Housing and Liquidity*

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July 15, 2012

Abstract
Housing, in addition to providing direct utility, facilitates credit transactions when home equity serves as collateral. We document big increases in home-equity loans coinciding with the start of the house-price boom, and suggest an explanation. When it is used as collateral, housing can bear a liquidity premium. Since liquidity is endogenous, even when fundamentals are deterministic and time invariant equilibrium house prices can display complicated patterns – including cyclic, chaotic and stochastic trajectories – some of which resemble bubbles. Our framework is tractable, with exogenous or with endogenous supply, and with exogenous or endogenous credit limits. Yet it captures several salient qualitative features of actual housing markets. Numerical work shows the model can also capture some, if not all, quantitative features, as well. The effects of monetary policy are also discussed.

JEL Classification: E44, G21, R21, R31
Keywords: Housing, Liquidity, Collateral, Bubbles

*All authors are affiliated with the Department of Economics at the University of Wisconsin-Madison. We thank many friends and colleagues for input, especially Charles Engle, Guillaume Rocheteau, Chao Gu, Derek Stacey, Dean Corbae and Gadi Barley. Wright acknowledges support from the NSF and the Ray Zemon Chair in Liquid Assets at the Wisconsin School of Business. The usual disclaimer applies.
1 Introduction

We study economies in which housing plays two roles. First, it provides direct utility. Second, houses are assets that can facilitate transactions when credit markets are imperfect: in the presence of limited commitment, it can be difficult for consumers to get unsecured loans, and this generates a role for home equity as collateral. This implies the equilibrium house price can bear a liquidity premium, as people are willing to pay more than the fundamental value (defined below), because home ownership provides security in the event one needs a loan. Since liquidity is at least partly endogenous, even when fundamentals are deterministic and time invariant equilibrium house prices can display complicated patterns – including cyclic, chaotic and stochastic trajectories – some of which resemble bubbles. Intuitively, prices of liquid assets are to some extent a matter of beliefs, and in a sense one might say housing has a certain moneyness, in that it ameliorates trading frictions.1

Our goal is to make these ideas precise and study their implications. This seems interesting primarily because it is consistent with experience since the turn of the millennium. It is commonly heard that there was a bubble in house prices during this period, which eventually burst, leading to all kinds of economic problems. It has also been noted that there was over the period a big increase in home equity loans. Reinhart and Rogoff (2009) contend financial developments2 allowed consumers “to turn their previously illiquid housing assets into ATM machines.” Ferguson (2008) also says this “allowed borrowers to treat their homes as cash machines,” and reports that between 1997 and 2006, US consumers withdrew an estimated $9 trillion from home equity. Greenspan and Kennedy (2007) find

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1 Yet houses are also very different from money, as we usually use the term, in ways that we discuss in detail below. By way of preview, on the supply side, in contrast to currency houses are produced by profit-maximizing agents. On the demand side, houses generate direct utility, which has several implications – e.g., there are no equilibria where the price is 0, making it harder to construct interesting dynamics; and agents with no credit limit typically do not divest themselves of housing, as they do with currency.

2 The key development they have in mind is securitization. As Holmstrom and Tirole (2011) say, “In the runup to the subprime crisis, securitization of mortgages played a major role ... by making [previously] nontradable mortgages tradable, led to a dramatic growth in the US volume of mortgages, home equity loans, and mortgage-backed securities in 2000 to 2008, partly in response to increased global demand for savings instruments.”
home equity withdrawal financed about 3% of personal consumption between 2001 and 2005, while Disney and Gathergood (2011) find a fifth of the recent growth in household debt is explained by house prices. Mian and Sufi (2011) estimate homeowners extracted 25 cents for every dollar increase in home equity, adding $1.25 trillion to household debt between 2002 and 2008. They also report that the loans were used for consumption, rather than, e.g., paying off credit card debt or buying financial assets, and were used more those who are younger and have lower credit scores, all consistent with our theory.

Figure 1 shows data for the US over the relevant period (exact data definitions and sources are given below; all figures are at the end of the paper). First, there are house prices, deflated in two ways. One divides by the CPI to correct for the purely nominal effect of inflation. The other divides by an index of rental rates, to correct for inflation plus changes in the demand for shelter relative to other goods and services. These data illustrate what people mean by a housing bubble: a dramatic run up followed by collapse in prices. We also show a measure of housing investment, normalized by GDP, to get an idea of what was happening to supply. And we show home equity loans, this time normalized three ways. The first again uses the CPI to control for purely nominal effects. The second divides by nominal GDP, to show an increase in home equity loans relative to general economic growth. The third divides by a measure of home equity, to make it clear that loans as a fraction of collateral increased, consistent with the above-mentioned financial innovation. While the exact number depends on which of the three series one uses, home equity loans increased a lot, from a stable value normalized to 0.3 in the 1990s, to somewhere between 0.7 and 1.0 at their peak.

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Definitions of the series in Figure 1 are as follows: Home Prices uses the FHFA Purchase Only price index. To turn it into a real variable, we divide by the CPI, or by the Rent index from BLS, with the real series normalized to 1 in 1993 so better fit on the chart. Loan data are from the Federal Reserve Flow of Funds Accounts. Home Equity Loans (HEL) are divided by CPI, by nominal GDP, and by Home Equity with the resulting series all normalized to 0.3 in 1993. Home Equity data is obtained by first subtracting home equity loans from mortgage loans to get closed-end mortgages, and then subtracting that from the Market Value of Homes, which includes the value of land, as constructed by Davis and Heathcote (2007). The Residential Fixed Investment data is from BEA, divided by nominal GDP, normalized to 0.5 in 1993.
The message to take away from this is the following: coinciding with the start of the house-price boom, there began a sizable increase in the real value of home equity loans, and an increase in housing investment; later prices drop fast, and investment falls, while home equity borrowing stays fairly high, at least for a while. This suggests to us that home equity lending is potentially relevant for understanding the episode. If one considers a house only as a consumer durable, with its value determined by its utility, the rent-price ratio should be roughly the sum of the discount and depreciation rates. There can be other costs and benefits of owning, including tax implications, but while these may affect the level of the rent-price ratio, as long as they are approximately constant, this should not generate the time series in Figure 1.4 Our position is that financial developments led to a bigger role for home equity in the credit market, this fueled an increase in the housing demand, and that led to an increase in price in the short run and quantity in the long run. Once one takes into account the liquidity role of home equity, it is possible to generate equilibria qualitatively, and to some extent quantitatively, consistent with experience.

Many people talk about bubbles, but it is not always obvious what they mean. As Case and Shiller (2003) say, “The term ‘bubble’ is widely used but rarely clearly defined. We believe that in its widespread use the term refers to a situation in which excessive public expectations of future price increases cause prices to be temporarily elevated.” Shiller (2011) more recently says “bubbles are social epidemics, fostered by a sort of interpersonal contagion. A bubble forms when the contagion rate goes up for ideas that support a bubble. But contagion rates depend on patterns of thinking, which are difficult to judge.” This may ring true, and phenomena like “excessive public expectations, social epidemics, and interpersonal contagion” are nothing if not bewitching. Here we instead emphasize the more pedestrian idea of liquidity, because we think we can make this precise. A bubble

4 Others have considered this point. Harding et al. (2007) estimate the depreciation rate on houses to be around 2.5 percent, so if the discount rate is around 3 percent, the rent-price ratio should be around 5. In Campbell et al. (2009) data, from 1975 to 1995 this ratio is indeed around 5; it declines to 3.7 in 2007.
for us is an asset price different from its fundamental value, which is the present value of holding the asset. This seems consistent with, e.g., Stiglitz (1990), who says “if the reason that the price is high today is only because investors believe that the selling price is high tomorrow – when ‘fundamental’ factors do not seem to justify such a price – then a bubble exists.” Here prices differ from fundamental values due to liquidity considerations.

In emphasizing credit frictions, we follow a large literature summarized in Gertler and Kiyotaki (2010) and Holmström and Tirole (2011), and related work surveyed by Nosal and Rocheteau (2011) and Williamson and Wright (2010), that tries to be explicit about the process of exchange by going into detail concerning how agents trade (bilateral, intermediated etc.), using which instruments (money, credit etc.) and at what terms (price taking, bargaining etc.). The direct antecedent of our approach is the body of work emanating from Kiyotaki and Moore (1997, 2005). There is a huge literature on asset pricing generally, of course; we empathize some differences between housing and other assets – e.g., typical assets generate dividend streams that enter your budget equation; houses enter your budget equation and your utility function. We show that this changes the circumstances under which bubbles exist, from one of low supply for other assets, to one of either low or high supply, depending on preferences, for housing. Relatedly, we show welfare may decrease with an increase in the housing stock, which typically does not happen with financial assets.

As for the literature on bubbles and liquidity, in general, there is too much to review here, so we refer readers to Farhi and Tirole (2011) for references. In terms of research on housing markets and recent US experience, there are several other papers that take seriously the precautionary or collateral function of home equity. A technical difference from some of this work is that we focus on fully rational agents, with homogenous beliefs, and indeed we can generate bubble-like equilibria under perfect foresight. We can do this with a fixed or with an endogenous supply of housing, which seems relevant since it has been suggested, e.g., by Shiller (2011), that “The housing-price boom of the 2000’s was little more than a
construction-supply bottleneck, an inability to satisfy investment demand fast enough, and was (or in some places will be) eliminated with massive increases in supply.” See also the important contribution by Glaeser et al. (2008). Since the housing literature in general is voluminous, and at the risk of neglecting some other important contributions, we can only cite a few examples that influenced our thinking on the issues.5

The paper is organized as follows. Section 2 lays out the environment. Section 3 discusses steady state. Section 4 presents dynamics. Section 5 endogenizes the supply of housing. Section 6 presents a monetary version of the model to study interplay between housing and inflation. The Appendix fills in some details of the monetary model, presents an extension that uses strategic (instead of axiomatic) bargaining, and endogenizes the credit limit.

