Bubble Trouble? Rational Storage, Mean Reversion and Runs in Commodity Prices.

Eugenio S. A. Bobenrieth[*] Juan R. A. Bobenrieth[†]
Brian D. Wright[‡]

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[*] Departamento de Economía Agraria, Instituto de Economía, and Finance UC, Pontificia Universidad Católica de Chile, ebobenrieth@uc.cl

[†] Departamento de Matemática, Universidad del Bío-Bío, jbobenri@ubiobio.cl

[‡] Department of Agricultural and Resource Economics, University of California, Berkeley, bwright@berkeley.edu

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Abstract

Recent volatility of prices of major commodities has generated a wide array of analyses and policy prescriptions that reflect a lack of consensus on the nature of the phenomenon and its implications for policy. A well-grounded annual model of a market for a storable staple product subject to random shocks to excess supply has been available since Gustafson (1958). The model has been considered incapable of explaining observed stochastic behavior of prices, including periods of large price "spikes" and "runs." Here we present an extension of the model in which price expectations are unbounded, and derive its implications for price time series and empirical tests of price behavior. In this model commodity value is equal to marginal consumption value, and hence the range of price behavior it can model rules out bubbles as defined in financial economics. We present versions of the model that exhibit behavior with episodes of price runs that could be characterized as "explosive" and might seem to be bubble-like. A given sample will yield returns consistent with mean reversion at sufficiently long holding periods, even though the long run expectation of price is infinite.

KEYWORDS: Bubbles, Mean Reversion, Storage.

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1 Introduction

Recent volatility of prices of major grains has generated a wide array of analyses and policy prescriptions that reflect a lack of consensus on the nature of the phenomenon and its implications for policy. An annual model of a market for a storable staple product subject to random shocks to excess supply has been available since Gustafson (1958), but though its basic logic, "buy low, sell high," is widely accepted as a wise aspiration, it has been considered incapable of explaining observed stochastic behavior of prices, including periods of large price "spikes" and "runs" that attract the concern of consumers and policy makers. There are two key reasons for the relative neglect of this model in analyses of recent commodity price behavior, each related to a perceived incompatibility of the implications of versions of the model with observed features of commodity prices.

One key reason is lack of empirical support for the model. It could not be seriously tested for more than three decades, due to lack of appropriate data and lack of a satisfactory estimation procedure. After a version of the Gustafson model was eventually tested (Deaton and Laroque 1992, 1995, 1996) via simulation of model solutions, and econometric estimation using the short available annual time series of global prices of a number of commodities, it was roundly rejected. Deaton (forthcoming, 2012) concluded that:

"We have a long-established theory - whose insights are deep enough that *some* part of them *must* be correct - which is wildly at odds with the evidence, and where it is far from obvious what is wrong [...]."

A second reason is the belief that if the storage model is specified to allow for bubble-like phenomena in which conditional price expectations can rise without bound, prices must behave in a very unrealistic fashion.

In the pioneering model of commodity price behavior with responsive supply of Scheinkman and Schechtman (1983, p.433), if price at zero harvest is infinite, and zero harvest has positive probability, then the conditional expectation of price is unbounded. They inferred that in this case "the model is exactly like an exhaustible resource model. Since stocks are always held, discounted price must exceed today's price by the marginal cost of storage. This seems very unrealistic behavior for the price of a producible commodity". Since continuously increasing price is something not observed in commodity markets (in contrast to price spikes, or price runs that eventually "crash"), their decision to restrict attention to models in which stocks carried to the next period are zero for available supply below some strictly positive level is understandable. In such models in the tradition of Gustafson (1958), (including Samuelson 1971, Gardner 1979, Newbery and Stiglitz 1981, and Wright and Williams 1982), the conditional expectation of price at far horizons is bounded at $(1 + r)p^*$, where p^* is the threshold price at which current carry-out stocks go to zero, and r is the one-year interest rate.

Several studies in finance have identified "mean reversion" variously defined, in asset prices, adding empirical support to the informal inference of Scheinkman and Schechtman (1983) that the standard model of storage must have positive probability of observation of current price exceeding the discounted value of price next period.

Recent tests have in some cases found price "exhuberance" in observations of aporadic runs of prices of securities rising faster than the rate of interest. Some authors (for example Phillips et al. 2011) have related such price behavior to former United States Federal Reserve Bank Chairman Greenspan's remark in December 1996 regarding "irrational exhuberance" of asset prices. This has led some researchers (for example Gilbert 2010) to look for similar behavior in commodity markets, though so far without very positive results.

After a brief review of the issues regarding consistency of the standard version of the model with observed time series of prices, we focus on the question of the capacity of the model to replicate stochastic bubble behavior. We draw on Scheinkman and Schechtman (1983) and Bobenrieth, Bobenrieth and Wright (2002) to derive new restrictions on price behavior, and simulate an example of price realizations in which the conditional expectation of price goes to infinity. We then establish some empirical implications for estimation of sample averages of returns from time series of prices, and relate these to findings of "mean reversion."

2 The Model

In this paper we use a stylized model of a market for a storable commodity such as a food grain with annual production, to reconsider the capacity of storage arbitrage to replicate key features of commodity price behavior identified in empirical studies.

We model a competitive market for a single storable consumption commodity such as a food grain in which time is discrete (annual) and all agents have rational expectations, in which the price process has an invariant distribution similar to that of Scheinkman and Schechtman (1983) and Bobenrieth, Bobenrieth and Wright (2002). The distribution of the harvest disturbance can have an atom at its minimum value, here normalized at zero, and price at zero consumption is infinite.

