Selecting Bidders Via Non-Binding Bids When Entry Is Costly

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September 25, 2012
PRELIMINARY DRAFT

1 Introduction

In many auctions, potential bidders have some private information about their willingness to pay prior to entry, and learn more after entry. In timber auctions, bidders know their costs but need to perform "cruises" to obtain estimates of the volume and composition of wood on the tracts. Similarly, in an oil and gas lease sale, bidders have initial estimates of the likelihood of finding hydrocarbons on tracts based on their experience in the area, but invest millions of dollars in seismic surveys before bidding. In the sale of complex financial or business assets, buyers know the value of possible synergies between the asset and their businesses, but need to perform due diligence before bidding; the legal and accounting costs of inspecting and verifying the quality of the asset often runs into the millions of dollars.

When entry is costly, sellers often have an incentive to restrict the number of bidders who are allowed to participate in the auction. Even if it is only the bidders who incur the entry costs, the seller bears some or all of these costs indirectly through their impact on bidder participation and bidding. If bidders have private information prior to entry, then the issue facing the seller is not only how many bidders to allow, but how to select the "right" bidders, i.e., those who are most likely to have the highest willingness to pay.

One way investment banks solve this problem is with "indicative" bids. A large number of potential buyers are contacted, and given the opportunity to submit non-binding proposals of price and other terms of sale. The bank uses these proposals to select the two or three most serious buyers, who then perform due diligence and participate in a standard auction.[1]

This paper studies how indicative bids can be used to select bidders. We consider a simple model, first studied by Ye (2007). Potential bidders are asked to simultaneously submit indicative bids if they are interested in participating in the sale. The two highest bidders are selected to participate in the auction. These bidders pay the entry cost, receive additional information about their values, and submit binding bids. We will refer to this mechanism as the indicative bidding mechanism.

How much information can the seller expect to obtain from indicative bids? If indicative bids are informative, the seller has every incentive to use them to select participants: revenue is highest when the participating bidders are those with the highest willingness to pay. However, the incentives for bidders to honestly report initial estimates of their willingness to pay are less clear. If entry costs are zero, then entry can be thought of as a free option, and we would expect indicative bidding to unravel to all bidders submitting the highest possible bid. But if entry costs are positive, then

[1] In a typical sale, 15 to 30 potential buyers might be contacted initially with a one-page description of the business for sale, called a"teaser"; five to ten might respond and eventually submit nonbinding proposals. The most serious bidders get face-to-face meetings with the seller, and an opportunity to revise their proposals. Typically no more than three are chosen to gain access to a "data room" containing contracts and financials of the business, perform due diligence, and submit final, binding bids.

bidders' incentives are partly aligned with those of the seller, since bidders want to avoid being selected if they are unlikely to win. Thus, low value bidders will try to separate themselves from high value bidders by bidding less than the highest possible bid. However, Ye (2007) shows that this separation cannot be perfect: a symmetric equilibrium with strictly-increasing bid functions does not exist.

Our central result is that indicative bids can indeed lead to partial separation of bidder types, helping the seller to select high-value bidders with greater likelihood. Fixing a discrete bid space (including an "opt out" bid), we show that if entry costs are substantial relative to the new information rents generated in the second stage, then the game has an (effectively) unique symmetric equilibrium. In the equilibrium, only a finite number of bids are used, which induces a finite partition of the space of bidders' private information. Bidders with types in the same element of the partition report the same bid, and bidders in higher elements report higher bids. Thus, the equilibrium leads to a partial sorting of the bidders.

The equilibrium has the flavor of the "cheap talk" equilibria of Crawford and Sobel (1982): indicative bids are monotonic in bidders' initial information, but only a finite number of different bids are used in equilibriumm, so different types of bidders "pool" on the same bid. But, while the sender-receiver game in that paper gives useful intuition for our setting, as noted in Ye (2007), indicative bids are not cheap talk. Indicative bids have exogenous meaning – higher bids advance over lower bids – and directly affect bidders' payoffs (by determining whether they advance), while cheap-talk messages in Crawford and Sobel derive meaning only from the receiver's interpretation of them. This means that the multiplicity of equilibrium in a typical cheap-talk game does not occur in our setting, since off-equilibrium-path messages cannot be deterred via adverse off-equilibrium beliefs. However, in our setting, if the set of messages allowed is unrestricted – in particular, if it is continuous – then no symmetric equilibrium exists at all; the seller can only extract useful information by limiting the bids available to a discrete set.

How well does the indicative bidding mechanism perform? The natural benchmark to compare to is an auction in which entry is unrestricted. Bidders decide on the basis of their private information whether or not to enter the auction, learn more about their values, and submit binding bids. When the number of potential bidders is large, we show that the indicative bidding mechanism yields both greater revenue and greater bidder surplus than a standard auction with unrestricted entry. Through numerical examples, we find that the indicative bidding mechanism also tends to give higher revenue, though not always higher total surplus, when the number of bidders is small.

This paper extends the theoretical literature on auctions with costly entry to environments in which bidders have private information. Most of the existing literature assumes no private information prior to entry and focusses on how the seller should restrict entry. Cremer, Spiegel, and Zheng (2009) characterize the optimal mechanism, and show that the seller can use entry fees and subsidies to extract all buyers' surplus. In the absence of such payments, Bulow and

Klemperer (2009) demonstrate that auctions are less efficient than sequential mechanisms but typically generate more revenue. Earlier papers include McAfee and McMillan (1987), Levin and Smith (1994), Burguet (1996), Menzes and Monteiro (2000), and Compte and Jehiel (2007).[2]

Samuelson (1985) was the first to study the impact of entry on auction outcomes in an environment where buyers already know their private values, and entry costs therefore consist of bid preparation costs rather than information acquisition costs. Ye (2007) considers the more general case in which entry involves information updating. He develops a two-stage auction mechanism, in which a first-round auction is used to allocate entry rights into the second-round auction; he shows that the entry rights auction has a unique, fully separating equilibrium only if the bids are binding. Hendricks, Pinkse, and Porter (2003) and Roberts and Sweeting (2011) estimate models of unrestricted, costly entry and private information based on bidding in oil and gas auctions and timber auctions, respectively.

The rest of the paper proceeds as follows. Section 2 introduces the model, and presents the result on existence and uniqueness of a symmetric equilibrium. We also give a numerical example to illustrate the structure of the equilibrium and some of its features. Section 3 gives comparative statics, examining the effect of changes in the environment on outcomes under the indicative bidding mechanism. Section 4 compares indicative bidding to standard auctions with unrestricted entry. We also analyze modifications of the indicative bidding mechanism such as reserve prices and bidder subsidies. Section 5 discusses the key assumptions of our model. Section 6 concludes.

2 Model

2.1 Environment

We use the "private value updating" version of Ye's (2007) model. There are $N \geq 3$ bidders. Bidders have private values, which consist of one component known before entry and a separate component learned only after the decision to participate is made. Specifically, bidder i has private value

$$s_i + t_i$$

for the object; bidders know s_i initially, but incur a cost c to learn t_i and participate in a sales mechanism. We think of c as the cost of due diligence – a bidder cannot participate in an auction to buy the object without incurring this cost.

We maintain the following assumptions throughout:

[2] See Klemperer (1999, 2004), Krishna (2002), Milgrom (2004), and Menzes and Monteiro (2005) for syntheses of the auction literature.

Assumption 1. *1.* $\{s_i, t_i\}$ *are independent,* s_i *are i.i.d.* $\sim U[0,1]$*, and* t_i *are i.i.d.* $\sim F_t$*, where* F_t *is an arbitrary distribution with (possibly degenerate) support in* $\Re^+$

2. $c < 1 + E\{t_i\}$

3. $c > E \max\{0, t_i - t_j\}$

The assumption that $\{s_i\}$ are independent of $\{t_i\}$ is of course made for tractability. It is perhaps most plausible in a natural resource setting, where s_i could be related to bidder i's costs of extracting the resource and t_i related to the quality or quantity of the resource itself. (In fact, as we note in the next remark, independence of the $\{t_i\}$ is not essential, although independent between s_i and t_i is.) The assumption that $c < 1+E\{t_i\}$ is simply the assumption that the game is non-trivial: that with positive probability, some bidder might find investigating the asset worthwhile. The last assumption, however, is much more material: it requires costs to be high enough that a bidder only wants to participate if he is either the only bidder, or if he has an initial advantage over his opponent.

Remark. One variation which we are also able to cover is if t_i are the same for all bidders. In that case, the assumption $c > E \max\{0, t_i - t_j\}$ is trivially satisfied. This corresponds to a setting where t_i is the objective quality of the asset, and is measured the same by everyone who examines it.

2.2 Mechanism

The game is played as follows. Some set of messages is available in the first round. Potential bidders learn their initial signals s_i, and choose which message to send. One message is "opt out", or decline to participate; the others are "opt-in" messages, and are fully-ordered. The two bidders who sent the highest opt-in messages advance to the second round, with ties broken randomly. Once these two bidders have been selected, they each pay a cost c, learn their second signal t_i, and participate in a standard ascending auction. If all but one bidder opts out, the one participating bidder still incurs the cost c, but wins the subsequent auction at price 0.[3] If all bidders opt out, the game ends with no sale.

Assumption 2. *The set of messages permitted in the first round is* $\{1, 2, \ldots, M\}$ *for some* $M \leq \infty$*, with message* 1 *being "opt out."*

As we discuss later, symmetric equilibrium will fail to exist when the set of allowed messages is continuous, or when there is not a lowest opt-in message. Once we assume this is not the case, it is without loss of generality to assume the set of messages is a subset of the positive integers.

[3] We consider the effect of a reserve price in a later section.

Since s_i is what each bidder knows at the time participation is determined, we refer to s_i as bidder i's type. We assume that the selected bidders play the dominant-strategy equilibrium of the English auction in the second stage. Thus, we can define two functions

$$\begin{aligned} f(h) &= -c + E_{t_i,t_j} \max\{0, h + t_i - t_j\} \\ g(h) &= -c + E_{t_i}\{h + t_i\} \end{aligned}$$

such that the interim expected payoff to a bidder with type s_i who advances to the second round against an opponent with type s_j is $f(s_i - s_j)$, and the interim expected payoff from advancing alone is $g(s_i)$. Note that Assumption 1 was that $f(0) < 0 < g(1)$. Note also that f and g are both continuous, g is strictly increasing, and f is weakly increasing everywhere and strictly increasing at least on $\Re^+$.[4]

2.3 Strategies and Payoffs

Since bidding in the auction itself is in dominant strategies, we focus on characterizing first-round play. A (possibly mixed) strategy is a mapping $\tau_i : [0,1] \to \Delta(\{1, 2, \ldots, M\})$ from types to probability distributions over the available messages, such that $\tau_i^k(s_i)$ is the probability with which bidder i sends message k when he has type s_i. Given a strategy τ_i and a type $h \in [0,1]$, we will let $\operatorname{supp} \tau_i(h) = \{k : \tau_i^k(h) > 0\}$ denote the support of $\tau_i(h)$; and for a set $H \subseteq [0,1]$, we will let $\operatorname{supp} \tau(H)$ denote $\cup_{h \in H} \operatorname{supp} \tau_i(h)$.

Given strategies by the other $N-1$ bidders, bidder i's expected payoff from sending message k is the probability of advancing to the second round if he sends message k, times the expected payoff from advancing. For $k \geq 2$, this can be written as

$$\begin{aligned} v(s_i, k) \quad = \quad & \Pr(\text{all other bidders opt out}) \cdot g(s_i) \\ & + \Pr(\text{all others send messages below } k\text{, but at least one does not opt out}) \\ & \quad \cdot E_{h \sim G_{<k}} f(s_i - h) \\ & + \textstyle\sum_{j=1}^{N-1} \Pr(j \text{ others send message } k\text{, the rest send messages below } k) \\ & \quad \cdot \tfrac{2}{j+1} \cdot E_{h \sim G_k} f(s_i - h) \\ & + \textstyle\sum_{j=1}^{N-1} \Pr(j-1 \text{ others send } k\text{, one sends a msg above } k\text{, the rest send msgs below } k) \\ & \quad \cdot \tfrac{1}{j} \cdot E_{h \sim G_{>k}} f(s_i - h) \end{aligned}$$

where $G_{<k}$, G_k, and $G_{>k}$ are the distribution of the type h of the other advancing bidder, conditional on each scenario and i being chosen to advance. Note that the second expression vanishes in the case $k = 2$; and of course, $v(s_i, 1) = 0$.

[4] $\max\{0, h + t_i - t_j\}$ is strictly increasing in h whenever $h + t_i - t_j > 0$, so f is strictly increasing unless h is below the minimum of the support of $t_i - t_j$. Since t_i and t_j have the same support, f is at least strictly increasing on $\Re^+$.

2.4 Equilibrium

Our solution concept is symmetric, perfect Bayesian equilibrium.[5] We will say that symmetric equilibrium is *effectively unique* if all symmetric equilibria differ only in the actions of a measure-zero set of types, and therefore all symmetric equilibria lead to the same outcome with probability 1.

We will show that for a given set of primitives – N, c, F_t, and the number of messages available M – a symmetric equilibrium exists, and is effectively unique. First, we show that a simple set of conditions – finiteness of the set of messages used, monotonicity of the strategy, indifference of the "marginal" types, and a "no-incentive-to-separate"-type condition on high-type bidders if there are unused messages available – are both necessary and sufficient for symmetric equilibrium.

Lemma 1. *Let $\{1, 2, \dots, M\}$ be the set of available messages for some $2 \leq M \leq \infty$, and let $\tau : [0, 1] \to \Delta(\{1, \dots, M\})$ be a first-round strategy. As above, let $v(h, k)$ denote the expected payoff to a bidder with type h sending message k, given all other bidders play strategy τ. Everyone playing τ constitutes a symmetric equilibrium if and only if there is some finite $m \leq M$ and some set of numbers*

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < \alpha_m = 1$$

such that:

1. $\operatorname{supp} \tau([0, \alpha_1)) = \{1\}$, *and* $\operatorname{supp} \tau((\alpha_{k-1}, \alpha_k)) = \{k\}$ *for* $k \in \{2, \dots, m\}$
2. $\operatorname{supp} \tau(\alpha_k) \subseteq \{k, k + 1\}$ *for* $k \in \{1, 2, \dots, m - 1\}$
3. $\operatorname{supp} \tau(1) \subseteq \{m, m + 1, m + 2, \dots, M\}$
4. $v(\alpha_k, k + 1) = v(\alpha_k, k)$ *for* $k \in \{1, 2, \dots, m - 1\}$
5. *either* $m = M$ *or* $v(1, m+1) \leq v(1, m)$, *and* $\operatorname{supp} \tau(1) = \{m\}$ *unless* $m < M$ *and* $v(1, m+1) = v(1, m)$

Proof of Sufficiency. We need to show that if bidder i's opponents are all playing τ, then τ is a best-response for bidder i.

First, note that since f and g are both continuous, $v(\,\cdot\,, k)$ is continuous for each k. Next, we show that among messages which might give different outcomes, v has strict single-crossing differences in message and type: if two messages k and $k' > k$ give a bidder different probabilities

[5] Like many entry games, this game may also have many asymmetric equilibria. For example, it is easy to specify primitives such that regardless of his initial signal s_i, any bidder finds it profitable to enter alone but unprofitable to enter against any opponent; in that case, for any j, the strategies "bidder j enters regardless of s_j; bidders $i \neq j$ never enter" is an equilibrium.

of being selected and $v(s_i, k') - v(s_i, k) = 0$, then $v(s_i', k') - v(s_i', k)$ is strictly positive for $s_i' > s_i$, and strictly negative for $s_i' < s_i$.

