Price Discovery

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Abstract

Within a novel model of correlated private values/costs (CPV) we investigate how well a particular market mechanism, the *buyer's bid double auction* (BBDA), performs in small and moderate sized markets: do well behaved equilibria exist and is there rapid convergence to efficient allocations and prices that accurately identify the market's underlying state. Using a combination of theorems and numerical experiments we establish that for two plausible stochastic specifications of costs/values simple equilibria do exist even in small markets. As a function of market size we bound traders' strategic behavior and prove rates of convergence to zero of (i) the allocation's inefficiency and (ii) the error in estimating the market's underlying state. These rates imply that even in moderate sized markets strategic behavior has an inconsequential effect on the estimate of the market state. We also show that the affiliation condition on traders' values and costs familiar from auction theory does not imply affiliation of bids and offers and therefore does not play the same pivotal role as it plays in auction theory in proving existence. Nevertheless, in the cases we consider, a a solution to the first order condition defines an equilibrium despite the failure of affiliation among bids and offers. We then extend our model to the case of correlated interdependent values (CIV). While so far we have been unable to obtain the same sequence of results as we have secured in the CPV case, we do show that the structure of equilibria in the CIV case is unchanged. As a consequence we can straightforwardly compute a substantial sample of equilibria for different market sizes and show that the BBDA's convergence properties do not change qualitatively with the introduction of interdependence.

1 Introduction

A market may have both allocative and informational purposes. This is particularly true in security and commodity markets that are the major centralized markets that collectively play an enormous

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role in modern economies. The allocative purpose is to redistribute goods among the traders so as to produce gains from trade. The informational purpose is to aggregate cost and benefit information of active traders into a price that other agents may then use to make decisions that depend on that price. In small markets traders have incentives to influence the market price in their favor. This behavior can cause misallocated goods and market prices that are biased relative to the underlying competitive price. In extreme cases the market may completely breakdown with no trade at all occurring. But in the simple double auction market that we study strategic incentives attenuate and improved allocational and informational performance result as the number of traders in the market becomes larger. In fact, as the number of traders becomes large, strategic incentives vanish, the allocation approaches the efficient allocation, and price reflects fundamentals accurately, i.e., the equilibrium converges to the fully revealing rational expectations equilibrium.

Our objective in this paper is to investigate the ability of a particular double auction mechanism to accomplish the allocational and informational purposes of centralized markets. Analysis of such markets is challenging: they may be small relative to the competitive ideal, buyers and sellers have private information concerning their values and costs, and traders' idiosyncratic values/costs may be correlated and, perhaps, interdependent. Interdependence means that if a trader's private value/cost were to become public, then that knowledge would enable every other trader to sharpen his estimate of his own cost/value. These characteristics lead us to study double auctions instead of standard auctions for two main reasons. First, double auctions are more faithful to the fundamental informational feature of centralized markets in the sense that they model private information on both the buyer and the seller sides. Second, the results from standard, one-sided auctions do not and can not carry over to markets with two-sided private information and the asymmetric strategies that buyers and sellers choose. Theorems from auction theory may provide useful analogies, but different methods are necessary in order to establish a double auction's properties and performance.

Four questions guide our inquiry. First, do well behaved equilibria exist and, if so, what are their structure in markets with small numbers of traders? Second, are the asymptotic results that establish rapid rates at which strategic behavior disappears and efficiency is approached indicative of the double auction's performance in small markets? Third, at what rate is informational accuracy achieved in double auctions? Fourth, does the introduction of interdependence change the performance of double auctions qualitatively, especially in small markets?

The answers we obtain for these questions are positive: in small markets equilibria exist, the rates in small markets at which their equilibria converge to efficiency and full information revelation approximate the asymptotic rates, and the presence of interdependence causes no significant change in performance. The last point is particularly significant because it implies that in equilibrium the realized market price of a small double auction market approximates the fully revealing rational expectations price.

In the formal language of auction theory, our initial focus is a correlated private values (CPV) model in which homogeneous, indivisible items are traded. After we gain a good understanding of this model we then generalize it to the correlated interdependent values (CIV) model and show that interdependence has little qualitative effect. Throughout there are m potential buyers, each of whom wishes to buy at most one item, and n potential sellers, each of whom has a single item to sell. Each trader's utility is quasilinear in his value/cost and money. The allocation problem of the market is to reassign the n items to the n traders who value them most highly. The informational problem is to estimate the underlying state of the market from the realized market price.

We address these problems with the *buyer's bid double auction* or *BBDA*.[1] After simultaneously collecting bids from buyers and asks from sellers, the BBDA sorts the bids/asks from lowest to highest and selects as the realized market price the upper endpoint of the interval of possible market-clearing prices. Thus, the market price in the BBDA is the $(m+1)^{\text{st}}$ order statistic of the $m+n$ bids/asks. Trade then occurs at this price.

This price setting rule implies that if a seller's ask is successful and results in him selling his unit of the good, then his ask is not the bid/ask that actually sets the market price. This is because a seller who trades must have submitted an ask a_j that is among the m smallest of $m+n$ bids/asks submitted; it cannot be the $(m+1)^{\text{st}}$ smallest. Therefore a seller has no strategic incentive to misrepresent his cost in order to affect his terms of trade. This simplifies the analysis.[2] By contrast, if a buyer submits a successful bid b_i, then not only will he receive a unit of the good, but b_i may be the $(m+1)^{\text{st}}$ smallest bid/ask and therefore determine price. This possibility gives buyer i an incentive to set b_i less than his value v_i in order to increase his expected gains from trade. The focus of our analysis is therefore the bidding behavior of buyers. Given honest reporting by each seller we seek for buyers an increasing strategy $B : \mathbb{R} \to \mathbb{R}$ that constitutes a Bayesian Nash equilibrium.

We introduce a novel process by which the values and costs of the traders are generated that contrasts with the standard specification in which values/costs are drawn from a compact segment of the real line. In our model the market's state $\mu \in \mathbb{R}$ is drawn from the diffuse, uniform prior on the real line. It is an improper probability distribution that may informally be thought of as the "uniform distribution over the entire real line." For each trader i, an idiosyncratic valuation term ε_i is independently drawn from a mean zero, symmetric distribution F. Trader i's value/cost is then $\mu + \varepsilon_i$. The underlying, highly uncertain common component μ creates correlation among traders' values/costs. If each trader privately observes his value/cost $v_i = \mu + \varepsilon_i$, then the environment is *correlated private values* (CPV). If instead each trader observes a noisy signal of his value/cost,

[1] The name originated with the bilateral double auction where "buyer's bid" refers to the fact that the buyer's bid is the price when trade occurs.

[2] That sellers have the dominant strategy of honestly reporting their costs does not mean, as Jackson and Swinkels (2005, footnote 6) suggest, that the BBDA can be analyzed "as a one-sided auction with a hidden reserve price schedule..." It is also the case that honest reporting as a seller's strategy is asymmetric relative to the equilibrium strategies of buyers. As discussed below, this asymmetry of behavior particularly complicates the CPV and the CIV cases and distinguishes the analysis of double auctions from auctions.

$\sigma_i = v_i + \delta_i = \mu + \varepsilon_i + \delta_i$, where δ_i is independently drawn from a mean zero, symmetric distribution G, then the environment is *correlated interdependent values*(CIV). Interdependence exists in the sense that trader i's signal carries information concerning the state μ and therefore can help every other trader j refine his estimate of his own value/cost.

Substantively the diffuse prior maximizes the traders' ex ante ignorance about the expected location of values/costs on the real line and the likelihood of different market prices. Thus, given any $\zeta \in \mathbb{R}$, the buyer ex ante cannot make any statement concerning the probability he will draw $v_i < \zeta$ or the probability he will draw $v_i > \zeta$; he is just completely uncertain. This maximally challenges the market mechanism's ability to discover the market clearing price, estimate the underlying state μ, and efficiently reallocate the supplied units that sellers bring to market. At the other extreme is a double auction market in an independent private values (IPV) environment in which each trader ex ante knows the state of the market μ with certainty and therefore knows the expected price.[3] Consequently in IPV markets, except when they are very small, a trader at the interim stage whose value/cost is not close to the expected price may predict with a high degree of certainty whether or not he will trade. This contrasts with our specifications of the CPV environment in which each buyer, no matter what v_i he draws, places a 0.5 probability that state μ exceeds his realized v_i and a 0.5 probability that μ is less than his realized v_i.

Methodologically the diffuse prior plays the *key* role in our analysis. The logic is this. In a Bayesian equilibrium both the structure of the model and other traders' equilibrium strategies are common knowledge among all traders as they receive their private signals of value (either v_i or c_j in the CPV environment or σ_i in the CIV environment) and decide what bid/ask to submit. The only information that is unique to each is his signal of value/cost. Each one's bid/ask is conditioned only on it. But every draw is the same in the sense that it gives identical information as to where other traders' values/costs are likely to be relative to his own draw. Specifically, if buyer i draws $v_i'' \in \mathbb{R}$ instead of $v_i' \in \mathbb{R}$, then his beliefs about another buyer j's value relative to his own do not change:

$$\begin{aligned}\Pr[v_j \in [v_i', v_i' + \zeta] | v_i'] &= \Pr[v_j \in [v_i'', v_i'' + \zeta] | v_i''], \\ \Pr[v_j \in [v_i' - \zeta, v_i'] | v_i'] &= \Pr[v_j \in [v_i'' - \zeta, v_i''] | v_i''].\end{aligned} \tag{1}$$

for any $\zeta \in \mathbb{R}^+$. Similar equations hold for his beliefs concerning any seller's cost. Thus in this sense the fundamentals of trader i's decision problem are invariant with respect to his signal.

Consider buyer i's equilibrium strategy $B_i : \mathbb{R} \to \mathbb{R}$. For any given $v_i \in \mathbb{R}$, buyer i underbids in order to influence price in his favor: the offset $\lambda_i(v_i) = v_i - B_i(v_i)$ is necessarily positive. Conceivably this offset may vary as v_i varies. This, even if such an equilibrium exists, seems implausible behaviorally because it requires all buyers' offsets to vary with their value even though

[3]In a IPV environment each trader's value/cost is independently drawn from a given distribution.

beliefs about each other's values/costs remain invariant in the manner equations (1) describe. It seems more reasonable that each buyer's offset remains fixed as his value varies. Further, if all buyers other than buyer i play strategies with fixed offsets, then buyer i's best response is also to play a fixed offset. Fixed offsets therefore extend the invariance in values/costs to bids/asks and reinforce the reasonableness of such strategies.

An *offset equilibrium* within our model has the analytical virtue that solving for buyer i's first order condition at value $v_i = 0$ for the offset λ_i at which his marginal utility equals zero gives, for every other $v_i \in \mathbb{R}$, the offset at which his marginal utility is zero. Moreover, checking that this solution to the first order condition is a global maximum and therefore an equilibrium is also simple: for $v_i = 0$, graph the buyer's utility and check that it is indeed a global maximum. If this is so for $v_i = 0$, then λ_i is also a global maximum for any $v_i \in \mathbb{R}$.

Exploiting this special but reasonable structure permits us to obtain the following sequence of results for the BBDA in the CPV environment.

Existence. We prove that for a broad class of distributions F a symmetric offset strategy B exists that is a solution to the first order conditions. We are unable to characterize analytically a condition on F that guarantees B is an equilibrium and demonstrate the source of the difficulty. A set of systematic numerical experiments for $m, n \in \{2, 3, \ldots, 20\}$, however, shows that if F is either the normal or Laplace distribution, then the offset solution is uniquely determined and an equilibrium.[4] Affiliation of bidders' values is commonly assumed in the analysis of auctions to insure the sufficiency of the first order approach. We prove in the case in which F is the normal distribution that affiliation does not play the same role in the theory of double auctions: though the values/costs of traders are affiliated in this case, each trader's cost/value is not affiliated with the order statistics of the other traders' bids/asks because of the asymmetry of behavior across the two sides of the market. While affiliation is an extremely useful sufficient condition in auction theory, it is neither useful nor necessary for existence of pure strategy equilibria in double auctions for small markets.

Non-existence and its resolution. Fully exploring the universe of distributions F that might describe the behavior of traders' idiosyncratic value/cost component ε_i and checking for each one whether or not equilibrium exists is beyond this paper's scope. Nevertheless we have determined what happens if F is the standard Cauchy. The reason we chose this distribution to check the robustness of our existence results is the same reason that statisticians commonly use it as a test case: the Cauchy distribution's density resembles the standard normal's density, but also has "fat tails" that can disrupt asymptotic sampling results. We show two results. First, for the bilateral

[4]These results complement the existence results of, first, Jackson and Swinkels (2005) showing that a possibly mixed equilibrium exists for double auctions in the CPV environment and, second, Fudenberg, Mobius, and Szeidl (2007) showing existence of increasing pure strategy equilibria in large markets.

case, in which the single buyer in effect makes a take-it-or-leave-it ask to the single seller, no equilibrium exists. The culprit is the non-existence of the Cauchy's first moment: the fat downward tail of the distribution implies that the marginal benefit to the buyer of lowering his bid and thereby driving down the price he pays when he buys always outweighs the marginal cost of decreasing the probability that he trades. But, second, in the multilateral case — two or more buyers and two or more sellers — a perfectly well behaved equilibrium exists for the BBDA.

This is a twist on the analysis of bilateral trade of Myerson and Satterthwaite (1983) who prove in an IPV model with compact support that trade is necessarily ex post inefficient in any equilibrium. Trade in contrast is indeterminant in our example because there are no equilibria. In both their model and in our model, however, increasing the number of traders resolves the failings of bargaining. We show numerically that an offset equilibrium exists in the case of the standard Cauchy distribution for a number of values of $m = n$ that are strictly greater than one. This sequence of equilibria exhibits the convergence rate that is proven for "regular" distributions in this paper. While the outcome of bilateral trade is indeterminant in this example, the market outcome in the BBDA appears to be uniquely determined and asserts itself immediately as bargaining transitions into a market. This example illustrates the robustness of the double auction and its ability to protect each bidder from a bid that is interim optimal and ex post catastrophic. Underlying this good behavior is a technical fact: even with fat tails interior order statistics are typically well behaved despite first and higher moments not existing.

Allocative efficiency. Let m, $n > 1$ be fixed integers and let $\tau \in \mathbb{N}^*$ index a sequence of markets with τm buyers and τn sellers. The equilibrium underbidding $v - B(v)$ by each buyer is $O(1/\tau)$ for all v and τ and in expectation results in too few units being traded.[5] As a consequence, the expected gains from trade that inefficiently fail to be achieved in the state μ as a fraction of the potential gains from trade in that state is $O(1/\tau^2)$. This quadratic rate agrees with rates that have previously been established for sufficiently large markets when values/costs, possibly correlated, are drawn from a compact support.[6]

Informational efficiency. Let $q = m/(m + n)$ and let ξ_q denote the q^{th} population quantile of the distribution F.[7] Let p^{BBDA} denote the price in the BBDA when buyers bid strategically and let

[5] For functions $f(t)$, $g(t) : \mathbb{N} \to \mathbb{R}_+$, $f(t)$ is $O(g(t))$ if there exists a constant $k \in \mathbb{R}$ such that

$$f(t) \leq kg(t) \textit{ for all } t \in \mathbb{N}^*,$$

and $f(t)$ is $\Theta(g(t))$ if there exists constants $k^{'}$, $k^{''}$ such that

$$k^{'} g(t) \leq f(t) \leq k^{''} g(t) \textit{ for all } t \in \mathbb{N}^*.$$

[6] See Satterthwaite and Williams (1989), Williams (1991), and Rustichini, Satterthwaite, and Williams (1994) for the IPV case and Cripps and Swinkels (2006) and Fudenberg, Mobius, and Szeidl (2007) for the CPV case.

[7] For $q \in (0, 1)$ the population quantile ξ_q of F is the unique solution to the equation $F(\xi_q) = q$.

p denote the price that would be determined in the BBDA if buyers (contrary to their self-interest) instead submitted their true values as their bids. Recalling that F is symmetric with zero mean, if all buyers honestly reported their values, then as the market becomes large p would converge to $\mu + \xi_q$. Given prior knowledge of F and therefore ξ_q, p can be used to estimate μ. The error $\left|p^{BBDA} - (\mu + \xi_q)\right|$ is bounded above by the sum

$$|p - (\mu + \xi_q)| + \left|p^{BBDA} - p\right|.$$

The first of these terms is the *sampling error*, which represents the error in the estimation that is inherent from the finiteness of the market. The second term is the *strategic error,* which is the error caused by buyer underbidding. The bound on buyer underbidding implies that the expected strategic error is $O(1/\tau)$; results from statistics imply that the expected sampling error is $\Theta(1/\sqrt{\tau})$. Thus the strategic error is of strictly higher order than the sampling error and therefore for larger markets has an insignificant impact upon the estimation of $\mu + \xi_q$ and μ.[8]

Asymptotic approximation of equilibrium strategies. As a complement to our analysis of equilibrium, we analyze a buyer's decision problem from an asymptotic perspective by determining the asymptotic limits to the probabilities in his first order condition for the selection of his bid. For fixed m and n, the solution to this *asymptotic first order condition* is a uniquely determined offset strategy. Using the asymptotic approach for the case in which F is the standard normal distribution we derive an approximate formula for a buyer's equilibrium offset as a function of m and n. This provides an approximation to equilibrium bidding behavior in all sizes of markets. As expected, it exhibits underbidding that is $O(1/\tau)$ in magnitude. Numerical investigation suggests that this approximation is quite good even in small markets. The formula is significant because there are very few examples of equilibria in the double auction literature.[9,10]

Correlated interdependent values. Ideally we would exactly replicate this sequence of results for the CPV environment for the CIV environment. This, however, has proved impossible for us, at least in this paper. But the diffuse prior in the CIV case does give the same invariance to

[8] We are unaware of other results in the double auction literature showing the rate at which informational efficiency is achieved. Vives (2011) in his recent study of supply function equilibria among firms that have private information concerning their marginal costs obtains analogous results.

[9] Three examples are as follows. The first example is the linear equilibrium to the split-difference bilateral double auction Chatterjee and Samuelson (1983) derived when the buyer's value and the seller's cost are independent and uniformly distributed on $[0, 1]$. The second is the linear equilibria for the BBDA that Satterthwaite and Williams (1989) and Williams (1991) constructed for arbitrary numbers of buyers and sellers in the independent private value environment. The third is the equilibria Kadan (2007) constructed for the bilateral case in a CPV environment.