2 The Environment

Each period in discrete time agents interact in two distinct markets. One is a frictionless centralized market, labeled AD for Arrow-Debreu. The other is a decentralized market, with explicit frictions detailed below, labeled KM for Kiyotaki-Moore. At each date t, in addition to labor ℓt, there are two nonstorable consumption goods xt and yt, plus housing ht. We assume ℓt, xt and ht are traded in the AD market, and yt in the KM market. The utility of a household is given by

\[ \lim_{T \to \infty} \mathbb{E} \sum_{t=0}^{T} \beta^t [U(x_t, y_t, h_t) - \ell_t], \]

5Here is a brief review. Carroll et al. (2003) study a precautionary motive in the demand for housing, and find evidence for it at moderate and higher income. Hurst and Stafford (2004) find that unemployment shocks and low asset positions increase households’ likelihood of using home-equity loans. Campbell and Hercowitz (2005) study a growth model where houses alleviate borrowing constraints. Arce and Lopez-Salido (2011) study a life-cycle model where some agents hold houses purely as a saving vehicle. Other work emphasizing the role of asset shortages includes Caballero and Krishnamurthy (2006) and Fostel and Geanakoplos (2008). Brady and Stinmel (2011) find the response to house price shocks has shifted since 1998, consistent with our timing assumptions. Aruoba et al. (2011) study the interaction between housing markets and inflation, and cite previous work along these lines. Other work on housing dynamics includes Burnside et al. (2011), Coulson and Fisher (2009), Ngai and Tenreyro (2009), Novy-Marx (2009), Piazzesi and Schneider (2009b,a) and Jaccard (2011). Liu et al. (2011) also also assume real estate can be used as collateral (but by producers, not consumers). See also Miao and Wang (2011) and Liu and Wang (2011). Finally, we mention that this paper is not about imperfect housing markets – houses here are traded in frictionless markets, like capital in standard growth theory. On frictional housing market models see Wheaton (1990), Albrecht et al. (2007), Head and Lloyd-Ellis (2010), and the references therein.
where $\beta \in (0, 1)$ and $A > 0$. A big gain in tractability comes from quasi-linear utility, although this can be generalized to some extent (e.g., Rocheteau et al. 2008). To ease the presentation, assume $U(x_t, y_t, h_t) = U(x_t, h_t) + u(y_t)$, where $U(\cdot)$ and $u(\cdot)$ satisfy all the usual assumptions, and normalize $U(0, 0) = u(0) = 0$.

For now there is a fixed stock of housing $H$. In terms of AD goods, $\ell_t$ can be converted one-for-one into $x_t$ (it is easy to use more general production functions). In terms of KM goods, some agents can produce $y_t$ using a technology summarized by the cost function $v(y_t)$. In some related papers, households buy $y_t$ from each other in the KM market and $v(y_t)$ is interpreted as the disutility of production; in other papers, households buy from firms or retailers. Although it does not matter for any interesting results, we follow the latter approach, with households buying $y_t$ from KM retailers. The retail technology works as follows: by investing at $t-1$ a fixed amount, normalized to 1, of the AD numeraire $x_{t-1}$, a retailer can at $t$ convert it into any amount $y_t \leq 1$ of the KM good and some amount $x_t = F(1 - y_t)$ of the AD good. The profit from this activity, conditional on selling $y_t$ in the KM market at $t$ for revenue $R_t$, measured in period $t$ numeraire, is $R_t + F(1 - y_t) - (1 + r)$, given the initial investment at $t - 1$ is repaid at $t$ at interest rate $1 + r = 1/\beta$.

Not all retailers earn the same payoff, since not all trade, in the KM market. Let $\alpha_f$ be the probability a retailer trades in KM, and $\alpha_h$ the probability a household trades. Also, assume $y \leq 1$ is not binding – as is the case, e.g., if $F'(0) = \infty$. Then expected profit is

$$\Pi_t = \alpha_f [R_t + F(1 - y_t)] + (1 - \alpha_f) F(1) - (1 + r)$$

$$= \alpha_f [R_t - v(y_t)] + F(1) - (1 + r),$$

where $v(y_t) \equiv F(1) - F(1 - y_t)$ is the opportunity cost of selling $y_t$. As in standard search theory (Pissarides 2000), if there is a $[0, 1]$ continuum of households and a $[0, N]$ continuum of retailers, the KM trading probabilities can be endogenized by $\alpha_f = \alpha(n)/n$.

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6 We assume here the limit in (1) exists; if not, one can use more advanced optimization techniques (see the discussion and citations in Rocheteau and Wright 2010).
and $\alpha_h = \alpha(n)$, where $\alpha(\cdot)$ comes from a standard matching technology and $n \leq N$ is the measure of firms that participate in the KM market. Firms may have to pay an entry cost, in addition to their initial investment in $x_t$, and $n$ can be determined by a standard free-entry condition. To make our main points, however, we assume this cost is small, and $1 + r < F(1)$, so that $n = N$. This implies $\alpha_h$ and $\alpha_f$ fixed constants.

For the issues at hand it is useful to have a frictional KM market. However, it is not necessary to invoke search per se. An alternative story that is equivalent for what we do is that households sometimes realize a demand for $y_t$ due to preference or opportunity shocks. Nice examples include the possibility that one has occasion to throw a party, or an opportunity to buy a boat at a good price; not-so-nice examples include the possibility that one has a medical emergency, or one’s boat sinks. Whether or not this is viewed in terms of search frictions, to show our results are robust, we consider various options for KM pricing, which as discussed below can be interpreted as allowing either bilateral or multilateral exchange. More significantly, in the KM market households have to buy on credit, since they have nothing to offer retailers by way of quid pro quo. One might object that one can always pay with cash; that would be playing right into our hands, however, as this is precisely what we consider in Section 6. For now, $y_t$ is acquired in exchange for debt $d_t$ to be retired in the next AD market.

Credit is limited by lack of commitment: households are free to default, albeit at the risk of punishment. At one extreme punishment can be so severe that credit is effectively perfect. At the other extreme is no punishment, not even exclusion from future credit, maybe because borrowers are anonymous, which completely rules out unsecured lending.

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7 The reason is the following (something we learned from Peralta-Alva et al. 2011). Our claim is that home equity lending is relevant for thinking about recent housing market experience. But if household consumption did not actually rise all that much, how can this be so important? The answer is that households care about the event where they *might* need a home equity loan, even if this is realized with probability less than 1 (precautionary demand). So we build a search-based version of the standard KM model, which we think may be interesting in its own right, and may have other interesting applications (not only for housing).

8 Given quasilinear utility, households are indifferent between one-period loans and longer-term debt if they are not credit constrained; if they are constrained, they actually prefer to pay off debt each period in AD so as to relax their constraints in KM.
In general, we impose a debt limit \( d \leq D = D(e_t) \) with \( e_t = \psi_t h_t \), where \( \psi_t \) is the price of \( h_t \) in terms of \( x_t \). Sometimes we focus on the linear case \( D(e_t) = D_0 + D_1 e_t \) with \( D_0 \geq 0 \leq D_1 \leq 1 \), but only to ease the presentation. It makes sense to consider \( D_1 < 1 \) if a creditor can seize only some assets after a default (e.g., he gets the house while the debtor absconds with the appliances), or if some value is lost to transaction costs. But \( D(e_t) > e_t \) is also possible when we have punishments beyond confiscating collateral. For now \( D(e_t) \) is exogenous. In the Appendix we show how to endogenize it as in Kehoe-Levine (1993) or Alvarez-Jermann (2000), both for the case where punishment for default involves taking away unsecured credit, and for the case where it involves taking way all future credit, and we show these alternatives differ in some interesting ways.

Let \( W_t(d_t, h_t) \) be a household’s value function entering the AD market with debt \( d_t \) and housing \( h_t \). Since \( d_t \) is paid off each period in AD, households start debt free in KM, where \( V_{t+1}(h_{t+1}) \) is the value function, next period. The household’s AD problem is then

\[
W_t(d_t, h_t) = \max_{x_t, h_t, h_{t+1}} \{ U(x_t, h_t) - A \ell_t + \beta V_{t+1}(h_{t+1}) \}
\]

\[
\text{st } x_t + \psi_t h_{t+1} = \ell_t + \psi_t h_t + T_t - d_t \text{ and } \ell_t \in [0, \bar{\ell}]
\]

where \( T_t \) denotes any other income, including transfers minus taxes, returns on investments etc. Since quasi-linear utility implies other income affects nothing except leisure, details of what enters \( T_t \) do not matter for the main results. Of course, we assume that wealth other than home equity – e.g., future wage income – cannot be used to secure loans, perhaps because it is hard to seize (in the language of Holmstrom and Tirole 2011, it is not pledgeable).

Before solving (3), we digress briefly to show how our approach is consistent with much research on the micro and macro economics of household production (Aruoba et al. 2012 provide a recent list of references). Suppose households value goods acquired on the market \( x^1_t \), and those produced at home \( x^2_t \). They also engage in market work \( \ell^1_t \), and home work
\( \ell_t^2 \). If \( x_t^2 = G_t(\ell_t^2, h_t) \) is the home production function, the generalization of (3) is

\[
W_t(d_t, h_t) = \max_{x_1, x_2, \ell_1^t, \ell_2^t, h_{t+1}} \left\{ U(x_1^t, x_2^t) - A_1 \ell_1^t - A_2 \ell_2^t + \beta V_{t+1}(h_{t+1}) \right\}
\]

\[
\text{st } x_1^t + \psi_t h_{t+1} = \ell_1^t + \psi_t h_t + T_t - d_t, \quad x_2^t = G_t(\ell_t^2, h_t) \quad \text{and } \ell_1^t + \ell_2^t \in [0, \bar{\ell}].
\]

Now \( h_t \) does not enter \( U(\cdot) \) directly, but indirectly as an input to \( G_t(\cdot) \) (one could have both, disaggregating home capital into, say, vacuum cleaners and plasma TV's). As is well known, we can substitute out \( x_2^t = G_t(\ell_t^2, h_t) \) and maximize out \( \ell_2^t \), to derive a reduced-form that depends only on market variables. Given this, although sometimes there are reasons for being more explicit about home production, here we assume \( h_t \) enters \( U(\cdot) \) directly.

Returning to the baseline specification, assuming \( \ell_t \in [0, \bar{\ell}] \) does not bind, we eliminate \( \ell_t \) using the budget equation to write

\[
W_t(d_t, h_t) = \psi_t h_t + T_t - d_t + \max_{x_t} \left\{ U(x_t^t, h_t) - x_t \right\} + \max_{h_{t+1}} \left\{ \beta V_{t+1}(h_{t+1}) - \psi_t h_{t+1} \right\}
\]

where we normalize the disutility of work to \( A = 1 \). Immediately this implies choices at \( t \), and in particular \( h_{t+1} \), are independent of \( (d_t, h_t) \), which simplifies the analysis because we do not have to keep track of the wealth distribution.\(^9\) The FOC’s from (5) are

\[
U_1(x_t^t, h_t) = 1 \quad \text{and } \psi_t = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}}.
\]

The envelope conditions are

\[
\frac{\partial W_t}{\partial d_t} = -1 \quad \text{and } \frac{\partial W_t}{\partial h_t} = U_2(x_t^t, h_t) + \psi_t,
\]

so that \( W \) is linear in debt (more generally, net worth), but not housing since \( h_t \) affects \( U(\cdot) \) directly and not only through the budget constraint.