To see this, first consider the case $k = 1$, so $v(s_i, k') - v(s_1, 1) = v(s_1, k')$. Since g is strictly increasing, and f is weakly increasing everywhere and strictly increasing on $\Re^+$, $v(\cdot, k')$ is weakly increasing, and strictly increasing *unless* all four of the following hold: (i) Pr(all other bidders opt out) $= 0$; (ii) $k' = 2$, or Pr(all others send messages below k', but at least one does not opt out) $=$ 0, or $G_{<k'}$ has support $\subseteq [s_i, 1]$; (iii) Pr(j others send message k', the rest send messages below k') $= 0$ or $G_{k'}$ has support $\subseteq [s_i, 1]$; (iv) Pr(j others send k', one sends a message above k', the rest send messages below k') $= 0$ or $G_{>k'}$ has support $\subseteq [s_i, 1]$. But if all four of these hold, then either message k' gives zero probability of advancing, or whenever bidder i advances by sending message k', he is against an opponent with type $s_j > s_i$, in which case $v(s_i, k') < 0$.

Next, consider $k \geq 2$. If we let $\mathbf{m}$ denote the vector of messages sent by bidder i's opponents, we can rewrite the last three lines of $v(s_i, k)$ as

$$E_{\mathbf{m}} \left\{ \Pr(i \text{ is selected} | k, \mathbf{m}) E_{s_j | i \text{ selected, } k, \mathbf{m}} f(s_i - s_j) \right\}$$

(The first term of $v(s_i, k)$ is the same for all $k > 1$, so it drops out in the difference $v(s_i, k') - v(s_i, k)$.) For each $\mathbf{m}$, sending a higher message weakly increases the probability of being selected, without changing the distribution of i's opponent conditional on being selected, since it is the distribution over the types of all competitors sending the highest message among $\mathbf{m}$. Thus, if k' and k give different probabilities of advancing, $v(s_i, k') - v(s_i, k)$ can be written as the difference in probability, times an expected value of $f(s_i - s_j)$ over the appropriate distribution of s_j. If that distribution has support $\subseteq [s_i, 1]$, then f is always negative and $v(s_i, k') - v(s_i, k) < 0$; otherwise, f is strictly increasing over some of the support of that distribution, and therefore the difference $v(s_i, k') - v(s_i, k)$ is strictly increasing in s_i.

Strict single-crossing differences and $v(\alpha_k, k+1) - v(\alpha_k, k) = 0$ (condition 4) means that bidder i weakly prefers message $k+1$ to k if and only if $s_i \geq \alpha_k$, and weakly prefers k to $k+1$ if and only if $s_i \leq \alpha_k$. This, along with transitivity (and $\alpha_k > \alpha_{k-1} > \cdots$), implies that if $s_i \geq \alpha_k$, bidder i weakly prefers message $k+1$ to all lower messages, and if $s_i \leq \alpha_k$, weakly prefers k to all higher messages up to m. This means that, by conditions 1-2, τ always selects a best-response among $\{1, 2, \ldots, m\}$ for bidders with types $s_i < 1$. If $M = m$, there are no other options; if $M > m$, then $v(1, m+1) - v(1, m) \leq 0$ (condition 5) and therefore $v(s_i, m+1) - v(s_i, m) \leq 0$ (single-crossing differences), and so no bidder gains by deviating to a message above m; so τ selects a best-response for all bidders with $s_i < 1$. Finally, for $s_i = 1$, $1 > \alpha_{m-1}$ implies a bidder with type 1 prefers message m to any lower message. If $v(1, m+1) = v(1, m)$, he is indifferent among all messages m and higher, and if $v(1, m+1) < v(1, m)$, the last condition says τ selects only message m. Thus, τ is always a best-response to τ, and therefore a symmetric equilibrium.

Outline of Proof of Necessity. The proof is given in the appendix. We show that in any symmetric equilibrium τ,

1. τ must be monotonic: if $k > k'$ and $k' \in \operatorname{supp}\tau(s_i)$, then $k \notin \operatorname{supp}\tau((s_i, 1])$
2. $\operatorname{supp}\tau([0,1))$ must be finite
3. no messages can be "skipped": if $k, k'' \in \operatorname{supp}\tau([0,1))$ and $k < k' < k''$, then $k' \in \operatorname{supp}\tau([0,1))$
4. $1 \in \operatorname{supp}\tau([0,1))$, and therefore $\operatorname{supp}\tau([0,1)) = \{1, 2, \ldots, m\}$ for some finite m

From there, we define α_k as the supremum of the set $\{h : k \in \operatorname{supp}\tau(h)\}$. The first three conditions of Lemma 1 then follow from monotonicity; the indifference conditions follow from continuity of $v(\,\cdot\,, k)$. Also by continuity, if $v(1, m+1) > v(1, m)$, then $v(s_i, m+1) > v(s_i, m)$ for a positive measure of s_i close to 1, so the last condition follows from $\operatorname{supp}\tau((\alpha_{m-1}, 1) = \{m\}$. □

This characterization of equilibrium leads to the following result:

Theorem 1. *Fix the primitives of the game – N, c, F_t, and the number of messages available M. A symmetric equilibrium exists, and is effectively unique. Further, given N, c, and F_t, there is a finite number $M^* \geq 2$ such that...*

1. *if $M < M^*$, all available messages are used in equilibrium*
2. *if $M \geq M^*$, only the first M^* messages are used with positive probability (and the equilibrium does not depend on M)*

Proof of Theorem 1. Given $\alpha = (\alpha_1, \ldots, \alpha_{m-1})$, the first condition of Lemma 1 pins down the strategy τ uniquely except on the set $\{\alpha_1, \ldots, \alpha_{m-1}, 1\}$, which has measure zero. This means that expected payoffs – and therefore whether the conditions $v(\alpha_k, k+1) = v(\alpha_k, k)$ and $v(1, m+1) \leq v(1, m)$ are satisfied – depend only on the thresholds α, not on other details of τ. This allows us to transform the question of equilibrium existence into a simple search for a number m and set of thresholds α satisfying these conditions.

Rewriting the expected payoff expression $v(s_i, 2)$ in terms of the thresholds α gives

$$\begin{aligned} v(s_1, 2) &= \alpha_1^{N-1} g(s_1) \\ &\quad + \textstyle\sum_{j=1}^{N-1} \binom{N-1}{j} (\alpha_2 - \alpha_1)^j \alpha_1^{N-1-j} \frac{2}{j+1} E(f(s_1 - h) | h \in (\alpha_1, \alpha_2)) \\ &\quad + \textstyle\sum_{j=1}^{N-1} (N-1)(1-\alpha_2) \binom{N-2}{j-1} (\alpha_2 - \alpha_1)^{j-1} \alpha_1^{N-1-j} \frac{1}{j} E(f(s_1 - h) | h > \alpha_2) \end{aligned}$$

If we plug in $s_1 = \alpha_1$ to set $v(\alpha_1, 2) = v(\alpha_1, 1) = 0$, the expected values become

$$\begin{aligned} E(f(\alpha_1 - h) | h \in (\alpha_1, \alpha_2)) &= \tfrac{\int_{\alpha_1}^{\alpha_2} f(\alpha_1 - h) dh}{\alpha_2 - \alpha_1} &= \tfrac{\int_{-(\alpha_2-\alpha_1)}^{0} f(h') dh'}{\alpha_2 - \alpha_1} \\ E(f(\alpha_1 - h) | h > \alpha_2) &= \tfrac{\int_{\alpha_2}^{1} f(\alpha_1 - h) dh}{1 - \alpha_2} &= \tfrac{\int_{-(1-\alpha_1)}^{-(\alpha_2-\alpha_1)} f(h') dh'}{1 - \alpha_2} \end{aligned}$$

and $v(\alpha_1, 2)$ depends only on α_1 and α_2. For reasons that will become apparent later, it will be convenient to write it as a function of $\alpha_2 - \alpha_1$ and $1 - \alpha_2$, so we will rewrite the first indifference condition accordingly as

$$u(\alpha_2 - \alpha_1, 1 - \alpha_2) \quad = \quad v(\alpha_1, 2) - v(\alpha_1, 1) \quad = \quad 0$$

Simplifying the combinatorial terms and evaluating the sums, this function u turns out to be

$$u(y, z) \quad = \quad d_1 g(1 - y - z) \quad + \quad d_2 \frac{\int_{-y}^{0} f(h)dh}{y} \quad + \quad d_3 \frac{\int_{-y-z}^{-y} f(h)dh}{z}$$

where

$$d_1 = (1 - y - z)^{N-1}, \quad d_2 = \frac{2}{N} \frac{(1-z)^N - (1-y-z)^N}{y} - 2(1 - y - z)^{N-1}, \quad d_3 = z \frac{(1-z)^{N-1} - (1-y-z)^{N-1}}{y}$$

Thought it is not immediately obvious, the three coefficients d_1, d_2 and d_3 are all non-negative (and have well-behaved limits as $y \to 0$). Later, we will use the fact that when $y+z = 1$, d_1 vanishes, and since the two integrals are negative and d_2 and d_3 can't both vanish, $u(y, z) < 0$ when $y + z = 1$.

For $k \geq 2$, the indifference condition is $v(\alpha_k, k + 1) - v(\alpha_k, k) = 0$. Rather than writing out $v(s_i, k + 1)$ and $v(s_i, k)$ and subtracting, it is simpler to directly derive their difference, based on considering all the scenarios of opponent types under which message $k + 1$ gives a strictly higher probability of being selected than message k. We can group these into four types of scenarios:

- those where $j \geq 2$ of i's opponents send message k and the rest send lower messages, so that sending message $k + 1$ increases the likelihood of being selected from $\frac{2}{j+1}$ to 1
- those where one of i's opponents sends a message above k, $j - 1 \geq 1$ send message k, and the rest send lower messages, in which case message $k + 1$ increases the likelihood from $\frac{1}{j}$ to 1
- those where $j \geq 2$ opponents send message $k + 1$ and the rest send lower messages, so that message $k + 1$ increases the likelihood from 0 to $\frac{2}{j+1}$
- those where one opponent sends a message higher than $k + 1$, $j - 1 \geq 1$ send message $k + 1$, and the rest send lower messages, so message $k+1$ increases the likelihood of advancing from 0 to $\frac{1}{j}$

For each scenario, we multiply the probability of the scenario, times the increase in probability from sending message $k + 1$ rather than k, times the expected payoff from advancing in that scenario, and calculate

$$
\begin{aligned}
v(s_i, k+1) - v(s_i, k) \;&=\; \textstyle\sum_{j=2}^{N-1} \binom{N-1}{j} (\alpha_k - \alpha_{k-1})^j \alpha_{k-1}^{N-1-j} (1 - \frac{2}{j+1}) E(f(s_i - h) | h \in (\alpha_{k-1}, \alpha_k)) \\
&\quad + \textstyle\sum_{j=2}^{N-1} (N-1)(1-\alpha_k) \binom{N-2}{j-1} (\alpha_k - \alpha_{k-1})^{j-1} \alpha_{k-1}^{N-1-j} (1 - \frac{1}{j}) E(f(s_i - h) | h > \alpha_k) \\
&\quad + \textstyle\sum_{j=2}^{N-1} \binom{N-1}{j} (\alpha_{k+1} - \alpha_k)^j \alpha_k^{N-1-j} \frac{2}{j+1} E(f(s_i - h) | h \in (\alpha_k, \alpha_{k+1})) \\
&\quad + \textstyle\sum_{j=2}^{N-1} (N-1)(1-\alpha_{k+1}) \binom{N-2}{j-1} (\alpha_{k+1} - \alpha_k)^{j-1} \alpha_k^{N-1-j} \frac{1}{j} E(f(s_i - h) | h > \alpha_{k+1})
\end{aligned}
$$

This time, plugging in $s_i = \alpha_k$ gives $v(\alpha_k, k+1) - v(\alpha_k, k)$ a function only of $\alpha_k - \alpha_{k-1}$, $\alpha_{k+1} - \alpha_k$, and $1 - \alpha_{k+1}$, which we will write as

$$
\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) \quad = \quad v(\alpha_k, k+1) - v(\alpha_k, k) \quad = \quad 0
$$

The function Δ turns out to be

$$
\Delta(x, y, z) \quad = \quad c_1 \frac{\int_0^x f(h) dh}{x} \quad + \quad c_2 \frac{\int_{-y}^0 f(h) dh}{y} \quad + \quad c_3 \frac{\int_{-y-z}^{-y} f(h) dh}{z}
$$

where

$$
\begin{aligned}
c_1 \quad &= \quad (1-y-z)^{N-1} + (1-x-y-z)^{N-1} - \tfrac{2}{N} \tfrac{(1-y-z)^N - (1-x-y-z)^N}{x} \\
c_2 \quad &= \quad \tfrac{2}{N} \tfrac{(1-z)^N - (1-y-z)^N}{y} - 2(1-y-z)^{N-1} - y \tfrac{(1-y-z)^{N-1} - (1-x-y-z)^{N-1}}{x} \\
c_3 \quad &= \quad z \tfrac{(1-z)^{N-1} - (1-y-z)^{N-1}}{y} - z \tfrac{(1-y-z)^{N-1} - (1-x-y-z)^{N-1}}{x}
\end{aligned}
$$

Again, all three coefficients c_1, c_2, and c_3 are nonnegative. The last two integrals are negative, and $\int_0^x f(h) dh$ is negative when x is small. In particular, $\int_0^x f(h) dh = 0$ has a unique solution on $x > 0$, which we will call $\underline{x}$. So except at the point $x = y = z = 0$, where all three coefficients vanish, $\Delta(x, y, z)$ is strictly negative when $x < \underline{x}$.

Finally, for $m < M$, consider the condition $v(1, m+1) - v(1, m) \leq 0$. Sending message $m+1$ advances for sure; sending message m advances for sure unless two or more opponents also send message m. So sending message $m+1$ instead of m increases the probability of advancing only in scenarios where, upon advancing, bidder i will face an opponent who sent message m, meaning his opponent will have type $h \in (\alpha_{m-1}, 1)$. This means that $v(1, m+1) \leq v(1, m)$ if and only if

$$
0 \quad \geq \quad E(f(1-h) | h \geq \alpha_{m-1}) \quad = \quad \frac{1}{1-\alpha_{m-1}} \int_{\alpha_{m-1}}^1 f(1-h) dh \quad = \quad \frac{1}{1-\alpha_{m-1}} \int_0^{1-\alpha_{m-1}} f(h') dh'
$$

This means that the problem of equilibrium existence reduces to the following: if we can find some m and α such that $u(\alpha_2 - \alpha_1, 1 - \alpha_2) = 0$ and $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) = 0$ for every $k \in \{2, 3, \ldots, m-1\}$, then

- if $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$, then any τ satisfying the support conditions of Lemma 1 is a symmetric equilibrium as long as $M \geq m$ messages are available.
- if $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$, then any τ satisfying the support conditions of Lemma 1 is a symmetric equilibrium if and only if exactly $M = m$ messages are available.

And of course, any equilibrium must correspond to an m and α satisfying the indifference conditions.

In fact, we will show that given the primitives of the environment N, c, and F_t, there is a finite number $M^* \geq 2$ such that:

1. for any $m < M^*$ ($m \geq 2$), a unique α exists satisfying the indifference conditions, but $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$
2. for $m = M^*$, a unique α exists satisfying the indifference conditions, and $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$
3. for $m > M^*$, no α exists satisfying the indifference conditions

Since, given m and α, any two τ satisfying the conditions of Lemma 1 differ only in the actions of the threshold types $\{\alpha_1, \ldots, \alpha_{m-1}, 1\}$, effective uniqueness follows from uniqueness of α.