[10] Examples of equilibrium obtained either through numerical calculation or theoretical derivation are important both for the insight that they give and because they provide a basis for experimentally testing the theory of double auctions. Experimental tests of the multilateral k-double auction include Kagel and Vogt (1993) and Cason and Friedman (1997).

each trader's decision problem that its presence gives to each buyer's decision problem in the CPV case. This invariance makes it easy to compute equilibria and do numerical experiments when F and G — the distributions respectively of each trader's idiosyncratic value/cost component ε_i and his noise component δ_i — are the normal distribution. These experiments confirm that in the CIV environment offset equilibria exist in small markets and converge towards allocative and informational efficiency at the same rates as is the case for equilibria in the CPV environment. In other words, this paper provides new theoretical evidence that fully rational strategic behavior in a market with two-sided private information and CIV does converge to a fully revealing rational expectations equilibrium (REE) as the number of traders increases from a small number to a large number.

Related work. This paper is therefore a step in developing an explicit theory as to how rational, self-seeking, non-cooperative buyers and sellers trading within a market institution can, as the number of traders becomes larger, lead to increasingly efficient allocations at a price that increasing accurately reveals the market's underlying fundamentals. Game theoretic approaches to this began with Vickrey (1961), Wilson (1977), Milgrom (1979), and Milgrom (1981) with their initial work on auctions. The two-sided nature of the problem was first recognized in the papers of Wilson (1985) and Gresik and Satterthwaite (1989) that began the theoretical study of multilateral double auctions. Within the IPV environment Satterthwaite and Williams (1989), Williams (1991), and Rustichini, Satterthwaite, and Williams (1994) established the quadratic rate of convergence to efficiency of the BBDA and the k-double auction. Relying on Jackson and Swinkels (2005) for existence, Cripps and Swinkels (2006) in a paper that is notable for allowing buyers multiple units of demand, sellers multiple units of supply, and a very general definition for the CPV environment study the efficiency of large double auctions and show that efficiency is approached at the same quadratic rate as it is in the IPV environment.

Reny and Perry (2006) in an ambitious paper investigate the existence and efficiency of double auction equilibria when traders' values/costs may be interdependent.[11] Under some technical conditions they prove that if the number of traders is sufficiently large, then an equilibrium in which each trader's strategy is increasing with respect to his private signal exists, the resulting allocation is efficient, and the realized market price p identifies the state of the market μ. A fully revealing REE exists if the market is large enough. This is a terrific result because, as they observe (p.1233), interdependence of values/costs "is the hallmark of environments in which rational expectations are required."

Reny and Perry's theorem, as with any theorem, has its limitations: it does not apply to small markets, it does not establish a rate at which equilibria converge to efficiency, and does not provide

[11] Their value/signal formulation as well as affiliation/interdependence structure is more general than is the case in our model.

insight as to what equilibrium strategies look like. Our results for CIV environments speak to these limitations by showing existence for small markets, demonstrating that convergence to efficiency is quadratic as is the case in both IPV and CPV environments, and showing that equilibrium strategies may take the simple form of a constant offset. In addition, if our model is modified while still maintaining the diffuse prior assumption, our techniques continue to apply and facilitate computing solutions to the first order conditions, checking that these solutions are equilibria or (equally interesting) are not equilibria, and determining their properties if it is an equilibrium.[12] This, not surprisingly, has a cost in terms of generality: our model is quite specific with its additive structure for values/costs and signals as well as its assumption that agents' prior on the underlying market state μ is diffuse. We thus see our paper as a complement and not as a substitute to Reny and Perry (2006), albeit in a less general informational framework.

Finally, it is well-known in the mechanism design literature that statistical dependence among privately observed types can be used to achieve higher levels of performance than in the case of independent types.[13] In Satterthwaite, Williams, and Zachariadis (2011), we generalize the McAfee and Reny (1992) mechanism for bilateral trade when the type space is compact to the multilateral, diffuse prior model considered here. Honest reporting is a Bayesian-Nash equilibrium in the McAfee-Reny mechanism and its equilibrium allocation is efficient. The truthful reporting by all traders means that strategic behavior does not cause any error in the aggregation of information by the market. This raises the following question. Because the adverse consequences of strategic behavior can be completely avoided with this designed mechanism, what is the purpose of studying an imperfect mechanism such as the BBDA? One reason is that the transfers of the designed mechanism are defined in terms of the distribution F, which renders it impractical.[14] The BBDA, in contrast, is defined without reference to the traders' probabilistic beliefs. In our paper we explore another reason: it may impose a sizeable ex post loss on a trader and it may require a substantial ex post monetary subsidy to operate. Our conclusion is that the designed approach

[12]There is an additional, technical reason why this paper complements Reny and Perry. One of their contributions is an example (see p. 1246-1248) illustrating how in CPV and CIV environments a trader's best response strategy may not be monotone increasing even if other buyers and sellers are playing increasing strategies. This possibility made it difficult for them to employ the fixed point theorem of Athey (2001) in proving existence. Their workaround is to show that if the market is sufficiently large, then every trader becomes almost a price taker and, as a consequence, has a monotone best response to any set of monotone strategies. But overcoming this difficulty and showing existence by making the market so large that traders' become essentially price-takers does not give insight into whether their example that best replies to monotone strategies are not necessarily monotone in small markets — where buyers' are definitely not price-takers —is the exception or the rule. Our results show that for small markets the set of specific CPV and CIV environments in which best replies to monotone strategies are monotone is not empty and includes some plausible examples.

[13]This observation originates in Myerson (1981), who constructed an example in which dependence among the private values of bidders allow the seller of an item to design an auction that extracts all of the possible revenue. Myerson (1981)'s example was developed into a general result on auctions by Crémer and McLean (1988). It was further developed as a general result in Bayesian mechanism design by McAfee and Reny (1992).

[14]This is the Wilson Critique of mechanism design, namely, that probabilistic beliefs are a way to model decision-making under uncertainty and are not a datum available for the practical definition of an economic mechanism. See Wilson (1987).

is a complicated and impractical exercise that achieves modest gains in comparison to a simple mechanism such as the BBDA.

Outline of the paper. This paper is organized as follows. The model is formally presented in Section 2. Section 3 presents the first order condition for a strategy $B : \mathbb{R} \to \mathbb{R}$ to define a Bayesian-Nash equilibrium, assuming that all sellers report honestly and all buyers use B. This first order condition is the basis for the convergence results and the computation of equilibrium in this paper. Section 4 derives from this first order condition (i) an invariance property of the first order condition, which reflects the invariance of a buyer's decision problem, and (ii) the existence of an offset strategy that solves the first order condition. Sections 5 presents theoretical results concerning the sufficiency of this first order condition. Section 6 computes the rates of convergence of buyer misrepresentation and the inefficiency that it causes to zero. In Section 7 we supplement the theoretical results on sufficiency with numerical results, and also illustrate numerically several properties of offset equilibria that were proven in previous sections. Section 8 contains our analysis of the "fat-tailed" case of the standard Cauchy distribution. Section 9 applies our bound on buyer misrepresentation to characterize the magnitude of strategic error in estimating a market parameter in comparison to sampling error that is inherent in the finiteness of a market. Section 10 analyzes the buyers' first order condition from an asymptotic perspective and produces a formula for buyer underbidding in arbitrary sizes of markets in the case of the standard normal distribution. Section 11 applies our analysis of the CPV case to the CIV case. Section 12 concludes. Many proofs are in the Appendices.

2 Model

The market consists of m potential buyers who seek to buy one unit of the good and n potential sellers each of whom is endowed with one unit of the good. The process that generates each buyer's and seller's valuation for the good is this. First, $\mu \in \mathbb{R}$ a common component to every buyer's value and each seller's cost is drawn from a diffuse, improper prior distribution that is uniform over the entire real line. Second, for each buyer $i \in \{1, 2, ..., m\}$ and for each seller $j \in \{1, 2, ..., n\}$ idiosyncratic terms ε_i and ε_j are drawn i.i.d. from distribution F on $\mathbb{R}$. These idiosyncratic components perturb the common component μ and create values $v_i = \mu + \varepsilon_i$ for buyers and costs $c_j = \mu + \varepsilon_j$ for sellers. Buyers and sellers are risk neutral. If a buyer i trades at price p then his utility is $v_i - p$, and if seller j trades at price p his utility is $p - c_j$. Buyers and sellers who do not trade receive zero utility.

Each trader privately learns his value v_i or cost c_j through a noiseless signal and computes from his signal his posterior distribution of μ: for buyers $F_\mu(\mu|v_i) = F(\mu - v_i)$ and analogously for sellers. His posterior on μ then enables him to compute his joint posterior of the other traders' values and costs. The purpose of this structure using a diffuse prior is to create maximal uncertainty about the market price and to reflect gains from trade that exist in markets due to idiosyncratic preferences as a result of, for example, differing portfolio positions of individual buyers and sellers.

The mechanism we use to realize these gains from trade is the buyer's bid double auction. In it buyers and sellers simultaneously announce their bids and asks that are then sorted in increasing order: $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(m)} \leq s_{(m+1)} \leq \cdots \leq s_{(m+n)}$. The traders who submit the highest n bids/asks are assigned the n objects available in the market while the buyers/sellers who submit the lowest m bids and asks are not assigned an object. Those buyers who are assigned an object pay the market clearing price $p = s_{(m+1)}$ and those sellers who are not assigned an object receive that market clearing price.[15]

We restrict attention to symmetric strategies amongst all buyers and all sellers. Furthermore, we note that because the market price is set at $s_{(m+1)}$ a seller never affects the price at which he sells. Therefore, a dominant strategy for him is to report his cost truthfully; let $\tilde{S}$ be this truthful strategy for sellers. Buyers can affect the market price when they buy and therefore may have an incentive to underbid. Let B be the buyers' symmetric strategy that maps values into bids. A pair of strategies $<\tilde{S}, B>$ is an equilibrium if, for each buyer $i \in \{1, \ldots, m\}$ and every $v_i \in \mathbb{R}$, $B(v_i)$ maximizes his expected utility given that all sellers play $\tilde{S}$ and all other buyers play B.

We make the following assumptions on the strategy B and the distribution F of the idiosyncratic terms ε_i and ε_j.

A1: B is strictly increasing.

A2: $\liminf_{v \to -\infty} B'(v) < \infty$.

A3: F is absolutely continuous on $\mathbb{R}$ with density f, which is positive and continuous on $\mathbb{R}$.

A4: The density f is symmetric about zero, i.e., $f(x) = f(-x)$ for all $x \in \mathbb{R}$.

A5: F satisfies

$$\lim_{x \to -\infty} \sup \frac{F(x)}{f(x)} < \infty. \tag{2}$$

We call an equilibrium $< \widetilde{S}, B >$ that complies with A1 and A2 on B as *regular*.

Given A3 and our exclusive focus on increasing strategies, ties between two or more traders' bids/asks are measure zero events and may be ignored. A4 implies that the mean and the median of F are both equal to zero and $F(x) = \overline{F}(-x)$ for all $x \in \mathbb{R}$, where $\overline{F} = 1 - F$. Notice that A5 is weaker than the assumption that the hazard rate $f(x)/F(x)$ is strictly decreasing, which is a common assumption in mechanism design.

[15]If there are ties at $s_{(m+1)}$ then available supply is first assigned to buyers and then to sellers. If supply is insufficient to supply those buyers at $s_{(m+1)}$ then the available supply is distributed amongst them using a fair lottery. Finally, if any further supply remains unassigned, then it is distributed back to the sellers at $s_{(m+1)}$ with a fair lottery. For more details see Satterthwaite and Williams (1989, pages 480-1).

Remark 1 *The normal distribution with mean* 0 *satisfies the assumptions on* F *(A3-A5) with A5 holding by virtue of the fact that the normal satisfies the monotone hazard rate condition. The Laplace distribution with mean* 0 *also satisfies the assumptions on* F*, while the Cauchy distribution does not satisfy A5.*

3 The focal buyer's expected utility and first order condition

Pick a buyer, whom we refer to as the focal buyer, and fix the n sellers' strategy at $\tilde{S}$ and the $m-1$ remaining buyers' strategy at B. Excluding the focal buyer's bid b, order the bids and asks of the remaining buyers and sellers into a vector: $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(m)} \leq s_{(m+1)} \leq \cdots \leq s_{(m+n-1)}$. This is the random vector against which the focal buyer, as a function of his v, chooses his b so as to maximize his expected utility. To streamline the notation, let x denote $s_{(m)}$, the m^{th} order statistic of the bid/ask vector, and let y denote $s_{(m+1)}$, the $(m+1)^{\text{st}}$ order statistic of the bid/ask vector. Then we have the following cases:

1. If $b < x < y$, then the price is $p = x$, the buyer fails to trade, and his utility is 0.
2. If $x < y < b$, then the price is $p = y$, the buyer trades, and his utility is $v - y$,
3. If $x \leq b \leq y$, then the price is $p = b$, the buyer trades, and his utility is $v - b$.

Given the value v, the focal buyer chooses his bid b to maximize his expected utility. Observe that bidding $b' = v$ dominates bidding $b' > v$. Conditional on the focal buyer's value v and the other m buyers' bidding strategy B, let $f_x(x|v)$, $f_y(y|v)$, and $f_{xy}(x,y|v)$ be the marginal density of x, the marginal density of y, and the joint density of x and y, respectively. These densities depend on v since the buyer uses his value to form his posterior beliefs over the other traders' valuations. The focal buyer's expected utility is

$$\begin{aligned}\pi(v,b|B) &= v\Pr[y<b|v] - \mathbb{E}\left[(v-y)\chi_{y<b}|v\right] + (v-b)\Pr[x\leq b\leq y|v]\\ &= v\Pr[y<b|v] - \mathbb{E}\left[y\chi_{y<b}|v\right] + (v-b)\Pr[x\leq b\leq y|v],\end{aligned}$$

where χ is the indicator function and

$$\begin{aligned}\Pr[y<b|v] &= \int_{-\infty}^{b} f_y(y|v)dy,\\ \mathbb{E}\left[y\chi_{y<b}|v\right] &= \int_{-\infty}^{b} y f_y(y|v)dy,\\ \Pr[x\leq b\leq y|v] &= \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}(x,y|v)dydx.\end{aligned}$$

Taking derivatives with respect to b and setting the result equal to zero yields the first order condition for b given v:

$$\begin{aligned}\pi_b(v,b|B) =&(v-b)\left[f_y(b|v)+\int_b^\infty f_{xy}(b,y|v)dy-\int_{-\infty}^b f_{xy}(x,b|v)dx\right]\\&-\Pr[x\leq b\leq y|v]\\=&(v-b)f_x(b|v)-\Pr[x\leq b\leq y|v]\\=&0.\end{aligned}\tag{3}$$

where $\pi_b = \partial\pi/\partial b$. In the first line $\int_b^\infty f_{xy}(b,y|v)dy = f_x\,(b|v)$ because $f_{xy}(x,y|v) = 0$ for all $y < x$. For the same reason, $\int_{-\infty}^b f_{xy}(x,b|v)dx = f_y\,(b|v)$. The third line is the focal buyer's marginal utility. Its first term is the focal buyer's expected gain from increasing his bid by Δb: he jumps over x with probability $f_x(b|v)\Delta b$, goes from non-trading to trading, and earns $v-b$, which gives an expected gain of $(v-b)f_x(b|v)\Delta b$. The second term is the expected cost of increasing his bid by Δb: with probability $\Pr[x\leq b\leq y|v]$ his bid b sets the price; increasing b by Δb forces the market clearing price he pays up by Δb, which gives an expected loss of $\Pr[x\leq b\leq y|v]\Delta b$.

Observe that the conditional independence of values and costs with respect to the common component μ implies that

$$\begin{aligned}f_x(b|v) &= \int_{-\infty}^\infty f_x(b|\mu)f_\mu(\mu|v)d\mu,\\ \Pr[x\leq b\leq y|v] &= \int_{-\infty}^\infty \Pr[x\leq b\leq y|\mu]f_\mu(\mu|v)d\mu,\end{aligned}$$

where $f_\mu(\mu|v)$ is the conditional on the buyer's signal v density of μ, $f_x(b|\mu)$ is the conditional on μ density of the other traders' m^{th} order statistic computed at the focal buyer's bid b, and $\Pr[x\leq b\leq y|\mu]$ is the conditional on μ probability that the focal buyer sets the price with his bid. Consequently the formulas for $f_x(b|\mu)$ and $\Pr[x\leq b\leq y|\mu]$ are the same as in Satterthwaite and Williams (1989a) with its analysis of the independent private values case.

To spell out the formulas for $f_x(b|\mu)$ and $\Pr[x\leq b\leq y|\mu]$ three probabilities, $M_{n,m}$, $K_{n,m}$, and $L_{n,m}$, must be defined. These three probabilities are conditional on μ and assume that in the sample of interest buyers each play the strategy B (with inverse B^{-1}), and sellers each truthfully ask their costs. In particular $M_{n,m}\,(b|B,\mu)$ is the probability that the focal buyer's bid b lies between $s_{(m)}$

and $s_{(m+1)}$ in a sample of $m-1$ buyers' bids and n sellers' asks:[16]

$$M_{n,m}(b|B,\mu) = \sum_{\substack{i+j=m \\ 0\leq i\leq m-1 \\ 0\leq j\leq n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} F\left(B^{-1}(b)-\mu\right)^{i} F\left(b-\mu\right)^{j}$$
$$\cdot \overline{F}\left(B^{-1}(b)-\mu\right)^{m-1-i} \overline{F}\left(b-\mu\right)^{n-j} \tag{4}$$

where i indexes buyers and j indexes sellers. Also note that, conditional on μ, the probability that a non-focal buyer with value $B^{-1}(b)$ bids at least b is $F\left(B^{-1}(b)-\mu\right)$ because the focal buyer sets his bid b freely and not in accordance with strategy B that the $m-1$ other buyers employ.

