induction. This will be useful when we do the welfare analysis in section 4. Let us first consider the continuation game starting in period 2 given that the current cutoff resulting from period 1 trade is k_1. If k_2 is the highest type that trades in period 2 and p_2 is the highest price offered, then the type k_2 seller must be indifferent between accepting p_2 or waiting for the information to be revealed, i.e. $p_2 = (1-\delta)k_2 + \delta(Ak_2 + B)$. Hence, given k_1, a buyer's expected profit conditional on trading in period 2 is given by

$$\pi_2(k_2; k_1) = \frac{k_2 - k_1}{1 - k_1}\left(\left(\frac{A}{2} - (1-\delta) - \delta A\right)k_2 + (1-\delta)B + \frac{A}{2}k_1\right).$$

In equilibrium, the zero expected profit condition $\pi_2(\kappa_2(k_1); k_1) = 0$ must hold because of competition between buyers in period 2, so that given k_1 the equilibrium cutoff of the continuation game in period 2 is given by

$$\kappa_2(k_1) = \frac{(1-\delta)B + \frac{A}{2}k_1}{1 - \delta - \frac{A}{2} + \delta A}. \tag{7}$$

Similarly, the players in period 1 will choose prices p_1 knowing that they will induce an equilibrium response in terms of cutoffs k_1 which takes into consideration, and must be consistent with, the optimal response $\kappa_2(k_1)$ of second period buyers and their implication in terms of prices. Let k_1 be the highest type willing to accept the highest first period offer p_1 when expecting p_2 to be offered the next period, that is $p_1 = (1-\delta)k_1 + \delta p_2$. Using the zero expected profit condition for period 1 and $\kappa_2(\cdot)$, one can solve for the unique equilibrium period 1 cutoff

$$k^*_1 = \frac{2B \cdot (A\delta - 2\delta + 2 - A) \cdot (1-\delta)}{2(1-\delta)(1-A)(A\delta - 2\delta + 2) + A^2}, \tag{8}$$

and by plugging this into (7) one gets the unique equilibrium period 2 cutoff

$$k^*_2 = \frac{2B \cdot (A\delta - 2\delta + 2) \cdot (1-\delta)}{2(1-\delta)(1-A)(A\delta - 2\delta + 2) + A^2}. \tag{9}$$

4 Private Offers

In this section, we will discuss the PBE outcomes in an opaque information structure and compare it to PBEs in the transparent market. We start with a couple of general results before we move to a full characterization of PBEs in the linear example.

4.1 General Results

With public offers, pure strategy equilibria are somehow easier to analyze because the continuation game does not change if a buyer today deviates from an equilibrium. First, note that (5) and (6) apply for private offers as well. In addition, we can show the following lemma that further characterizes pure strategy equilibria with private offers.

Lemma 4 *(i) In any pure strategy PBE with private offers, there is a mass of seller types who trade in each period.*
(ii) $p_t(\cdot)$ and $\pi_t(\cdot; k_{t-1})$ are differentiable for all $t \leq T-1$.

Proof. (i) Let us consider a pure strategy PBE with private offers that results in cutoffs $(k_1^*, \ldots, k_{T-1}^*)$. We prove the statement by backward induction. In period $T-1$, expected profits are given by $\pi_{T-1}(k; k_{T-2}^*) = \frac{1}{1-F(k_{T-2}^*)} \int_{k_{T-2}^*}^{k} (v(c) - \delta v(k) - (1-\delta)k) f(c) dc$. Since $v(k_{T-2}^*) - \delta v(k_{T-2}^*) - (1-\delta)k_{T-2}^* > 0$, there exists an $\epsilon > 0$ so that

$$v(k_{T-2}^*) - \delta v(k_{T-2}^* + \epsilon) - (1-\delta)(k_{T-2}^* + \epsilon) > 0.$$

Hence, $\pi_{T-1}(k_{T-2}^* + \epsilon; k_{T-2}^*) > 0$, so $k_{T-1}^* > k_{T-2}^*$.

Next, let us assume $k_t^* < \cdots < k_{T-1}^*$. We prove $k_t^* < k_{t+1}^*$ by contradiction. Let us assume $k_{t-1}^* = k_t^*$. Then, a period t buyer can deviate by offering $p_{t+1}(k)$ because all seller types who would have accepted in period $t+1$, accept in period t and even some better seller types accept which guarantees him positive expected profits.

(ii) In period $T-1$, differentiability of the price and expected profit function follows immediately from differentiability of $v(k)$. In all other periods, $\pi_t(\cdot; k_{t-1})$ is differentiable because $p_t(k) = (1-\delta)k + \delta p_{t+1}(k_{t+1}^*)$ where $p_{t+1}(k_{t+1}^*)$ is a constant (unlike with public offers). ■

This lemma implies that with private offers, in a pure strategy equilibrium with cutoffs $(k_1^*, \ldots, k_{T-1}^*)$, in every period $t \leq T-1$ the following zero-profit condition must be satisfied

$$\mathbb{E}^F[v(c) | [k_{t-1}^*, k_t^*]] = p_t(k_t^*). \tag{10}$$

Now, we are ready to show two main results about the relation and differences of pure strategy PBEs in the two information structures. First, we show that for any pure strategy PBE with private offers, the associated equilibrium cutoffs must correspond to equilibrium cutoffs with public offers. The reason is that although buyers do not observe previous prices with private offers, in equilibrium, their beliefs about previous prices must be correct. However, the opposite implication is not true as the second result demonstrates. With private offers, pure strategy equilibria are potentially harder to sustain because a buyer's marginal cost of increasing the price offer is lower with private offers than with public offers. In particular, we prove that with private offers and for large discount factors, no pure strategy equilibria exist. This is formalized in the following theorem.

Theorem 2 *(Private offers)*
(i) Equilibrium cutoffs (and prices) in any pure strategy equilibrium with private offers correspond to equilibrium cutoffs (and prices) in a pure strategy equilibrium with public offers.
(ii)There exists a $\delta^* < 1$ such that if $\delta > \delta^*$ there is no pure strategy equilibrium with private offers.

To best understand the intuition behind (ii), note that with public offers, the seller has a stronger incentive to reject the offer than if the offer had been made privately. Suppose one of the buyers make an off-equilibrium high offer, with public offers the seller gains additional reputation of her type being high by rejecting this offer. Where the strength of her signal is endogenously determined by the amount of money she left on the table. Hence, her continue value would increase upon a rejection. Instead, with private offers, she cannot use the off-equilibrium higher offer as a signal, so her continuation value remains constant and hence, she is more temped to take it. This implies that for all δ pure strategies are going to be harder to sustain with private offers. In other words, the supply curve is less elastic with public offers than with private offers at the equilibrium cutoffs as illustrated in figure 3 for an example with three periods, uniformly distributed costs, $v(c) = \frac{c+1}{2}$ and $\delta \in \{0.7, 0.8, 0.9\}$. As a result, in order to attract a marginally higher type, the buyers have to pay more in the public offer case. Hence, from the buyer's perspective, the marginal effect on her expected profit from a deviation from a public offer PBE to higher prices/cutoffs is always higher with private offers than with public offers.

The marginal effect of a buyer's deviation from a PBE on his expected profit is also affected by the discount factor δ but in a non-monotonic way. Recall that for any pure-strategy equilibrium cutoffs $(k_1^*, \ldots, k_{T-1}^*)$ a buyer's expected profit conditional on buying the good is given by (6) and hence, the marginal profit of buyers at time $t < T-1$ is given by

$$\frac{\partial}{\partial k_t}\pi_t(k_t; k_{t-1}^*)\big|_{k_t=k_t^*} = \underbrace{\frac{f(k_t^*)}{1-F(k_{t-1}^*)}\left(v(k_t^*) - p_t(k_t^*)\right)}_{\text{marginal benefit}} - \underbrace{\frac{\partial p_t}{\partial k}(k_t^*)\frac{F(k_t^*) - F(k_{t-1}^*)}{1-F(k_{t-1}^*)}}_{\text{marginal cost}}. \tag{11}$$

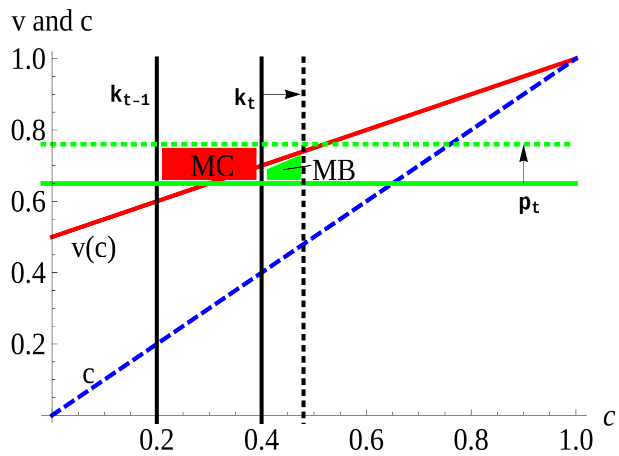


Figure 2: MB and MC of deviating from equilibrium

Note that this expression represents the net marginal benefit (NMB) of deviation of a buyer for both information structures, while $\frac{\partial p_t}{\partial k}(k_t^*)$ differs across information structures.[13] Suppose a buyer wanted to deviate with a higher price in order to sell to an equal number of additional types under both information structures. He would have the same marginal benefit (MB) in terms of additional surplus from the extra type he now trades with. Yet, there is a difference in the cost side. With private offers, supply is more elastic which means that the price he would need to offer with private offers, in order to attract the additional seller types, must increase less than with public offers. Note in the equation above that the marginal cost (MC) is the mass of types that trade $\frac{F(k_t^*)-F(k_{t-1}^*)}{1-F(k_{t-1}^*)}$ times the price change $\frac{\partial p_t}{\partial k}(k_t^*)$ which is just the slope of the supply curve. Therefore, the MC of this deviation with private offers (MC^i) is lower than the MC with public offers (MC^p). Hence, the net marginal benefit of a deviation with private offers $NMB^i = MB - MC^i$ is greater than the net marginal benefit of a deviation with public offers $NMB^p = MB - MC^p$ (for $\delta < 1$):

$$NMB^i > NMB^p \ for \ \delta < 1.$$

To show that the equilibrium fails with private offers because there is a profitable deviation for a buyer, we need to show that $NMB^i > 0$. [14] Note first that

$$\lim_{\delta \to 1} NMB^p = \lim_{\delta \to 1} NMB^i = 0$$

since for $\delta \to 1$ everything collapses to no trade until T. In order to isolate this effect from the other effects, we consider instead the average NMB per type trading. For private offers, the average

[13] With public offers $\frac{\partial p_t}{\partial k}(k_t^*)$ might not always exist but we are only using it to provide an intuitive explanation. We only need differentiability in the private offer case for the proof of theorem 2 and with private offers $\frac{\partial p_t}{\partial k}(k_t^*)$ always exists.

[14] Note that since we started with a PBE with public offers, $NMB^p < 0$ must hold.

MC is just given by $\frac{\partial p_t}{\partial k}(k_t^*) = 1 - \delta$ which converges to zero as $\delta \to 1$ and the zero-profit condition (10) must be satisfied. However, we can show in the formal proof in the Appendix that the average MB per type trading converges to a strictly positive number. More precisely, we show

$$\lim_{\delta \to 1} \frac{1}{F(k_t^*) - F(k_{t-1}^*)} \left(v(k_t^*) - \mathbb{E}[v(c)|[k_{t-1}^*, k_t^*]] \right) = \frac{v'(0)}{2f(0)} > 0.$$

Hence we can establish that there exists a $\delta^* < 1$ such that for $\delta > \delta^*$ there is no pure strategy equilibrium with private offers. Note that with public offers, MC must converge to a positive number greater than $\frac{v'(0)}{2f(0)}$ in order to outset the MB and make sure that no deviation is profitable.