The full construction – finding M^*, finding α given $m \leq M^*$, showing it is unique, showing $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$ if and only if $m = M^*$, and showing no such α exists for $m > M^*$ – is given in the appendix. Here, we give a brief outline to give intuition. The key to the construction is that for $k \geq 2$, when we write the difference $v(\alpha_k, k+1) - v(\alpha_k, k)$ as $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1})$, the resulting function Δ is continuous in its three arguments, and away from $(0, 0, 0)$, is strictly single-crossing in all three arguments – from below in the first, and from above in the other two.[6] This means that given y and z, there is at most a unique $x > 0$ such that $\Delta(x, y, z) = 0$; and if we let $x^*(y, z)$ denote this value (when it exists), x^* is increasing in y and z. This therefore means that given α_k and α_{k+1}, there is a unique value of α_{k-1} satisfying the indifference condition at α_k; and that as the differences $1 - \alpha_{k+1}$ and $\alpha_{k+1} - \alpha_k$ grow, so does the difference $\alpha_k - \alpha_{k-1}$. Similarly, when $v(\alpha_1, 2) - v(\alpha_1, 1)$ is written as $u(\alpha_2 - \alpha_1, 1 - \alpha_2)$, the function u is continuous and strictly single-crossing from above in both its arguments.

So now fix m, and construct α as follows to satisfy the indifference conditions. For $t \geq 0$, define $\alpha_m(t) = 1$ and $\alpha_{m-1}(t) = 1 - t$. Let $\alpha_{m-2}(t)$ be the unique value satisfying the indifference condition at $\alpha_{m-1}(t)$, that is, the unique solution to $0 = \Delta(\alpha_{m-1}(t) - \alpha_{m-2}(t), \alpha_m(t) - \alpha_{m-1}(t), 1 - \alpha_m(t))$. Construct $\alpha_{m-3}(t)$ similarly from $\alpha_{m-2}(t)$ and $\alpha_{m-1}(t)$, and so on down the line until $\alpha_1(t)$. Let $\bar{t}$ be the maximal value of t at which each step of the construction is well-defined and $\alpha_1(t) \geq 0$. Calculate $u(\alpha_2(0) - \alpha_1(0), 1 - \alpha_2(0))$ and suppose it is strictly positive.

[6]That is, if $(x, y, z) \neq 0$ and $\Delta(x, y, z) = 0$, then for $x' \neq x$, $\Delta(x', y, z) - \Delta(x, y, z)$ is strictly positive for $x' > x$, strictly negative for $x' < x$; and likewise if $y' \neq y$ and $z' \neq z$, $\Delta(x, y', z) - \Delta(x, y, z)$ has the opposite sign as $y' - y$ and $\Delta(x, y, z') - \Delta(x, y, z)$ the opposite sign as $z' - z$. Δ is also differentiable in all its arguments, although this is mostly useful in showing the single-crossing properties.

By construction (using the single-crossing properties of Δ), as t increases, each difference $\alpha_k(t) - \alpha_{k-1}(t)$ increases, so $\alpha_2(t) - \alpha_1(t)$ and $1 - \alpha_2(t)$ both increase. It is easy to show that for t close to $\bar{t}$, i.e., $\alpha_1(t)$ close to 0, $u(\alpha_2(t) - \alpha_1(t), 1 - \alpha_2(t)) < 0$. Since u is continuous and single-crossing from above in both its arguments, there is a unique t^* at which $u(\alpha_2(t^*) - \alpha_1(t^*), 1 - \alpha_2(t^*)) = 0$; let $\alpha = (\alpha_1(t^*), \ldots, \alpha_{m-1}(t^*))$, and it satisfies all the indifference conditions.

On the other hand, if the initial construction failed – if $\alpha_j(0)$ could not be found to satisfy the indifference condition at $\alpha_{j+1}(0)$ for some j, or if $u(\alpha_2(0) - \alpha_1(0), 1 - \alpha_2(0))$ was not positive – then m was too high; define M^* as the maximal value of m for which the initial construction works. We show that the construction works for every $m \leq M^*$ and fails for every $m > M^*$; and that for each $m \leq M^*$, the resulting α is the only one that satisfies all the indifference conditions. The second part of Assumption 1 – that $g(1) > 0$ – ensures that the construction works for $m = 2$, and so $M^* \geq 2$.

For a given M, this means there is a unique m and α such that the indifference conditions are satisfied and either $m = M$ or $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$. By the sufficiency part of Lemma 1, any τ satisfying the support conditions is an equilibrium, so an equilibrium exists. And all τ satisfying the support conditions differ only in the actions of a measure-zero set of types $\{\alpha_1, \ldots, \alpha_{m-1}, 1\}$, which along with uniqueness of α gives (by the necessity part of Lemma 1) effective uniqueness of equilibrium. $\square$

2.5 A Numerical Example

Next, we show a numerical example to illustrate the construction and the properties of the resulting equilibria. Let $N = 3$, $c = 0.03$, and let F_t be the degenerate distribution at 0, so $t_i = t_j = 0$ and $f(h) = -c + \max\{0, h\}$.

First, we try the construction with $m = 6$. To begin, we let $\alpha_5 = \alpha_6 = 1$. We then calculate $\alpha_4 = 0.94$, because $\Delta(0.06, 0, 0) = 0$. Then $\alpha_3 = 0.7893$, because $\Delta(0.1507, 0.06, 0) = 0$. Then $\alpha_2 = 0.5243$, and $\alpha_1 = 0.1291$. Next, we calculate $u(\alpha_2 - \alpha_1, 1 - \alpha_2)$, and find it to be $-0.0139 < 0$, so the construction fails and we know $M^* < 6$.

Next, we try it for $m = 5$. This time, $\alpha_4 = \alpha_5 = 1$, $\alpha_3 = 0.94$, $\alpha_2 = 0.7893$, and $\alpha_1 = 0.5243$. This time, $u(\alpha_2 - \alpha_1, 1 - \alpha_2) = 0.1178 > 0$, so the construction succeeds, so $M^* = 5$.

Next, we let $\alpha_4(t) = 1 - t$, with $\alpha_3(t)$, $\alpha_2(t)$, and $\alpha_1(t)$ adjusting to satisfy the indifference conditions. Figure 1 shows how the thresholds move, with t on the x-axis. The red curve is $u(\alpha_2(t) - \alpha_1(t), 1 - \alpha_2(t))$; note that it starts positive, ends negative, and crosses zero once: at $\alpha_4 = 0.9681$, $\alpha_3 = 0.8505$, $\alpha_2 = 0.6251$, and $\alpha_1 = 0.2742$. This gives us our α satisfying all the indifference conditions.

Figure 2 shows the same exercise for the same example, but with $m = 4$, 3, and 2.

This gives us four equilibria to consider: one when only 2 messages are allowed, one when 3

Figure 1: Construction of α when $m = 5$

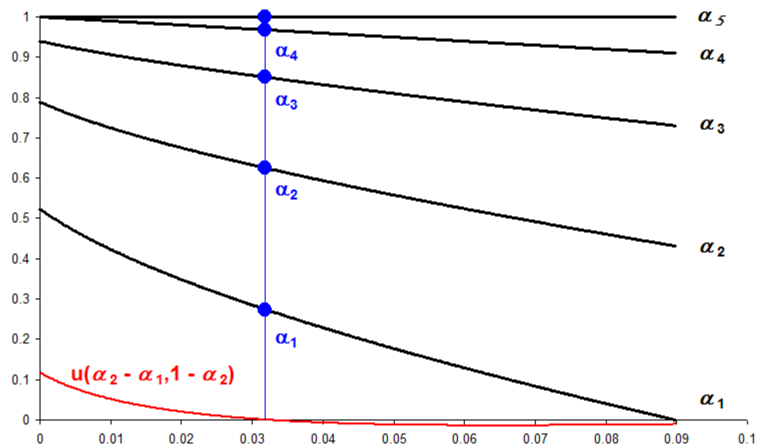
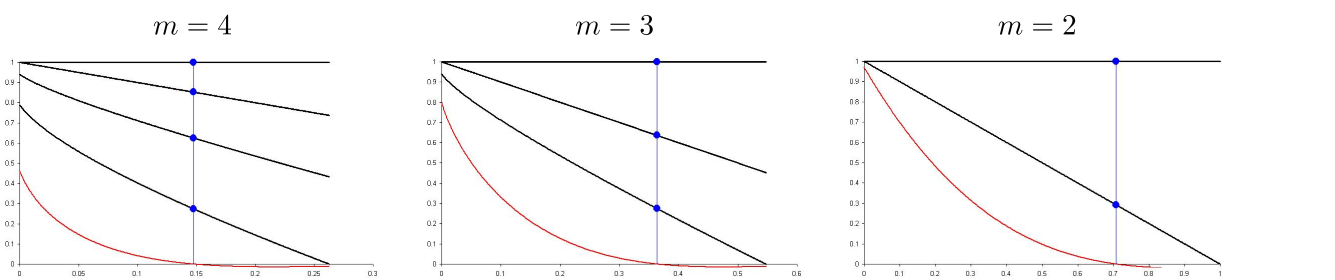


Figure 2: Construction of α when $m = 4$, 3, 2



are, one when 4 are, and one when 5 or more messages are available. Figure 3 shows how the set of types $[0, 1]$ gets divided up into messages, depending on the number of messages available M.

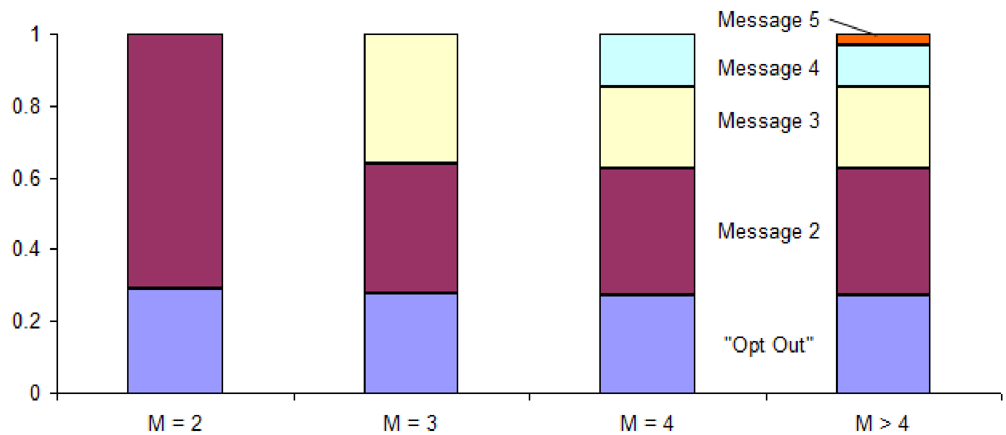
For each of these games, we can calculate expected revenue, bidder surplus, and total surplus:

Messages Available	2	3	4	> 4
Revenue	41.92	45.26	45.52	45.53
Bidder Surplus	24.08	22.49	22.37	22.37
Total Surplus	65.99	67.76	67.89	67.89

There are several things worth noting about this example:

Revenue is increasing in the number of messages available. This is consistent across all numerical examples we have done, and we believe it is likely true generally, although we have not been able to prove it. The intuition is clear: when M is higher, α_1 is lower, so the seller is less

Figure 3: Equilibrium partition of $[0, 1]$ for various M



likely to get zero revenue due to fewer than two bidders opting in; and since the intervals $[\alpha_{k-1}, \alpha_k]$ are compressed with more messages, the bidders sort more effectively, and the selected bidders are more likely to be those with the two highest signals, increasing revenue.

It therefore appears that the seller does not benefit from restricting the set of messages more than necessary.

Total surplus is also increasing in the number of messages available. This too is consistent across all our numerical examples, although the intuition is less unambiguous: a lower participation threshold α_1 means more participation costs incurred, which offsets some of the efficiency gains from more reliably selecting the highest-type bidders.

Bidder surplus, however, is decreasing in the number of messages available. Bidders benefit from a higher opt-in threshold, since they benefit from being the only entrant in an auction; and they benefit from wider intervals, since they benefit from facing the "wrong" opponent (one with a lower signal) when they advance. This seems to outweigh the cost of a bidder with a high signals being less sure of advancing due to wider intervals. This too is consistent across all our numerical examples.

When $m = M^*$, there is finer separation at the top of the type space. Figure 3 shows that when $M \geq M^* = 5$, there is finer separation among high types than low types: the intervals over which bidder types pool are narrower at the top and get wider as types get lower,

$$1 - \alpha_4 \quad < \quad \alpha_4 - \alpha_3 \quad < \quad \alpha_3 - \alpha_2 \quad < \quad \alpha_2 - \alpha_1$$

In fact, this can be proven to always hold when $m = M^*$, that is, in the "most revealing" equilibrium (the equilibrium of the game when at least M^* messages are available). However, it is not always true when $m = M < M^*$, as the $m = 3$ case of the example illustrates.[7]

3 Comparative Statics

3.1 How M^* Changes With Primitives

In a loose sense, we can interpret M^* as the maximal amount that the seller is able to discriminate among bidders in equilibrium. That is, a high value of M^* indicates bidders pooling into lots of small intervals sending each message, so that the bidders selected are likely to be those with the highest types; a low value of M^* indicates less separation among bidder types. It is therefore natural to think about how the primiitives of the game affect M^*.

It turns out, changes that favor the bidders, tend to increase the amount of separation that can be achieved in equilibrium. Fewer bidders, lower participation costs, or more information to be learned during due diligence (leading to higher expected information rents in the second-stage auction) all lead to increases in M^*.[8]

Theorem 2. *1. M^* is decreasing in N.*

2. As long as Assumption 1 continues to hold, M^* is decreasing in c, and increases when the variability of F_t is increased via a mean-preserving spread.

The proof, in the appendix, is fairly straightforward. We show the function Δ is single-crossing in N from above: if $\Delta^{(N=n)}(x, y, z) = 0$, then $\Delta^{(N=n-1)}(x, y, z) > 0$. This means that each "step" of the construction of α involves a narrower interval, and more steps can therefore be taken before the lower boundary is reached. Similarly, a decrease in c, or a mean-preserving spread applied to F_t, increase f, therefore increasing Δ, and therefore likewise allowing smaller "steps" in the construction.

3.2 Characterizing Equilibrium When N is Large

It is not too hard to look at what happens to our game as the number of potential bidders gets large.

Theorem 3. *When N is large, $M^* = 2$ and equilibrium entry is below the efficient level.*

[7] The proof is straightforward, by induction (using monotonicity of x^*), when $1 - \alpha_{m-1} < \alpha_{m-1} - \alpha_{m-2}$ holds as the base case. When $m = M^*$, $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$ so $1 - \alpha_{m-1} < \underline{x}$, while $\alpha_{m-1} - \alpha_{m-2} = x^*(1 - \alpha_{m-1}, 0) > x^*(0, 0) = \underline{x}$; but when $m < M^*$, $1 - \alpha_{m-1} > \underline{x}$ and the base case may not hold.

[8] Recall that this is only for the case $f(0) < 0$, where interim payoffs to an advancing bidder are only positive when he is ahead. In the opposite case $f(0) > 0$, we expect these results to be reversed: our intuition is that in general, more messages can be used when $f(0)$ is close to 0.

