THEORY OF CAPITAL ALLOCATION

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Abstract

We demonstrate that firms should allocate capital to lines of business based on marginal default values. The marginal default value for a line of business is the derivative of the value of the firm's option to default with respect to the scale of the line. Marginal default values give a unique allocation of capital that adds up exactly, regardless of the joint probability distribution of line-by-line returns. Capital allocations follow from the conditions for the firm's optimal portfolio of businesses. The allocations are systematically different from allocations based on VaR or contribution VaR. We set out practical applications, including implications for regulatory capital standards.

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CAPITAL ALLOCATION

1. Introduction

This paper presents a general procedure for allocating capital. We focus on financial firms, but the procedure also works for non-financial firms that operate in a mix of safe and risky businesses. The efficient capital allocation for a line of business depends on its *marginal default value*, which is the derivative of the value of the firm's option to default (its *default put*) with respect to a change in the scale of the business. Marginal default values are unique and add up exactly.

Consider a financial firm that diversifies across different activities and asset classes (lines of business), which may include lending, trading and market making, investment banking, asset management and retail services, such as credit-card operations. Financing comes from debt (including deposits if the firm is a bank) and from risk capital, which is primarily common equity. (We will provide a more precise definition of "capital" later.)

If the firm can identify capital requirements by lines of business, then it can allocate its capital back to the businesses. Accurate allocations are important. If insufficient capital is allocated to risky lines of business, then the risks that these businesses impose on the firm as a whole and on lenders or counterparties will not be properly accounted for. Incentives and compensation will be distorted. Capital allocation is also important if capital is limited. Limited capital has a shadow price if the firm is forced to pass up positive-NPV investments.

Capital allocation is required to assess the cost and profitability of each line of business and, therefore, to price products and services. If capital is costly, then the more capital a product or service requires, the higher the break-even price. We show how capital allocations should be "priced" and charged back to lines of business.

Capital allocation is also necessary to calculate the net benefits of hedging or securitization. Suppose, for example, that a credit-default swap transaction can cancel out 20% of the risk of a bank's loan portfolio, freeing up 20% of the capital that the portfolio would otherwise require. The bank must then compare the costs of hedging to the value of the capital released. To do that, the bank has to know how much capital was properly allocated to the loan portfolio in the first place.

Capital is costly for two reasons. First, returns to equity are subject to corporate income tax. (In corporate finance, one would say that returns to equity, unlike debt, do not generate interest tax shields.) Second, additional capital may increase agency costs and monitoring costs borne by shareholders. Of course capital has benefits too, especially for financial firms that require sound credit to transact with counterparties. If such a firm's credit is doubtful, then counterparties may demand full collateral or take their business elsewhere. But capital is *not* costly simply because shareholders demand a higher expected rate of return than creditors. Modigliani and Miller uncovered this fallacy more than 50 years ago.

We derive efficient capital allocations with and without imposing capital constraints and also for financial firms subject to risk-based regulatory capital requirements. The allocations always satisfy two requirements. First, *no risk-shifting* or cross-subsidies: Capital should be allocated so that a marginal change in the composition of the firm's portfolio of lines of business does not affect the credit quality of the firm's liabilities. Second, *no internal arbitrage*: capital should be allocated so that it is not possible to add value merely by shifting capital from one line of business to another.

Our capital allocation procedure works for any joint probability distribution of returns. The key assumption is sufficiently complete financial markets, so that lines of business have well-defined market values.

Capital allocations based on marginal default values differ systematically from allocations based on value at risk (VaR). For example, capital allocations proportional to VaR would allocate zero capital to a risk-free asset. Efficient allocation procedures assign *negative* capital to risk-free and some low-risk assets. Such assets should not be charged for capital; they should be rewarded, because at the margin they *reduce* the value of the firm's default put.

Perhaps our results about capital allocation and regulation would be just incremental if the theory of capital allocation were well understood. But a literature search for general principles does not yield clear answers. Allocations based on value-at-risk (VaR) ignore diversification across lines of business and therefore do not add up properly. The proper procedures for allocating this diversification benefit are not obvious, and it appears that various procedures are used in practice.[1] Contribution VaRs, which depend on the covariances or betas of line-by-line returns vs. returns for the firm as a whole, do add up. See Saita (1999) and Stulz (2003), for example.[2] But we will

[1] See Helbekkmo (2006), for example.

[2] Contribution VaRs appear in Froot and Stein (1998, pp. 67-68), Stoughton and Zechner (2007), Saita (1999), Stulz (2003, pp. 99-103) and no doubt in other places. The label varies: synonyms for "contribution" include "marginal," for example in Saita and Stulz. Others refer to "incremental VaR," which is not the same thing. Incremental VaR is the discrete change in VaR from adding or subtracting an asset or business from the bank's overall portfolio. Merton and Perold (1993), Perold (2005) and Turnbull (2000) focus on incremental VaR.

show why allocations based on contribution VaRs are not consistent with maximizing value.

The academic and applied literature on VaR and risk management is enormous. See Jorion (2006) and Stulz (2003), for example. Prior work on capital allocation is much more limited. Merton and Perold (1993) and Perold (2005) are probably the best places to start. These papers focus on decisions to add or subtract an entire line of business, and conclude that a bank should not attempt to allocate its total capital back to lines of business. We disagree with this conclusion, but mostly agree with how they set up the capital-allocation problem. They define "risk capital" as the present-value cost of acquiring complete insurance against negative returns on the firm's net assets—the value of a one-period at-the-money (forward) put, which has an exercise price equal to the current value of the net assets plus one period's interest at the risk-free rate. We start with the firm's default put, which is almost the same thing. (The value of the default put equals the cost of insurance for the firm's debt and other liabilities, not for insuring its net assets, as in Merton and Perold's setup.)

Froot and Stein (1998) consider capital allocation, but their main interest is how banks invest capital, not how to allocate an existing stock of capital to a portfolio of existing businesses.[3] They show that value-maximizing banks will act as if risk-averse, even in perfect financial markets, if investment opportunities are uncertain and raising equity capital on short notice is costly. They discuss contribution VAR and the problems of implementing risk-adjusted return on capital (RAROC). They do not consider default, however. Turnbull (2000) extends this line of research, introducing default risk.

[3] Froot (2007) builds on this model to analyze risk allocation in the insurance industry.

Stoughton and Zechner (2007) add a focus on information and agency costs internal to the firm.

This paper is not about the optimal level of capital, either from a private or social point of view.[4] We assume that equity capital is costly for tax reasons or because of agency or monitoring costs, but we do not model the tradeoff of these costs vs. the costs created by default risk. We assume that management has already considered this tradeoff and decided on an acceptable level of credit quality, which we will define as the ratio of default-put value to the firm's debt and other liabilities. Credit quality is especially important for financial firms dealing with counterparties who do not want to absorb credit risk or bear the cost to monitor it.

This paper extends Myers and Read (2001), who analyze capital (surplus) allocation for insurance companies.[5] Principles are similar here, although proofs are now general and applications are not limited by the special characteristics of insurance. Myers and Read considered capital allocation for a fixed portfolio of lines of insurance with joint lognormal or normal distributions. Here we derive efficient capital allocations for firms with a fixed portfolio of businesses and also for a firm that chooses its *optimal* portfolio of businesses. We solve the portfolio problem for three cases: (1) available capital is fixed, (2) the amount of capital is an unconstrained decision variable and (3) the firm is subject to risk-based regulatory capital requirements. Efficient capital allocations satisfy the no risk-shifting and no internal arbitrage requirements in all three cases.

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[4] See Kashyap et al. (2010), Miles et al. (2013) and Admati et al. (2010).

[5] Follow-on articles in the insurance literature include Cummins, Lin, and Phillips (2006), Grundl and Schmeiser (2007), Zanjani (2010) and Bauer and Zanjani (2012).

Section 2 of this paper defines capital and proves that marginal default values "add up" and support a unique allocation. We show how differences in marginal default values can by offset by compensating capital allocations. Section 3 derives optimal portfolios of lines of businesses as solutions to constrained optimization problems. Capital allocations follow from the conditions for the optimum portfolio.