We now describe what happens in KM. When a trading opportunity arises, retailers (or sellers) produce \( y_t \) and households (or buyers) consume \( y_t \) in return for which the latter

\(^9\)To be clear, this requires that the constraint \( \ell_t \in [0, \bar{\ell}] \) is slack. More generally, people with very low or high net worth may be unable to set \( \ell_t \) high or low enough to settle all debt or get to their preferred \( h_{t+1} \) in one period. But if we start with a distribution of \( h_t \) and \( d_t \) that is not too disperse, relative to \( [0, \bar{\ell}] \), households can settle all debt and acquire the same \( h_{t+1} \) each period, without borrowing or lending across AD markets.
issue debt \( d_t \leq D(e_t) \) coming due in the next AD market. The terms of trade \( (y_t, d_t) \) can be determined in many ways, but we begin with competitive Walrasian pricing.\(^{10}\) To ease

the presentation, assume \( N \alpha_f = \alpha_h \), so the measures of buyers and sellers are equal. For

buyers, the trade surplus is \( S_{bt} = u(y_t) + W(d_t, h_t) - W(h_t) = u(y_t) - d_t \), since \( W(\cdot) \) is linear

in \( d_t \) by (7). They maximize \( S_{bt} \) subject to \( d_t = p_t y_t \leq D_t \), taking as given the price \( p_t \) and
debt limit \( D_t \). Similarly, sellers maximize \( S_{st} = p_t y_t - v(y_t) \). If the credit constraint is slack,
equilibrium is \( y_t = y^* \), where \( u'(y^*) = v'(y^*) \), and \( p^* = v'(y^*) \). If \( d^* = v'(y^*) y^* \leq D_t \) then
equilibrium is \( (y_t, d_t) = (y^*, d^*) \). But if \( d^* > D_t \) then equilibrium is given by \( p_t = v'(y_t) \)
from the sellers’ FOC, and \( y_t = D_t / p_t \) from the buyers’ constraint. Thus, \( d^* > D_t \) implies

a debt-constrained equilibrium, where \( d_t = D_t \) and \( y_t \) solves \( v'(y_t) y_t = D_t \).

For future reference let \( g(y_t) \equiv v'(y_t) y_t \), and note that \( g'(y_t) > 0 \), so that when \( d^* > D_t \)

we can write \( y_t = g^{-1}(D_t) < y^* \). Then we illustrate the results in the preceding paragraphs
in Figure 2, and formalize them as follows:

**Proposition 1** Let \( y^* \) and \( p^* = v'(y^*) \) be the equilibrium ignoring \( d_t \leq D_t \), and let \( d^* = g(y^*) \). KM equilibrium at \( t \) is given by:

\[
y_t = \begin{cases} 
  g^{-1}(D_t) & \text{if } D_t < d^* \\
  y^* & \text{if } D_t \geq d^* 
\end{cases} \quad \text{and } \quad d_t = \begin{cases} 
  D_t & \text{if } D_t < d^* \\
  d^* & \text{if } D_t \geq d^* 
\end{cases}
\]

Equation (8)

As an alternative trading mechanism, suppose we pair off buyers and sellers, and let

them bargain bilaterally. One option is to use the generalized Nash solution,

\[
(y_t, d_t) = \arg \max_{S_{bt} S_{st}} S_{st}^{1-\theta} \quad \text{st } \quad d_t \leq D_t,
\]

where \( \theta \) is the bargaining power of the buyer. One can show (see Lagos and Wright 2005)
the outcome is the same as (8), except that instead of \( g(y) = v'(y) y \), we redefine

\[
g(y) = \frac{\theta v(y) u'(y) + (1-\theta) u(y) v'(y)}{\theta u'(y) + (1-\theta) v'(y)}
\]

Equation (9)

\(^{10}\)Even if one adopts the search interpretation of our KM market, to motivate Walrasian pricing, one can

make trade multilateral as in Lucas-Prescott search models of the labor market, as opposed to bilateral as in Mortenson-Pissarides search models.

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Nash says \( d^* = g(y^*) = \theta v(y^*) + (1 - \theta) u(y^*) \), while Walras says \( d^* = v'(y^*) y^* \), but in either case Proposition 1 holds as stated. Another option is Kalai’s (1977) bargaining solution, which has certain advantages in imperfect-credit models (Aruoba et al. 2007),

\[
(y_t, d_t) = \text{arg max } S_{bl} \text{ st } S_{bl} = \theta [u(y_t) - v(y_t)] \text{ and } d_t \leq D_t.
\]

This is the same as Nash iff the constraint is slack, but in any case satisfies (8) with

\[
g(y) = \theta v(y) + (1 - \theta) u(y).
\]

As another option Appendix A provides a strategic bargaining game consistent with (8).

In each case, the equilibrium outcome is given by (8) for a particular \( g(\cdot) \), and we can write the KM value function as

\[
V_t(h_t) = W(0, h_t) + \alpha_h \{ u[y(\psi_t h_t)] - d(\psi_t h_t) \},
\]

where it is understood that \( y(\psi_t h_t) \) and \( d(\psi_t h_t) \) are given by (8) with \( D_t = D(\psi_t h_t) \). Using (7)-(8) we derive

\[
\frac{\partial V_t}{\partial h_t} = U_2(x_t, h_t) + \psi_t + \alpha_h \left[ u'(y) y'(\psi_t h_t) - d'(\psi_t h_t) \right] \psi_t
\]

\[
= U_2(x_t, h_t) + \psi_t + \alpha_h D_1 \mathcal{L}[y(\psi_t h_t)] \psi_t
\]

where we define the liquidity premium

\[
\mathcal{L}(y) \equiv \begin{cases} 
  u'(y)/g'(y) - 1 & \text{if } y < y^* \smallskip 
  0 & \text{otherwise} \end{cases}.
\]

Inserting these results into the FOC for \( h_{t+1} \) in (6) and using \( 1 + r = 1/\beta \), we get the Euler equation for housing

\[
r \psi_t = U_2(x_{t+1}, h_{t+1}) + (\psi_{t+1} - \psi_t) + \alpha_h D_1 \mathcal{L}[y(\psi_t h_t)] \psi_{t+1}.
\]
The three terms on the RHS describe the benefits of a bigger house: 1) it provides more utility; 2) it generates more capital gains or losses; and 3) it secures more credit. Setting $h_t = H$ and using the FOC $U_1(x_t, H) = 1$ to define $x = X(H)$, (14) defines a univariate difference equation in the price of housing, $\psi_t = \Psi (\psi_{t+1})$. An equilibrium is any sequence $\{\psi_t\}$ solving $\psi_t = \Psi (\psi_{t+1})$, as long as it is nonnegative and bounded (to satisfy the transversality condition; see Rocheteau and Wright 2010). Given $\{\psi_t\}$, we can easily recover all the other variables, $e_t = \psi_t H$, $D_t = D (e_t)$, $y_t = y (e_t)$ etc.

3 Steady State Equilibrium

A stationary equilibrium, or steady state, is a constant solution to $\psi = \Psi (\psi)$. In steady state there are no capital gains, and (14) becomes

$$r \psi = U_2 [X(h), h] + \alpha_h D_1 \mathcal{L} [y (e)] \psi.$$ (15)

One can interpret this as the long-run demand for housing, with slope

$$\frac{\partial h}{\partial \psi} = \frac{U_{11} (U_2 - \alpha_h \psi^2 h \mathcal{L}' y')}{(U_{11} U_{22} - U_{12}^2 + U_{11} \alpha_h \psi^2 \mathcal{L}' y')} \psi.$$ (16)

after simplification. Suppose $\mathcal{L}' (y) < 0$. Then $\partial h / \partial \psi < 0$ and demand slopes downward. Even without $\mathcal{L}' (y) < 0$, one can prove $\partial h / \partial \psi < 0$ using the method in Wright (2010), but to avoid technicalities it is easiest to assume $\mathcal{L}' (y) < 0$. In any case, we have the result:

**Proposition 2** With a fixed supply $H$, steady state equilibrium is unique.

Proposition 2 highlights a difference between housing and currency: in monetary models, whenever there exists a steady state where currency is valued there coexists one where it is not; but when cash is replaced by home equity as an instrument to facilitate transactions steady state is unique. We emphasize that this makes it harder to generate interesting dynamics (e.g., we cannot focus on paths that transit between two steady states, or oscillate

---

12 One can prove $\mathcal{L}' (y) < 0$ for many mechanisms, including Walras and Kalai. One cannot prove it for Nash bargaining without additional assumptions, like decreasing absolute risk aversion, or $\theta$ close to 1.
between points close to two steady states). In terms of substantive results, if \( e = \psi H > e^* \) then \( \mathcal{L}(e) = 0 \). In this case, (15) implies \( \psi = \psi^* \equiv U_2 [X(H), H] / r \), where we define the fundamental price \( \psi^* \) by the present value of the marginal utility of living in house \( H \) forever. But if \( e < e^* \), then \( \mathcal{L}(y) > 0 \) and (15) implies \( \psi > \psi^* \). In this case, houses bear a premium that we call a bubble. Rather than debating the semantics of this word, we prefer to stick to the economics, which is plainly this: when credit constraints are relevant, agents will pay a premium for any asset that relaxes them.

In monetary theory there are related results, in similar models, for equity in fixed supply and neoclassical capital, as well as currency (Geromichalos et al. 2007; Lagos and Rocheteau 2008). Houses are also different from these assets. With other assets in fixed supply, e.g., there is a liquidity premium iff supply is low. With home equity, although it is still true that there is a liquidity premium iff \( e \) is low, \( e \) can be low either when \( H \) is low or when \( H \) is high, depending on the elasticity of demand. Thus,

\[
\frac{de}{dH} \approx - H \left( U_{22} U_{11} - U_{21}^2 \right) - U_2 U_{11},
\]

where \( A \approx B \) means that \( A \) and \( B \) take the same sign. For \( U(x, h) = \tilde{U}(x) + h^{1-\sigma} / (1 - \sigma) \), e.g., \( \sigma < 1 \) implies housing bears a liquidity premium when \( H \) is low, and \( \sigma > 1 \) implies it bears a premium when \( H \) is high.