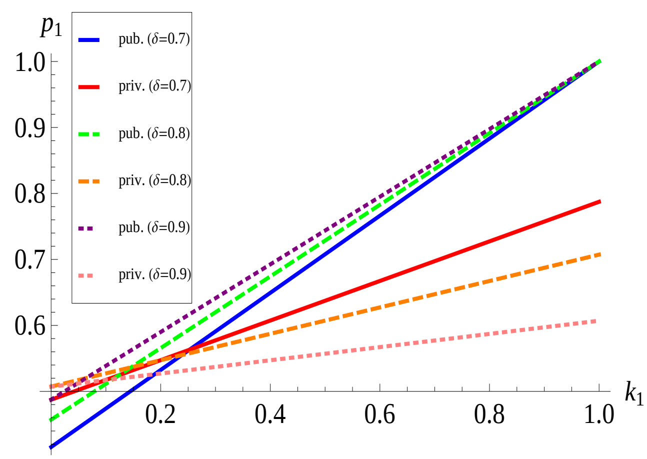
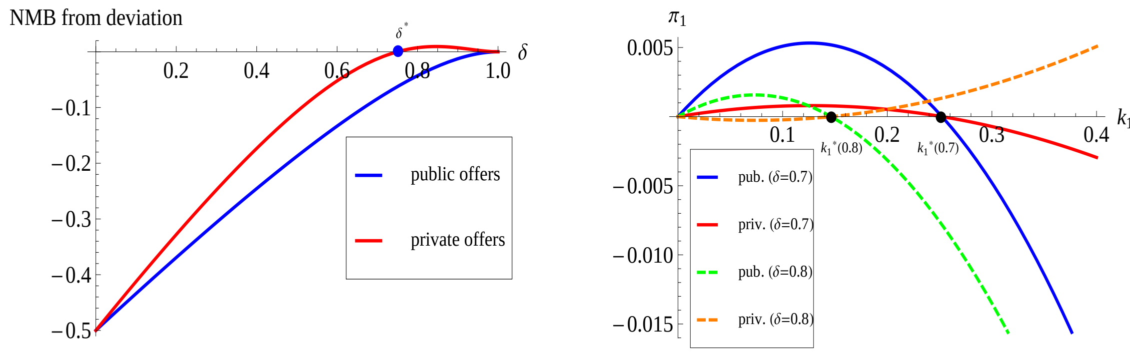


Figure 3: Inverse Supply $p_1(k)$ for $T = 3$, $v(c) = \frac{1+c}{2}$, $F_1(c) = c$

The MB and MC of attracting a marginally higher seller type (as we have defined it above) is qualitatively illustrated in figure 2. The MC is crucially affected by the supply curve if all buyers offer prices resulting in cutoffs (k_1^*, k_2^*) which we have delineated in figure 3 for $T = 3$, $v(c) = \frac{1+c}{2}$ and $F(c) = c$ for different values of δ in both information structures. Note that with private offers, the supply curve (faced by period 1 buyers) has a kink at k_2^* because period 2 buyers will offer $p_2(k_2^*)$ no matter what period 1 buyers do. Hence, the best outside option of sellers with $c < k_2^*$ is to buy in period 2 and for all sellers $c > k_2^*$ is to wait until period 3 in which the information is revealed. With public offers, period 2 prices are smoothly adjusted to period 1 offers, so there is no kink in the supply curve. In the graph one can nicely see how the supply curves become more elastic at the period 1 equilibrium cutoffs $k_1^* < k_2^*$ as δ increases, so an extra penny offered attracts more sellers if sellers are more patient. With private offers, $\frac{\partial p_t}{\partial k}|_{k^*(\delta)} = 1 - \delta$ can become arbitrarily small as δ increases because tomorrow's price p_{t+1} is not affected by today's buyers' actual decisions, but for public offers, the slope of the supply curve is bounded away from zero, so that the MC does not overweight the MB which will converge to zero as $\delta \to 1$. Note that in period $T - 1$, today's price actually affects tomorrow's price $p_T(k_{T-1}) = v(k_{T-1})$.



(a) NMB of increasing the price at k_1^* as a function of δ

(b) $\pi_1(k)$

Figure 4: Profitability of deviations from public offer equilibrium for $T = 3$, $A = B = 0.5$, $F_1(c) = c$

The NMB for a buyer to deviate from a given PBE with public offers, as a function of δ, is illustrated in panel (a) of Figure 4 for the same example. One can see how the NMB for private offers is always higher and it becomes positive for high δ before it converges to zero. Note that the difference between the two curves is solely driven by the different elasticities of supply curves in the two information structures. Finally, Figure 4 (b) shows the expected profits of a buyer conditional on receiving the good in the first period as a function of the cutoff k_1 corresponding to the buyer's price offer p_1 for the same example. With public offers, Bertrand competition pushes the equilibrium cutoff up to $k_1^*(\delta)$ since as long as expected profits are positive, buyers want to set higher prices in order to outbid the competitor. However, given the competitor is offering $p_1\left(k_1^*(\delta)\right)$ deviations to higher prices would lead to losing money and deviations to lower prices would never be accepted. For low δ this is still the case with private offers and hence the public offers equilibrium is also an equilibrium with private offers. Instead, when δ is sufficiently high (e.g. for $\delta = 0.8$) it becomes profitable for a buyer to increase the price given the other buyer is offering $p\left(k_1^*(\delta)\right)$. Hence, the pure strategy equilibrium with public offers collapses with private offers.

Remark 2 *A natural question one can ask is whether a similar result holds if T is held fixed, but the time intervals, Δ are made smaller. In that case, the discount factor from one period to the other $e^{-r\Delta}$ goes to 1, but the number of periods increases at the same time $N = \frac{T}{\Delta}$. We will show in section 6 that for small Δ, no pure strategy PBE exists with private offers.*

4.2 The Linear Case

The problem with the general setting is that the set of pure strategy equilibria can possibly be very rich. Hence, in the following, we will focus on the case with linear valuation functions and uniform cost distribution, so that there are unique pure strategy equilibrium cutoffs with public offers. In addition, the expressions are somewhat more tractable. From now on, let $v(c) = Ac + B$, $A, B > 0$, $A + B = 1$ and $F(x) = x$, $x \in [0, 1]$.

First, we show that with private offers there is a $\delta^* = 1 - \frac{A}{2}$ that partitions the space $[0, 1]$ of discount factors into two fundamentally different regions. For $\delta < \delta^*$, there are no mixed PBEs, i.e. the unique PBE is the one that corresponds to the pure strategy PBE with public offers, while for $\delta > \delta^*$, there is no pure strategy PBE. Moreover, we can show that in any mixed PBE, mixing has to be discrete in periods $t > 1$. Using these results, we are able to fully characterize mixed strategy PBEs for the three period case and thereby show their existence for $\delta > \delta^*$ in order to finally make welfare statements in section 5. [15]

Theorem 3 *(Private offers; linear) With private offers, $v(c) = Ac + B$ and uniformly distributed costs, there exists $\delta^* = 1 - \frac{A}{2}$ such that*
(i) for $\delta < \delta^*$, there are only pure strategy PBEs with prices that coincide with the unique equilibrium prices in the public offers game, i.e. there are no PBEs in which buyers play mixed strategies, and
(ii) for $\delta > \delta^*$, there is no pure strategy equilibrium.

Recall that

$$\pi_t(k_t^*; k_{t-1}^*) = \frac{k_t^* - k_{t-1}^*}{1 - k_{t-1}^*} \left[\frac{A}{2} k_t^* + \frac{A}{2} k_{t-1}^* + B - p_t(k_t^*) \right].$$

Hence, with private offers, by the chain rule and the zero profit condition (10), the marginal net benefit of a buyer of deviating to a higher price is given by

$$\frac{\partial}{\partial k_t} \pi_t(k_t; k_{t-1}^*)\big|_{k_t = k_t^*} \quad = \quad \frac{k_t^* - k_{t-1}^*}{1 - k_{t-1}^*} \left[\frac{A}{2} - (1 - \delta) \right]$$

which is just the linear version of (11). For $\delta \leq 1 - \frac{A}{2}$, the pure strategy equilibrium with private offers coincides with the equilibrium with public offers since $\frac{\partial \pi_t}{k_t}(k_t^*; k_{t-1}^*) \leq 0$, i.e. none of the buyers has an incentive to marginally increase the price offer. Because of the quadratic structure of $\pi_t(\cdot; k_{t-1}^*)$, this implies that buyers do not want to deviate to any higher price offers. In contrast, for $\delta > 1 - \frac{A}{2}$, $\frac{\partial \pi_t}{k_t}(k_t^*; k_{t-1}^*) > 0$, so it is a profitable deviation for a buyer, to increase the price a little bit if offers are made privately. Hence, the pure strategy PBE cutoffs in the public offers setting are not supported by a PBE with private offers.

With a linear valuation function and uniform cost, we can additionally show that for $\delta < 1 - \frac{A}{2}$ the pure strategy equilibrium prices are the unique equilibrium outcome with private offers. Hence, $1 - \frac{A}{2}$ partitions the interval $[0, 1]$ of discount factors into a region in which there are only pure PBEs and a region in which there are no pure strategy PBEs with private offers. In order to prove this statement, we need to understand the structure of mixed strategy equilibria if offers are private.

[15] We will briefly discuss the case $\delta = \delta^*$ in section 5.

Therefore, let us look at the uniform and linear analogue of the buyer's expected profit function given by (4):

$$\pi_t(k_t; K_{t-1}^m) = \int_0^{k_t} \int_0^{c} \frac{1}{1-k} dK_{t-1}^m(k) \left(Ac + B - p_t(k_t)\right) dc. \tag{12}$$

In order to have mixing on an interval $[a, b]$ in period t, buyers must be indifferent between all prices that result in cutoffs in $[a, b]$. Hence, $\pi_t(k; K_{t-1}^m) = 0$ for all $k \in [a, b]$. However, this requires that there must be continuous mixing in previous periods on that interval since otherwise the expected profit function is piecewise quadratic and non-constant as can be seen in the formal proof in the appendix. The intuition is that expected profits can be kept at a non-negative level for higher prices if the current cutoff type is high with some probability. The formal proof can be found in the appendix. Furthermore, by induction we can conclude that there must be continuous mixing on $[a, b]$ in period 1 whenever there is continuous mixing in any later period $t > 1$ on $[a, b]$. It turns out in the proof in the appendix that this leads to a contradiction since this would imply that in period 2 cutoffs must be distributed according to $K_2(k) = \frac{\delta - 1 + \frac{A}{2}}{\delta \frac{\partial}{\partial k_2} p_2(k)}$ on $[a, b]$ which is a proper cdf if and only if $\delta > 1 - \frac{A}{2}$ and $p_2(k)$ is constant. In order to see that p_2 must be constant, we need the following lemma which shows that prices are convex in cutoffs.

Lemma 5 *(Convex prices; private offers) With private offers, $p_t(k)$ is differentiable and*

$$\frac{\partial}{\partial k_t} p_t(k_t) = 1 - \sum_{s=1}^{T-t-1} \delta^s \left(\prod_{u=t+1}^{t+s-1} K_u(k_t) \right) \cdot (1 - K_{t+s}(k_t)) - \delta^{T-t-1} \cdot (1 - A) \cdot \left(\prod_{u=t+1}^{T-1} K_u(k_t) \right) (> 0)$$

is nondecreasing.

Proof. We argue by backward induction over t. In period $T-1$, $p_{T-1}(k_{T-1}) = \delta\left(Ak_{T-1} + B\right) + (1-\delta)k_{T-1} = (1 - \delta + \delta A)\, k_{T-1} + \delta B$ is differentiable and $\frac{\partial}{\partial k_{T-1}} p_{T-1} = 1 - \delta + \delta A$. Given p_{t+1} is differentiable with a nondecreasing derivative, (2) is differentiable and

$$\frac{\partial}{\partial k_t} p_t(k_t) = \delta K_{t+1}(k_t) \frac{\partial}{\partial k_{t+1}} p_{t+1}(k_t) + (1 - \delta)$$

is nondecreasing and piecewise constant. ■

Recall that with private offers, today's actual prices do not affected future price offers. Nevertheless, in order to attract a marginally higher seller type today, the buyers have to compensate the seller for valuing the good more and for the opportunity to sell in later periods which affects today's price intricately if future buyers play mixed strategies. If in equilibrium, a type k seller is not attracted by future prices with some probability α, then attracting this marginal type requires a price that takes into account that in the last period he can get $v(k) = Ak + B$. Hence, attracting this type requires to increase the price by the discounted αA in addition to $(1 - \delta)$ which solely compensates the marginally higher type for valuing the good more. Since the probability of no agreement before the last period α is increasing in the type, the marginal price increase that is necessary for attracting a type k seller is increasing in k or put differently, p_{t+1} is weakly convex. This shows how past and future cutoffs affect today's expected profit functions of the buyers in very different ways. Broadly speaking, past cutoffs boost today's expected profits while future cutoffs harm today's expected profits. This insight will also play a crucial role when we construct a mixed strategy equilibrium for $T = 3$.

Remark 3 *It is interesting to note that the convexity of prices can be generalized to the case when $v(\cdot)$ is a weakly convex function. Nevertheless, a buyer's expected profit function does not have the nice piecewise quadratic structure, so that the following analysis is less tractable than in the linear case.*

We have summarized some properties of mixed strategy equilibria with private offers in the following proposition.

Proposition 1 *(Mixed Strategy Equilibria; Linear Case) With private offers and $\delta > 1 - \frac{A}{2}$ all mixed strategy equilibria must satisfy the following properties.*
(i) In all periods $t > 1$, buyers mix between at most countably many prices.
(ii) If buyers in period 1 mix continuously between prices that result in cutoffs in an interval (a, b), then buyers in periods $t > 1$ do not choose any price that results in a cutoff in (a, b) with positive probability.

The technical implication of this proposition is that expected profits of buyers $\pi_t(k_t; K^m_{t-1})$ in periods $t < T$ are continuous and piecewise quadratic in k_t wherever period 1 buyers do not mix continuously. Moreover, all kinks of π_t correspond to cutoffs in previous or future periods. At every cutoff of a future period, π_t has a downward kink because the convexity of $p_t(k_t)$ and the shape of π_t can change from a parabola that is open above to a parabola that is open below (as in figure 5) since the coefficient in front of k_t^2 is decreasing. Moreover, it is worth noting that all pieces of $\pi_{T-1}(\cdot; K^m_{T-2})$ are parabolas that are open below since $\frac{A}{2} - (1 - \delta) - \delta A < 0$.