Production is subject to a common exogenous i.i.d. disturbance $\omega \in [0, \bar{\omega}]$, $0 < \bar{\omega} < \infty$. The distribution of ω is of the form $\alpha L_d + (1-\alpha) L_c$, where $\alpha \in [0,1]$, L_d is a discrete distribution with a unique atom at 0, and L_c is an absolutely continuous distribution, with continuous derivative when restricted to its support $[0, \bar{\omega}]$.

Assume that there is a continuum of identical producers, a continuum of identical storers, and a continuum of identical consumers; each of the three has total measure one. There is a one-period lag between the producers' choice of effort $\lambda \geq 0$ and output of the commodity $\omega' \lambda$, where ω' is next period's harvest shock. Cost of effort is given by a function $g : \mathbb{R}_+ \to \mathbb{R}_+$, with $g(0) = 0$, $g'(0) = 0$, and $g'(\lambda) > 0$, $g''(\lambda) > 0$ for all $\lambda > 0$. Storers can hold any non-negative amount of available supply from one year to the next, and then these stocks are all available for consumption or for further storage.

We replace the key assumption of Scheinkman and Schechtman (1983) that the physical storage cost function is strictly convex and its derivative appears additively in the Euler equation with the assumption that the physical storage cost function is zero; the sole cost of storage is the cost of capital invested. Given storage x and effort λ, the next period's total available supply is $z' \equiv x + w' \lambda$. Producers and storers are risk neutral and have a common constant discount factor $\delta \equiv 1/(1+r)$, where $r > 0$ is the discount rate.

The utility function of the representative consumer $U : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, once continuously differentiable, strictly increasing and strictly concave. It satisfies $U(0) = 0,\ U'(0) = \infty$. The inverse consumption demand curve, with zero income elasticity, is then $f = U'$.[1] We assume U has a finite upper bound, and thus total revenue $cf(c)$ is also bounded, and that the expectation of f with respect to L_c is finite.[2] The perfectly competitive market yields the same solution as the surplus maximization problem. The Bellman equation for the surplus problem is:

$$\nu(z) = \max_{x,\lambda}\{U(z-x) - g(\lambda) + \delta E[\nu(z')]\}, \qquad \text{subject to}$$

$$z' = x + \omega'\lambda,$$

$$x \geq 0, \quad z - x \geq 0, \quad \lambda \geq 0,$$

where $E[.]$ denotes the expectation with respect to next period's productivity shock ω'.

By standard results (see for example Stokey and Lucas with Prescott, 1989), ν is continuous, strictly increasing, strictly concave, and the optimal storage and effort functions $x(z)$ and $\lambda(z)$ are single valued and continuous.

Consumption and price are given by the functions $c(z) \equiv z - x(z),\ p(z) \equiv f(z - x(z))$.

The storage and effort functions x and λ satisfy the Euler conditions:

$$(1) \qquad f(z - x(z)) \geq \delta E[\nu'(x(z) + \omega'\lambda(z))], \qquad \text{with equality if } x(z) > 0,$$

$$(2) \qquad g'(\lambda(z)) \geq \delta E[\omega'\nu'(x(z) + \omega'\lambda(z))], \qquad \text{with equality if } \lambda(z) > 0,$$

and the envelope condition $\nu'(z) = f(z - x(z))$.

Given initial available supply $z > 0$, if the probability of zero productivity shock, α, is strictly positive, condition (1) implies that $z' > 0$ and $x(z') > 0$, and this arbitrage condition holds with equality in the current period and for the indefinite future. When positive, storage $x(z)$ is strictly increasing with z, and effort $\lambda(z)$ is decreasing with z. Note that $p(0) = f(0) = \infty$.

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[1] As discussed in footnote 1 of Scheinkman and Schechtman 1983, specification of a quasilinear utility function is one way to incorporate income, in the setting of general equilibrium models that generate the same set of equilibria as this partial equilibrium specification.

[2] This guarantees that for a model with harvest disturbances with distribution L_c there is a finite threshold price above which discretionary stocks are zero.

Define available supply at time t as z_t. Given arbitrary fixed $z_0 > 0$, the function that yields the supremum of the support of z_{t+1} is $\hat{z}(z_t) \equiv x(z_t) + \lambda(z_t)\bar{\omega}$. From the fact that there exists a unique fixed point z^* of $\hat{z}$ such that $\hat{z}(z) < z$ for all $z > z^*$, we conclude that $z_t \leq \bar{z} \equiv \max\{z_0, \max\{\hat{z}(z) : 0 \leq z \leq z^*\}\}$, for all $t \geq 0$. Then a suitable state space is $S \equiv [0, \bar{z}]$. Storage takes values in the set $[0, \bar{x}]$, where $\bar{x} \equiv x(\bar{z})$.

3 Empirical Relevance Revisited

3.1 Failure to match observed high price correlations

Recent empirical results (Cafiero et al. 2011) have shown, first, that a simple alternate specification version of the numerical model Deaton and Laroque (1992), parameterized to generate realistic levels of price variation, can generate the high levels of serial correlation observed in commodity prices. Second, they show further that the application of Deaton and Laroque's econometric approach, modified to improve its numerical accuracy, using their own data set, can yield empirical results that are consistent with observed levels of price variation and autocorrelation for seven major commodities. In a subsequent paper, Cafiero et al. (2012) derive maximum likelihood estimates that impose no more assumptions than the previous pseudo-maximum likelihood estimates, for the sugar global market, and obtain even better results.