Similarly, $K_{n,m}\left(b|B,\mu\right)$ is the probability that the focal buyer's bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers' bids and $n-1$ sellers' asks:

$$K_{n,m}(b|B,\mu) = \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} F\left(B^{-1}(b)-\mu\right)^{i} F\left(b-\mu\right)^{j}$$
$$\cdot \overline{F}\left(B^{-1}(b)-\mu\right)^{m-1-i} \overline{F}\left(b-\mu\right)^{n-1-j}. \tag{5}$$

Finally, $L_{m,n}\left(b|B,\mu\right)$ is the probability that the focal buyer's bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-2$ buyers' bids and n sellers' asks:

$$L_{n,m}(b|B,\mu) = \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-2 \\ 0\leq j\leq n}} \begin{pmatrix} m-2 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} F\left(B^{-1}(b)-\mu\right)^{i} F\left(b-\mu\right)^{j}$$
$$\cdot \overline{F}\left(B^{-1}(b)-\mu\right)^{m-2-i} \overline{F}\left(b-\mu\right)^{n-j}. \tag{6}$$

Given these formulas, the focal buyer's marginal utility when his value is v, his bid is b, each other buyer is playing strategy B, and each seller is truthfully asking his cost is

$$\begin{aligned}
\pi_b\left(v,b|B\right) =&(v-b)f_x(b|v) - \Pr[x\leq b\leq y|v] \\
=&(v-b)\int_{-\infty}^{\infty}\left[nf(b-\mu)K_{n,m}\left(b|B,\mu\right)\right]f\left(\mu-v\right)d\mu \\
&+(v-b)\int_{-\infty}^{\infty}\left[(m-1)f\left(B^{-1}\left(b\right)-\mu\right)L_{n,m}\left(b|B,\mu\right)\left(B^{-1}\right)'\left(b\right)\right]f\left(\mu-v\right)d\mu \\
&-\int_{-\infty}^{\infty}M_{n,m}\left(b|B,\mu\right)f\left(\mu-v\right)d\mu.
\end{aligned} \tag{7}$$

[16]So, $M_{n,m}\left(b|B,\mu\right) = \Pr[x\leq b\leq y|\mu]$.

The function $(B^{-1})'(b)$ is the derivative at b of the non-focal buyers' inverse bidding function $B^{-1\cdot}$ It's value is the rate at which a non-focal buyer's value v increases as his bid increases. Also, in the formula, we use the equalities $f_\mu(\mu|v) = f(\mu - v)$ and $f_v(v|\mu) = f(v - \mu)$ that follow from the diffuse prior assumption.

To understand this formula, suppose the focal buyer with value v increases his bid from $B(v) = b$ to $b + \Delta b$. Conditional on μ, consider in order the three terms on the right hand side of (7).

1. Pick a seller j and his ask c_j. The term $nf(b-\mu)K_{n,m}(b|B,\mu) \times \Delta b$, is the probability that the buyer's Δb increase in his bid causes him to jump over seller j's ask c_j and go from not trading at b to trading at $b+\Delta B$. This switch to trading requires that $c_j = s_{(m)}$ in the sample of $m+n-1$ bids/asks the focal buyer faces. Because the focal buyer jumps over seller j, $f(b-\mu) \times \Delta b$ is the probability $c_j \in (b, b+\Delta b)$ and $K_{n,m}(b|B,\mu)$ is the probability that, excluding seller j and the focal buyer, the bids/asks of the other $m-1$ buyers and $n-1$ sellers bracket c_j making it is the marginal ask $s_{(m)}$ that the focal buyer must jump in order to trade. Finally, the probability $f(b-\mu)K_{n,m}(b|B,\mu) \times \Delta b$ that j is both marginal and jumped over is multiplied by n because there are n sellers who could be the marginal seller j.

2. Pick a seller k. The term $(m-1)f(B^{-1}(b)-\mu)L_{n,m}(b|B,\mu)\,\Delta b/\left(B^{-1}\right)'(b)$ is the probability that the focal buyer by increasing his bid b by Δb jumps over buyer k's bid $b_k = B(v_k)$ and, moreover, $b_k = s_{(m)}$ in the vector of bids/asks the focal buyer faces. As a result the focal buyer goes from not trading to trading. Derivation of this event's probability exactly parallels the derivation of the first term in (7) with one important difference: the probability buyer k's bid b_k is in the interval $(b, b+\Delta b)$ is $\Delta b f\left(B^{-1}(b)-\mu\right)\left(B^{-1}\right)'(b)$, not $f(b-\mu)\Delta b$. The reason is that, $B(v_k) \in [b, b+\Delta b)$ if and only if $v_k \in [B^{-1}(b), B^{-1}(b+\Delta b)]$. The probability of this event is $F[B^{-1}(b+\Delta b)] - F[B^{-1}(b)]$, which can be approximated with the first degree Taylor series:

$$\begin{aligned}\Delta b \frac{\partial}{\partial b} F(B^{-1}(b)-\mu) &= \Delta b f\left(B^{-1}(b)-\mu\right)\frac{d}{db}B^{-1}(b)\\ &= \Delta b f\left(B^{-1}(b)-\mu\right)(B^{-1})'(b).\end{aligned}$$

3. As explained above, the term $M_{n,m}(b|B,\mu)$ is the probability that the focal buyer's bid lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of $m-1$ buyers each playing strategy B (with inverse B^{-1}) and n sellers each asking their costs. Thus, for the full sample of m bids (including the focal buyer's bid of b) and the n asks, $M_{n,m}(b|\mu)$ is the probability that the focal buyer's bid b equals $s_{(m+1)}$ and therefore sets the price. Consequently increasing b by Δb increases his expected cost of successfully trading by $M_{n,m}(b|B,\mu)\,\Delta b$,

Finally, the unobservable common component μ must be integrated out of each of the three terms using the conditional density $f_\mu(\mu|v) = f(\mu - v)$.

4 Properties of regular, symmetric solutions to the first order condition

In this section we investigate the properties of regular, symmetric solutions to the focal buyer's first order condition (FOC). Specifically, we focus on solutions B such that, for all $v \in \mathbb{R}$,

$$\pi_b(v, b|B)|_{b=B(v)} = 0, \tag{8}$$

where equation (7) defines $\pi_b(v, b|B)$ and $<\tilde{S}, B>$ satisfies A1 and A2, so that if $<\tilde{S}, B>$ turns out to be an equilibrium, then it is a regular equilibrium. We consider the case of $n \geq 1$ sellers and $m \geq 2$ buyers. The assumption that there are at least two buyers permits the analysis of a buyer's first order condition as a differential equation in his strategy B. The section's main result is that an offset solution to the first order condition always exists. In order to prove this we first establish that the vector field that the first order condition implies is invariant with respect to translation along the 45° diagonal of the (v, b) plane. This intermediate result follows from the fact — stated in the introduction — that each buyer's decision problem in choosing his bid b is identical irrespective of where on $\mathbb{R}$ his value falls.

4.1 Invariance of the vector field that the FOC defines

Fix the distribution F that satisfies A3—A5. Let B satisfy A1 and A2 and be a fixed point to the first order condition (8) in the following sense: the focal buyer's optimal strategy is B conditional on the other $m-1$ buyers playing strategy B and the n sellers truthfully asking their costs. The pair $<\tilde{S}, B>$ is then a symmetric solution to the first order condition and, if an appropriate sufficiency condition is satisfied, it is a symmetric equilibrium. Deriving B is facilitated by working with the vector field $\overrightarrow{V}$ that the first order condition (8) defines on the space $W = \{(v, b)\,|v > b\} \subset \mathbb{R}^2$.

That we are seeking a fixed point in B enables us to make an important simplification in the formulas for $K_{n,m}$, $L_{n,m}$, and $M_{n,m}$. If the focal buyer has value v, then in equilibrium he bids $B(v) = b$ and, consequently, $B^{-1}(b) = v$. Therefore within formula (4) for $M_{n,m}(b|B, \mu)$ the factors being summed become a function of v, b, and μ instead of a function of b, B, and μ:

$$\begin{aligned}
&F\left(B^{-1}(b) - \mu\right)^i F(b-\mu)^j\, \overline{F}\left(B^{-1}(b) - \mu\right)^{m-1-i} \overline{F}(b-\mu)^{n-j} \\
&= F(v-\mu)^i F(b-\mu)^j\, \overline{F}(v-\mu)^{m-1-i} \overline{F}(b-\mu)^{n-j}.
\end{aligned}$$

Thus the dependence on B drops out. Define

$$M_{n,m}^{*}(v,b|\mu)=\sum_{\substack{i+j=m\\0\leq i\leq m-1\\0\leq j\leq n}}\left(\begin{array}{c}m-1\\i\end{array}\right)\left(\begin{array}{c}n\\j\end{array}\right)F\left(v-\mu\right)^{i}F\left(b-\mu\right)^{j}\overline{F}\left(v-\mu\right)^{m-1-i}\overline{F}\left(b-\mu\right)^{n-j}, \tag{9}$$

and in parallel fashion define $K_{n,m}^{*}\left(v,b|\mu\right)$ and $L_{n,m}^{*}\left(v,b|\mu\right)$. Similarly on the second line of (7) the density $f\left(B^{-1}\left(b\right)-\mu\right)$ becomes $f\ \left(v-\mu\right)$.

Consequently $<\tilde{S},B>$ is a symmetric solution to the first order condition (8) if, for all pairs $(v,b)\in W$ such that $b=B\left(v\right)$,

$$\begin{aligned}\pi_{b}\left(v,b|B\right)|_{b=B(v)}=&(v-b)\int_{-\infty}^{\infty}\left[nf(b-\mu)K_{n,m}^{*}\left(v,b|\mu\right)\right]f\left(\mu-v\right)d\mu\\&+\left(v-b\right)\left(B^{-1}\right)'\left(b\right)\int_{-\infty}^{\infty}\left[(m-1)f\left(v-\mu\right)L_{n,m}^{*}\left(v,b|\mu\right)\right]f\left(\mu-v\right)d\mu\\&-\int_{-\infty}^{\infty}M_{n,m}^{*}\left(v,b|\mu\right)f\left(\mu-v\right)d\mu.\\=&0.\end{aligned}$$

Solve this equation for $\left(B^{-1}\right)'\left(b\right)$ to obtain

$$V'(v,b)=\frac{\int_{-\infty}^{\infty}M_{n,m}^{*}f(v-\mu)d\mu-(v-b)\int_{-\infty}^{\infty}nf(b-\mu)K_{n,m}^{*}f\left(\mu-v\right)d\mu}{(v-b)\int_{-\infty}^{\infty}(m-1)f(v-\mu)L_{n,m}^{*}f(\mu-v)d\mu} \tag{10}$$

where $V'(v,b)\equiv\left(B^{-1}\right)'\left(b\right)$ and we have omitted the arguments of $M_{n,m}^{*}$, $K_{n,m}^{*}$, and $L_{m,n}^{*}$ for notational brevity. We define $\overrightarrow{V}$ to be the vector field that, for $(v,b)\in W$, has value $\overrightarrow{V}\left(v,b\right)\equiv\left(1,V'\left(v,b\right)\right)$.

Observe that the value of $\overrightarrow{V}$ depends only on the point $\left(v,b\right)$. The interpretation of its value at a given point $(v'',b'')\in W$ is that if $<\tilde{S},B>$ is a symmetric solution to the first order condition in which (i) all m buyers follow the identical strategy B and (ii) $b''=B\left(v''\right)$, then the derivative of B evaluated at v'' has value $1/B'\left(v''\right)=V'\left(v'',b''\right)$. Thus solutions to the first order condition (8) can be traced out by following the vector field $\overrightarrow{V}$ using any point $(v,b)\in W$ as a starting point.[17] The following lemma establishes the invariance property of $\overrightarrow{V}$.

Lemma 1 *For any $(v,b)\in\mathbb{R}^{2}$ and any $\rho\in\mathbb{R}$, the vector field $\overrightarrow{V}$ satisfies*

$$\overrightarrow{V}(v,b)=\overrightarrow{V}(v+\rho,b+\rho).$$

Proof. Let $b^{*}=b+\rho$, $v^{*}=v+\rho$ and $\mu^{*}=\mu-\rho$. The first step is to notice from formulas

[17]A standard result in differential equations is that a solution $v=B^{-1}\left(b\right)$ to (10) exists that goes through any given point $(v,b)\in W$. See, for example, Arnold (1973, Theorem 7.1).

(4)-(6) and (9) that the values of $M^*_{n,m}(v,b|\mu)$, $K^*_{n,m}(v,b|\mu)$ and $L^*_{n,m}(v,b|\mu)$ are determined by the differences $b-\mu$ and $v-\mu$. The following formulas thus hold:

$$\begin{aligned}
M^*_{n,m}(v^*,b^*|\mu) &= M^*_{n,m}(v,b|\mu^*),\\
K^*_{n,m}(v^*,b^*|\mu) &= K^*_{n,m}(v,b|\mu^*),\\
L^*_{n,m}(v^*,b^*|\mu) &= L^*_{n,m}(v,b|\mu^*).
\end{aligned}$$

Now consider formula (10). We have

$$\begin{aligned}
&V'(v^*,b^*)\\
&= \frac{\int_{-\infty}^{\infty}\left[M^*_{n,m}(v^*,b^*|\mu) - nf(b^*-\mu)(v^*-b^*)K^*_{n,m}(v^*,b^*|\mu)\right] f(v^*-\mu)d\mu}{\int_{-\infty}^{\infty}(m-1)f(v^*-\mu)(v^*-b^*)L^*_{n,m}(v^*,b^*|\mu)f(v^*-\mu)d\mu}\\
&= \frac{\int_{-\infty}^{\infty}\left[M^*_{n,m}(v,b|\mu^*) - nf(b^*-\mu)(v^*-b^*)K^*_{n,m}(v,b|\mu^*)\right] f(v^*-\mu)d\mu}{\int_{-\infty}^{\infty}(m-1)f(v^*-\mu)(v^*-b^*)L^*_{n,m}(v,b|\mu^*)f(v^*-\mu)d\mu}\\
&= \frac{\int_{-\infty}^{\infty}\left[M^*_{n,m}(v,b|\mu^*) - nf(b-\mu^*)(v-b)K^*_{n,m}(v,b|\mu^*)\right] f(v-\mu^*)d\mu}{\int_{-\infty}^{\infty}(m-1)f(v-\mu^*)(v-b)L^*_{n,m}(v,b|\mu^*)f(v-\mu^*)d\mu}\\
&= \frac{\int_{-\infty}^{\infty}\left[M^*_{n,m}(v,b|\mu) - nf(b-\mu)(v-b)K^*_{n,m}(v,b|\mu)\right] f(v-\mu)d\mu}{\int_{-\infty}^{\infty}(m-1)f(v-\mu)(v-b)L^*_{n,m}(v,b|\mu)f(v-\mu)d\mu}\\
&= V'(v,b)
\end{aligned}$$

where the fourth line follows from a change of variable in the integrals in which $\mu+\rho$ replaces μ. ■

4.2 Existence of an offset solution to the first order condition

An offset strategy B for a buyer is a function of the form

$$B(v) = v - \lambda$$

for some constant $\lambda \in \mathbb{R}$. Notice that an offset strategy satisfies A1 and A2. We consider in this section the existence of offset equilibria of the form $<\tilde{S}, B>$ in which each seller j uses his dominant strategy of asking his cost c_j and each buyer i uses the same offset strategy $B(v_i) = v_i - \lambda$. The first order condition (8) holding almost everywhere is a necessary condition for a strictly increasing function B to define an equilibrium of the form $<\tilde{S}, B>$.[18] We prove in this section that there exists an offset strategy B in the market with $n \geq 1$ sellers and $m \geq 2$ buyers that satisfies the first order condition (10). The proof relies on the Intermediate Value Theorem and the following lemma.

[18]The reason is that every strictly increasing function is differentiable almost everywhere, see, for instance, Royden (1988, p. 96).

Lemma 2 *Consider any $\delta > 0$. There exists a constant $L^*(F, \delta) > 0$ such that*

$$v - b > L^*(F, \delta) \Rightarrow V'(v, b) < \delta. \tag{11}$$

The lemma's proof is in Appendix A. It implies that

$$\lim_{b \to -\infty} \inf V'(v, b) \leq 0$$

for all v.

Theorem 1 *For $n \geq 1$ and $m \geq 2$, a constant $\lambda = \lambda(F, n, m) \in \mathbb{R}$ exists such that the strategy*

$$B(v) = v - \lambda \tag{12}$$

satisfies the first order condition (10) at all $v \in \mathbb{R}$.

Proof. Fix the value of v^*. Lemma 2 implies that $V'(v^*, b) < 1$ for b sufficiently small. It is shown below that $V'(v, v) = \infty$ at any $v \in \mathbb{R}$. Applying the continuity of the distribution F and its density f, the Intermediate Value Theorem implies the existence of a value of b^* at which $V'(v^*, b^*) = 1$. Define $\lambda = v^* - b^*$. The invariance property established in Lemma 1 implies that

$$V'(v^* + \rho, v^* - \lambda + \rho) = V'(v^*, v^* - \lambda) = 1$$

for all $\rho \in \mathbb{R}$. The line

$$(v^* + \rho, v^* - \lambda + \rho)_{\rho \in \mathbb{R}}$$

is therefore a solution curve of the vector field $\overrightarrow{V}$. This is depicted in Figure 1, which illustrates the normalized vector field at the points of interest.

It remains to be shown that $V'(v, v) = \infty$ at any $v \in \mathbb{R}$. Returning to (9), notice that

$$M^*_{n,m}(v, v|\mu) = \sum_{\substack{i+j=m \\ 0 \leq i \leq m-1 \\ 0 \leq j \leq n}} \binom{m-1}{i} \binom{n}{j} F(v - \mu)^{i+j}\, \overline{F}(v - \mu)^{m+n-1-i-j}$$

is nonzero for all $v, \mu \in \mathbb{R}$. The numerator of $V'(v, b)$ in (10) thus approaches a nonzero real number as b converges upward to v while its denominator converges to zero. ■

Figures 2 (a)-(d) depict the normalized vector field in the case of F being the standard normal distribution in the case of τm buyers and τn sellers (for $\tau = 2, 4, 8, 16$ and $m = n = 1$). Notice the invariance of the vector field as it is translated on the $45°$ diagonal. Inspection of the figures suggests that the computed offset is unique.

