Ambiguous Business Cycles[*]

COSMIN ILUT
Duke University

MARTIN SCHNEIDER
Stanford University and NBER

This draft: January 2012

Abstract

This paper considers business cycle models with agents who dislike both risk and ambiguity (Knightian uncertainty). Ambiguity aversion is described by recursive multiple priors preferences that capture agents' lack of confidence in probability assessments. While modeling changes in risk typically requires higher-order approximations, changes in ambiguity in our models work like changes in conditional means. Our models thus allow for uncertainty shocks but can still be solved and estimated using first-order approximations. In our estimated medium-scale DSGE model, a loss of confidence about productivity works like 'unrealized' bad news. Time-varying confidence emerges as a major source of business cycle fluctuations.

1 Introduction

How do changes in aggregate uncertainty affect the business cycle? The standard framework of quantitative macroeconomics is based on expected utility preferences and rational expectations. A change in uncertainty is then typically modeled as an *anticipated change in risk, followed by a change in the magnitude of realized shocks.* Indeed, expected utility agents think about the uncertain future in terms of probabilities. An increase in uncertainty is described by the anticipated increase in a measure of risk (for example, the conditional variance of a shock or the conditional probability of a disaster). Moreover, rational expectations implies that agents' beliefs coincide with those of the econometrician (or model builder). As a result, an anticipated increase in risk is followed on average by unusually large shocks, reflecting the higher variance of shocks or the larger likelihood of disasters.

This paper studies business cycle models with agents who are averse to ambiguity (Knightian uncertainty). Ambiguity averse agents do not think in terms of probabilities – they lack the confidence to assign probabilities to all relevant events. An increase in uncertainty may then correspond to a loss of confidence that makes it more difficult to assign probabilities. Formally, we describe preferences using multiple priors utility (Gilboa and Schmeidler (1989)). Agents act *as if* they evaluate plans using a worst case probability drawn from a set of multiple beliefs. A loss of confidence is captured by an increase in that set of beliefs. It could be triggered, for example, by worrisome information about the future. Conversely, an increase in confidence is captured by a shrinkage of the set of beliefs – agents might learn reassuring information that moves them closer toward thinking in terms of probabilities. In either case, agents respond to a change in confidence if the worst case probability used to evaluate actions also changes.

The paper proposes a simple and tractable way to incorporate ambiguity and shocks to confidence into a business cycle model. At every date, agents' set of beliefs about an exogenous shock, such as an innovation to productivity, is parametrized by an interval of means centered around zero. A loss of confidence is captured by an increase in the width of the interval; in particular, the "worst case" mean becomes worse. Conversely, an increase in confidence is captured by a narrowing of the interval and thereby a better worst case mean. Since agents take actions based on the worst case mean, a change in confidence works like a news shock: an agent who gains (loses) confidence responds *as if* he had received good (bad) news about the future.

The difference between a change in confidence and news is that the latter is followed, at least on average, by a shock realization that validates the news. For example, bad news is on average followed by bad outcomes. A loss of confidence, however, need not be followed by a bad outcome. Information that triggers a loss of confidence affects agents' perception of future shocks, described by their (subjective) set of beliefs. It does not directly affect properties of the distribution of realized shocks (such as the direction or magnitude of the shocks). A connection between confidence and realized shocks occurs only if the two are explicitly assumed to be correlated.

We study ambiguity and confidence shocks in economies that are essentially linear. The key property is that the worst case mean that supports agents' equilibrium choices can be written as a linear function of the state variables. It implies that equilibria can be accurately characterized using first order approximations. In particular, we can study agents' responses to changes in uncertainty, as well as time variation in uncertainty premia on assets, without resorting to higher order approximations. This is in sharp contrast to the case of changes in risk, where higher order solutions are critical.

The effects of changes in confidence are what one would expect from changes in uncertainty. For example, a loss of confidence about productivity generates a wealth effect – due to more uncertain wage and capital income in the future, and also a substitution effect since the return on capital has become more uncertain. The net effect on macroeconomic aggregates depends on the details of the economy. In our estimated medium scale DSGE model, a loss of confidence generates a recession in which consumption, investment and hours decline together. In addition, a loss of confidence generates increased demand for safe assets, and opens up a spread between the returns on ambiguous assets (such as capital) and safe assets. Business cycles driven by changes in confidence thus give rise to countercyclical spreads or premia on uncertain assets.

To quantify the effects of ambiguity shocks in driving the US business cycle, we incorporate ambiguity averse households into an otherwise standard medium scale DSGE model based on Christiano et al. (2005) and Smets and Wouters (2007). Agents view neutral productivity shocks as ambiguous and their confidence about productivity varies over time, a type of "uncertainty shock". Even though uncertainty shocks are present, standard linearization methods can be used to solve the model and ease its Bayesian estimation. The main results are that the estimated confidence process (i) is persistent and volatile, (ii) has large effects on the steady state of endogenous variables, with welfare costs of ambiguity about 15% of steady state consumption, (iii) accounts for a sizable fraction of the variance in output - about 30% at business cycle frequencies and about 55% overall, (iv) generates positive comovement between consumption, investment and hours worked. It is this positive comovement that distinguishes confidence shocks from other shocks and leads the estimation to assign an important role to confidence in driving the business cycle.

We emphasize that changes in confidence can also generate booms that look "exuberant", in the sense that output and employment are unusually high and asset premia are unusually low, while fundamentals such as productivity are close to their long run average. An exuberant boom occurs in our model whenever there is a persistent shift in confidence above its mean. While agents in the model behave *on average* as if they are pessimistic, what matters for the nature of booms and busts is how variables move relative to model-implied averages. Confidence jointly moves asset premia and economic activity, thus generating booms and slumps. Moreover, the fact that agents behave as if they are pessimistic on average helps explain the magnitude of average asset premia, which are puzzlingly low in rational expectations models.

While we do not assume rational expectations, we do impose discipline on belief sets in the estimation by connecting them to the data generating process. To describe agents' perception of a shock, we decompose the estimated shock distribution into an ambiguous

3

and a risky component. For the risky component, agents know the probabilities. For the ambiguous component, they know only the long run empirical moments. When forecasting in the short run, however, agents realize that the data are consistent with many distinct models. Discipline comes from requiring that admissible beliefs make "good enough" forecasts on average, under any data generating process that is consistent with the long run moments. More models are "good enough" in this sense if the ambiguous component of the data is more variable. As a result, movements in confidence about a shock are effectively bounded by the variability of the shock.

As an additional check on the size and time variation of the estimated belief sets, we compare the implied range of output growth and inflation forecasts – a measure of ambiguity perceived about these variables – to the interquartile range of SPF survey forecasts for output growth and inflation. Disagreement of survey forecasters is often used as a measure of uncertainty since disagreement among experts plausibly reflects uncertainty about what the right model of the future is. We find both similar magnitudes (mean forecast ranges fluctuate between 0.5 and 2%) and qualitatively similar behavior. For example, both forecast dispersion and implied ambiguity were low in the boom years of the mid 1990s and increased at the start of recessions and especially in the 2008 Financial Crisis.

Our paper is related to several strands of literature. The decision theoretic literature on ambiguity aversion is motivated by the Ellsberg Paradox. Ellsberg's experiments suggest that decision makers' actions depend on their confidence in probability assessments – they treat lotteries with known odds differently from bets with unknown odds. The multiple priors model describes such behavior as a rational response to a lack of information about the odds. To model intertemporal decision making by agents in a business cycle model, we use a recursive version of the multiple priors model that was proposed by Epstein and Wang (1994) and has recently been applied in finance (see Epstein and Schneider (2010) for a discussion and a comparison to other models of ambiguity aversion). Axiomatic foundations for recursive multiple priors were provided by Epstein and Schneider (2003).

Hansen et al. (1999) and Cagetti et al. (2002) study business cycles models with robust control preferences. The "multiplier" preferences used by these authors assume a smooth penalty function for deviations of beliefs from some reference belief. Models of changes in uncertainty with robust control thus typically use tools developed for models of changes in risk, such as higher order approximations (for example, Bidder and Smith (2011)). Multiple priors utility is not smooth when belief sets differ in means. As a result, it allows for first order effects of uncertainty that do not arise under expected utility. This is why linear approximations are sufficient to study dynamics in response to uncertainty shocks.

Some of the mechanics of our model are reminiscent of rational expectations models with

signals about future "fundamentals" (for example Beaudry and Portier (2006), Christiano et al. (2008), Schmitt-Grohe and Uribe (2008), Jaimovich and Rebelo (2009), Blanchard et al. (2009),Christiano et al. (2010a) and Barsky and Sims (2011)). On impact, the response to a loss of confidence about productivity in our model resembles the response to noise that is mistaken for bad news about productivity in a model with noisy signals. The difference is that noise in a signal only matters for agents' decisions if the typical signal contains enough news. Put differently, noise matters for the business cycle only if news shocks are also present and sufficiently important. In contrast, confidence shocks and their role in the business cycle need not be connected to news shocks.[1]

Confidence shocks can affect agents' actions (and hence the business cycle) even if they are uncorrelated with shocks to "fundamentals" (such as productivity) at all leads and lags. They share this feature with noise shocks as well as with sunspots (for example Farmer (2009)), stochastic bubbles (Martin and Ventura (2011)) and shocks to higher order beliefs (Angeletos and La'O (2011)). At the same time, confidence shocks differ from those other shocks in that they alter agents' perceived uncertainty of fundamentals. This is typically reflected in asset premia. Depending on the application, this need not be true for the other types of shocks.

Recent work on changes in uncertainty in business cycle models has focused on changes in realized risk – looking either at stochastic volatility of aggregate shocks (see for example Fernández-Villaverde and Rubio-Ramirez (2007), Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2010), Basu and Bundick (2011) and the review in Fernández-Villaverde and Rubio-Ramírez (2010)), time-varying probability of aggregate disaster (Gourio (2011)) or at changes in idiosyncratic volatility in models with heterogeneous firms (Bloom et al. (2009), Arellano et al. (2010), Bachmann et al. (2010) and Christiano et al. (2010b)). We view our work as complementary to these approaches. In particular, confidence shocks can generate responses to uncertainty – triggered by news, for example – that is not connected to later realized changes in risk.

The paper proceeds as follows. Section 2 reviews multiple priors utility and shows how ambiguity about the mean of a bet entails first order effects of uncertainty. Section 3 analyzes a stylized business cycle model with ambiguity about productivity. Section 4 presents a general framework for adapting business cycle models to incorporate ambiguity aversion. Here we show how uncertainty can be studied using linear techniques. Section 5 describes the estimation of a medium scale DSGE model for the US.

---

[1]This distinction between noise, news and confidence and its observable implications are discussed further in Section 3 below.

2 Recursive multiple priors

Ellsberg (1961) showed that there is a behaviorally meaningful distinction between risk (uncertainty with known odds or objectively given probabilities) and ambiguity (unknown odds). For example, many people strictly prefer to bet on an urn that is known to contain an equal number of black and white balls than on an urn of unknown composition. Gilboa and Schmeidler (1989) showed that Ellsberg-type behavior can be derived from a model of rational choice with axiomatic foundations. Their axioms allow for a "preference for knowing the odds" that is ruled out under expected utility.

For our business cycle application, we use a recursive version of the multiple priors model. Uncertainty is represented by a period state space S. One element $s \in S$ is realized every period, and the history of states up to date t is denoted by $s^t = (s_0, ..., s_t)$. Preferences order uncertain streams of consumption $C = (C_t)_{t=0}^{\infty}$, where $C_t : S^t \to \Re^n$ and n is the number of goods. Utility for a consumption process $C = \{C_t\}$ is defined recursively by

$$U_t\left(C; s^t\right) = u\left(C_t\right) + \beta \min_{p \in \mathcal{P}_t(s^t)} E^p\left[U_{t+1}\left(C; s^t, \tilde{s}_{t+1}\right)\right], \tag{2.1}$$

where $\mathcal{P}_t\left(s^t\right)$ is a set of probabilities on S that govern the distribution of next period state $\tilde{s}_{t+1}$.

Utility after history s^t is given by felicity from current consumption plus expected continuation utility evaluated under a "worst case" belief. The worst case belief is drawn from a set $\mathcal{P}_t\left(s^t\right)$ that may depend on the history s^t. The primitives of the model are the felicity u, the discount factor β and the entire process of one-step-ahead belief sets $\mathcal{P}_t\left(s^t\right)$. Expected utility obtains as a special case if all sets $\mathcal{P}_t\left(s^t\right)$ contain only one belief. More generally, a nondegenerate set of beliefs captures the agent's lack of confidence in probability assessments; a larger set $\mathcal{P}_t\left(s^t\right)$ says that the agent is less confident after having observed history s^t.

Discussion

The maxmin representation (2.1) is implied by a preference for knowing the odds. Gilboa and Schmeidler assume that one can observe choice among state contingent consumption *lotteries*, as in Anscombe and Aumann (1963) axiomatization of the (subjective) expected utility model. Lotteries are a source of "objective" uncertainty (known odds) whereas for the state s there are no objectively given probabilities. Observing choice between state contingent lotteries can then identify whether or not agents deal with uncertainty about s *as if* they think in terms of probabilities.

In particular, the necessary conditions for an expected utility representation in Anscombe

and Aumann (1963) include the independence axiom. It says that C is strictly preferred to $\tilde{C}$ if and only if any lottery between C and some other plan D is also strictly preferred to a lottery with the same probabilities on $\tilde{C}$ and D. It implies that an agent who is indifferent between two consumption plans C and $\tilde{C}$ never strictly prefers a lottery between those two plans. This latter implication contradicts Ellsberg-type behavior. For example, if one plan is a hedge for the other then forming a lottery assigns known probabilities to otherwise ambiguous contingencies.[2] Put differently, randomizing between two indifferent ambiguous plans that hedge each other can transform ambiguity into risk, resulting in a strictly more desirable plan.

Gilboa and Schmeidler show that a maxmin representation of utility follows if independence is replaced by two alternative axioms. *Uncertainty aversion* says that a lottery between indifferent plans is weakly preferred to either plan. The axiom thus allows (but does not require) strict preference for the lottery, weakening independence. *Certainty independence* says that the independence axiom holds as long as D is constant. This axiom says that strict preference for a lottery between indifferent plans can occur only if in fact one plan hedges the other. Randomizing with constant plans is not helpful because constant plans cannot be hedges. For example, a lottery between an ambiguous plan and its certainty equivalent is never strictly preferred to the certainty equivalent (and hence the plan itself). The latter property is shared by the multiple priors and expected utility models.

Epstein and Schneider (2003) provide foundations for the intertemporal model (2.1). They consider a family of conditional preference orderings, one for each history s^t. Each conditional preference ordering satisfies the Gilboa-Schmeidler axioms, suitably modified for multiperiod plans. Moreover, conditional preferences at different histories are connected by dynamic consistency.[3] The setup of the model also implies consequentialism, that is, utility from a consumption plan at history s^t depends only on the consumption promised in future histories that can still occur after s^t has been realized. Both dynamic consistency and consequentialism are properties that are also satisfied by the standard time separable expected utility model.

First order effects of uncertainty

Consider the effect of a small increase in uncertainty at a point of certainty. Fix a constant consumption bundle $\bar{C}$, a date t and a nonconstant function f that maps states

---

[2] For example, suppose C pays one if state s occurs and zero otherwise, and $\tilde{C}$ is a perfect hedge for C, that is, it pays one if s does not occur and zero otherwise. Then a lottery between C and $\tilde{C}$ is purely risky and may thus be preferable to the ambiguous plans C and $\tilde{C}$.

[3] In addition to the recursive representation (2.1), the model also allows for a "sequence" representation where minimization at every node is over probabilities over future sequences of state s and where those probabilities are updated measure-by-measure by Bayes' rule.

into consumption bundles – a "bet" on the state one period ahead at $t+1$. Construct an uncertain consumption plan C by setting $C_\tau = \bar{C}$ for all $\tau \neq t+1$, but $C_{t+1} = \bar{C} + \alpha f(\tilde{s}_{t+1})$ where α is a scalar. We expand utility around certainty ($\alpha = 0$). For $\alpha > 0$, we obtain

$$
\begin{aligned}
U\left(C; s^t\right) &= U\left(\bar{C}; s^t\right) + \beta \min_{p \in \mathcal{P}_t(s^t)} E^p[u\left(\bar{C} + \alpha f(\tilde{s}_{t+1})\right)] \\
&= U\left(\bar{C}; s^t\right) + \beta \left(\alpha u'\left(\bar{C}\right) E^{p^0}[f\left(\tilde{s}_{t+1}\right)] + \frac{1}{2}\alpha^2 u''\left(\bar{C}\right) E^{p^0}\left[f\left(\tilde{s}_{t+1}\right)^2\right] + ... \right), \quad (2.2)
\end{aligned}
$$

where higher order terms are omitted and p^0 is the belief that achieves the minimum. A similar expansion holds for $\alpha < 0$, but the minimizing belief p^0 may be different since consumption depends differently on the state s.