Section 4 presents examples of portfolio optimization and capital allocation assuming that returns are normally distributed. Section 5 considers practical applications and implications. We derive an adjusted present value (APV) rule for valuing investment in a line of business, and we show how capital allocations should be priced and charged back to lines of business. We contrast our results with allocations based on VaR or contribution VaR. Also we summarize implications for bank capital regulation. Risk-based capital requirements impose an additional constraint that inevitably distorts internal capital allocations and investments. The distortions are *not* from "regulatory arbitrage," that is, from substitution of risky for safer assets within a line of business. Section 6 recaps the paper's main findings and notes areas for further work.

2. Default Values and Capital Allocation

Start with a financial firm's market-value balance sheet:

<u>Assets</u>	<u>Liabilities and equity</u>
Assets (A_i)	Debt or other liabilities (L)
Default Put (P)	Equity (E)
Franchise value (G)	

The lines of business A_i are assumed marked to market. The firm's "franchise value," which includes intangible assets and the present value of future growth opportunities, is entered as G. We assume for simplicity that franchise value disappears (G = 0) if the firm defaults.[6]

The *default-risk free* value of debt or other liabilities, including deposits if the firm is a bank, is L. Other liabilities could include insurance contracts, letters of credit, swap agreements, unfunded pension liabilities or (for an industrial firm) fixed-price contracts to pay suppliers. We do *not* assume that debt or other liabilities are default-risk free, however. We have simply moved default risk to the left side of the balance sheet as the default-put value P. For now we assume no third-party financial guarantees or other credit backup. Therefore the default put value P translates directly to the credit spread demanded by lenders and counterparties. We comment on deposit guarantees and regulatory capital requirements in Section 5.[7]

We define the maturity of the default put as one period. If the firm defaults, the payoff to the put equals the shortfall of the end-of-period asset value from the end-of-period payment due to lenders and counterparties, including interest. Defining the length of the period is an issue for practice. A financial firm would probably update capital allocations frequently, which suggests a period length of a month, quarter or year. On the other hand, the firm may issue longer-term debt or enter longer-term transactions with

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[6] We could generalize by introducing a residual franchise value G_{min} in default. The availability of G_{min} to satisfy creditors would reduce the value of the default put.

[7] Third-party credit backup does not complicate our analysis if acquired at market value, that is, in zero-NPV transactions. The backup absorbs default risk otherwise borne by debt-holders or other counterparties. The firm could pay for backup from the additional cash raised by issuing debt on more favorable terms. Equity value and capital would not change. Deposit insurance creates two problems, however. First, the insurance has been offered at low premiums, thus subsidizing risky banks. Second, deposit insurers are government agencies that may not be as well-equipped as private investors to monitor and prevent risk-shifting.

counterparties. These transactions may require a longer view of the firm's credit quality and a longer-term put.[8]

Equity (E) is the market value of equity, defined as common stock plus issues of preferred stock or subordinated debt that count as capital. The firm's capital C is not the same thing as its equity, however. The capital-account balance sheet is:

Assets	Liabilities and capital
Assets (A)	Debt or other liabilities (L)
	Capital (C)

Capital is $C = A - L$, the difference between the market value of the firm's assets and the *default-risk free* value of its liabilities. Capital C is the "cushion" of assets A over promised payments to creditors or counterparties. Capital is not a pot of cash held in a reserve account or money market securities. It is a measure of how much equity (or of some types of subordinate debt) that is at risk to protect debt and other liabilities.

The capital-account balance sheet is close to a book balance sheet, because it does not show default value P or the intangible assets or future growth opportunities in G. Thus capital is not equity at market value. If it were, a firm could increase its capital simply by increasing asset risk and the risk of default and thus forcing down the market value of the firm's liabilities.

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[8] Our analysis works for any put maturity, with one qualification. We assume that the amount of capital is set at the start of the period and not added to or withdrawn during the period. If the put maturity is long, say three years, then the value of the put will depend on whether the firm is able and willing to raise additional capital if asset value declines in years one or two. Valuing the put becomes a much more complicated dynamic problem if the firm also has to decide on the optimal policy for replacing possible losses in its capital account.

The asset portfolio consists of two or more assets (lines of business) with start-of-period values A_i. Thus $A = \sum_i A_i$. The value of the default put is:

$$P = PV\left[\max\left\{0,(R_L L - R_A A)\right\}\right], \tag{1}$$

where R_L is the gross return to a dollar of debt or other liabilities (one plus the promised interest rate) and R_A is the uncertain gross return on the bank's assets. All returns are assumed to be uncertain except for R_L.[9] The end-of-period promised payoff to liability holders, including interest, is $R_L L$. With complete markets, the present value of the default put is:

$$P = \int_Z \left[R_L L - R_A A\right]\pi(z)\,dz\,, \tag{2}$$

where $\pi(z)$ is a state-price density in the default region Z. This region consists of all outcomes where assets fall short of liabilities and hence the put is in the money.

Each state z is a unique point in the default region Z. Each point is a combination of returns on the assets (R_i), which generate a portfolio return of $R_A A$. The valuation Eq. (2) sums across the continuum of states, with the payoff in each state z multiplied by the state-price density $\pi(z)$. Note that the states are identified by asset returns and that the state prices $\pi(z)$ are fixed. Therefore an extra dollar delivered in state z by asset A_i has exactly the same present value as an extra dollar delivered by A_j. The valuation formula sums across states.

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[9] We take L as fixed. $R_L L$ is the exercise price of the default put. We could allow for uncertain liabilities, for example insurance contracts, as in Myers and Read (2001), who define marginal default values with respect to liabilities rather than assets. But in our paper it's easier to think of a risky liability as a short position in a risky asset.

Define the *marginal default value* of asset *i* as $p_i = \partial P/\partial A_i$, the partial derivative of overall put value P with respect to A_i. We can show that these marginal default-option values add up exactly and uniquely. The sum of the products of each asset and its marginal default value equals the default value of the bank as a whole.

Proposition 1. The default value P can be expressed as an asset-weighted sum of marginal default values p_i:

$$P = \sum_i p_i A_i \tag{3}$$

A proof for the two-asset case is provided in Appendix 1. Generalization to three or more assets is straightforward. Note that Eq. (3) requires no assumptions about the probability distribution of returns. The only assumption is sufficiently complete markets, so that the assets A_i have well-defined market values.

The capital ratio for the firm as a whole is $c \equiv \frac{C}{A}$. Therefore, $L = (1-c)A$ and Eq. (2) can be modified as:

$$P = \int_Z A[R_L(1-c) - R_A]\,\pi(z)\,dz \tag{4}$$

It's clear from this valuation formula that an across-the-board expansion of assets and liabilities (with *c* constant) will result in a proportional increase in overall default value. Given *c*, $p \equiv \frac{\partial P}{\partial A}$ is a constant for any proportional change, regardless of the size of the change.[10]

[10] We assume that the probability distribution of returns to a line of business does not change as the line is expanded. The assumption could be violated in some circumstances, for example if expansion allowed addition of diversifying assets, thus reducing the variance of the rate of return

Expansion of a single line of business will also affect P, but not proportionally. Therefore we also allow capital ratios to vary by line. Define the capital ratio for line *i* as c_i. Default value is:

$$\begin{aligned} P &= \int_Z \left[\left(\sum_i (1-c_i) A_i \right) R_L - \left(\sum_i A_i R_i \right) \right] \pi(z) dz \\ &= \sum_i \int_Z A_i \left[(1-c_i) R_L - R_i \right] \pi(z) dz \end{aligned} \tag{5}$$

The default value per unit of assets is

$$p = \frac{P}{A} = \sum_i \int_Z a_i \left[(1-c_i) R_L - R_i \right] \pi(z) dz, \tag{6}$$

where $a_i \equiv \frac{A_i}{A}$. The marginal default values are

$$p_i \equiv \frac{\partial P}{\partial A_i} = \int_Z \left[(1-c_i) R_L - R_i \right] \pi(z) dz = \frac{\partial p}{\partial a_i} \tag{7}$$

Our adding-up result still holds. Also, an increase in the marginal capital allocation c_i always decreases the exercise price of bank's default put and reduces its value. Therefore, we can offset differences in p_i by compensating changes in the capital allocations c_i.

We derive optimal capital allocations in the next section. But Eq. (7) gives a preview of one result. For a risk-free asset, where $R_i = R_L$, marginal default value is negative at any positive capital allocation c_i.

$$p_i = \int_Z \left[-c_i R_L \right] \pi(z) dz < 0. \tag{8}$$

We will show that optimal capital-adjusted marginal default values must be all positive and equal across lines. Thus risk-free assets must be given a *negative* capital allocation c_i. Other low-risk assets may also get negative allocations. Note the contrast to allocations based on VaR or contribution VaR. For example, the contribution VaR for a safe asset is zero, since the covariance of the safe return with the bank's overall return is zero.