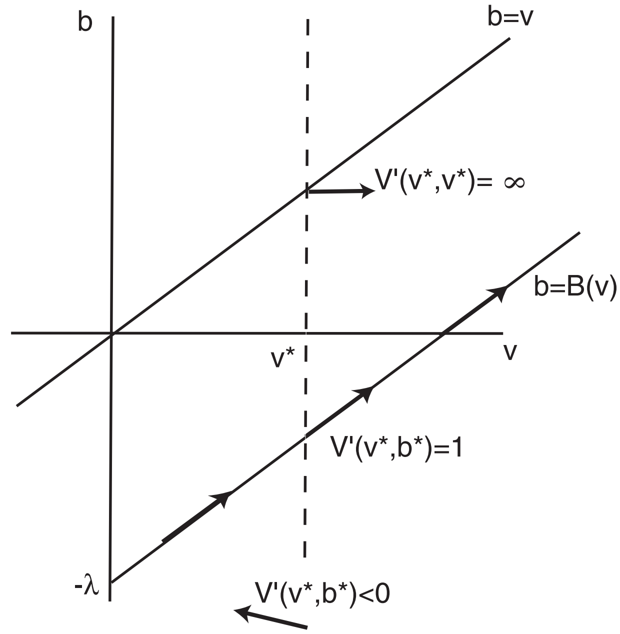


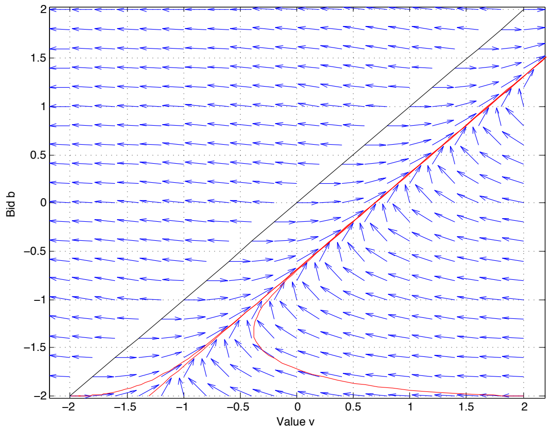
Figure 1: This solution curve is the graph of the strategy $B(v) = v - \lambda$.

5 Sufficiency of the first order approach

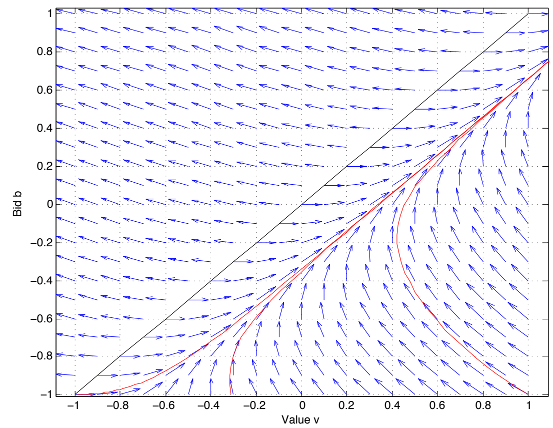
Existence of an offset solution to a buyer's first order condition leaves open the question as to whether or not such a solution (or any other solution to the first order condition) defines an equilibrium. This section addresses the sufficiency of the first order approach, an issue that is complicated in our model by the statistical dependence of values and costs and the asymmetry of buyers underbidding and sellers reporting truthfully. The most straightforward approach, which is both necessary and sufficient, is to graph expected utility and check that it is maximized at the FOC's solution. Assuming an offset solution, the invariance of a buyer's decisions problem implies that optimality of the offset solution for all v can be checked at a single, specific value of v. This makes this approach feasible and we use it in all our numerical experiments.

A second approach is to derive a condition on the fundamentals of the model that ensures sufficiency of the first order approach. We now derive a condition of this type that parallels the derivation in Milgrom and Weber (1982) of a similar condition for the sufficiency of the first order approach in auctions. We then discuss the limitation of this condition and, in particular, the reasons why statistical affiliation of traders' costs and values cannot play the same useful role in double auctions as it does in auctions.

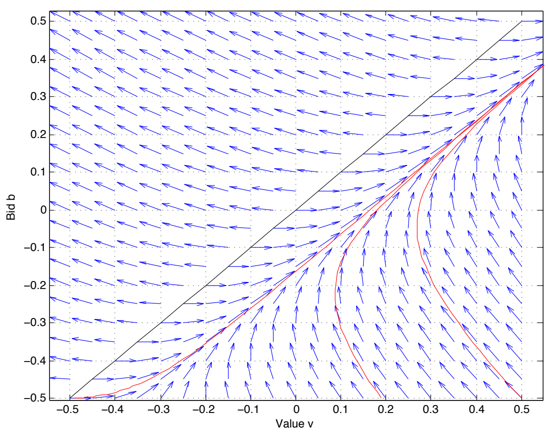
Recall that $x = s_{(m)}$ and $y = s_{(m+1)}$ are the m^{th} and the $(m+1)^{\text{st}}$ order statistics in the sample of $m+n-1$ bids and asks of the $m-1$ buyers (excluding the focal buyer) playing the strategy B



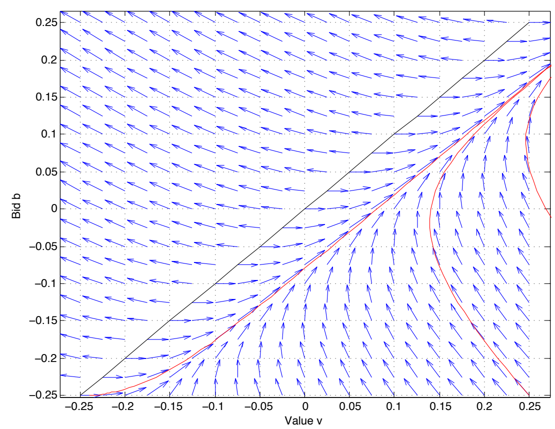
(a) $\tau = 2$



(b) $\tau = 4$



(c) $\tau = 8$



(b) $\tau = 16$

Figure 2: Solving the differential equation for different values of τ ($m = n = 1$).

and the n sellers playing the truthful strategy $\tilde{S}$.

Theorem 2 *Suppose for all $v \in \mathbb{R}$ the function B solves the buyer's first order condition (3) and satisfies assumptions A1 and A2. A sufficient condition for $<\tilde{S}, B>$ to be an equilibrium for the BBDA with $m \geq 2$ buyers and $n \geq 1$ sellers is that, for every $b \in \mathbb{R}$, the ratio*

$$\mathcal{R}(v, b, B, m, n) = \frac{\Pr\left(x < b < y|v\right)}{f_x\left(b|v\right)} \tag{13}$$

is decreasing in $v > b$.

Proof. Function B satisfies at all $(v, b) \in W$ the first order condition (3) such that for $b = B(v)$ we have

$$\pi_b(v, b|B) = (v - b) f_x(b|v) - \Pr[x \leq b \leq y|v] = 0 \tag{14}$$

where $W = \{(v,b)\,|v > b\} \subset \mathbb{R}^2$. A sufficient condition for B to be a symmetric equilibrium is for $(v,b) \in W$,

$$\left\{\begin{array}{c} \pi_b(v,b|B) \geq 0 \text{ if } b < B(v) \\ \pi_b(v,b|B) = 0 \text{ if } b = B(v) \\ \pi_b(v,b|B) \leq 0 \text{ if } B(v) < b < v \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \pi_b(v,b|B) \geq 0 \text{ if } B^{-1}(b) < v, \\ \pi_b(v,b|B) = 0 \text{ if } B^{-1}(b) = v, \\ \pi_b(v,b|B) \leq 0 \text{ if } v < B^{-1}(b) < B^{-1}(v)\,. \end{array}\right\}.$$

The left-hand column of conditions is the first derivative test on the variable b for a fixed value of v. The equivalent conditions in the right-hand column are obtained by applying the inverse mapping B^{-1} to the variables. They are useful below when we fix the bid b and interpret v as the variable.

Observe that the right column is satisfied if, for any $b \in \mathbb{R}$, $\pi_b(v,b|B)$ is increasing in $v \geq b$ and equals zero at $v = B^{-1}(b)$. Rewrite the focal buyer's marginal utility (14) as

$$\begin{aligned} \pi_b(v,b|B) &= (v-b) f_x(b|v) - \Pr[x \leq b \leq y|v] \\ &= \left[(v - B^{-1}(b)) + (B^{-1}(b) - b)\right] f_x(b|v) - \Pr[x \leq b \leq y|v] \\ &= f_x(b|v) \left\{ (v - B^{-1}(b)) + \left[(B^{-1}(b) - b) - \frac{\Pr[x \leq b \leq y|v]}{f_x(b|v)} \right] \right\} \end{aligned} \tag{15}$$

for any $(v,b) \in W$. We focus on the term in the curly brackets in (15), which determines the sign of $\pi_b(v,b|B)$. The first term is increasing in v and equals zero at $v = B^{-1}(b)$. That the second term equals zero at $v = B^{-1}(b)$ is confirmed by substituting $B^{-1}(b)$ for v into it and noting that the result equals zero because B solves the first order condition (14):

$$\begin{aligned} & f_x(b|B^{-1}(b)) \left\{ (B^{-1}(b) - b) - \frac{\Pr[x \leq b \leq y|B^{-1}(b)]}{f_x(b|B^{-1}(b))} \right\} \\ &= (B^{-1}(b) - b) f_x(b|B^{-1}(b)) - \Pr[x \leq b \leq y|B^{-1}(b)] \\ &= 0. \end{aligned}$$

Finally, the second term is increasing in v if the fraction in this expression (which is $\mathcal{R}(v,b,B,m,n)$) is decreasing in v. ■

The requirement that $\mathcal{R}(v,b,B,m,n)$ is decreasing in $v > b$ serves the same purpose for the BBDA as the requirement that the inverse hazard rate, for any b,

$$\mathcal{S}(v,b,B,m,1) = \frac{\Pr\left(s_{(m-1)} < b|v\right)}{f_{s_{(m-1)}}(b|v)} \tag{16}$$

is decreasing in v serves within the theory of first price auctions with correlated values.[19] In each

[19] See Milgrom and Weber (1982, p. 1106-08). In the auction setting v denotes the value of the focal buyer and $s_{(m-1)}$ denotes the largest value among the values of the other $m-1$ buyers who are competing to buy the single item being auctioned.

case the numerator is the marginal expected cost to a buyer from increasing his bid and thereby driving up the price that he pays and the denominator is the marginal probability of acquiring an item by increasing his bid. Milgrom and Weber (1982, p. 1107-08) show that in such auctions $\mathcal{S}$ decreasing in v is sufficient for a solution B to the buyer's first order condition to be an equilibrium to the auction. They also show that expression (16) is necessarily decreasing if the m buyers' values are statistically affiliated. That affiliation of buyers' values implies satisfaction of the sufficient condition (16) is useful because a substantial body of research exists establishing conditions under which common multivariate distributions generate affiliated random vectors. As a consequence affilation is a crucial part of auction theory's foundation.

Affiliation, however, cannot serve the same role in establishing sufficiency of the first order approach in double auctions. While it is straightforward in our model to impose conditions on F (such as the monotone hazard rate property) that insure that the values/costs of traders are affiliated, we next show below that it is not in general true that the ordered bids/asks of traders are affiliated.

Specifically, assume that (for a fixed value of the state μ) the components of the random vector of buyers' values and sellers' costs $(v_1, \ldots, v_m, c_1, \ldots, c_n)$ are affiliated. This is true whenever F has a monotone hazard rate, as in the normal distribution. Following Milgrom and Weber (1982, pp.1118-1189) and Tong (1990, pp. 130-132), we know that: (i) the components of the vector of ordered values and costs $(t_{(1)}, t_{(2)}, \ldots t_{(n+m)})$ are affiliated; (ii) assuming the use of the offset strategy $B(v) = v - \lambda$ by buyers, the components of the random vector of bids/asks $(v_1 - \lambda, \ldots, v_m - \lambda, c_1, \ldots, c_n)$ are affiliated; and (iii) the components of the random vector of ordered bids/asks $(s_{(1)}, s_{(2)}, \ldots s_{(n+m)})$ are *associated,* which is a strictly weaker relationship among random variables than affiliation.[20]

Theorem 3 *If $m \geq 2$ and $n \geq 1$, F is the standard normal distribution, and buyers use any offset strategy, then a buyer's value is not affiliated with the ordered bids/asks of others.*

See Appendix C for the proof.

This theorem should be contrasted with our numerical results that follow in Sections 7 and 8, wherein offset equilibria that appear to satisfy A1 and A2 are computed in small markets. The failure of affiliation to hold — even in equilibrium — among ordered bids and asks impedes a proof of sufficiency of equilibrium; it does not appear, however, to impede the existence and computation of well-behaved equilibria for common F.[21]

[20] In the BBDA with one seller ($n = 1$) and multiple buyers the seller's ask is his cost. On the surface this appears to be a standard auction, albeit with the twist of a hidden reservation value, to which the analysis of Milgrom and Weber (1982) may be applied. This approach, however, fails because for that analysis a buyer's value must be affiliated with the ordered statistics of that hidden reservation value and other buyers' bids. Theorem 3 establishes that this needed affiliation does not hold in general.

[21] Reny and Perry (2006) foresaw the problems that are caused for proving existence of equilibrium in double auctions by the failure of affiliation. This is an obstacle in small markets that is avoided in Reny and Perry (2006)

A weakness of condition (13) is that it is not reduced to an interpretable condition directly in terms of F. The expanded formula for $\mathcal{R}$ in the case of an offset strategy suggests the difficulty of this task:

$$\frac{\int_{-\infty}^{\infty} M_{n,m}(b|B,\mu)f(v-\mu)\,d\mu}{\int_{-\infty}^{\infty}\left[nK_{n,m}(b|B,\mu)f(b-\mu)+(m-1)L_{n,m}(b|B,\mu)f(b+\lambda-\mu)\right]f(v-\mu)\,d\mu}.$$

or

$$\frac{\int_{-\infty}^{\infty} M^*_{n,m}(v,b|\mu)f(v-\mu)\,d\mu}{\int_{-\infty}^{\infty}\left[nK^*_{n,m}(v,b|\mu)f(b-\mu)+(m-1)L^*_{n,m}(v,b|\mu)f(b+\lambda-\mu)\right]f(v-\mu)\,d\mu}.$$

The integration, which arises from existence of the common component, together with the formulas for $M^*_{n,m}$, $K^*_{n,m}$ and $L^*_{n,m}$ with their complex summations across event probabilities stymies reduction to a condition directly in terms of F.

As a substitute in the case of offset solutions, for any given choice of F, it is simple numerically to check our condition (13) for a solution to be an equilibrium. Specifically, numerically check that $\mathcal{R}(v,0,B,m,n)$ is decreasing for all $v\in\mathbb{R}^+$ where the functions $M^*_{n,m}$, $K^*_{n,m}$ and $L^*_{n,m}$ are evaluated at $(v,0|\mu)$. The invariance property that Lemma (1) establishes allows us to fix $b=0$ and only check that $\mathcal{R}$ is decreasing on $v\in\mathbb{R}^+$. It is not necessary to check that, for every $b\in\mathbb{R}$, $\mathcal{R}$ is decreasing for all $v>b$. This establishes the following corollary:

Corollary 1 *Suppose for all $v\in\mathbb{R}$ the offset strategy $B(v)=v-\lambda$ solves the buyer's first order condition (3) and that, for $b=0$,*

$$\mathcal{R}(v,0,B,m,n)=\frac{\Pr(x<0<y|v)}{f_x(0|v)} \tag{17}$$

is decreasing in v for all $v\in\mathbb{R}^+$. Then B is an offset equilibrium.

Note that the offset λ satisfies the equation $\mathcal{R}(\lambda,0,B,m,n)=\lambda$.

The exception to being unable to reduce the requirement that $\mathcal{R}$ is decreasing in v to a simple condition on F is the bilateral case, $m=n=1$. There the (sole) buyer's utility as a function of his bid b and his value v is

$$\pi(v,b)=(v-b)F_{c|v}(b|v),$$

where $F_{c|v}$ is the distribution function of the seller's cost c given the buyer's value v. Hence the first order condition is

$$\pi_b(v,b)=0\Rightarrow v-b=\frac{F_{c|v}(b|v)}{f_{c|v}(b|v)}. \tag{18}$$

Strengthen A5 to:

through their focus upon the case of a large number of traders. Our paper complements Reny and Perry (2006) by showing that for the cases we investigate the failure of affiliation is not a problem even in small markets.

A5′: F has a monotone hazard rate, i.e., $f(x)/F(x)$ is decreasing for all $x \in \mathbb{R}$.

This allows us to prove the following result.

Theorem 4 *Under assumptions A3-A4 & A5′ the first order condition (18) has a unique, increasing, continuous solution that globally maximizes the buyer's utility, and is the unique equilibrium strategy B of the bilateral BBDA. Furthermore, B takes the simple form of an offset strategy,*

$$B(v) = v - \lambda,$$

where λ is a positive constant.

The proof, which is in Appendix B, consists of showing how our assumptions map into the assumptions in Kadan (2007, Theorem 1). The proof's key is the affiliation of the random vector (μ, v, c) that follows from Assumption A5′.

In summary, Theorem 1 shows the existence of an offset solution to the first order condition. Corollary 1 provides a numerically tractable means to check if such a solution in fact defines an equilibrium. In Section 7 we report that $\mathcal{R}$ is decreasing in v for F normal or Laplacian when B is an offset solution. Together these three pieces establish a main contribution of the paper: for small markets well behaved pure strategy equilibria exist for at least some common distributions F.

6 Convergence to truthful revelation and efficiency

Given a constant $\varphi \geq 1$ and assumptions A3-A5 on F, the offset solution converges to the buyers' truthful strategy $\tilde{B}(v) = v$ for all sequences of markets such that

$$\frac{1}{\varphi} \leq \frac{m}{n}, \frac{n}{m} \leq \varphi \tag{19}$$

as $m, n \to \infty$. More specifically, we prove that the misrepresentation, $v - B(v)$, in any regular equilibrium strategy B is $O(1/m)$ and, second, we prove that convergence to ex post efficiency is $O\left(1/m^2\right)$. Observe that the bound (19), parameterized by $\varphi > 1$, on the admissible ratio of buyers and sellers implies $O(1/n) = O(1/m) = O(1/(n \wedge m))$.

Theorem 5 *Consider any $\varphi \geq 1$ and values of $m \geq 2$ and $n \geq 1$ that satisfy (19). There exists a constant $K(F, \varphi)$ such that any regular equilibrium strategy B in the market with n sellers and m buyers satisfies the inequality*

$$v - B(v) \leq \frac{K(F, \varphi)}{m} \tag{20}$$

for all $v \in \mathbb{R}$.