We now compare the welfare effects of small changes in uncertainty by varying α (the scale of the bet) near zero. To isolate the effect of uncertainty, we focus on bets f such that for *any* small $\alpha \neq 0$, the agent is strictly worse off than at certainty ($\alpha = 0$). In the case of a risk averse agent with expected utility (u is strictly concave and $\mathcal{P}_t(s^t) = \{p^0\}$) this simply means that the bet f is actuarially fair, or $E^{p^0}[f(\tilde{s}_{t+1})] = 0$.[4] Consider now the magnitude of the utility loss from an increase in risk as α moves away from zero. The first order term in (2.2) vanishes, but the second term is negative because of risk aversion. We have restated the familiar result that, under expected utility, changes in risk have second order effects on utility near certainty.

Consider next a multiple priors agent. To make the argument as simple as possible, assume that this agent is risk neutral, so that all higher order terms in (2.2) vanish. In order for the agent to be worse off than at certainty for any small $\alpha \neq 0$ we must have

$$
\min_{p \in \mathcal{P}_t(s^t)} E^p f\left(\tilde{s}_{t+1}\right) < 0 < \max_{p \in \mathcal{P}_t(s^t)} E^p f\left(\tilde{s}_{t+1}\right). \quad (2.3)
$$

In other words, there is ambiguity about whether the bet is fair. The effects of uncertainty are now driven only by the first order terms. For example, if $\alpha > 0$ then the worst case mean is the negative lower bound, whereas if $\alpha < 0$ the worst case mean is the positive upper bound. Changes in ambiguity due to changes in the interval of means (2.3) thus have first order effects on welfare near certainty. More generally, if the agent is also risk averse, then higher order also terms matter, but changes in welfare near certainty are still dominated by the first order terms.

---

[4]Indeed, if f were not fair and had nonzero mean then for small enough α of the same sign as the mean, the first order effect in (2.2) dominates and we obtain higher utility than at $\alpha = 0$. Conversely, taking fair bets always lowers utility because u is strictly concave.

3   A stylized example

To illustrate the role of ambiguity in business cycles, we consider a stylized business cycle model. Our main criterion for this model is simplicity. We abstract from internal propagation of shocks through endogenous state variables such as capital or sticky prices or wages. For uncertainty about productivity to have an effect on labor hours and output, we assume that labor has to be chosen before productivity is known. This introduces an intertemporal decision that depends on both risk and ambiguity. In fact, with the special preferences and technology we choose, the effects of both ambiguity and risk can be read off a loglinear closed form solution, which facilitates comparison.

A representative agent has felicity over consumption and labor hours

$$U\left(C,N\right) = \frac{1}{1-\gamma}C^{1-\gamma} - \beta N$$

where γ is the coefficient of relative risk aversion (CRRA) or equivalently the inverse of the intertemporal elasticity of consumption (IES). Agents discount the future with the discount factor β. Setting the marginal disutility of labor equal to β simplifies some algebra below by eliminating constant terms.

Output Y_t is made from labor N_t according to the linear technology.

$$Y_t = Z_t N_{t-1},$$

where $\log Z_t$ is random. The fruits of labor effort made at date $t-1$ thus only become available at date t. One interpretation is that goods have to be stored for some time before they can be consumed. It may be helpful to think of the period length as very short, such as a week.

For simplicity, we assume that log productivity $z_t = \log Z_t$ is serially independent and normally distributed. The productivity process takes the form

$$z_{t+1} = \mu_t - \frac{1}{2}\sigma_u^2 + u_{t+1} \tag{3.1}$$

Here u is an iid sequence of shocks, normally distributed with mean zero and variance σ_u^2. The sequence μ is deterministic and unknown to agents – its properties are discussed further below.

Preferences are given by (2.1) with felicity as above. Agents perceive the unknown component μ_t to be ambiguous. We parametrize their one-step-ahead set of beliefs at date t by a set of means $\mu_t^p \in [-a_t, a_t]$. Here a_t captures agents' lack of confidence in his probability

assessment of productivity z_{t+1}. We allow confidence itself to change over time to reflect, for example, news agents receive. We assume an AR(1) process for a_t:

$$a_{t+1} = (1 - \rho_a)\,\bar{a} + \rho_a a_t + \varepsilon_{t+1}^a \tag{3.2}$$

with $\bar{a} > 0$ and $0 < \rho_a < 1$. The lack of confidence a_t thus reverts to a long run mean $\bar{a}$. Periods of low $a_t < \bar{a}$ represent unusually high confidence in future productivity, whereas $a_t > \bar{a}$ describes periods of unusual lack of confidence. We further assume that ε_t^a is independent of u_t.[5]

Consider now the Bellman equation of the social planner problem

$$V(Y, a) = \max_N \left\{ U(Y, N) + \beta \min_{\mu^p \in [-a, a]} E^p \left[V\left(e^{\tilde{z}} N, \tilde{a}\right) \right] \right\}$$

where tildes indicate random variables and where the conditional distribution of $\tilde{z}$ under belief p is given by (3.1) with $\mu_t = \mu^p$. The transition law of the exogenous state variable a is given by (3.2).

It is natural to conjecture that the value function is increasing in output. The "worst case" belief, p_0 say, then has mean $\mu^{p_0} = -a$. Combining the first order condition for labor with the envelope condition, we obtain

$$\beta = E^{p_0} \left[\beta \left(\tilde{Z} N\right)^{-\gamma} \tilde{Z} \right] \quad \text{with } \mu^{p_0} = -a \tag{3.3}$$

The constant marginal disutility of labor is equal to the marginal product of labor, weighted by future marginal utility because labor is chosen one period in advance.

3.1 The effect of uncertainty on hours

With our special preferences and technology, optimal hours are independent of current productivity (or output). Taking logarithms and using normality of the shocks, we can write the decision rule for hours as

$$n = -\left(1/\gamma - 1\right) \left(a + \frac{1}{2}\gamma\sigma_u^2 \right) \tag{3.4}$$

---

[5]We assume that a is an exogenous persistent process, interpreted as the cumulative effect of news that affect confidence. Epstein and Schneider (2007) and Epstein and Schneider (2008) present a model of learning under ambiguity that shows how updating affects confidence and Ilut (2009) considers a model of of updating about a perpetually changing hidden state. A key feature of those models is that confidence moves slowly with signals. We thus view a as a reasonable stand-in to examine the dynamics of confidence in a business cycle setting. More generally, it could also be interesting to allow for correlation between innovations to confidence and other shocks. This is omitted here for simplicity.

The equation describes the effect of uncertainty on aggregate hours. Uncertainty works the same way whether it is ambiguity, as measured by a, or risk, as measured by the product of the quantity of risk σ_u^2 and risk aversion γ.

As usual, an increase in uncertainty has wealth and substitution effects. Consider first an increase in risk. On the one hand, higher risk lowers the certainty equivalent of future production, which, in the absence of ambiguity, is given by $N \exp(-\frac{1}{2}\gamma\sigma_u^2)$. Other things equal, the resulting wealth effect leads the planner to reduce consumption of leisure and increase hiring. However, higher risk also lowers the risk adjusted return on labor. Other things equal, the resulting substitution effect leads the planner to reduce hiring. The net effect depends on the curvature in felicity from consumption, determined by γ. With a strong enough substitution effect, an increase in risk lowers hiring.

Consider now an increase in ambiguity. When a increases, the planner acts *as if* expected future productivity has declined. Mechanically, an increase in ambiguity thus entails wealth and substitution effects familiar from the analysis of news shocks. The interpretation of these effects, however, is the same as in the risk case. On the one hand, higher ambiguity lowers the certainty equivalent of future production, which, in the absence of risk, is given by $N \exp(-a_t)$. On the other hand, higher ambiguity lowers the uncertainty-adjusted return on labor. Again, with a strong enough substitution effect an increase in uncertainty lowers hiring.

Given separable felicity and the iid dynamics of z_t, inspection of the Bellman equation shows that the value function depends on current output only through the utility of consumption – the other terms depend only on the state variable a_t, not on current productivity or past hours. It follows that the value function is increasing in output, verifying our conjecture above. Below, we argue that the "guess-and-verify" approach to finding the worst case belief that we have used here to solve the planner problem is applicable much more widely.

The complete dynamics of the model are then given by the productivity equation (3.1) as well as

$$
\begin{aligned}
y_t &= z_t + n_{t-1}, \\
n_t &= -\left(1/\gamma - 1\right)\left(a_t + \frac{1}{2}\gamma\sigma_u^2\right), \\
a_t &= (1 - \rho_a)\,\bar{a} + \rho_a a_{t-1} + \varepsilon_t^a,
\end{aligned}
$$

The economy is driven by productivity and ambiguity shocks. Productivity shocks temporarily change output but have no effect on hours. In contrast, ambiguity shocks have persistent effects on both hours and output.

With a strong enough substitution effect ($1/\gamma > 1$), a loss of confidence (an increase in a) generates a recession. During that recession, productivity is not unusually low. Hours are nevertheless below steady state: since the marginal product of labor is more uncertain, the planner finds it optimal not to make people work. Conversely, an unusual increase in confidence – a drop of a_t below its long run mean – generates a boom in which employment and output are unusually high, but productivity is not. In other words, a phase of low realizations of a_t will look to an observer like a wave of optimism, where output and employment surge despite average productivity realizations.

3.2   Decentralization

Suppose that agents have access to a set of contingent claims. Write $q_t(\tilde{z}, \tilde{a})$ for the date t price of a claim that pays one unit of consumption at date $t+1$ if $(z_{t+1}, a_{t+1}) = (\tilde{z}, \tilde{a})$ is realized and denote the spot wage by w_t. The agent's date t budget constraint is

$$C_t + \int q_t(\tilde{z}, \tilde{a}) \, \theta_t(\tilde{z}, \tilde{a}) \, d(\tilde{z}, \tilde{a}) = w_t N_t + \theta_{t-1}(z_t, a_t),$$

where $\theta_t(\tilde{z}, \tilde{a})$ is the amount of claims purchased at t that pays off one unit of the consumption good if $(\tilde{z}, \tilde{a})$ is realized at $t+1$. Since aggregate labor is determined one period in advance, this set of contingent claims completes the market – claims on $(\tilde{z}, \tilde{a})$ can be used to form any portfolio contingent on the aggregate state (Y, a).

Assume that there are time invariant functions for prices $q(\tilde{z}, \tilde{a}; Y, a)$ and $w(Y, a)$ as well as aggregate labor $N(Y, a)$ that depend only on the aggregate state (Y, a). Assume further that the agent knows those price functions. The Bellman equation is

$$W(\theta, Y, a) = \max_{C, N, \theta'(.)} \left\{ U(C, N) + \beta \min_{\mu^p \in [-a, a]} E^p \left[W\left(\theta'(\tilde{z}, \tilde{a}), e^{\tilde{z}} N(Y, a), \tilde{a} \right) \right] \right\}$$
$$w(Y, a) N + \theta = C + \int q(\tilde{z}, \tilde{a}; Y, a) \, \theta'(\tilde{z}, \tilde{a}) \, d(\tilde{z}, \tilde{a})$$

Conjecture again that utility depends positively on the state variable Y. The worst case mean is then once more $\mu^p = -a$ and the maximization problem becomes standard.

In particular, prices are related to the agent's marginal rates of substitution through Euler equations. Letting $f^\mu(\tilde{z}, \tilde{a}|a)$ denote the conditional density of the exogenous variables (z, a)

implied by (3.1) and (3.2) with $\mu_t = \mu$, we have

$$w = \beta C\left(\theta, Y, a\right)^\gamma \tag{3.5}$$

$$q\left(\tilde{z}, \tilde{a}; Y, a\right) = \beta f^{-a}\left(\tilde{z}, \tilde{a}|a\right)\left(\frac{C\left(\theta'\left(\tilde{z}, \tilde{a}\right), e^{\tilde{z}} N\left(Y, a\right), \tilde{a}\right)}{C\left(\theta, Y, a\right)}\right)^{-\gamma} \tag{3.6}$$

The wage is equal to the marginal rate of substitution of consumption for hours. State prices are equal to the marginal rates of substitutions of current for future consumption. Importantly, state prices are based on the worst case conditional density f^{-a}. This is how ambiguity aversion contributes to asset premia and how it shapes firms' decisions in the face of uncertainty.

For simplicity, we consider two-period lived firms that hire workers only at date t and sell output only at date $t + 1$. To pay the wage bill at date t, they issue contingent claims which they subsequently pay back out of revenue at date $t + 1$. The profit maximization problem is

$$\max_{N, \theta(.)} \int q\left(\tilde{z}, \tilde{a}; Y, a\right)\left(e^{\tilde{z}} N - \theta\left(\tilde{z}, \tilde{a}\right)\right) d\left(\tilde{z}, \tilde{a}\right)$$
$$\text{s.t. } wN = \int q\left(\tilde{z}, \tilde{a}; Y, a\right) \theta\left(\tilde{z}, \tilde{a}\right) d\left(\tilde{z}, \tilde{a}\right)$$

As usual, the financial policy of the firm is indeterminate. Substituting the constraint into the objective, the first order condition with respect to labor equates the wage to the marginal product of labor

$$w = \int q\left(\tilde{z}, \tilde{a}; Y, a\right) e^{\tilde{z}} d\left(\tilde{z}, \tilde{a}\right) \tag{3.7}$$

Since labor is chosen one period in advance, the marginal product of labor involves state prices, which in turn reflect uncertainty perceived by agents. Substituting for prices from (3.5)-(3.6), we find that the planner's first order condition for labor (3.3) must hold in any equilibrium.

From the first order conditions, wages and state prices can be solved out in closed form. Let $Q^f\left(Y, a\right) = \int q\left(\tilde{z}, \tilde{a}; Y, a\right) d\left(\tilde{z}, \tilde{a}\right)$ denote the price of a riskless bond. We can then write

$$w\left(Y, a\right) = \beta Y^\gamma,$$
$$Q^f\left(Y, a\right) = \beta Y^\gamma \exp\left(a + \gamma\sigma_u^2\right),$$
$$q\left(\tilde{z}, \tilde{a}; Y, a\right) = Q^f\left(Y, a\right) f^0\left(\tilde{z}, \tilde{a}|a\right) \exp\left(-\frac{1}{2}\sigma_u^2\left(\frac{a}{\sigma_u^2} + \gamma\right)^2 - \left(\frac{a}{\sigma_u^2} + \gamma\right)\left(\tilde{z} - \frac{1}{2}\sigma_u^2\right)\right) \tag{3.8}$$

13

where f^0 is the density of the exogenous variables $(\tilde{z}, \tilde{a})$ if $\mu_t = 0$.

With utility linear in hours, labor supply is perfectly elastic at a wage tied to current output. Since output does not react to uncertainty shocks on impact, neither does the wage. Uncertainty shocks are transmitted to the labor market because asset prices affect labor demand. The bond price increases with both ambiguity and risk. Intuitively, either type of uncertainty encourages precautionary savings and thereby lowers the riskless interest rate

$$r^f(Y, a) = -\log Q^f(Y, a) = -\log \beta - \gamma \log Y - a - \gamma \sigma_u^2$$

The price of a claim on a particular state $(\tilde{z}, \tilde{a})$ in formula (3.8) is equal to the riskless bond price multiplied by an "uncertainty neutral" density. We have written the latter as the density for $\mu_t = 0$ times an exponential term that collects uncertainty premia.

If agents do not care about either type of uncertainty ($a = \gamma = 0$), then uncertainty premia are zero and the exponential term is one. More generally, the relative price of a "bad" state (that is, lower productivity $\tilde{z}$) is higher when confidence is lower (or a is higher). Intuitively, when confidence is lower, then agents value the insurance provided by claims on bad states more highly. This change in relative prices also affects firms' hiring decision. Indeed, since firms can pay out more in good (high $\tilde{z}$) states, a loss of confidence that makes claims on good states less valuable increases firms' funding cost. Conversely, an increase in confidence makes claims on good states more valuable; lower funding costs then induce more hiring.

The functional form of the state price density is that of an affine pricing model with "time-varying market prices of risk" (that is, time varying coefficients multiplying the shocks). This type of pricing model is widely used in empirical finance. Here time variation in confidence drives the coefficient $a/\sigma_u^2 + \gamma$ on the shock $\tilde{z}$ and thus permits a structural interpretation of the functional form. A convenient feature of affine models is that conditional expected returns on many interesting assets are linear functions of the state variables. Consider, for example, a claim to consumption next period. Its price and excess return are

$$Q^c(Y, a) = \beta E^{-a}\left[\left(e^{\tilde{z}} N(Y, a)\right)^{1-\gamma} Y^\gamma\right] = Q^f(Y, a) N(Y, a) \exp\left(-a - \gamma \sigma_u^2\right)$$
$$r^e(\tilde{z}, Y, a) = \log\left(e^{\tilde{z}} N(Y, a)\right) - \log Q^c(Y, a) - \log Q^f(Y, a)$$
$$= \tilde{z} + a + \gamma \sigma_u^2$$

Long run average excess returns have an ambiguity and a risk component. Moreover, the conditional expected excess return depends positively on a. In other words, a loss of confidence not only generates a recession, but also increases the *conditional* premium on

the consumption claim.