Thus we are now in a position to consider the relevance of the Gustafson model for interpreting and testing recent claims regarding the behavior of commodity prices. In particular, we address in this paper claims that grain markets display "mean reversion," or that they have recently been disrupted by "bubbles," (Gilbert 2010) or by "exhuberant" behavior (Phillips et al. 2011), and by the popular notion that such claims can be resolved, at least in principle, from observed price behavior.

The model tested by Deaton and Laroque (1992, 1995, 1996) and Cafiero et al. (2011, 2012) assumes linear demand, with stocks that go to zero at a finite price. To address questions about mean reversion, speculative runs and related phenomena, we have adopted a demand specification that, if $\alpha > 0$, does not impose mean reversion at high prices, and allows for unbounded price expectations.[3] Thus our model is capable of producing behavior that includes conditional expectations of prices that go to infinity as the horizon recedes, as observed by Scheinkman and Schechtman (1983). But, is this extension of the model of any empirical relevance to actual price behavior in commodities such as grains?

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[3] As for the linear case, questions have been raised about the realism of the behavior of prices in that model.

3.2 Behavior of the model with unbounded conditional price expectations

Scheinkman and Schechtman (1983) stated that price behavior when $\alpha > 0$ and price at zero consumption is infinite is very unrealistic for a producible commodity, since it is "exactly like that of a natural resource model in which discounted price rises by the marginal cost of storage." Thus we start by considering a case of the model which is in fact a natural resource model.

3.2.1. The deterministic finite natural resource model

If $\alpha = 1$ then our model, which has no storage cost, is the deterministic Hotelling model of consumption of a finite resource with unbounded price. Standard results are that price rises monotonically at the rate of interest, so that discounted future prices equal the current price. Such price behavior is indeed inconsistent with actual stochastic evolution of prices for commodities such as food grains. Do prices in the model with $0 < \alpha < 1$ behave exactly like this?

3.2.2. The stochastic model with unbounded price expectations

Intertemporal storage arbitrage implies that, in the model with $0 < \alpha < 1$, $\{\delta^t p_{m+t}\}_{t \geq 0}$ is a martingale and $\{\, E_m[\delta^t p_{m+t}] \ : \ t \geq 0\} = p_m$, where $E_m[.]$ denotes the expectation conditional on the price at time m, p_m. Indeed the conditional expectation of price behaves exactly as the price in the deterministic natural resources model discussed above. But in this stochastic model the price path does not follow its expectation, contrary to the inference of Scheinkman and Schechtman (1983). Nor does the statement of Bessembinder et al. (1995 p. 362) that the path of conditional expectations at different horizons "describes several points on the path that investors expect the spot price will take" hold for this model. To the contrary, as the horizon recedes, the path of realized prices eventually drifts down and away from the rising profile of conditional expectations, any fraction of which becomes an upper bound on that realized path.

The sequence of probability measures of prices conditional on any initial price p_m converges to a unique invariant measure, uniformly in p_m, and consequently the sequence of discounted prices converges in probability to zero, uniformly in p_m.[4]

More precisely:

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[4] If $0 \leq \alpha < 1$, the sequence of probability measures of z_t, $\{\gamma_t\}_{t=0}^{\infty}$, converges in the total variation norm to a unique invariant probability γ_*, regardless of the value of z_0. The idea of the proof for the case $0 \leq \alpha < 1$ can be found in Bobenrieth, Bobenrieth and Wright (2002). It follows immediately that the sequence of probability measures of prices $\{\gamma_t c^{-1} f^{-1}\}_{t=0}^{\infty}$ converges in the total variation norm to the unique invariant probability measure $\gamma_* c^{-1} f^{-1}$. Note that $\text{Prob}[p_t \geq y] = (\gamma_t c^{-1} f^{-1})([y, \infty])$, where $p_t = f(c(z_t))$ is the price at time t. $H_t(y) \equiv \text{Prob}[p_t \geq y]$ converges uniformly to a unique invariant upper c.d.f. H_*, with $\lim_{p \to \infty} H_*(p) = 0$. If $0 \leq \alpha < 1$, then the support of the invariant distribution of prices is an interval $[\underline{p}, \infty]$ with $0 < \underline{p} < \infty$.

Theorem 1: *Let $\alpha < 1$. Given $\beta > 0$ and $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that for any price realization p_m,*

$$\textit{Prob}\left[\delta^t p_{m+t} < \beta \mid p_m\right] \;\geq\; 1 - \varepsilon, \quad \forall\, t \geq T.$$

Theorem 1 implies that for any sample size $N \in \mathbb{N}$, given any finite sequence of realized initial prices $\{p_m, p_{m+1}, \cdots, p_{m+N-1}\}$, we have the following bound on the joint probability of the gross discounted relative price changes from each initial price in the sample, beyond a finite T', where T' is independent of the finite sequence of initial price realizations:

$$\text{Prob}\left[\frac{\delta^t p_{m+t}}{p_m} < \beta,\ \frac{\delta^t p_{m+1+t}}{p_{m+1}} < \beta,\ \cdots,\ \frac{\delta^t p_{m+N-1+t}}{p_{m+N-1}} < \beta \;\mid\; p_{m+N-1}\right] \;\geq\; 1 - \varepsilon,$$

for all $t \geq T'$.[5]

The existence of a unique invariant distribution which is a global attractor implies for this price process that, with probability one, the sequence of price realizations is dense on the support $[\underline{p}, \infty]$ of the invariant distribution. The infinite sequence of price realizations visits every neighborhood of every price in the support, no matter how high, infinitely often, almost surely. Given this fact, the following proposition regarding discounted prices might not be surprising:

Proposition 1: *Let $\alpha < 1$. For any given price realization p_m, for arbitrary positive real number D, there exists a horizon $d \in \mathbb{N}$, such that:*

$$\textit{Prob}[\delta^t p_{m+t} > D \mid p_m] > 0, \quad \forall\, t \geq d.$$

For the case $0 < \alpha < 1$, the maximum of the support of the conditional distribution of discounted price goes to infinity as the horizon increases, in contrast to the case for the standard Gustafson model with bounded price, where the maximum goes to zero. To prove Proposition 1, we need Proposition 2, which might seem counter-intuitive given Proposition 1.