If capital allocations are constant ($c_i = c$), marginal default values p_i will vary across lines of business. A firm that allocates capital in proportion to assets, despite varying marginal default values, is forcing some businesses to cross-subsidize others. This contaminates performance measurement, incentives, compensation, pricing and decisions about securitization and hedging. The remedy is to vary capital allocation depending on marginal default values, so that each business's capital-adjusted contribution to default value is the same. In other words, capital should be allocated to satisfy the no risk-shifting principle of capital allocation: a marginal change in the composition of the firm's portfolio of lines of business does not affect the credit quality of the firm's liabilities.

Before moving in the next section to optimal portfolios, we note two further results. First, marginal default values can be expressed as the sum of a scale term and a business-composition term:

$$p_i = p + \frac{\partial p}{\partial a_i}\left(1 - a_i\right) \tag{9}$$

where $a_i \equiv \frac{A_i}{A}$. The first term p is the change in default value due to an increase in A, the overall scale of the firm's assets, ignoring any change in the composition of its assets.

The second term captures the change in p due to a change in the composition of the asset portfolio $\partial p/\partial a_i$. The partial derivatives of the unit default value p and the marginal default values p_i with respect to the allocations c_i are:

$$\frac{\partial p}{\partial c_i} = -\int_Z a_i R_L \pi(z) dz = a_i \left(\frac{\partial p}{\partial c}\right) \tag{10}$$

$$\frac{\partial p_i}{\partial c_i} = -\int_Z R_L \pi(z) dz = \frac{\partial p}{\partial c} \tag{11}$$

Second, the valuation expressions can be simplified by defining $\Pi_z(R_x) \equiv \int_Z R_x \pi(z) dz$. For example, $\Pi_z(R_L)$ is the present value of a safe asset's return (but only in the in-the-money region Z, like the payoff on a cash-or-nothing put triggered by default). Write marginal default value p_i as:

$$p_i = (1 - c_i)\Pi_z(R_L) - \Pi_z(R_i) \tag{12}$$

The overall default value is

$$p = (1 - c)\Pi_z(R_L) - \Pi_z(R_A) \tag{13}$$

Here $(1 - c)\Pi_z(R_L)$ is the present value of the exercise price of the default put, received only if the put is exercised. $\Pi_Z(R_A)$ is the present value of the asset given up if the put is exercised. The difference between these two values is the value of the put.

The present values $\Pi_Z(R_X)$ have exact analytic solutions if returns are normally distributed—see Section 4. Our results and procedures do not depend on specific probability distributions, however, so we use this more general notation.

Combining Eqs. (12) and (13), the relationship between the marginal default value for a line of business and the default value for the firm as a whole is

$$p_i - p \;=\; -\left(c_i - c\right)\Pi_Z\left(R_L\right) \;-\; \left(\Pi_Z\left(R_i\right) \;-\; \Pi_Z\left(R_A\right)\right) \tag{14}$$

We derive capital allocation formulas from Eq. (14) and the first principle of capital allocation (no risk-shifting): capital should be allocated so that a marginal change in the composition of the firm's business portfolio does not affect the credit quality of the firm's liabilities. We measure credit quality by the ratio of the value of the default put to the value of default-free liabilities, so this condition is $p_i \equiv \frac{\partial P}{\partial A_i} = \frac{P}{L}\left(\frac{\partial L}{\partial A_i}\right) = \left(\frac{P}{L}\right)\left(1 - c_i\right)$.

Putting these results together gives the following formula for allocating capital:

$$\mathrm{c_i} = \mathrm{c} + \frac{\Pi_z(\mathrm{R_A}) - \Pi_z(\mathrm{R_i})}{\Pi_z(\mathrm{R}_L) - \mathrm{P/L}}. \tag{15}$$

Next we consider how a financial firm should (in principle) choose its optimal portfolio of lines of business. Capital allocation formulas will also follow from the conditions for the optimum.

3. Portfolio Optimization and Capital Allocation

Assume that the management of a financial firm calculates the optimum *portfolio* of lines of business. Financial firms solve this problem implicitly when they set strategy, launch new lines of business or force major restructurings.

Management that plans for the long run would choose optimal capital structure at the same time as the optimal portfolio, trading off the costs of additional equity (tax and agency and monitoring costs) against the benefits of reduced liabilities and better credit.

We assume that the firm has considered this tradeoff and decided on a minimum level of credit quality, defined as the ratio of the value of the default put to the value of default-free liabilities: $P(A,C) \leq \alpha L$. The firm then optimizes subject to this *credit-quality constraint*. This constraint is critical. If we did *not* impose it, the optimization would allow for shifting of credit risk and associated costs from the firm to its creditors and counterparties.

For example, a financial firm may decide that it needs a single-A credit rating in order to transact with counterparties and set α accordingly. (Notice that α determines put value and the credit spread that lenders or counterparties require.) Or management may solve the optimization problem at several different levels of credit quality, and then decide which credit quality provides the greatest value.

The objective is to maximize the market value of the firm. Capital has a tax cost of τ per dollar per period. (We will just refer to tax costs, but τ could also cover other costs of contributing and maintaining capital.) For simplicity we assume the optimal portfolio is chosen once or for one period only.[11]

We assume that debt can be raised at market rates in zero-NPV transactions.[12] That is, debt markets are competitive and that lenders and depositors are fully informed. Thus the firm must pay interest rates that fairly compensate lenders for the default risk they bear.[13]

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[11] We hold franchise value and growth opportunities G constant. Dynamics are more complicated. Froot and Stein (1998) introduce some dynamics of bank capital structure decisions.

[12] Therefore, we leave out a sources = uses of cash or investment = financing constraint, because the shadow price of the constraint would be zero.

[13] The interest rate paid to depositors includes the value of transaction and other services provided "free of charge" by the bank.

3.1. Portfolio Optimization with a Credit Quality Constraint

The decision variables are the amounts of capital C and liabilities L, plus A_i, the amount of assets held in line i. Total assets are $A = \sum_i A_i$. The profit margin for line i, expressed in present-value units, is $m_i(A_i)$, with decreasing returns to scale.[14]

Capital is to be allocated line by line at rates c_i, with $C = \sum_i c_i A_i$. The capital ratio for the firm is a weighted average of the line-of-business capital ratios: $c = \sum_i c_i x_i$ where $x_i \equiv \frac{A_i}{A}$. We know that $P = \sum_i p_i A_i$ and $L = \sum_i (1-c_i) A_i$. If we assume the credit quality constraint is binding, the Lagrange function is:

$$V(\vec{A}, \vec{c}, \lambda) = \sum_i \int m_i(A_i) dA_i - \tau c_i A_i + \lambda \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) \tag{16}$$

The conditions for an optimum are:

(16a) $$\frac{\partial V}{\partial A_i} = m_i(A_i) - \tau c_i + \lambda \left(\alpha (1-c_i) - p_i \right) = 0$$

(16b) $$\frac{\partial V}{\partial c_i} = - A_i \left[\tau + \lambda \left(\alpha + \frac{\partial p_i}{\partial c_i} \right) \right] = 0$$

(16c) $$\frac{\partial V}{\partial \lambda} = \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) = 0$$

The first condition (16a) tells us that, at an optimum, the line-of-business margins are equal to the product of the cost of capital τ and the line-of-business capital allocation rates: $m_i(A_i^*) = \tau c_i^*$, where asterisks indicate optimum values. (We show below that

[14] Margins will probably depend on credit quality, for example because of collateral requirements or costs imposed by nervous counterparties. But credit quality is held constant in this optimization so long as the credit quality constraint is binding. Therefore we do not express $m_i(A_i)$ as a function of α.

the last term in condition (16a) is zero at the optimum.) Thus the ratio of the margin to the capital allocation rate is the same for all lines.