3.3 Comparison to news and noise shocks

We have seen that confidence shocks in our model work like unrealized news shocks with a bias. In this subsection we compare our model with confidence shocks to a rational expectations model with news and noise shocks. We show that the two models can be distinguished by considering either quantity moments or asset price data. To this end, we study a version of the stylized model above in which agents receive noisy signals about future productivity.

Suppose that, at date t, agents observe a signal about productivity one period ahead. The joint dynamics of the signal and productivity itself are

$$s_t = z_{t+1} + \sigma_s \varepsilon_{s,t}$$
$$z_{t+1} = -\frac{1}{2} (1 - \pi) \sigma_u^2 + u_{t+1}$$

where the noise $\varepsilon_{s,t}$ is uncorrelated with all other shocks and $\pi = \sigma_u^2 / (\sigma_u^2 + \sigma_s^2)$. Since the conditional variance of $var(z_{t+1}|s_t)$ is $(1 - \pi) \sigma_u^2$ the constant in the second equation ensures that $E[Z_{t+1}|s_t] = 1$. The parameter π indicates how good the signal is: if $\pi = 1$ then the signal reveals tomorrow's productivity (a pure news shock) whereas $\pi = 0$ says that the signal is worthless (a pure noise shock).

Optimal hours are now

$$n_t = (1/\gamma - 1) \left(\pi s_t - \frac{1}{2} \gamma (1 - \pi) \sigma_u^2 \right)$$

With a strong enough substitution effect, agents work more if a good signal arrives. Of course, the signal could either reflect good future productivity (news) or a positive realization of ε_s (noise). In the news case, the shock will be followed by realized high productivity, but in the noise case this need not happen (and in fact will not happen on average). Ambiguity shocks are thus similar to noise shocks in that can they affect actions, but not realized fundamentals.

Nevertheless, it is straightforward to distinguish the news & noise model from the ambiguity model above. First, consider a simple regression of log productivity on log hours. In a large sample, we obtain a slope coefficient

$$\hat{\beta} = \frac{cov(z_{t+1}, n_t)}{var(n_t)} = \frac{(1/\gamma - 1) \pi \sigma_u^2}{(1/\gamma - 1)^2 \pi^2 (\sigma_u^2 + \sigma_s^2)} = (1/\gamma - 1)^{-1}$$

In other words, if news matters for employment ($\gamma \neq 1$), then employment must forecast productivity. In contrast, the ambiguity model above does not predict such a relationship. The point here is that while ambiguity shocks work mechanically like noise shocks, noise shocks can matter only if there are also enough news shocks, and the presence of news will be reflected in the regression coefficient.

A second important difference between a news & noise model and the ambiguity model above lies in the predictions for asset prices. The price formula (3.8) shows that an econometrician who observes data from the ambiguity model will recover a pricing kernel that features "time varying market prices of risk". Such time variation would be reflected for example in predictability regressions of excess returns on forecasting variables. Moreover, prices changes can be dominated by changes in confidence which can be uncorrelated with changes in expected fundamentals, thus leading to "excess volatility". In contrast a news model predicts that risk premia are constant and asset prices move mostly with expected fundamentals.

3.4   Bounding ambiguity by measured volatility

Consider now the connection between the true dynamics of log productivity z in (3.1) and the agents' set of beliefs. In our model, productivity consists of two components, the iid shock u that agents view as risky, and the deterministic sequence μ that agents view as ambiguous. In line with agents' lack of knowledge about μ, we do not impose a particular sequence as "the truth". Instead, we restrict only the long run average and variability of μ, and thereby also of productivity z. We then develop a bound on the process a_t that ensures that the belief set is "small enough" relative to the variability in the data observed by agents. For quantitative modeling, the bound imposes discipline on how much the process a_t can vary relative to the volatility in the data measured by the modeler.

Consider first the long run behavior of μ. Let I denote the indicator function and let $\Phi\left(., m, s^2\right)$ denote the cdf of a univariate normal distribution with mean m and variance s^2. We assume that the empirical distribution of μ converges to that of an iid normal stochastic process with mean zero and variance σ_μ^2. Formally, we require that, for any integers $k, \tau_1, ..., \tau_k$ and real numbers $\bar{\mu}_1, ..., \bar{\mu}_k$,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} I\left(\left\{\mu_{t+\tau_j} \leq \bar{\mu}_j; \quad j = 1, .., k\right\}\right) = \prod_{j=1}^{k} \Phi\left(\bar{\mu}_j; 0, \sigma_\mu^2\right).$$

In particular, we assume that for almost every realization of the shocks u, the empirical second moment $\frac{1}{T} \sum_{t=1}^{T} \mu_t u_t$ converges to zero.

For example, if we were to observe μ and record the frequency of the event $\{\mu_t \leq \bar{\mu}\}$ then that frequency would converge to $\Phi\left(\bar{\mu}, 0; \sigma_\mu^2\right)$. For a two-dimensional example, the frequency of the event $\{\mu_t \leq \bar{\mu}_1, \mu_{t+\tau} \leq \bar{\mu}_2\}$ is assumed to converge to $\Phi\left(\bar{\mu}_1, 0; \sigma_\mu^2\right) \Phi\left(\bar{\mu}_2, 0; \sigma_\mu^2\right)$. Similarly, recording the frequency of an event that jointly restricts elements of μ spaced in time as described by the τ_js always delivers in the long run the cdf of an iid multivariate normal distribution. At the same time, almost every draw from an iid normal process with mean zero and variance σ_μ^2 would deliver a sequence μ_t that satisfies the condition.

We also require that the ambiguous component in the data is not systematically related to the risky component. In particular, we assume that for almost every realization of the shocks u, the empirical second moment $\frac{1}{T}\sum_{t=1}^{T} \mu_t u_t$ converges to zero. This has implications for the long run empirical distribution of log productivity z. Indeed, given a true sequence μ that satisfies the above condition, then for almost every realization of the shocks u the empirical mean $\frac{1}{T}\sum_t z_t$ converges to zero, the empirical variance $\frac{1}{T}\sum_t z_t^2$ converges to $\sigma_z^2 = \sigma_\mu^2 + \sigma_u^2$, and the empirical autocovariances at all leads and lags converge to zero. In other words, to an econometrician who sees a large sample, the data look like white noise (that is, uncorrelated with mean zero and variance σ_z^2) regardless of the true sequence μ.

If an econometrician fits a covariance stationary statistical model to the productivity data, he thus recovers an iid process with mean zero and variance σ_z^2. Ambiguity averse agents look at the data differently. Even though they know the limiting properties of μ and hence of z, when they make decisions at date t, they are concerned that they do not know the current conditional mean μ_t needed to forecast z_{t+1}. They understand that statistical tools cannot help them learn μ_t in real time. They deal with their lack of knowledge at date t by behaving *as if* they minimize over a set of forecasting rules (that is, a set of one-step-ahead conditional probabilities) indexed by the interval $[-a_t, a_t]$. It makes sense to assume that this interval should be smaller the less variable the data are (lower σ_z^2) and, in particular, the less variability in the data is attributed to ambiguity as opposed to risk (lower σ_μ^2).

We thus develop a bound on the process a_t, denoted $a^{\max}$, that is increasing in σ_μ^2. The basic idea is that even the boundary forecasts indexed by $\pm a^{\max}$ should be "good enough" in the long run. To define "good enough", we calculate the frequency with which one of the boundary forecasting rules is the best forecasting rule in the interval $[-a^{\max}, a^{\max}]$. The forecasting rule with mean $a^{\max}$ is the best rule at date t if its mean $a^{\max} - \frac{1}{2}\sigma_u^2$ is closest to the true conditional mean $\mu_t - \frac{1}{2}\sigma_u^2$, that is, if $\mu_t \geq a^{\max}$. Similarly, the rule $-a^{\max}$ is the best rule if $\mu_t \leq -a^{\max}$. We now require that the frequency with which μ_t falls outside the interval $[-a^{\max}, a^{\max}]$, thus making the boundary forecasts the best forecasts, converges in the long run to a number $\alpha \in (0, 1)$. Given our assumption on the long run behavior of μ

above, the bound is defined by

$$\Phi\left(a^{\max}; 0, \sigma_\mu^2\right) = \alpha/2$$

The number α determines the tightness of the bound. For example, $\alpha = 5\%$ implies $a^{\max} \approx 2\sigma_\mu$.

The bound $a^{\max}$ restricts the variability in the worst case mean relative to measured volatility in the data. Suppose the variance of the productivity is measured to be σ_z^2. Denote by $\rho = \sigma_\mu^2/\sigma_z^2$ the share of the variability in the data that agents attribute to ambiguity. Then with $\alpha = 5\%$ we require $a_t \leq 2\sqrt{\rho}\sigma_z$. The bound is tighter if less of the variability in the data is due to ambiguity. In the extreme case of $\rho = 0$, the process a_t must be identically equal to zero – agents treat all variability in z as risk. In practice, the bound dictates parameter restrictions on the law of motion for a_t. In a discrete time model, we cannot impose exactly that $a_t \in [0, 2\sqrt{\rho}\sigma_z]$. However, small enough volatility of ε_t^a in (3.2) ensures that those conditions are virtually always satisfied – this is the approach we follow in our quantitative work below.

It is interesting to compare how risk and ambiguity affect the long run behavior of business cycles variables in our simple model. Consider, for example, the empirical mean and variance of output in a large sample

$$\bar{y} = -\frac{1}{2}\sigma_u^2 - (1/\gamma - 1)\left(\bar{a} + \frac{1}{2}\gamma\sigma_u^2\right)$$
$$\bar{\sigma}_y^2 = \sigma_z^2 + (1/\gamma - 1)^2 var\left(a_t\right)$$

Here we have used the law of large numbers for u together with our assumptions on μ, which imply that the long run moments are the same for every possible sequence μ. The bound puts discipline on the role of ambiguity in explaining business cycles. For example, suppose that we assume $\bar{a} > 3\sqrt{var\left(a_t\right)}$ and $\bar{a} + 3\sqrt{var\left(a_t\right)} < 2\sqrt{\rho}\sigma_z$ in order to keep a almost always in the interval $[0, 2\sqrt{\rho}\sigma_z]$. Together these conditions imply that $var\left(a_t\right) < (\rho/9)\sigma_z^2$, which in turn bounds the share of $\bar{\sigma}_y^2$ that can be contributed by time-varying ambiguity.

4   General framework

4.1   Environment & equilibrium

We consider economies with many individuals $i \in I$ and we assume Markovian dynamics. The econometrician's model of the exogenous state $s \in S$ is a Markov chain with

transition probabilities $p^*(s_t)$. Agent i's preferences are of the form (2.1) with primitives $(\beta_i, u^i, \{\mathcal{P}^i(s_t)\})$. Given preferences, it is helpful to write the rest of the economy in fairly general notation that many typical problems can be mapped into. Consider a recursive competitive equilibrium. The vector X collects endogenous state variables that are predetermined as of the previous date. Let A^i denote a vector of actions taken by agent i. One action is the choice of consumption and we write $c^i(A^i)$ for agent $i's$ consumption bundle when his action is A^i. Finally, let Y denote a vector of other endogenous variables not chosen directly by any agent – this vector will typically include prices, but perhaps also variables such as government transfers.

The technology and market structure are summarized by a set of reduced form functions or correspondences. A recursive competitive equilibrium consists of action and value functions A^i and V^i, respectively, for all agents $i \in I$, as well as a function describing the other endogenous variables Y. We also write A for the collection of all actions $(A^i)_{i \in I}$ and A^{-i} for the collection of all actions except that of agent i. All functions take as argument the state (X, s) and satisfy

$$W^i(A, X, s; p) = u^i\left(c^i\left(A^i\right)\right) + \beta_i E^p\left[V^i(x'(X, A, Y(X, s), s), s')\right] \quad ; i \in I \qquad (4.1)$$

$$A^i(X, s) = \arg \max_{A^i \in B^i(Y(X,s), A^{-i}, X, s)} \min_{p \in \mathcal{P}^i(s)} W^i(A, X, s; p) \qquad i \in I \qquad (4.2)$$

$$V^i(X, s) = \min_{p \in \mathcal{P}^i(s)} W^i(A(X, s), X, s; p) \qquad (4.3)$$

$$0 = G(A(X, s), Y(X, s), X, s) \qquad (4.4)$$

The first equation simply defines the agent's objective in state (X, s) if his belief over the next exogenous state is p; the function x' describes the transition of the endogenous state variables. The second and third equation provide agent i's optimal policy and value function; B^i is the agent's budget set correspondence. The function G summarizes all other contemporaneous relationships such as market clearing, the government budget constraint or the optimality conditions of firms. There are enough equations in (4.4) to determine all endogenous variables Y.

The equations make explicit only the problems of individuals – agents who maximize utility – since this is where ambiguity aversion enters directly. The problems of firms can typically be subsumed into equation (4.4) and the transition function x'. In particular, this is true for models in which firms maximize shareholder value, as in our stylized example above. Indeed, shareholder value depends on state prices that can be taken to be elements of Y. Firm actions can be elements of Y or X (the latter if they are state variables, for example prices set for some time) and the firm value can be an element of X. Firms' optimality

conditions as a function of state prices (cf. (3.7) in our example) are contained in G and x'. As in the example, ambiguity affects firms as it is reflected in prices.

We have explicitly split the endogenous variables into A, Y and X to clarify the effect of the minimization step in (4.2) on the choice of A. If there is only one transition density $p(s)$ that is the same for all $i \in I$ (and thus no minimization step), then the system can typically be written as a single functional equation. We assume that this is possible here as well. Let w denote the entire vector of endogenous variables chosen the period before the exogenous state s is realized. It includes not only the endogenous state X, but also past actions. We assume that, for given p, there is a function H such that the functional equation

$$E^p \left[H \left(w, W(w, s), W(W(w, s), s') \right) | s \right] = 0 \qquad (4.5)$$

has a solution

$$W(w, s) := \left(x'(X, A(X, s), Y(X, s), s), A(X, s), Y(X, s) \right)$$

Note that the state variable X is the only element of w that affects W since A and Y were assumed to not directly affect what happens next period. The general notation here does not fully exploit this feature of the problem. Nevertheless, writing things with one vector w will make it easier to describe how the model is solved once the worst case belief is known.

4.2   Characterizing optimal actions & equilibrium dynamics

Characterizing equilibrium consists of two tasks. First, we need to find the endogenous variables A and Y as functions of the state (X, s). Second, we want to describe the dynamics of the system when the evolution of the state is driven by the econometrician's transition density p^*. The first task involves solving for worst case beliefs. For every state (X, s), there is a measure $p^{0i}(X, s)$ that achieves the minimum for agent i in (4.2). Since the minimization problem is linear in probabilities, we can replace $\mathcal{P}^i_t$ by its convex hull without changing the solution. The minimax theorem then implies that we can exchange the order of minimization and maximization in the problem (4.2). It follows that the optimal action A^i is the same as the optimal action if the agent held the probabilistic belief $p^{0i}(X, s)$ to begin with. In other words, for every equilibrium of our economy, there exists an economy with expected utility agents holding beliefs p^{0i} that has the same equilibrium.

The observational equivalence just described suggests the following guess-and-verify procedure to compute an equilibrium with ambiguity aversion:

1. guess the worst case beliefs p^{0i}

2. solve the model assuming that agents have expected utility and beliefs p^{0i} (that is, find the functions A and Y solving (4.1)-(4.4) given p^{0i} or the function W solving (4.5))

3. compute the value functions V^i

4. verify that the guesses p^{0i} indeed achieve the minima in (4.3) for every i.

Turn now to the second task. Suppose we have found the optimal action functions A as well as the response of the endogenous variables Y and hence the transition for the states X. We are interested in stochastic properties of the equilibrium dynamics that can be compared to the data. We characterize the dynamics in the standard way by calculating (or simulating) moments of the economy under the true distribution of the exogenous shocks p^*. The only unusual feature is that this true distribution need not coincide with the distributions p^{0i} that are used to compute optimal actions.

4.3 First order effects of uncertainty

We now specialize the process of belief sets $\mathcal{P}$ to capture random changes in confidence that have first order effects. We assume that the family of distributions of the state next period can be represented as

$$s_{t+1} = E^p[s_{t+1}|s_t] + \varepsilon_{t+1}^s; \qquad p \in \mathcal{P}(s_t) \tag{4.6}$$

where the distribution of the innovation ε_{t+1}^s is independent of p. The restriction here is that there is no ambiguity about conditional moments other than the mean. At the same time, the distribution of ε^s may depend on the state s. For example, there could be heteroskedastic shocks.

One example for (4.6) is provided by the belief structure in our simple model (3.1)-(3.2). In that model, ambiguity is about mean productivity, and confidence is a component of s that is uncorrelated with productivity itself. Using the notation of this section, we have $s = (z, a)'$ and

$$s_{t+1} = \begin{pmatrix} \mu^p - \frac{1}{2}\sigma_u^2 \\ (1 - \rho_a)\bar{a} + \rho_a a_t \end{pmatrix} + \begin{pmatrix} u_{t+1} \\ \varepsilon_{t+1}^a \end{pmatrix}; \qquad \mu^p \in [-a_t, a_t] \tag{4.7}$$

Here only the first component of the conditional mean $E^p[s_{t+1}|s_t]$ – the one corresponding to productivity itself – depends on p through the mean shifter μ^p. In contrast, the evolution of confidence is not ambiguous.