For the discussion that follows, given a price realization p_m, let $E_m[.]$ denote the expectation conditional on p_m.

Proposition 2: *Let $\alpha < 1$. Given any price realization p_m, the sequence of discounted prices, $\{\delta^t p_{m+t}\}_{t \geq 0}$, goes to zero, almost surely (as $t \to \infty$).*

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[5] In the proof, presented in the Appendix, we use the facts that the Markov operator is stable and quasicompact, and that given any initial price, any neighborhood of infinity, and any integer k, the price process visits that neighborhood in a time that is some multiple of k, with positive probability.

Proof of Proposition 2: The Euler condition for storage arbitrage (1) implies that, if $\alpha > 0$, $\{\delta^t p_{m+t}\}_{t\geq 0}$ is a martingale and $\sup\{\, E_m[\delta^t p_{m+t}] \ : \ t \geq 0\,\} = p_m < \infty$. In the case $\alpha = 0$, $\{\delta^t p_{m+t}\}_{t\geq 0}$ is a supermartingale and $\sup\{\, E_m[\delta^t p_{m+t}] \ : \ t \geq 0\,\} = p^* < \infty$. In both cases, by the Martingale Convergence Theorem (due to Doob) we conclude that $\delta^t p_{m+t} \to Y$ a.s. (as $t \to \infty$), where Y is a real random variable. By Theorem 1, $\delta^t p_{m+t} \to 0$ in probability (as $t \to \infty$), and hence $Y = 0$ almost surely. *Q.E.D.*

Proof of Proposition 1: For the nontrivial case $0 < \alpha < 1$, we prove the result by contradiction. If not, there exist a price realization p_m, a real number $D > 0$ and a sequence of natural numbers $\{t_k\}_{k\in\mathbb{N}} \uparrow \infty$ with $\text{Prob}[\delta^{t_k} p_{m+t_k} > D \mid p_m] = 0$, for all t_k. Therefore $\delta^{t_k} p_{m+t_k} \leq D$ a.s., for all t_k. Then the Lebesgue dominated convergence theorem and the fact that $\lim_{t_k\to\infty} \delta^{t_k} p_{m+t_k} = 0$ a.s. imply that $\lim_{t_k\to\infty} E_m[\delta^{t_k} p_{m+t_k}] = 0$, a contradiction to $E_m[\delta^{t_k} p_{m+t_k}] = p_m > 0$, for all t_k. *Q.E.D.*

If $0 < \alpha < 1$, we have that $E_m[\delta^t p_{m+t}] = p_m, \ \forall\, t \geq 0$. Nevertheless, Proposition 2 states $\{\delta^t p_{m+t}\}_{t\geq 0}$, converges to zero almost surely, implying that $\{E_m[\delta^t p_{m+t}]\}_{t\geq 0}$ does not converge to the expectation of the almost sure limit of $\{\delta^t p_{m+t}\}_{t\geq 0}$. As a consequence, the sequence of discounted prices is not uniformly integrable.

Proposition 2 is easy to understand in a model with $\alpha = 0$, but if $0 < \alpha < 1$, how can the discounted price be going to zero, almost surely, if there is positive probability that discounted price exceeds D at any sufficiently far horizon? The explanation hinges on the distinction drawn above between a profile of expectations conditional on a price realization and the path of realizations. By Proposition 2, with probability one, for any given path of discounted price realizations there is a time beyond which that path is permanently below D. But by Proposition 1, there is no finite horizon beyond which all paths possible from date m are below D. In fact, at any finite horizon, with positive probability price rises at a rate greater than the discount rate r, continuously within that horizon. Although any path of discounted price realizations eventually remains permanently below D, before it does so, it can exceed any given arbitrary high finite bound. It is recognition of such a possibility that keeps $E_m[\delta^t p_{m+t}]$ equal to p_m as the horizon, and the probability that the discounted price will be below D at that horizon, both increase.

Proposition 2 implies that, given a price realization p_m, the sample mean and sample variance of a discounted price sequence go to zero almost surely, that is:

$$N^{-1}\sum_{t=0}^{N-1} \delta^t p_{m+t} \to 0 \quad \text{a.s.} \qquad (\text{as } \ N \to \infty)\ , \qquad \text{and}$$

$$N^{-1}\sum_{t=0}^{N-1}\left[\delta^t p_{m+t} - N^{-1}\sum_{j=0}^{N-1}\delta^j p_{m+j}\right]^2 \to 0 \quad \text{a.s.} \qquad (\text{as } \ N \to \infty).$$

Thus the estimators are consistent with respect to the first two moments of the limiting distribution of discounted price. For the case $0 < \alpha < 1$, the sample average of discounted price realizations starting at any price realization p_m, is eventually permanently below any arbitrary positive fraction of the profile of expectations, conditional on p_m, of discounted price. Nevertheless the variance of the distribution of discounted price, conditional on p_m, goes to infinity as $t \to \infty$.