4.3 Existence and Full Characterization of Equilibria for $T = 3$

We have already shown that for $\delta < 1 - \frac{A}{2}$, the unique equilibrium with private offers coincides with the pure strategy equilibrium with public offers characterized above. From now on, we will assume $\delta > 1 - \frac{A}{2}$ to focus on the most interesting case in which equilibria differ across information structures. A full characterization of the equilibria with private offers for general T is a daunting task. Hence, we will only provide a characterization for $T = 3$. There are possibly multiple mixed strategy equilibria, yet all of the equilibria share some common properties. In particular, in any equilibrium, the second period strategies and first period expected cutoff type are the same across equilibria. We end this section by fully characterizing the equilibrium for the case in which the buyers mix only between a non offer and one serious offer (i.e. 2 cutoffs) in the first period and in the process establish the existence of equilibria. We will use these results in order to make a welfare comparison between private and public offers in the section 5.

The first proposition fully characterizes the strategies chosen by period 2 buyers. Moreover, it states the interesting fact that buyers in period 1 must make a non-offer that is accepted by no seller type with positive probability. The key of the proof is to reveal that the expected profit functions must look like in figures 5 (b) and 6.

Proposition 2 *In any mixed strategy equilibrium, the following two must hold:*
(i) Period 2 buyers mix between exactly two prices that result in the two cutoffs given by

$$\underline{k_2} = \frac{B(1-\delta)}{A\delta - \delta + 1 - \frac{A}{2}}, \qquad \overline{k_2} = \frac{B(1-\delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{A}{2}},$$

where $\underline{k_2}$ *is chosen with probability* $q \equiv \frac{\frac{A}{2}-(1-\delta)}{\delta(A\delta+1-\delta)}$.
(ii) Period 1 buyers mix between 0 *and cutoffs that lie between* $\underline{k_2}$ *and* $\overline{k_2}$ *where* 0 *is chosen with positive probability.*

Proof. We have organized the proof in three steps. First, we show in step 1 that buyers in period 2 mix between exactly two prices and the first part of (ii). Step 2 discusses the second part of (ii), i.e. that there must be non-offers with positive probability in period 1. Finally, in step 3 we can pin down the exact values of $\underline{k}_2$ and $\overline{k}_2$.

Step 1: Period 2 buyers mix between exactly two prices resulting in cutoffs $\underline{k}_2, \overline{k}_2$ *and period 1 cutoffs must be in* $\{0\} \cup [\underline{k}_2, \overline{k}_2]$.

First, note that with $T = 3$, buyers in period 1 and 2 must mix between at least two cutoffs. The reason is that if buyers in period 1 would play pure strategies, then there is a unique price at which buyers in period 2 make zero profits, i.e. the unique Bertrand equilibrium in that period contains only pure strategies of the buyers. If buyers in period 2 played pure strategies in equilibrium, then the same argument holds for period 1 expected profits. Since we have already established in Theorem 3 that if $\delta > 1 - \frac{A}{2}$ there cannot be pure strategy equilibria, there must be mixing in both periods.

Let us first consider the continuation game in period 2 given beliefs about the current cutoffs represented by the cdf K_1. With $T = 3$, the indifference condition of the seller in period $T - 1$ (3) can be written as

$$p_2(k_2) = (1-\delta)k_2 + \delta(Ak_2 + B)$$

which represents the price that is accepted by all seller types in $[0, k_2]$ and rejected by all other seller types in period 2. Then, buyers' expected profits in period 2 (conditional on receiving the good), if they choose a price of $p_2(k_2)$, are given by

$$\pi_2(k_2; K_1) = \int_0^{k_2} \left(\int_0^c \frac{1}{1-k_1} dK_1(k_1) \right) (Ac + (1-\delta)B - k_2(\delta A + 1 - \delta))dc.$$

Note that π_2 is continuous and at the smallest element $k_1^m < 1$ in the support of K_1, for all ϵ small enough we have

$$\begin{aligned}
\frac{\partial \pi_2}{\partial k_2}(k_2; K_1)\big|_{k_2 = k_1^m + \epsilon} &= \int_0^{k_1^m+\epsilon} \frac{1}{1-k_1} dK_1(k_1)(1-\delta)\left(B + (k_1^m + \epsilon)(A-1)\right) \\
&\quad - \int_{k_1^m}^{k_1^m+\epsilon} \left(\int_0^c \frac{1}{1-k_1} dK_1(k_1) \right) dc(A\delta + 1 - \delta) \\
&> \int_0^{k_1^m+\epsilon} \frac{1}{1-k_1} dK_1(k_1) \left[(1-\delta)\left(B + k_1^m(A-1)\right) - \epsilon(2(A\delta + 1 - \delta) - A)\right] \\
&> 0,
\end{aligned}$$

so in equilibrium, period 2 buyers do not choose prices that result in a cutoff type smaller or equal to k_1^m with positive probability since if they did increasing the price a little bit would be a profitable deviation for any buyer. In particular, in any equilibrium, seller types close to zero trade in period 2, that is in the continuation game starting at period 2 it must hold that $K_2(k_2) = 0$ for small k_2.

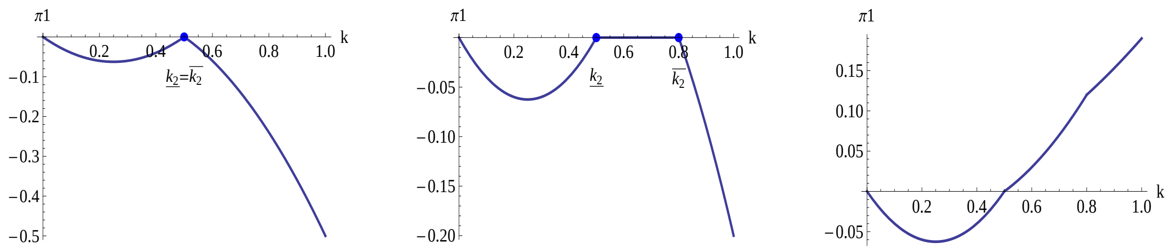
Let us now analyze the behavior of agents in period 1. By the Reverse-Skimming Property and (2), we have that in period 1, a price of

$$p_1(k_1) = \delta \left[\left(\int_{k_1}^1 p_2(k_2) dK_2(k_2) \right) + K_2(k_1) p_2(k_1) \right] + k_1(1-\delta) \tag{13}$$

is accepted by types in $[0, k_1]$ and rejected by all other seller types. By Proposition 1, buyers in period 2 can only mix between discretely many cutoffs, so the support of K_2 is discrete and $p_1(\cdot)$ is piecewise linear, continuous and by lemma 5, it is also weakly convex. Buyers' expected profits in period 1 if they choose a price of $p_1(k_1)$ are given by

$$\pi_1(k_1; 0) = k_1 \cdot \left(\frac{A}{2} k_1 + B - p_1(k_1) \right).$$

$\pi_1(\cdot)$ is continuous, piecewise quadratic and at any period 2 cutoff it has a "downward" kink (that is the slope is dropping discontinuously) because of the convexity of p_1. Hence, in equilibrium, period 1 expected profits must qualitatively look like one of the graphs in figure 5. Note that for small k_1, $p_1(k_1) = \delta \int_{k_1}^1 p_2(k_2) dK_2(k_2) + k_1(1-\delta)$ because $K_2(k_2) = 0$ for small k_2. Hence, the parabola most to the left must be open above because $\frac{A}{2} - (1-\delta) > 0$. We have already argued that buyers must mix between at least two prices in every period, so we can exclude the possibility of the expected profit function in period 1 having a shape as in figure 5 (c). Hence, one can see



(a) Mixing between two cutoffs (b) Mixing on an interval (c) No mixing

Figure 5: Possible shapes of buyers' profits in period 1

that there exist period 2 cutoffs $0 < \underline{k}_2 \leq \overline{k}_2 < 1$ such that period 1 buyers choose only prices with positive probability that are in $\{0\} \cup [\underline{k}_2, \overline{k}_2]$.

Using these insights about π_1, we can conclude that π_2 is piecewise quadratic on $[0,1] \setminus [\underline{k}_2, \overline{k}_2]$ where the coefficient in front of $(k_2)^2$ is negative as a multiple of $\frac{A}{2} - (1-\delta) - \delta A < 0$. Hence, all pieces of π_2 are open below. At every cutoff that is chosen with positive probability in period 1, π_2 has a kink. Hence, period 2 expected profits are qualitatively as in figure 6. Note however, that π_2 does not have to be piecewise quadratic in $[\underline{k}_2, \overline{k}_2]$ as in figure 6.

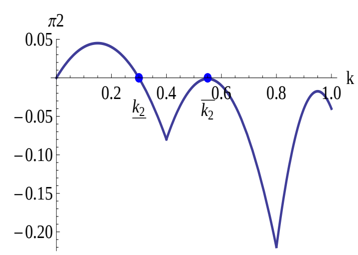


Figure 6: Qualitative shapes of buyers' expected profits in period 2

Next, we will argue that π_1 must look like in figure 5 (b). Let us first assume that none of the pieces of π_1 is constant and equal to zero as is the case in figure 5 (a). Then, in a mixed strategy equilibrium, buyers in period 1 mix between exactly two prices that result in cutoff types 0 and $k_1 = \underline{k}_2 = \overline{k}_2$, respectively. Moreover, k_1 must be a cutoff type in period 2, because it corresponds to a kink of π_1. Thus, we can conclude $\pi_1(k_1) = \pi_2(k_1) = 0$ and $\pi_2(k) \leq 0$ for all $k \geq k_1$. In addition, π_2 has its only kink at k_1, so buyers do not mix between prices in period 2, but choose a price with probability one that results in a cutoff k_1. This contradicts Theorem 3. Hence, there cannot be an equilibrium where none of the pieces of π_1 is constant and equal to zero. Finally, we can conclude that period 2 buyers must mix between exactly two cutoffs $\{\underline{k}_2, \overline{k}_2\}$ only. This can be seen as follows: One can infer directly from Proposition 1 (ii) that period 2 buyers do not choose prices that result in cutoffs in $(\underline{k}_2, \overline{k}_2)$. Moreover, because $\pi_1(k) = 0$ on $\{0\} \cup [\underline{k}_2, \overline{k}_2]$ only, π_2 can have kinks in that region only. Hence, $\pi_2(\underline{k}_2) = \pi_2(\overline{k}_2) = 0$, $\pi_2(k) \leq 0$ for $k \geq \underline{k}_2$ and the fact that π_2 is piece-wise quadratic on $[0, \underline{k}_2] \cup [\overline{k}_2, 1]$ with parabolas that are open below imply that $\pi_2(k) > 0$ for $k \in (0, \underline{k}_2)$ and $\pi_2(k) < 0$ for $k \in (\overline{k}_2, 1]$.

Thus, in any equilibrium the support of K_1 is a subset of $\{0\} \cup [\underline{k}_2, \overline{k}_2]$ and the support of K_2 is $\{\underline{k}_2, \overline{k}_2\}$ for some $\underline{k}_2, \overline{k}_2 \in (0, 1]$. Let $K_2(\underline{k}_2) = q$ and $K_1(0) = r$, noting that we already know from Lemma 6 that $q = \frac{\delta - 1 + \frac{A}{2}}{\delta(1-\delta+\delta A)} \neq 0$.

Step 2: In any mixed strategy equilibrium, there must be non-offers with positive probability in period 1, i.e. $r > 0$.

Let us assume $r = 0$ and let us denote the smallest element in the support of K_1 by $\underline{k} < 1$. Note that $Ak + (1-\delta)B - k(\delta A + (1-\delta)) = (1-\delta)(k(A-1)+B) \geq (1-\delta)B(1-k)$ which is strictly positive for $B > 0$ and $k < 1$. Hence, there exists an $\epsilon > 0$ such that $A\underline{k} + (1-\delta)B - (\underline{k}+\epsilon)(\delta A + (1-\delta)) > 0$. Then, $\pi_2(\underline{k} + \epsilon) > 0$ which is a contradiction to $\underline{k}_2 < \underline{k}$ being in the support of K_2.

Step 3: $\underline{k}_2 = \frac{B(1-\delta)}{A\delta - \delta + 1 - \frac{A}{2}}$ *and* $\overline{k}_2 = \frac{B(1-\delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{A}{2}}$

In equilibrium, it must hold that $\pi_2(\underline{k}_2) = 0$, that is

$$\int_0^{\underline{k}_2} Ac + (1-\delta)B - \underline{k}_2(\delta A + 1 - \delta) dc = \underline{k}_2 \cdot \left(\frac{A}{2}\underline{k}_2 + (1-\delta)B - \underline{k}_2(\delta A + 1 - \delta) \right) = 0$$

which is equivalent to $\underline{k}_2 = \frac{B(1-\delta)}{A\delta - \delta + 1 - \frac{A}{2}}$. For $\overline{k}_2$, we use that $\pi_1(\overline{k}_2) = 0$ since this is equivalent to

$$\overline{k}_2 \cdot \left(\frac{A}{2}\overline{k}_2 + (1-\delta^2)B - \delta(\delta(A-1)+1)\overline{k}_2 - (1-\delta)\overline{k}_2 \right) = 0$$

because $K_2(\overline{k}_2) = 1$. Hence, $\overline{k}_2 = \frac{B(1-\delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{A}{2}}$. ∎

Even though the equilibrium strategy in period 1 is not unique, all equilibrium strategies have some properties in common. In particular, the expected cutoff type is constant across equilibria as the following lemma shows.