Proof. When $M^* > 2$, an equilibrium with three messages exists when $M = 3$; we will show this is impossible for N sufficiently large. If three messages are used, then $\alpha_2 - \alpha_1 = x^*(1-\alpha_2, 0) \geq \underline{x}$, where $\underline{x}$ was defined earlier as the positive value where $\int_0^{\underline{x}} f(h)dh = 0$. This of course means that $\alpha_1 \leq 1 - \underline{x}$. With $m = 3$, the indifference condition for α_1 is

$$\begin{aligned} 0 &= \alpha_1^{N-1} g(\alpha_1) + \left(\tfrac{2}{N} \tfrac{\alpha_2^N - \alpha_1^N}{\alpha_2 - \alpha_1} - 2\alpha_1^{N-1}\right) \tfrac{\int_{-(1-\alpha_1)}^0 f(h)dh}{1-\alpha_1} + d_3 \tfrac{\int_{-(1-\alpha_1)}^{-(\alpha_2-\alpha_1)} f(h)dh}{1-\alpha_2} \\ &< \alpha_1^{N-1} g(1) + \left(\tfrac{2}{N} \tfrac{\alpha_2^N - \alpha_1^N}{\alpha_2 - \alpha_1} - 2\alpha_1^{N-1}\right) f(0) \end{aligned}$$

since the third term in the first line is negative, $g(\alpha_1) < g(1)$ and $\frac{\int_{-(1-\alpha_1)}^0 f(h)dh}{1-\alpha_1} \leq f(0)$, and both have positive coefficients. Rearranging, dividing by $\alpha_1^{N-1} f(0)$, and letting $x = \frac{\alpha_2}{\alpha_1}$ gives

$$\frac{g(1)}{-f(0)} + 2 \quad > \quad \frac{2}{N} \frac{x^N - 1}{x - 1} \quad = \quad 2 \frac{\sum_{j=0}^{N-1} x^j}{N} \quad \geq \quad 2x^{\frac{N-1}{2}}$$

Since $x = \frac{\alpha_2}{\alpha_1} = 1 + \frac{\alpha_2 - \alpha_1}{\alpha_1} \geq 1 + \frac{\underline{x}}{\alpha_1} > 1 + \underline{x}$ is bounded away from 1, the right-hand side grows without bound as N increases, so this inequality cannot hold for N too large; so for N large, $M^* = 2$.

To see that entry is below the efficient level, first recall from our construction earlier that $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$ if and only if $m = M^*$. When $M^* = 2$, then, $\int_0^{1-\alpha_1} f(h)dh \leq 0$. In equilibrium, a bidder who opts in, gets selected, and learns he was not the only one to opt in, gets expected payoff $\frac{1}{1-\alpha_1} \int_{\alpha_1}^1 f(s_i - h)dh = \frac{1}{1-\alpha_1} \int_0^{1-\alpha_1} f(1 - s_i + h)dh$, which is nonpositive for $s_i = 1$ and strictly negative for $s_i < 1$; so with probability 1, when multiple bidders opt in, each gets negative expected payoff.

Now, in a normal English auction, entry does not impose a net externality. (If bidder i enters and wins, his surplus – the difference between his own and the second-highest valuation – is exactly his contribution to total surplus. If he raises the price but does not win, this simply transfers surplus from the winner to the seller.) This means that when either zero or one other bidder opts in, bidder i's entry decision in the indicative-bidding game does not impose a net externality. When two or more other bidders opt in, however, entry imposes a *positive* externality. Think of the effect of i's entry in two steps: first, the game expands from an English auction with two bidders to an English auction with three (i, and the two bidders who would have been selected had i opted out); then, one of the three bidders is randomly selected to get sent home. If i is sent home, his entry had no effect. If not, the net externality he imposes is the negative of the payoff expected by the bidder who was sent home because i arrived. But when $M^* = 2$, with probability 1, that bidder was expecting negative payoff, so the net externality is positive.

So now take a bidder with type very close to, but just below, the participation threshold. The payoff he would get from opting in is just barely negative; the externality he would impose is strictly positive; so having him enter would increase total surplus. □

Since M^* is discrete and the inefficient participation result holds whenever $M^* = 2$, this is not only a "limit result" in the usual sense: there is a finite N above which M^* is exactly 2 and entry is equilibrium is guaranteed to be inefficiently low. However, we can also characterize what happens in the limit as $N \to \infty$, which will prove useful in comparing indicative bidding to other possible mechanisms. When $m = M^* = 2$, the same argument as in the first part of the proof (but with $\alpha_2 = 1$) establishes that α_1 cannot be bounded away from 1 as N grows, so $\lim_{N\to\infty} \alpha_1 = 1$. However, we can show that $\lim_{N\to\infty} \alpha_1^N \in (0,1)$. With $m = 2$, the indifference condition at α_1 simplifies to

$$0 \quad = \quad \alpha_1^{N-1} g(\alpha_1) + \left(\frac{2}{N} \frac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1} \right) \frac{\int_{-(1-\alpha_1)}^0 f(h)dh}{1-\alpha_1}$$

Let $\bar{f} = \frac{1}{1-\alpha_1} \int_{-(1-\alpha_1)}^0 f(h)dh$. Now, $\frac{2}{N} \frac{1-\alpha_1^N}{1-\alpha_1} = \frac{2}{N} \sum_{i=0}^{N-1} \alpha_1^i \leq 2$, so the indifference condition implies $0 \geq \alpha_1^{N-1} g(\alpha_1) + (2 - 2\alpha_1^{N-1})\bar{f}$, which is impossible if $\alpha_1^N \to 1$; so $\lim \alpha_1^N < 1$.

Next, we show $\lim \alpha_1^N > 0$. The indifference condition is $\frac{2}{N} \frac{1-\alpha_1^N}{1-\alpha_1}(-\bar{f}) = \alpha_1^{N-1} g(\alpha_1) + 2\alpha_1^{N-1}(-\bar{f})$, which is equivalent to

$$2\alpha_1 \frac{-\bar{f}}{g(\alpha_1) - 2\bar{f}} \quad = \quad N(1-\alpha_1) \frac{\alpha_1^N}{1-\alpha_1^N}$$

As $N \to \infty$, $\alpha_1 \to 1$, $\bar{f} \to f(0)$, and $v(\alpha_1) \to v(1)$, so the left-hand side goes to $\frac{-2f(0)}{g(1)-2f(0)} \in (0,1)$. As for the right side, if $\alpha_1^N \to 0$, $N(1-\alpha_1)\frac{\alpha_1^N}{1-\alpha_1^N} \to 0$ as well,[9] giving a contradiction since the left-hand side does not go to 0, so $\alpha_1^N \not\to 0$. If α_1^N has a limit in $(0,1)$, then $-\ln(\alpha_1^N)\frac{\alpha_1^N}{1-\alpha_1^N}$ has the same limit as $N(1-\alpha_1)\frac{\alpha_1^N}{1-\alpha_1^N}$,[10] and therefore

$$\lim_{N\to\infty} \alpha_1^N \quad = \quad Q^{-1}\left(\frac{-2f(0)}{g(1)-2f(0)} \right)$$

where $Q(h) \equiv \frac{-h\ln(h)}{1-h}$. (Note that Q is strictly increasing, and maps $[0,1]$ to $[0,1]$, so this inverse is well-defined.)

Having a statement for $\lim_{N\to\infty} \alpha_1^N$ will allow us to get statements for revenue and total surplus in the limit, which will be useful for future results. For example, if we let $x = Q^{-1}\left(\frac{-2f(0)}{g(1)-2f(0)} \right) = \lim \alpha_1^N$, then $\lim N(1-\alpha_1) = \lim(-\ln(\alpha_1^N)) = -\ln(x)$, and so $\lim N(1-\alpha_1)\alpha_1^{N-1} = -x\ln(x)$. So in the limit, there is probability x that no bidders opt in, probability $-x\ln(x)$ that one does, and probability $1 - x + x\ln(x)$ that two or more do. In the limit, all bidders opting in have signals $s_i \approx 1$, so we can easily state expected revenue and surplus for each case as well, giving us "almost-closed-form" expressions for limit outcomes. (We will use this in the next section to prove the comparisons to other mechanisms.)

[9] The Taylor expansion of $\ln(\alpha_1)$ around 1 gives $1 - \alpha_1 \leq -\ln(\alpha_1)$, and therefore $N(1-\alpha_1) \leq -\ln(\alpha_1^N)$; so if $\alpha_1^N \to 0$, since $-\ln(\alpha_1^N)\frac{\alpha_1^N}{1-\alpha_1^N} \to 0$ (by L'Hopital's Rule), $N(1-\alpha_1)\frac{\alpha_1^N}{1-\alpha_1^N} \to 0$ as well.

[10] Letting $h = \lim \alpha_1^N$ and applying L'Hopital's Rule to $\lim \frac{1-h^{1/N}}{1/N}$ gives $\lim N(1-\alpha_1) = -\ln(h)$.

4 Other Mechanisms

4.1 Comparison to Auctions Without Indicative Bids

A natural benchmark to compare our game to is what would happen if we did not ration entry, but simply announced an ascending auction and allowed bidders to choose whether or not to participate. We assume the timing is the same: bidders learn s_i; bidders then decide simultaneously whether to particpiate; and those bidders who choose to participate, incur the cost c, learn their second signal t_i, and participate in the auction. This is sort of a hybrid between the entry game of Levin and Smith, where there is no s_i and so bidders have no private information when deciding whether to enter and play a symmetric mixed strategy equilibrium; and the entry game of Samuelson, where there is no t_i, so bidders know their valuations when they decide whether to enter and play a symmetric cutoff equilibrium.

For $j \geq 1$, let

$$v_j(h,\gamma) = -c + E_{s_2,\ldots,s_{j+1} \sim i.i.d.U[\gamma,1]} E_{t_1,t_2,\ldots,t_{j+1}} \max\{0, h + t_1 - \max_{i \neq 1}\{s_i + t_i\}\}$$

and let $v_0(h,\gamma) = g(h)$, so that $v_j(h,\gamma)$ denotes the expected payoff to bidder 1 from participating, given signal $s_1 = h$, and given that j other bidders, with signals drawn independently from the uniform distribution on $[\gamma, 1]$, also participate. It is easy to show that in our setting (with Assumption 1), there is a (effectively) unique symmetric equilibrium, where bidders enter when their signal s_i is above a cutoff γ, and γ satisfies

$$0 \quad = \quad \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j}(1-\gamma)^j v_j(\gamma,\gamma)$$

Like in our game, as $N \to \infty$, $\gamma \to 1$, and γ^N stays away from 0 and 1. We can compare the outcome with messages to the outcome without:

Theorem 4. *When N is sufficiently large, the auction with indicative bids gives strictly more entry, higher revenue, and greater bidder surplus than a standard auction without indicative bids.*

Remark. The rankings in Theorem 4 need not hold for small N. In examples we calculated – with F_t degenerate and $N = 3$ and 4 – we found that the indicative-bidding game gave higher revenue than the pure auction when at least M^* messages were available, but not always otherwise. Bidder surplus, on the other hand, was higher in our game than the pure auction when only 2 messages were available, but often not when more than 2 were used in equilibrium.

Proof of Theorem 4. *Part 1: Greater Entry.* In the game without indicative bids, we can rewrite the indifference condition for entry as

$$\gamma^{N-1} g(\gamma) + (1 - \gamma^{N-1}) m = 0$$

where m is the expected value of $v_j(\gamma, \gamma)$, conditional on $j \geq 1$. $g(\gamma) \leq g(1)$, and $m \leq f(0)$ (because facing multiple opponents is worse than facing one, and the marginal entrant always has weakly lower type than his opponents), so

$$\gamma^{N-1} g(1) + (1 - \gamma^{N-1}) f(0) \quad \geq \quad 0 \quad \longrightarrow \quad \gamma^{N-1} \quad \geq \quad \tfrac{-f(0)}{g(1) - f(0)}$$

so $\lim_{N\to\infty} \gamma^N = \lim_{N\to\infty} \gamma^{N-1} \geq \frac{-f(0)}{g(1)-f(0)}$. We will show that this implies $\lim_{N\to\infty} \gamma^N > \lim_{N\to\infty} \alpha_1^N$, so the probability that no bidder enters is higher in the game without indicative bids; as well as $\lim_{N\to\infty}(\gamma^N + N\gamma^{N-1}(1-\gamma)) > \lim_{N\to\infty}(\alpha_1^N + N\alpha_1^{N-1}(1-\alpha_1))$, so the probability that either zero or one bidder enters is higher.

First, suppose $\lim_{N\to\infty} \gamma^N \leq \lim_{N\to\infty} \alpha_1^N$. Then

$$\tfrac{-f(0)}{g(1)-f(0)} \quad \leq \quad Q^{-1}\left(\tfrac{-2f(0)}{g(1)-2f(0)}\right)$$

Letting $\xi \equiv \frac{g(1)}{-f(0)} > 0$, and recalling that Q is strictly increasing,

$$Q\left(\tfrac{1}{\xi+1}\right) \quad \leq \quad \tfrac{2}{\xi+2}$$

A bit of algebra shows this is equivalent to $\ln(\xi+1) - \frac{2\xi}{\xi+2} \leq 0$, which turns out to be impossible for $\xi > 0$. ($\ln(1) - \frac{0}{2} = 0$, and $\ln(\xi+1) - \frac{2\xi}{\xi+2}$ is strictly increasing in ξ, so it's positive above 0.) This gives a contradiction, proving $\lim \gamma^N > \lim \alpha_1^N$, so all bidders opting out is more likely in the pure auction without indicative bids.

Finally, as noted in the last section, if $\alpha_1^N \to x$, $N(1-\alpha_1)\alpha_1^{N-1} \to -x\ln(x)$, and similarly $N(1-\gamma)\gamma^{N-1} \to -y\ln(y)$ if $y = \lim \gamma^N$. But $-h\ln(h) + h$ is increasing, so $y > x$ implies $-y\ln(y) + y > -x\ln(x) + x$, so the probability that either zero or one bidder enters is higher as well in the auction without indicative bids.

Part 2: Greater Bidder Surplus. We assume here that F_t is continuous (no point masses); a separate proof applies if it is not. We will show that total bidder surplus goes to 0 in both games, at rate $\frac{1}{N}$; but N times total bidder surplus has a higher limit in the auction with indicative bids.

In the game with indicative bids, we know that bidders with types $s_i < \alpha_1$ get zero payoff, and bidders with types $s_i > \alpha_1$ get

$$v(s_i, 2) \quad = \quad \alpha_1^{N-1} g(s_i) + \left(\tfrac{2}{N}\tfrac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1}\right)\tfrac{\int_{-(1-\alpha_1)}^{0} f(s_i - \alpha_1 + h)dh}{1-\alpha_1}$$

Since $v(\alpha_1, 2) = 0$, $v(s_i, 2) = v(s_i, 2) - v(\alpha_1, 2)$, so

$$\begin{aligned} v(s_i, 2) \quad &= \quad \alpha_1^{N-1}(g(s_i) - g(\alpha_1)) + \left(\tfrac{2}{N}\tfrac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1}\right)\tfrac{\int_{-(1-\alpha_1)}^{0}(f(s_i-\alpha_1+h) - f(h))dh}{1-\alpha_1} \\ &= \quad (s_i - \alpha_1)\left[\alpha_1^{N-1}\tfrac{g(s_i)-g(\alpha_1)}{s_i-\alpha_1} + \left(\tfrac{2}{N}\tfrac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1}\right)\tfrac{\int_{-(1-\alpha_1)}^{0}\frac{f(s_i-\alpha_1+h)-f(h)}{s_i-\alpha_1}dh}{1-\alpha_1}\right] \end{aligned}$$

Recall that $g(h) = -c + h + E(t_i)$ is differentiable everywhere, with derivative 1. As for $f(h) = -c + E\max\{0, h + t_i - t_j\}$, note that $\max\{0, h + t_i - t_j\}$ has derivative 1 when $h + t_i - t_j > 0$ and derivative 0 when $h + t_i - t_j < 0$. This means the derivative of $E\max\{0, h + t_i - t_j\}$ is equal to the probability that $h + t_i - t_j$ is positive; for F_t massless, this is continuous and so f is differentiable; for h close to 0, by symmetry, this is equal to $\frac{1}{2}$.