The example suggests a way to specify simple but rich families of beliefs that are compatible with (4.6). Start from a vector u of fundamental shocks that people feel

ambiguous about. In addition to productivity, this set might contain policy shocks. Next, define a subvector a of s that has the same length as u to capture confidence about u. Finally, parametrize the set of beliefs by an interval for each fundamental shock, centered at zero and bounded by $|a|$. In principle, confidence could be different for different fundamental shocks. One could imagine, for example, that ambiguity about fiscal and monetary policy is correlated, but quite different from ambiguity about technology. This is because the flow of news that drives confidence is likely to be different for those shocks.

Risk shocks and ambiguity

While the simple model is homoskedastic, we emphasize that (4.6) also allows for nonlinear specifications that link changes in ambiguity and changes in volatility. For example, consider a variation on (4.7) with stochastic volatility in productivity that feeds back to agents' perception of ambiguity. Let $s = \left(z, \sigma_{z,t}^2 \right)$

$$s_{t+1} = \begin{pmatrix} \mu^p - \frac{1}{2}\sigma_{z,t}^2 \\ (1 - \rho_\sigma)\,\bar{\sigma}_z^2 + \rho_\sigma \sigma_{z,t}^2 \end{pmatrix} + \begin{pmatrix} \sigma_{z,t}\varepsilon_{t+1}^z \\ \varepsilon_{t+1}^\sigma \end{pmatrix}; \qquad R_t^p = \frac{(\mu^p)^2}{2\sigma_{z,t}^2} \le \eta$$

where the parameters of the stochastic volatility process are chosen such that the variance "almost never" becomes negative. With normal distributions, R_t^p represents the entropy of a belief with mean μ^p relative to a benchmark belief with mean zero. The inequality says that in periods of high turbulence (high $\sigma_{z,t}^2$), there is also more ambiguity about the conditional mean, reflected in a wider interval for that parameter.

4.4   Essentially linear economies

The computation and interpretation of equilibria is particularly simple if the conditional mean of the exogenous state is linear under both the econometrician's belief and the worst case belief, that is,

$$E^{p^*}[s_{t+1}|s_t] = \bar{s}^* + \Phi^*(s_t - \bar{s}^*) \tag{4.8}$$

$$E^{p^0}[s_{t+1}|s_t] = \bar{s}^0 + \Phi^0(s_t - \bar{s}^0) \tag{4.9}$$

For the econometrician's belief, this is a common assumption. For the worst case belief, it is an implicit restriction on the model that must be checked by guess-and-verify. However, this type of guess is natural if the family of means $E^p[s_{t+1}|s_t]$ is linear in s_t. For example, in the example (4.7), it is natural to conjecture that the worst case mean is $\mu^{p_0} = -a_t$. We now describe how the above guess-and-verify method works and how the equilibrium can be analyzed if (4.9) holds.

Finding the equilibrium law of motion

For a given belief p^0, step 2 of the procedure – finding the equilibrium law of motion – amounts to finding a solution $W(w, s)$ to (4.5). We look for an approximate solution by linearization. Define the *worst case steady state* $\bar{w}$ by

$$H\left(\bar{w}^0, \bar{w}^0, \bar{w}^0, \bar{s}^0, \bar{s}^0\right) = 0$$

Intuitively, this is where w would converge if the law of motion of the exogenous state were (4.6) with the worst case conditional mean (4.9). The actual law of motion will typically be different if the true conditional mean differs from the worst case. The worst case steady state, just like the worst case belief, should be viewed only as a tool to describe agents' responses to uncertainty.

Denote the deviation from the worst case steady state by $\hat{w}_t^0 := w_t - \bar{w}^0$ and $\hat{s}_t^0 := s_t - \bar{s}^0$ and perform a first order Taylor expansion of H in (4.5) around this steady state to obtain

$$E^{p^0}\left[\alpha_{-1}\hat{w}_{t-1}^0 + \alpha_0\hat{w}_t^0 + \alpha_1\hat{w}_{t+1}^0 + \delta_0\hat{s}_t^0 + \delta_1\hat{s}_{t+1}^0 \,|\, s_t\right] = 0,$$

where $\alpha_{-1}, \alpha_0, \alpha_1, \delta_0, \delta_1$ are constants determined by equilibrium conditions. Together with (4.9), this is a familiar system of expectational difference equations. We use time subscripts to make the notation comparable to other such equations in the literature (as for example in Christiano (2002)). The timing is that w_{t-1} contains endogenous variables determined one period before the exogenous state s_t is realized and w_t corresponds to $W(w_{t-1}, s_t)$, determined once s_t is known. Under the usual regularity conditions, the method of undetermined coefficients delivers a solution

$$\hat{w}_t^0 = \rho\hat{w}_{t-1}^0 + \nu\hat{s}_t^0. \tag{4.10}$$

It is important to note that the econometrician's transition density p^* has not been used to find this solution. This is because agents do not know the econometrician's belief about the data (we do not impose rational expectations). Instead, agents' behavior is driven by their worst case belief p^0. Nevertheless, standard tools from solving expectational difference equations under rational expectations can be used to find the above approximation to the equilibrium law of motion W.

Step 3 of the guess-and-verify procedure computes agents' value functions under the worst case belief, V^{0i} say. Nonlinearity of the value functions could be important here; it is thus useful to compute value functions using higher order approximations around $(\bar{w}^0, \bar{s}^0)$. Finally, step 4 verifies the guess by solving the minimization problem in (4.2) for the mean E^p. In our applications, this amounts to checking monotonicity of a function. Indeed,

suppose beliefs are given by (4.7) and the guess is $\mu^p = E^{p_0}[z_{t+1}|s_t] = -a_t$. It is verified by checking whether for any $X, A, s = (z, a)$ and a', the function

$$\tilde{V}(z') := V^0(x'(X, A, Y(X, s)), s, z', a'), z', a')$$

is strictly increasing.

Characterizing equilibrium dynamics

Consider now the dynamics of the model from the perspective of the econometrician. Agents' response to ambiguity leads to actions and hence equilibrium outcomes given by (4.10). At the same time, the exogenous state moves according to the equation in (4.8), in which the steady state equals $\bar{s}^*$. Suppose the volatility of the shocks is negligible. First order effects of ambiguity imply that the resulting *zero risk* steady state is typically not equal to $(\bar{s}^0, \bar{w}^0)$. Instead, it is given by $\bar{s} = \bar{s}^*$ and

$$\bar{w} - \bar{w}^0 = \rho(\bar{w} - \bar{w}^0) + \nu(\bar{s}^* - \bar{s}^0) \tag{4.11}$$

Mechanically, the dynamic system behaves *as if* it has been displaced from $(\bar{s}^0, \bar{w}^0)$ so as to make the impulse response of w_t take on the value $\bar{w}$ in both the first and second period after the shock. The condition in (4.11) states that when the economy's time t initial condition is equal to $(w_{t-1} = \bar{w}, s_t = \bar{s}^*)$, the linearized equilibrium conditions generate a time t solution $W(w_{t-1}, s_t)$ that maintains the economy at its initial condition $\bar{w}$. Put differently, the zero risk steady state reflects the response of agents who observe $\bar{s}^*$ at date t and whose ambiguity aversion leads them to act *as if* the exogenous state will converge to $\bar{s}^0$.

Denote by $\hat{w}_t := w_t - \bar{w}$ and $\hat{s}_t := s_t - \bar{s}^*$ the deviations from the zero risk steady state. Combining (4.10) and (4.11), those deviations follow the law of motion

$$
\begin{aligned}
\hat{w}_t &= \hat{w}_t^0 + \bar{w}^0 - \bar{w} \\
&= \rho(\hat{w}_{t-1}^0 + \bar{w}^0 - \bar{w}) + \nu(\hat{s}_t^0 + \bar{s}^0 - \bar{s}^*) \\
&= \rho\hat{w}_{t-1} + \nu\hat{s}_t
\end{aligned}
$$

In other words, the actual movement of the endogenous variables around $\bar{w}$ when displaced by actual shocks $\hat{s}_t$ is the same as their movement around the worst case steady state when displaced by shocks $\hat{s}_t^0$.

5 An estimated model with ambiguity

This section describes the model that we use to describe the US business cycles. The model is based on a standard medium scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007). Many elements of our model are standard in the literature (for example in Del Negro et al. (2007), Christiano et al. (2008), Schmitt-Grohe and Uribe (2008) and Justiniano et al. (2011)). What is new is that decision makers are ambiguity averse. We defer the description of the details of the model that are not related to ambiguity to Appendix 6.1.

5.1 Setup: parametrizing ambiguity

Preferences reflect both ambiguity aversion and internal habit formation. As in Section 4, utility is defined over uncertain streams of consumption $\overrightarrow{C} = \left(\overrightarrow{C}_t\right)_{t=0}^{\infty}$ where date t consumption $\overrightarrow{C}_t : S^t \to \Re^2$ includes two goods, the final good and leisure. Agent $i's$ felicity function is:

$$u^i\left(\overrightarrow{C}_t, \overrightarrow{C}_{t-1}\right) = \log(C_t - \theta C_{t-1}) - \frac{\psi_L}{1+\sigma_L} h_{i,t}^{1+\sigma_L}.$$

where C_t denotes individual consumption of the final good, $h_{i,t}$ denotes a specialized labor service supplied by the household and θ is controls internal habit formation.

Utility follows a recursion similar to (2.1):

$$U_t\left(\overrightarrow{C}; s^t\right) = u^i\left(\overrightarrow{C}_t, \overrightarrow{C}_{t-1}\right) + \beta \min_{p \in \mathcal{P}_t(s^t)} E^p\left[U_{t+1}\left(\overrightarrow{C}; s_t, s_{t+1}\right)\right], \tag{5.1}$$

The sets of beliefs $\mathcal{P}_t(s^t)$ reflect ambiguity about the transitory productivity level Z_{t+1}. The technology shock Z_t enters into the production of intermediate goods as described in detail in Appendix 6.1. From the perspective of the econometrician, the dynamics for Z_t is:

$$\log Z_t = \rho_z \log Z_{t-1} + \sigma_z z_t^x. \tag{5.2}$$

We parametrize the sets $\mathcal{P}_t(s^t)$ of one-step ahead conditional beliefs about future technology by an exogenous component a_t that captures time-varying ambiguity:

$$\log Z_{t+1} = \rho_z \log Z_t + \sigma_z z_{t+1}^x + \mu_t \tag{5.3}$$

$$\mu_t \in [-a_t, -a_t + 2|a_t|] \tag{5.4}$$

$$a_{t+1} = (1 - \rho_a)\overline{a} + \rho_a a_t + \sigma_a a_{t+1}^x \tag{5.5}$$

where the shocks z^x and a^x are standard normal iid shocks. As in Section 4, we assume that

the agent knows the evolution of a_t, but that he is not sure whether the conditional mean of $\log Z_{t+1}$ is really $\rho_z \log Z_t$. Instead, the agent allows for a range of intercepts. If a_t is higher, then the agent is less confident about the mean of $\log Z_{t+1}$ – his belief set is larger.

A convenient feature shared by both the model here and the simple model of section 3 is that it is fairly easy to see what the worst case scenario is, or which μ_t solves the minimization problem in (5.1). Indeed, the environment (given by B, x' u and G in the general formulation of section 4) is such that, under expected utility and rational expectations, a first order solution is known to provide a satisfactory approximation to the equilibrium dynamics. Moreover, it can be checked that the value function, under expected utility, is increasing in Z_t. This monotonicity implies that the worst case scenario belief that solves the minimization problem in (5.1) is given by the lower bound of the set $[-a_t, -a_t + 2|a_t|]$. Intuitively, it is natural that the agents take into account that the worst case is always that the mean of productivity innovations is as low as possible. According to the worst case belief p^0, technology thus evolves as

$$\log Z_{t+1} = \rho_z \log Z_t + \sigma_z z^x_{t+1} - a_t \tag{5.6}$$

Choosing ambiguity parameters

Time variation in ambiguity is governed by the three parameters $\overline{a}, \rho_a$ and σ_a. Two considerations matter for selecting a prior over these parameters. The first is technical: we would like the interval for μ remains centered around zero which is true only if a remains nonnegative. Unfortunately, nonnegativity is incompatible with a linear law of motion for a. We thus require parameters such that the unconditional mean $\overline{a}$ is more than three unconditional standard deviations away from zero:

$$\overline{a} \geq 3 \frac{\sigma_a}{\sqrt{1 - \rho_a^2}}. \tag{5.7}$$

As a result, the probability that a becomes negative is less than .15%, and any negative as will be small. Any a close to zero will thus represent a small set of belief that is close to having a single mean close to zero - a very confident agent.

The second consideration is that we want to bound the lack of confidence by the measured variance of the shock that agents perceive as ambiguous. In section 3.4 we argued that a reasonable upper bound for a is given by $2\sqrt{\rho}\sigma_z$, where $\rho \in [0, 1]$ is the share of the variability in the data that agents attribute to ambiguity. When $\rho = 1$ we obtain the largest upper bound, i.e. $a_t \leq 2\sigma_z$. Again we cannot enforce the bound exactly, but assume that it is

violated with probability .13%:

$$\bar{a} + 3\frac{\sigma_a}{\sqrt{1 - \rho_a^2}} \leq 2\sigma_z. \tag{5.8}$$

In preliminary estimations of the model, we find that when the three ambiguity parameters $\bar{a}, \rho_a$ and σ_a are separately estimated the implied unconditional volatility of the a_t process is so large that it implies very frequent negative realizations to a_t. We thus restrict attention to the subset of the parameter space in which (5.7) is binding. It is helpful to write $\bar{a} = n\sigma_z$ with $n \in [0, 1]$ because of (5.8). We estimate two ambiguity parameters ρ_a and n, together with the other parameters of the model, including σ_z. We can then compute $\bar{a} = n\sigma_z$ and infer σ_a from (5.7).

5.2 Estimation and Data

The solution of our model with ambiguity follows the general steps described in section 4.4; details are in Appendix 6.3. The linearity of the state space representation of the model and the assumed normality of the shocks allow us to estimate the model using standard Bayesian methods as discussed for example in An and Schorfheide (2007) and Smets and Wouters (2007). We estimate the posterior distribution of the structural parameters by combining the likelihood function with prior information. The likelihood is based on the following vector of observable variables:

$$\left[\Delta \log Y_t^G, \Delta \log I_t, \Delta \log C_t, \log L_t, \log \pi_t, \log R_t, \Delta \log P_{I,t}\right]$$

where Δ denotes the first difference operator. The vector of observables refers to data for US on GDP growth rate, investment growth rate, consumption growth rate, log of hours per capita, log of gross inflation rate, log of gross short term nominal interest rate and price of investment growth rate. The sample period used in the estimation is 1984Q1-2010Q1. The data sources are described in Appendix 6.4. In the state space representation we do not allow for a measurement error on any of the observables.

We now discuss the priors on the structural parameters. The only parameter we calibrate is the share of government expenditures in output which is set to match the observed empirical ratio of 0.22. The rest of the structural parameters are estimated. The priors on the parameters not related to ambiguity and thus already present in the standard medium scale DSGE are broadly in line with those adopted in previous studies (e.g. Christiano et al. (2010b) and Justiniano et al. (2011)). The prior for each of the autocorrelation parameter of the shock processes is a Beta distribution with a mean of 0.5 and a standard deviation

of 0.15. The prior distribution for the standard deviation of the 7 fundamental shocks is an Inverse Gamma with a mean of 0.01 and a standard deviation of 0.01.

Regarding the ambiguity parameters, we follow the argument in section 5.1 and estimate the two parameters n and ρ_a. The prior on the scaling parameter n is a Beta distribution with mean 0.5 and standard deviation equal to 0.25. The prior is loose and it allows a wide range of plausible values. The prior on ρ_a follows the pattern of the other autocorrelation coefficients and is a Beta distribution with a mean of 0.5 and a standard deviation of 0.15.

The prior and posterior distributions are described in Table 2. The posterior estimates of our structural parameters that are unrelated to ambiguity are in line with previous estimations of such medium scale DSGE models (Del Negro et al. (2007), Smets and Wouters (2007), Justiniano et al. (2011), Christiano et al. (2010b)). These parameters imply that there are significant 'frictions' in our model: price and wage stickiness, investment adjustment costs and internal habit formation are all substantial. The estimated policy rule is inertial and responds strongly to inflation but also to output gap and output growth. Given that these parameters have been extensively analyzed in the literature, we now turn attention to the role of ambiguity in our estimated model.