Proposition 3 *(Constant Expected Cutoff) Let* $T = 3$ *and* $\delta > 1 - \frac{A}{2}$*. In any mixed strategy equilibrium, the expected period 1 cutoff is constant. In particular,*

$$\int_0^{\overline{k}_2} k dK_1(k) = \frac{(1-\overline{k}_2)\left(1 - \frac{\delta}{1+\delta}\overline{k}_2\right)}{1 - \overline{k}_2 \cdot \frac{(1+\delta)(1-\delta+A\delta) - \frac{A}{2}}{(1+\delta)(1-\delta+A\delta)}} + \frac{1+2\delta}{1+\delta}\overline{k}_2 - 1. \tag{14}$$

Moreover, the following must hold

$$\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k) = \frac{1}{1 - \overline{k}_2 \cdot \frac{(1+\delta)(1-\delta+A\delta) - \frac{A}{2}}{(1+\delta)(1-\delta+A\delta)}}. \tag{15}$$

These statements follow from the fact that $\pi_2(\overline{k}_2) = 0$ and $\pi_2(K) \le 0$ for all $K \ge \overline{k}_2$. The key of the proof is to apply Fubini's theorem and to note that the zero profit condition at $\overline{k}_2$ is a function of $\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)$ and $\int_0^{\overline{k}_2} k dK_1(k)$ and that the second condition, that expected profits are non-positive for cutoffs greater than $\overline{k}_2$, pins down $\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)$ uniquely. The economic intuition is that given that $\pi_2(\overline{k}_2) = 0$, a buyer's period 2 expected profit for prices that result in higher cutoffs than $\overline{k}_2$ should only depend on the average distribution of remaining types given by the density $\overline{f}(c) = \int \frac{1}{1-k_1} dK_1$ since $k_1 < \overline{k}_2$ with probability one. Note that the conditions we have found are necessary but not sufficient for PBE outcomes with private offers. In particular, with the above period 2 outcomes, $\pi_1(\underline{k}_2) = 0$ must not be satisfied. As a last step, we prove that with private offers, a PBE indeed exists by constructing an equilibrium in which the buyers in period 1 mix between exactly two cutoffs.

Proposition 4 *(Existence) For $T = 3$ and $\delta > 1 - \frac{A}{2}$, a PBE always exists.*

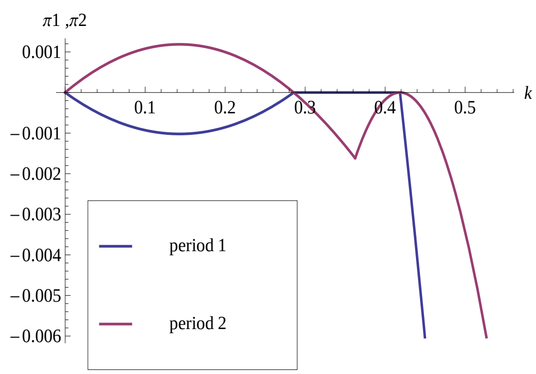


Figure 7: Profits in period 1 and 2 with $T = 3$ and $v(c) = 0.5 + 0.5c$

Figure 7 illustrates for $v(c) = 0.5+0.5c$ the expected profit functions π_1 and π_2 in the equilibrium that we have constructed in the proof of proposition 4 in the appendix. It highlights how period 1 cutoffs must corresponds to kinks of π_2 and period 2 cutoffs must correspond to kinks of π_1. This is, however, not a unique equilibrium. In particular, there can potentially be equilibria in which period 1 buyers mix between $\{0\}$ and prices in $(\underline{k}_2, \overline{k}_2)$, so that (14) and (15) are satisfied.

5 Efficiency of Dark versus Transparent Markets

Given the multiplicity of equilibria with private offers and the difficulty of constructing equilibria we will focus our efficiency comparisons on the linear case with $T = 3$. We believe our results likely generalize beyond this case but our inability to obtaining a tractable characterization prevents us from formally establishing such results. The main reason we think the result extends is that the economic force behind the results does not depend on the model being linear or the horizon being short. The driver for our result is that private offers lack the signaling value of public offers and hence sellers are always more willing to accept a private offer. Since the inefficiency arises because there is too little trade, eliminating the possibility to signal for the seller helps to generate more trade, which in turn increases efficiency.

First, we show that with private offers, there is more trade after each period. Since the expected cutoff is constant in periods 1 and 2, this result holds for all perfect Bayesian equilibria in the private offers case.

Proposition 5 *(Expected Trade) Let $T = 3$. Expected trade up to period 1 and 2, respectively, is always greater with private offers than with public offers.*

In period 1, there is more trade because, sellers are more inclined to accept a price offer with private offers than with public offers because they do not have a signaling incentive with private offers. In order to keep the period 1 supply small enough, so that period 1 buyers do not make positive expected profits with high price offers, period 2 buyers must set relatively high prices in period 2. This creates high period 2 cutoffs. On the other hand, period 1 buyers must make non-offers with a positive but small probability in order to cap period 2 expected profits without decreasing the expected trade in period 1 too much. This proposition points to the fact that with private offers efficiency is enhanced. However, the expected total surplus also depends on the second moment of cutoffs generated by the mixed strategies of buyers, so that an additional step is necessary in order to establish a clean welfare comparison. Let $G(x) = \frac{A-1}{2}x^2 + Bx$. Then, with public offers, total expected surplus is given by

$$\begin{aligned} V(\delta, A, B, \text{public}) &= \int_0^{k_1^*} (A-1)c + Bdc + \delta \int_{k_1^*}^{k_2^*} (A-1)c + Bdc + \delta^2 \int_{k_2^*}^{1} (A-1)c + Bdc \\ &= (1-\delta)G(k_1^*) + \delta(1-\delta)G(k_2^*) + \delta^2 G(1). \end{aligned}$$

and with private offers, total expected social surplus is given by

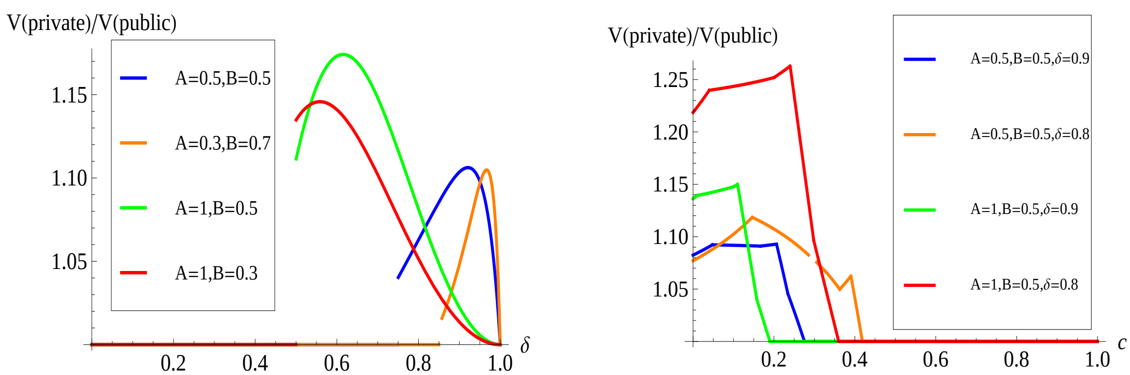
$$\begin{aligned} V(\delta, A, B, \text{private}) &= \int_0^1 \int_0^1 \int_0^{k_1} (A-1)c + Bdc + \delta \int_{k_1}^{\max\{k_1,k_2\}} (A-1)c + Bdc \\ &\quad + \delta^2 \int_{\max\{k_1,k_2\}}^{1} (A-1)c + Bdc \, dK_1(k_1) \, dK_2(k_2) \\ &= \int_0^1 \int_0^1 (1-\delta)G(k_1) + \delta(1-\delta)G(\max\{k_1,k_2\}) \, dK_1(k_1) \, dK_2(k_2) + \delta^2 G(1) \end{aligned}$$

if period 1 cutoffs and period 2 cutoffs are distributed according to K_1 and K_2, respectively. Note that if $A-1 > 0$, then it would follow immediately that private offers are more efficient by Jensen's inequality. For $A-1 < 0$ we need an additional technical step in order to prove the result. Since we assume $A + B = 1$ throughout this paper, we are in the case in which $A - 1 < 0$. We will argue, however, in chapter 6 that this result holds in greater generality.

Theorem 4 *(Efficiency) With $T = 3$, linear valuations and uniformly distributed costs, the unique equilibrium with public offers is weakly Pareto-dominated by any equilibrium with private offers. If $\delta > 1 - \frac{A}{2}$, then the unique equilibrium with public offers is strictly Pareto-dominated by any equilibrium with private offers.*

This theorem is a very clear-cut statement. No matter which equilibrium agents end up playing in the private offer environment, the equilibrium outcome will lead to higher efficiency than in the unique equilibrium of the public offer environment. Hence, at least with linear valuations and uniformly distributed types, the opaque environment strictly dominates the transparent environment.

Additionally, it would be natural to ask whether efficiency gains are monotone in the type or discount factors. The following graphs illustrate the welfare comparison for the equilibrium with private offers in which buyers mix between exactly two prices in each period.



(a) Average

(b) By type

Figure 8: Social surplus ratio $\frac{V(private)}{V(public)}$ if buyers mix between 2 prices in both periods

The next proposition follows immediately from figure 8.

Proposition 6 *i) The efficiency gains from making offers privately are generally non-monotonic in the type.*
ii) The average gains are generally non-monotonic in δ.

It is striking that in figure 8 (a), there seems to be a discontinuity in the welfare comparison at $1 - \frac{A}{2}$ when we consider the private offer equilibrium in which all buyers mix between exactly two prices. There is indeed a discontinuity for all equilibria. Let us therefore look at the limit as $\delta \to 1 - \frac{A}{2}$. One can easily show (simply by calculation) that in the limit, the expected period 1 cutoff with private offers $\mathbb{E}^{K_1}[k_1]$ (if period 1 buyers mix according to K_1) is equal to $\underline{k}_2$ which is greater than the public offer period 1 cutoff k_1^*:

$$\lim_{\delta \to 1-\frac{A}{2}} \underline{k}_2 = \lim_{\delta \to 1-\frac{A}{2}} \mathbb{E}^{K_1}[k_1] > \lim_{\delta \to 1-\frac{A}{2}} k_1^*.$$

Moreover, with private offers, the period 2 cutoff is $\overline{k}_2$ with probability one, i.e. buyers do not mix in the limit, and $\overline{k}_2$ is equal to the public offer period 2 cutoff k_2^*:

$$\lim_{\delta \to 1-\frac{A}{2}} \overline{k}_2 = \lim_{\delta \to 1-\frac{A}{2}} k_2^* \text{ and } \lim_{\delta \to 1-\frac{A}{2}} q = 0.$$

Hence, in the limit, there is *strictly* less expected trade in period 1 with public offers and period 2 cutoffs and prices coincide with both information structures. In the limit, period 1 expected profits with private offers are zero for all cutoffs smaller than $\overline{k}_2$, i.e. period 1 buyers are indifferent between all prices lower than $p_1(\overline{k}_2)$. The reason is that the marginal revenue from increasing the price is $A/2$ and the marginal cost is $1 - \delta = \frac{A}{2}$. Hence, there is a rich set of period 1 buyer strategies that constitute an equilibrium. It includes the PBE resulting in the public offer PBE outcome (k_1^*, k_2^*), as well as the limiting private offer PBEs, i.e. the continuity of the correspondence that maps δ to the set of PBEs is not violated, but the smallest equilibrium efficiency level with private offers for $\delta > 1 - \frac{A}{2}$ is strictly bounded away from the efficiency level resulting from the public offer PBE. As an illustrative example, Figure 5 depicts the expected profit functions $\pi_1(k)$ for private and public offers and $\pi_2(k)$ for public offers.

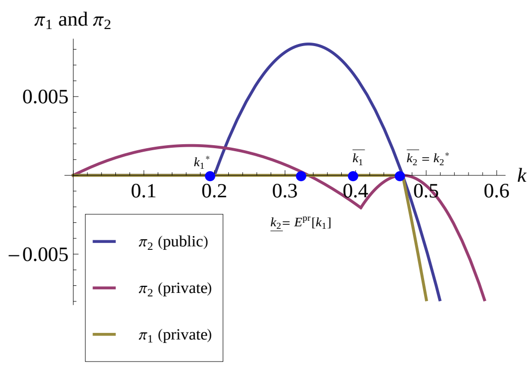


Figure 9: Profits as $\delta \to 1 - \frac{A}{2}$ with $T = 3$ and $v(c) = 0.5 + 0.5c$

6 Robustness and discussion

In this section, we discuss the role and restrictiveness of the assumptions that we have made throughout the paper. First, we examine the assumption that there are strictly positive gains from trade for all types $c < 1$ and no gains from trade for the highest type $c = 1$. Moreover, for the main results of the paper, we have assumed that the number of periods before T is fixed and we were interested in how the equilibria change as the seller becomes more patient, i.e. as δ changes. A related question to ask is how public and private offer equilibria relate to each other if the length of periods become smaller. We will conclude this section by presenting an analogue to theorems 1 and 2 as the commitment level changes. In particular, we show that for small levels of commitment, there is never a pure strategy equilibrium with private offers.