Because of this, welfare \( W \) (average utility) may fall actually as \( H \) increases, which is generally not the case for financial assets in these models. Here, if \( H \) increases AD utility rises, but in examples KM utility may falls by enough to make overall utility lower. Intuitively, if demand is elastic we may want liquid assets to be scarce, because this makes them more valuable, and that ameliorates credit frictions. Relatedly, while \( \psi \) and \( e \) always increase with \( \alpha_h \), it is not clear how they change with \( D_1 \), since higher \( D_1 \) makes home equity more useful as collateral, but also means a given amount of collateral goes further. We summarize these results as follows:
Proposition 3 Suppose $h = H$ is fixed. If $e > e^*$ in steady state then $\psi = \psi^*$, but if $e < e^*$ then $\psi > \psi^*$. We can have $e < e^*$ either when $H$ is low or when $H$ is high, depending on utility, and we can have $\partial W/\partial H < 0$. Also, $\psi$ is increasing in $\alpha_h$, but can be increasing or decreasing in $D_1$.

4 Cyclic, Chaotic and Stochastic Equilibria

Consider first deterministic equilibria, given by nonnegative and bounded solutions to

$$
\psi_t = \Psi(\psi_{t+1}) = \frac{U_2[X(H),H] + \psi_{t+1} + \alpha_D D_1 L[y(\psi_{t+1} H)]}{1 + r} \psi_{t+1}.
$$

The first observation is that any interesting dynamics must emerge from liquidity considerations, which show up in the nonlinear term $L[y(\psi_{t+1} H)]$. To see this, set $\alpha_h$ or $D_1$ to 0. Then (17) is a linear difference equation, which can be rearranged as

$$
\psi_{t+1} = -U_2[X(H),H] + (1 + r) \psi_t.
$$

This has a unique steady state at the fundamental price $\psi^*$. This is also the unique equilibrium, as any solution to difference equation (18) other than $\psi_t = \psi^* \forall t$ has $\psi_t \to \infty$ or $\psi_t \to -\infty$. So there are no interesting dynamics when the liquidity motive is inoperative.

When $\alpha_D D_1 > 0$, however, as long as $H \psi_{t+1} < e^*$ we have $L[y(\psi_{t+1} H)] > 0$, and nonlinearity kicks in. We analyze this in $(\psi_{t+1}, \psi_t)$ space, where it is natural to think of $\psi_t$ as a function of $\psi_{t+1}$, because given $\psi_{t+1}$ demand for $h_t$ is a (single-valued) function and market clearing pins down $\psi_t$. However, as usual, there can be multiple values of $\psi_{t+1}$ for which this mapping yields the same $\psi_t$, so the inverse $\psi_{t+1} = \Psi^{-1}(\psi_t)$ can be a correspondence. Of course $\Psi$ and $\Psi^{-1}$ cross on the 45° line at the unique steady state. Textbook methods (e.g., Azariadis 1993) tell us that whenever $\Psi^{-1}$ and $\Psi$ cross off the 45° line there is a cycle of period 2 - i.e., a solution $(\psi^1, \psi^2)$ to $\psi^2 = \Psi(\psi^1)$ and $\psi^1 = \Psi(\psi^2)$, or a fixed point of $\Psi^2$, that is nondegenerate in that $\psi^1 \neq \psi^2$ - and this happens whenever $\Psi$ has a slope less than $-1$ on the 45° line. In a 2-cycle equilibrium, even though fundamentals
are constant, $\psi$ oscillates between $\psi^1$ and $\psi^2$ as a self-fulfilling prophecy: a nonstationary house-price bubble.

Before discussing the economic intuition, consider more generally $n$-cycles, which are nondegenerate solutions to $\psi = \Psi^n (\psi)$. We show that $n$ cycles exist for different $n$ by way of example (there is no claim that exotic equilibria exist for all parameters, only that they may exist). It is obvious from (17) is that we need $\alpha_b D_1$ to be relatively big, so the nonlinear part of the difference equations matters, and what one finds in practice is that we need reasonably big risk aversion and low $\beta$. Consider $H = 1$, $v (y) = y$, $D (e) = e$ and

$$U (x, h) = \tilde{U} (x) + \kappa \frac{h^{1-\sigma}}{1-\sigma} \quad \text{and} \quad u (y) = \eta (y + e)^{1-\gamma} - e^{1-\gamma}.$$

Cyclic equilibria can exist for some examples, described in Table 1, for various KM mechanisms, including Walrasian pricing, axiomatic bargaining and strategic bargaining. This is relevant because in some models one gets cycles for certain mechanisms but not others.\(^\text{14}\)

Example 1, with Walrasian pricing, has a unique equilibrium, the steady state, as shown in Figure 3. Example 2, also with Walrasian pricing but different parameters, as also shown in Figure 3, has a 2-cycle in addition to the steady state equilibrium. In this case, $y^* = 1.0833$, the constraint binds iff $\psi < y^*$, and this happens in alternate periods. Although we do not show the graphs (they look similar), Examples 3 and 4 also have 2-cycles, with Kalai and strategic bargaining. So the results do not require a particular pricing protocol, but emerge generally from liquidity considerations. Example 2 also has a 3-cycle, with $\psi^1 = 0.8680 < y^*$, $\psi^2 = 1.5223 > y^*$ and $\psi^3 = 1.1134 > y^*$. When a 3-cycle exists, by the Sarkovskii theorem and Li-Yorke theorem (again see Azariadis 1993), there exist cycles of all orders, and chaotic dynamics, where chaos is a nonnegative bounded solution \{\psi_t\} to (17) with the property that $\psi_s \neq \psi_t \forall s \neq t$.\(^\text{13}\)

\(^{13}\)The form of $\tilde{U}$ is irrelevant for all the results. Also, $\sigma$ is irrelevant, since it vanishes from $\partial U / \partial h$ when $h = H = 1$. The role of $\varepsilon$ in $u (y)$ is merely to force $u (0) = 0$.

\(^{14}\)In a Kehoe-Levine model, e.g., Gu et al. (2012) prove cycles exist for generalized Nash but not Kalai or buyer-take-all bargaining. Our results emerge from nonlinearity in the nature of liquidity, not from something induced by nonlinear pricing, like that coming from Nash bargaining in Gu et al.
Mechanism Walras Walras Kalai Game Walras Walras

\[ \begin{align*}
\alpha_h & \quad 0.5 & 0.5 & 0.9 & 0.5 & 0.99 & 0.99 \\
\beta & \quad 0.8 & 0.6 & 0.6 & 0.8 & 0.95 & 0.95 \\
\theta & \quad n/a & n/a & 0.9 & 0.6 & n/a & n/a \\
\kappa & \quad 0.125 & 0.3333 & 0.1 & 0.125 & 0.1 & 0.1 \\
\sigma & \quad n/a & n/a & n/a & n/a & 4 & 4 \\
\eta & \quad 1.5125 & 3.2479 & 0.5882 & 3.0368 & 1.027 & 1.028 \\
\gamma & \quad 2 & 7 & 9 & 8 & 8 & 16 \\
\varepsilon & \quad 0.1 & 0.1 & 0.5 & 0.1 & 0.0001 & 0.0001
\end{align*} \]

<table>
<thead>
<tr>
<th>Table 1: Parameter Values for the Examples</th>
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While many people talk about housing bubbles, without abandoning at least some basic axioms, like rationality, or at least rational expectations, producing formal models with such self-fulfilling prophecies is nontrivial. Our simple stationary economy displays complicated dynamics even though agents have perfect foresight. House prices fluctuate because houses convey liquidity, which is to some extent a matter of beliefs. Consider a 2-cycle. Suppose at \( t \) agents believe that \( \psi_{t+1} \) will be high. Then liquidity will not be so scarce at \( t+1 \), and this lowers the amount people are willing to pay for it at \( t \). Thus, a low \( \psi_t \) is consistent with market clearing given a high \( \psi_{t+1} \). By the same (i.e., opposite) logic a high \( \psi_{t+1} \) is consistent with low \( \psi_{t+2} \), and so on. Agents are willing to pay more for \( H \) at \( t+1 \) when they know the price will drop at \( t+2 \) precisely because the price drop means liquidity will soon become scarce, and that makes it currently dear. Hence the prices of liquid assets can oscillate. Cycles of order \( n > 2 \) and the extreme case of chaos are more complicated self-fulfilling prophecies, but the intuition is similar: there is a tendency, or at least a possibility, for equilibrium house prices to fluctuate, even though there always coexists a steady state equilibrium where they do not.

While these examples suffice to make a point, they do not resemble recent experience very well. Loosely speaking, prices tend to go up and down rather too regularly, without the stereotypical bubble pattern of a prolonged boom followed by collapse. To address this, consider an equilibrium from Example 5, illustrated in Figure 4. The blue line is the 45°
line, the red curve gives $\psi_{t+1}$ as a correspondence of $\psi_t$, where we zoom in around steady state $\psi^*$. In this case, $\psi^*$ corresponds to two values of $\psi_{t+1}$. One equilibrium is $\psi_{t+1} = \psi^*$, where the price stays put. But there is another, where we jump to $\psi_{t+1} = \psi_1 > \psi^*$, then set off on an increasing trajectory shown by the dashed line. Along this path $\psi$ keeps increasing at a rate just below the rate of time preference; this cannot continue forever, however, without violating transversality. Therefore, when we reach $\psi_2$ in Figure 4, in the equilibrium under construction, the price drops to $\psi_3$, then converges in an oscillating path back to $\psi^*$. This very much does display the stereotypical pattern of boom and bust.

Mission accomplished? No. The parameters used to construct Example 5 are not great – while it does have a reasonable $\beta = 0.95$, this comes at the expense of a big coefficient of relative risk aversion $\gamma = 8$. Lowering $\gamma$ requires lowering $\beta$ if we want to generate complicated dynamics. This does not mean that no one could produce a more realistic example; we can, however, prove the following point (suggested by Charles Engle) about any perfect foresight equilibrium. Rearrange (17) as

$$
\frac{\psi_{t+1} - \psi_t}{\psi_t} = r - \frac{U_2 [X(H), H]}{\psi_t} - \frac{\alpha_k \psi_{t+1} D_L \mathbb{E} [y(\psi_{t+1} H) \mid \psi_t]}{\psi_t}.
$$

The capital gain on the LHS is bounded above by the rate of time preference, since on the RHS we have $r$ minus two positive terms, the service flow and the liquidity premium. There is no way for these dynamics to generate capital gains exceeding $r$, as one sometimes sees in the data, since this would open up opportunities for arbitrage-like profits.