The theorem extends the following lemma, which implies that buyer misrepresentation is $O(1/m)$ at values of v for which $B'(v)$ exists and satisfies $B'(v) \leq 1/\delta$.

Lemma 3 *Consider any $\varphi \geq 1$ and values of $m \geq 2$ and $n \geq 1$ that satisfy (19). For any $\delta > 0$, there exists a constant $K(F, \varphi, \delta)$ such that $V'(v, b) \geq \delta$ only if*

$$v - b \leq \frac{K(F, \varphi, \delta)}{m}. \tag{21}$$

The proof of Lemma 3 is in Appendix A. We now apply it to prove Theorem 5.

Proof. Since B is a regular equilibrium strategy assumption A2 implies that we can choose a value of $\delta \in (0, 1)$ such that

$$\lim \inf_{v \to -\infty} B'(v) < \frac{1}{\delta}. \tag{22}$$

Let $K(F, \varphi)$ equal the constant $K(F, \varphi, \delta)$ whose existence Lemma 3 guarantees. Lemma 3 states that the bound $v - B(v) \leq \frac{K(F,\varphi)}{m}$ holds at values of v for which $V'(v, B(v)) \geq \delta$ or, equivalently, for values of v at which $B'(v)$ exists and satisfies $B'(v) \leq \frac{1}{\delta}$.

We now extend this bound to all values of v at which B is differentiable. Pick a v^* at which B is differentiable and let $\Delta(v^*)$ denote the set of values of $v < v^*$ at which $B'(v)$ exists and is less than $1/\delta$. This set is nonempty by virtue of (22). Let $\overline{v} = \sup \Delta(v^*)$ and notice that $B'(v) \geq 1/\delta$ for $v \in (\overline{v}, v^*)$ at which B' exists. The bound (20) holds at $\overline{v}$ because it holds at each $v \in \Delta(v^*)$ and $\sup_{v \in \Delta(v^*)} B(v) \leq B(\overline{v})$ since, by assumption A1, B is strictly increasing. Then

$$\begin{aligned} v^* - \overline{v} &= \int_{\overline{v}}^{v^*} dv \\ &< \int_{\overline{v}}^{v^*} \frac{1}{\delta} dv \\ &\leq \int_{\overline{v}}^{v^*} B'(v) dv \\ &\leq B(v^*) - B(\overline{v}), \end{aligned}$$

where the integral on the third line exists because a strictly increasing function is differentiable almost everywhere. Rearranging implies $v^* - B(v^*) \leq \overline{v} - B(\overline{v})$ and, since bound (20) holds at $\overline{v}$,

$$v^* - B(v^*) \leq \frac{K(F, \varphi)}{m}.$$

The final step extends bound (20) to values of v^* at which $B'(v^*)$ does not exist. There exists an increasing sequence $(v^t)_{t \in \mathbb{N}^*}$ such that $B'(v^t)$ exists for each $t \in \mathbb{N}^*$ and

$$\lim_{t \to \infty} v^t = v^*. \tag{23}$$

The bound (20) holds at each v^t in the sequence, i.e.,

$$v^t - B(v^t) \le \frac{K(F,\varphi)}{n \wedge m}.$$

We have

$$\lim_{t\to\infty} B(v^t) \le B(v^*) \tag{24}$$

because B is increasing. The bound therefore holds at v^* because of (23) and (24). ■

For a symmetric equilibrium $<\tilde{S}, B>$ in the market with m buyers and n sellers, relative inefficiency is defined as

$$1 - \frac{\textit{exp. gains from trade in } <\tilde{S}, B>}{\textit{the expected potential gains from trade}} \tag{25}$$

or equivalently,

$$\frac{\textit{exp. potential gains from trade that are not achieved in } <\tilde{S}, B>}{\textit{exp. potential gains from trade}}. \tag{26}$$

Theorem 6 *Consider any $\varphi \ge 1$ and values of $m \ge 2$ and $n \ge 1$ that satisfy (19). There exists a constant $\kappa(F,\varphi)$ such that the relative inefficiency of any regular equilibrium $<\tilde{S}, B>$ in the market with $m \ge 2$ buyers and $n \ge 1$ sellers is no more than $\kappa(F,\varphi)/m^2$.*

The proof of Theorem 6 is in Appendix A. The theorem establishes the same bound on relative inefficiency in each state $\mu \in \mathbb{R}$. It is thus reasonable to state that ex ante relative inefficiency is $O\left(m^{-2}\right)$ despite the fact that μ ex ante does not have a proper probability distribution.

Remark 2 *Note that if we start from a fixed number m of buyers and n of sellers, and then consider a sequence of markets with τm buyers and τn sellers for $\tau \in \mathbb{N}^*$ then the rates reported in Theorems and 5 and 6 become $O(1/\tau)$ and $O(1/\tau^2)$.*

7 A Computational Analysis for the Case of the Standard Normal Distribution

This section presents the results of a computational analysis of our model in the case of $F = \Phi$, the cumulative for the standard normal distribution $\mathcal{N}(0,1)$. Our calculations suggest that the offset solution to the buyer's first order condition in this case is unique in each size of market, determines an equilibrium of the BBDA, and defines the only symmetric smooth, regular equilibrium of the BBDA.

Table 1 presents several equilibrium quantities. We consider τm buyers and τn sellers and $m = n = 1$ so that the first column gives the number of traders on each side of the market. Column

2 computes the solution to the equation $V'(v, v-\lambda) = 1$. GFT in column 3 is the expected potential gains from trade, and GFT_{BBDA} in column 4 is the expected gains from trade achieved when buyers use the offset equilibrium λ shown in column 2.[22]

τ	λ	GFT	GFT_{BBDA}	$(GFT - GFT_{BBDA})/GFT$
2	0.6896	1.3265	1.2221	0.0795
4	0.3374	2.9008	2.8535	0.0163
8	0.1556	6.0812	6.0653	0.0026
16	0.0762	12.4604	12.4516	0.0007

Table 1: Column 2 is the equilibrium offset solution, column 3 is the expected gains from trade, and column 4 is the expected gains from trade achieved in the BBDA, all for different sizes of market τ reported in column 1 ($m = n = 1$).

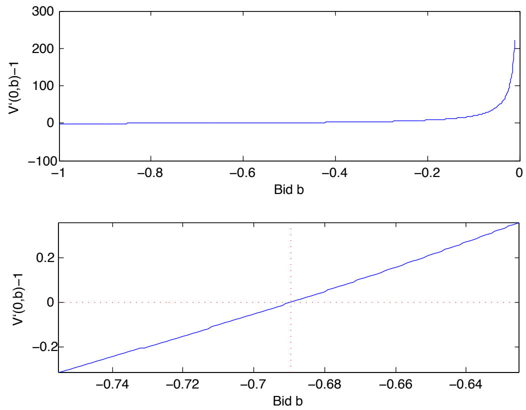
The offset as expected from Theorem 5 vanishes at a rate of $O(1/\tau)$.[23] The difference $(GFT - GFT_{BBDA})/GFT$ in column 5 is the ex ante relative expected loss to the traders from using the inefficient BBDA. The values in column 5 decrease by a factor of roughly four each time that τ doubles as expected from Theorem 6, which shows that relative inefficiency is $O(1/\tau^2)$.

Figures 3 (a)-(d) depict the calculations of the optimal offset of column 1 in Table 1, by graphing the difference $V'(v, b) - 1$ in the case of $v = 0$, $b \in (0, 1]$ and $\tau = 2, 4, 8, 16$. The invariance of the vector field implies that the calculations in the case of $v = 0$ are representative of this difference at other values of v (i.e., $V'(0, b) - 1 = V'(v, v + b) - 1$ for all $v \in \mathbb{R}$). The right side of the graphs corresponds to the point $(v, b) = (0, 0)$ in Figure 1, and moving leftward by decreasing b corresponds to moving vertically down the b-axis in Figure 1. At $(v, b) = (0, 0)$, the normalized vector field is horizontal because $V'(0, 0)$ is infinite; one can see $V'(0, b) - 1$ growing large in Figures 3 (a) - (d) as b decreases to 0. The scale used on the top graph in each figure to show the rapid increase in $V'(0, b)$ as b nears 0 makes it difficult to see that $V'(0, b) - 1$ declines steadily across the interval $(0, 1]$. This is shown over a restricted interval near the offset solution λ from Table 1 in the bottom graphs of the four figures. All four graphs show that $V'(0, \lambda) - 1 = 0$ has a unique solution, i.e., there is a unique offset solution to the buyer's first order condition.

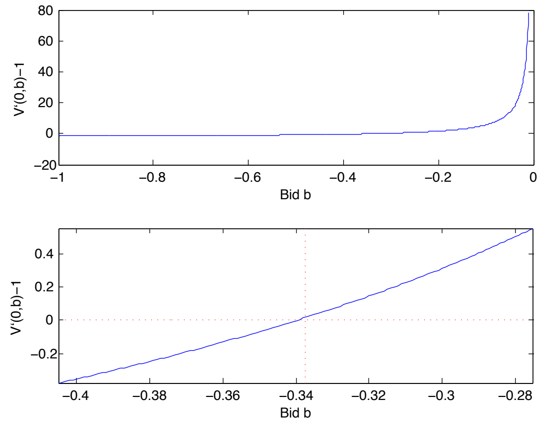
We next demonstrate that the offset solution to the first order condition defines an equilibrium in the BBDA. Figures 4 (a)-(d) depict the maximization of the selected buyer's utility at the offset value from Table 1 together with the strict monotone decline of his marginal utility, which changes from positive to negative at this offset value, for $\tau = 2, 4, 8, 16$. It is assumed in these figures that the selected buyer has value $v = 0$ and that all other buyers use the offset strategy that is presented in Table 1. The graphs numerically demonstrate in each size of market that the selected buyer's unique best response to the offset solution to the first order condition is this offset solution itself.

[22] GFT & GFT_{BBDA} are calculated in a fixed state μ using a Monte Carlo method. GFT has the same value for all μ, while GFT_{BBDA} has the same value for all μ when all buyers use the same offset strategy.

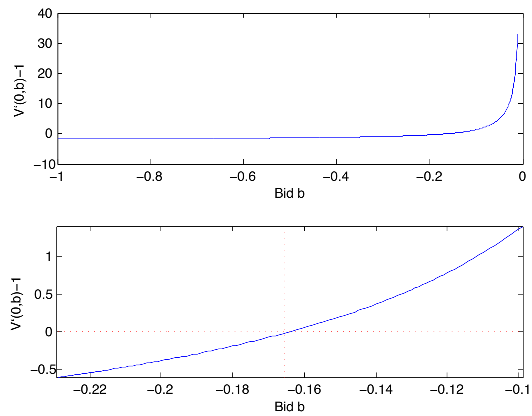
[23] In the bilateral case for $F = \Phi$ we have that $c|v \sim \mathcal{N}(v, 2)$ and the unique offset solution is 1.0620.



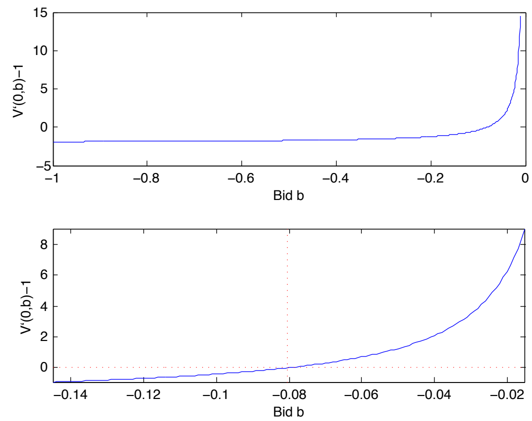
(a) $\tau = 2$



(b) $\tau = 4$



(c) $\tau = 8$

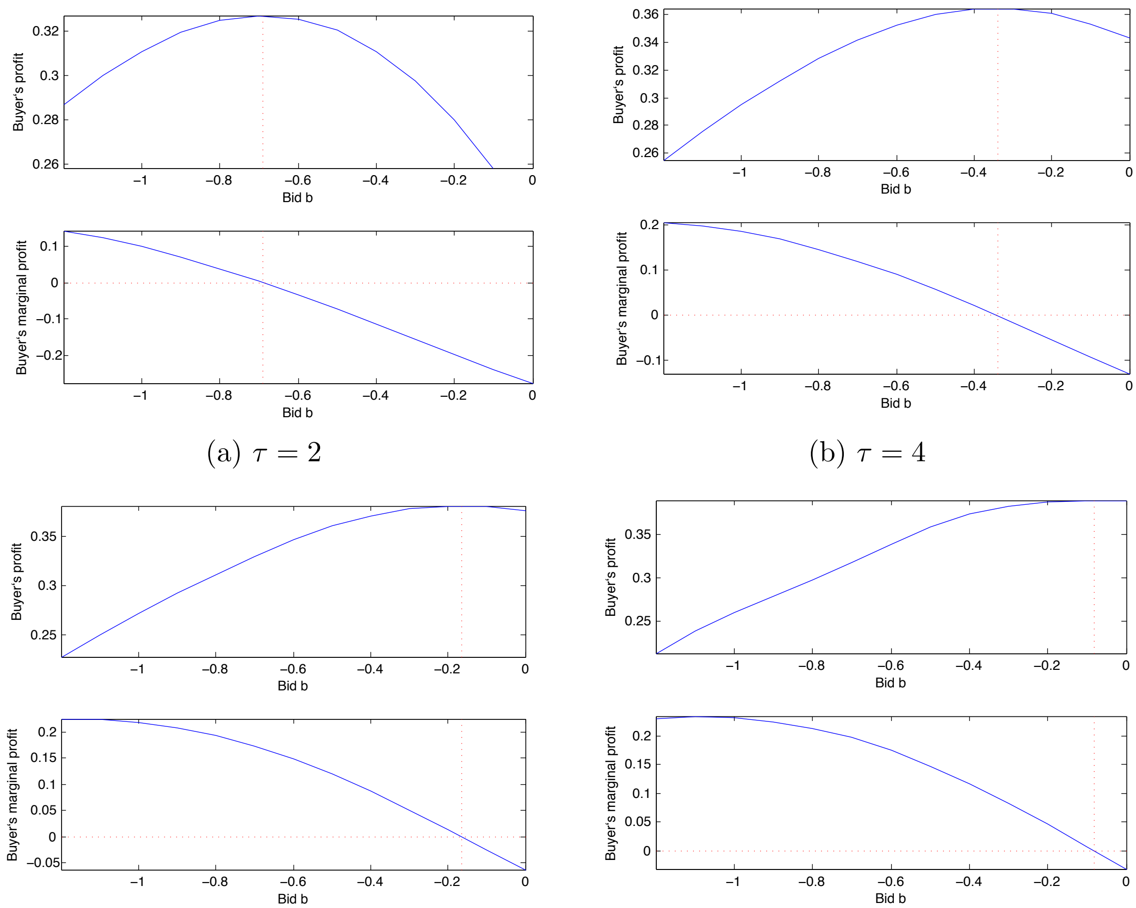


(d) $\tau = 16$

Figure 3: Focal buyer's vector field minus one vs his bid b, for $v = 0$, F the standard normal distribution, and different size of market τ ($m = n = 1$).

The graphs therefore suggest that the offset solution to the buyer's first order condition defines an equilibrium of the BBDA.

We conclude this example by arguing that the offset equilibrium is in fact the unique symmetric, smooth and regular equilibrium strategy for buyers in the BBDA in the case of the standard normal distribution. While the formal results in this paper do not require uniqueness, it is interesting that calculations suggest that it holds in this benchmark example. The bottom graphs in Figures 3 (a) - (d) suggest that $V'(0, b)$ is strictly decreasing in $-b$. Applying the invariance of the vector field, this is also true for $V'(v, b)$. The following argument shows that this fact implies that the offset solution is the only solution to the first order conditions that defines a smooth and regular equilibrium strategy for buyers. Consider starting at a point in Figure 1 between the line $v = b$



(a) $\tau = 2$

(b) $\tau = 4$

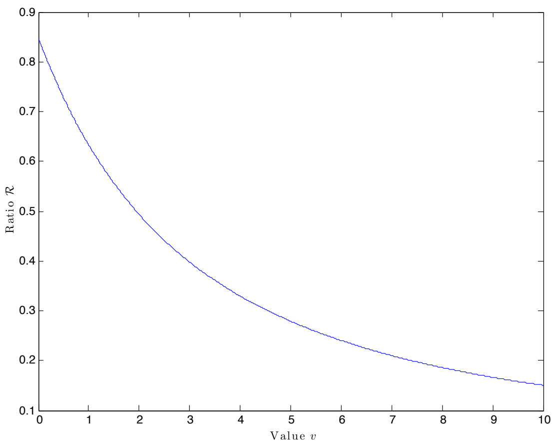
(c) $\tau = 8$

(d) $\tau = 16$

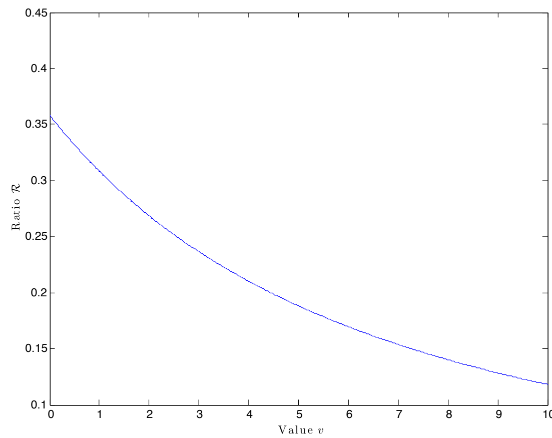
Figure 4: Focal buyer's utility and marginal utility vs his bid b, for $v = 0$, F the standard normal distribution, and different market size τ ($m = n = 1$). Other sellers report honestly, and other buyers use the offset reported in Table 1.

and the line determined by the offset solution $V'(v, v - \lambda) = 1$ and then solving the buyer's first order condition by tracing out a solution curve to the normalized vector field. Because $V'(v, b)$ is decreasing in $-b$, the vector field flattens as b nears v; a solution traced in the direction of decreasing b therefore runs into the line $v = b$ and fails to define a strategy for all values v. Starting at a point on the axis $v = 0$ below the offset solution and moving in the direction of decreasing b, a solution curve either (i) decreases off to $b = -\infty$ if $V'(v, b)$ remains positive along the curve or (ii) reverses direction at a point at which $V'(v, b)$ changes from positive to negative. A solution in case (i) violates the bound (20) that is established in the proof of Theorem 5 for all regular solutions to the first order condition. It must therefore fail to be regular. In case (ii), the solution fails to define a strategy for the entire real line.