5.3   Results: steady state

Ambiguity has important effects on both the steady state and the business cycle of our model. Consider first the steady state. The posterior mode of the structural parameters of ambiguity implies that the steady state level of ambiguity is

$$\bar{a} = n\sigma_z = 0.963 \times 0.0045 = 0.00435,$$

which means that the agent is on average concerned about a one-step ahead future technology level that is 0.435% lower than the true technology, normalized to 1. In the long run, the agent expects the technology level to be Z^*, which solves:

$$\log Z^* = \rho_z \log Z^* - \bar{a}.$$

For the estimated $\rho_z = 0.955$, we get that $Z^* = 0.903$. Thus, the ambiguity-averse agent expects under his worst case scenario evaluation the long run mean technology to be approximately 9% lower than the true mean. Based on these estimates and using (5.7), we can directly find that the standard deviation of the innovations to ambiguity is $\sigma_a = 0.000405$.

Our interpretation of the reason we find a relatively large $\bar{a}$ is the following: the estimation prefers to have a large σ_a because the ambiguity shock provides a channel in the model that

delivers dynamics that seem to be favored by the data. Indeed, as detailed in the next section, the ambiguity shocks generate comovement between variables that enter in the observation equation. This is a feature that is strongly present in the data and is not easily captured by other shocks. Given the large role that the fit of the data places on the ambiguity shock, the implied estimated σ_a is relatively large. Because of the constraint on the size of the mean ambiguity in (5.7), this results also in a large required steady state ambiguity. Thus, given also the estimated σ_z, the posterior mode for n is relatively large. The picture that comes out of these estimates is that ambiguity is large in the steady state, it is volatile and persistent.

The estimated amount of ambiguity has substantial effects on the steady state of endogenous variables.[6] To describe these effects we perform the following calculations.[7] We fix all the estimated parameters of the model at their posterior mode but change the standard deviation of the transitory technology shock, σ_z, from its estimated value of $\overline{\sigma}_z = 0.0045$ to being equal to 0. When $\sigma_z = 0$, then the level of ambiguity $\overline{a}$ is also equal to 0.[8] By reporting the difference between the steady states with $\sigma_z = \overline{\sigma}_z > 0$ and with $\sigma_z = 0$ we calculate the steady state effect of fluctuations in transitory technology that goes through the estimated amount of ambiguity. In Table 1 we present the net percent difference of some variables of interest between the two cases, i.e. for a variable X we report $100[X_{SS(\sigma_z=\overline{\sigma}_z)},/X_{SS(\sigma_z=0)}-1]$, where $X_{SS(\sigma_z=\overline{\sigma}_z)}$ and $X_{SS(\sigma_z=0)}$ are the steady states of variable X under $\sigma_z = \overline{\sigma}_z$ and respectively $\sigma_z = 0$.[9]

Table 1: Steady state percent difference from zero fluctuations

Variable	$Welfare$	$Output$	$Capital$	$Consumption$	$Hours$	$Nom.Rate$
	-13.1	-15	-14	-16.4	-14.8	-42.5

As evident from Table 1, the effect of fluctuations in the transitory technology shock that goes through ambiguity is very substantial. Output, capital, consumption, hours are all significantly smaller, by about 15%, when $\sigma_z = \overline{\sigma}_z$. The nominal interest rate is smaller by 42%, which corresponds to the quarterly steady state interest rate being lower by 73 basis points. Importantly, the welfare cost of fluctuations in this economy is also very large, of

[6]As described in Section 4.4, we refer to the steady state of our linearized model as a 'zero risk steady state', in which the variances of the shocks only have an effect on the endogenous variables through ambiguity.

[7]The quantitative implications of the model are reported by evaluating parameters at the posterior mode.

[8]As evident from equation (5.8) or from the discussion of Section 3.4, since $0 \leq \overline{a} \leq \sigma_z$, then $\sigma_z = 0$ implies $\overline{a} = 0$. Intuitively, when there is no observed variability in the z_t process, then there cannot be any perceived ambiguity about its one-step ahead mean.

[9]For welfare, we report the difference in terms of steady state consumption under $\sigma_z = 0$.

about 13% of steady state consumption.

The steady state effects of fluctuations in technology that go through ambiguity are much larger than what it is implied by the standard analysis featuring only risk, as for example in Lucas (1987). By standard analysis we mean the strategy of shutting down all the other shocks except the transitory technology and computing a second order approximation of the model in which there is no ambiguity but $\sigma_z = \overline{\sigma}_z$. For such a calculation, we find that the welfare cost of business cycle fluctuations is around 0.01% of steady state consumption. The effects on the steady state values of the other variables reported in Table 1 is negligible.

5.4 Results: business cycle fluctuations

In this section we analyze the role of time-varying ambiguity in generating business cycles. We highlight the role of ambiguity by discussing three main points: a theoretical variance decomposition of variables; a historical variance decomposition based on the smoothed shocks and impulse responses experiments. We conclude by comparing our model implications for forecast dispersion to survey data.

Variance decomposition

Table 3 reports the theoretical variance decomposition of several variables of interest. For each structural shock we compute the share of the total variation in the corresponding variable that is accounted by that shock at two horizons: one is at the business cycle frequency which incorporates periodic components with cycles between 6 and 32 quarters, as in Stock and Watson (1999). The second is at a long-run horizon which is the theoretical variance decomposition obtained by solving the dynamic Lyapunov equation characterizing the law of motion of the model. These two shares are reported in the first two rows of Table 3. In the third row, we also report for comparison the variance decomposition in an estimated model without ambiguity.

At business cycle frequency the ambiguity shock accounts for about 27% of GDP variability. It simultaneously explains a large share of real variables such as consumption (52%), investment (14%), hours (31%) and less for inflation (2%) and the nominal interest rate (7%). The long-run theoretical decomposition implies that the shock is even more important. It explains about 55% of GDP variability and it is a significant driver of the other variables: consumption (62%), investment (51%), hours (52%), inflation (29%) and the nominal interest rate (38%). Based on these two sets of numbers we can conclude that the ambiguity shock is an important factor of business cycle fluctuations while also having a low-frequency component that magnifies its role in the total variance decomposition. The simultaneous large shares of variation explained by ambiguity suggest that time-variation in the agents'

confidence about technology shocks can be a unified source of macroeconomic variability.

For comparison, we can analyze the estimated model without ambiguity. The business cycle frequency variance decomposition for a model that sets the level of ambiguity to zero, i.e. $n = 0$, is reported in the third row of Table 3. There the largest share of GDP variability is explained by the marginal efficiency of investment shock, confirming the results of many recent studies, such as Christiano et al. (2010b) and Justiniano et al. (2011). Introducing time-varying ambiguity reduces the importance of the other shocks, except for the transitory technology shock Z_t, in explaining the decomposition of the level of observed variables. The reduction in effects are especially strong for the marginal efficiency of investment and growth rate shocks. With ambiguity, the shock Z_t becomes more important. The reason is that ambiguity enters in the model indirectly through the variance of σ_z. Thus, with ambiguity the estimated variance of Z_t affects the likelihood evaluation through two channels: one direct, through the shock Z_t, and one indirect through the variance of the shock a_t.

Impulse responses

We now turn to analyzing the impulse responses for the ambiguity shock in the estimated model. As suggested already in the discussion, an increase in ambiguity generates a recession, in which hours worked, consumption and investment fall. It is also worth emphasizing that the impulse response is symmetric: a fall in ambiguity, compared to steady state, generates an economic boom in which variables comove positively. The fact that this shock predicts comovement between these variables is an important feature that helps explain why the estimation prefers in the likelihood maximization such a shock.

Before we proceed further, let us define the excess return on capital. The return on capital R_t^K is defined in equation (6.18) in Appendix. The ex-post excess return ex_{t+1} is then defined as the difference between the realized return on capital and the nominal interest rate, i.e. $ex_{t+1} := R_{t+1}^K - R_t$. As detailed in Appendix, from the two intertemporal optimality conditions with respect to capital and bonds, it follows that in the linearized equilibrium, R_t equals $E_t^{p^0} R_{t+1}^K$. Thus, under the worst case expectations, $E_t^{p^0} ex_{t+1} = 0$. However, the average realized R_{t+1}^K will be in general different than $E_t^{p^0} R_{t+1}^K$. The reason is that the former is obtained under the true data generating process in which the worst case mean a_t does not materialize on average in the realization of z_{t+1}, while $E_t^{p^0} R_{t+1}^K$ represents the average R_{t+1}^K if a_t would materialize.

Figure 3 plots the responses to a one standard deviation increase in ambiguity for the estimated model.[10] On top of the mentioned comovement in macro aggregates, the model also predicts a fall in the price of capital, a fall in the real interest rate and a countercyclical

---

[10]The impulse response is plotted as percentage deviations from the 'zero risk steady state' of the model.

ex-post excess return.[11]

We briefly explain these results. The main intuition in understanding the effect of the ambiguity shock is to relate it to its interpretation of a news shock. An increase in ambiguity makes the agent act under a more cautious forecast of the future technology. From an outside observer that analyzes the agent's behavior, it seems that this agent acts under some negative news about future productivity. This negative news interpretation of the increase in ambiguity helps explain the mechanics and economics of the impulse response. As described in detail in Christiano et al. (2008) and Christiano et al. (2010a), in a rational expectations model, a negative news about future productivity can produce a significant bust in real economy while simultaneously generating a fall in the price of capital. This result is reflected in our impulse response. In our model, the negative news is on average not materialized, because nothing changed in the true process for technology, as shown in the first panel of Figure 3. However, because of the persistent effect of ambiguity, the economy continues to go through a prolonged recession.

The ex-post excess return, defined above as ex_{t+1}, is positive following the period of the initial increase in ambiguity and it is positive persistently along the path. The explanation for the countercyclical excess returns is that the negative expectation about future productivity does not materialize in the true data generating process, so ex-post capital pays more. Thus along this path of higher ambiguity, R^K_{t+1} is systematically higher than $E^{p^0}_t R^K_{t+1}$. The ex-post excess return reflects a rational uncertainty premium that ambiguity averse agents require to invest in the uncertain asset.

Historical shock decomposition

We conclude the description of the role of ambiguity shocks in business cycle fluctuations by discussing the historical variance decomposition and the smoothed shocks that result from the estimated model. In Figure 1 we plot the Kalman filter estimate of the ambiguity shock a_t, as a deviation from its steady state value. The figure first shows that ambiguity is very persistent. After an initial increase around 1991, which also corresponds to an economic downturn, the level of ambiguity was low and declining during the 90's, reaching its lowest values around 2000. It then increases back to levels close to steady state until 2005. Following a few years of relatively small upward deviations from its mean, ambiguity spikes starting in 2008. Ambiguity rapidly increases, so that throughout 2008 it doubles over each quarter. Ambiguity reaches its peak in 2008Q4 when it is 8 times larger than its 2008Q1 value. The figure shows a dotted vertical line at 2008Q3, which corresponds to the Lehman Brothers bankruptcy. Our model interprets the period following 2008Q1 as one in which ambiguity

---

[11]By price of capital we mean the Tobin's marginal q. For the equation determining the evolution of q_t in the scaled version of our economy, see equation (6.20) in Appendix.

about future productivity has increased dramatically.

Based on these smoothed path of ambiguity shocks we can now calculate what the model implies for the historical evolution of endogenous variables. In Figure 2 we compare, as deviations from steady state, the observed data with the counterfactual historical evolution for the growth rate of output, consumption, investment and the level of hours worked when the ambiguity shock is the only shock active in the model economy. The ambiguity shock implies a path for variables that comes close to matching the data, especially for output, consumption and hours. The model implied path of investment is less volatile but the correlation with the observed data is still significantly large. It is interesting that the ambiguity shock helps explain some of the business cycle frequency of these variables but also the low-frequency component as present in hours worked.[12]

The ambiguity shock generates the three large recessions observed in this sample. Indeed, if we analyze the smoothed path of the shock in Figure 2, the time-varying ambiguity helps explain the recession of the 1991, the large growth of the 1990's (as a period of low ambiguity), and then the recession of 2001. Given that the estimated ambiguity still continues to rise through 2005, the model misses by predicting a more prolonged recession than in the data, where output picks up quickly. The rise in ambiguity in 2008 predicts in the model that output, investment and consumption fall. The model matches the fall in consumption, but fails to generate a large fall in investment. It is important to highlight that the ambiguity shock implies that in the model consumption and investment comove. Indeed, in the historical decomposition, recessions are times when both of these variables fall. This is an important effect because standard shocks that have been recently found to be quantitatively important (as for example in Christiano et al. (2010b) and Justiniano et al. (2011)), such as the marginal efficiency of investment or intertemporal preference shocks imply a weak, and most often a negative comovement between these two components.

Survey dispersion data

Our estimation treats confidence about productivity as a latent variable with its range restricted by the measured volatility of productivity. The results suggest that time variation in confidence is an important source of observed economic fluctuations. We provide external validation for our model by comparing model-implied confidence to popular measures of confidence based on the dispersion of survey forecasts. In particular, we look at data from the Survey of Professional Forecasters on one quarter ahead projections for Q/Q real GDP growth and inflation, expressed in annualized percentage points. Our measure of dispersion is the interquartile range (the difference between the 75th percentile and the 25th percentile).

---

[12]Usually the low-frequency movement in hours worked is attributed to exogenous labor supply shocks, corresponding to shocks to ψ_L in our model. See for example Justiniano et al. (2011).

We also construct model-implied measures of confidence about GDP growth and inflation. Agents' belief sets about one-quarter-ahead technology shocks are given by (5.3)-(5.5). Agents also know the structure of the economy, so that their set of forecast means about any variable X_{t+1} can be read off the (linearized) solution of the model. Our measure of (lack of) confidence about X_{t+1} is the range of forecasts, $R_t X_{t+1}$, implied by the belief set:

$$R_t X_{t+1} := \left| E^{a_t} X_{t+1} - E^{-a_t} X_{t+1} \right| = 2|E^{a_t}(X_{t+1}|s_t = 0)|, \tag{5.9}$$

where $E^{a_t}(X_{t+1}|s_t = 0)$ denotes the conditional expectation of X_{t+1} evaluated at a_t and at values of the other state variables, denoted here by s_t, equal to zero. The latter equality in (5.9) follows from the linearity of the law of motion for X_{t+1}. Finally, we use the absolute value operator to maintain a positively valued range.

Table 4 reports summary statistics. The first column shows that the mean range of forecasts is very similar in magnitude to the SPF interquartile range for both inflation and real GDP growth. The second column says that the variability of the range of forecasts is similar to the SPF interquartile range for inflation, while it is about half the SPF interquartile range for growth. The third column shows the correlation coefficient between the measures: it is significantly positive for both inflation and growth. Figure 4 plots the range of forecasts implied by the belief set (solid line) against the SPF interquartile range (dashed line). The top panel shows real GDP growth and the bottom one shows inflation. The main disconnect between the model and the data is the high SPF real growth forecast dispersion in the late 1980s. Otherwise, the time path generated by the ambiguity shock a_t matches qualitatively the alternative measures of confidence obtained from survey forecasts.

References

AN, S. AND F. SCHORFHEIDE (2007): "Bayesian Analysis of DSGE Models," *Econometric Reviews*, 26, 113–172.

ANGELETOS, G. AND J. LA'O (2011): "Decentralization, Communication, and the Origins of Fluctuations," NBER Working Papers 17060.

ANSCOMBE, F. AND R. AUMANN (1963): "A Definition of Subjective Probability," *Annals of mathematical statistics*, 199–205.

ARELLANO, C., Y. BAI, AND P. KEHOE (2010): "Financial Markets and Fluctuations in Uncertainty," FRB of Minneapolis, Research Department Staff Report.

BACHMANN, R., S. ELSTNER, AND E. SIMS (2010): "Uncertainty and Economic Activity: Evidence from Business Survey Data," NBER Working Papers 16143.

BARSKY, R. AND E. SIMS (2011): "Information, Animal Spirits, and the Meaning of Innovations in Consumer Confidence," *The American Economic Review*, forthcoming.

BASU, S. AND B. BUNDICK (2011): "Uncertainty Shocks in a Model of Effective Demand," Boston College Working Papers 774.

BEAUDRY, P. AND F. PORTIER (2006): "Stock Prices, News, and Economic Fluctuations," *American Economic Review*, 96, 1293–1307.

BIDDER, R. AND M. SMITH (2011): "Robust Control in a Nonlinear DSGE Model," Unpublished manuscript, San Francisco Fed.

BLANCHARD, O., J. L'HUILLIER, AND G. LORENZONI (2009): "News, Noise, and Fluctuations: An Empirical Exploration," NBER Working Papers 15015.

BLOOM, N., M. FLOETOTTO, AND N. JAIMOVICH (2009): "Really Uncertain Business Cycles," Unpublished manuscript, Stanford University.

CAGETTI, M., L. P. HANSEN, T. SARGENT, AND N. WILLIAMS (2002): "Robustness and Pricing with Uncertain Growth," *Review of Financial Studies*, 15, 363–404.

CALVO, G. (1983): "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12, 383–398.

CHRISTIANO, L. (2002): "Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients," *Computational Economics*, 20, 21–55.

CHRISTIANO, L., M. EICHENBAUM, AND C. EVANS (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113.