Concede defeat? Not quite. Consider sunspot equilibria – i.e., outcomes that fluctuate stochastically over time even though fundamentals are deterministic. While there can exist complicated sunspot equilibria, a simple stationary example takes the following form: whenever the price is $\psi^1$, it jumps to $\psi^2 > \psi^1$ with probability $\lambda^1$ and stays put with probability $1 - \lambda^1$; and whenever it is $\psi^2$, it falls back to $\psi^1$ with probability $\lambda^2$. Agents have rational expectations (i.e., they know $\lambda^1$ and $\lambda^2$). It is not hard to show such equilibria exist, with
stochastic price fluctuations, even though fundamentals are constant.\textsuperscript{15} The intuition is the same as that given above for a deterministic cycles, but quantitatively the results can differ.

To show this Example 6 displays an equilibrium where before $t = 4$, $\psi_t = \psi^* = 0.5255$. From $t = 4$ to 8, every period there is a probability of jumping to a deterministic path transitioning between $\psi = 0.5350$ and $\psi^*$, and a probability of going to a higher price. These probabilities change each period, and agents know this. After $t = 8$, everything is deterministic again. One realization has $\psi$ increasing at 9\% per year for 5 years, then collapsing, and oscillating back to $\psi^*$. This example has $\gamma = 16$, which we do not claim is realistic. But it makes the point that sunspots overcome the bound on capital gains given by (19). If we want huge capital gains, naturally they must occur with a low probability, so a realization where $\psi$ increases a lot several periods in a row is a rare event. In the example it can be expected to happen with probability around 11\%, or about once a century, which seems reasonable, since even if bubbles are recurrent events, as Reinhart and Rogoff (2009) suggest, this does not mean they happen that often.

To close this section, we reiterate that while our results are reminiscent of some findings in monetary economics, they are not the same.\textsuperscript{16} Yes, $H$ and $M$ are both assets, and both can facilitate transactions. But models of fiat currency generally have multiple steady states, and our model has a unique steady state. This makes it more difficult to construct interesting dynamics, but to us that seems better that assuming houses are intrinsically useless. Part of the motivation for this project came from an example of Narayana Kocherlakota, where what he called housing could exhibit somewhat interesting dynamics – at least, an equilibrium where price jumps stochastically from $\psi > 0$ to $\psi = 0$, although once

\textsuperscript{15}There are different ways to prove the existence of sunspot equilibria. Azariadis and Guesnerie (1986), e.g., note that in the limit when $\lambda^1 = \lambda^2 = 1$ the sunspot equilibrium described in the text reduces to a 2-cycle, which already have been shown to exist. By continuity, proper sunspot equilibria exist. Using continuity means the sunspot equilibria that are guaranteed to exist are similar to 2-cycles, however. So we construct examples directly, not based on this continuity argument. See Trejos and Wright (2012) for a recent discussion of different ways to construct sunspot equilibria in related models.

\textsuperscript{16}Even if they were, so what? It is like saying that the results are special cases of labor economics, since we use search, or consumption theory, because we have Euler equations. To be clear, we do not claim to be inventing mathematical tools here, only applying them to the topical issue of housing markets.
it goes down it never comes back. In his example, housing really is a fiat object, with no fundamental utility. Our \( H \) has intrinsic value as shelter, ruling out equilibria where \( \psi = 0 \), or where \( \psi_t \to 0 \) either stochastically or deterministically. Another way \( H \) differs from \( M \) is that it can produced by profit-maximizing private agents, something to which we now turn, because we think the supply side is an interesting part of recent housing market experience.

## 5 Endogenous Supply

We introduce a technology for home building, where producing \( \Delta h_t \) units of housing requires an input of \( c(\Delta h_t) \) units of AD numeraire. The construction business, like other AD activity, is assumed to perfectly competitive, and in equilibrium

\[
\psi_t = c' [h_{t+1} - (1 - \delta)h_t],
\]

where \( \delta \) is a depreciation rate. The households’ AD problem is unchanged, except now \( e_t = \psi_t (1 - \delta) h_t \), and the housing Euler equation becomes

\[
(1 + r) \psi_t = U_2 [X(h_{t+1}), h_{t+1}] + \psi_{t+1} (1 - \delta) + \alpha_h D_1 (1 - \delta) \mathcal{L}[y(h_{t+1}\psi_{t+1})] \psi_{t+1}. \tag{21}
\]

In steady state, (20)-(21) can be written

\[
(r + \delta) \psi = U_2 [X(h), h] + \alpha_h (1 - \delta) D_1 \mathcal{L}[y(e)] \psi \tag{22}
\]

\[
\psi = c' (\delta h), \tag{23}
\]

where (22) is the straightforward generalization of long-run demand (15), while (23) is a long-run supply relation. Combining (22) and (23), we get

\[
r + \delta = \frac{U_2 [x(h), h]}{c' (\delta h)} + \alpha_h (1 - \delta) D_1 \mathcal{L}[(1 - \delta) c' (\delta h) h].
\]

The RHS goes to \( \infty \) as \( h \) goes to 0, and vice-versa, and it is strictly decreasing. Hence, there is still a unique steady state. And as in Section 4, we can again get a liquidity premium when the now endogenous supply of housing is high or when it is low, depending on elasticities.
As a special case, the model with $H$ fixed is recovered by making supply vertical. At the other extreme, if it is horizontal the price is pinned down by (the constant) marginal cost. But except for the very special case of horizontal supply, the results derived earlier survive.

**Proposition 4** The results in Propositions 2-3 all hold with $H$ endogenous.

Moving beyond steady states, an equilibrium is defined by a nonnegative and bounded path for $\{h_t, \psi_t\}$ satisfying (20)-(21). One should anticipate the existence of interesting dynamics in this bivariate system, given the univariate results. Instead of an exhaustive analysis of the dynamics here we instead use the system to organize a particular narrative concerning recent events. As the story goes, at the start of the episode in question, financial innovation gave households easier access to home equity loans – this is what it means to say they “turned their previously illiquid housing assets into ATM machines.” This stimulate a demand increase, and hence a run up followed by a down turn in house prices. We now show this can happen, in two distinct ways, without appealing to sunspots. One is to consider parameter values that imply a large set of equilibria, and try to select one that resembles the data. The other relies on financial innovation occurring gradually over time.

### 5.1 A Self-fulfilling Prophecy

Suppose at $t = 1$ the economy is in steady state, where $D(\epsilon) = D_1 \epsilon$ with $D_1$ small. To make a stark point, consider an unexpected, once-and-for-all, innovation at $t = 2$ that increases $D_1$. For some parameter values we get the analog of saddle-path stability.\(^{17}\) In this case, there is a unique equilibrium, where $\psi$ jumps at $t = 2$ and monotonically declines to a new steady state, as construction gradually raises the housing stock to meet higher demand. For other parameters, the system displays a classic indeterminacy, where the steady state is a sink. In this case, there are many perfect foresight equilibrium transition paths to the

\(^{17}\)Exactly as in the standard neoclassical growth model, we have one predetermined variable (quantity) and one jump variable (price), so determinacy requires steady state to be a saddle.
new steady state. This means $\psi$ can jump after the change in $D_1$ to any value in some interval, before beginning the transition. This gives us some freedom to pick a path that looks something like the episode in question.

One such transition is shown in Figure 5, constructed under Walrasian pricing, using parameters such that $y \leq D(e)$ is binding, and verifying numerically that both eigenvalues are real and less than 1 at the new steady state. This equilibrium looks like the data in Figure 1, starting around 2000, in the following sense. Of course they do not look *exactly* the same, but the paths are similar in that: prices first soar then tumble, whether we measure them by the price relative to numeraire or by the price-rent ratio; home equity loans go up, and stay up, as households take advantage of the financial innovation; and construction rises, then drops, as we approach the new steady state. Note that home equity lending rises quickly, even though the housing stock takes its time adjusting due to increasing marginal construction costs, because the price jump makes $e_t = \psi_t h_t$ rise even before the quantity $h_t$ rises, naturally.

What is perhaps less obvious is that welfare increases over the period, as shown by the black path in Figure 5. Financial development, formalized here as an increase in $D_1$, is a good thing for the economy because it relaxes credit constraints, even though it can lead to a path that resembles a bubble, complete with collapse. Now in the real world, some agents have had the bad fortune to buy high and sell low; this paper is not about redistributive effects (on that note see, e.g., Kiyotaki et al. 2011). For the representative household, financial development, like innovation in other areas, such as transportation, can be a good thing on the whole even if some people get hurt.

5.2 A Determinate Transition?

This above example of a boom-and-bust housing market equilibrium has at least two issues. First, it relies on extreme parameters, and in particular it uses a very low $\beta = 0.6$. Second, it
relies on a multiplicity that makes it difficult to say what theory predicts – we looked at one of many equilibria; there are others that look different. Moreover, it seems a stretch to think that there was a once-and-or-all innovation that is well captured by an unanticipated shock to $D_1$. More likely, financial innovation occurred more gradually and was implemented over several years. Agents might have expected some developments, while others were perhaps more of a surprise. We now construct an equilibrium resembling the data without appealing to multiplicity, based on our earlier result that says an increase in $D_1$ can reduce the price because it means less housing goes a longer way in satisfying one’s need for collateral. Thus, the ultimate price drop in the following example is due to housing ultimately becoming so good as collateral that liquidity premium is not so high.

The outcome hinges on what agents know and when they know it. If we take the ratio of home equity credit to the total value of residential fixed assets from 1996 to 2010, then divide it by $\alpha_h$, we construct an empirical measure of $D_1$. For this exercise we set $\alpha_h = 0.1$, the approximate proportion of home owners with home equity credit. The $D_1$ series constructed in this way has three major changes. From 2000 to 2003, $D_1$ changes gradually from around 0.11 to 0.20. In 2004, $D_1$ jumps from 0.2 to 0.26. Then in 2008 it suddenly increases from 0.26 to around 0.33. Suppose in 1996 that households expect $D_1$ to go up in the future. Precisely, they predict that $D_1$ stays constant at 0.11 until 2000, then starts to increase gradually to 0.2, where it will stay. But when 2003 comes around, the increase in $D_1$ is more than expected. Now they believe $D_1$ will increase to 0.26 and stay there. After four years, however, it keeps increasing until households can borrow up to 1/3 of their home equity. In 2008, suppose they believe $D_1$ will jump to 0.8 and stay there forever.