The behavior of the price path is related to the profile of conditional expectations at time m by the following theorem:

THEOREM 2: *Let* $0 < \alpha < 1$. *Given any price realization* p_m, *with probability one, for any* $1 \leq l < \infty$, *there exists a finite time* $\tau(l)$, *which depends on the sequence of price realizations, such that:*

$$\frac{E_m[p_{m+t}]}{l} > p_{m+t}, \quad \forall\ t \geq \tau(l),$$

implying that

$$p_{m+t} = o(E_m[p_{m+t}]), \quad a.s.$$

PROOF OF THEOREM 2: By Proposition 2, $\delta^t p_{m+t} \to 0$ (as $t \to \infty$), with probability one. Therefore, given any l, $1 \leq l < \infty$, there exists a time $\tau(l)$ that satisfies $\delta^t p_{m+t} \cdot l < p_m = \delta^t E_m[p_{m+t}], \quad \forall\ t \geq \tau(l)$. *Q.E.D.*

Theorem 2 defines a sequence of upper bounds on the path of price realizations. Note that the profile of conditional expectations $E_m[p_{m+t}]$ is itself an upper bound beyond some date $\tau(1)$. Any given fraction of the profile of expectations conditional on initial price is an upper bound on any price realized beyond some fixed horizon, with probability one.

4 Price Behavior in this Model: Do We See Bubbles?

The behavior of price expectations and realizations in the model is illustrated in the example in Figure 1. The profile of conditional expectations, $E_m[p_{m+t}]$, rises to infinity at the discount rate. A possible sequence of price realizations is illustrated as a series of dots beginning at p_m.[6] After period 23, all the realizations of price lie below $E_m[p_{m+t}]$. The curve $E_m[p_{m+t}]/2$ shows another bound at half the price expectations is effective beginning at date 39. It is obvious that further bounds generated by successively higher values of l would imply that the long-run rate of increase of realized price is strictly lower than the discount rate, 4%, even though the storage arbitrage condition (1) holds, with equality, each period, and that price runs of any finite length, understood as sequences of prices rising faster than the interest

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[6] Bobenrieth, Bobenrieth and Wright (2008) offers a foundation for a strategy for numerical solution of marginal values in cases where they are unbounded.

rate, recur infinitely often along the path of realizations, almost surely. Figure 2 shows the logarithms of the same price series, dramatizing the runs of price increases greater than the rate of interest.

These figures show that runs of prices rising for several years at a rate greater than the rate of interest before crashing, denoted "explosive" by Phillips et al. (2011), and and fulfilling the the empirical ex post criterion for identification of bubbles in grain prices enunciated by Timmer (2008), are consistent with our equilibrium model with rational expectations. In this model, they do not signify the disruptive effects of irrational speculation, but rather the dampening effect of storage that prevents sharper price jumps, but with declining effectiveness if low harvests persist.

Discussions of volatile grain price behavior often raise the issue of price bubbles, frequently without defining the term. Brunnermeier (2008) includes a key feature of most definitions of finance economists when he states that "Bubbles refer to asset prices that exceed an asset's fundamental value because current owners believe they can resell the asset at an even higher price." Are price runs characteristic of price behavior in our model, as illustrated in Figures 1 and 2, consistent with this definition? In our model we have assumed no convenience yield, and the law of one price holds. In that setting, when storage is positive, the value of a commodity such as food grain equals its value in consumption, as indicated in the envelope condition after equations (1) and (2) above. Storage is a one period investment, so its "fundamental" is the gross annual return, which derives its marginal value from its marginal value in consumption. Thus in our model bubbles consistent with Brunnermeier's definition cannot occur. In this model, they do not signify the disruptive effects of rational or irrational speculation, but rather the dampening effect of storage that prevents sharper price jumps, but with declining effectiveness, during episodes of repeated low harvests.

5 Some Empirical Implications

Implications of the model for the empirical behavior of sample averages of returns on the stocks, held over specific intervals, are summarized in the following theorem:

THEOREM 3: *Let* $0 < \alpha < 1$. *With probability one, for any given path of price realizations* $\{p_t\}_{t\geq 0}$, *for any* $n \in \mathbb{N}$ *and for any* $\beta > 0$, *there exist* $J = J(\{p_t\}_{t\geq 0}, n, \beta) \in \mathbb{N}$, $k = k(\{p_t\}_{t\geq 0}, J, \beta) \in \mathbb{N}$, $k > n$, *and* $K = K(\{p_t\}_{t\geq 0}, k, \beta) \in \mathbb{N}$, $K > J$, *such that:*

$$\text{(i)}\quad J^{-1}\sum_{t=0}^{J-1}\left[\frac{\delta^n p_{t+n} - p_t}{p_t}\right] \in (-\beta,\, \beta),$$

$$\text{(ii)}\quad J^{-1}\sum_{t=0}^{J-1}\left[\frac{\delta^k p_{t+k} - p_t}{p_t}\right] \in (-1,\, -1+\beta), \qquad \textit{and}$$

(iii) $K^{-1}\sum_{t=0}^{K-1}\left[\frac{\delta^k p_{t+k}-p_t}{p_t}\right] \in (-\beta,\, \beta).$