Condition (16b) is the *no internal arbitrage* principle of capital allocation: if capital allocation rates are set correctly, it will not be possible to add value simply by reallocating capital from one line of business to another. The marginal product of capital is the same in all lines. To see this, recall that $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$. Therefore, (16b) can be written as

$$\frac{\partial V}{\partial c_i} = -A_i\left[\tau + \lambda\left(\alpha + \frac{\partial p}{\partial c}\right)\right] = 0 \tag{17}$$

We can also find the shadow price on the credit constraint from condition (2): $\lambda = \frac{\tau}{\alpha + \frac{\partial p}{\partial c}}$.

Condition (16c)—the credit quality constraint—implies that the marginal default values by line of business bear the same relationship to the line-of-business capital allocation rates as the default value for the firm bears to the capital ratio: $p_i = \alpha(1-c_i)$ and $p = \alpha(1-c)$. This is our no risk-shifting condition with the firm's credit quality set to α. Therefore, efficient line-by-line capital allocation rates require:[15]

$$\frac{p_1}{1-c_1} = \frac{p_2}{1-c_2} = \cdots = \frac{p}{1-c} = \alpha. \tag{18}$$

[15] This result may appear inconsistent with Myers-Read (2001), who conclude that marginal default values in all lines of business must be equal to avoid cross subsidies. But they define marginal default values with respect to liabilities, not assets. Our result is equivalent to the condition that marginal default values with respect to *liabilities* are the same in all lines.

Eq. (18) is not yet a recipe for calculating capital allocations, because the marginal default value p_i depends on c_i, the capital allocation to line i, which we have not yet determined. Eq. (18) just says that capital-adjusted marginal default values must all be the same when expressed as a fraction of liabilities. (Dividing p_i by $1-c_i$ gives the ratio of marginal default value to the debt used at the margin to finance assets in line of business i.)

3.2 Portfolio Optimization with a Capital Constraint

Capital is a decision variable for long-run planning. But capital is likely to be fixed over weeks, months or quarters. We can add a capital constraint $C \le \overline{C}$ to the Lagrange function:

$$V\left(\vec{A},\vec{c},\lambda\right) = \sum_i \int m_i(A_i)dA_i - \tau c_i A_i + \lambda\left(\alpha \sum_i (1-c_i)A_i - \sum_i p_i A_i\right) + \kappa\left(\overline{C} - \sum_i c_i A_i\right) \quad (19)$$

The conditions for an optimum with this additional constraint are:

(19a) $$\frac{\partial V}{\partial A_i} = m_i(A_i) - (\tau+\kappa)c_i + \lambda\left(\alpha(1-c_i) - p_i\right) = 0$$

(19b) $$\frac{\partial V}{\partial c_i} = -A_i\left[\tau + \kappa + \lambda\left(\alpha + \frac{\partial p_i}{\partial c_i}\right)\right] = 0$$

(19c) $$\frac{\partial V}{\partial \lambda} = \left(\alpha \sum_i (1-c_i)A_i - \sum_i p_i A_i\right) = 0$$

(19d) $$\frac{\partial V}{\partial \kappa} = \overline{C} - \sum_i c_i A_i = 0$$

If both constraints are binding, the margin in each line of business is equal to the product of the capital allocation rate and the "all-in" cost of capital $\tau+\kappa$. That is,

$m_i(A_i*) = (\tau + \kappa)c_i^*$. The shadow price on the capital constraint is $\kappa = \frac{m(A^*)}{c*} - \tau$. The shadow price on the credit constraint is $\lambda = \frac{\tau + \kappa}{\alpha + \frac{\partial p}{\partial c}}$. As before, $p_i = \alpha(1 - c_i^*)$.

3.3. Capital Allocation

We know from Section 2 that capital allocations should be set line by line using Eq. (15). Eq. (18) requires $\frac{p_i}{1-c_i} = \frac{p}{1-c} = \alpha$. The formula for efficient capital allocation becomes:[16]

$$c_i = c + \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_L) - \alpha} \tag{20}$$

Here the constrained level of credit quality α substitutes for the general credit quality measure P/L in Eq. (15).

Thus the capital allocated to line of business i depends on the firm's overall capital ratio c, on credit quality α and on the difference in default payoff values for the overall firm vs. the line of business. The allocation does not depend directly on the investment in line i, but only indirectly, because decisions about investments, capital and capital allocation are made jointly when the firm optimizes. But for capital allocation, the joint optimization need deliver only the overall capital ratio c and the overall default payoff value $\Pi_Z(R_A)$.

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[16] Here we make a simplifying assumption that investing an additional dollar in line i does not affect the default payoff value $\Pi_Z(R_i)$. This is the case if decreasing returns to investment come from an increasing cost of achieving a fixed probability distribution of the returns R_i. For example, we could specify the present value margin as $m_i(A_i) = \Pi(R_i) - e_iA_i$, where $\Pi(R_i)$ is the present value of returns in all states of nature, not just the default region Z, and e_i is a positive cost of expanding line i. Capital would be measured after e_iA_i is paid for.

Marginal capital allocations for an asset or line of business therefore depend on the present value of its returns in default, that is, on the present value of its returns as distributed across the default region Z. If its returns are "riskier" than the overall portfolio return R_A in region Z—that is, worth *less* than the overall portfolio return in that region—then $c_i > c$. If its returns are relatively "safe" in region Z—worth *more* than the overall return in default—then $c_i < c$. The capital ratio for line *i* does *not* depend on the line's marginal effect on the *probability* of default. It depends on the *value* of the line's payoff in default.

Thus capital can be allocated depending on the marginal default value of each line of business, where marginal default value is the derivative of the value of the firm's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. Differences in marginal default values can be offset by differences in marginal capital allocations. Cross-subsidies are avoided if capital allocations are set so that capital-adjusted marginal default values are the same for all lines, as in Eq. (18). Each line's capital ratio should depend on the value of the line's payoffs in default. The procedure of setting marginal capital requirements to equalize capital-adjusted marginal default values follows from optimization of the firm's portfolio of businesses.

4. Default Values and Capital Allocation for Joint Normal Distributions

If asset returns are normally distributed, the return to a portfolio of assets is normally distributed too. This allows closed-form formulas for marginal default values and capital allocations. The formulas illustrate how our capital allocation procedures work and what the allocations depend on. The formulas will also make it relatively easy to construct and interpret numerical examples.[17]

The default value P depends on the market value of assets A, the market value of default-free liabilities L, and on σ_A, the standard deviation of end-of-period asset returns per unit of assets. The present value of the default option is:

$$P(A,L,\sigma_A) = (L-A)\mathrm{N}\{y\} + \sigma_A A \mathrm{N}'\{y\} \tag{21}$$

where N is the cumulative distribution function for a standard normal variable and $y = \dfrac{L-A}{\sigma_A A}$. Capital is defined as the market value of assets less the market value of default-free liabilities, so the present value of the default option can also be expressed as a function of assets and capital: $P(A,C,\sigma_A) = -C\,\mathrm{N}\{y\} + \sigma_A A\,\mathrm{N}'\{y\}$, where $y = \dfrac{C}{\sigma_A A}$. The default value per unit of assets is a function of the capital-to-asset ratio:

$$p\,(c,\sigma_A) = -c\,\mathrm{N}\{y\} + \sigma_A\,\mathrm{N}'\{y\}, \tag{22}$$

where $y = \dfrac{c}{\sigma_A}$.

Remember from Eq. (9) that $p_i = \dfrac{\partial P}{\partial A_i} = p + \dfrac{\partial p}{\partial a_i}(1-a_i)$. The change in p due to a change in the composition of the portfolio ($\partial p / \partial a_i$) can also be written as

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[17] The assumption that returns are normally distributed, which we make for illustrative purposes, may strike some readers as inconsistent with the limited liability. Whereas shareholders have limited liability, however, this is not necessarily true for the business units of a corporation. Consider the "desks" of a trading firm. As JP Morgan's "whale" trades demonstrated recently, the desks can lose billions. See Merton (1997), who begins with the assumption that the surplus of the firm's assets is normally distributed and shows that the value of the firm's equity is log-normally distributed.

$\frac{\partial p}{\partial a_i} = \frac{\partial p}{\partial c}\frac{\partial c}{\partial a_i} + \frac{\partial p}{\partial \sigma_A}\frac{\partial \sigma_A}{\partial a_i}$. But $\frac{\partial c}{\partial a_i} = \frac{c_i - c}{1 - a_i}$ and $\frac{\partial \sigma_A}{\partial a_i} = \frac{\sigma_{iA} - \sigma_A^2}{\sigma_A(1 - a_i)}$, so the marginal default value for each line of business is:

$$p_i = p + \frac{\partial p}{\partial c}(c_i - c) + \frac{\partial p}{\partial \sigma_A}\left(\frac{\sigma_{iA} - \sigma_A^2}{\sigma_A}\right) \tag{23}$$

where σ_{iA} is the covariance of the return on line of business *i* with the portfolio return.