The transition is displayed figure 6, with blue showing prices, red home equity loans and green residential investment (normalized to fit on the chart). In this example, housing prices peaks in 2007 at about 11% above the price in 1996, and the price-rent ratio undergoes a similar change. This is about a quarter of the actual change in the data, which leads us
to conclude that the story can account for a sizable fraction but certainly not the majority of the observations of interest. After 2007, the price drops suddenly by around 7%. Home equity loans triple over this period, about the same as in data. So, in this example, with $D_1$ recovered from the data, the model generates an equilibrium path for price that is qualitatively similar to the data, but the boom is not as big quantitatively, with the equilibrium price path accounting for about a quarter of the actual change. We can generate a big drop in price after 2007.

All of the examples are meant to be just that – examples, as opposed to more serious calibrations. More work could be done to better estimate expectations and to fine tune the parameter values, of course. The intention here is to illustrate how dynamic economic analysis can be used to think about the observations in question. There are pitfalls along the way. When one tries to use a specification with multiple equilibria, there is always a question of selection. When one tries to confront the data with a model that has a unique equilibrium, given expectations about structural changes, we have to ask what people knew and when did they know it. Still, we find it useful to see how far these models can go in terms of generating time series that look at least somewhat like the data.

6 Housing, Money and Banking

In the model, so far, households put up home equity as collateral for consumption loans. This is consistent in a stylized way with experience, but in reality, more typically households use home equity to borrow cash from banks, then use the cash to buy consumption goods. Here we model this explicitly, not only for the sake of realism, but to introduce a role for monetary policy, which some people think affects housing markets.\footnote{Aruoba et al. (2012) give references to a literature on housing and inflation. Li and Li (2010) give references to other models where real assets are used to secure cash loans.} We assume for the sake of this discussion that money is the only means of payment accepted in KM. As is standard in modern monetary theory, this can be formalized using the assumption that KM...
trade is anonymous. Thus, if a seller does not know the identity of an individual buyer, and if the buyer were to try to secure a loan with equity in a house, he could put up a claim on a nonexistent house, the house of a stranger, one that is under water, etc. Nevertheless, buyer have relationships with their banks, which know their identities and keep records of the actions, making bank loans possible while retail loans are not.

For simplicity, housing stock is again fixed at $H$ and $\delta = 0$. Also, each period, with probability $\alpha_h$ a household wants to consume the KM good and with probability $1 - \alpha_h$ does not, but conditional on wanting to consume a household trades with probability 1. At the start of each period, before KM convenes, households have access to cash they brought in from the previous period, and can also access a financial market, FM. We describe FM in terms of intermediaries called banks that work as follows. Households that want to consume in the KM (borrowers) may withdraw cash from banks to increase their purchasing power, while those that do not (depositors) keep their money in the bank. Settlement occurs in the next AD market. While FM is competitive, we maintain the assumption of limited commitment: households can renege on bank loans, and hence home equity is used as collateral. It is not important, and cannot be determined, who carries money out of the AD into the next period, since it can be reallocated in FM. Therefore, we assume all currency is deposited in banks at the end of each AD, and those that want to consume in KM withdraw cash – generally, more than their deposits – while the rest do not.\textsuperscript{19}

The generalized AD value function is $W_t(d_t, h_t, m_t)$, where a household’s portfolio now consists of debt, housing and money in the bank. The FM value function is $J_{t+1}(h_{t+1}, m_{t+1})$, given debt is paid off in the AD. The AD problem is

\[
W_t(d_t, h_t, m_t) = \max_{x_t, \ell_t, h_{t+1}, m_{t+1}} \{ U(x_t, h_t) - \ell_t + \beta J_{t+1}(h_{t+1}, m_{t+1}) \}
\]

subject to

\[
x_t + \psi_t h_{t+1} + \phi_t m_{t+1} = \ell_t + \psi_t h_t + T_t + \phi_t m_t - d_t
\]

\textsuperscript{19}This is the model of banking in Berentsen et al. (2007). It is similar to the classic model of Diamond and Dybvig (1983), in that banking provides liquidity insurance, except that our banks deal in cash.
where $\phi_t$ is the value of a dollar in terms of the AD numeraire. Eliminating $\ell_t$ and taking FOC's, we get

$$U_1(x_t, h_t) = 1, \quad \psi_t = \beta \frac{\partial J_{t+1}}{\partial m_{t+1}} \quad \text{and} \quad \phi_t = \beta \frac{\partial J_{t+1}}{\partial m_{t+1}}.$$  \hspace{1cm} (24)

Hence, $(x_t, h_{t+1}, m_{t+1})$ is independent of $(d_t, h_t, m_t)$ and $W_t$ is linear in wealth.

The FM value function satisfies

$$J_t(h_t, m_t) = \alpha_h \max_{\hat{m}_t} V_t[B(1 + \rho_t)(\hat{m}_t - m_t) \phi_t, h_t, \hat{m}_t] + (1 - \alpha_h) W_t[-(1 + \rho_t)m_t \phi_t, h_t, 0]$$

subject to $(1 + \rho_t)(\hat{m}_t - m_t) \phi_t \leq D(\psi_t h_t),$ where $\rho_t$ is the interest rate and $D(\psi_t h_t)$ the borrowing limit on bank loans. Thus, with probability $\alpha_h$ the household increases $m_t$ to $\hat{m}_t$, spends it in KM, and has a real obligation in the next AD of $(1 + \rho_t)(\hat{m}_t - m_t) \phi_t$; and with probability $1 - \alpha_h$ they leave their money in the bank, skip KM, and enter the next AD with deposits (a negative liability) of $-(1 + \rho_t)m_t \phi_t$. Here $V_t$ is the KM value function conditional on wanting to consume,

$$V_t(d_t, h_t, m_t) = u(y_t) - \phi_t \hat{m}_t + W_t(d_t, h_t, m_t)$$  \hspace{1cm} (25)

where, again, an abstract mechanism determines $g(y_t) = \phi_t \hat{m}_t$.

The outcome depends on whether the FM borrowing constraint, $(1 + \rho_t)(\hat{m}_t - m_t) \phi_t \leq D(\psi_t h_t)$, binds. In Case 1, where it does not bind, the FOC for $\hat{m}$ is

$$-(1 + \rho_t) \phi_t + \frac{\partial V_t}{\partial \hat{m}_t} = 0.$$  \hspace{1cm} (26)

Using (25), this reduces to $L(y_t) = \rho_t$, which sets the marginal benefit of liquidity to the loan rate. In terms of the choice of money coming out of the AD, as opposed to coming out of the FM, the FOC for $m_{t+1}$ leads to the Euler equation

$$(1 + r) \phi_t = (1 + \rho_{t+1}) \phi_{t+1}.$$  \hspace{1cm} (27)

Let $i_t$ be the nominal interest rate that makes agents willing to give up a dollar in AD at $t$ and get back $1 + i_t$ dollars in AD at $t + 1$.\(^{20}\) The standard Fisher Equation says

\(^{20}\)We are thinking a book entry on some balance sheet here, not a “tangible” nominal bond that might be traded in the KW market. To say it differently, this nominal asset is illiquid.
1 + i_t = \phi_t / \beta \phi_{t+1}, \text{ since } \phi_t / \phi_{t+1} = 1 + \pi_t \text{ is inflation and } 1 / \beta = 1 + r_t \text{ is the real interest rate. Hence (27) says } \rho_{t+1} = i \text{ in this case, and the Euler equation for housing is}

\[(1 + r) \psi_t = \psi_{t+1} + U_2 (x_{t+1}, h_{t+1}).\]

When the borrowing constraint is slack, housing is priced fundamentally.

Consider next Case 2, where the borrowing constraint binds. Then households who want to consume in KM borrow to the limit in FM, and

\[J_t(h_t, m_t) = \alpha_h V_t \left[ D(\psi_t h_t), h_t, m_t + \frac{D(\psi_t h_t)}{1 + \rho_t} \phi_t \right] + (1 - \alpha_h) W_t \left[-(1 + \rho_t) \phi_t m_t, h_t, 0 \right].\]

In this case, the Euler equation for money is

\[(1 + r) \phi_t = \alpha_h [L(y_{t+1}) + 1] \phi_{t+1} + (1 - \alpha_h) \rho_{t+1} \phi_{t+1}\]

(28)

and the Euler equation for housing is

\[(1 + r) \psi_t = U_2 (x_{t+1}, h_{t+1}) + \psi_{t+1} + \frac{\alpha_h D_1 \psi_{t+1}}{1 + \rho_{t+1}} [L(y_{t+1}) - \rho_{t+1}] ,\]

(29)

which has an extra term, compared to (14), because now housing relaxes the binding borrowing constraint.

There are two subcases in Case 2, where the constraint binds. In Case 2a, bank lending exhausts the deposits, so there is no idle cash sitting in the vault, and we need \( \rho_{t+1} > 0 \) to clear the loan market. This yields \( g(y_{t+1}) = D(\psi_{t+1} h_{t+1}) / (1 - \alpha_h) \). In Case 2b, when all \( \alpha_h \) borrowers borrow to their limit we do not exhaust deposits, which means \( \rho_{t+1} = 0 \). Then the Euler equation for housing is

\[(1 + r) \psi_t = \psi_{t+1} + U_2 (x_{t+1}, h_{t+1}) + D_1 \psi_{t+1} i.\]

Two conditions are relevant to determine which case obtains. One is the individual debt limit: can individuals in FM borrow as much as they want? If \( D(e) \) is low, they are constrained in FM and \( H \) bears a premium. The other is an aggregate condition: can deposits satisfy all \( \alpha_h \) of the borrowers? If so, there is idle cash and \( \rho = 0 \).
So there are three possibilities: 1) the aggregate and individual limits are slack; 2a) the individual limit binds but the aggregate is slack; and 2b) both bind. Since the outcome depends on $D(\psi,t) = D_0 + D_1\psi,t$, we partition $(D_0, D_1)$ space into regions according to which case obtains. In Appendix B we define $D_1 = B_1(D_0)$ and $D_1 = B_2(D_0) \geq B_1(D_0)$, two downward sloping curves in $(D_0, D_1)$ space. Figure 7 shows that for large $D_0$ and $D_1$, we get Case 1 where borrowing limits do not bind; as $D_0$ and $D_1$ decrease, we move to Case 2a, where deposits are exhausted and $\rho \in (0, i)$; and finally, as $D_0$ and $D_1$ decrease further we move to Case 2b, with idle deposits and $\rho = 0$.