PROOF OF THEOREM 3: For $j \in \mathbb{N}$ and for $t \in \mathbb{N}\cup\{0\}$, let $Y_{t+j} \equiv \frac{\delta^j p_{t+j}-p_t}{p_t}$. The arbitrage equation for storage (1) implies that there exists $\bar{p} \geq p(x^j(z_t))$, $\bar{p}$ depends on z_t, such that $\delta^j\alpha_1^j\bar{p} = p_t$, where α_1 is the size of the atom at zero of the distribution of ω, and $x^j \equiv x \circ x \circ \cdots \circ x$ (j times). Therefore,

$$-1 \leq Y_{t+j} \leq \delta^j\frac{\bar{p}}{p_t} = \frac{1}{\alpha_1^j}.$$

The arbitrage equation (1) also implies $E_t[Y_{t+j}] = 0$. It follows that the sequence $\{X_t\}_{t\geq 0}$, where $X_t \equiv Y_{t+j}$, is uniformly bounded, and $\sum_{i=1}^{\infty}\sup_t|\mathrm{Cov}(X_t,\, X_{t-i})| < \infty$. A strong law of large numbers (see Davidson 1994, p.297) implies that

$$\lim_{N\to\infty} N^{-1}\sum_{t=0}^{N-1}\left[\frac{\delta^j p_{t+j}-p_t}{p_t}\right] = 0, \quad \text{a.s.} \tag{2}$$

Evaluating (2) for $j = n$ we conclude that there exists $J \in \mathbb{N}$ such that (i) holds. For this J, by Proposition 2,

$$\lim_{k\to\infty} J^{-1}\sum_{t=0}^{J-1}\left[\frac{\delta^k p_{t+k}-p_t}{p_t}\right] = -1, \quad \text{a.s.,}$$

establishing (ii) for large enough k. Finally, evaluating (2) for $j = k$ we obtain $K,\ K > J$, satisfying (iii). *Q.E.D.*

Expression (i) of Theorem 3 shows that the average excess rate of return on stocks held over n periods is greater than a given, arbitrary $-\beta$, for a sufficiently large sample size J, as implied by a strong law of large numbers.[7] Expression (ii) states that, with the same sample of initial holding dates, if we increase the holding interval sufficiently, to k periods, (and increase the sample size by $k-n$ periods to accommodate the extended lead), the average gross discounted return is within an arbitrary β of a total loss. At this sample size, the sample average (ii) could be considered a downward-biased estimator of the expected $k-$period rate of increase in price, which in this model is constant. Thus for any sample of prices of any given length, one can find a sufficiently far horizon such that the estimated average return

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[7] A similar result is confirmed (on a very different time scale) for daily returns for wheats on the Kansas City and Minneapolis grain exchanges in Bobenrieth (1996).

can be taken to imply "mean reversion," as defined for example in Bessembinder et al. (1995) even if the behavior of prices does not exhibit mean reversion, as in the stationary model considered here. Expression (*iii*) reflects the fact that the sample average for the longer holding period approaches the conditional expectation for that horizon, when the sample size is sufficiently increased.

Comparison of results (*i*) through (*iii*) has another interpretation, more relevant for estimation of the long-run return on storage from any given time zero. As the horizon is increased, the discounted present value of price realizations conditional on any price p_t in the sample of size J in (*i*) eventually converges, along the path of realizations, to a neighborhood of zero in finite time, as stated in Proposition 2. From this point of view, comparison of (*ii*) with (*i*) reflects the convergence of the gross discounted value to its almost sure limit of a one hundred percent loss over the holding period, as the latter goes to infinity. But (*iii*) shows an increase in the average excess rate of return back to an arbitrary neighborhood of the conditional expectation of zero when sufficient observations are added to include some that have high rates of price increase through the fixed horizon. Note that (*iii*) does not imply that an initial investment at time zero can be restored to profitability if held for a sufficiently long time.

6 Conclusions

Recent high volatility of prices of major commodities has generated a wide array of analyses and policy prescriptions reflecting the continued failure of economists to approach a consensus on the nature of the phenomenon and its implications for policy. The remarkable work of Gustafson (1958) introduced a market model that derived numerically the storage demand consistent with given consumer demand, yield distribution, cost of storage and interest rate, given maximization of expected profits with an infinite horizon. The standard annual model shows why prices tend to be skewed, and why they do not closely reflect production shocks. Recent empirical results confirm that it can, contrary to previous claims, also match the high price correlations seen annual prices of major commodities. However it assumes bounded price expectations, and thus cannot address the behavior of prices if expectations are unbounded, as in some models of speculative behavior.

Here we have presented an extension of the model in which price expectations are unbounded, and derived its implications for price time series and empirical tests of price behavior. In this model commodity value is equal to marginal consumption value, and hence the range of price behavior it can model rules out bubbles as defined in financial economics (for example Brunnermeier 2008). We present versions of the model that exhibit behavior with episodes of price runs that could be characterized as "explosive" (see Gilbert 2010) and might seem to be bubble-like. Time series of prices with conditional expectations that go to infinity are consistent with stationary behavior indistinguishable from that produced by a version of the standard model with bounded conditional price expectations.

The stationary price process that we have examined reveals the importance of distinguishing any given profile of conditional price expectations from the path of price realizations. The rate of increase of any profile of conditional price expectations in our model is constant at the discount rate, while the realized price at a sufficiently far horizon is bounded below any given positive fraction of the profile of expectations conditional on the current price. A given sample will yield returns consistent with mean reversion at sufficiently long holding periods, even though the long run expectation of price is infinite.

REFERENCES

Bessembinder, H., J. F. Coughenour, P. J. Seguin and M. Smaller (1995): "Mean Reversion in Equilibrium Asset Prices," *The Journal of Finance,* 50(1), 361-375.