CHRISTIANO, L., C. ILUT, R. MOTTO, AND M. ROSTAGNO (2008): "Monetary Policy and Stock Market Boom-Bust Cycles," ECB Working Papers 955, European Central Bank.

——— (2010a): "Monetary Policy and Stock Market Booms," NBER Working Papers 16402.

CHRISTIANO, L., R. MOTTO, AND M. ROSTAGNO (2010b): "Financial Factors in Economic Fluctuations," Working Papers 1192, European Central Bank.

DEL NEGRO, M., F. SCHORFHEIDE, F. SMETS, AND R. WOUTERS (2007): "On the Fit of New Keynesian Models," *Journal of Business and Economic Statistics*, 25, 123–143.

ELLSBERG, D. (1961): "Risk, Ambiguity, and the Savage Axioms," *The Quarterly Journal of Economics*, 643–669.

EPSTEIN, L. AND T. WANG (1994): "Intertemporal Asset Pricing under Knightian Uncertainty," *Econometrica*, 62, 283–322.

EPSTEIN, L. G. AND M. SCHNEIDER (2003): "Recursive Multiple-Priors," *Journal of Economic Theory*, 113, 1–31.

———— (2007): "Learning Under Ambiguity," *Review of Economic Studies*, 74, 1275–1303.

———— (2008): "Ambiguity, Information Quality, and Asset Pricing," *Journal of Finance*, 63, 197–228.

———— (2010): "Ambiguity and Asset Markets," *Annual Review of Financial Economics*, forthcoming.

ERCEG, C. J., D. W. HENDERSON, AND A. T. LEVIN (2000): "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics*, 46, 281–313.

FARMER, R. (2009): "Confidence, Crashes and Animal Spirits," NBER Working Papers 14846.

FERNÁNDEZ-VILLAVERDE, J., P. GUERRÓN-QUINTANA, J. RUBIO-RAMÍREZ, AND M. URIBE (2010): "Risk Matters: The Real Effects of Volatility Shocks," *The American Economic Review*, forthcoming.

FERNÁNDEZ-VILLAVERDE, J. AND J. RUBIO-RAMIREZ (2007): "Estimating Macroeconomic Models: A Likelihood Approach," *Review of Economic Studies*, 74, 1059–1087.

FERNÁNDEZ-VILLAVERDE, J. AND J. RUBIO-RAMÍREZ (2010): "Macroeconomics and Volatility: Data, Models, and Estimation," NBER Working Papers 16618.

FISHER, J. (2006): "The Dynamic Effects of Neutral and Investment-specific Technology Shocks," *Journal of Political Economy*, 114, 413–451.

GILBOA, I. AND D. SCHMEIDLER (1989): "Maxmin Expected Utility with Non-unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

GOURIO, F. (2011): "Disasters Risk and Business Cycles," *The American Economic Review*, forthcoming.

HANSEN, L., T. SARGENT, AND T. TALLARINI (1999): "Robust Permanent Income and Pricing," *Review of Economic Studies*, 66, 873–907.

ILUT, C. (2009): "Ambiguity Aversion: Implications for the Uncovered Interest Rate Parity Puzzle," Unpublished manuscript, Duke University.

JAIMOVICH, N. AND S. REBELO (2009): "Can News about the Future Drive the Business Cycle?" *The American Economic Review*, 99, 1097–1118.

JUSTINIANO, A. AND G. PRIMICERI (2008): "The Time-Varying Volatility of Macroeconomic Fluctuations," *The American Economic Review*, 98, 604–641.

JUSTINIANO, A., G. PRIMICERI, AND A. TAMBALOTTI (2011): "Investment Shocks and the Relative Price of Investment," *Review of Economic Dynamics*, 14, 101–121.

LUCAS, R. (1987): *Models of Business Cycles*, Basil Blackwell Oxford.

MARTIN, A. AND J. VENTURA (2011): "Economic Growth with Bubbles," *The American Economic Review*, forthcoming.

SCHMITT-GROHE, S. AND M. URIBE (2008): "What's News in Business Cycles," NBER Working Papers 14215.

SMETS, F. AND R. WOUTERS (2007): "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *The American Economic Review*, 97, 586–606.

STOCK, J. AND M. WATSON (1999): "Business Cycle Fluctuations in US Macroeconomic Time Series," *Handbook of Macroeconomics*, 1, 3–64.

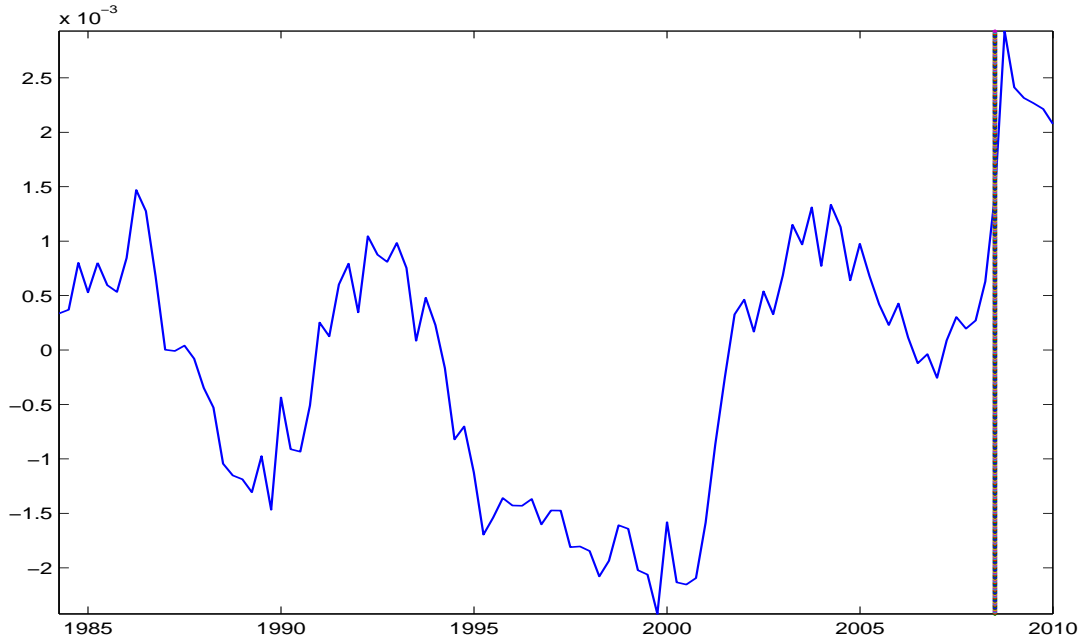Figure 1: Estimated ambiguity shock
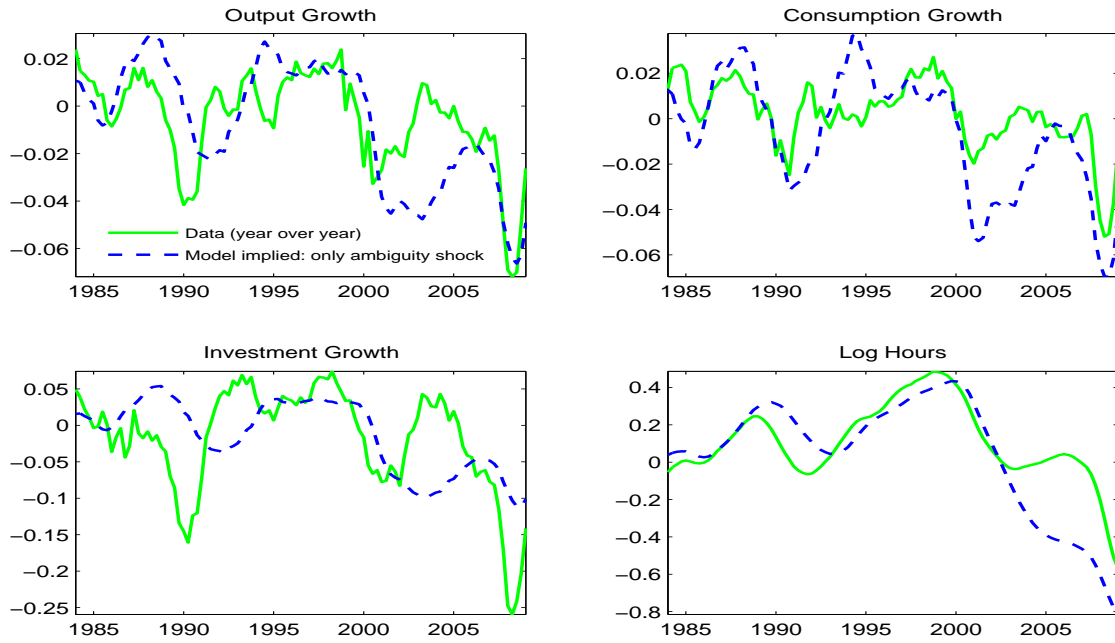

Figure 2: Historical shock decomposition

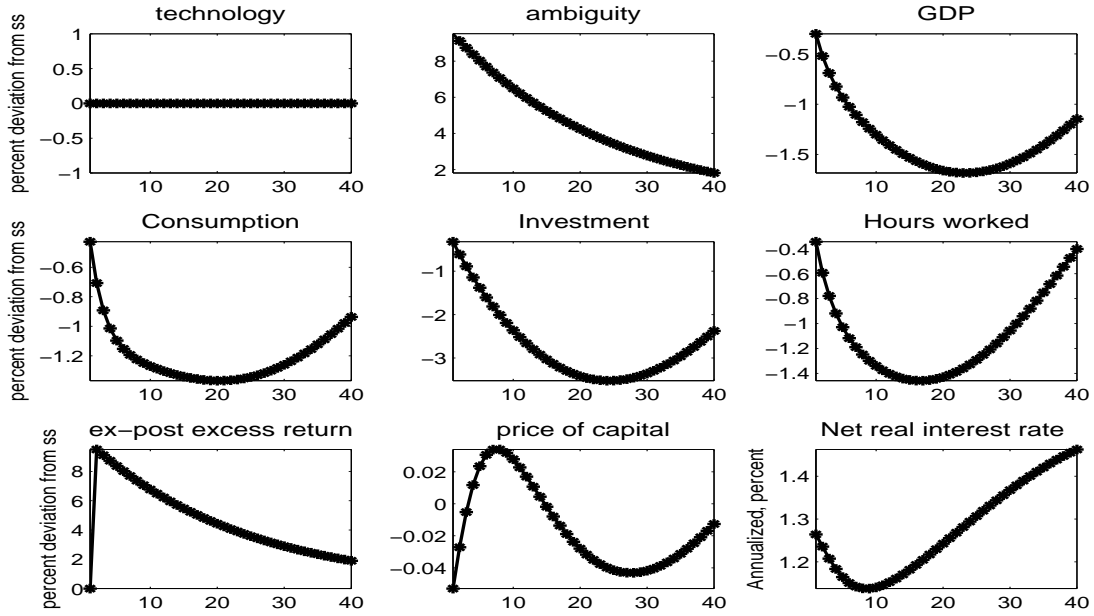Figure 3: Impulse response: positive shock to ambiguity
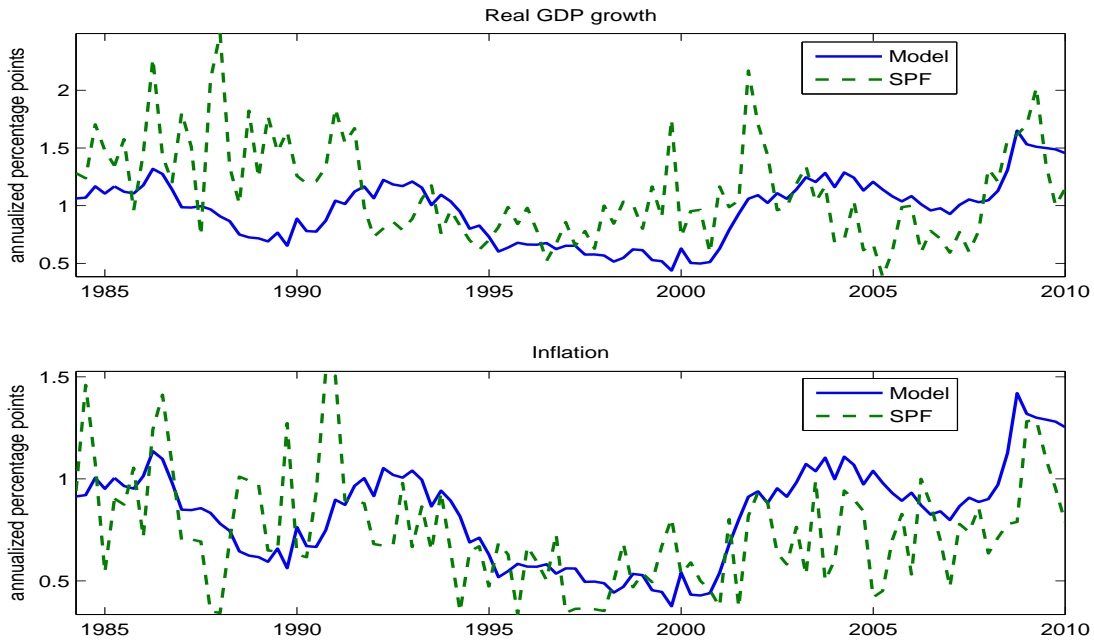

Figure 4: Model implied and SPF range of forecasts

Table 2: Priors and Posteriors for structural parameters

Parameter	Description	Prior			Posterior		
		Type[a]	Mean	St.dev	Mode	[.5 ,	.95][b]
α	Capital share	B	0.4	0.02	0.322	0.291	0.353
δ	Depreciation	B	0.025	0.002	0.0237	0.0206	0.0279
$100(\beta^{-1}-1)$	Discount factor	G	0.3	0.05	0.353	0.2586	0.4728
$100(\mu_\epsilon^*-1)$	Growth rate	N	0.4	0.1	0.5	0.4	0.6
$100(\mu_\Upsilon-1)$	Price of inv. growth	N	0.4	0.1	0.46	0.43	0.49
$100(\bar{\pi}-1)$	Net inflation	N	0.6	0.2	0.85	0.66	1.17
ξ_p	Calvo prices	B	0.5	0.1	0.743	0.681	0.841
ξ_w	Calvo wages	B	0.5	0.1	0.938	0.912	0.953
S''	Investment adj. cost	G	10	5	13.92	6.157	27.962
ϑ	Capacity utilization	G	2	1	1.959	0.433	4.279
a_π	Taylor rule inflation	N	1.7	0.3	2.09	1.771	2.473
a_y	Taylor rule output	N	0.15	0.05	0.059	0.013	0.188
a_{gy}	Taylor rule growth	N	0.15	0.05	0.209	0.116	0.294
ρ_R	Taylor rule smoothing	B	0.5	0.15	0.808	0.751	0.842
λ_f-1	SS price markup	N	0.2	0.05	0.22	0.134	0.314
λ_w-1	SS wage markup	N	0.2	0.05	0.135	0.069	0.22
θ	Internal habit	B	0.5	0.1	0.661	0.535	0.729
σ_L	Disutility of labor	G	2	1	1.886	1.64	2.288
n	Level ambiguity	B	0.5	0.25	0.963	0.827	0.999
ρ_z	Transitory technology	B	0.5	0.15	0.955	0.928	0.974
$\rho_{\mu_\epsilon^*}$	Persistent technology	B	0.3	0.15	0.132	0.014	0.509
ρ_ζ	Efficiency of investment	B	0.5	0.15	0.494	0.351	0.722
ρ_{λ_f}	Price mark-up	B	0.5	0.15	0.907	0.62	0.961
ρ_g	Government spending	B	0.5	0.15	0.954	0.923	0.977
ρ_{μ_Υ}	Price of investment	B	0.5	0.15	0.957	0.929	0.983
ρ_a	Level Ambiguity	B	0.5	0.15	0.96	0.936	0.981
σ_z	Transitory technology	IG	0.01	0.01	0.0045	0.0041	0.0058
$\sigma_{\mu_\epsilon^*}$	Persistent technology	IG	0.01	0.01	0.0044	0.0029	0.0064
σ_ζ	Efficiency of investment	IG	0.01	0.01	0.0183	0.016	0.0231
σ_{λ_f}	Price mark-up	IG	0.005	0.01	0.0102	0.007	0.033
σ_g	Government spending	IG	0.01	0.01	0.0195	0.017	0.0236
σ_{μ_Υ}	Price of investment	IG	0.01	0.01	0.003	0.0026	0.0034
σ_{ϵ_R}	Monetary policy	IG	0.005	0.01	0.0015	0.0013	0.0017

a B refers to the Beta distribution, N to the Normal distribution, G to the Gamma distribution, and IG to the Inverse-gamma distribution.

b Posterior percentiles obtained from 2 chains of 200,000 draws generated using a Random walk Metropolis algorithm. We discard the initial 50,000 draws and retain one out of every 5 subsequent draws.