The option delta ($\partial p/\partial c$) and vega ($\partial p/\partial \sigma_A$) are:

$$\frac{\partial p}{\partial c} = -\,\mathrm{N}\{y\} \tag{24a}$$

$$\frac{\partial p}{\partial \sigma_A} = \mathrm{N}'\{y\} \tag{24b}$$

The option delta is negative, so the higher the capital allocation, the lower the marginal default value. The option vega is positive, so the higher the covariance of returns, the higher the marginal default value.

We combine the constraint on credit quality from Eq. (18) and the expression for marginal default value in Eq. (23) and solve for the capital allocation for line *i*:

$$c_i = c - \left(\frac{\partial p}{\partial c} + \alpha\right)^{-1}\left(\frac{\partial p}{\partial \sigma_A}\right)\left[\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A}\right] \tag{25}$$

Thus the marginal capital allocations in the normal case depend on the delta and vega of the default put and on the difference between σ_{iA}, the covariance of asset i's return with the overall return, and the variance of the overall return σ_A^2. Riskier assets ($\sigma_{iA} > \sigma_A^2$) must be allocated extra capital ($c_i > c$). Safer assets ($\sigma_{iA} < \sigma_A^2$) require less capital ($c_i < c$). Safe or low-risk assets have negative capital allocations.

4.1. Numerical Examples

Consider a firm selecting a portfolio of businesses from scratch. The firm can invest in either one or both of two lines of business. Let $x_1 = x$ and $x_2 = 1 - x$ be the proportions of total assets invested in the two lines. Assume that asset returns are normally distributed. Then the risk of the firm's assets σ_A depends on the standard deviations of asset returns σ_1 and σ_2, the correlation of returns ρ and the asset allocation x. The firm's liabilities are riskless except for the possibility of default.

The goal is to maximize APV, defined here as the NPV of the investments less the cost of capital. The NPV of each line i ($i = 1, 2$) can be expressed as the product of the amount invested A_i and an average margin $\overline{m}_i\left(A_i\right)$. Both lines of business are subject to decreasing returns to scale, that is, $m_i'\left(A_i\right) < 0$.

We have to impose a constraint on credit quality because our objective function (specification of the APV) includes the *cost* of capital but not any *benefits*. An unconstrained solution would set capital to zero. But if the optimal solution includes positive investment in at least one line ($A^* > 0$), then the credit quality constraint will be binding.

Suppose the firm wishes to operate with a default value that does not exceed a fraction α of the present value of liabilities: $P \leq \alpha L$. The optimal capital ratio—the minimum capital ratio consistent with this credit quality constraint—is completely determined by the minimum credit quality α and the risk of the portfolio risk σ. The risk of the portfolio is determined by the standard deviations of asset returns σ_1 and σ_2, the correlation of asset returns ρ, and the asset allocation (x). Therefore, the optimal capital ratio is $c^* = c^*\left(\alpha, \sigma\left(x\right)\right)$.

Total assets at any fixed combination of the two lines of business can be found by solving for the point at which the marginal NPV is equal to the cost of capital:

$$m_x\left(A^*, x\right) = x m_1\left(xA^*\right) + \left(1-x\right) m_2\left(\left(1-x\right)A^*\right) = \tau c^*\left(\alpha, \sigma\left(x\right)\right) \qquad (26)$$

Given assets A^* and capital ratio c^*, we can calculate capital C, default-free liabilities L, and the default value P:

$$C = c^*\left(\alpha, \sigma\left(x\right)\right) A^*\left(x\right) \qquad (27a)$$

$$L = \left(1 - c^*\left(\alpha, \sigma\left(x\right)\right)\right) A^*\left(x\right) \qquad (27b)$$

$$P = \alpha L \qquad (27c)$$

The marginal default values (p_i) and capital allocation rates (c_i) by line of business can be calculated using Eqs. (23) and (25).

To obtain illustrative numerical results, we assume that the standard deviation of asset returns for Line 1 and 2 are 10% and 30%, respectively, and the returns are uncorrelated. The gross margin for Line 1 is 2% minus 0.0001% of Line 1 assets, and the gross margin for Line 2 is 3% minus 0.0001% of Line 2 assets. Line 1, in other words, is "low risk, low return" and Line 2 is "high risk, high return." The tax cost of capital is $\tau = 3\%$. Assume the firm seeks to maintain credit quality $\alpha = 1\%$.

The adjusted present value of the firm's portfolio of investments depends on the amount of assets acquired A, the asset allocation x, the net present value per dollar of investment in each line, the amount of capital needed to meet the credit quality constraint, and the cost of capital τ. The APV is $A\left[x\overline{m}_1(xA)+(1-x)\overline{m}_2((1-x)A)-\tau c(x)\right]$.

Let x^* be the asset allocation at which the APV of the portfolio is a maximum. At the optimum, the ratio of the marginal NPV to capital is the same in both lines of business and is equal to the cost of capital: $\frac{m_1(A_1)}{c_1^*(x^*)} = \frac{m_2(A_2)}{c_2^*(x^*)} = \tau$. Also, the marginal profit in both lines is zero:

$$m_1(A_1) - \tau c_1^*(x^*) = m_2(A_2) - \tau c_2^*(x^*) = 0. \tag{28}$$

The numerical results are summarized in Table 1. The first column contains the results at the optimal asset allocation, where the APV of the firm's investments is at the maximum. The columns to the right give results for allocations ranging from 100% in Line 1 (0% in Line 2) to 100% in Line 2 (0% in Line 1). In all cases the constraint on credit quality is binding. If the firm chooses to operate as a stand-alone Line-1 business (with asset risk equal to 10%), it will have assets of \$17,130 and require capital of \$1,639. Its capital ratio will be 9.57%. If the firm chooses to operate as a stand-alone Line-2 business (with asset risk equal to 30%), it will have assets of \$14,146 and require capital of \$7,476. Its capital ratio will be 52.85%.

The optimum asset allocation calls for investing 54.46% of assets in Line 1, which is relatively safe, and 45.54% in Line 2. Portfolio asset risk is 14.71% and the capital ratio is 17.66%. The firm has assets of \$38,205 and requires capital of \$6,749. Capital allocations are - 2.69% (yes, that is a minus) of assets in Line 1 and 42.00% of assets in Line 2. This means - \$559 of capital is allocated to Line 1 and \$7,308 of capital is allocated to Line 2. The APV of the portfolio is $368, which is equal to the $570 NPV minus the cost of capital (3% of $6,749).

The market value of the firm is lower at all other asset allocations. For example, with 100% of assets invested in Line 1, the APV is only $147. At this point the covariance of the return on Line 1 with the return on the portfolio is equal to the variance of the return on the portfolio—Line 1 *is* the portfolio. The capital allocation to Line 1 is just the portfolio capital ratio: 9.57%. In contrast, the covariance of the return on Line 2 with the return on the portfolio is zero. The capital allocation to Line 2 is less than zero: –6.27%. Thus, not only is the gross profitability (per-dollar net present value) of an investment in Line 2 greater than the gross profitability of an investment Line 1, the risk of marginal investment in Line 2 is less than the risk of marginal investment in Line 1.

Risk and profitability change as investment in Line 2 expands and investment in Line 1 contracts. With 90% of assets invested in Line 1 (10% in Line 2), the covariance of the return on Line 2 is greater than zero, and the covariance of the return on Line 1 is equal to the variance of the portfolio return. As a result, the capital allocation to Line 2 increases from –6.27% to 8.76% and the capital allocation to Line 1 decreases from 9.57% to 8.76%. At this asset allocation the co-variances and marginal default values are equal. The marginal profitability of Line 2 has declined due to the increase in its marginal portfolio risk and capital allocation.

At the optimal asset allocation, the higher gross profitability of Line 2 is exactly offset by its higher capital cost. Thus the marginal profitability of Line 1 and the marginal profitability of Line 2 are zero at the optimum.

How could headquarters implement the optimal portfolio? It could simply allocate –\$559 of capital to Line 1 and \$7,308 of capital to Line 2. Each line would be charged the 3% cost on any additional capital sought by either line. Neither line would want to expand, because expansion would push marginal profitability into negative territory.