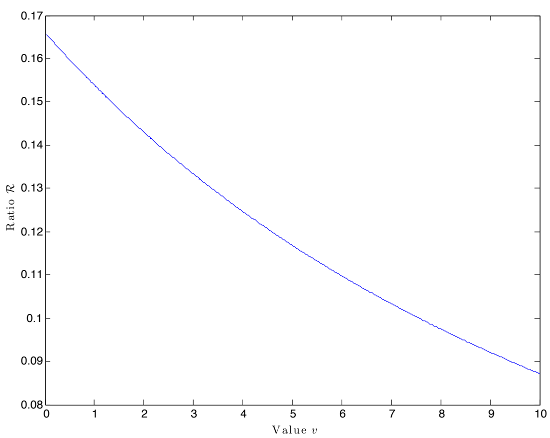
Figures 2 (a)-(d) depict this uniqueness of equilibrium in the case of the standard normal distribution and for different size of market $\tau = 2, 4, 8, 16$ ($m = n = 1$). Each graph is centered on the point $v = b = 0$. The line $v = b$ corresponds to honest revelation. The line parallel and below this line corresponds to the offset solution to the buyer's first order condition for the value of the offset given in Table 1. As in the argument above, a solution curve to the vector field that lies above the offset solution converges into the line $v = b$ and thus fails to define a strategy for all values of v. A solution curve below the offset solution turns back on itself as b decreases at a point where $V'(v, b)$ changes from positive to negative and thus also fails to define a strategy for all values of v. The offset solution is thus the only solution to the buyer's first order condition that defines a strategy across the entire range of buyer values.



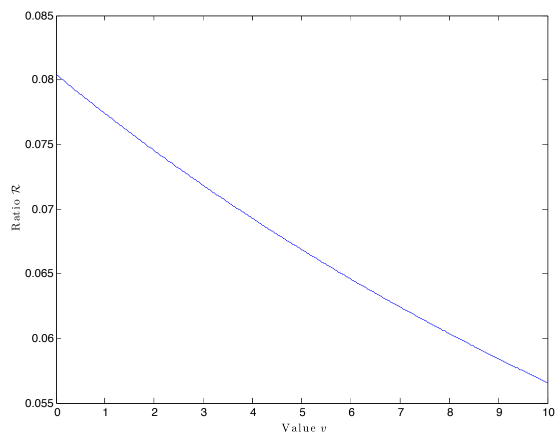
(a) $\tau = 2$



(b) $\tau = 4$



(c) $\tau = 8$



(d) $\tau = 16$

Figure 5: Focal buyer's sufficiency condition vs his value v, for $v - b = \lambda$, F the standard normal distribution, and different size of market τ ($m = n = 1$).

Finally, Figures 5 (a) - (d) depict the ratio $\mathcal{R}$, as given by equation (17), for different values of

v and $b = 0$. The figures suggest that the sufficient condition that $\mathcal{R}$ is decreasing in v as derived in Corrolary 1 is satisfied in the case of the normal distribution.

8 Robustness Check: The Case of the Cauchy Distribution

Assumption A5 bounds the effect of the downward tail of the distribution F upon a focal buyer's marginal analysis. The Cauchy distribution does not satisfy A5 and hence our convergence proofs do not apply in this case. We investigate the BBDA in this section in the case of the standard Cauchy distribution as way to test the dependence of our results upon A5. The Cauchy distribution is commonly used in statistics to test the robustness of results because (i) the graph of its density is bell-shaped and similar in appearance to the density of a normal distribution, but (ii) moments of all orders (including expected value and variance) are not defined for the Cauchy distribution. Point (ii) is due to the the "fat tails" of the Cauchy distribution, which is the reason that it does not satisfy A5. The importance of fat tails in the field of finance also supports the testing of a model of a market mechanism by using the Cauchy distribution.

We find that while the Cauchy distribution is indeed pathological in the bilateral case, the BBDA resolves this pathology in the multilateral case with as few as two traders on each side of the market. A buyer in the BBDA equates the marginal benefit from lowering his bid and thereby decreasing his expected price when he trades to the marginal cost of losing a profitable trade with the lower bid. We first show in the bilateral case that the marginal benefit from lowering his bid always exceeds the marginal cost. This reflects the fat downward tail of the Cauchy distribution, i.e., there always remains a substantial probability of trading for a buyer no matter how much he lowers his bid. As a consequence, a buyer always has the incentive to lower his bid further and further, and so an optimal bid does not exist for him. For $\tau = 2$, 4, 8, and 16 ($m = n = 1$), however, the computation of a buyer's equilibrium offset is straightforward, a buyer's marginal expected utility changes sign from positive to negative at this computed solution, and the computed equilibrium offsets exhibit the $O(1/\tau)$ rate of convergence. This numerical evidence is discussed after Theorem 7, which concerns the bilateral case.

These results are significant for two reasons. First, they suggest that our convergence results hold more generally than we are able to prove in our theorems. Assumption A5 is thus effective but coarse as a sufficient condition for our convergence results. Second, the literature on multilateral double auctions is motivated by the inefficiency of bilateral trade due to strategic behavior as proven by Myerson and Satterthwaite (1983). This inefficiency has been shown to diminish rapidly as a market increases in size. This example points to a different sort of failure in trading in the bilateral case — nonexistence of equilibrium — that is quickly resolved as the number of traders increases. While the outcome of trading may be indeterminant in the case of bilateral bargaining, we can compute a well-defined and well-behaved outcome in the multilateral case. The ability of a market to resolve the indeterminacy of bargaining is particularly striking in this example because of the

Cauchy distribution's pathological properties.

Theorem 7 *Consider the BBDA in the case in which $m = n = 1$ and F is the standard Cauchy distribution (i.e., location parameter* 0 *and scale parameter* 1*). Given honest reporting by the seller, a buyer's marginal expected utility is negative for all bids b below the buyer's value.*

The proof is by direct calculation using the formula for the Cauchy cumulative distribution and density. It is in Appendix B.

Table 2 presents the calculated values of the equilibrium offset λ_{Cauchy} in the case of the standard Cauchy distribution for markets with τ traders on each side. Its value roughly falls by half as τ doubles, reflecting its $O(1/\tau)$ rate of convergence. Column 3 displays the ratio between λ_{Cauchy} and $\lambda_{Gaussian}$, the equilibrium offset in the case of the standard normal distribution from Table 1. In the bilateral case of $m = 1$, the value $\lambda_{Cauchy} = \infty$ reflects Theorem 7, while $\lambda_{Gaussian} = 1.0632$ reflects Theorem 4 (see Footnote 23). While the ratio is infinite in the bilateral case, the equilibrium offsets become comparable in size across the two distributions as soon as the number of traders increases to 2 on a side. Equilibrium bidding in the multilateral case is thus comparable for the two distributions. The fat tails of the Cauchy distribution distinguish it from the normal distribution; while the fat downward tail is crucial in the bilateral case, its impact is bounded and diminishing in significance as the number of traders increases. The tail of the distribution matters less and less in a buyer's decision problem as the number of traders increases as it becomes more likely that the market price will be near the median value/cost. Figure 6 (a)-(d) depict the graph of the focal buyer's marginal utility in the Cauchy case, which changes from positive to negative at the computed offset. Furthermore, Figure 7 (a)-(d) asks further support of the uniqueness of the computed offset by plotting the vector field of the focal buyer $V'(0, b)$ minus 1, and showing how it is uniquely zero at the computed offset. Sufficiency of the first order approach thus appears to hold in the case of the Cauchy distribution even though calculations suggest that this distribution does not satisfy condition (13) for sufficiency of Theorem 2, see Figure 8 (a)-(d).

As a final point, the equilibrium offset does not uniquely maximize a focal buyer's expected utility in the case of the Cauchy distribution. Because of the fat downward tail, the expected price faced by a focal buyer conditional upon his value in this case is negatively infinite when sellers report honestly and the other buyers bid less than their values.[24] As a consequence, all bids $b < v$ provide the focal buyer with an infinite expected utility. In what sense does the computed offset represent utility maximization and equilibrium? The answer lies in the use of a distribution with

[24]The argument for large m is as follows. The distribution of the state μ conditional upon the focal buyer's value v is Cauchy with location parameter v and scale parameter 1. The symmetry of the Cauchy distribution implies that state μ is less than v with probability 1/2. Because of the fat downward tail of the Cauchy distribution, the expected value of μ conditional on being below v is $-\infty$. Regardless of how the focal buyer bids, the price in the BBDA is bounded above by the $(m+1)^{\text{st}}$ smallest value/cost among the $2m - 1$ values/costs of the other traders. This is true because the other buyers bid less than their values. This order statistic converges in distribution to μ as m increases, see Rothenberg, Franklin, and Tilanus (1964)).

an infinite support as an approximation of a finite world. Truncating the Cauchy distribution to the support $[-c, c]$ for $c > 0$ and scaling its density so that its integral over this interval equals one produces a distribution for which a buyer's equilibrium offset and corresponding expected utility are well-defined. This equilibrium offset converges to the value presented in Table 2 as c increases to infinity. Though the limit of expected utility is infinite, the marginal analysis in the Cauchy case derives a meaningful limit of the solution to a buyer's decision problem in the finite case.

τ	λ_{Cauchy}	$\lambda_{Cauchy}/\lambda_{Gaussian}$
2	1.1656	1.6903
4	0.4936	1.4630
8	0.2235	1.4364
16	0.1053	1.3819

Table 2: Comparison of the equilibrium offsets when traders' idiosyncratic noise is Cauchy and Gaussian distributed. Column 1 is the market size τ ($m = n = 1$), Column 2 is the equilibrium offset in the Cauchy case, and Column 3 is the ratio of the equilibrium offsets of the Cauchy and the Gaussian cases.

9 Strategic Error versus Sampling Error

Markets serve both allocative and informative purposes, and strategic behavior by traders can disrupt a market's effectiveness in both respects. We have explored in previous sections the diminishing effect of strategic behavior upon allocative efficiency as the market increases in size. In this section, we demonstrate that strategic behavior is similarly limited in its effect upon a market's ability to aggregate information that is dispersed among its traders.

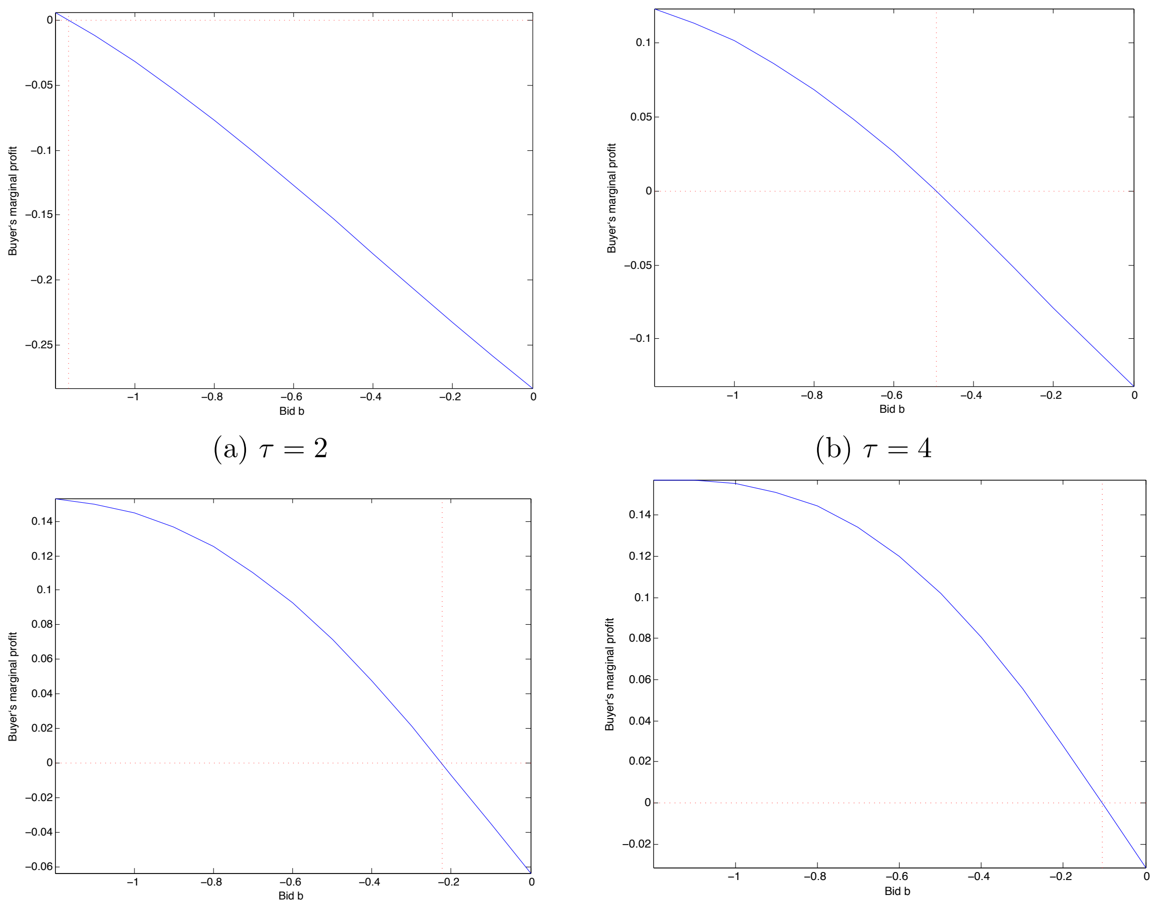
The informational objective here is to estimate the underlying state μ from the market price, for once it is determined, the distribution of values and costs is properly specified. Some notation is needed for this section. For a fixed number m of buyers and n of sellers, we consider markets with τm buyers and τn sellers for $\tau \in \mathbb{N}^*$. Let

$$q = \frac{m}{m+n}$$

and let ξ_q denote the q^{th} population quantile of F, i.e.,

$$F(\xi_q) = q.$$

Let $t_{(\tau m+1)}$ denote the $(\tau m + 1)^{\text{st}}$ order statistic among $\tau(n + m)$ values/costs and $s_{(\tau m+1)}$ the $(\tau m+1)^{\text{st}}$ order statistic among τm bids and τn asks, assuming that buyers use a regular equilibrium strategy and sellers report honestly. The value $s_{(\tau m+1)}$ is the market price in the BBDA. We are



(a) $\tau = 2$

(b) $\tau = 4$

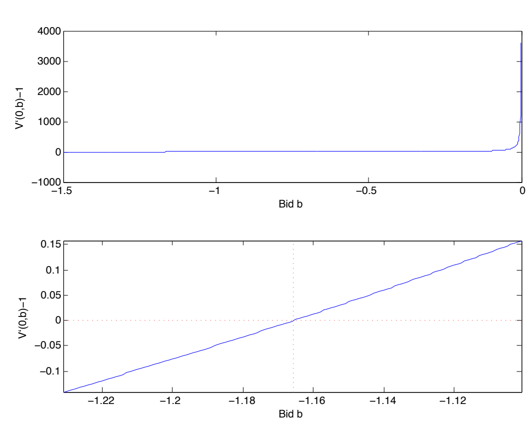
(c) $\tau = 8$

(d) $\tau = 16$

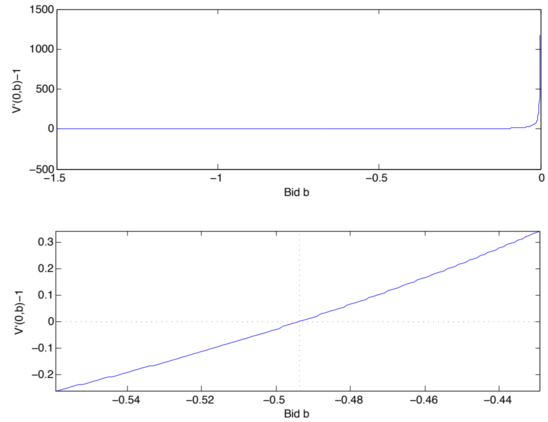
Figure 6: Focal buyer's marginal utility vs his bid b, for $v = 0$, F the standard Cauchy distribution, and different size of market τ ($m = n = 1$).

interested in $s_{(\tau m+1)}$ as an estimate of $\mu + \xi_q$, which is the population quantile given the state μ. It is assumed in our Bayesian game analysis that a trader knows τm, τn and F; he therefore can compute ξ_q, which means that the market price $s_{(\tau m+1)}$ enables a trader to estimate the underlying state μ as $s_{(\tau m+1)} - \xi_q$. This is consistent with Manski (2006), who showed that a competitive market price estimates a quantile of the underlying distribution of private signals that depends upon the relative sizes of supply and demand. Our contribution here is to incorporate strategic behavior into the analysis and then to assess the magnitude of its impact upon the estimation of the population quantile of interest.[25]

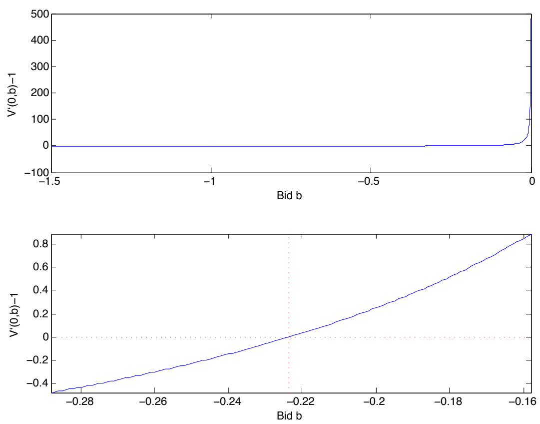
[25]This is a weak notion of identifying μ from the market price $s_{(rm+1)}$. A common objective in a prediction market, for instance, is to produce a price that itself estimates an unknown parameter, such as the probability that a particular event will occur in the future. The success of a prediction market depends upon the design of the contract that is traded, which is beyond the scope of this paper. The price $s_{(rm+1)}$ does approximate μ in the special case of $m = n$, which we explore numerically below.