Table 3: Theoretical variance decomposition

Shock\Variable	Output	Cons.	Invest.	Hours	Inflation	Int. rate
TFP Ambiguity (a_t)	27.2	52.1	14.4	31.1	2	7.4
	(55.4)	(62.7)	(51.4)	(52.1)	(29.6)	(38.5)
	[-]	[-]	[-]	[-]	[-]	[-]
Transitory technology (z_t)	12.1	13.5	9.6	2.5	23.8	15.9
	(5.5)	(5)	(5.7)	(6.5)	(15.2)	(10.5)
	[4.1]	[7.3]	[3.1]	[3.1]	[17.2]	[15.4]
Persistent technology $(\mu^*_{\epsilon,t})$	5.9	5.7	5.3	10.4	5.1	1.9
	(8.1)	(7.1)	(8.4)	(9.3)	(7.7)	(6.7)
	[18.8]	[37.4]	[12.8]	[29.8]	[14.2]	[5.9]
Government spending (g_t)	3.4	2.1	0.22	3.3	0.75	1.4
	(0.6)	(0.3)	(0.1)	(0.8)	(0.5)	(0.8)
	[4.8]	[4.1]	[0.3]	[3.9]	[0.3]	[0.7]
Price mark-up $(\lambda_{f,t})$	13.1	12.5	13.3	14.4	61.6	46.8
	(8.4)	(8.6)	(8.4)	(8.9)	(32.1)	(18.9)
	[16.2]	[27.2]	[14.2]	[15.2]	[65.8]	[58.2]
Monetary policy $(\epsilon_{R,t})$	3.6	5.7	2.3	4.1	1	13.1
	(1.7)	(1.8)	(1.7)	(1.8)	(1.2)	(6.3)
	[4.1]	[8.8]	[2.1]	[4]	[0.3]	[11.4]
Price of investment $(\mu_{\Upsilon,t})$	1.7	0.5	2.3	1.6	0.3	0.7
	(5.1)	(4.2)	(6.1)	(5)	(3.3)	(4.4)
	[2.2]	[1.3]	[2.2]	[1.8]	[0.1]	[0.4]
Efficiency of investment (ζ_t)	32.8	7.6	52.5	32.3	5.3	12.6
	(15)	(10.1)	(18)	(15.4)	(10.3)	(13.9)
	[49.6]	[13.8]	[65.1]	[42.1]	[2]	[7.7]

Note: For each variable, the first two rows of numbers refer to the variance decomposition in the estimated model with ambiguity. The first row is the business cycle frequency and the second row is the long-run decomposition. The third row, in squared brackets, refers to the business cycle frequency decomposition in the estimated model without ambiguity.

Table 4: Model implied and SPF range of forecasts

Variable	Mean	St.dev	Correlation
Inflation SPF	0.74	0.27	-
Inflation Model	0.82	0.24	0.44 (0.27, 0.58)
Real growth SPF	1.12	0.43	-
Real growth Model	0.96	0.27	0.25 (0.06, 0.42)

Note: The correlation is between the SPF and model implied range of forecasts. In parantheses we report the 95% confidence interval.

6 Appendix

6.1 Structure of estimated model

In this section we describe the structure of the estimated model in Section 5.

The goods sector

The final output in this economy is produced by a representative final good firm that combines a continuum of intermediate goods $Y_{j,t}$ in the unit interval by using the following linear homogeneous technology:

$$Y_t = \left[\int_0^1 Y_{j,t}^{\frac{1}{\lambda_{f,t}}} dj \right]^{\lambda_{f,t}},$$

where $\lambda_{f,t}$ is the markup of price over marginal cost for intermediate goods firms. The markup shock evolves as:

$$\log(\lambda_{f,t}/\lambda_f) = \rho_{\lambda_f} \log(\lambda_{f,t-1}/\lambda_f) + \lambda_{f,t}^x,$$

where $\lambda_{f,t}^x$ is $i.i.d.N(0, \sigma_{\lambda_f}^2)$. Profit maximization and the zero profit condition leads to the following demand function for good j:

$$Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \tag{6.1}$$

The price of the final good is:

$$P_t = \left[\int_0^1 P_{j,t}^{\frac{1}{1-\lambda_{f,t}}} dj \right]^{(1-\lambda_{f,t})}.$$

The intermediate good j is produced by a price-setting monopolist using the following production function:

$$Y_{j,t} = \max\{Z_t K_{j,t}^\alpha \left(\epsilon_t H_{j,t} \right)^{1-\alpha} - \Phi \epsilon_t^*, 0\}, \tag{6.2}$$

where Φ is a fixed cost and $K_{j,t}$ and $H_{j,t}$ denote the services of capital and homogeneous labor employed by firm j. Φ is chosen so that steady state profits are equal to zero. The intermediate goods firms are competitive in factor markets, where they confront a rental rate, $P_t \tilde{r}_t^k$, on capital services and a wage rate, W_t, on labor services. The variable ϵ_t is a technology shock with a covariance stationary growth rate. The variable Z_t is a transitory technology shock. It is stationary from the perspective of the econometrician, but it is perceived to be ambiguous by agents, as described in Section 5.1.

The fixed costs grow with the exogenous variable, ϵ_t^* :

$$\epsilon_t^* = \epsilon_t \Upsilon^{\left(\frac{\alpha}{1-\alpha}t\right)},$$

with $\Upsilon > 1$. If fixed costs were not growing, then they would eventually become irrelevant. We specify that they grow at the same rate as ϵ_t^*, which is the rate at which output grows. Note that the growth of ϵ_t^*, i.e. $\mu_{\epsilon,t}^* \equiv \Delta \log(\epsilon_t^*)$, exceeds that of ϵ_t, i.e. $\mu_{\epsilon,t} \equiv \Delta \log(\epsilon_t)$:

$$\mu_{\epsilon,t}^* = \mu_{\epsilon,t} \Upsilon^{\frac{\alpha}{1-\alpha}}.$$

This is because we have another source of growth in this economy, in addition to ϵ_t. In particular, we posit a trend decrease in the price of investment. We discuss this process as well as the representation for Z_t further below. The stochastic growth rate evolves as:

$$\log(\mu_{\epsilon,t}^*/\mu_\epsilon^*) = \rho_{\mu_\epsilon^*} \log(\mu_{\epsilon,t-1}^*/\mu_\epsilon^*) + \mu_{\epsilon,t}^{x*},$$

where $\mu_{\epsilon,t}^{x*}$ is $i.i.d. N(0, \sigma_{\mu_\epsilon^*}^2)$ and μ_ϵ^* is the steady state growth rate of the economy.

We now describe the intermediate good firms pricing opportunities. Following Calvo (1983), a fraction $1 - \xi_p$, randomly chosen, of these firms are permitted to reoptimize their price every period. The other fraction ξ_p cannot reoptimize and set $P_{it} = \bar{\pi} P_{i,t-1}$, where $\bar{\pi}$ is steady state inflation. The jth firm that has the opportunity to reoptimize its price does so to maximize the expected present discounted value of the future profits:

$$E_t^{p^0} \sum_{s=0}^\infty (\beta\xi_p)^s \frac{\lambda_{t+s}}{\lambda_t} \left[P_{j,t+s} Y_{j,t+s} - W_{t+s} H_{j,t+s} - P_{t+s} \tilde{r}_{t+s}^k K_{j,t+s} \right], \qquad (6.3)$$

subject to the demand function (6.1), where λ_t is the marginal utility of nominal income for the representative household that owns the firm. It should be noted that the expectation operator in these equations is the expectation under the worst case belief p^0. This is because state prices in the economy reflect ambiguity.

There are perfectly competitive "employment agencies" that aggregate the households specialized labor inputs $h_{i,t}$ into a homogeneous labor service according to the following function:

$$H_t = \left[\int_0^1 (h_{i,t})^{\frac{1}{\lambda_w}} di \right]^{\lambda_w},$$

where λ_w is the constant markup of wages over the household's marginal rate of substitution. These employment agencies rent the homogeneous labor service H_t to the intermediate goods firms at the wage rate W_t. In turn, these agencies pay the wage $W_{i,t}$ to the household

supplying labor of type i. Similarly as for the final goods producers, profit maximization and the zero profit condition lead to the following demand function for labor input of type i:

$$h_{i,t} = H_t \left(\frac{W_t}{W_{i,t}} \right)^{\frac{\lambda_w}{\lambda_w - 1}}. \tag{6.4}$$

We follow Erceg et al. (2000) and assume that the household is a monopolist in the supply of labor by providing $h_{i,t}$ and it sets its nominal wage rate, $W_{i,t}$. It does so optimally with probability $1 - \xi_w$ and with probability ξ_w is does not reoptimize its wage. In case it does not reoptimize, it sets the wage as:

$$W_{i,t} = \bar{\pi} \mu_\epsilon^* W_{i,t-1}.$$

Households

The household accumulates capital subject to the following technology:

$$\bar{K}_{t+1} = (1 - \delta)\bar{K}_t + \left[1 - S \left(\zeta_t \frac{I_t}{I_{t-1}} \right) \right] I_t, \tag{6.5}$$

where ζ_t is a disturbance to the marginal efficiency of investment with mean unity, $\bar{K}_t$ is the beginning of period t physical stock of capital, and I_t is period t investment. The function S reflects adjustment costs in investment. The function S is convex, with steady state values of $S = S' = 0, S'' > 0$. The specific functional form for $S(.)$ that we use is:

$$S \left(\zeta_t \frac{I_t}{I_{t-1}} \right) = \exp \left[\sqrt{\frac{S''}{2}} \left(\zeta_t \frac{I_t}{I_{t-1}} - 1 \right) \right] + \exp \left[-\sqrt{\frac{S''}{2}} \left(\zeta_t \frac{I_t}{I_{t-1}} - 1 \right) \right] - 2. \tag{6.6}$$

The marginal efficiency of investment follows the process:

$$\log(\zeta_t) = \rho_\zeta \log(\zeta_{t-1}) + \zeta_t^x,$$

where ζ_t^x is $i.i.d.N(0, \sigma_\zeta^2)$.

Households own the physical stock of capital and rent out capital services, K_t, to a competitive capital market at the rate $P_t \tilde{r}_t^k$, by selecting the capital utilization rate u_t:

$$K_t = u_t \bar{K}_t.$$

Increased utilization requires increased maintenance costs in terms of investment goods per unit of physical capital measured by the function $a(u_t)$. The function $a(.)$ is increasing and

convex, $a(1) = 0$ and u_t is unity in the nonstochastic steady state. We assume that $a''(u) = \vartheta r^k$, where r^k is the steady state value of the rental rate of capital. Then, $a''(u)/a'(u) = \vartheta$ is a parameter that controls the degree of convexity of utilization costs. In the linearized equilibrium, only ϑ matters for dynamics. The specific form for $a(u_t)$ that we use is

$$a(u_t) = \frac{1}{2}r^k\vartheta u_t^2 + r^k(1-\vartheta)u_t + r^k(\frac{1}{2}\vartheta - 1). \tag{6.7}$$

The ith household's budget constraint is:

$$P_tC_t + P_t\frac{I_t}{\mu_{\Upsilon,t}\Upsilon^t} + B_t = B_{t-1}R_{t-1} + P_t\overline{K}_t[\tilde{r}_t^k u_t - a(u_t)\Upsilon^{-t}] + W_{i,t}h_{i,t} + X_{i,t} - T_tP_t \tag{6.8}$$

where B_t are holdings of government bonds, R_t is the gross nominal interest rate, $X_{i,t}$ is the net cash inflow from participating in state contingent securities at time t and T_t is net lump-sum taxes. We assume that the cost in consumption units of one unit of investment goods is $(\Upsilon^t\mu_{\Upsilon,t})^{-1}$. The stationary component of the relative price of investment follows:

$$\log(\mu_{\Upsilon,t}) = \rho_{\mu_\Upsilon}\log(\mu_{\Upsilon,t-1}) + \mu_{\Upsilon,t}^x,$$

where $\mu_{\Upsilon,t}^x$ is $i.i.d.N(0,\sigma_{\mu_\Upsilon}^2)$.

The government

The market clearing condition for this economy is:

$$C_t + \frac{I_t}{\mu_{\Upsilon,t}\Upsilon^t} + G_t = Y_t^G, \tag{6.9}$$

where G_t denotes government expenditures and Y_t^G is our definition of measured GDP, i.e. $Y_t^G \equiv Y_t - a(u_t)\Upsilon^{-t}\overline{K}_t$. We model government expenditures as $G_t = g_t\epsilon_t^*$, where g_t is a stationary stochastic process. The fiscal policy is Ricardian. The government finances G_t by issuing short term bonds B_t and adjusting lump sum taxes T_t. The law of motion for g_t is:

$$\log(g_t/g) = \rho_g\log(g_{t-1}/g) + g_t^x,$$

where g_t^x is.$i.d.d.N(0,\sigma_g^2)$.

The nominal interest rate R_t is set by a monetary policy authority according to:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R}\left[\left(\frac{\pi_t}{\bar{\pi}}\right)^{a_\pi}\left(\frac{Y_t^G}{Y_t^*}\right)^{a_y}\left(\frac{Y_t^G}{\mu_\epsilon^*Y_{t-1}^G}\right)^{a_{gy}}\right]^{1-\rho_R}\exp(\epsilon_{R.t}),$$

where $\epsilon_{R.t}$ is a monetary policy shock $i.i.d.N(0,\sigma_{\epsilon_R}^2)$, $\bar{\pi}$ is the constant inflation target, R is

the steady state nominal interest rate target equal to $\bar{\pi}\mu^*_\epsilon/\beta$ and Y^*_t is the level of output along the deterministic growth path.

6.2 Equilibrium conditions for the estimated model

Here we describe the equations that characterize the equilibrium of the estimated model in Section 5. To solve the model, we first scale the variables in order to induce stationarity. As mentioned in section 6.1, the model has two sources of growth: a stochastic trend in neutral technology and a deterministic trend in the price of investment goods. The real variables are scaled as follows:

$$\epsilon^*_t = \epsilon_t \Upsilon^{\left(\frac{\alpha}{1-\alpha}t\right)}, c_t = \frac{C_t}{\epsilon^*_t}, y_t = \frac{Y_t}{\epsilon^*_t}, g_t = \frac{G_t}{\epsilon^*_t}$$

$$\bar{k}_{t+1} = \frac{\bar{K}_{t+1}}{\epsilon^*_t \Upsilon^t}, i_t = \frac{I_t}{\epsilon^*_t \Upsilon^t}, \lambda_{z,t} = \lambda_t P_t \epsilon^*_t.$$

where λ_t is the Lagrange multiplier on the household budget constraint in (6.8). The scaling here indicates that because of the deterministic trend in the price of investment goods, the capital stock and investment grow at a faster rate than output and consumption.

Let μ_t be the Lagrange multiplier on the capital accumulation equation in (6.5) and define the nominal price of capital expressed in units of consumption goods as

$$Q_{\bar{K},t} = \frac{\mu_t}{\lambda_t}$$

Price variables are then scaled as:

$$q_t = \frac{Q_{\bar{K},t}}{\Upsilon^{-t}P_t}, r^k_t = \frac{\tilde{r}^k_t}{\Upsilon^{-t}}, \widetilde{w}_t = \frac{W_t}{\epsilon^*_t P_t}$$

We will also make use of other scaling conventions:

$$\mu^*_{\epsilon,t} = \mu_{\epsilon,t}\Upsilon^{\frac{\alpha}{1-\alpha}}, \ p^*_t = (P_t)^{-1}\left(\int_0^1 P_{j,t}^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}}dj\right)^{\frac{1-\lambda_{f,t}}{\lambda_{f,t}}}, \ w^*_t = (W_t)^{-1}\left(\int_0^1 W_{i,t}^{\frac{\lambda_w}{1-\lambda_w}}di\right)^{\frac{1-\lambda_w}{\lambda_w}},$$

where the index i refers to households and the index j to monopolistically competitive firms.

We now present the nonlinear equilibrium conditions characterizing the model, in scaled form. The expectation operator in these equations, $E^{p^0}_t$, is the one-step ahead conditional expectation under the worst case belief p^0. The latter is described by equation (5.6).