6.1 Gap at the top $v(1) > 1$

Throughout the paper, we have assumed that $v(1) = 1$. This assumption together with continuity and monotonicity of $v(c)$ guarantees that in equilibrium, in each period $t < T$, there is a positive mass of high type sellers who do not want to sell. The reason is that the highest type $c = 1$ never wants to trade with buyers who are only willing to pay a price equal to the expected valuation of the good which is always smaller than 1. Hence, there is no price with which buyers make non-negative expected profit and which all seller types should accept. As a result, we did not have to worry about off-equilibrium beliefs of the buyers about the seller type when an offer that is supposed to be accepted by everyone is rejected. This type of freedom in choice of off-equilibrium beliefs can lead to a huge multiplicity of equilibria.

Nevertheless, all results can easily be generalized to settings with $v(1) > 1$ if one of the following two conditions is additionally satisfied.

1. Either we need that the adverse selection problem is severe enough to guarantee that trade does not end with probability one before period $T - 1$ or

2. we need some restrictions on off-equilibrium beliefs.

In order to make the adverse selection problem strong enough, we have to restrict attention to high discount factors $[\overline{\delta}, 1]$ for some $\overline{\delta}$ since if the seller is patient enough he does not want to sell off the good too quickly. Hence, the equilibrium we characterized is the unique one. One possible restrictions on off-equilibrium beliefs that would sustain our equilibrium construction would be, for example, that if an offer that should be accepted by all seller types is rejected, then the belief of buyers in future periods coincide with the beliefs of buyers at the beginning of the period in which this off-equilibrium deviation took place. That way, it is never profitable for the seller to reject the equilibrium offer.

Finally, note that making these alternative assumptions, our results indeed extend to cases in which $A + B > 1$. However, the extremely cumbersome computations would obscure the main ideas of the paper if we only assumed $A + B \geq 1$. In figure 8 we have included some examples with $A + B > 1$ as an illustration.

6.2 No Gap at the bottom ($v(0) = 0$)

We made the assumption that $v(0) > 0$ to make sure that there is always some trade in period 1. If we have no gap at the bottom then it is possible for trade to completely unravel in all periods before information is revealed. The following lemma shows this effect in a setting with linear valuations and uniform cost distribution. It is interesting to note that this result holds true with both private and public offers and for all δ.

Proposition 7 *(No gap case) Let us assume $B = 0, A > 1$ and $F(c) = c$. With private and with public offers, the unique equilibrium outcome is that no seller type trades before T.*

6.3 High frequency of trade

One natural question to ask is how public and private offers differ as commitment goes to zero. Let us therefore consider a setting with a finite time horizon T, periods of length Δ and $\delta = e^{-r\Delta}$ where r is the rate at which sellers discount. We are interested in the case $\Delta \to 0$. The existence of pure strategy equilibria is an immediate corollary of theorem 1.

Corollary 1 *For any $\Delta > 0$ there exists a pure strategy equilibrium with public offers.*

Moreover, using a similar argument as in theorem 2, one can show the following theorem.

Theorem 5 *If $f(c), v'(c) > 0$ for all $c \in [0, 1]$, then there exists a Δ^* so that for all $\Delta < \Delta^*$ there is no pure strategy equilibrium with private offers.*

The intuition for this result is, that the non-existence of pure strategy equilibria with private offers is driven by the high elasticity of the supply curve faced by buyers. Locally, this elasticity only depends on the discounting from one period to the next period because the cutoff seller's trade-off is only between buying today or tomorrow. Hence, the number of periods is not essential for the marginal net benefit of deviating, but rather the marginal time preference of the seller.

Appendix

A Proofs

A.1 Model

Proof. (Lemma 1; Reverse-Skimming Property) We show that in an arbitrary period $t \leq T - 1$ the Reverse-Skimming property holds. First note, that with private offers, the strategy of period t buyers is independent of the price history while with public offers, it may be a function of past prices. Let us fix a history of prices $(p_s)_{s \leq t-1}$ and strategies of buyers (that are functions of past prices with public offers). The benefit of a buyer of type c of waiting until a future period $\tau < T-1$ and accepting a price p_τ (in the support of the pricing strategy of period τ buyers) instead of accepting today's price p_t is given by $\delta^{\tau-t}(p_\tau - c) - (p_t - c)$ is increasing in c. For period $T-1$, $\delta^{T-1-t}(v(c) - c) - (p_t - c)$ is also increasing in c. Hence, if a seller of type c accepts today's price, then every type $c' < c$ must also be willing to accept today's offer. Hence, buyers in every period accept offers according to a cutoff strategy $k_t(p)$. ■

Proof. (Lemma 2; Inverse supply) (i) (Private offers) With private offers, beliefs of buyers are independent of price histories. Hence, the continuation game in an equilibrium is unaffected by past offers. We argue by backward induction.

Base of induction: In period T-1, a seller of type c accepts an offer p if and only if $p \geq \delta v(c) + (1-\delta)c \equiv p_{T-1}(c)$. p_{T-1} is increasing and continuous because v is increasing and continuous. Hence, $p_{T-1}(k_{T-1})$ is the unique price that results in a cutoff k_{T-1} in period $T-1$.

Induction hypothesis: Let us now assume, there exists an increasing and continuous inverse supply function $p_{t+s}(\cdot) = k_{t+s}^{-1}(\cdot)$ $(s > 0,\ s+t \leq T-1)$ for periods after t. Hence, the pricing strategies of buyers in periods $t+1, \ldots, T-1$ can be mapped one-to-one to distributions of cutoffs. We denote the cdfs of cutoffs that result from the pricing strategies in the continuation game after period t by K_s, $s = t+1, \ldots, T-1$.

Induction step: In period t, the continuation payoff of a seller c who rejects an offer in period t is given by

$$\begin{aligned} V(c) &= \textstyle\sum_{u=1}^{T-t} \left(\delta^u \left(\prod_{v=t+1}^{t+u-1} K_v(c) \right) \cdot \int_c^1 (p_{t+u}(k_{t+u}) - c) dK_{t+u}(k_{t+u}) \right) + \\ & \quad \delta^{T-t+1}(v(c) - c) \left(\prod_{u=t+1}^{T-1} K_u(c) \right) + c \\ &= \delta \left[\left(\int_c^1 p_{t+1}(k_{t+1}) dK_{t+1}(k_{t+1}) \right) + K_{t+1}(c) p_{t+1}(c) \right] + (1-\delta)c. \end{aligned} \tag{16}$$

Since p_{t+1} is increasing, $V(c)$ is increasing and continuous and $p_t(c) = V(c)$ defines an inverse demand function, i.e. $p_t(k)$ is the unique price that results in a cutoff k.

(ii) (Public offers) We show first that an inverse supply function $p_t(\cdot)$ exists in every equilibrium and that the following statement about arbitrary continuation games must hold. Let us consider an arbitrary continuation game after period t that results in cutoffs $(k_{t+1}, \ldots, k_{T-1})$, i.e. $k_s = k_s(p_s)$ $(s = t+1, \ldots, T-1)$ if p_s is the highest price offer in period s, then these cutoffs do not have to be monotone over time. However, there is no trade in a period s if $k_s < k_{s'}$ for $s' < s$, so the price in that period does not affect the continuation payoff of any seller type. Let $u_i \in \{1, 2, \ldots\}$ be such that $(k_t, k_{t+u_1}, \ldots, k_{t+\sum_{i=1}^n u_i})$ be the largest subsequence with $k_{t+1} \leq k_{t+1+u_1} \leq \cdots \leq k_{t+\sum_{i=0}^n u_i}$ $(u_0 = 1)$. Then, we show that

$$p_{t+\sum_{i=1}^j u_i} \left(k_{t+\sum_{i=1}^j u_i} \right) = \delta^{u_{j+1}} p_{t+\sum_{i=1}^{j+1} u_i} \left(k_{t+\sum_{i=1}^{j+1} u_i} \right) + \left(1 - \delta^{u_{j+1}}\right) k_{t+\sum_{i=1}^{j+1} u_i} \tag{17}$$

for all $i = 1, \ldots n-1$. We again argue by backward induction.

Base of induction: First, note that in period $T-1$, independently of the history, sellers of type c accept a price offer p if $p > \delta v(c) + (1-\delta)c \equiv p_{T-1}(c)$ and reject the price p if $p < p_{T-1}$. Hence, $p_{T-1}(k_{T-1})$ is the unique price that results in a cutoff type k_{T-1}. p_{T-1} is continuous and increasing.

Induction hypothesis: There exist inverse supply functions $p_s(k)$ for all periods $s \leq t+1$ and (17) holds.

Induction step: Now, we will show that for every k_t there exists a unique $p_t(k_t)$ that results in a cutoff k_t and that if the cutoff $k_t \in \left[k_{t+\sum_{i=0}^{j} u_i}, k_{t+\sum_{i=0}^{j+1} u_i}\right)$, then

$$p_t(k_t) = \delta^{\sum_{i=0}^{j+1} u_i} p_{t+\sum_{i=0}^{j+1} u_i}\left(k_{t+\sum_{i=0}^{j+1} u_i}\right) + \left(1 - \delta^{\sum_{i=0}^{j+1} u_i}\right) k_t. \tag{18}$$

Let us take an arbitrary p_t that results in a cutoff k_t in equilibrium. Then, given this cutoff, buyers form consistent beliefs and a continuation game with cutoffs $(k_{t+1}, \ldots, k_{T-1})$ is realized. For a seller of type $c \in \left[k_{t+\sum_{i=0}^{j} u_i}, k_{t+\sum_{i=0}^{j+1} u_i}\right)$, the continuation payoff if she rejects the offer p_t is given by

$$\delta^{\sum_{i=0}^{j+1} u_i} p_{t+\sum_{i=0}^{j+1} u_i}\left(k_{t+\sum_{i=0}^{j+1} u_i}\right) + \left(1 - \delta^{\sum_{i=0}^{j+1} u_i}\right) c.$$

By the induction hypothesis, this continuation payoff is continuous in c on the whole type space and increasing in c. (In particular, it is piecewise linear with kinks at future cutoffs.) Thus, p_t must be equalt to this continuation payoff for type k_t. Hence, in equilibrium, there is only one price p_t that is accepted by all seller types $c < k_t$ and rejected by all seller types $c > k_t$ and (18) must hold. This concludes the induction.

Next, we show the second part of the lemma. Let us assume that in a pure strategy equilibrium $(k_1, \ldots, k_{T-1})$ there is a period t in which $k_t < k_{t-1}$, i.e. the period t buyers make non-offers that are not accepted by any remaining types. Then, if instead of offering $p_t(k_t)$, they would offer $p_t(k_{t-1})$ future beliefs of buyers and hence, the continuation game remains unchanged. Moreover, for previous periods $s < t$ the price offer in period s is irrelevant since all seller types smaller than k_{s-1} trade in period $s-1$ anyway. Hence, one can transform the equilibrium to an equilibrium $(k'_1, \ldots, k'_{T-1})$ that results in the same outcome and such that $k'_1 \leq \cdots \leq k'_{T-1}$. In this equilibrium, for all t $p'_t(k'_t) = p'_{t+1}(k'_{t+1}) + (1-\delta)k'_t$ if p'_t is the inverse supply function in this equilibrium.

■

A.2 Public Offers

Proof. (Theorem 1) (i): It remains to show that the functions $c_t^*(\cdot)$ inductively defined for $t = 1, \ldots, T-1$ by (**??**) and (**??**) are left-continuous.

Step 1: If $\pi(k; k_{t-1})$ is left-continuous in k, then c_t^* is increasing

First note, that because of left-continuity of $\pi_t(\cdot; k_{t-1})$, we either have $\pi_t(c_t^*(k_{t-1}), k_{t-1}) > 0$ or $\pi_t(c_t^*(k_{t-1}), k_{t-1}) = 0$. Moreover, note that $\pi_t(k; k_{t-1})$ is always differentiable in k_{t-1}. Let us consider an arbitrary k_{t-1} and an infinitesimal increase in k_{t-1}. If $\pi_t(c_t^*(k_{t-1}), k_{t-1}) > 0$, there exists an $\epsilon > 0$ so that $\pi_t(c_t^*(k_{t-1}), k_{t-1} + \gamma) > 0$ for all $\gamma < \epsilon$. Hence, $c_t^*(k_{t-1} + \gamma) > c_t^*(k_{t-1})$ for all $\gamma < \epsilon$. On the other hand, if $\pi_t(c_t^*(k_{t-1}), k_{t-1}) = 0$, then

$$\begin{aligned}
&\frac{\partial}{\partial k_{t-1}} \pi_T(k; k_{t-1})\big|_{k=c_t^*(k_{t-1})} \\
= \;& \frac{f(k_{t-1})}{1 - F(k_{t-1})} \cdot \left[\frac{1}{1 - F(k_{t-1})} \int_{k_{t-1}}^{c_t^*(k_{t-1})} (v(c) - p_t(c_t^*(k_{t-1})))f(c)dc - (v(k_{t-1}) - p_t(c_t^*(k_{t-1}))) \right] \\
= \;& -\frac{f(k_{t-1})}{1 - F(k_{t-1})} (v(k_{t-1}) - p_t(c_t^*(k_{t-1}))) > 0
\end{aligned}$$

because if we had $v(k_{t-1}) - p_t(c_t^*(k_{t-1})) \geq 0$, then $\int_{k_{t-1}}^{c_t^*(k_{t-1})} (v(c) - p_t(c_t^*(k_{t-1})))f(c)dc > 0$ because v is increasing and this is a contradiction to the zero-profit assumption $\pi_t(c_t^*(k_{t-1}), k_{t-1}) = 0$. Hence, $c_t^*(\cdot)$ is increasing at k_{t-1}.