Now consider the impact of monetary policy, in terms of changes in the nominal rate $i$. In Case 1, where households are unconstrained and $\rho = i$, houses are priced fundamentally at $\psi^*$. In Case 2a, where $\rho \in (0, i)$ house prices may go up or down with $i$, depending on parameters, although they must go down under a condition given in Proposition 5 below.21 In Case 2b, where $\rho = 0$, real house prices rise with $i$, because higher nominal rates make money less valuable, so households want to move into their portfolios more $H$ and less $M$, leading to higher real house prices. Details are in Appendix B, and here we simply summarize the key results:

**Proposition 5** There is a unique steady state equilibrium, and it satisfies:

1) if $B_2(D_0) < D_1$ then $\rho = i$, $\psi = U_2/r$ and $\partial \psi / \partial i = 0$;

2a) if $B_1(D_0) < D_1 < B_2(D_0)$ then $\rho \in (0, i)$ and $\partial \psi / \partial i < 0$ iff $[1 + L(y)] g(y)$ is increasing in $y$;

2b) if $B_1(D_0) > D_1$ then $\rho = 0$, $\psi = U_2 / (r - iD_1)$ and $\partial \psi / \partial i > 0$.

The main finding in terms of policy is that higher nominal rates have ambiguous effects on the housing market, but we provide precise conditions concerning how they depend on parameters, including the tightness of borrowing constraints. Figure 8 illustrates the

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21This is similar to a condition in Li and Li (2010), although they only have only financial assets, and not housing.
outcome for an example. House prices, shown in blue, first increase, then decrease with the exogenous nominal rate $i$, until they eventually become independent of $i$. The endogenous interest rate paid on deposits and charged on loans by commercial banks is shown in red. Also shown in green is KM consumption $y$. Other results can be derived – e.g., we can verify analytically that $y$ must decrease with $i$ in all three equilibria. One can also generalize this monetary version of model to have endogenous $H$, one can consider optimal policy etc. We leave this to future work. The goal of this Section is mainly to show that, once we have a liquidity-based theory of housing markets, we can naturally think about the effects of monetary policy on housing markets in a new light.

7 Conclusion

In this project we studied economies where houses, in addition to providing utility as shelter, can be used as collateral for consumption loans, either directly through retailers, or through commercial banks. This formalizes some ideas mentioned in the Introduction about recent financial innovations allowing consumers to “treat their homes as cash machines.” In equilibrium, housing can bear a liquidity premium, so that prices are above fundamental values. We showed this implies that equilibrium house prices can exhibit a variety of dynamic patterns, some of which look somewhat like bubbles, while maintaining consistency with observations on home equity loans and housing investment. We provided a variety of numerical examples to illustrate the points, using various different pricing mechanisms, and imposing either perfect foresight or rational expectations. While some of the examples seem qualitatively similar to actual housing markets, future work might investigate in more detail how well this kind of theory can do quantitatively.

To summarize, housing markets can display interesting price dynamics in several ways. First, there is an inherent tendency for the price of any liquid asset, including housing, to oscillate, giving rises to deterministic equilibrium cycles. As an extreme case, we proved
that house prices can be chaotic. These results make the point that housing markets can fluctuate even if fundamentals are constant, but they do not resemble the classic bubble pattern of a long runup followed by a collapse. So we provided a different example with exactly this pattern, as a perfect foresight equilibrium, based on the nonlinear dynamics inherent in liquidity. However, in this example, as we proved must be the case in any deterministic equilibrium, capital gains are bounded by the rate of time preference. Hence we considered sunspot equilibria, where markets fluctuate randomly even if fundamentals are deterministic. In these equilibria the expected capital gain is bounded, but it is possible to have realizations with large capital gains several periods in a row. Of course this happens with a low probability, but that is not inconsistent with historical experience.

These results were for a model with fixed supply $H$. We then endogenized $H$ by introducing competitive home building (the model with fixed supply is a special limiting case, as is the other extreme, where price is pinned down by constant marginal cost). Using this version, one can try to capture quantity as well as price movements, and we presented two examples. In one, steady state was a sink in price-quantity space, so that after a one-time financial innovation there are a great many equilibrium paths, and we showed how we can select some that look something like recent experience. A lesson from that example is that bubble experiences can consistent with welfare gains that can last long after the bubble breaks. In the other example, steady state was a saddle, with a unique equilibrium path. We tried to calibrate the dynamics of exogenous financial innovation a little more carefully here, to see how much of actual price movements could be captured. That example easily accounts for 1/4 of the price runup, as well as the collapse, while maintaining consistency with home equity loans and housing investment over the period. We also showed how monetary policy can affect house prices, although there is more to do on that dimension.

For the main analysis, we assumed exogenous credit limits. In Appendix C we show how to endogenize the unsecured borrowing limit $D_0$, as is standard in the relevant literature,
by assuming defaulters can be punished. In one version we take away defaulters unsecured
credit in the future, but they can still use collateralized loans, since the originators of these
loans are not worried about default. In another version we take away all future credit from
defaulters. The main results go through with endogenous credit limits, and some new results
emerge – e.g., in the version we take away all future credit from defaulters, the existence
of secured credit makes it easier to sustain unsecured credit, so one might say that the
two types of loans are complements. All these applications and examples were intended to
illustrate that there are many productive ways to think about housing markets and credit
markets using liquidity-based theory, and using this approach allows us to generate, without
deviating from rational expectations, equilibria with some qualitatively interesting patterns.
We would like to know the extent to which one can do better quantitatively; a more careful
empirical study is relegated to future work.
Appendix A
Consider the following extensive-form bargaining game:

**Stage 1:** The seller offers \((y_t, d_t)\).

**Stage 2:** The buyer responds by accepting or rejecting, where:

- accept implies trade at these terms;
- reject implies they go to stage 3.

**Stage 3:** There is a coin toss, such that:

- with probability \(\theta\), the buyer makes a take-it-or-leave-it offer;
- with probability \(1 - \theta\), the seller makes a take-it-or-leave-it offer.

We impose that any offer must satisfy \(d_t \leq D_t\). We claim there is a unique SPE, characterized by acceptance of the initial Stage 1 offer, given by

\[
(y_t, d_t) = \arg \max_{S_{st}} \text{st } S_{st} = \theta [u(\bar{y}_t) - \bar{d}_t] \text{ and } d \leq D_t,
\]

(30)

where \((\bar{y}_t, \bar{d}_t)\) is the offer a buyer would make if (off the equilibrium path) we were to reach Stage 3.

The first observation is that, off the equilibrium path, if bargaining were to go to Stage 3 and the buyer got to make a final offer, he would offer \((\bar{y}, \bar{d})\) where:

\[
\bar{y} = \begin{cases} 
    v^{-1}(D_t) & \text{if } D_t < v(y^*) \\
    y^* & \text{if } D_t \geq v(y^*)
\end{cases}
\]

\[
\bar{d} = \begin{cases} 
    D_t & \text{if } D_t < v(y^*) \\
    v(y^*) & \text{if } D_t \geq v(y^*)
\end{cases}
\]

There are four possible cases: 1) the constraint \(d \leq D\) is slack at the initial and final offer stage; 2) it binds in the initial but not the final offer stage; 3) it binds in both; and 4) it binds in the final but not the initial offer stage. It is easy to check that case 4 cannot arise, so we are left with three.

**Case 1:** In the final offer stage, if the buyer proposes, his problem is

\[
\max_{y,d} \{u(y) - d\} \text{ st } d = v(y),
\]

with solution \(y = y^*\) and \(d = v(y^*)\). If the seller proposes the buyer gets no surplus, so the buyer’s expected surplus before the coin flip is \(\theta [u(y^*) - v(y^*)]\). Therefore, in the initial offer stage, the seller’s problem is

\[
\max_{y,d} \{d - v(y)\} \text{ st } u(y) - d = \theta [u(y^*) - v(y^*)],
\]
with solution $y = y^*$ and $d = d^* = (1 - \theta) u(y^*) + \theta v(y^*)$. Since $d^* > v(y^*)$, this case occurs iff $D > d^*$.

Case 2: The buyer’s expected payoff before the coin flip is again $\theta [u(y^*) - v(y^*)]$, but at the initial offer stage the constraint binds, so the seller solves

$$\max_y \left\{ D - v(y) \right\} \text{ st } u(y) - D = \theta [u(y^*) - v(y^*)].$$

The solution satisfies $u(y) = D + \theta [u(y^*) - v(y^*)]$ and $d = D$. This case occurs iff $v(y^*) < D < d^*$.

Case 3: In the final offer stage, if the buyer proposes, his problem is

$$\max_y \left\{ u(y) - D \right\} \text{ st } D = v(y).$$

This implies $y = v^{-1}(D)$, and his expected surplus before the coin flip is $\theta [u \circ v^{-1}(D) - D]$. At the initial offer stage, the seller’s problem is

$$\max_y \left\{ D - v(y) \right\} \text{ st } u(y) - D = \theta [u \circ v^{-1}(D) - D].$$

The solution satisfies $u(y) = \theta u \circ v^{-1}(D) + (1 - \theta) D$ and $d = D$. This case occurs iff $D < v(y^*)$ and $D < u(y^*) - \theta u \circ v^{-1}(D) + \theta D$, the last inequality coming from the observation that, at the first stage, if the constraint is slack, the buyer pays $u(y^*) - \theta u \circ v^{-1}(D) + \theta D$ to get $y^*$. This last inequality is equivalent to $(1 - \theta) D < u(y^*) - \theta u \circ v^{-1}(D)$, which always holds if $D < v(y^*)$.

To sum up, $d = D$ if $D < d^*$ and $d = d^*$ id $d^* \leq D$; and $y$ is given by

$$y = \begin{cases} 
    u^{-1} \left[ \theta u \circ v^{-1}(D) + (1 - \theta) D \right] & \text{if } D < v(y^*) \\
    u^{-1} \left[ D + \theta [u(y^*) - v(y^*)] \right] & \text{if } v(y^*) < D < d^* \\
    y^* & \text{if } D > d^*
\end{cases}$$

It is easy to check $y = g^{-1}(D)$ is differentiable and strictly increasing for $D < d^*$.