46

Goods production 1. the real marginal cost of producing one unit of output, m_t:

$$m_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{\left(r_t^k\right)^{\alpha}\widetilde{w}_t^{1-\alpha}}{Z_t} \tag{6.10}$$

2. marginal cost must also satisfy another condition: namely, that m_t must equal the cost of renting one unit of capital divided by the marginal productivity of capital (the same is true for labor):

$$m_t = \frac{r_t^k}{\alpha Z_t \left(\Upsilon\frac{\mu_{\epsilon,t}^* l_t}{k_t}\right)^{1-\alpha}}, \tag{6.11}$$

where we used that the labor to capital ratios will be the same for all firms. The aggregate homogeneous labor, l_t, can be written in term of the aggregate, h_t, of household differentiated labor, $h_{i,t}$:

$$l_t := \int_0^1 H_{j,t}dj = (w_t^*)^{\frac{\lambda_w}{\lambda_w-1}}h_t, \text{ where } h_t := \int_0^1 h_{i,t}di.$$

3. Conditions associated with Calvo sticky prices:

$$p_t^* = \left[(1-\xi_p)\left(\frac{K_{p,t}}{F_{p,t}}\right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} + \xi_p\left(\frac{\bar{\pi}}{\pi_t}p_{t-1}^*\right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}}\right]^{\frac{1-\lambda_{f,t}}{\lambda_{f,t}}} \tag{6.12}$$

$$K_{p,t} = \lambda_{f,t}\lambda_{z,t}y_t m_t + \beta\xi_p E_t^{p^0}\left(\frac{\bar{\pi}}{\pi_{t+1}}\right)^{\frac{\lambda_{f,t+1}}{1-\lambda_{f,t+1}}}K_{p,t+1} \tag{6.13}$$

$$F_{p,t} = \lambda_{z,t}y_t + \beta\xi_p E_t^{p^0}\left(\frac{\bar{\pi}}{\pi_{t+1}}\right)^{\frac{1}{1-\lambda_{f,t+1}}}F_{p,t+1} \tag{6.14}$$

$$K_{p,t} = F_{p,t}\left[\frac{1-\xi_p\left(\frac{\bar{\pi}}{\pi_t}\right)^{\frac{1}{1-\lambda_{f,t}}}}{(1-\xi_p)}\right]^{1-\lambda_{f,t}} \tag{6.15}$$

Households 1. Marginal utility of consumption (FOC wrt c_t):

$$\lambda_{z,t} = \frac{\mu_{\epsilon,t}^*}{c_t\mu_{\epsilon,t}^* - bc_{t-1}} - E_t^{p^0}\frac{b\beta}{c_{t+1}\mu_{\epsilon,t+1}^* - bc_t} \tag{6.16}$$

2. Capital accumulation decision (FOC wrt $\bar{k}_{t+1}$):

$$\lambda_{z,t} = E_t^{p^0}\frac{\beta}{\pi_{t+1}\mu_{\epsilon,t+1}^*}\lambda_{z,t+1}R_{t+1}^k \tag{6.17}$$

where the return on capital is defined as:

$$R_t^k = \frac{u_t r_t^k - a(u_t) + (1-\delta)q_t}{\Upsilon q_{t-1}}\pi_t \tag{6.18}$$

and physical capital accumulates following:

$$\bar{k}_{t+1} = \frac{(1-\delta)\bar{k}_t}{\Upsilon \mu_{\epsilon,t}^*} + \left[1 - S\left(\frac{\zeta_t i_t \mu_{\epsilon,t}^* \Upsilon}{i_{t-1}}\right)\right]i_t \tag{6.19}$$

3. Investment decision (FOC wrt i_t):

$$\lambda_{z,t}\frac{1}{\mu_{\Upsilon,t}} = \lambda_{z,t}q_t\left[1 - S\left(\frac{\zeta_t i_t \mu_{\epsilon,t}^* \Upsilon}{i_{t-1}}\right) - S'\left(\frac{\zeta_t i_t \mu_{\epsilon,t}^* \Upsilon}{i_{t-1}}\right)\frac{\zeta_t i_t \mu_{\epsilon,t}^* \Upsilon}{i_{t-1}}\right] + \tag{6.20}$$
$$+ \beta E_t^{p^0}\frac{\lambda_{z,t+1}}{\mu_{\epsilon,t+1}^* \Upsilon}q_{t+1}S'\left(\frac{\zeta_t i_t \mu_{\epsilon,t}^* \Upsilon}{i_{t-1}}\right)\zeta_{t+1}\left(\frac{i_{t+1}\mu_{\epsilon,t+1}^* \Upsilon}{i_t}\right)^2.$$

4. Bond decision (FOC wrt B_t):

$$\lambda_{z,t} = E_t^{p^0}\frac{\beta}{\pi_{t+1}\mu_{\epsilon,t+1}^*}\lambda_{z,t+1}R_t \tag{6.21}$$

5. Conditions associated with Calvo sticky wages:

$$w_t^* = \left[(1-\xi_w)\left(\frac{\psi_L K_{w,t}}{\tilde{w}_t F_{w,t}}\right)^{\frac{\lambda_w}{1-\lambda_w(1+\sigma_L)}} + \xi_w\left(\frac{\bar{\pi}\mu_\epsilon^*}{\pi_{w,t}}w_{t-1}^*\right)^{\frac{\lambda_w}{1-\lambda_w}}\right]^{\frac{1-\lambda_w}{\lambda_w}} \tag{6.22}$$

$$\pi_{w,t} = \pi_t \mu_{\epsilon,t}^*\frac{\tilde{w}_t}{\tilde{w}_{t-1}} \tag{6.23}$$

$$F_{w,t} = l_t\frac{\lambda_{z,t}}{\lambda_w} + \beta\xi_w\left(\mu_\epsilon^*\right)^{\frac{1}{1-\lambda_w}}E_t^{p^0}\left(\frac{1}{\pi_{w,t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{\bar{\pi}^{\frac{1}{1-\lambda_w}}}{\mu_{\epsilon,t+1}^*\pi_{t+1}}F_{w,t+1} \tag{6.24}$$

$$K_{w,t} = l_t^{1+\sigma_L} + \beta\xi_w E_t^{p^0}\left(\frac{\bar{\pi}}{\pi_{w,t+1}}\mu_\epsilon^*\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)}K_{w,t+1} \tag{6.25}$$

$$K_{w,t} = \tilde{w}_t F_{w,t}\frac{1}{\psi_L}\left[\frac{1 - \xi_w\left(\frac{\bar{\pi}\mu_\epsilon^*}{\pi_{w,t}}\right)^{\frac{1}{1-\lambda_w}}}{1 - \xi_w}\right]^{1-\lambda_w(1+\sigma_L)} \tag{6.26}$$

6. Capital utilization (FOC wrt u_t) :

$$r_t^k = a'(u_t) \tag{6.27}$$

Monetary policy   1. Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\bar\pi}\right)^{a_\pi} \left(\frac{y_t^G}{y^G}\right)^{a_y} \left(\frac{y_t^G \mu_{\epsilon,t}^*}{y_{t-1}^G \mu_\epsilon^*}\right)^{a_{gy}}\right]^{1-\rho_R} \exp(\epsilon_{R.t}) \tag{6.28}$$

Resource constraint   1. Production function:

$$(p_t^*)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left\{Z_t \left(\frac{u_t \overline{k}_t}{\Upsilon \mu_{\epsilon,t}^*}\right)^\alpha l_t^{1-\alpha} - \Phi\right\} = y_t \tag{6.29}$$

2. Resource constraint:

$$y_t = c_t + \frac{i_t}{\mu_{\Upsilon,t}} + g_t + \frac{a(u_t)\overline{k}_t}{\mu_{\Upsilon,t}\Upsilon \mu_{\epsilon,t}^*} \tag{6.30}$$

3. Definition of GDP:

$$y_t^G = c_t + \frac{i_t}{\mu_{\Upsilon,t}} + g_t \tag{6.31}$$

The 22 endogenous variables to be determined are:

$$c_t, i_t, y_t, y_t^G, l_t, \overline{k}_{t+1}, u_t, \lambda_{z,t}, q_t, r_t^k, R_t^k, m_t, R_t, p_t^*, \pi_t, F_{p,t}, K_{p,t}, w_t^*, F_{w,t}, K_{w,t}, \tilde{w}_t, \pi_{w,t}.$$

We have listed 22 equations above, from (6.10) to (6.31).

6.3   Solution method

Here we describe the solution method for the estimated model presented in Section 5. The logic follows the general formulation in section 4.4, where we show how we solve essentially linear economies with ambiguity aversion. The solution involves the following procedure. First, we solve the model as a rational expectations model in which the worst case scenario expectations are correct on average. The equations describing the equilibrium conditions under these expectations were presented in the Appendix 6.2. Second, we take the equilibrium decision rules formed under ambiguity and then characterize the dynamics under the econometrician's law of motion for productivity described in equation (5.2).

Let w_t denote the endogenous variables and s_t the exogenous variables. For notational purposes, split the vector s_t into the technology shock $z_t := \log Z_t$, the ambiguity variable a_t and the rest of the exogenous variables, $\widetilde{s}_t$, expressed in logs, of size n. In the case of our

estimated model $n = 6$. Under the worst case belief

$$z_{t+1} = \rho_z z_t + z_{t+1}^x - a_t \tag{6.32}$$

We can summarize our procedure for finding the equilibrium dynamics in the following steps:

1. Find the deterministic 'worst case steady state'. Here we take the steady state values of exogenous variables $\bar{s}^0$. This vector includes setting $a_t = \bar{a}$, $\tilde{s}_t = \tilde{s}_{n\times1}^0$ and finding the steady state technology level of the process in (6.32). The latter is

$$\bar{z}^o = -\frac{\bar{a}}{1 - \rho_z}$$

Using $\bar{s}^0 := (\tilde{s}_{n\times1}^0, \bar{z}^o, \bar{a})$ one can compute the 'worst case steady state' of the endogenous variables. This can be done by analytically solving the equilibrium conditions presented in section 6.2 evaluated at a deterministic steady state $\bar{s}^0$. Denote these steady state values of these endogenous variables as a vector $\bar{w}_{m\times1}^0$.

2. Linearize the model around the 'worst case steady state'. Denote the deviation from the worst case steady state by $\hat{w}_t^0 := w_t - \bar{w}^0$ and $\hat{s}_t^0 := s_t - \bar{s}^0$. Posit a linear equilibrium law of motion:

$$\hat{w}_t^0 = A\hat{w}_{t-1}^0 + B\hat{s}_t^0 \tag{6.33}$$

and specify the linear evolution of the exogenous variables:

$$\hat{s}_t^0 := \begin{bmatrix} \widehat{\tilde{s}}_t \\ \widehat{z}_t \\ \widehat{a}_t \end{bmatrix} = P \begin{bmatrix} \widehat{\tilde{s}}_{t-1} \\ \widehat{z}_{t-1} \\ \widehat{a}_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ z_t^x \\ a_t^x \end{bmatrix}$$

where $\Xi_t := \begin{bmatrix} \varepsilon_t & z_t^x & a_t^x \end{bmatrix}'$ denotes the innovations to the exogenous variables s_t with $\Xi_t \sim N(0, \Sigma)$. Importantly for us,

$$E^{p^0} \hat{s}_{t+1}^0 = P\hat{s}_t^0 + E^{p^0}\Xi_{t+1}, \tag{6.34}$$

where $E^{p^0}\Xi_{t+1} = 0$. To reflect the time t worst case belief about $\widehat{z}_{t+1}$, recall (6.32) so the matrix P satisfies the restriction:

$$P = \begin{bmatrix} \rho_{n\times n} & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix} \tag{6.35}$$

where ρ is a matrix reflecting the autocorrelation structure of the elements in $\widetilde{s}_t$.

To solve for matrices A and B, we can use any standard solution techniques of forward looking rational expectations model. In particular here we follow the method of undetermined coefficients of Christiano (2002). Let the linearized equilibrium conditions, presented in nonlinear form in Appendix 6.2, be restated in general as:

$$E^{p^0}\left[\alpha_{-1}\hat{w}^0_{t-1} + \alpha_0\hat{w}^0_t + \alpha_1\hat{w}^0_{t+1} + \delta_0\hat{s}^0_t + \delta_1\hat{s}^0_{t+1}|s_t\right] = 0 \tag{6.36}$$

where $\alpha_{-1}, \alpha_0, \alpha_1, \delta_0, \delta_1$ are constants determined by the equilibrium conditions. Substitute the posited policy rule into the linearized equilibrium conditions to get:

$$0 = \left(\alpha_{-1}A^2 + \alpha_0 A + \alpha_1\right)w_{t-1} + \left(\alpha_{-1}AB + \alpha_{-1}BP + \alpha_0 B + \delta_0 P + \delta_1\right)s_t +$$
$$+ \left(\alpha_{-1}B + \delta_0\right)\widetilde{E}_t\Xi_{t+1}$$

Thus, as in Christiano (2002), A is the matrix eigenvalue of matrix polynomial:

$$\alpha(A) = \alpha_{-1}A^2 + \alpha_0 A + \alpha_1 = 0 \tag{6.37}$$

and B satisfies the system of linear equations:

$$F = (\delta_0 + \alpha_{-1}B)P + \left[\delta_1 + (\alpha_{-1}A + \alpha_0)B\right] = 0 \tag{6.38}$$

3. Consider now the dynamics of the model from the perspective of the econometrician. Agents' response to ambiguity leads to actions and hence equilibrium outcomes given by (6.33), where A and B are determined by (6.37) and (6.38). At the same time, the exogenous state z_t moves according to the equation

$$z_t = \rho_z z_{t-1} + z^x_t \tag{6.39}$$

so the steady state of z equals $z^* = 0$. Thus, we have to correct for the fact that, from the perspective of the agent's worst case beliefs at $t-1$, the average innovation of the technology shock at time t is not equal to 0. Comparing (6.39) and (6.32), the average innovation is then equal to a_{t-1} :

$$z_t = E^{p^0}z_t + z^x_t + a_{t-1}. \tag{6.40}$$

3.a) Find the 'zero risk steady state'. Take the steady state of the exogenous variables under the econometrician's belief $\bar{s}^* := (\widetilde{s}^0_{n\times 1}, z^*, \bar{a})$ which differs from $\bar{s}^0$ only in the element corresponding to the steady state technology level z^*. Then, the zero risk steady state is the

fixed point $\bar{w}$ that solves

$$\bar{w} - \bar{w}^0 = A\left(\bar{w} - \bar{w}^0\right) + B\left(\bar{s}^* - \bar{s}^0\right) \tag{6.41}$$

where the difference $\bar{s}^* - \bar{s}^0 = \begin{bmatrix} 0_{n \times 1} & z^* - \bar{z}^o & 0 \end{bmatrix}'$. Thus, $\bar{w}$ can be analytically found as:

$$\bar{w} = \bar{w}^0 + B(\bar{s}^* - \bar{s}^0)\left(I - A\right)^{-1}$$

3.b) Dynamics around the 'zero risk steady state'.

Denote by $\hat{w}_t := w_t - \bar{w}$ and $\hat{s}_t := s_t - \bar{s}^*$ the deviations from the zero risk steady state. Combining (6.33) and (6.41), those deviations follow the law of motion

$$\begin{aligned}
\hat{w}_t &= \hat{w}_t^0 + \bar{w}^0 - \bar{w} \\
&= A\left(\hat{w}_{t-1}^0 + \bar{w}^0 - \bar{w}\right) + B\left(\hat{s}_t^0 + \bar{s}^0 - \bar{s}^*\right) \\
&= A\hat{w}_{t-1} + B\hat{s}_t \tag{6.42}
\end{aligned}$$

We want to characterize the equilibrium dynamics under the econometrician's belief in which the worst case belief a_{t-1} is not materialized in the z_t realization. That means that the law of motion for the exogenous states under the econometrician belief is

$$\hat{s}_t = \begin{bmatrix} \rho_{n \times n} & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_a \end{bmatrix} \hat{s}_{t-1} + \Xi_t. \tag{6.43}$$

As presented in equation (6.40), we can then describe this evolution by adding an average innovation of a_{t-1} to the expected value of z_t formed under the time $t - 1$ worst case belief. The latter expectation is formed using the matrix P as defined in (6.35), so we can write (6.43) as:

$$\hat{s}_t = P\hat{s}_{t-1} + \begin{bmatrix} 0_{n \times n} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{s}_{t-1} + \Xi_t \tag{6.44}$$

6.4   Data sources

The data used to construct the observables are:

1. Real Gross Domestic Product, BEA, NIPA table 1.1.6, line 1, billions of USD, in 2005 chained dollars.

2. Gross Domestic Product, BEA, NIPA table 1.1.5, line 1, billions of USD, seasonally adjusted at annual rates.

3. Personal consumption expenditures on nondurable goods, BEA, NIPA table 1.1.5, line 5, billions of USD, seasonally adjusted at annual rates

4. Personal consumption expenditures on services, BEA, NIPA table 1.1.5, line 6, billions of USD, seasonally adjusted at annual rates

5. Gross private domestic investment, fixed investment, nonresidential and residential, BEA, NIPA table 1.1.5, line 8, billions of USD, seasonally adjusted at annual rates.

6. Personal consumption expenditures on durable goods, BEA, NIPA table 1.1.5, line 4, billions of USD, seasonally adjusted at annual rates.

7. Nonfarm business hours worked, BLS PRS85006033, seasonally adjusted at annual rates, index 1992=100.

8. Civilian noninstitutional population over 16, BLS LNU00000000Q.

9. Effective Federal Funds Rate. Source: Board of Governors of the Federal Reserve System.

We then perform the following transformations of the above data to get the observables:

10. GDP deflator: (2) / (1)

11. Real per capita GDP: (1) / (8)

12. Real per capita consumption: [(3)+(4)] / [(8)*(10)]

13. Real per capita investment: [(5)+(6)] / [(8)*(10)]

14. Per capital hours: (7) / (8)

15. Relative price of investment: We use the price index for consumption expenditures on durable goods (BEA, NIPA table 1.1.4, line 4) and price index for fixed investment (BEA, NIPA table 1.1.4, line 8). We follow the methodology proposed in Fisher (2006). An appendix detailing the procedure used in the construction of this series is available from the authors upon request.