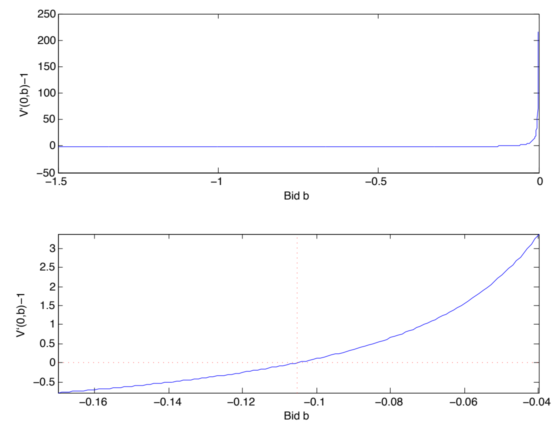
(a) $\tau = 2$



(b) $\tau = 4$



(c) $\tau = 8$



(d) $\tau = 16$

Figure 7: Focal buyer's vector field minus one vs his bid b, for $v = 0$, F the standard Cauchy distribution, and different size of market τ ($m = n = 1$).

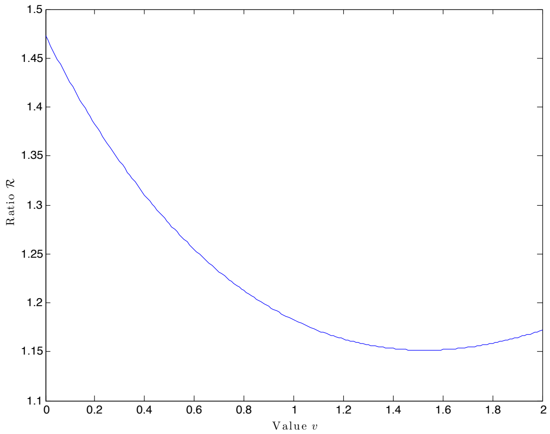
The *absolute error* in the estimation of $\mu + \xi_q$ using $s_{(\tau m+1)}$ is

$$\left| s_{(\tau m+1)} - \xi_q - \mu \right| .$$

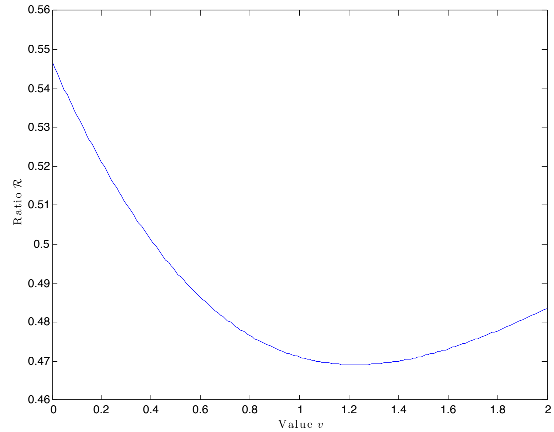
We bound this above as

$$\begin{aligned} & \left| s_{(\tau m+1)} - t_{(\tau m+1)} + t_{(\tau m+1)} - \xi_q - \mu \right| \\ \leq \quad & \left| s_{(\tau m+1)} - t_{(\tau m+1)} \right| + \left| t_{(\tau m+1)} - \xi_q - \mu \right| . \end{aligned} \tag{27}$$

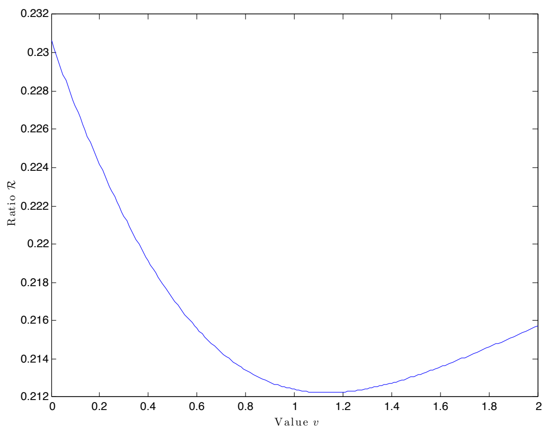
The term $\left| s_{(\tau m+1)} - t_{(\tau m+1)} \right|$ is the *strategic error*, which captures the effect of strategic underbidding of buyers upon the estimation of $\mu + \xi_q$. The term $\left| t_{(\tau m+1)} - \xi_q - \mu \right|$ is the *sampling error*, which reflects the fact that a sample of $\tau\,(m+n)$ values costs does not perfectly reflect the popu-



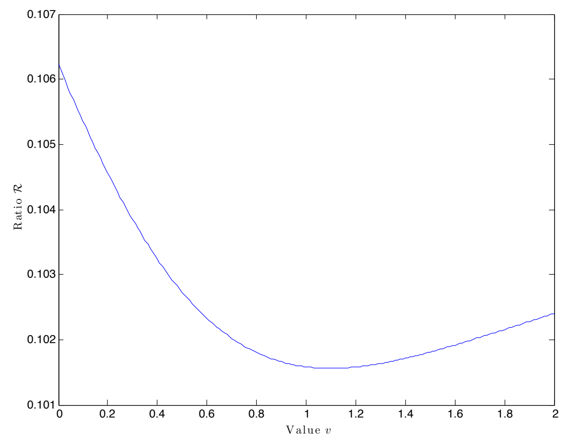
(a) $\tau = 2$



(b) $\tau = 4$



(c) $\tau = 8$



(d) $\tau = 16$

Figure 8: Focal buyer's sufficiency condition vs his value v, for $v - b = \lambda_{Cauchy}$, F the standard Cauchy distribution, and different size of market τ ($m = n = 1$).

lation. Assuming that buyers use a regular equilibrium strategy in each size of market, Theorem 5 implies that

$$\left| s_{(\tau m+1)} - t_{(\tau m+1)} \right| \leq \frac{K(F,\varphi)}{\tau m},$$

i.e., strategic error is a random variable that is $O(1/\tau)$ given the fixed values of m and n. The sampling error in contrast is a random variable distributed on the half line $[0, \infty)$; it can be arbitrarily small or large. The goal in the remainder of this section is to demonstrate that the sampling error on average, however, is of strictly larger order than the strategic error. Asymptotically, the effect of strategic error upon the estimation of $\mu + \xi_q$ will be shown to vanish completely because of the dominating effect of the sampling error, despite the variability of the sampling error.

Theorem 8 *Suppose that the distribution F satisfies the regulatity condition on its tail given by*

$$\lim_{x\to\infty} \inf \frac{-\log(1-F(x))}{\log x} > 0. \tag{28}$$

For fixed m and n and for $\tau \in \mathbb{N}^*$, we consider sequences of equilibria $<\tilde{S}, B>$ in markets with τm buyers and τn sellers in which $\tilde{S}$ denotes honest revelation and B is regular. For each value of the state μ, the sampling error satisfies

$$E\left[\left|t_{(\tau m+1)} - \xi_q - \mu\right| |\mu\right] \in \Theta\left(\frac{1}{\sqrt{\tau}}\right),$$

i.e., there exist constants $\underline{\sigma}(m,n,F)$, $\overline{\sigma}(m,n,F) > 0$ such that

$$\frac{\underline{\sigma}}{\sqrt{\tau}} \leq E\left[\left|t_{(\tau m+1)} - \xi_q - \mu\right| |\mu\right] \leq \frac{\overline{\sigma}}{\sqrt{\tau}} \tag{29}$$

for all $\tau \in \mathbb{N}^*$.

The result is proven for a fixed value of μ, which we set equal to zero in the proof below with no loss in generality. As in Theorem 6, it is reasonable to claim that $E\left[\left|t_{(\tau m+1)} - \xi_q - \mu\right|\right]$ is $\Theta(1/\sqrt{\tau})$ despite the improper distribution of μ because the same terms $\underline{\sigma}/\sqrt{\tau}$, $\overline{\sigma}/\sqrt{\tau}$ bound $E\left[\left|t_{(\tau m+1)} - \xi_q - \mu\right| |\mu\right]$ for each value of μ. We first note that $t_{(\tau m+1)} - \xi_q$ is asymptotically normal after a suitable rescaling, i.e.,

$$\lim_{\tau\to\infty} \Pr\left(\left(t_{(\tau m+1)} - \xi_q\right) \frac{f(\xi_q)\sqrt{\tau(m+n)}}{\sqrt{q(1-q)}} \leq t\right) = \Phi(t),$$

where Φ denotes the standard normal with mean 0 (see, for example, Arnold, Balakrishnan, and Nagaraja (1992, Thm. 8.5.1)). The random variable

$$\left(t_{(\tau m+1)} - \xi_q\right) \frac{f(\xi_q)\sqrt{\tau(m+n)}}{\sqrt{q(1-q)}}$$

thus converges in distribution to the standard normal. The theorem rests upon the convergence of the first absolute moment of this statistic to the corresponding first absolute moment of the limiting standard normal distribution. Condition (28) is sufficient for this convergence. It is satisfied by the normal distribution with mean 0 and variance σ^2, the Laplace distribution with location parameter 0 and scale parameter $b > 0$, and the Cauchy distribution with location parameter 0 and scale parameter $\gamma > 0$.[26]

[26] We verify condition (28) for these distributions using an alternative sufficient condition that is derived as follows.

Proof. Condition (28) insures that[27]

$$\lim_{\tau\to\infty} E\left[\left|t_{(\tau m+1)}-\xi_q\right|\middle|\,\mu=0\right]\cdot\frac{f(\xi_q)\sqrt{\tau(m+n)}}{\sqrt{q(1-q)}}=E_\Phi[|x|]. \tag{30}$$

From (30) it is clear that for sufficiently large τ,

$$\frac{1}{2\sqrt{\tau}}\cdot K\cdot E_\Phi[|x|]<E\left[\left|t_{(\tau m+1)}-\xi_q\right|\middle|\,\mu=0\right]<\frac{2}{\sqrt{\tau}}\cdot K\cdot E_\Phi[|x|],$$

where K denotes the constant

$$K=\frac{\sqrt{q(1-q)}}{f(\xi_q)\sqrt{(m+n)}}.$$

The existence of the constants $\underline{\sigma}(m,n,F)$, $\overline{\sigma}(m,n,F)$ follows immediately. ■

Sampling error is a random variable whose magnitude can be arbitrarily large or small. This contrasts with strategic error, which is $O(1/\tau)$ in every sample of τm values and τn costs. The expected value of sampling error is $\Theta(1/\sqrt{\tau})$, however; it is of strictly larger order than sampling error. The significance of this point is captured by the following theorem, which states that the asymptotic distribution of $s_{(\tau m+1)}$ is the same as the asymptotic distribution of $t_{(\tau m+1)}$. The impact of strategic behavior upon the market price as an estimate of the population quantile $\mu+\xi_q$ thus vanishes completely in an asymptotic analysis because it is dominated by the randomness of sampling. A sufficiently large market thus aggregates information as if buyers were telling the truth instead of acting strategically.

Theorem 9 *For fixed m and n and for $\tau\in\mathbb{N}^*$, we consider sequences of equilibria $<\tilde{S},B>$ in markets with τm buyers and τn sellers in which $\tilde{S}$ denotes honest revelation and B is regular. For each value of μ,*

$$t_{(\tau m+1)},s_{(\tau m+1)}\sim AN\left(\mu+\xi_q,\frac{mn}{\tau(m+n)^3f^2(\xi_q)}\right),$$

Applying L'Hôpital's rule twice,

$$\lim_{x\to\infty}\frac{-\log(1-F(x))}{\log x}=\lim_{x\to\infty}\frac{\frac{f(x)}{1-F(x)}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{xf(x)}{1-F(x)}-\lim_{x\to\infty}\frac{xf'(x)+f(x)}{-f(x)}=\lim_{x\to\infty}\frac{xf'(x)}{-f(x)}-1.$$

The second application of L'Hopital's rule is appropriate only if $\lim_{x\to\infty}xf(x)=0$. If it is instead the case that $\lim_{x\to\infty}xf(x)>0$, then we can stop at that point and conclude that

$$\lim_{x\to\infty}\frac{-\log(1-F(x))}{\log x}>0.$$

It is therefore clear that

$$\lim_{x\to\infty}\frac{xf'(x)}{-f(x)}>1$$

is a sufficient condition for F to satisfy (28).

[27] This result originates in Anderson (1982). We apply here Shorack and Wellner (1986, Theorem 4, p. 475) in the case of $g(x)=|x|$. Condition (28) above is assumption (12) of this theorem and its assumption (14) holds for $g(x)=|x|$ with $M=1$ and $x^*=3$.

i.e., both $t_{(\tau m+1)}$ and $s_{(\tau m+1)}$ are asymptotically normal with the same mean and variance.

The proof is in the Appendix E. The theorem implies that $t_{(\tau m+1)}$ and $s_{(\tau m+1)}$ are each asymptotically unbiased, consistent, and asymptotically normal estimators of $\mu + \xi_q$. The result concerning $t_{(\tau m+1)}$ is well-known and was mentioned above. The contributions of the theorem are (i) the asymptotic normality of $s_{(\tau m+1)}$ and (ii) the fact that $t_{(\tau m+1)}$ and $s_{(\tau m+1)}$ have exactly the same asymptotic distribution. This is true even though $E\left[t_{(\tau m+1)} - s_{(\tau m+1)} \,|\mu\right]$ is strictly positive for all μ.

9.1 Numerical Example

A numerical example suggests that the adverse effect of strategic behavior upon information aggregation can be insignificant, even in markets with small numbers of traders. We compute the market clearing price $t_{(\tau m+1)}$ in the BBDA with no strategic misrepresentation by buyers and the price $s_{(\tau m+1)}$ in the BBDA given the use of an equilibrium offset strategy by buyers (for $m = n = 1$).[28] These calculations are carried out for four values of τ, $m = n = 1$, $\mu = 0$ and $F = \Phi$, the standard normal distribution. The assumption of $m = n = 1$ implies that the population quantile of interest is $\xi_{m/(m+n)} = \xi_{1/2} = 0$, the median of Φ. The market price $s_{(\tau m+1)}$ thus directly estimates the underlying state $\mu = 0$ in this example.

Table 3 compares the variances of $t_{(\tau m+1)}$ and $s_{(\tau m+1)}$ with the asymptotic variance that these random variables share, namely

$$\frac{mn}{\tau(m+n)^3 f^2(\xi_q)} = \frac{1}{8\tau\phi^2\,(0)}.$$

It is notable how close the values of the three variances are to each other even for these small values of τ. This suggests that the asymptotic limit can be meaningful for approximations involving $t_{(\tau m+1)}$ or $s_{(\tau m+1)}$ even in relatively small markets.

Because $\mu = \xi_{1/2} = 0$, the absolute error in this example is $\left|s_{(\tau m+1)}\right|$ and the sampling error is $\left|t_{(\tau m+1)}\right|$, while the strategic error remains equal to $\left|s_{(\tau m+1)} - t_{(\tau m+1)}\right|$. Table 4 presents the expected values of absolute error, sampling error and strategic error for different values of τ. The absolute error differs from the sum of the sampling and strategic errors because of the use of the absolute value in (27). As τ doubles, the absolute and sampling errors decrease by a factor of approximately $\sqrt{2}$ while the strategic error decreases by a factor of approximately 2, which reflects the respective rates of convergence ($1/\sqrt{\tau}$ and $1/\tau$) of these errors. Finally, the table suggests

[28] The calculations are made using a Monte Carlo method: (i) a 2τ-vector of i.i.d. random variables is drawn from $\mathcal{N}(0,1)$; (ii) the first τ are identified as buyers' values and the last τ as sellers' costs so that the vector has the form $(v_1, \ldots, v_\tau, c_1, \ldots, c_\tau)$; (iii) $t_{(\tau m+1)}$ is computed by ordering the components of this vector; (iv) the equilibrium offset $\lambda(\tau)$ that is calculated in Table 1 is subtracted from each value v_i to produce the vector of bids and asks $(b_1, \ldots, b_\tau, c_1, \ldots, c_\tau)$, whose components are then ordered to determine $s_{(\tau m+1)}$. These steps are repeated, which allows the computation of the sample variances and absolute first moments of interest.

that strategic error can be insignificant relative to sampling error in estimating the state μ even in small markets.

τ	$\text{var}\left(t_{(\tau m+1)}\right)$	$\text{var}\left(s_{(\tau m+1)}\right)$	$\frac{1}{8\tau\phi^2(0)}$
2	0.3646	0.3834	0.3927
4	0.1887	0.1901	0.1963
8	0.0954	0.0958	0.0981
16	0.0482	0.0483	0.0491

Table 3: The variances of $t_{(\tau m+1)}$, $s_{(\tau m+1)}$ and their common asymptotic limit for various sizes τ of markets.

τ	Exp. Abs. Error $\mathbb{E}\left[\left\|s_{(\tau m+1)}\right\|\right]$	Exp. Sampling Error $\mathbb{E}\left[\left\|t_{(\tau m+1)}\right\|\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\|t_{(\tau m+1)} - s_{(\tau m+1)}\right\|\right]$
2	0.4926	0.5329	0.3198
4	0.3457	0.3644	0.1679
8	0.2474	0.2549	0.0819
16	0.1756	0.1782	0.0381

Table 4: Absolute, sampling and strategic errors for various sizes τ of markets.

10 An Asymptotic Analysis of a Buyer's Decision Problem

The first order condition (7) is complicated and provides little insight directly into a buyer's decision problem. We now analyze this equation by identifying the asymptotic values of its probabilities. The goal of an asymptotic analysis is to replace a complicated calculation with a simpler formula, thereby identifying what is most fundamental in the problem and facilitating an approximate solution. This goal is accomplished here; this asymptotic analysis in fact models how a sophisticated buyer might approach his decision problem in the BBDA, as opposed to using (7).

Identifying the asymptotic probabilities in the first order condition produces the *asymptotic first order condition*; we call its solution the *asymptotic offset*, which approximates the equilibrium offset that solves (7). Numerical calculations presented below suggest that the asymptotic offset approximates the equilibrium offset quite well even in small markets for the case of the standard normal distribution. More importantly, the asymptotic offset is an explicit formula in the market size that exhibits the rate of convergence of the equilibrium offset to zero. The formula for the asymptotic offset is noteable because there are few closed-form examples of equilibria in the k-double auction literature, which has impeded experimental testing of the theory.[29] Finally, the

[29]In particular, as mentioned in the Introduction, we are aware of no closed-form examples of equilibria in the multilateral case of correlated private values considered in this paper.

fact that this approximation improves in large markets is significant because it is exactly these markets in which is most difficult to compute the equilibrium offset.