Step 2: $c_t^*(\cdot)$, $p_t(\cdot)$ and $\pi_t(\cdot; k_{t-1})$ are left-continuous

We argue by backward induction in t. $p_{T-1}(\cdot)$ is left-continuous because v is continuous and hence, $\pi_{T-1}(k_{T-2}; k)$ is left-continuous in k. (It is even continuous.) Let $k_{T-2}^{(n)} \uparrow k_{T-2}$. Then, $c_{T-1}^*(k_{T-2}^{(n)}) \leq c_{T-1}^*(k_{T-2})$ for all n and $c_{T-1}^*(k_{T-2}^{(n)})$ is an increasing sequence by step 1. Hence, $\lim_{n\to\infty} c_{T-1}^*(k_{T-2}^{(n)})$ exists. We will show next that $\lim_{n\to\infty} c_{T-1}^*(k_{T-2}^{(n)}) = c_{T-1}^*(k_{T-2})$. Therefore, consider an arbitrary sequence $k^{(m)} \uparrow c_{T-1}^*(k_{T-2})$ such that $\pi_{T-1}(k^{(m)}; k_{T-2}) > 0$ (which must exist by definition of c_{T-1}^*). Then, for any m, there exists an $n(m)$ such that $\pi_{T-1}(k^{(m)}; k_{T-2}^{(n)}) > 0$ for all $n \geq n(m)$ because $\pi_{T-1}(k; \cdot)$ is continuous for all k. Hence, $k^{(m)} \leq c_{T-1}^*(k_{T-2}^{(n(m))}) \leq c_{T-1}^*(k_{T-2}) = \lim_{m\to\infty} k^{(m)}$. Hence, $\lim_{n\to\infty} c_{T-1}^*(k_{T-2}^{(n)}) = \lim_{m\to\infty} c_{T-1}^*(k_{T-2}^{(n(m))}) = c_{T-1}^*(k_{T-2})$ and thus, $c_{T-1}^*(\cdot)$, $p_{T-2}(\cdot)$ and $\pi_{T-2}(\cdot; k_{T-3})$ are left-continuous.

Let now assume that $c_{t+1}(\cdot)$, $p_t(\cdot)$ and $\pi_t(\cdot; k_{t-1})$ are left-continuous. Hence, $c_t^*(\cdot)$ is increasing by step 1. The rest of the argument works analogously to above, so that $c_t^*(\cdot)$, $p_{t-1}(\cdot)$ and $\pi_{t-1}(\cdot; k_{t-2})$ are left-continuous for all t.

(ii) is proven in the main part of the paper. ■

A.3 Private Offers

Proof. (Theorem 2) (i) We need to show that any pure strategy equilibrium with private offers is a pure strategy equilibrium with public offers. Let us fix a pure strategy equilibrium with private offers. First, note that buyers are making nonnegative expected profits in equilibrium and given the buyers' strategies and beliefs, the seller applies the same acceptance rule with public offers.

However, with public offers, the price is more sensitive to a change in cutoffs. Hence, if a deviation from the equilibrium prices was not profitable with private offers, it is certainly not profitable with public offers.

(ii) Let us denote for a given δ the set of all possible equilibrium cutoffs of a private offer game by $PE^{private}(\delta)$ and for a public offer game by $PE^{public}(\delta)$, respectively. Let us assume that for any δ^* there exists a $\delta > \delta^*$ such that $(k_1^*(\delta), \ldots, k_T^*(\delta)) \in PE^{private}(\delta)$. Hence, there exists a sequence $\delta_n \to_{n\to\infty} 1$ and a corresponding sequence $(k_1^*(\delta_n), \ldots, k_T^*(\delta_n)) \in PE^{private}(\delta_n)$. Let us define a function $\delta \mapsto (k_1(\delta), \ldots, k_T(\delta))$ such that $k_t(\delta_n) = k_t^*(\delta_n)$ for all t and infinitely many n (which is possible because $k_t^*(\delta) \to_{\delta\to 1} 0$ for all t) and k_t increasing, continuously differentiable and $k_{t-1} < k_t$ for all t. Recall that by (11) at any equilibrium $(k_1^*, \ldots, k_{T-1}^*) \in PE^{private}(\delta)$ the marginal profit of buyers at time $t < T-1$ is given by (11), i.e. with private offers

$$\frac{\partial}{\partial k_t}\pi_t(k_t^*; k_{t-1}^*) \quad = \quad \frac{F(k_t^*) - F(k_{t-1}^*)}{1 - F(k_{t-1}^*)}\left[f(k_t^*)\left(\frac{v(k_t^*)}{F(k_t^*) - F(k_{t-1}^*)} - \frac{\mathbb{E}[v(c)|[k_{t-1}^*, k_t^*]]}{F(k_t^*) - F(k_{t-1}^*)}\right) - (1-\delta)\right]$$

by the zero profit condition (10). Note that since $k_t(\cdot)$ are continuously differentiable, by applying l'Hopital's lemma twice, we can write

$$\begin{aligned}
&\lim_{\delta\to 1}\frac{v(k_t(\delta))(F(k_t(\delta)) - F(k_{t-1}(\delta))) - \int_{k_{t-1}(\delta)}^{k_t(\delta)} v(c)f(c)dc}{(F(k_t(\delta)) - F(k_{t-1}(\delta)))^2} \\
= \; &\lim_{\delta\to 1}\frac{v'(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta}(F(k_t(\delta)) - F(k_{t-1}(\delta))) - v(k_t(\delta))f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta} + v(k_{t-1}(\delta))f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}}{2(F(k_t(\delta)) - F(k_{t-1}(\delta)))\left(f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}\right)} \\
= \; &\lim_{\delta\to 1}\frac{v'(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta}}{2\left(f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}\right)} \\
&+\frac{f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}}{f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}} \cdot \frac{v(k_{t-1}(\delta)) - v(k_t(\delta))}{2(F(k_t(\delta)) - F(k_{t-1}(\delta)))} \\
= \; &\lim_{\delta\to 1}\frac{v'(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta}}{2\left(f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}\right)} \\
&+\frac{f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}}{f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta}} \cdot \frac{v'(k_{t-1}(\delta))\frac{\partial k_{t-1}(\delta)}{\partial\delta} - v'(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta}}{2\left(f(k_t(\delta))\frac{\partial k_t(\delta)}{\partial\delta} - f(k_{t-1}(\delta)\frac{\partial k_{t-1}(\delta)}{\partial\delta})\right)} \\
= \; &\frac{v'(0)\frac{\partial k_t(1)}{\partial\delta}}{2f(0)\left(\frac{\partial k_t(1)}{\partial\delta} - \frac{\partial k_{t-1}(1)}{\partial\delta}\right)} - \frac{f(0)\frac{\partial k_{t-1}(1)}{\partial\delta}\cdot v'(0)\left(\frac{\partial k_t(1)}{\partial\delta} - \frac{\partial k_{t-1}(1)}{\partial\delta}\right)}{2f(0)^2\left(\frac{\partial k_t(1)}{\partial\delta} - \frac{\partial k_{t-1}(1)}{\partial\delta}\right)^2} \\
= \; &\frac{v'(0)}{2f(0)} > 0
\end{aligned}$$

if $f(0) \geq 0$, because the equilibrium cutoffs converge all to 0 as $n \to \infty$. We can conclude

$$\lim_{\delta\to 1}\left[f(k_t^*)\left(\frac{v(k_t)}{F(k_t) - F(k_{t-1})} - \frac{\mathbb{E}[v(c)|[k_{t-1}, k_t]]}{F(k_t) - F(k_{t-1})}\right) - (1-\delta)\right] > 0.$$

Thus, there exists a n^* such that for all $n > n^*$ $(k_1^*(\delta_n), \ldots, k_{T-1}^*(\delta_n)) \notin PE^{Private}(\delta_n)$ which is a contradiction. Hence, there must exist a $\delta^* < 1$ such that for all $\delta > \delta^*$, there does not exist a pure strategy equilibrium with private offers. ■

Proof. (Theorem 3 and Proposition 1) We have already shown above that with public offers there is a unique pure strategy equilibrium, while with private offers, there exists a pure strategy equilibrium if and only if $\delta < 1 - \frac{A}{2}$. In the following, we analyze mixed strategy equilibria in the private offer case. In particular, we show that for $\delta < \delta*$ there are no mixed strategy PBEs. First, we show that there cannot be "too much" mixing in a mixed strategy equilibrium in the private offer case.

Lemma 6 *(Mixing in period 1) If $\delta < 1 - \frac{A}{2}$, buyers in period 1 mix at most between countably many cutoffs. If $\delta > 1 - \frac{A}{2}$ and expected period 1 profit $\pi_1(k;0) = 0$ for all $k \in (a,b)$, then any $k \in (a,b)$ cannot be in the support of $K_2, \ldots, K_T$ since it must hold that $K_2(k) = \frac{\delta - 1 + \frac{A}{2}}{\delta \frac{\partial}{\partial k_2} p_2(k)}$.*

Proof. (Lemma 6) In period 1, expected profit is given by

$$\pi_1(k_1;0) = k_1 \cdot \left[\frac{A}{2}k_1 + B - p_1(k_1)\right].$$

If buyers mix between all cutoffs $k \in (a,b)$, then it must hold that $\delta\left(\int_{k_1}^1 p_2(k_2)dK_2(k_2) + K_2(k_1)p_2(k_1)\right) + k_1(1-\delta) = \frac{A}{2}k_1 + B$ or equivalently

$$\delta\left(\int_{k_1}^1 p_2(k_2)dK_2(k_2) + K_2(k_1)p_2(k_1)\right) + k_1(1-\delta) = \left(\delta - \left(1 - \frac{A}{2}\right)\right)k_1 + B.$$

Note that the left hand side of the identity must be nondecreasing in k_1, so if $\delta < 1 - \frac{A}{2}$, then there cannot be mixing on (a,b) in period 1. If $\delta \geq 1 - \frac{A}{2}$, then the left hand side is differentiable, so the right hand side must be differentiable, so that

$$K_2(k) = \frac{\delta - \left(1 - \frac{A}{2}\right)}{\delta \frac{\partial}{\partial k_2} p_2(k)}$$

on $k \in (a,b)$. Since K_2 is a cdf, $\frac{\partial}{\partial k_2}p_2(k)$ cannot be increasing on (a,b), so that by lemma 5 $\frac{\partial}{\partial k_2}p_2(k)$ must be constant on (a,b). This implies that the support of K_2 is disjoint from (a,b) and because $\frac{\partial}{\partial k_2}p_2(k)$ must be constant on (a,b), also the intersection of the supports of $K_3, \ldots, K_{T-1}$ and (a,b) must be empty. ■

Lemma 7 *(Discrete mixing in periods $t > 1$) In any period $t > 1$, buyers mix between at most countably many cutoffs, that is $K_2, \ldots, K_{T-1}$ are step functions.*

Proof. (Lemma 7) Let us argue by induction. If buyers in period t mix between more than countably many cutoffs, then there exists an interval (a,b) such that $\pi_t(k_t)=0$ for all $k_t\in(a,b)$ and we have on (a,b) that $\int_0^{k_t}\int_0^c \frac{1}{1-k}dK_{t-1}^m(k)\,(Ac+B-p_t(k_t))\,dc=0$. After applying integration by parts and setting $H(k_t)\equiv\int_0^{k_t}\left(\int_0^c\int_0^x\frac{1}{1-k}dK_{t-1}^m(k)dx\right)dc$, one can see that this is equivalent to the ordinal differential equation

$$AH'(k_t)k_t-AH(k_T)=H'(k_t)\left(p_t(k_t)-B\right).$$

Thus, we can conclude that

$$H(k_t)\equiv\int_0^{k_t}\left(\int_0^c\int_0^x\frac{1}{1-k}dK_{t-1}^m(k)dx\right)dc \quad=\quad const\cdot\exp\left(\int_0^{k_t}\frac{1}{z-\frac{p_t(z)-B}{A}}dz\right)$$

and by Fubini's Theorem $H(k_t)=\int_0^{k_t}\frac{k_t-k}{2(1-k)}dK_{t-1}^m(k)$ which is increasing because $\frac{k_t-k}{2(1-k)}>0$ for $0<k<k_t$. Thus, the cdf $K_{t-1}^m(k)$ must be strictly increasing everywhere on (a,b). By lemma 6, buyers in period 2 cannot mix on an interval (a,b) and an induction argument shows that there cannot be mixing on an interval in any period $t>1$. ■

Hence, for $\delta<1-\frac{A}{2}$, the pure strategy equilibrium is the unique equilibrium while for $\delta>1-\frac{A}{2}$ all mixed strategy equilibria can only have discrete mixing in periods $t>1$ and if there is mixing on an interval (a,b) in period 1, then no cutoff in (a,b) is chosen with positive probability in periods $t>1$.