If the firm started with a non-optimal asset mix, headquarters could simply charge the cost of capital shown in the appropriate non-optimal column of Table 1. One line or the other would face negative marginal profitability and move to relinquish capital, which would free up the other line to expand. Both lines would be content only at the optimum. Thus the optimum could, in principle, be achieved in a decentralized setting.

Pricing for each line's product or service would be determined by operating costs plus the cost of capital at the optimum. Compensation would be determined by the APVs, which measure the lines' contributions to firm value.

5. Applications and Implications

We recommend that financial firms allocate capital to lines of business based on marginal default values. One can think of the allocation procedure in two steps. First identify each line's marginal impact on the value of the firm's default put. (The value of the default put will be small for well-capitalized firms, but nevertheless positive. The default put will show up in credit spreads demanded by lenders and counterparties.) Some lines of business will have larger marginal impacts than others. Second, calibrate the marginal capital allocated to each line of business so that capital-adjusted marginal default values are the same for all lines. See Eq. (15) or Eq. (20), which hold for any joint probability distribution of line-by-line returns, and Eq. (25) for the joint normal distribution.

Our allocations are fundamentally different than allocations based on VaR or contribution VaR. They are derived from the conditions for an optimal portfolio of lines of business. Capital charges based on our allocations could be used to implement an optimal portfolio in a decentralized setting.

Capital charges proportional to VaR are not consistent with the conditions for an optimum and would distort or destroy the optimum in a decentralized setting. For example, we have showed that risk-free or low-risk lines of business should get *negative* capital allocations. Allocations based on VaR would be zero or positive.

We have also showed how capital should be "priced" and charged back to line of business. If total capital is held fixed, the "all-in" cost of a dollar of capital equals $\tau+\kappa$, that is, the sum of the tax or other costs of holding equity (τ) and the shadow price of the constraint on total capital (κ).

Notice that the all-in cost is *not* an interest rate or "cost of equity" on the allocated capital. Suppose that the managers of line 1 are considering expanding assets by $100 million. First they should calculate the NPV of the expansion using a cost of capital matched to the market risk of future returns in line 1. In other words, they should respect the corporate finance principle that the discount rate depends on the risk of the investment, not on the risk of the firm's overall portfolio or on the interest rate that the firm pays to borrow. The cost of capital depends on the use of funds, not the source. Second, they should subtract $\tau+\kappa$ times $100 million from the NPV. Suppose NPV = +$10 million. If $\tau = .01$ and $\kappa = .02$, the adjusted present value of the expansion is APV = $10 - .03×100 = +$7 million.

Our capital-allocation procedures apply for any joint probability distribution of line-by-line returns. Computing marginal default values and capital allocations is straightforward in principle. We have used Monte Carlo simulation to perform numerical experiments (not reported) with a variety of probability distributions. But the default put is a deep out-of-the-money option, and the lower tail is the part of the joint distribution about which the least information is available. Thus information, not computation, is the challenge for practice.

Capital allocations should vary depending on composition of the firm's portfolio. Thus the same line of business can have a higher capital allocation in firm B than in firm A if the line is a larger fraction of B's portfolio and/or more highly correlated with returns on B's other lines than A's other lines.

5.1 Capital Allocation for Non-financial Corporations

We have focused on financial firms, but our results also apply to non-financial corporations.

In corporate finance, the tax-adjustment term in APV is usually expressed as a tax advantage of debt rather than a tax cost of equity. NPV is calculated at an opportunity cost of capital, as if the investment were all-equity financed, and the present value of interest tax shields is then added. See Myers (1974) and Brealey, Myers and Allen (2013), Ch. 19. The interest tax shields depend on the amount of debt supported by the investment. In our setting, where the firm is allocating capital, NPV should be calculated as if the investment were 100% financed by "tax-free" financing, i.e. debt and other liabilities, and 0% by equity capital. The tax cost of the equity capital required to support the investment is then subtracted. These setups are of course equivalent, two sides of the same coin.

The theory of optimal capital structure usually solves (hypothetically) for the target debt-to-value ratio that maximizes firm value. Our model suggests that the firm should not target a debt ratio, but credit quality, defined as a credit spread or rating. The debt ratio required to meet the credit quality target will then vary as the risks of lines of business change and as the firm's portfolio of businesses evolves.

Consider a conglomerate with divisions in several industries, some safe, some risky. The conglomerate's managers could assign more "debt capacity" per dollar of assets in the safer divisions. It could equally well decide on its overall capital structure, determined by a target credit quality (or debt rating), and then allocate less equity capital, and therefore more debt, to the safer divisions. The safer divisions could get lower weighed average costs of capital (WACCs), not only because of lower business risk, but also because of more debt capacity and higher interest tax shields. The divisional debt ratios could be computed from our model. Of course our model says that divisions are "safer" if their returns are less volatile and less correlated with returns on other divisions. The debt capacity of a division depends on the conglomerate's portfolio of divisions.

5.2 Regulatory Capital Requirements

Regulators do not allocate capital, but they do set risk-based capital requirements, which sounds like nearly the same thing. Therefore we consider whether a regulator could set risk-based capital requirements that (1) limit the size of the regulated firm's default put and (2) still allow the firm to operate at its portfolio optimum. We will refer to banks, although our conclusions apply to any financial firm subject to prudential regulation. We are *not* here concerned with "regulatory arbitrage," which is the substitution of risky for safer assets *within* a line of business. Regulatory arbitrage is a difficult but distinct problem.[18]

Assume that the bank can raise additional capital if necessary. The regulator sets out to enforce better credit quality than the bank would choose on its own. This can be done in two ways. First, the regulator could require a lower ratio of default-put value to the bank's debt, including deposits, and other liabilities. This means setting $\hat{\alpha} < \alpha$ in Eq. (16). (The "hat" indicates a parameter or variable determined by the regulator.) The bank would then choose its optimal portfolio subject to the tighter credit-quality constraint.

This regulatory strategy seems ideal, because there would be no distortion of the bank's investments or internal capital allocations. The practical problem is for the regulator to monitor and observe credit quality and to demand additional capital if the credit-quality constraint (using $\hat{\alpha}$) is not met. The credit spreads demanded by the bank's creditors and counterparties could assist the regulator. But monitoring credit quality cannot be outsourced to creditors and counterparties if the bank is too big to fail or if creditors and counterparties will be bailed out in a crisis. In that case observed credit spreads will understate the value of the put absorbed by the regulator or government.

Second, the regulator could set risk-based capital requirements line by line. This

[18] Palia and Porter (2003) include a good description of risk-weighted capital requirements and regulatory capital arbitrage. See Gordy (2003) for the theoretical basis for risk-weighted capital requirements.

intervention changes the firm's optimization to:

$$V\left(\vec{A},\vec{c},\lambda\right) = \sum_i \int m_i\left(A_i\right)dA_i - \tau c_i A_i + \lambda\left(\alpha\sum_i\left(1-c_i\right)A_i - \sum_i p_i A_i\right) + \hat{\kappa}\left(\sum_i \hat{c}_i A_i - \sum_i c_i A_i\right)$$

(29)

Notice that the risk-based capital requirements do not replace the bank's internal capital allocations c_i. The bank still wants to allocate capital efficiently, subject to the regulatory constraint.

The conditions for an optimum with risk-based capital requirements are:

(29a) $$\frac{\partial V}{\partial A_i} = m_i\left(A_i\right) - \tau c_i + \lambda\left(\alpha\left(1-c_i\right) - p_i\right) + \hat{\kappa}\left(\hat{c}_i - c_i\right) = 0$$

(29b) $$\frac{\partial V}{\partial c_i} = -A_i\left[\tau + \kappa + \lambda\left(\alpha + \frac{\partial p_i}{\partial c_i}\right)\right] = 0$$

(29c) $$\frac{\partial V}{\partial \lambda} = \left(\alpha\sum_i\left(1-c_i\right)A_i - \sum_i p_i A_i\right) = 0$$

(29d) $$\frac{\partial V}{\partial \hat{\kappa}} = \sum_i \hat{c}_i A_i - \sum_i c_i A_i = 0$$

This optimization is similar to the capital-constrained optimization in Eq. (19) except that the regulatory capital requirement $\hat{C} = \sum_i \hat{c}_i A_i$ replaces the fixed capital amount $\overline{C}$. The "all-in" cost of capital is now $\tau + \hat{\kappa}\left(\hat{c}_{\hat{i}} - c_i\right)$. The shadow price on the capital constraint is $\hat{\kappa} = \frac{m\left(A*\right)}{c*} - \tau$. The shadow price on the credit constraint is $\lambda = \frac{\tau + \hat{\kappa}}{\alpha + \frac{\partial p}{\partial c}}$. As before, $p_i = \alpha\left(1 - c_i*\right)$.