As $N \to \infty$, $\alpha_1 \to 1$, so $s_i - \alpha_1 \to 0$ for all $s_i \in [\alpha_1, 1]$; replacing the difference terms with derivatives, in the limit,

$$\begin{aligned} v(s_i, 2) &= (s_i - \alpha_1) \left[\alpha_1^{N-1} g'(\alpha_1) + \left(\frac{2}{N} \frac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1} \right) \frac{\int_{-(1-\alpha_1)}^{0} f'(h)dh}{1-\alpha_1} \right] \\ &= (s_i - \alpha_1) \left[\alpha_1^{N-1} + \left(\frac{2}{N} \frac{1-\alpha_1^N}{1-\alpha_1} - 2\alpha_1^{N-1} \right) \frac{1}{2} \right] \\ &= (s_i - \alpha_1) \frac{1-\alpha_1^N}{N(1-\alpha_1)} \end{aligned}$$

Now, the ex-ante expected surplus of a single bidder is the expected value over s_i of $\max\{0, v(s_i, 2)\}$, which is

$$\int_0^{\alpha_1} 0 ds_1 + \int_{\alpha_1}^1 v(s_1, 2) ds_1 = \frac{1}{2}(1-\alpha_1)^2 \frac{1-\alpha_1^N}{N(1-\alpha_1)}$$

Multiplying by N, the combined expected surplus of all N bidders is $U = \frac{1}{2}(1-\alpha_1)(1-\alpha_1^N)$.

Next, we do the same calculation for the auction without indicative bids. This time (suppressing the second argument of v_j), a bidder's expected payoff given signal $s_i \geq \gamma$ is

$$\begin{aligned} u_{nm}(s_i) &= \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j} (1-\gamma)^j v_j(s_i) \\ &= \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j} (1-\gamma)^j (v_j(s_i) - v_j(\gamma)) \\ &= (s_i - \gamma) \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j} (1-\gamma)^j \frac{v_j(s_i) - v_j(\gamma)}{s_i - \gamma} \\ &\approx (s_i - \gamma) \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j} (1-\gamma)^j v_j'(\gamma) \end{aligned}$$

in the limit. Now by the same logic as before, if we think of v_j as the expectation over $\{t_i\}$, v_j' is the probability that, given j opponents, bidder i is ahead; when all signals are within $[\gamma, 1]$ and $\gamma \to 1$, this is $\frac{1}{j+1}$. So in the limit,

$$u_{nm}(s_i) = (s_i - \gamma) \sum_{j=0}^{N-1} \binom{N-1}{j} \gamma^{N-1-j} (1-\gamma)^j \frac{1}{j+1}$$

This sum simplifies: since $\binom{N-1}{j}\frac{1}{j+1} = \frac{(N-1)!}{j!(N-1-j)!}\frac{1}{j+1} = \frac{(N-1)!}{(j+1)!(N-(j+1))!} = \frac{1}{N}\binom{N}{j+1}$,

$$\begin{aligned} u_{nm}(s_i) &= (s_i-\gamma)\textstyle\sum_{j=0}^{N-1}\binom{N-1}{j}\gamma^{N-1-j}(1-\gamma)^j\frac{1}{j+1} \\ &= (s_i-\gamma)\frac{1}{N}\frac{1}{1-\gamma}\textstyle\sum_{j=0}^{N-1}\binom{N}{j+1}\gamma^{N-1-j}(1-\gamma)^{j+1} \\ &= (s_i-\gamma)\frac{1}{N}\frac{1}{1-\gamma}\textstyle\sum_{j'=1}^{N}\binom{N}{j'}\gamma^{N-j'}(1-\gamma)^{j'} \\ &= (s_i-\gamma)\frac{1}{N}\frac{1}{1-\gamma}(1-\gamma^N) \end{aligned}$$

Integrating over s_i (and with payoff 0 for types $s_i < \gamma$), and then multiplying by the number of bidders N, gives ex-ante surplus to all bidders as

$$U_{nm} \;=\; N\tfrac{1}{2}(1-\gamma)^2\tfrac{1}{N(1-\gamma)}(1-\gamma^N) \;=\; \tfrac{1}{2}(1-\gamma)(1-\gamma^N)$$

Now, since $\alpha_1 \to 1$ and $\gamma \to 1$, total bidder surplus is going to 0 in both games. However, if we consider N times total bidder surplus (just to get the leading term), we get

$$\begin{aligned} \lim_{N\to\infty} NU &= \lim \tfrac{1}{2}N(1-\alpha_1)(1-\alpha_1^N) &= \lim \tfrac{1}{2}(-\ln(\alpha_1^N))(1-\alpha_1^N) &= -\tfrac{1}{2}(1-x)\ln(x) \\ \lim_{N\to\infty} NU_{nm} &= \lim \tfrac{1}{2}N(1-\gamma_1)(1-\gamma_1^N) &= \lim \tfrac{1}{2}(-\ln(\gamma_1^N))(1-\gamma_1^N) &= -\tfrac{1}{2}(1-y)\ln(y) \end{aligned}$$

where $x = \lim \alpha_1^N$ and $y = \lim \gamma_1^N$. But $-\frac{1}{2}(1-h)\ln(h)$ is decreasing in h; so since $x < y$, bidder surplus is higher in the game with indicative bids.

Part 3: Greater Revenue. Since we just showed that in the limit, bidder surplus is 0 in either game, it suffices to show that the limit of expected total surplus is strictly higher in the indicative-bidding game than in the pure auction. Beginning with the pure auction at its equilibrium participation level, we can contemplate transitioning to our game via two steps. First, while keeping the participation level the same, change the rules so that when more than two bidders try to participate, all but two are sent home; and then second, lower the participation threshold from the pure-auction equilibrium level to the indicative-bidding equilibrium level, which we showed above to be lower. We claim each of these changes strictly increases total surplus; since bidder surplus is zero in the limit, the seller captures the increase.

To see why "sending bidders home" increases total surplus, note that in an English auction, bidder participation does not impose any net externality: ex post, a bidder's payoff is exactly his contribution to total surplus. So if bidders are sent home, the change in total surplus is the negative of the payoff they anticipated receiving. When N is large, α_1 is close to 1, so all bidders who participate have signals close to 1; this means when two or more bidders participate, each bidder is expecting negative payoff, so sending one home increases total surplus.

What's left, then, is to show that once we have already switched over to the message game, lowering the participation threshold to the equilibrium level, from a level higher than that, increases total surplus. In the limit, we can write total surplus as the probability that exactly one bidder enters, times the surplus when he does, which is $g(1)$; plus the probability two or more bidders try to enter, times the surplus when exactly two participate. From the no-externalities observation above, the total surplus when two bidders with signals approximately 1 participate is $g(1) + f(0)$: total surplus would have been $g(1)$ had only one shown up, and the second to arrive gets payoff $f(0)$. Given the probabilities, total surplus, as a function of $x = \lim \alpha_1^N$, is

$$\begin{aligned} TS &= x \times 0 + (-x \ln(x))g(1) + (1 - x - (-x \ln(x)))(g(1) + f(0)) \\ &= (1 - x)g(1) + (1 - x + x \ln(x))f(0) \\ \frac{d}{dx} TS &= -g(1) - f(0) + f(0) + \ln(x) f(0) \\ &= -g(1) + \ln(x) f(0) \end{aligned}$$

If we can show this is negative whenever x is above the equilibrium level, then decreasing the participation threshold increases total surplus.

Dividing by $-f(0)$ and recalling that $\xi = \frac{g(1)}{-f(0)}$, $\frac{dTS}{dx}$ has the same sign as $-\xi - \ln(x)$; since this is decreasing in x, if it's negative when x is at the equilibrium level, it's negative above the equilibrium level. So it suffices to show that $\frac{-x \ln(x)}{1-x} = \frac{2}{\xi+2}$ implies $-\xi - \ln(x) < 0$, which turns out to just be algebra.[11] Thus, in the limit, total surplus increases when x falls from the pure-auction equilibrium participation level to the indicative-bidding equilibrium level; so switching to indicative bidding strictly increases total surplus, and bidder surplus remains 0 in the limit, so revenue increases. □

4.2 Reserve Prices and Subsidies

We've seen that as N gets large, there remains a nonvanishing chance that exactly one bidder opts in, and buys the asset at price 0. Thus, it might seem that the seller could increase revenue by using a reserve price. However, this intuition turns out to be wrong. When N is large, bidder surplus is effectively zero, so the seller is capturing all surplus; and as noted above, participation is below the level that maximizes total surplus. A reserve price would further depress participation, leading to lower total surplus and lower revenue. On the other hand, subsidizing participation would increase participation, increasing total surplus, and with N large, the seller would capture this additional surplus:

Theorem 5. *When N is sufficiently large,*

1. *any positive reserve price would strictly reduce both revenue and total surplus*

2. *a small bidder subsidy would strictly increase both revenue and total surplus*

[11] Suppose toward contradiction that $\frac{-x \ln(x)}{1-x} = \frac{2}{\xi+2}$ but $-\ln(x) > \xi$. Then $\frac{-x \ln(x)}{1-x} = \frac{2}{\xi+2} > \frac{2}{-\ln(x)+2}$, meaning $x(\ln(x))^2 - 2x \ln(x) > 2 - 2x$. This turns out to be impossible, because $x(\ln(x))^2 - 2x \ln(x) + 2x - 2$ is 0 at $x = 1$ and is increasing, so it is negative on $x < 1$.

The proof is in the appendix. Of course, the subsidy would have to be smaller than $-f(0)$ to prevent the structure of equilibrium from breaking down; but a sufficiently small subsidy (whether paid only when two or more bidders participate, or paid also when a lone bidder opts in) would be strictly beneficial.

4.3 Allowing More Than Two Bidders To Participate

For tractability, our model of indicative bidding has assumed that the second-stage auction is limited to two bidders. Of course, indicative bids could be used to select the three most eager bidders, or some larger number, to participate. Intuitively, we expect the economics of the resulting game to be similar, but the details of the model are substantially more complex. (When two bidders advance, the indifference condition for a threshold-value bidder, $\Delta(\cdot,\cdot,\cdot)$, is the sum of three expected-value terms; if three bidders could advance, it would contain six, since there would be six relevant combinations of which regions of the type space the bidder's two opponents are from.)

However, since our focus has been on the case where $f(0) < 0$ – a bidder only wants to advance when he is alone or against an opponent weaker than himself – it is natural to limit the auction to two bidders. In fact, efficiency would dictate only selecting one; the second bidder is of course required to generate revenue, but more would likely be wasteful. Similar to the large-N result above comparing indicative bidding to a pure auction, it is easy to show that when $f(0) < 0$ and N is sufficiently large, the seller would earn lower revenue by allowing 3 bidders (or a larger number) into the auction than by limiting it to two.

4.4 Auctioning Entry Rights

Ye (2007), after noting that non-binding bids can never perfectly select the bidders with the highest signals, proposes instead using binding first-round bids to auction the right to participate in the second round. In our setting, with the efficient number of entrants being one, this could be done by simply auctioning off, in the "first" round, the right to claim the good for free in the second round. Such an auction would indeed have a symmetric, monotonic equilibrium, and would always lead to the efficient entry outcome – the bidder with the highest initial signal being the only one to incur the entry cost.

However, we feel such a mechanism would be vulnerable in a particular way. Rather than the model in this paper, consider a variation where a buyer's value is $s_i \cdot V$, where s_i is bidder i's private information and V is the objective quality of the good, perfectly observed by anyone performing due diligence. If the distribution of V was fixed, then a one-winner "entry rights" auction would result in the efficient outcome (the bidder with the highest s_i winning the right to the prize), based on the expected value $E(V)$. However, if the seller knew V, then high-value sellers would be unhappy accepting a price based on the ex-ante distribution of V, and would want to change to a mechanism where price was set after V was learned. Worse, if worthless assets could easily be generated, there could be a rush of entry by sellers with worthless goods, hoping to capitalize on prices based on $E(V)$; sellers with legitimate assets would have an even stronger incentive to distinguish themselves. A mechanism using indicative bids avoids this problem: since no money is committed until after due diligence, sellers with worthless goods would have no incentive to enter.

4.5 More Bidder Subsidies

Ye (2007) also observed that, in the case where F_t is degenerate (his *game without value updating*), by exactly subsidizing the participation cost c for the bidders selected to advance to the second round, a seller could indeed create a game with a symmetric, monotonic equilibrium, allowing him to always select the two highest-type bidders for the second round. (Conditional on being the marginal entrant, a bidder expects zero profits in the auction, since he has the second-highest signal, but has his costs covered, so he is indifferent about whether to enter and therefore willing to reveal his exact signal truthfully.) In fact, when c is sufficiently small, such a strategy leads to higher revenue than indicative bidding without a subsidy, although as c grows, this ordering reverses. (In the limit where N is large, the increase in revenue from subsidizing two bidders, instead of using indicative bids, is positive but small while c is small, and becomes strongly negative as c grows.) However, there is no analog to this strategy when F_t is non-degenerate.

5 Discussion and Extensions

5.1 Restriction of Message Space

As discussed above, the equilibria considered in Theorem 1 only exist when the set of available messages is discrete. That is, if there was a message available "in between" 1 and 2, some bidders would find it profitable to deviate to that message; the seller must exogenously prohibit bidders from "bidding fractions." In fact, if the set of permitted messages is continuous, there is no symmetric equilibrium.[12]

This highlights a key difference between indicative bids and cheap talk in a standard sender-receiver setting, a la Crawford and Sobel (1982). In a true cheap-talk game, messages have no "natural" exogenous meaning, only the information the receiver infers from them. This allows us to rule out deviations to off-equilibrium-path beliefs via disadvantageous off-equilibrium-path beliefs, allowing for multiple equilibria and making the set of available messages irrelevant (as long as it is big enough). In our game, however, there are no beliefs: the messages are assumed to be

[12] This follows easily from two steps of the proof of necessity of Lemma 1 – that only finitely many messages can be used in equilibrium by bidders with types $s_i < 1$, and that no two messages used in equilibrium can have another message between them not used in equilibrium.

ordered, and the seller committed to selecting the bidders sending the highest messages. Thus, off-equilibrium-path messages still have exogenous meaning, and equilibrium can only be constructed by limiting the available set of messages.

If the seller was free to select whichever two bidders he wanted, our equilibria (along with pessimistic beliefs about bidders sending off-equilibrium-path messages) would still be equilibria with a continuous message space, but would fail Farrell's (1993) condition of neologism-proofness: for each $k \in \{1, \ldots, m-1\}$, there would be an interval around α_k which was self-signaling.[13]

Even with a discrete set of messages, in our game, the only equilibrium which remains an equilibrium when any "extra messages" are available is the most-informative ($m = M^*$), because this is the only one where high-type bidders have no incentive to deviate to unused high messages under which they would advance for sure. This condition is analogous to the "no incentive to separate" (NITS) condition introduced by Chen, Kartik and Sobel (2008) as a refinement to the set of cheap-talk equilibria. (Under NITS, the lowest-type sender prefers his equilibrium payoff to credibly revealing his true type.) Chen, Kartik and Sobel show that only the most-informative of the equilibria of the standard sender-receiver game satisfy NITS, just like only the most-informative of our equilibria remain equilibria when extra messages are available.

5.2 Extending our model beyond $s_i \sim U[0,1]$

For tractability, we assumed that $s_i \sim U[0,1]$. In fact, this can be relaxed, but not all the way. Recall the indifference condition

$$\Delta(x,y,z) \quad = \quad c_1 \frac{\int_0^x f(h)dh}{x} \quad + \quad c_2 \frac{\int_{-y}^0 f(h)dh}{y} \quad + \quad c_3 \frac{\int_{-y-z}^{-y} f(h)dh}{z}$$

The main thing we need to preserve is the ability to show Δ single-crossing in x (from below) and y and z (from above), so that our construction of unique α for each $m \leq M^*$ remains valid.