Appendix B

Here we verify the results in Proposition 5, and derive

$$B_1(D_0) = \begin{cases} 
    \frac{r [g(\tilde{y})(1 - \alpha_h) - D_0]}{i g(\tilde{y})(1 - \alpha_h) - i D_0 + HU_2} & \text{if } D_0 < g(\tilde{y})(1 - \alpha_h) \\
    0 & \text{if } D_0 > g(\tilde{y})(1 - \alpha_h)
\end{cases}$$

$$B_2(D_0) = \max \left\{ \frac{r [g(\tilde{y})(1 - \alpha_h)(1 + i) - D_0]}{HU_2}, 0 \right\}$$

where $\tilde{y}$ and $\bar{y}$ satisfy $L(\tilde{y}) = i/\alpha_h$ and $L(\bar{y}) = i$. 

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Case 1: The borrowing constraint is not binding. In steady state, we have

\[ i = \mathcal{L}(y), \quad \rho = i \]

\[ r \psi = U_2 [X(H), H] \]

\[ g(y) < \frac{D_0 + D_1 \psi H}{(1 + \rho)(1 - \alpha_h)}. \]

The last condition comes from two observations: when \( \rho > 0 \), to clear the market we must have \( g(y) = \phi_t M_t / \alpha_h \), as borrowing exhausts deposits; and when the borrowing constraint is not binding, \( (1 - \alpha_h) \phi_t M_t < \alpha_h (D_0 + D_1 \psi H) / (1 + \rho) \). We can easily see that this equilibrium exists if

\[ g(\bar{y}) < \frac{D_0 + D_1 \psi H}{(1 - \alpha_h)(1 + \rho)} \]

with \( \psi = U_2 / r \), or \( D_1 > g(\bar{y})(1 + i)(1 - \alpha_h) - D_0 \). Uniqueness follows immediately. Furthermore, \( \partial \psi / \partial i = 0 \) and \( \partial y / \partial i < 0 \).

Case 2: The borrowing constraint is binding. In steady state,

\[ i = \alpha_h \mathcal{L}(y) + (1 - \alpha_h) \rho \quad (31) \]

\[ r \psi = \alpha_h [\mathcal{L}(y) - \rho] \frac{\psi D_1}{1 + \rho} + U_2 [X(H), H] \quad (32) \]

\[ g(y) = \phi_t M_t + \frac{D_0 + D_1 \psi H}{1 + \rho}. \quad (33) \]

We now consider the subcases separately.

Case 2a: If \( \rho > 0 \), market clearing and a binding borrowing constraint imply

\[ \phi_t M_t = \frac{\alpha_h (D_0 + D_1 \psi H)}{(1 - \alpha_h)(1 + \rho)}. \quad (34) \]

Using (31), we get \( \rho = (i - \alpha_h \mathcal{L}) / (1 - \alpha_h) \). This, (34) and (33) yield

\[ \psi = \frac{g(y)[1 + i - 1 - \alpha_h \mathcal{L}(y)] - D_0}{D_1 H}. \]

Substituting these into (32), we get

\[ \frac{r}{D_1} = \frac{\alpha_h \mathcal{L}(y) - i}{1 + i - \pi [1 + \alpha_h \mathcal{L}(y)]} + \frac{HU_2 [X(H), H]}{g(y)[1 + i - \pi - \pi \alpha_h \mathcal{L}(y)] - D_0} \equiv \Phi(y). \quad (35) \]

The RHS is decreasing in \( y \), so there is at most one solution. Note in this subcase \( \rho < \mathcal{L}(y) \), implying \( 0 < \rho < i \). This and (31) imply \( \alpha_h i < \mathcal{L}(y) < i \). Consequently, this equilibrium
exists iff (35) has a solution in \((\bar{y}, \tilde{y})\), where \(L(\bar{y}) = i/\alpha_h\) and \(L(\tilde{y}) = i\). This requires \(\Phi(\bar{y}) > r/D_1\) and \(\Phi(\tilde{y}) < r/D_1\), or \(B_1(D_0) < D_1 < B_2(D_0)\). One can derive
\[
\frac{\partial y}{\partial i} \approx -\frac{gD_1(L(y) + 1)\alpha_h}{\psi(1 + \rho)^2} - \frac{g\psi}{(1 + \rho)^2} = 0,
\]
\[
\frac{\partial \rho}{\partial i} \approx -\frac{\alpha_hL'D_1g}{(1 + \rho)\psi} + \frac{U_2(X(H), H)g'}{\psi^2} > 0,
\]
\[
\frac{\partial \psi}{\partial i} \approx -D_1\alpha_h [gL' + g(L(y) + 1)] \approx -\frac{d}{dy}[L(y) + 1]g(y).
\]
Therefore, \(\partial \psi/\partial i < 0\) iff \([L(y) + 1]g(y)\) is increasing.

Case 2b: If \(\rho = 0\), steady state is characterized by
\[
i/\alpha_h = L(y) \quad \text{(36)}
\]
\[
r\psi = i\psi D_1 + U_2[X(H), H] \quad \text{(37)}
\]
\[
g(y) > \frac{D(\psi H)}{1 - \alpha_h}. \quad \text{(38)}
\]
In this subcase, (36) determines \(y\) and (37) determines \(\psi\). This equilibrium exists iff (38) holds given \(y\) and \(\psi\), which leads to \(B_1(D_0) > D_1\). It is obvious in this case that \(\partial y/\partial i < 0\) and \(\partial \psi/\partial i > 0\).

Appendix C
Suppose \(D_1\) is exogenous, and consider making \(D_0\) endogenous. Debtors always repay secured loans, given \(D_1 \leq 1\), but may renege on unsecured loans. We consider two types of punishment: 1) take away a defaulter’s unsecured credit; 2) take away all future credit. To simplify the exposition, let \(U(x, h) = U(x) + v(h)\) and \(\alpha_h = 1\). Then AD consumption is fixed, and we normalize \(U(X) - X = 0\) (by adding a constant to utility) with no loss of generality.

The KM value function along the equilibrium (no-fault) path and after a deviation (default) are
\[
V^E = \frac{1}{1 - \beta} \{u(y^E) + v(h^E) - d^E\}
\]
\[
V^D = \frac{1}{1 - \beta} \{u(y^D) + v(h^D) - d^D\},
\]
where \((y^E, h^E, d^E)\) and \((y^D, h^D, d^D)\) both depend on \(D_0\), and on the punishment. The repayment constraint is
\[
v(h^E) - d_0 + \beta V^E \geq v(h^E) + \psi(h^E - h^D) + \beta V^D.
\]
Combining these, the maximum unsecured debt \( \tilde{D}_0 \) one is willing to repay is

\[
\tilde{D}_0 = T \left( D_0 \right) \equiv \frac{1}{r} \left\{ u \left( y^E \right) + v \left( h^E - d^E \right) \right\} - \frac{1}{r} \left\{ u \left( y^D \right) + v \left( h^D - d^D \right) \right\} - \psi \left( h^E - h^D \right),
\]

where

\[
y^E = \min \left\{ g^{-1} \left( D_0 + D_1 \psi H \right), y^* \right\}, \quad d^E = \min \left\{ g \left( y^* \right), D_0 + D_1 \psi \right\},
\]

\[
r\psi = v' \left( H \right) + \psi D_1 \mathcal{L} \left( y^E \right).
\]

Exactly as in Alvarex-Jermann, an equilibrium credit limit is a fixed point of \( T \left( D_0 \right) \).

**Punishment 1:** Given \( \psi, y^D, h^D \) and \( d^D \) we solve

\[
y^D = \min \left\{ g^{-1} \left( D_1 \psi h^D \right), y^* \right\}, \quad d^D = \min \left\{ g \left( y^* \right), D_1 \psi h^D \right\},
\]

\[
r\psi = v' \left( h^D \right) + \psi D_1 \mathcal{L} \left( y^D \right).
\]

Obviously \( T \left( 0 \right) = 0 \). Moreover, as \( D_1 \to 0, T' \left( 0 \right) \to \mathcal{L} \left( 0 \right) /r \). If \( D_1 \) is sufficiently small and \( \mathcal{L} \left( 0 \right) > r \), there exists an equilibrium with unsecured credit. If \( D_1 \) is too big, no unsecured credit can exist.

**Punishment 2:** Now \( y^D = 0, h^D = v'^{-1} \left( r\psi \right) \leq h^E = H, d^D = 0 \) and

\[
T \left( D_0 \right) = \frac{1}{r} \left[ u \left( y^E \right) + v \left( h^E - d^E - v \left( h^D \right) \right) \right] - \psi \left( h^E - h^D \right).
\]

Notice \( T \left( 0 \right) > 0 \) and, if \( D_0 \) is big enough that non-defaulters are not constrained, \( T \left( D_0 \right) \) is constant. So a strictly positive endogenous unsecured credit limit always exists. Also, \( T \) is increasing in \( D_1 \). To see this, if non-defaulters are constrained,

\[
\frac{\partial T \left( D_0 \right)}{\partial D_1} = \frac{1}{r} \mathcal{L} \left( y^E \right) H \frac{dD_1 \psi}{dD_1} - \frac{d\psi}{dD_1} \left( H - h^D \right).
\]

As \( d\psi/dD_1 = \psi \left( \mathcal{L} + \psi D_1 H \mathcal{L}' y' \right) / \left( r - D_1 \mathcal{L} - D_1^2 \psi \mathcal{L}' y' \mathcal{H} \right) \), we have

\[
\frac{\partial T \left( D_0 \right)}{\partial D_1} = \psi h^D \mathcal{L} - \psi^2 D_1 H \mathcal{L}' y' \left( H - h^D \right) / \left( r - D_1 \mathcal{L} - D_1^2 \psi \mathcal{L}' y' H \right) > 0.
\]

If non-defaulters are not constrained, \( \partial T \left( D_0 \right) / \partial D_1 = 0 \). So \( \partial T \left( D_0 \right) / \partial D_1 \geq 0 \). Since \( T \) is weakly increasing in \( D_1 \), the equilibrium credit limit is too. In this case, secured credit helps unsecured credit.
References


Figure 1: Housing Sector and Home Equity Loans

Figure 2: Trading Mechanism
Figure 3: Housing Price Dynamics

Figure 4: Price Dynamics
Figure 5: Housing Price Dynamics: One Period Unexpected Change in Financial Market

Figure 6: Housing Price Dynamics: Transition Path Trigered by Gradual Change in Financial Market