Let us assume that there exists a mixed equilibrium for $\delta<1-\frac{A}{2}$ and let us denote the smallest period in which buyers use a mixed strategy by $\underline{t}$. Then, buyers' expected profit in period $\underline{t}$ is given by

$$\pi_{\underline{t}}(k_{\underline{t}};k_{\underline{t}-1})=\frac{k_{\underline{t}}-k_{\underline{t}-1}}{1-k_{\underline{t}-1}}\cdot\left(B-p_{\underline{t}}(k_{\underline{t}})+A\frac{k_{\underline{t}}+k_{\underline{t}-1}}{2}\right)$$

and it is piece-wise quadratic where by lemma 5 the coefficient in front of the quadratic part of $\pi_{\underline{t}}(k_{\underline{t}})$ is given by

$$\begin{aligned}&\frac{A}{2}-1+\delta+\sum_{s=2}^{T-\underline{t}-1}\delta^s\left(\prod_{u=\underline{t}+1}^{\underline{t}+s-1}K_u(k_{\underline{t}})\right)\cdot(1-K_{\underline{t}+s}(k_{\underline{t}}))+\delta^{T-\underline{t}-1}\cdot(1-A)\cdot\left(\prod_{u=\underline{t}+1}^{T-1}K_u(k_{\underline{t}})\right)\\ =\;&\frac{A}{2}-1+\delta-\sum_{s=2}^{T-\underline{t}-1}\delta^{s-1}(1-\delta)\left(\prod_{u=\underline{t}+1}^{\underline{t}+s-1}K_u(k_{\underline{t}})\right)-\delta^{T-\underline{t}-1}\left(1-\delta(1-A)\right)\left(\prod_{u=\underline{t}+1}^{T-1}K_u(k_{\underline{t}})\right)<0\end{aligned}$$

which is decreasing in $k_{\underline{t}}$. Hence, buyers in period $\underline{t}$ must play a pure strategy in equilibrium which is a contradiction. ■

Proof. (Proposition 3) First, note that in any equilibrium it must hold that $\pi_2(\overline{k}_2) = 0$ and for all $K > \overline{k}_2$, $\pi_2(K) \leq 0$, i.e.

$$\int_0^{\overline{k}_2} \int_0^c \frac{1}{1-k} dK_1(k)(Ac + B - ((1-\delta+A\delta)\overline{k}_2 + \delta B))dc = 0$$
$$\int_0^K \int_0^c \frac{1}{1-k} dK_1(k)(Ac + B - ((1-\delta+A\delta)K + \delta B))dc \leq 0 \;\; \forall\, K > \underline{k}_2.$$

Let us first simplify the first equality. By applying Fubini's Theorem and then, noting that $\frac{\overline{k}_2-k}{1-k} = 1 + \frac{\overline{k}_2-1}{1-k}$ and $\frac{\overline{k}_2^2-k^2}{1-k} = 1 + k + \frac{\overline{k}_2^2-1}{1-k}$, we can deduce

$$\begin{aligned}
&\int_0^{\overline{k}_2} \int_0^c \frac{1}{1-k} dK_1(k)(Ac + B - ((1-\delta+A\delta)\overline{k}_2 + \delta B))dc \\
= \;& \frac{A}{2}\int_0^{\overline{k}_2} \frac{\overline{k}_2^2 - k^2}{1-k} dK_1(k) + ((1-\delta)B - (1-\delta+A\delta)\overline{k}_2)\int_0^{\overline{k}_2} \frac{\overline{k}_2 - k}{1-k} dK_1(k) \\
= \;& \frac{A}{2} + (1-\delta)B - (1-\delta+A\delta)\overline{k}_2 + \frac{A}{2}\int_0^{\overline{k}_2} k dK_1(k) \\
& + \int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)\left(\left(\overline{k}_2^2 - 1\right)\frac{A}{2} + (\overline{k}_2 - 1)((1-\delta)B - (1-\delta+A\delta)\overline{k}_2)\right) \\
= \;& \frac{A}{2}\cdot\left(1 - \frac{1+2\delta}{1+\delta}\overline{k}_2 + \int_0^{\overline{k}_2} k dK_1(k) + (\overline{k}_2 - 1)\left(1 - \frac{\delta}{1+\delta}\overline{k}_2\right)\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)\right).
\end{aligned}$$

Thus, in equilibrium, the following must hold

$$1 - \frac{1+2\delta}{1+\delta}\overline{k}_2 + \int_0^{\overline{k}_2} k dK_1(k) = (1-\overline{k}_2)\left(1 - \frac{\delta}{1+\delta}\overline{k}_2\right)\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k). \tag{19}$$

In order to simplify the second inequality, we can use that $\pi_2(\overline{k}_2) = 0$, and see that for $K > \overline{k}_2$

$$\begin{aligned}
&\int_0^K \int_0^c \frac{1}{1-k} dK_1(k)(Ac + B(1-\delta) - (1-\delta+A\delta)K)dc \\
= \;& \int_0^{\overline{k}_2} \int_0^c \frac{1}{1-k} dK_1(k)dc(\overline{k}_2 - K)(1-\delta+A\delta) + \int_{\overline{k}_2}^K \int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)(Ac + B(1-\delta) - (1-\delta+A\delta)K)dc \\
= \;& (K - \overline{k}_2)\Bigg[\int_0^{\overline{k}_2} \frac{1}{1-k} dK_1(k)\left(\frac{A}{2}\overline{k}_2 + B(1-\delta) - \underbrace{\left(1-\delta+A\delta-\frac{A}{2}\right)}_{>0}K\right) \\
& - \int_0^{\overline{k}_2} \int_0^c \frac{1}{1-k} dK_1(k)dc(1-\delta+A\delta)\Bigg]
\end{aligned}$$

is quadratic in K and the parabola is open below. The parabola has a zero at $\overline{k}_2$ and we will show in the following that it cannot have another zero. If $\pi_2(k') = 0$ for a $k' > \overline{k}_2$, then π_2 is positive on $(\overline{k}_2, k')$ which cannot hold in equilibrium. If the parabola (if it was extended to values smaller than $\overline{k}_2$) has a zero at a $k' < \overline{k}_2$ and if the support of K_1 does not contain $(\overline{k}_2 - \epsilon, \overline{k}_2)$ for a $\epsilon > 0$, then $\pi_2(k) > 0$ for $k \in (\overline{k}_2 - \epsilon, \overline{k}_2)$ which leads to a contradiction. Finally, if there is continuous mixing on some $(\overline{k}_2 - \epsilon, \overline{k}_2)$, then since the slope from the right of π_2 is negative at π_2, the slope from the left must also be negative because

$$\begin{aligned}
\frac{\partial}{\partial k_2}\pi_2(k_2) &= \frac{\partial}{\partial k_2}\int_0^{k_2}\int_0^{c}\frac{1}{1-k}dK_1(k)(Ac-(1-\delta+A\delta)k_2+B(1-\delta))dc \\
&= \int_0^{k_2}\frac{1}{1-k}dK_1(k)(1-\delta)(Ak_2-k_2+B)-(1-\delta+A\delta)\int_0^{k}\int_0^{c}\frac{1}{1-k}dK_1(k)dc
\end{aligned}$$

and $k_2(A-1)+B>0$. This again cannot hold in equilibrium. As a result, the parabola can only have one zero $\overline{k}_2$ and it follows from by plugging in the value of $\overline{k}_2$ calculated in proposition 2 that

$$\begin{aligned}
&\frac{\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)\left(\frac{A}{2}\overline{k}_2+B(1-\delta)\right)-\int_0^{\overline{k}_2}\int_0^{c}\frac{1}{1-k}dK_1(k)dc(1-\delta+A\delta)}{\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)(1-\delta+A\delta-\frac{A}{2})} = \overline{k}_2 \\
\Leftrightarrow\quad & \frac{A}{2}\overline{k}_2+B(1-\delta)-\frac{\int_0^{\overline{k}_2}\int_0^{c}\frac{1}{1-k}dK_1(k)dc(1-\delta+A\delta)}{\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)} = \frac{B(1-\delta^2)\left(1-\delta-\frac{A}{2}+A\delta\right)}{\left(1-\delta^2-\frac{A}{2}+A\delta^2\right)} \\
\Leftrightarrow\quad & (1-\delta)B\cdot\frac{\frac{A}{2}}{\left(1-\delta^2-\frac{A}{2}+A\delta^2\right)(1-\delta+A\delta)} = \frac{\int_0^{\overline{k}_2}\int_0^{c}\frac{1}{1-k}dK_1(k)dc}{\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)} \\
\Leftrightarrow\quad & \overline{k}_2\cdot\frac{\frac{A}{2}}{1-\delta^2+A\delta+A\delta^2} = \frac{1+(\overline{k}_2-1)\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)}{\int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k)} \\
\Leftrightarrow\quad & \int_0^{\overline{k}_2}\frac{1}{1-k}dK_1(k) = \frac{1}{\frac{\frac{A}{2}\overline{k}_2}{1-\delta^2+A\delta+A\delta^2}-\overline{k}_2+1}
\end{aligned}$$

This proves (15). Plugging (15) into (19), shows (14). ■

Proof. (Proposition 4) Let us assume that buyers in period 1 mix between cutoffs $\{0, \overline{k_1}\}$ only with $\overline{k_1} \in (0,1)$, $K_1(k_1 = 0) = r$ and $K_1(k_1 = \overline{k_1}) = 1 - r$ for some $r \in (0,1)$. Then expected profit in period 2 is given by

$$\pi_2(k_2) = \begin{cases} r\cdot k_2\cdot\left(\frac{A}{2}k_2+(1-\delta)B-k_2\left(1-\delta+\delta A\right)\right) & \text{if } k_2 < \overline{k_1} \\ (r-1)\frac{\overline{k_1}}{1-\overline{k_1}}\left(\frac{A}{2}\overline{k_1}+(1-\delta)B-k_2\left(1-\delta+\delta A\right)\right) & \\ +k_2\left(\frac{A}{2}k_2+(1-\delta)B-k_2\left(1-\delta+\delta A\right)\right)\left(r+\frac{1}{1-\overline{k_1}}(1-r)\right) & \text{if } k_2 > \overline{k_1}. \end{cases}$$

Note that both parts are quadratic in k_2 and that π_2 is continuous everywhere. Moreover, note that in both parts the coefficient in front of $(k_2)^2$ is negative. The first part is equal to zero if $k_2 \in \left\{0, \frac{B(1-\delta)}{1-\delta+\delta A-\frac{A}{2}}\right\}$ and the second part must only have one zero in equilibrium, i.e. it must hold that the discriminant is zero.

$$\begin{aligned}&\left[(1-\delta)B\left(r+\tfrac{1}{1-\overline{k_1}}(1-r)\right)-(r-1)\tfrac{\overline{k_1}}{1-\overline{k_1}}\left(1-\delta+\delta A\right)\right]^2=\\&(2A-4+4\delta(1-A))\left(r+\tfrac{1}{1-\overline{k_1}}(1-r)\right)(r-1)\tfrac{\overline{k_1}}{1-\overline{k_1}}\left(\tfrac{A}{2}\overline{k_1}+(1-\delta)B\right)\end{aligned} \tag{20}$$

and the null is at

$$\overline{k_2}=-\frac{(1-\delta)B\left(r+\frac{1}{1-\overline{k_1}}(1-r)\right)-(r-1)\frac{\overline{k_1}}{1-\overline{k_1}}\left(1-\delta+\delta A\right)}{(A-2+2\delta(1-A))\left(r+\frac{1}{1-\overline{k_1}}(1-r)\right)}. \tag{21}$$

If we denote the two zeros of π_2 by $\underline{k_2}\equiv\frac{B(1-\delta)}{1-\delta+\delta A-\frac{A}{2}}$ and $\overline{k_2}$, then it must hold

$$0\le\underline{k_2}<\overline{k_1}<\overline{k_2}\le 1,$$

if we want to make sure that there is mixing in period 2. Hence, we have $K_2\left(k_2=\underline{k_2}\right)=q$ and $K_2\left(k_2=\overline{k_2}\right)=1-q$ and expected profit in period 1 is given by

$$\pi_1(k_1)=\begin{cases}k_1\left[\left(\frac{A}{2}-(1-\delta)\right)k_1+B(1-\delta^2)-\delta\left[q\underline{k_2}\left(A\delta+1-\delta\right)+(1-q)\overline{k_2}\left(A\delta+1-\delta\right)\right]\right] & \text{if } k_1<\underline{k_2}\\ k_1\left[\left(\frac{A}{2}-(1-\delta)\right)k_1+B(1-\delta^2)-\delta\left[qk_1\left(A\delta+1-\delta\right)+(1-q)\overline{k_2}\left(A\delta+1-\delta\right)\right]\right] & \text{if } \underline{k_2}<k_1<\overline{k_2}\\ k_1\left[\left(\frac{A}{2}-(1-\delta)\right)k_1+B(1-\delta^2)-\delta k_1\left(A\delta+1-\delta\right)\right] & \text{if } \overline{k_2}<k_1\end{cases}$$