If s_i were drawn from some distribution G, we could interpret x, y and z as the values of the CDF, i.e., as $x = G(\alpha_k) - G(\alpha_{k-1})$, $y = G(\alpha_{k+1}) - G(\alpha_k)$, and $z = 1 - G(\alpha_{k+1})$ (and as $y = G(\alpha_2) - G(\alpha_1)$ and $z = 1 - G(\alpha_2)$ in the expression of u). (Thus, in the construction, when we increase z, we would now imagine the interval x adjusting to contain the same probability weight, rather than the same length.) The effects of each variable on the sign of Δ (or u) through the *coefficients* c_1, c_2, c_3 (and d_1, d_2, d_3 in the case of u) would be unchanged. However, a change in z would now change the value of the other integrals $\frac{\int_0^x f(h)dh}{x}$ and $\frac{\int_{-y}^0 f(h)dh}{y}$: by moving the intervals

[13] Meaning, all types within that interval – and only those types within that interval – would prefer revealing themselves to be within that interval to any equilibrium message. If the seller was not constrained in his choice of two bidders for the second round, revenue maximization depends on choosing bidders with the highest signals, so a revenue-maximizing seller would best-respond to such a neologism by treating it as a message "between" messages k and $k+1$.

(α_{k-1}, α_k) and (α_k, α_{k+1}) while maintaining their probability weight, the intervals could be getting wider or narrower in actual size, and the conditional expectations could be growing or shrinking.

It turns out, if the distribution G had a decreasing density function, all the effects would go in the "correct direction": increasing z would decrease the integrals corresponding to $\int_0^x f(h)dh$ and $\int_{-y}^0 f(h)dh$, decreasing Δ, and increasing y decreases the integral corresponding to $\int_0^x f(h)dh$, also decreasing Δ; so all our results go through. However, if the density function corresponding to G is sometimes increasing, then the effect of y and z on Δ become unclear, and our proofs would no longer be valid.

5.3 Focus on the case $f(0) < 0$

Our assumption that $f(0) < 0$ is an important one for tractability. Returning to the indifference condition above, if $f(0) < 0$, then the second and third terms are always negative, so if $\Delta \approx 0$, the first term must be positive. However, if $f(0) > 0$, then the first term is always positive; for $\Delta = 0$, the third term would have to be negative, but it is not obvious ex-ante whether the middle term is positive or negative. This makes characterizing the properties of Δ much harder. Similarly, when $f(0) > 0$, it is not unambiguous whether the "opt out" message is used in equilibrium: if the bottom interval is narrow, it may not be, although if the bottom interval is wide, then it would be. The point is, there are more possibiltiies to consider when $f(0) > 0$, and we are unsure of how to proceed.

We did solve some numerical examples with $f(0) > 0$, however, and the general structure of equilibrium appeared to be very similar; we just can't say whether we were finding all the equilibria. Two interesting differences: when $f(0) < 0$, M^* is increasing when $f(0)$ increases (through a decrease in c or a wider distribution of t); when $f(0) > 0$, the opposite is true. That is, we believe more separation is generally possible when $f(0)$ is closer to 0, not when it is larger absolutely. (If $f(0)$ were sufficiently large, all bidders would want to opt in regardless of the competition, and equilibrium would degenerate to all bidders playing the highest available message.) Similarly, when $f(0) < 0$ and $M \geq M^*$, we get more separation at the top, and intervals getting wider as we move down; when $f(0) > 0$, we expect this to be the opposite, with more separation at the bottom and the most pooling among the higher types.

6 Conclusion

We have shown that indicative bids convey some information about the bidders' types when the bid space is discrete. The result is often greater efficiency and higher revenues than an auction with unrestricted entry, particularly when the number of bidders is large.

In implementing the indicative bidding mechanism, the seller does not have to know the underlying cost and information structure. It is enough to specify an ordered set of discrete bids, designating the lowest bid as "opt out"; the bidders will then endogeneously pool on a finite number of the lowest bids. The seller could further restrict the number of bids, but it appears that she has no incentive to do so: in numerical examples, we have found that both revenue and efficiency appear to always increase in the number of bids allowed, up to the maximum number than can be used in equilibrium. However, we have not yet been able to prove this result generally or find a counterexample.

An alternative implementation is to present the bidders with a partition of the space of their estimates, and ask them to report the interval in which their estimate lies. This implementation relies less heavily on bidders' ability to deduce the equilibrium of the game, and more heavily on the seller's knowledge of the game. The seller needs to know the primitives to compute the correct thresholds, in order to induce truthful reporting by the bidders.

We have chosen a simple game form to show that indicative bids can be informative and useful to the seller. We believe this finding is robust to a wider class of indicative bidding mechanisms, but have not pursued this issue or the related issue of optimality in this paper. The format of the auction used in the second stage will clearly matter: buyers who advance to the auction have different beliefs about each other conditional on the outcome of the indicative bid stage. These beliefs do not matter in second-price auction because bidding is in dominant strategies, but they will affect equilibrium bidding behavior in a first-price auction, greatly complicating the derivation of the payoffs in the indicative bidding stage. A more promising direction for future research may be sequential mechanisms. A seller could ask the buyers to submit their indicative bids sequentially. She could also use the outcome of the message game to determine the order in which she asks buyers who advance to participate and bid.

Appendix – Omitted Proofs

A.1 Omitted Proof of Necessity in Lemma 1

We begin by establishing properties that must hold in any symmetric equilibrium.

In any symmetric equilibrium, strategies are monotonic.

We want to show that if τ is a symmetric equilibrium strategy, then for any two types $s_i' > s_i$ and any two messages $k' > k$, if $k' \in \operatorname{supp} \tau(s_i)$, then $k \notin \operatorname{supp} \tau(s_i')$.

We showed in the text that if k' and k give different probabilities of advancing, $v(s_i, k') - v(s_i, k)$ is strictly single-crossing from below in s_i: so if $k' \in \operatorname{supp} \tau(s_i)$ and a bidder with type s_i therefore weakly prefers k' to k, a bidder with type $s_i' > s_i$ strictly prefers k' to k, and so never plays k.

The only remaining case we need to rule out, then, is if k and k' give the same probability of advancing, and $k' \in \operatorname{supp} \tau(s_i)$ and $k \in \operatorname{supp} \tau(s_i')$. In such a case, every bidder type $s_i'' \in (s_i, s_i')$ would also have to send a message giving the same probability of advancing. (Since $s_i'' < s_i'$, s_i'' can't send a message with higher probability of advancing than k; and since $s_i'' > s_i$, s_i'' can't send a message with lower probability than k'.) But the interval (s_i, s_i') has positive measure; and it's impossible for a positive measure of types to all have the same probability of advancing unless they all send the same message, which finishes the proof.

In any symmetric equilibrium, the set of messages sent by bidders with types $s_i \neq 1$ is finite.

By assumption, $f(0) < 0$, and f is continuous; let $\epsilon > 0$ solve $f(\epsilon) = 0$. We will show that for any interval $[s, s + \epsilon)$, at most two messages are sent by bidders with signals in that interval, which limits the number of messages used (other than by bidders with type $s_i = 1$) to at most $2\lceil \frac{1}{\epsilon} \rceil$. Suppose at least three messages were used by bidders with types in $[s, s + \epsilon)$. Let $k_1 < k_2 < k_3$ be three messages used by bidders in that interval, chosen such that a positive measure of bidders either use message k_3 or use messages between k_2 and k_3, so that messages k_2 and k_3 give different probabilities of advancing. We will show all bidders in the interval strictly prefer message k_2 to k_3, giving a contradiction.

By monotonicity of strategies, since k_1 is sent by some bidder with type $s_i \geq s$, all bidders sending message k_2 or higher have types greater than s. Now consider the cases where sending message k_3 gives bidder i a higher probability of advancing than sending message k_2. This can only occur when the second-highest message sent by i's opponents is at least k_2 (otherwise i would have advanced for sure), which means the highest message of i's opponents must also be at least k_2; which means whenever sending k_3 gives i a higher probability of advancing, he advances against an opponent with type $s_j > s$. But since $s_i < s + \epsilon$, this means $s_i - s_j < \epsilon$ in all the cases where sending the higher message caused i to advance; but then $f(s_i - s_j) < f(\epsilon) \leq 0$, so bidder i prefers not to advance in all those cases. So sending k_2 is strictly better than k_3.

In any symmetric equilibrium, the strategies played by types $s_i \neq 1$ are the m lowest messages available, for some finite m.

First, note that the opt-out message must be played with positive probability. If not, it would be impossible to advance alone, and all messages used in equilibrium would give a positive probability of advancing; together, these would imply that bidders with signals close to 0 earn negative payoffs, when they could deviate to message 1 and get 0.

Second, we show that no messages are "skipped" in equilibrium. Let $\mathcal{K}$ be the set of messages that are sent with positive probability. Suppose there were messages $k < k' < k''$, with $k, k'' \in \mathcal{K}$ but $k' \notin \mathcal{K}$. Suppose k and k'' are adjacent in $\mathcal{K}$, i.e., there are no messages $k''' \in \mathcal{K}$ with $k < k''' < k''$. By monotonicity, there must be some type s such that bidders below type s use message k or lower, and bidders above type s use k'' or higher; by continuity, bidders with type s must be indifferent between k and k''.

From bidder i's point of view, sending message k'' instead of message k increases the probability of advancing in two scenarios: when his second-highest competitor sends message k'', and when his second-highest competitor sends message k. If the second-highest opposing message is k'', the highest is at least k''; by monotonicity, this means that when i advances in this scenario, he is up against a stronger opponent with $s_j \geq s = s_i$, and (since $f(0) < 0$) earns negative expected payoff. This means that if he is indifferent between the two messages, he must earn strictly positive payoff in expectation in the second scenario, when the second-highest competitor sends k. By deviating to message k', he would get the best of both worlds: always advancing when his second-highest competitor sent message k, but never advancing when he sent k'', making k' a profitable deviation. So no messages are "skipped" in equilibrium, and (from above) a finite number are played, meaning $\mathcal{K}$ must be the lowest m messages available for some m.

This, along with monotonicity, implies that when the set of available messages is $\{1, 2, \ldots, M\}$, then in any symmetric equilibrium, there is an $m \leq M$ and a series of thresholds

$$0 \;=\; \alpha_0 \;<\; \alpha_1 \;<\; \alpha_2 \;<\; \cdots \;<\; \alpha_{m-1} \;<\; \alpha_m \;=\; 1$$

such that for each k, bidders with types $s_i \in (\alpha_{k-1}, \alpha_k)$ send message k and bidders with with α_k send either k or $k+1$ or mix between the two.[14]

[14] In the "generic" case, bidders with type $s_i = 1$ will send message m, but that there is a nongeneric case where bidders with type $s_i = 1$ are indifferent between m and any unused, higher message, and may therefore mix among all messages at least as high as m.

In any symmetric equilibrium, "threshold types" $s_i = \alpha_k$ are indifferent between two messages.

Each $v(\,\cdot\,, k)$ is continuous, so $v(\,\cdot\,, k+1) - v(\,\cdot\,, k)$ is continous; so if bidders with types s_i just below α_k play message k (and therefore weakly prefer k to $k+1$), and bidders with types just above s_i play $k+1$ (and therefore weakly prefer $k+1$ to k), then $v(\alpha_k, k+1) - v(\alpha_k, k) = 0$.

In any symmetric equilibrium, if unused messages $m' > m$ exist, they must be unprofitable for bidders with $s_i < 1$.

Equilibrium also requires that if there are any unused messages higher than m, deviations to those messages are not profitable. If $m = M$ (the number of available messages), of course, no such messages exist. If $m < M$, however, we need to rule out deviations to unused messages $m' > m$. A bidder with type $s_i > \alpha_{m-1}$ who sent such a message would advance for certain; by sending m, he advances for certain except when multiple opponents also send message m, in which case he advances with probability less than 1. So a deviation to a message m' would increase his probability of advancing only against an opponent with type $s_j \in (\alpha_{m-1}, 1)$. If $E_{s_j \in (\alpha_{m-1},1)} f(s_i - s_j) > 0$, such a deviation would be profitable. So ruling out such a deviation requires that

$$0 \quad \geq \quad \int_{\alpha_{m-1}}^{1} f(s_i - s_j) ds_j \quad = \quad \int_0^{1-\alpha_{m-1}} f(s_i - 1 + h) dh$$

for every $s_i \in (\alpha_{m-1}, 1)$, or that $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$.

A.2 Omitted Construction from Proof of Theorem 1

As noted in the text, it will suffice to show that given N, c, and F_y, we can find an $M^* \geq 2$ such that:

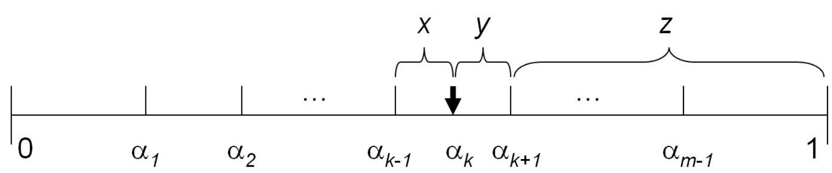
- For any $m \leq M^*$, there is a unique α satisfying $v(\alpha_k, k+1) = v(\alpha_k, k)$ for every k
- For any $m > M^*$, there is no such α
- For $m < M^*$ and α satisfying the indifference conditions, $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$; for $m = M^*$ and α satisfying the indifference conditions, $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$.

Then Theorem 1 follows. When $M < M^*$, at the candidate equilibrium α for each $m \leq M$, $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$, and so the only actual equilibrium requires $m = M$; when $M \geq M^*$, only the candidate equilibrium with $m = M$ satisfies the requirements. Once the right m and α are found, any strategy τ satisfying the first three conditions of the lemma is an equilibrium; and any strategies satisfying these three conditions differ only in the actions of bidders with types $s_i \in \{\alpha_1, \ldots, \alpha_m\}$, so if α is unique, equilibrium is effectively unique.

The remainder of the proof has five parts: showing the construction of M^*; showing the construction of α for a given $m \le M^*$; showing the uniqueness of that α; showing that no such α exists when $m > M^*$; and showing that at that α, $\int_0^{1-\alpha_{m-1}} \le 0$ if and only if $m = M^*$.

Part 1: Calculating M^*

First, note the arguments of the function Δ: they are $\alpha_k - \alpha_{k-1}$, $\alpha_{k+1} - \alpha_k$, and $1 - \alpha_{k+1}$ (which is zero at $m = k - 1$, since $\alpha_m = 0$). Thus, when we talk about $\Delta(x, y, z)$, its arguments are as follows:



Similarly, the arguments of u are $y = \alpha_2 - \alpha_1$ and $z = 1 - \alpha_2$. In the appendix, we establish crucial properties of Δ and u:

Lemma 2. *1. Away from $(x, y, z) = (0, 0, 0)$, Δ is strictly single-crossing in x from below, and in y and z from above.*

2. $u(y, z)$ is strictly single-crossing from above in both y and z.

3. If $\Delta(1 - y - z, y, z) \ge 0$ then $u(y, z) \ge 0$, and $u(y, z) > 0$ if $y + z < 1$.

As noted in the text, other than at $(0, 0, 0)$, Δ is negative when $x < \underline{x}$, where $\underline{x}$ is the positive solution to $\int_0^{\underline{x}} f(h)dh = 0$. Lemma 2 part 1 implies that as x increases, Δ crosses 0 at most once on $(0, 1 - y - z]$, from below. Given (y, z) with $y + z < 1$, we can therefore define a function $x^*(y, z)$ as the unique value of $x \in (0, 1 - y - z]$ satisfying $\Delta(x, y, z) = 0$ when such a solution exists, and $+\infty$ otherwise. This means that $\alpha_{k-1} = \alpha_k - x^*(\alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1})$ is the unique value of α_{k-1} satisfying $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) = 0$ when a solution exists, and $-\infty$ otherwise. The fact that Δ is decreasing in y and z when $\Delta = 0$ implies that x^* is strictly increasing in both its arguments.