Suppose that the risk-based capital requirements succeed in forcing higher credit quality, so that the bank's credit-quality constraint is slack and $\lambda = 0$. Then the bank will choose investment in line i by solving

$$\frac{\partial V}{\partial A_i} = m_i(A_i) - \tau c_i + \hat{\kappa}(\hat{c}_i - c_i) = 0 \tag{30}$$

The bank will underinvest in line i where $\hat{c}_i > c_i$ and over-invest where $\hat{c}_i < c_i$, depending on how tight the regulatory constraint is and the size of its shadow price $\hat{\kappa}$.

This regulatory strategy will *always* distort the bank's investments, unless the regulator could set the capital requirements exactly equal to the internal capital allocations than the bank would choose if subject to a tighter regulatory constraint on credit quality. The regulator would have to know the bank's lines of business, including possible new lines, margins $m_i(A_i)$ by line and the joint probability distribution of line-by-line returns. But that omniscient regulator could achieve the same result more simply and directly by tightening the credit-quality constraint and then tracking credit quality. In that case the risk-adjusted capital requirements would be at best redundant.

The idea that risk-based capital requirements could be non-distorting becomes more incredible when one realizes that efficient capital allocations depend on the composition of the bank's portfolio of lines of businesses and the correlations of the lines' returns.[19] Custom capital requirements would be required for each bank. Also regulators would have to set *negative* risk-based capital requirements for safe and many low-risk assets.

It may be that risk-based capital requirements are "better than nothing." The real

[19] Gordy (2003) derives conditions in a VaR setting for risk-based capital requirements that do not depend on the composition of the bank's portfolio of lines of business. The conditions are extremely restrictive, however. For example, there must be a single common factor driving line-by-line returns.

issue is whether they are better than setting constraints on overall credit quality or assuring bulletproof credit quality by requiring all banks to hold much more capital than has been customary.[20] The tax cost of holding additional capital could easily be offset by allowing banks to hold matching amounts of assets tax-free. [21]

6. Conclusions

We argue that capital can and should be allocated based on the marginal default value of each line of business, where marginal default value is the derivative of the value of the bank's default put with respect to a change in the scale of the business. Capital allocations are relevant for pricing, performance measurement, incentives, compensation, and trading and hedging decisions.

Capital allocations based on marginal default values add up exactly. This adding-up result requires complete markets, complete enough that the bank's assets and default put option have well-defined market values, but does not require any restrictions on the joint probability distributions of returns. Allocations will be sensitive to distributional assumptions, however.

Differences in marginal default values across lines of business should be cancelled out by offsetting differences in marginal capital allocations. We calculate the resulting capital allocations from the conditions for an optimal portfolio of lines. The allocations are systematically different from allocations proportional to VaR or contribution VaR. For example, VaR measures allocate zero capital to risk-free assets.

[20] See Admati and Hellwig (2013).

Our allocations are always negative for risk-free assets and often negative for low-risk assets.

We derive capital allocations from the conditions for an optimal portfolio of lines of business. Of course no bank or financial firm solves an explicit mathematical program to determine its optimal portfolio at the start of every period. Usually the firm takes its existing portfolio as fixed or considers gradual marginal changes. Our capital allocation procedures also work for any fixed portfolio of businesses.

Sometimes a bank or financial firm has to decide whether to add or subtract a line of business or a significant block of assets—see Merton and Perold (1993) and Perold (1995). The decision hinges on whether the bank is better off with or without the business or assets. Capital allocations "with" are not the same as "without." All capital allocations can change after a discrete investment. The only general way to evaluate discrete changes is to compare value with vs. without, using different capital allocations.

If the discrete change is small relative to the bank's overall assets, allocations for existing lines can be a good approximation if the bank has many existing lines of business and if the line of business that is changed is not too large. Allocations for a business that is expanded or contracted can be very sensitive to the magnitude of the change, but allocations to existing businesses can be much more stable and for practical purposes may not have to be adjusted frequently.[22]

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[21] Note that the present value of an income tax paid on an investment return does not depend on the risk of the investment. See the "Myers theorem" in Derrig (1994).

[22] Myers and Read (2001) perform experiments showing that allocations to existing lines change slowly when new lines are added and subtracted. We have run similar experiments (not reported) with similar results. Of course these results are reassuring only if portfolio composition changes gradually.

Consider a proposal to add an entirely new business. The new business's present value is reduced by the cost of the capital allocated to it. The investment is worthwhile if its APV is positive, taking the mix of existing businesses as constant. APV equals NPV minus the all-in cost of allocated capital, which includes the tax or other costs of holding capital and a shadow price if the amount of capital is constrained.[23] The amount of capital allocated increases steadily as the scale of the new business increases. Thus capital allocation is a source of decreasing returns to investment.[24] Optimal scale (holding existing assets constant) is reached when APV, net of the all-in cost of allocated capital is zero.

We have assumed tax costs of holding capital. Bank capital is also said to be costly because of agency and information and monitoring costs. See Merton and Perold (1993) and Perold (2005), for example. These costs are less clear. For example, if the bank is not fully transparent, additional capital should add value, not reduce it, because banks do business with counterparties who depend on the bank's credit. If the number of counterparties is large, the total cost of counterparties' due diligence and continuing credit tracking of the bank can be significant. These costs are passed on to the bank as less favorable terms on the banks' transactions. A bank with more capital, other things equal, imposes lower costs on counterparties and should be more profitable. Thus lack of transparency is an argument for more capital, not less.

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[23] If raising equity capital is feasible but incurs transaction costs, the marginal transaction costs should be charged against APV in place of the shadow price on the capital constraint.

[24] Line-by-line APVs could not be used to construct the optimal overall mix of business, however. The APV of each business would depend on the order in which candidate businesses were evaluated. This problem is highlighted by Merton and Perold (1993) and Perold (2005).

We have yet to see a good explanation for agency costs of bank capital. Are they costs of free cash flow, where managers are reluctant to curtail investment and release cash to shareholders? Adding debt in place of equity is regarded as a treatment or cure for this free-cash-flow problem. But too much debt could force managers to disinvest inefficiently *early*.[25] There is no reason to believe that more debt and less equity always add value. There is no reason to believe that more capital in a bank always generates more agency costs. For example, a cushion of extra capital can protect franchise value and forestall regulatory intervention if the bank suffers temporary losses. We believe that the agency costs of bank capital have to be thought through much more carefully.

[25] Lambrecht and Myers (2008) show how debt can lead to too much "discipline" and to *under*investment.

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Appendix 1

Observation 1 The default value P can be allocated uniquely across assets, proportional to the assets' marginal default values p_i:

$$P = \sum_{i=1}^{M} p_i A_i .$$

A proof of the observation for the two-asset case is below. Generalization to M assets is straightforward.

The amount of debt and other liabilities depends on A and a parameter c, the *capital ratio*, which measures the amount of capital that the bank puts up to back its liabilities. The capital ratio is a choice made by the bank or its regulators. For now we take c as constant across the bank's lines of business, with $L = (1 - c)(A_1 + A_2)$. Therefore:

$$\begin{aligned} P &= \int_Z [R_L(1-c)(A_1 + A_2) - R_1A_1 - R_2A_2]\pi(z)dz \\ &= A_1\int_Z [R_L(1-c) - R_1]\pi(z)dz + A_2\int_Z [R_L(1-c) - R_2]\pi(z)dz. \end{aligned}$$

Changes in A_1 and A_2 affect limits of integration at the boundary of the default region Z. These marginal effects can be left out, however, because the put payoff on the boundary is zero.[26] Thus p_1 and p_2 are:

$$p_1 = \frac{\partial P}{\partial A_1} = \int_Z \pi(z)[R_L(1-c) - R_1]dz,$$

[26] Even if there were value effects from shifts of the boundary, we can show that the effects would cancel. Appendix 1 in Myers and Read (2001) shows how boundary changes cancel.

$$p_2 = \frac{\partial P}{\partial A_2} = \int_Z \pi(z)[R_L(1 - c) - R_2]dz.$$

Multiply p_1 and p_2 by the respective asset values A_1 and A_2 to get the adding-up result, $p_1A_1 + p_2A_2 = P$.