Bobenrieth, E. S. A. (1996): "Commodity Prices Under Time-Heterogeneous Shocks Density," Ph.D. Thesis.University of California at Berkeley.

Bobenrieth, E. S. A., J. R. A. Bobenrieth, and B. D. Wright (2002): "A Commodity Price Process With a Unique Continuous Invariant Distribution Having Infinite Mean," *Econometrica,* 70, 1213-1219.

——— (2008): "A Foundation for the Solution of Consumption-Saving Behavior with a Borrowing Constraint and Unbounded Marginal Utility," *Journal of Economic Dynamics & Control,* 32, 695-708.

Brunnermeier, M. K. (2008): *"Bubbles."* New Palgrave Dictionary of Economics, Second Edition.

Cafiero, C., E. S. A. Bobenrieth, J. R. A. Bobenrieth, and B. D. Wright (2011): "The Empirical Relevance of the Competitive Storage Model," *Journal of Econometrics,* 162, 44-54.

——— (2012): "Can Simple Storage Arbitrage Explain Commodity Price Dynamics? Evidence from Sugar Prices," Working Paper FAO, Pontificia Universidad Católica de Chile, Universidad del Bío-Bío, and University of California at Berkeley.

Davidson, J. (1994): *Stochastic Limit Theory.* Oxford University Press.

Deaton, A. (2012): *"Puzzles and paradoxes: a life in applied economics."* in Michael Szenberg and Lall Ramrattan, eds., Eminent Economists II, Cambridge University Press.

Deaton, A., and G. Laroque (1992): "On the Behaviour of Commodity Prices," *Review of Economic Studies,* 59, 1-23.

——— (1995): "Estimating a Nonlinear Rational Expectations Commodity Price Model with Unobservable State Variables," *Journal of Applied Econometrics,* 10, S9-S40.

——— (1996): "Competitive Storage and Commodity Price Dynamics," *Journal of Political Economy,* 104, 896-923.

FUTIA, C. A. (1982): "Invariant Distributions and Limiting Behavior of Markovian Economic Models," *Econometrica,* 50, 377-408.

GARDNER, B. L. (1979): *Optimal Stockpiling of Grain.* Lexington Books, Lexington, Mass.

GILBERT, C. L. (2010): "Speculative Influences on Commodity Futures Prices 2006-2008," UNCTAD Discussion paper.

GUSTAFSON, R. L. (1958): "Carryover Levels for Grains," USDA Technical Bulletin 1178, Washington, DC: United States Department of Agriculture.

NEWBERY, D. M., AND J. E. STIGLITZ (1981): *The Theory of Commodity Price Stabilization: A Study in the Economics of Risk.* New York : Oxford university Press.

PHILLIPS, P. C. B., Y. WU, AND J. YU (2011): "Explosive Behavior in the 1990s NASDAQ: When Did Exuberance Escalate Asset Values?," *International Economic Review,* 162, 44-54.

SAMUELSON, P. (1971): "Stochastic Speculative Price," *Proceedings of the National Academy of Sciences,* 68, 335-337.

SCHEINKMAN, J. A., AND J. SCHECHTMAN (1983): "A Simple Competitive Model with Production and Storage," *Review of Economic Studies,* 50, 427-441.

STOKEY, N. L., R. E. LUCAS, JR., WITH E. C. PRESCOTT (1989): *Recursive Methods in Economic Dynamics.* Cambridge, MA: Harvard University Press.

TIMMER, C. P. (2008): "Causes of High Food Prices." Asian Development Outlook Update.

WRIGHT, B. D., AND J. C. WILLIAMS (1982): "The Economic Role of Commodity Storage," *The Economic Journal,* 92, 596-614.

YOSIDA, K., AND S. KAKUTANI (1941): "Operator-Theoretical Treatment of Markoff's Process and Mean Ergodic Theorem," *Annals of Mathematics,* 42, 188-228.

APPENDIX. PROOF OF THEOREM 1: Consider the probability of the complement,

$$\text{Prob}\left[\delta^t p_{m+t} \geq \beta \mid p_m\right] = \text{Prob}\left[p_{m+t} \geq \frac{\beta}{\delta^t} \mid p_m\right] = \mu_t\left(\left[\frac{\beta}{\delta^t}, \infty\right]\right),$$

where μ_t is the probability measure of the price at time $m+t$, conditional on p_m. Furthermore,

$$\mu_t\left(\left[\frac{\beta}{\delta^t}, \infty\right]\right) \;\leq\; |\mu_t - \mu_*| \;+\; \mu_*\left(\left[\frac{\beta}{\delta^t}, \infty\right]\right),$$

where μ_* is the invariant probability measure of the price process and $|\cdot|$ denotes the total variation norm.

The transition probability of the price process satisfies, with respect to the point ∞, what is called in Futia a Generalized Uniqueness Criterion (Futia, 1982, p.390). In addition, the corresponding Markov operator L is stable and quasicompact (Theorems 4.6 and 4.10 in Futia, 1982, p.394 and p. 397). Using Theorem 3.6 in Futia (1982, p.390), and Theorem 4 in Yosida and Kakutani (1941, p.200), we obtain the following conclusion : independent of p_m, there exist constants $M > 0, \eta > 0$, such that :

$$||(L^*)^t \;-\; L_1^*|| \;\leq\; \frac{M}{(1+\eta)^t} \qquad \forall\; t \in \mathbb{N},$$

where L^* is the adjoint of the Markov operator L, L_1^* is a continuous linear operator, the image of which consists precisely of the fixed points of L^*, and $||\cdot||$ is the operator norm. Therefore, if δ_{p_m} denotes the unit point mass at p_m, then :

$$|\mu_t - \mu_*| \;=\; |(L^*)^t(\delta_{p_m}) - L_1^*(\delta_{p_m})| \;\leq\; ||(L^*)^t \;-\; L_1^*|| \;\leq\; \frac{M}{(1+\eta)^t} \quad \forall\; t \in \mathbb{N}.$$

Finally, since μ_* has no atom at infinity, we have that $\lim_{t\to\infty} \mu_*\left(\left[\frac{\beta}{\delta^t}, \infty\right]\right) \;=\; 0.$

Q.E.D.