We now construct M^* via the following algorithm:

1. Let $\beta_0 = \beta_1 = 1$

2. Starting with $k = 2$, and as long as it is positive, define β_k recursively by

$$\beta_k \quad = \quad \beta_{k-1} - x^*(\beta_{k-2} - \beta_{k-1}, 1 - \beta_{k-2})$$

   (By definition, $x^*(\beta_{k-2} - \beta_{k-1}, 1 - \beta_{k-2}) \le \beta_{k-1}$ unless it is $+\infty$, so $\beta_k \ge 0$ unless it is $-\infty$.)

3. Define $\overline{M}$ as the highest value of k such that $\beta_k \geq 0$, i.e., the unique value of k such that $x^*(\beta_{k-2} - \beta_{k-1}, 1 - \beta_{k-2}) \leq \beta_{k-1}$ but $x^*(\beta_{k-1} - \beta_k, 1 - \beta_{k-1}) = +\infty$. Note that $\overline{M}$ is finite: is $\Delta(x, \cdot, \cdot) < 0$ when $x \in (0, \underline{x})$, $x^*(\cdot, \cdot) \geq \underline{x}$, so $\beta_k \leq \beta_{k-1} - \underline{x}$ and therefore $\overline{M} \leq \frac{1}{\underline{x}} + 2$.

4. Set $M^* = \overline{M} + 1$ if $u(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) > 0$, and $M^* = \overline{M}$ otherwise.

Note that $M^* \geq 2$: even if the algorithm terminated immediately (if $\beta_2 = -\infty$) then $\overline{M} = 1$, and in step 4 we would calculate

$$u(\beta_0 - \beta_1, 1 - \beta_1) \quad = \quad u(0,0) \quad = \quad v(1)$$

because the coefficients d_2 and d_3 are both zero at $(y, z) = (0, 0)$. Our first assumption was that $v(1) > 0$, so $M^* = \overline{M} + 1 = 2$.

Part 2: Finding α Satisfying Conditions 5 and 6 Given $m \leq M^*$

We will extend our construction from before, adding a new parameter t which takes values between 0 and some $\bar{t} \leq 1$.

1. Define $\beta_0(t) = 1$, and $\beta_1(t) = 1 - t$

2. For $k = 2, 3, \ldots, m-1$, define $\beta_k(t)$ recursively by

$$\beta_k(t) \quad = \quad \beta_{k-1}(t) - x^*(\beta_{k-2}(t) - \beta_{k-1}(t), 1 - \beta_{k-2}(t))$$

   or $\beta_k(t) = -\infty$ if $\beta_{k-1}(t) = -\infty$.

3. Since $m \leq M^*$, note that at least at $t = 0$, $(\beta_0(t), \beta_1(t), \ldots, \beta_{m-1}(t))$ are all greater than $-\infty$ (and therefore nonnegative), since they match the $\{\beta_0, \beta_1, \ldots\}$ found in the first algorithm, which didn't terminate until $\overline{M} \geq M^* - 1 \geq m - 1$.

4. From the properties of x^*, note that as t increases (and therefore $\beta_1(t) = 1 - t$ decreases), $x^*(\beta_0(t) - \beta_1(t), 1 - \beta_0(t)) = x^*(t, 0)$ increases continuously, and therefore $\beta_2(t) = \beta_1(t) - x^*(t, 0)$ decreases continuously, until it hits 0 at which point x^* jumps to $+\infty$ and $\beta_2(t)$ jumps to $-\infty$. Similarly, as $\beta_1(t) - \beta_2(t)$ increases continuously and $\beta_2(t)$ decreases continuously, $\beta_3(t)$ decreases continuously, until it hits 0. Likewise with $\beta_4(t)$, and so on. Let $\bar{t}$ be the value of t at which $\beta_{m-1}(t)$ hits 0. So $(\beta_1(t), \beta_2(t), \ldots, \beta_{m-1}(t))$ are all decreasing continuously on $[0, \bar{t})$, and $\beta_{k-1}(t) - \beta_k(t)$ is continuously increasing over that range for each k.

5. Since $m \leq M^*$, note that $u(\beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) > 0$. If $m = M^*$ and M^* was found via the "first scenario" above ($M^* = \overline{M} + 1$), then

$$\begin{aligned} u(\beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) &= u(\beta_{M^*-2} - \beta_{M^*-1}, 1 - \beta_{M^*-2}) \\ &= u(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) \\ &> 0 \end{aligned}$$

by definition. On the other hand, if $m < M^*$ or $m = M^*$ and M^* was found via the "second scenario" above ($M^* = \overline{M}$), it means that

$$\begin{array}{rcl} \beta_m = \beta_{m-1}(0) - x^*(\beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) & \geq & 0 \\ & \downarrow & \\ \Delta(\beta_m - \beta_{m-1}(0), \beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) & = & 0 \\ & \downarrow & \\ \Delta(1 - (\beta_{m-2}(0) - \beta_{m-1}(0)) - (1 - \beta_{m-2}(0))), \beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) & \geq & 0 \\ & \downarrow & \\ u(\beta_{m-2}(0) - \beta_{m-1}(0), 1 - \beta_{m-2}(0)) & > & 0 \end{array}$$

6. On the other hand, at $\bar{t}$, $\beta_{m-1}(\bar{t}) = 0$ (by definition), and so $(\beta_{m-2}(\bar{t}) - \beta_{m-1}(\bar{t})) + (1 - \beta_{m-2}(\bar{t})) = 1 - \beta_{m-1}(\bar{t}) = 1$; it's easy to show that $u(y, z) < 0$ when $y + z = 1$, and so $u(\beta_{m-2}(\bar{t}) - \beta_{m-1}(\bar{t}), 1 - \beta_{m-2}(\bar{t})) < 0$.

7. So now define a function $U : [0, \bar{t}] \to \Re$ by $u(t) = u(\beta_{m-2}(t) - \beta_{m-1}(t), 1 - \beta_{m-2}(t))$. Since $\beta_{m-1}(\cdot)$ and $\beta_{m-2}(\cdot)$ are continuous on $[0, \bar{t}]$, so is U; and we just showed that $U(0) > 0$ and $U(\bar{t}) < 0$. Further, both arguments of u in the definition of $U(t)$ are strictly increasing in t; and u is decreasing in both its arguments when it is at zero; so there must be a single value of t at which $U(t) = 0$. Call that value t^*.

8. Let $\alpha_0 = 0$; $\alpha_1 = \beta_{m-1}(t^*)$; $\alpha_2 = \beta_{m-2}(t^*)$; and so, up to $\alpha_{m-1} = \beta_1(t^*)$ and $\alpha_m = \beta_0 = 1$. By construction, $u(\alpha_2 - \alpha_1, 1 - \alpha_2) = 0$ – this is how t^* was defined. And by construction, $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) = 0$ – this is how $\beta_{m-k+1}(t)$ was defined recursively. So we have our α satisfying conditions 5 and 6.

Part 3: Uniqueness of α Given m

We already showed that $x^*(y, z)$ is strictly increasing in both its arguments. Suppose there were two strings,

$$\begin{array}{ccccccccccccc} 0 & = & \alpha_0 & < & \alpha_1 & < & \alpha_2 & < & \cdots & < & \alpha_{m-1} & < & \alpha_m & = & 1 \\ 0 & = & \gamma_0 & < & \gamma_1 & < & \gamma_2 & < & \cdots & < & \gamma_{m-1} & < & \gamma_m & = & 1 \end{array}$$

both satisfying conditions 5 and 6 of Lemma 1. By assumption, $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) = 0$ for each k, meaning $\alpha_k - \alpha_{k-1} = x^*(\alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1})$, and similarly $\gamma_k - \gamma_{k-1} = x^*(\gamma_{k+1} - \gamma_k, 1 - \gamma_{k+1})$.

First, suppose $\alpha_{m-1} = \gamma_{m-1}$. In that case,

$$\begin{array}{ccccccc} \alpha_{m-2} & = & \alpha_{m-1} - x^*(\alpha_m - \alpha_{m-1}, 1 - \alpha_m) & = & \gamma_{m-1} - x^*(\gamma_m - \gamma_{m-1}, 1 - \gamma_m) & = & \gamma_{m-2} \\ \alpha_{m-3} & = & \alpha_{m-2} - x^*(\alpha_{m-1} - \alpha_{m-2}, 1 - \alpha_{m-1}) & = & \gamma_{m-2} - x^*(\gamma_{m-1} - \gamma_{m-2}, 1 - \gamma_{m-1}) & = & \gamma_{m-3} \end{array}$$

and likewise $\alpha_{m-4} = \gamma_{m-4}$ and so on; so $\alpha = \gamma$ and uniqueness has not been violated.

So suppose WLOG that $\alpha_{m-1} > \gamma_{m-1}$. (If $\alpha_{m-1} < \gamma_{m-1}$, switch the labels α and γ.) Since x^* is increasing in both its arguments,

$$\alpha_{m-1} - \alpha_{m-2} \quad = \quad x^*(1-\alpha_{m-1}, 0) \quad < \quad x^*(1-\gamma_{m-1}, 0) \quad = \quad \gamma_{m-1} - \gamma_{m-2}$$

which, along with $\alpha_{m-1} > \gamma_{m-1}$, means $\alpha_{m-2} > \gamma_{m-2}$. So then

$$\alpha_{m-2} - \alpha_{m-3} \quad = \quad x^*(\alpha_{m-1} - \alpha_{m-2}, 1-\alpha_{m-1}) \quad < \quad x^*(\gamma_{m-1} - \gamma_{m-2}, 1-\gamma_{m-1}) \quad = \quad \gamma_{m-2} - \gamma_{m-3}$$

and therefore $\alpha_{m-3} > \gamma_{m-3}$; and so on. Repeating the argument, we get $\alpha_2 > \gamma_2$ and $\alpha_2 - \alpha_1 < \gamma_2 - \gamma_1$; since u is decreasing in both its arguments when it is close to 0, $u(\alpha_2 - \alpha_1, 1 - \alpha_2) > u(\gamma_2 - \gamma_1, 1 - \gamma_1)$, contradicting the assumption that both are equal to 0.

Part 4: Non-existence of α when $m > M^*$

In our calculation of M^* above, we set $\beta_0 = \beta_1 = 1$ and $\beta_k = \beta_{k-1} - x^*(\beta_{k-2} - \beta_{k-1}, 1 - \beta_{k-2})$, with $\overline{M}$ being where this string terminated because $x^*(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) = +\infty$; we then set M^* equal to either $\overline{M}$ or $\overline{M}+1$. We will first show non-existence of α when $m > \overline{M}+1$, then for $m = \overline{M}+1$ under the conditions where $M^* = \overline{M}$.

For the first case, suppose that $m > \overline{M}+1$, and that there was a string $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = 1$ such that $\Delta(\alpha_k - \alpha_{k-1}, \alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1}) = 0$, or $\alpha_k - \alpha_{k-1} = x^*(\alpha_{k+1} - \alpha_k, 1 - \alpha_{k+1})$, for every $k = 2, \ldots, m-1$. Since $\alpha_{m-1} < 1 = \beta_1$ and x^* is increasing in both its arguments, so

$$\alpha_{m-1} - \alpha_{m-2} \quad = \quad x^*(1 - \alpha_{m-1}, 0) \quad > \quad x^*(0,0) \quad = \quad \beta_1 - \beta_2$$

and so $\alpha_{m-2} < \beta_2$. Similarly

$$\alpha_{m-2} - \alpha_{m-3} \quad = \quad x^*(\alpha_{m-1} - \alpha_{m-2}, 1 - \alpha_1) \quad > \quad x^*(1 - \beta_2, 0) \quad = \quad \beta_2 - \beta_3$$

and so $\alpha_{m-3} < \beta_3$, and so on. Repeating, we get $\alpha_{m-(\overline{M}-1)} < \beta_{\overline{M}-1}$ and $\alpha_{m-(\overline{M}-1)} - \alpha_{m-\overline{M}} > \beta_{\overline{M}-1} - \beta_{\overline{M}}$.

Now, we defined $\overline{M}$ as the step at which our algorithm for calculating the next β terminated; so by definition, $x^*(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) = +\infty$. On the other hand, if $m > \overline{M}+1$, there must be a positive $\alpha_{m-(\overline{M}+1)} = \alpha_{m-\overline{M}} - x^*(\alpha_{m-(\overline{M}-1)} - \alpha_{m-\overline{M}}, 1 - \alpha_{m-(\overline{M}-1)}) > 0$. Together, these contradict the result that x^* is increasing in both its arguments.

For the other case, $m = \overline{M}+1 > M^*$, the same argument establishes that $\alpha_{m-(\overline{M}-1)} < \beta_{\overline{M}-1}$ and $\alpha_{m-(\overline{M}-1)} - \alpha_{m-\overline{M}} > \beta_{\overline{M}-1} - \beta_{\overline{M}}$. Since there are only $\overline{M}+1$ messages used, $\alpha_{m-\overline{M}} = \alpha_1$ and $\alpha_{m-(\overline{M}-1)} = \alpha_2$, so α satisfying condition 5 means

$$u(\alpha_2 - \alpha_1, 1 - \alpha_1) = u(\alpha_{m-(\overline{M}-1)} - \alpha_{m-\overline{M}}, 1 - \alpha_{m-(\overline{M}-1)}) = 0$$

But $M^* = \overline{M}+1$ implies that

$$u(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) \leq 0$$

so together these contradict u strictly decreasing in both arguments when it is close to 0.

Part 5: $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$ **if and only if** $m = M^*$

We will show this with two contradictions: we will show that if either $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$ and $m < M^*$, or $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$ and $m = M^*$, then if α satisfies all the other indifference conditions, $u(\alpha_2 - \alpha_1, 1 - \alpha_2) \neq 0$.

Note that when $y = z = 0$, the coefficients c_2 and c_3 in Δ vanish and $\Delta(x, 0, 0) = \frac{c_1}{x} \int_0^x f(h)dh$; so $x^*(0, 0)$ is the unique positive solution to $\int_0^x f(h)dh = 0$, which we earlier labeled $\underline{x}$. Of course, this means $\int_0^{1-\alpha_{m-1}} f(h)dh > 0$ if and only if $1 - \alpha_{m-1} > \underline{x}$. And in our earlier construction of $(\beta_0, \beta_1, \ldots)$, $\beta_2 = 1 - x^*(0, 0) = 1 - \underline{x}$.

For the first one,

$$\begin{array}{rcl}
\int_0^{1-\alpha_{m-1}} f(h)dh & > & 0 \\
 & \downarrow & \\
\alpha_{m-1} & < & \beta_2 \\
 & \downarrow & \\
\alpha_{m-1} - \alpha_{m-2} \;=\; x^*(1 - \alpha_{m-1}, 0) & > & x^*(1 - \beta_2, 0) \;=\; \beta_2 - \beta_3 \\
 & \downarrow & \\
\alpha_{m-2} & < & \beta_3 \\
 & \downarrow & \\
\alpha_{m-2} - \alpha_{m-3} \;=\; x^*(\alpha_{m-1} - \alpha_{m-2}, 1 - \alpha_{m-1}) & > & x^*(\beta_2 - \beta_3, 1 - \beta_2) \;=\; \beta_3 - \beta_4 \\
 & \downarrow & \\
\alpha_{m-3} & < & \beta_4
\end{array}$$

and so on. If $m = M^*$, this means $\alpha_2 < \beta_{M^*-1}$ and $\alpha_2 - \alpha_1 > \beta_{M^*-1} - \beta_{M^*}$. But then since u is decreasing in both arguments, $u(\alpha_2 - \alpha_1, 1 - \alpha_2) = 0 < u(\beta_{M^*-1} - \beta_{M^*}, 1 - \beta_{M^*-1})$. But this contradicts the construction of M^*, which required either that $M^* = \overline{M}$ and $u(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) \leq 0$, or that $M^* = \overline{M} + 1$ and therefore β_{M^*} not exist to satisfy $\beta_{M^*-1} - \beta_{M^*} = x^*(\beta_{M^*-2} - \beta_{M^*-1}, 1 - \beta_{M^*-2})$.