Appendix 2

This appendix provides detailed description of the variables presented in each row in Table 1.

Asset allocation	Proportion of total assets invested in Line 1 ("x")
Asset risk	Standard deviation of returns on a portfolio with proportion x of assets invested in Line 1 & balance invested in Line 2
Capital ratio	Ratio ("c") of capital ("C") to assets ("A") required to achieve minimum credit quality (default-to-liability ratio).
Assets	Value of total assets in portfolio
Line 1	Value of assets invested in Line 1 ("A_1" where $A_1 = x*A$)
Line 2	Value of assets invested in Line 2 ("A_2" where $A_2 = (1-x)*A$)
Liabilities	Present value of default-free liabilities ("L" where L = A - C)
Capital	Total capital (C = c*A)
Default value	Present value of option to default ("P"); obtained via risk-neutral valuation under the assumption of normal return distributions
APV	Present value of portfolio after deducting the cost of risk capital ($APV = NPV - \tau*C$)
NPV	Net present value of investments
Line 1	Net present value of investment in Line 1
Line 2	Net present value of investment in Line 2
All-in cost of capital	The sum of the market cost of capital (τ) and the internal shadow price of capital
Default-to-liability ratio	The present value of the default put expressed as a ratio to the present value of default-free liabilities
Default-to-asset ratio	The present value of the default put expressed as a ratio to the present value of assets
Default-to capital ratio	The present value of the default put expressed as a ratio to capital
Variance A	The variance of returns on the portfolio
Covariance 1,A	The covariance of returns on Line 1 with returns on the portfolio
Covariance 2,A	The covariance of returns on Line 2 with returns on the portfolio
Marg. default value Line 1	Marginal default value for Line 1 ("p_1") calculated using Eq. (23)
Marg. default value Line 2	Marginal default value for Line 2 ("p_2") calculated using Eq. (23)
Capital allocation Line 1	Capital allocation rate for Line 1 ("c_1") calculated using Eq. (25)
Capital allocation Line 2	Capital allocation rate for Line 2 ("c_2") calculated using Eq. (25)
Capital	Equal to the product of the minimum capital ratio and total assets (C = c*A)
Line 1	Equal to the product of the Line 1 capital allocation rate and Line 1 assets ($C_1 = c_1*A_1$)
Line 2	Equal to the product of the Line 2 capital allocation rate and Line 2 assets ($C_2 = c_2*A_2$)
Capital charge Line 1	Equal to the product of the market cost of capital and Line 1 capital ($= \tau*C_1$)
Capital charge Line 2	Equal to the product of the market cost of capital and Line 2 capital ($= \tau*C_2$)
APV Line 1	Adjusted present value of Line 1 is net present value of Line1 less the cost of allocated capital ($APV_1 = NPV_1 - \tau*C_1$)
APV Line 2	Adjusted present value of Line 2 is net present value of Line1 less the cost of allocated capital ($APV_2 = NPV_2 - \tau*C_2$)
Marginal profit Line 1	Marginal profit reflects the all-in cost of capital (including shadow price)
Marginal profit Line 2	Marginal profit reflects the all-in cost of capital (including shadow price)

Table 1 - Capital Allocation with Constraint on Credit Quality

This table presents numerical examples for a firm selecting a two-line portfolio with a constraint on the credit quality (default-to-liability ratio). We assume that the firm maintains a default-to-liability ratio of 1%. The standard deviations of asset returns for Lines 1 and 2 are 10% and 30%, respectively; and the returns are uncorrelated. The per-dollar net present values (gross margins) are 2% minus 0.0001% of Line 1 assets and 3% minus 0.001% of Line 2 assets, respectively. The cost of capital is τ = 3%. See Appendix 2 for a detailed description of the variables in each row.

Asset allocation	**54.46%**	**100%**	**90%**	**80%**	**70%**	**60%**	**50%**	**40%**	**30%**	**20%**	**10%**	**0%**
Asset risk	14.71%	10.00%	9.49%	10.00%	11.40%	13.42%	15.81%	18.44%	21.21%	24.08%	27.02%	30.00%
Capital ratio	17.66%	9.57%	8.76%	9.57%	11.85%	15.32%	19.74%	24.93%	30.82%	37.38%	44.67%	52.85%
Assets	38,205	17,130	22,404	28,133	33,528	37,315	38,157	35,616	30,612	24,686	19,024	14,146
Line 1	20,806	17,130	20,164	22,506	23,470	22,389	19,079	14,247	9,184	4,937	1,902	0
Line 2	17,399	0	2,240	5,627	10,058	14,926	19,079	21,370	21,428	19,749	17,122	14,146
Liabilities	31,457	15,492	20,441	25,442	29,556	31,598	30,626	26,737	21,178	15,459	10,527	6,671
Capital	6,749	1,639	1,963	2,691	3,972	5,717	7,531	8,880	9,434	9,227	8,498	7,476
Default value	315	155	204	254	296	316	306	267	212	155	105	67
NPV	570	196	265	350	445	534	590	596	555	484	403	324
Line 1	200	196	200	197	194	197	200	183	142	87	36	0
Line 2	371	0	65	153	251	336	390	413	413	397	367	324
APV	368	147	206	269	326	362	364	330	272	207	148	100
All-in cost of capital	3%	3%	3%	3%	3%	3%	3%	3%	3%	3%	3%	3%
Default-to-liability ratio	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
Default-to-asset ratio	0.82%	0.90%	0.91%	0.90%	0.88%	0.85%	0.80%	0.75%	0.69%	0.63%	0.55%	0.47%
Default-to-capital ratio	4.66%	9.45%	10.41%	9.45%	7.44%	5.53%	4.07%	3.01%	2.24%	1.68%	1.24%	0.89%
Variance A	0.0216	0.0100	0.0090	0.0100	0.0130	0.0180	0.0250	0.0340	0.0450	0.0580	0.0730	0.0900
Covariance 1,A	0.0054	0.0100	0.0090	0.0080	0.0070	0.0060	0.0050	0.0040	0.0030	0.0020	0.0010	0.0000
Covariance 2,A	0.0410	0.0000	0.0090	0.0180	0.0270	0.0360	0.0450	0.0540	0.0630	0.0720	0.0810	0.0900
Marginal default value Line 1	1.03%	0.90%	0.91%	0.94%	0.97%	1.01%	1.04%	1.08%	1.13%	1.18%	1.25%	1.34%
Marginal default value Line 2	0.58%	1.06%	0.91%	0.78%	0.68%	0.61%	0.56%	0.53%	0.51%	0.49%	0.48%	0.47%
Capital allocation Line 1	-2.69%	9.57%	8.76%	6.40%	3.07%	-0.60%	-4.39%	-8.35%	-12.73%	-17.90%	-24.58%	-34.39%
Capital allocation Line 2	42.00%	-6.27%	8.76%	22.24%	32.33%	39.21%	43.87%	47.12%	49.48%	51.20%	52.36%	52.85%
Capital	6,749	1,639	1,963	2,691	3,972	5,717	7,531	8,880	9,434	9,227	8,498	7,476
Line 1	-559	1,639	1,767	1,440	720	-135	-838	-1,190	-1,169	-884	-468	0
Line 2	7,308	0	196	1,251	3,252	5,852	8,369	10,070	10,602	10,111	8,965	7,476
Capital charge Line 1	-17	49	53	43	22	-4	-25	-36	-35	-27	-14	0
Capital charge Line 2	219	0	6	38	98	176	251	302	318	303	269	224
APV Line 1	216	147	147	154	172	201	225	219	177	113	50	0
APV Line 2	151	0	59	115	154	161	139	111	95	94	98	100
Marginal profit Line 1	0.00%	0.00%	-0.28%	-0.44%	-0.44%	-0.22%	0.22%	0.83%	1.46%	2.04%	2.55%	3.03%
Marginal profit Line 2	0.00%	3.19%	2.51%	1.77%	1.02%	0.33%	-0.22%	-0.55%	-0.63%	-0.51%	-0.28%	0.00%