As we have shown above, in equilibrium we must have $q=\frac{\delta-1+\frac{A}{2}}{\delta(A\delta+1-\delta)}$, $\underline{k_2}=\frac{B(1-\delta)}{1-\delta+\delta A-\frac{A}{2}}$ and $\overline{k_2}=\frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{A}{2}}$ and $\overline{k_1},r$ must solve the system of equations given by (21) and (20). Indeed we can find a solution r and

$$\overline{k_1}=\frac{2B(1-\delta)(-2\delta^3+2\delta^3A+4\delta^2A-2\delta^2+2\delta-A+2)}{(-2\delta^2+2\delta^2A+2\delta A-A+2)(-2\delta^2+2\delta^2A+2-A)}$$

such that $0\le\underline{k_2}<\overline{k_1}<\overline{k_2}\le 1$. ■

A.4 Efficiency of Dark versus Transparent Markets

Proof. (Proposition 5) In period 1, we can easily calculate the difference between the expected cutoff with private offers and the private offer period 1 cutoff using (8), (9), Proposition 3 and

Proposition 2 and see that it is positive:

$$\begin{aligned}
\int_0^{\overline{k}_2} kdK_1(k) - k_1^* &= \frac{(1-\frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{A}{2}})\left(1-\frac{\delta}{1+\delta}\frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{A}{2}}\right)}{1-\overline{k}_2\cdot\frac{(1+\delta)(1-\delta+A\delta)-\frac{A}{2}}{(1+\delta)(1-\delta+A\delta)}} + \frac{1+2\delta}{1+\delta}\frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{A}{2}} - 1 \\
&\quad - \frac{2B\cdot(A\delta-2\delta+2-A)\cdot(1-\delta)}{2(1-\delta)(1-A)(A\delta-2\delta+2)+A^2} \\
&= \frac{(1-A)(1-\delta)(-4+4A-A^2+8\delta-6A\delta+2A^2\delta-4\delta^2+2A\delta^2+2A^2\delta^2)}{(2-A-2\delta^2+2A\delta^2)(4-4A+A^2-8\delta+10A\delta-2A^2\delta+4\delta^2-6A\delta^2+2A^2\delta^2)} \\
&= \frac{1}{2}\overline{k}_2\cdot\frac{2A(1-\delta)-(A-1)^2(1-\delta)^2-3(1-\delta)^2+\delta^2 A}{(4-4A+A^2)(1-2\delta+\delta^2)+2A\delta-2A\delta^2+A^2\delta^2} \\
&\geq \frac{1}{2}\overline{k}_2\cdot\frac{(1-\delta)^2-(A-1)^2(1-\delta)^2+\delta^2 A}{(4-4A+A^2)(1-2\delta+\delta^2)+2A\delta-2A\delta^2+A^2\delta^2} > 0.
\end{aligned}$$

for $\delta > 1-\frac{A}{2}$ and $A+B=1$. In period 2, the difference between the cutoffs in the two information structures is given by

$$\begin{aligned}
\int_0^1 kdK_2(k) - k_2^* &= q\underline{k}_2 + (1-q)\overline{k}_2 - \frac{2B\cdot(A\delta-2\delta+2)\cdot(1-\delta)}{2(1-\delta)(1-A)(A\delta-2\delta+2)+A^2} \\
&= \frac{\frac{A}{2}-(1-\delta)}{\delta(A\delta+1-\delta)}\frac{B(1-\delta)}{A\delta-\delta+1-\frac{A}{2}} + \left(1-\frac{\frac{A}{2}-(1-\delta)}{\delta(A\delta+1-\delta)}\right)\frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{A}{2}} \\
&\quad - \frac{2B\cdot(A\delta-2\delta+2)\cdot(1-\delta)}{2(1-\delta)(1-A)(A\delta-2\delta+2)+A^2} \\
&= -\frac{A^2B(-1+\delta)(-2+A+2\delta)}{(1-\delta+A\delta)(2-A-2\delta+2A\delta)((1-\delta)^2(2-A)^2+2A\delta(1-\delta)+A^2\delta^2)}.
\end{aligned}$$

Note that this function is zero if and only if the numerator is zero which is a quadratic function in δ. It is easy to check that the numerator is zero if and only if $\delta \in \left\{1-\frac{A}{2}, 1\right\}$. Hence, we can conclude that the function is positive in $\left[1-\frac{A}{2}, 1\right]$ because it is positive at $\delta = 1-\frac{A}{4}$ and there are no singularities for $\delta \in [0,1]$. ■

Proof. (Theorem 4) The key of the proof for $\int_0^1 G(k_1)dK_1(k_1) \geq G(k_1^*)$ is to note that $G(x) = \frac{A-1}{2}x^2 + Bx$ is increasing for $A-1>0$, $G(0)=0$ and to calculate the certainty equivalent of the cutoffs in the private offers case. Given that with private offers, period 1 cutoffs are in $[0,\overline{k}_2]$ and given that they are given by lemma 3,

$$\begin{aligned}
\int_0^1 G(k_1)dK_1(k_1) &\geq \frac{G(\overline{k}_2)}{\overline{k}_2}\int_0^1 k_1 dK_1(k_1) \\
&= \frac{(1-A)^2(1-\delta)(1+2\delta)(1-\delta^2+A\delta^2)}{(2-A-2\delta^2+2A\delta^2)^2} \\
&\geq G(k_1^*) = \frac{2(2-A)B^2(1-\delta)^2(2-A-4\delta+4A\delta+2\delta^2-3A\delta^2+A^2\delta^2)}{((2-A)^2(1-\delta)^2+2A\delta-2A\delta^2+A^2\delta^2)^2}
\end{aligned}$$

for $A+B=1$. In period 2, it holds that

$$\int_0^1 \int_0^1 G(\max\{k_1,k_2\}) dK_1(k_1) dK_2(k_2) \leq \int_0^1 G(k_2) dK_2(k_2) = qG(\underline{k}_2) + (1-q)G(\overline{k}_2)$$

and in $[\underline{k}_2, \overline{k}_2]$ we have

$$\begin{aligned} qG(\underline{k}_2) + (1-q)G(\overline{k}_2) &= (1-A)^2(1-\delta) \cdot \Bigg[\frac{4-2A+2A^2-A^3+\delta(-8+12A-5A^2+A^3)}{(1-\delta+A\delta)(2-A-2\delta+2A\delta)^2(2-A-2\delta^2+2A\delta^2)} + \\ &\quad \frac{\delta^2(-6A+7A^2-A^3)+\delta^3(8-16A+8A^2)+\delta^4(-4+12A-12A^2+4A^3))}{(1-\delta+A\delta)(2-A-2\delta+2A\delta)^2(2-A-2\delta^2+2A\delta^2)} \Bigg] \\ &\geq G(k_2^*) = 2(1-A)^2(1-\delta)(2-2\delta+A\delta) \cdot \\ &\quad \frac{2-2A+A^2-4\delta+5A\delta-A^2\delta+2\delta^2-3A\delta^2+A^2\delta^2}{4-4A+A^2-8\delta+10A\delta-2A^2\delta+4\delta^2-6A\delta^2+2A^2\delta^2)^2}. \end{aligned}$$

Hence we can conclude that $V(\delta, A, B, public) \geq V(\delta, A, B, private)$. ■

A.5 Robustness and Discussion

Proof. (Proposition 7) If $B=0$, then we can assume $1 < A$ and $\frac{A}{2} < 1-\delta^2+\delta^2 A$. With public offers, the best response function in period 2 is given by

$$k_2(k_1) = \begin{cases} \frac{\frac{A}{2}}{1-\delta-\frac{A}{2}+\delta A} k_1 & \text{if } k_1 < 1-2(1-\delta)\frac{A-1}{A} \\ 1 & \text{otherwise} \end{cases}.$$

and expected profit in period 1 is given by $\pi_1(k_1; k_2(\cdot)) = k_1 \cdot \left(\left(\frac{A}{2} - (1-\delta) \right) k_1 - \delta(1-\delta+\delta A) k_2(k_1) \right)$. Hence, it is easy to check that $k_1 = 0$ maximizes π_1 because

$$\frac{A}{2} - (1-\delta) - \delta(1-\delta+\delta A)\frac{\frac{A}{2}}{1-\delta-\frac{A}{2}+\delta A} < \delta A - \delta(1-\delta+\delta A)\frac{\frac{A}{2}}{1-\delta-\frac{A}{2}+\delta A} < 0.$$

With private offers, the only equilibrium candidate is the one without trade in periods 1 and 2. This is an equilibrium because

$$\pi_1(k_1; 0) = k_1^2 \cdot \left(\frac{A}{2} - (1-\delta^2) - \delta^2 A \right) \qquad \text{and}$$
$$\pi_2(k_2; 0) = k_2^2 \cdot \left(\frac{A}{2} - (1-\delta) - \delta A \right)$$

which is maximized by $k_1 = 0$ and $k_2 = 0$. ■

Proof. (Theorem 5) Analogously to the proof of theorem 2, we argue by contradiction. Let us assume that there exists a sequence $\Delta_n \to 0$ so that there exists a pure strategy equilibrium with private offers for all Δ_n. For any Δ, let us denote equilibrium cutoffs with private offers by

$(k^*_{\Delta}, k^*_{2\Delta} \ldots, k^*_{T-\Delta}) \equiv (k^*_{\Delta}(\Delta), k^*_{2\Delta}(\Delta) \ldots, k^*_{T-\Delta}(\Delta))$. Let t_Δ be a multiple of Δ. A period t buyer's expected profit with $t \in (t_\Delta, t_\Delta + \Delta]$ is given by

$$\pi_t(k^*_{t_\Delta}, k^*_{t_\Delta-\Delta}) = \frac{F(k^*_{t_\Delta}) - F(k^*_{t_\Delta-\Delta})}{1 - F(k^*_{t_\Delta-\Delta})} \cdot \left[\mathbb{E}[v(c)|[k^*_{t_\Delta-\Delta}, k^*_{t_\Delta}]] - p_{t_\Delta}(k^*_{t_\Delta})\right].$$

Then, for $t_\Delta < (T-1)\Delta$ marginal expected profit of buyers is given by

$$\begin{aligned}\frac{\partial}{\partial k_{t_\Delta}}\pi_t(k_{t_\Delta}; k^*_{t_\Delta-\Delta})\big|_{k_{t_\Delta}=k^*_{t_\Delta}} = & \frac{F(k^*_{t_\Delta}) - F(k^*_{t_\Delta-\Delta})}{1 - F(k^*_{t_\Delta-\Delta})} \cdot \\ & \left[f(k^*_{t_\Delta})\left(\frac{v(k^*_{t_\Delta})}{F(k^*_{t_\Delta}) - F(k^*_{t_\Delta-\Delta})} - \frac{\mathbb{E}[v(c)|[k^*_{t_\Delta-\Delta}, k^*_{t_\Delta}]]}{F(k^*_{t_\Delta}) - F(k^*_{t_\Delta-\Delta})}\right) - (1 - e^{-r\Delta})\right]\end{aligned}$$

Let us consider an arbitrary subsequences $\Delta_{n(m)} \to 0$ so that $\lim_{m\to\infty} k^*_{t_{\Delta_{n(m)}}-\Delta_{n(m)}} \equiv \underline{c}$ and $\lim_{m\to\infty} k^*_{t_{\Delta_{n(m)}}} \equiv \overline{c}$ exist. Note that $\overline{c} \geq \underline{c}$. If $\overline{c} > \underline{c}$, then for all $t \in (t_{\Delta_{n(m)}}, t_{\Delta_{n(m)}} + \Delta_{n(m)}]$

$$\lim_{m\to\infty} \frac{\partial}{\partial k_{t_{\Delta_{n(m)}}}}\pi_t(k_{t_{\Delta_{n(m)}}}; k^*_{t_{\Delta_{n(m)}}-\Delta_{n(m)}})\big|_{k_{t_{\Delta_{n(m)}}}=k^*_{t_{\Delta_{n(m)}}}} = \frac{f(\overline{c})\,(v(\overline{c}) - \mathbb{E}[v(c)|[\underline{c}, \overline{c}]])}{1 - F(\underline{c})} > 0,$$

so there must exist an m large enough such that there is a profitable deviation for buyers in periods $t < (T-1)\Delta_{n(m)}$, i.e. such that $(k^*_{\Delta_{n(m)}}, \ldots, k^*_{T-\Delta_{n(m)}})$ is not an equilibrium with private offers which is a contradiction. If $\overline{c} = \underline{c}$, then by L'Hopital's lemma, we can show analogously to theorem 2 that

$$\frac{v(k^*_{t_{\Delta_{n(m)}}})(F(k^*_{t_{\Delta_{n(m)}}}) - F(k^*_{t_{\Delta_{n(m)}}-\Delta_{n(m)}})) - \int_{k^*_{t_{\Delta_{n(m)}}-\Delta_{n(m)}}}^{k^*_{t_{\Delta_{n(m)}}}} v(c)f(c)dc}{(F(k^*_{t_{\Delta_{n(m)}}}) - F(k^*_{t_{\Delta_{n(m)}}-\Delta_{n(m)}}))^2} \to_{m\to\infty} \frac{v'(0)}{2f(0)} > 0$$

because $f(0) \geq 0$ and $v'(0) > 0$. [16] We can conclude that in the limit $m \to \infty$, the marginal expected profit converges to a strictly positive number. Hence, for sufficiently large m, there is a profitable deviation for buyers. ■

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