On the other hand, suppose $\int_0^{1-\alpha_{m-1}} f(h)dh \leq 0$, so $\alpha_{m-1} \geq \beta_2$. The reverse arguments give $\alpha_{m-k} \geq \beta_{1+k}$ and $\alpha_{m-k} - \alpha_{m-k-1} \geq \beta_{1+k} - \beta_{2+k}$ for each k. Setting $k = m - 2$,

$$u(\beta_{m-1} - \beta_m, 1 - \beta_{m-1}) \;\leq\; u(\alpha_2 - \alpha_1, 1 - \alpha_2) \;=\; 0$$

However, the construction of M^* gave $u(\beta_{m-1} - \beta_m, 1 - \beta_{m-1}) > 0$ for any $m < M^*$: because either (case 1) $m = M^* - 1 = \overline{M}$ and $u(\beta_{\overline{M}-1} - \beta_{\overline{M}}, 1 - \beta_{\overline{M}-1}) > 0$; or (case 2) $m < \overline{M}$, which means that β_{m+1} was well-defined, which implies that $\Delta(1 - y - z, y - z) \geq 0$ for $y = \beta_{m-1} - \beta_m$ and $z = 1 - \beta_{m-1}$, which implies (by part 3 of Lemma 2) $u(\beta_{m-1} - \beta_m, 1 - \beta_{m-1}) > 0$. □

A.3 Proof of Theorem 2

First, we will show that $u(y,z)$ and $\Delta(x,y,z)$ both increase when c falls or when a mean-preserving spread is applied to F_t. Both g and f are decreasing in c. As for F_t, g depends on F_t only through $E(t_i)$, which is unaffected by a mean-preserving spread. $f(h) = E\max\{-c, -c+h+t_i-t_j\}$ is the expected value of a convex function of t_i, and therefore increases when a MPS is applied to the distribution of t_i; and is likewise the expected value of a convex function of t_j, and therefore increases when a MPS is applied to the distribution of t_j. Since u and Δ are essentially weighted-averages of f (in the case of Δ) or f and g (in the case of u), both are decreasing in c, and both increase when a MPS is applied to F_y.

As for N, let $u^{(n)}$, $\Delta^{(n)}$, $c_i^{(n)}$, and $d_i^{(n)}$ denote the relevant term when $N=n$; we will show that if $x>0$ and $\Delta^{(n)}(x,y,z)=0$, then $\Delta^{(n-1)}(x,y,z)\geq 0$; and likewise if $u^{(n)}(y,z)>0$, then $u^{(n-1)}(y,z)>0$.

One part of the proof of Theorem 1 – in particular, the proof of part 2 of Lemma 2, that where it crosses 0, Δ is strictly decreasing in z – involved a proof (given in the separate appendix) that

$$\frac{\partial c_1^{(n)}/\partial z}{c_1^{(n)}} \quad\leq\quad \frac{\partial c_3^{(n)}/\partial z}{c_3^{(n)}} - \frac{1}{z} \quad \textit{and} \quad \frac{\partial c_1^{(n)}/\partial z}{c_1^{(n)}} \quad\leq\quad \frac{\partial c_2^{(n)}/\partial z}{c_2^{(n)}}$$

Taking derivatives, it turns out that

$$\frac{\partial c_1^{(n)}}{\partial z} \;=\; -(n-1)c_1^{(n-1)} \quad , \quad \frac{\partial c_2^{(n)}}{\partial z} \;=\; -(n-1)c_2^{(n-1)} \quad , \quad \frac{\partial c_3^{(n)}}{\partial z} \;=\; -(n-1)c_3^{(n-1)} + \frac{c_3^{(n)}}{z}$$

which gives

$$\frac{c_1^{(n-1)}}{c_1^{(n)}} \quad\geq\quad \frac{c_3^{(n-1)}}{c_3^{(n)}} \quad \textit{and} \quad \frac{c_1^{(n-1)}}{c_1^{(n)}} \quad\geq\quad \frac{c_2^{(n-1)}}{c_2^{(n)}}$$

This means that if

$$0 \;=\; \Delta^{(n)}(x,y,z) \;=\; c_1^{(n)}\frac{\int_0^x f(h)dh}{x} + c_2^{(n)}\frac{\int_{-y}^0 f(h)dh}{y} + c_3^{(n)}\frac{\int_{-y-z}^{-y} f(h)dh}{z}$$

then multiplying by $\frac{c_1^{(n-1)}}{c_1^{(n)}}$ gives

$$\begin{aligned}
0 \;&=\; c_1^{(n-1)}\frac{\int_0^x f(h)dh}{x} + \frac{c_1^{(n-1)}}{c_1^{(n)}}c_2^{(n)}\frac{\int_{-y}^0 f(h)dh}{y} + \frac{c_1^{(n-1)}}{c_1^{(n)}}c_3^{(n)}\frac{\int_{-y-z}^{-y} f(h)dh}{z} \\
&\leq\; c_1^{(n-1)}\frac{\int_0^x f(h)dh}{x} + \frac{c_2^{(n-1)}}{c_2^{(n)}}c_2^{(n)}\frac{\int_{-y}^0 f(h)dh}{y} + \frac{c_3^{(n-1)}}{c_3^{(n)}}c_3^{(n)}\frac{\int_{-y-z}^{-y} f(h)dh}{z} \\
&=\; c_1^{(n-1)}\frac{\int_0^x f(h)dh}{x} + c_2^{(n-1)}\frac{\int_{-y}^0 f(h)dh}{y} + c_3^{(n-1)}\frac{\int_{-y-z}^{-y} f(h)dh}{z} \\
&=\; \Delta^{(n-1)}(x,y,z)
\end{aligned}$$

We similarly showed, in the proof of Lemma 2 part 3, that $\frac{\partial d_1^{(n)}/\partial z}{d_1^{(n)}} \leq \frac{\partial d_2^{(n)}/\partial z}{d_2^{(n)}}$ and $\frac{\partial d_1^{(n)}/\partial z}{d_1^{(n)}} \leq \frac{\partial d_3^{(n)}/\partial z}{d_3^{(n)}}$, which leads via the same steps to the result for u: if $u^{(n)}(y,z) > 0$, then $u^{(n-1)}(y,z) > 0$.

So now let Δ, u, M^*, and so on refer to some set of "original" primitives c, F_t, and N, and $\widetilde{\Delta}$, $\widetilde{u}$, $\widetilde{M}^*$, etc., refer to a "new" set of primitives arrived at by either reducing c, applying a mean-preserving spread to F_t, or reducing N. By construction, M^* is the largest number such that when $\beta_1 = \beta_0 = 1$, $\beta_2 = x^*(0,0)$, $\beta_3 = \beta_2 - x^*(1-\beta_2, 0)$, $\beta_4 = \beta_3 - x^*(\beta_2 - \beta_3, 1-\beta_2)$, etc., β_{M^*-1} is well-defined and $u(\beta_{M^*-2} - \beta_{M^*-1}, 1 - \beta_{M^*-2}) > 0$. But if $\widetilde{\Delta}(x,y,z) \geq 0$ whenever $\Delta(x,y,z) \geq 0$, then $\widetilde{x}^*(y,z) \leq x^*(y,z)$. So if we repeat the same construction with the new parameters, we get $\widetilde{\beta}_{k-1} - \widetilde{\beta}_k \leq \beta_{k-1} - \beta_k$ and $\widetilde{\beta}_k \geq \beta_k$ for each k. This means $\widetilde{\beta}_{M^*-1}$ is well-defined, and since $u(y,z)$ is single-crossing from above in both its arguments,

$$\begin{array}{rcl} u(\beta_{M^*-2} - \beta_{M^*-1}, 1 - \beta_{M^*-2}) & > & 0 \\ & \downarrow & \\ u(\widetilde{\beta}_{M^*-2} - \widetilde{\beta}_{M^*-1}, 1 - \widetilde{\beta}_{M^*-2}) & > & 0 \\ & \downarrow & \\ \widetilde{u}(\widetilde{\beta}_{M^*-2} - \widetilde{\beta}_{M^*-1}, 1 - \widetilde{\beta}_{M^*-2}) & > & 0 \end{array}$$

so $\widetilde{M}^* \geq M^*$, completing the proof.

A.4 Proof of Theorem 5

In the limit as $N \to \infty$, we know that α_1^N goes to x, where x solves $\frac{-x\ln(x)}{1-x} = \frac{2}{\xi+2}$ and $\xi = \frac{g(1)}{-f(0)}$, the payoff ratio of bidder "success" (a bidder with $s_i \approx 1$ entering and finding himself alone) to "failure" (a bidder with $s_i \approx 1$ entering and finding himself against an opponent with $s_j \approx 1$). Let $\overline{g} = -c + 1 + E(t_i)$.

Consider first a reserve price $r < 1 + \min \operatorname{supp}(t_i)$, which (in the limit, when only bidders with $s_i \approx 1$ enter) never prevents a sale and never binds when two bidders participate; it can therefore be thought of as a change to the outcome when one bidder enters, to give expected payoff $\overline{g} - r$ (rather than $\overline{g}$) to the bidder. We can think of this as reducing $g(1)$ and giving the seller the difference $\overline{g} - g(1)$; since this reduces $g(1)$, it reduces ξ, therefore increasing x. Instead of changing $g(1)$ and considering the effect on x, though, it is easier to work in reverse: we imagine setting a reserve price just high enough to have a particular effect on x, calculate the change in $\xi = \frac{g(1)}{-f(0)}$ that would allow, and then look at the effect of both those changes on revenue.

Solving $\frac{-x\ln(x)}{1-x} = \frac{2}{\xi+2}$ for ξ gives $\xi = -2\frac{1-x}{x\ln(x)} - 2$, so

$$\begin{aligned} d\xi &= \left[\tfrac{2}{x\ln(x)} + 2\tfrac{1-x}{(x\ln(x))^2}\left(\ln(x) + \tfrac{x}{x}\right)\right] dx \\ &= \left[\tfrac{2}{x\ln(x)} + 2\tfrac{1-x}{(x\ln(x))^2} + 2\tfrac{\ln(x)}{(x\ln(x))^2} + 2\tfrac{-x\ln(x)}{(x\ln(x))^2}\right] dx \\ &= 2\tfrac{1-x+\ln(x)}{(x\ln(x))^2} dx \end{aligned}$$

(Note that this is negative.) Next, we think about revenue. If nobody enters, revenue is 0; if one bidder enters with $s_i \approx 1$, as noted above, revenue is $\overline{g} - g(1)$. If two bidders enter with $s_i \approx s_j \approx 1$, revenue is $\overline{g} - f(0)$. (This is not obvious. If only one bidder enters an auction with no reserve price, his payoff is $\overline{g}$, as is total surplus. If a second bidder enters, he imposes no externality, so total surplus is $\overline{g} + f(0)$, but now both bidders are earning $f(0)$, so the payoff of the seller must be $\overline{g} + f(0) - 2f(0) = \overline{g} - f(0)$.) Since these events occur with probabilities x, $-x\ln(x)$, and $1 - x + x\ln(x)$, respectively, expected revenue is

$$\begin{aligned} \textit{revenue} &= -x\ln(x)(\overline{g} - g(1)) + (1 - x + x\ln(x))(\overline{g} - f(0)) \\ &= x\ln(x)(g(1) - f(0)) + (1-x)(\overline{g} - f(0)) \\ &= -f(0)\left[x\ln(x)(\xi+1) + (1-x)(\overline{\xi}+1)\right] \end{aligned}$$

where $\overline{\xi} = \frac{\overline{g}}{-f(0)}$. Differentiating, the incremental effect of increasing x (through a positive reserve price and the corresponding reduction in z) is

$$\begin{aligned} d(\textit{revenue}) &= -f(0)\left[(\ln(x) + \tfrac{x}{x})(\xi+1)dx + x\ln(x)d\xi - (\overline{\xi}+1)dx\right] \\ &= -f(0)\left[(\ln(x)+1)(\xi+1)dx - (\overline{\xi}+1)dx + x\ln(x)\cdot 2\tfrac{1-x+\ln(x)}{(x\ln(x))^2}dx\right] \\ &= -f(0)\left[\ln(x)(\xi+1) - (\overline{\xi} - \xi) + 2\tfrac{1-x+\ln(x)}{x\ln(x)}\right] dx \end{aligned}$$

Now, we do two substitutions. From $\frac{-x\ln(x)}{1-x} = \frac{2}{\xi+2}$, we get $\xi = 2\frac{1-x}{-x\ln(x)} - 2$, as well as $\frac{1-x}{x\ln(x)} = -\frac{\xi+2}{2}$, so

$$\begin{aligned} d(\textit{revenue}) &= -f(0)\left[\ln(x)\left(2\tfrac{1-x}{-x\ln(x)} - 2 + 1\right) - (\overline{\xi} - \xi) - 2\tfrac{\xi+2}{2} + 2\tfrac{\ln(x)}{x\ln(x)}\right] dx \\ &= -f(0)\left[-2\tfrac{1-x}{x} - \ln(x) - (\overline{\xi} - \xi) - \xi - 2 + \tfrac{2}{x}\right] dx \\ &= -f(0)\left[-(\overline{\xi} - \xi) - \ln(x) - \xi\right] dx \\ &= -f(0)\left[-\overline{\xi} - \ln(x)\right] dx \end{aligned}$$

Next, we will show that $\ln(x)+\xi>0$, so that when $\xi\leq\overline{\xi}$, the derivative above is negative. Now, $\xi+\ln(x)=2\frac{1-x}{-x\ln(x)}-2+\ln(x)$, but to show this is positive, it will be easier to substitute in $h=-\ln(x)$:

$$\xi+\ln(x) \;=\; 2\frac{1-e^{-h}}{he^{-h}}-2-h \;>\; 0 \;=\; \frac{2}{h}\left[e^h-1-h-\frac{1}{2}h^2\right]$$

since $h=-\ln(x)>0$ and $e^h=\sum_{j=0}^{\infty}\frac{h^j}{j!}>1+h+\frac{1}{2}h^2$. So $\xi+\ln(x)>0$, which means that when $\xi\leq\overline{\xi}$, $-(\overline{\xi}-\xi)-\xi-\ln(x)<0$, so revenue is strictly decreasing as x increases. This means that when reserve price is non-negative (so $g(1)\leq\overline{g}$ and therefore $\xi\leq\overline{\xi}$), an increase in reserve price decreases revenue.

(In the case of a reserve price above the minimum of the support of $1+t_i$, which therefore sometimes prevents trade (when either one or two bidders participate), the effect is even worse. The reserve price reduces $g(1)$ and $f(0)$, reducing ξ and therefore increasing x; but since total surplus is reduced, the seller does not capture all of the surplus diverted from the bidders. While the algebra is trickier (since $f(0)$ changes), the result is the same: a higher reserve price reduces revenue.)

We saw that $\xi+\ln(x)>0$ as long as $x<1$, and therefore revenue is *strictly* decreasing in reserve price at zero reserve ($\xi=\overline{\xi}$). By continuity, this means that for ξ slightly *above* $\overline{\xi}$, revenue is still increasing as reserve price falls – even after reserve price crosses 0 and becomes negative. This means that a commitment to make a small payment to the bidder in the event exactly one bidder opts in – provided the payment is sufficiently small – must *increase* expected revenue. While a payment only in the event that exactly one bidder opts in is not the usual way to implement a subsidy, a subsidy paid to all selected bidders, but calibrated to have the same expected value, would have the same effect on entry (as well as the same direct effect on the seller's profit), and would therefore be outcome-equivalent. So a sufficiently small subsidy paid to participating bidders would strictly increase revenue.

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