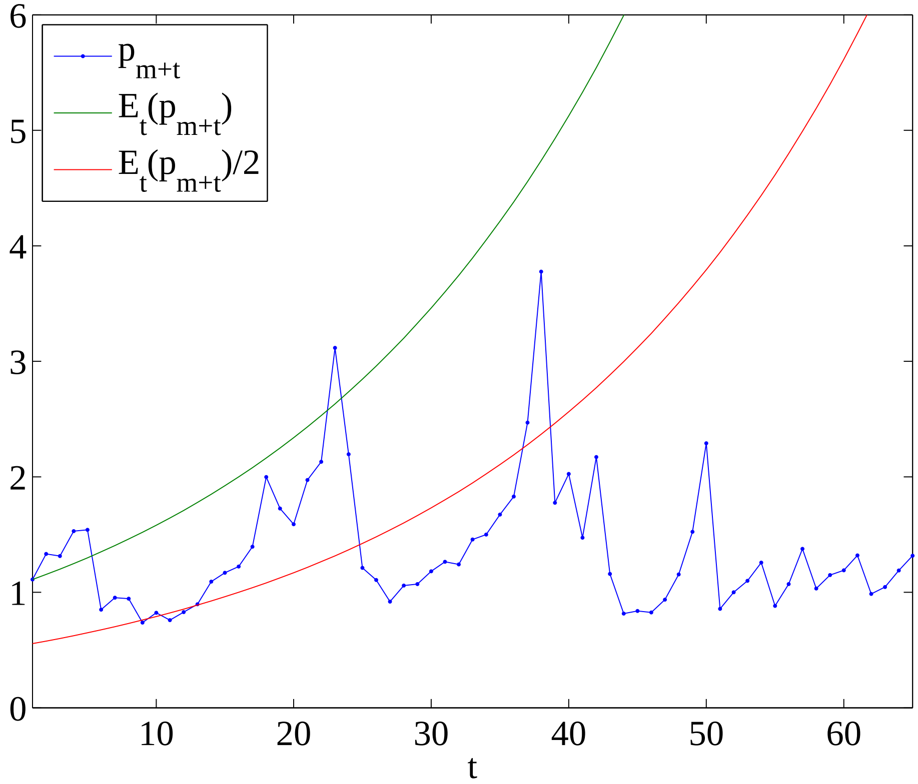

FIGURE 1.

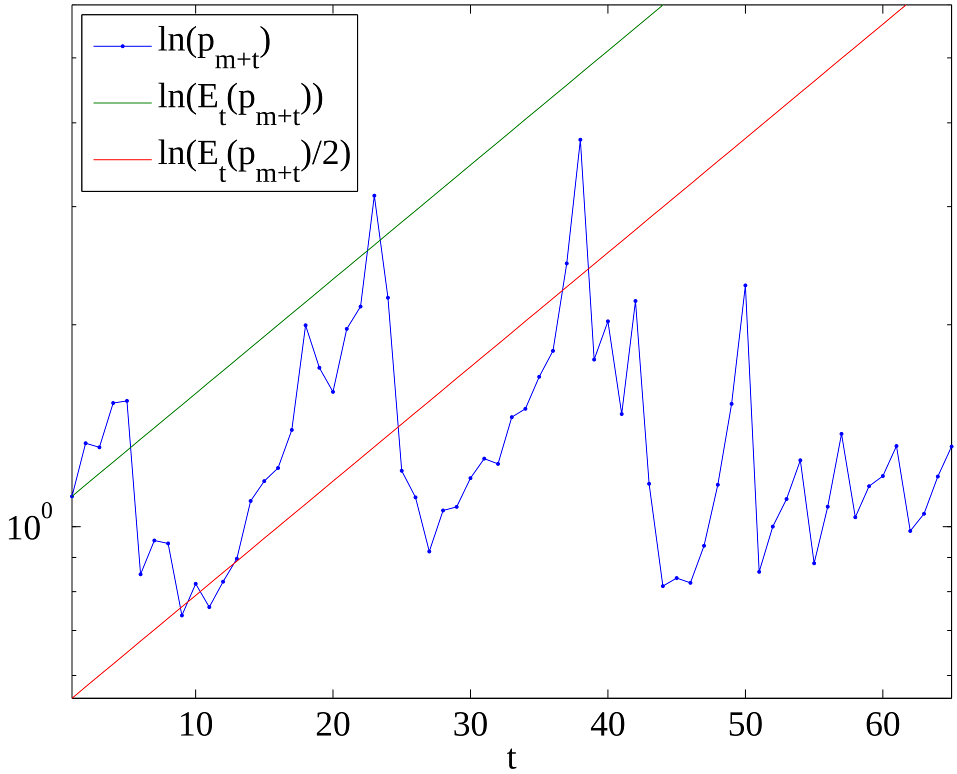

Figure 2.