The formal asymptotic analysis appears in the proof of the theorem below, which is deferred to Appendix D. We summarize the argument here. For fixed m, n and for $\tau \in \mathbb{N}^*$, let w_τ denote the difference $s_{(\tau m+1)} - s_{(\tau m)}$ in a sample of τn sellers' asks and $\tau m - 1$ buyers' bids. Rewrite the focal buyer's first order condition as

$$\begin{aligned}
0 &= \pi_b(v,b) = (v-b) f_{s_{(\tau m)}}(b|v) - \Pr[s_{(\tau m)} \leq b \leq s_{(\tau m+1)}|v] \\
&= (v-b) f_{s_{(\tau m)}}(b|v) - \Pr[b - s_{(\tau m)} > 0 \ \&\ b - s_{(\tau m)} < w_\tau |v] \\
&= (v-b) \int_{-\infty}^{\infty} f_{s_{(\tau m)}}(b|\mu) f_\mu(\mu|v) d\mu \\
&- \int_{-\infty}^{\infty} \Pr[b - s_{(\tau m)} > 0 \ \&\ b - s_{(\tau m)} < w_\tau |\mu] f_\mu(\mu|v) d\mu.
\end{aligned}$$

The asymptotic distributions of $s_{(\tau m)}$ and w_τ are now substituted into this formula: a suitable rescaling of w_τ is asymptotically exponential (see Siddiqui (1960)) while Theorem (13) (presented in Appendix E) states that $s_{(\tau m)}$ conditional on μ is asymptotically normal with mean $\xi_q + \mu$.[30] The asymptotic first order condition for general distributions F is then

$$\begin{aligned}
0 = \ & (v-b) \tilde{f}_{s_{(\tau m)}}(b|v) \\
& - \frac{2\Delta}{\tau} \int_{-\infty}^{\infty} \frac{1}{1 - \frac{2\Delta}{\Lambda} x} \tilde{f}_{s_{(\tau m)}}(x + \xi_q + \mu|\mu)\, f(v - b + \xi_q + x) dx \\
& - O\left(\frac{1}{\tau^2}\right) \cdot \tilde{f}_{s_{(\tau m)}}(b|v),
\end{aligned} \tag{31}$$

where $\tilde{f}_{s_{(\tau m)}}$ denotes the asymptotic density of $s_{(\tau m)}$ and Δ, Λ are constants that depend on m, n, τ and ξ_q. Formulas for these constants are given by (74) and (73) in Appendix D.

At this point, we restrict attention to the standard normal distribution ($f = \phi$) in order to reduce the integral in (31) and thereby solve the problem in closed form. A Taylor's approximation is applied using the fact that the focal buyer realizes that his strategy is $O(1/\tau)$. These steps produce the asymptotic first order condition for the standard normal case

$$0 = (v-b) \tilde{f}_{s_{(\tau m)}}(b|v) \tag{32}$$

$$- \left(\frac{1}{(m+n)\phi(\xi_q)} \frac{1}{\tau + C(m,n)} + O\left(\frac{1}{\tau^2}\right) \right) \tilde{f}_{s_{(\tau m)}}(b|v) \Leftrightarrow$$

$$0 = (v-b) - \left(\frac{1}{(m+n)\phi(\xi_q)} \frac{1}{\tau + C(m,n)} + O\left(\frac{1}{\tau^2}\right) \right), \tag{33}$$

[30] As before $q = m/(m+n)$ and $\xi_q = \{y : F(y) = q\}$.

where

$$C(m,n) = \frac{mn - (m+n)^2\phi^2(\xi_q)}{(m+n)^3\phi^2(\xi_q)}.$$

The unique solution of (33) is the asymptotic offset

$$\tilde{\lambda}(\tau) = \frac{1}{(m+n)\phi(\xi_q)} \cdot \frac{1}{\tau + C(m,n)} + O\left(\frac{1}{\tau^2}\right). \tag{34}$$

While (31), (33), are (34) are derived under the assumption that $v - B(v)$ is $O(1/\tau)$, these equations do not otherwise depend upon the strategy B. This is true because the magnitude of underbidding by the other buyers vanishes sufficiently fast that it is inconsequential in the focal buyer's decision problem for large τ: the asymptotic distributions of the τm^{th} and $(\tau m+1)^{\text{st}}$ order statistics are the same regardless of whether the sample is bids/asks or true values/costs. In effect, the asymptotic analysis justifies the focal buyer choosing his bid under the hypothesis that the other buyers bid honestly. This substantially simplifies his decision problem. Notice also that the asymptotic offset $\tilde{\lambda}(\tau)$ is a constant that is independent of v. An offset strategy thus uniquely solves the buyer's asymptotic first order condition, which motivates our focus upon this form of strategy in all sizes of markets.

Theorem 10 *Consider a sequence of markets with τm buyers and τn sellers for $\tau \in \mathbb{N}^*$ and the special case in which $F = \Phi$ ($f = \phi$). If the other buyers employ a regular strategy $B_\tau(\cdot)$ such that $v - B_\tau(v)$ is of order $O(1/\tau)$ for all $v \in \mathbb{R}$, then $\tilde{\lambda}(\tau)$ as given in (34) is the unique solution to the focal buyer's asymptotic first order condition (32). The asymptotic offset $\tilde{\lambda}(\tau)$ is further approximated by the simpler formula*

$$\lambda_{\text{approx}}(\tau) = \frac{1}{(m+n)\phi(\xi_q)} \frac{1}{\tau}, \tag{35}$$

in the sense that

$$\lim_{\tau\to\infty} \frac{\tilde{\lambda}(\tau)}{\lambda_{\text{approx}}(\tau)} = 1.$$

Numerical Example. In the case of $m = n = 1$, we have that $q = 1/2$, $\xi_{1/2} = 0$, $f(0) = \phi(0) = \sqrt{1/(2\pi)}$, and $C(1,1) = \pi/4 - 1/2$, so that

$$\lambda_{\text{approx}}(\tau) = \sqrt{\frac{\pi}{2}} \frac{1}{\tau}.$$

Fig. 9 depicts this approximate solution λ_{approx} as compared to the corresponding numerical calculation $\lambda(\tau)$ obtained from the first order condition (7).Table 10 reports the error and the relative error in this approximation. We see that the (relative) error in the approximation diminishes fast so that the approximate solution performs well even for modest market sizes.

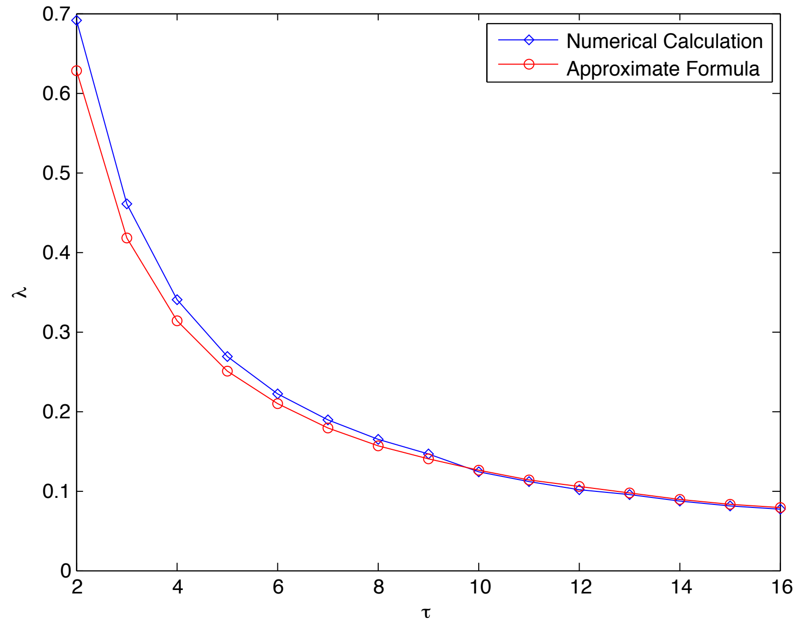


Figure 9: Approximate Solution.

τ	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	0.6896	0.6267	0.0629	0.0913
4	0.3374	0.3133	0.0241	0.0713
8	0.1640	0.1567	0.0073	0.0447
16	0.0805	0.0783	0.0022	0.0269

Table 5: Approximate Solution.

11 Correlated Interdependent Values

We extend our model in this section to the case in which traders have interdependent values/costs and correlated private signals (CIV). As before there are m buyers and n sellers whose values/costs are generated as follows: buyer i's value is $v_i = \mu + \varepsilon_i$ and seller j's cost is $c_j = \mu + \varepsilon_j$ where the state μ is drawn from the uniform improper prior on $\mathbb{R}$ and the terms ε_i, ε_j are i.i.d. according to the distribution F on $\mathbb{R}$. In this section, however, a trader does not know his value/cost at the time that he chooses his bid/ask. Instead each trader privately observes a noisy signal concerning his value/cost prior to his participation in the BBDA. The signals are generated as follows: buyer

i's signal σ_i equals

$$\sigma_i = v_i + \delta_i = \mu + \varepsilon_i + \delta_i,$$

and seller j's signal σ_j equals

$$\sigma_j = c_j + \delta_j = \mu + \varepsilon_j + \delta_j,$$

where the noise components δ_i, δ_j are i.i.d. according to the distribution G on $\mathbb{R}$. We assume that G is absolutely continuous with density $g > 0$ on $\mathbb{R}$ and mean zero. The signals are therefore unbiased estimators of a trader's value/cost: $\mathbb{E}[v_i|\sigma_i] = \sigma_i$ and $\mathbb{E}[c_j|\sigma_j] = \sigma_j$. The presence of the noise components make traders' values/costs interdependent: if buyer i could somehow learn seller j's signal σ_j, then his expected value $\mathbb{E}[v_i|\sigma_i, \sigma_j] \neq \sigma_i$ varies with σ_j. The reason is that σ_j allows i to update his estimate of μ, the result of which causes him to revise his estimate of his own value v_i.

The BBDA, however, requires traders to submit simultaneously and privately their asks/bids, which prevents each from securing direct information about others' signals. But immediately after the simultaneous submission of asks/bids, each trader does observe the realized market price p and resulting alllocation of the n objects available in the market. This p provides new information about the magnitude of his own signal relative to the signals of the other traders because the realized price is either the m^{th} or the $(m+1)^{\text{st}}$ order statistic of the asks/bids the other traders submit.

A rational trader anticipates the revelation of this information at the interim stage, i.e., after he knows his signal but before the market price and allocation are realized. His choice of ask/bid, at the interim stage, essentially matters only in the unlikely event that his planned ask/bid happens to be very close to the realized price. This is because small adjustments in his ask/bid in that case determine whether he trades or not. However, although this effect were true for buyers also in the CPV case, an additional effect is present for both sellers and buyers in the CIV case. Essentially adding the noise components δ enriches each trader's decision problem with the issue of inferring from the event of trading. To see this recall that since the market price is the $(m+1)^{\text{st}}$ order statistic of the submitted asks/bids it asks a lot of information about the common value μ; better knowledge of μ allows traders to better estimate their own value/cost. These anticipatory adjustments allow them to (partially) protect against ex post regret, i.e., the well known winner's or loser's curse.

Hence, while the BBDA continues to give a seller the incentive to set his ask a_j equal to his expectation of his cost c_j, in general this expectation no longer equals σ_j as a result of what he anticipates learning from the realized price. Similarly, while the BBDA continues to give a buyer the incentive to set his bid b_i below his expectation of v_i in general this expectation no longer equals his signal σ_i.

Our goal in this section is not to provide a full treatment of the CIV case. We feel this is redundant given the richness of the results provided in the CPV case thus far and the natural connection between the CIV and CPV cases. To this effect we wish to highlight the main points of departure from the CPV case. Basically the lack of traders' knowledge of their value/cost and the requirement for anticipatory inferences leads us to the introduction of the *price taking bid/ask* in the next subsection. There we intuitively — and without resorting to strict arguments — describe how a trader's decision problem can be decomposed into two stages, and how his price taking bid/ask plays the same role in the CIV case that a trader's value/cost played in the CPV case. We also exhibit that the price taking bid/ask is not, in general, equal to a trader's signal for reasons that have to do with the relative sizes of supply and demand. Then in subsection 11.2 we present the first order approach not only for buyers (as in the CPV case) but also for sellers. The first order approach allows us to formalize some of the intuition we built in subsection 11.1 and also provides the necessary machinery for the numerical results of subsection 11.3. The numerical results show how simple offset equilibria exist in this case as well, further validating their importance and prevalence in this environment.

11.1 Some Intuition

Let us begin with the sellers' problem, which is simpler. The first step for a focal seller — given m and n, the strategies of the other traders, and the underlying joint distribution of their signals and values/costs — is to construct the function $\mathbb{E}[c|\sigma, x]$ where x is the m^{th} order statistic of the asks of the $n-1$ remaining sellers, and the bids of all m buyers. To trade a seller's ask has to be less than x. We call the focal seller's ask, a, *pivotal* when a happens to equal x, i.e., when the focal seller's ask is equal to the market price. For signal $\sigma \in \mathbb{R}$, define the seller's *price-taking ask* to be the ask a_{PT} that solves the fixed point equation:

$$a_{\text{PT}} = \mathbb{E}[c|\sigma, x = a_{\text{PT}}], \tag{36}$$

i.e., given σ, he asks exactly his expected value for the good in the event that his ask happens to be pivotal.

The second step for the focal seller, given his expected cost a_{PT}, is to decide on the ask he will actually report. This is easy for him because the trading institution is the BBDA and as a seller he cannot affect the price at which he trades. His optimal strategy is therefore to ask a_{PT}, his value conditional on σ and being pivotal. Doing so is the essence of price taking. Given a market price p, then the seller wants to sell his unit of supply if and only if his expected cost is less than or equal to the realized price. In the BBDA if $a_{\text{PT}} < x$, then x becomes the realized price p, the focal seller trades, and earns positive utility. If $a_{\text{PT}} = x$, then x becomes the price p, whether the focal seller trades depends on the realization of a random draw, but in any case he is indifferent whether or not he trades. If $a_{\text{PT}} > x$, then the focal seller does not trade, which is fine since the return from trading would be negative. By reporting a_{PT} he obtains exactly the outcome he seeks even though he must report his ask prior to the price being realized. Notice that asking a_{PT} is *not* a dominant strategy since its computation relies on the m^{th} order statistic of other traders' ask/bids, which in turn depends on their equilibrium strategies.

Now focus on buyers. Paralleling the focal seller's computation of a_{PT}, a focal buyer computes b_{PT} as a function of his signal $\sigma \in \mathbb{R}$. This gives him his *price-taking bid* which solves the fixed point equation

$$b_{\text{PT}} = \mathbb{E}[v|\sigma, x = b_{\text{PT}}], \tag{37}$$

i.e., the right hand side is the updated expectation over his value, given his signal and being pivotal. Note that here, however, x is the m^{th} order statistic of a different population than for a seller: the population for the focal buyer are the asks of all n sellers, and the bids of the $m-1$ remaining buyers. For a buyer the second step is more complicated because, as in the CPV case, he can affect the price and so acts strategically to choose his bid b that is less than his price-taking bid b_{PT} and that maximizes his expected utility.[31] Observe that if — counterfactually — buyers and sellers both reported their price-taking asks and bids, then there would be no asymmetry between a focal buyer and a focal seller. Specifically if a buyer and a seller each received the identical signal σ, then the seller's price-taking ask a_{PT} would equal the buyer's price-taking bid b_{PT}.

The implications of interdependence on the traders' price-taking asks and bids are most easily seen in the limiting case where we have τm buyers and τn sellers and $\tau \to \infty$ while the ratio of $m/(m+n) = q$ is held fixed. If all traders report their price-taking asks/bids, which is optimal in the limit, then the price as well as the relevant order statistics converge to $\mu + \xi_q$ where $\tilde{H}(\xi_q) = q$ and $\tilde{H}$ is the mixture distribution (normalized around the common component μ) of traders' price-taking asks and bids. Now ξ_q is a function of the relative size of the market. Given that both F and G are zero mean, if $q > 0.5$ (more buyers than sellers), then $\xi_q > 0$. If $q = 0.5$, then $\xi_q = 0$ and if $q < 0.5$ then $\xi_q < 0$.

Thus, for instance, if $q > 0.5$, then in the limit the market clearing price would converge to $\mu + \xi_q > \mu$ if all traders reported their signals honestly. But each trader with $\sigma \approx \mu + \xi_q$ understands that in expectation he received a positive noise term δ and therefore his signal overestimates his value/cost. Hence for such traders $a_{\text{PT}} < \sigma$ for sellers and $b_{\text{PT}} < \sigma$ for buyers, i.e., all traders correct their signals downward before reporting their bids/asks. Similarly if $q < 0.5$, then all traders correct their signals upward. Only when $q = 0.5$ would traders not correct their signals in the limit; $\mu + \xi_q = \mu$, their signals equal their price taking asks and bids, and realized price is μ, the median of the reported bids/asks.

[31]The artificiality of describing the decision as a two step process occurs here. For the buyer choosing the bid b less than b_{PT} changes his calculation of his price-taking bid. Consequently choosing the bid b and computing the function b_{PT} must be done simultaneously. This becomes explicit when we present the solution to the buyer's first order condition in the next subsection.

11.2 First Order Approach

We formally derive the first order conditions for the strategies S and B to define an equilibrium in the CIV case in Appendix F. Our objective below is to present these first order conditions as they compare to the conditions derived in the CPV case in Section 3.

11.2.1 Buyer's Expected Utility and First Order Condition

Given signal σ, the focal buyer chooses his bid b to maximize his expected utility. Let x denote the m^{th} and y the $(m+1)^{\text{st}}$ order statistics in a sample of $m-1$ buyers using strategy B and all n sellers using strategy S. Conditional on the focal buyer's signal σ, let $f_\mu(\mu|\sigma)$ denote the density of the state μ conditional on the signal σ, and let $f_x^B(x|\cdot)$, $f_y^B(y|\cdot)$, and $f_{xy}^B(x,y|\cdot)$ be the conditional densities of x, y, and the joint of x and y respectively. We index the densities and the utility in the CIV case with a superscript B to distinguish them from the densities in a focal seller's problem as discussed below. This distinction was not needed in the CPV case because the strategy of each seller was straightforward to characterize. The focal buyer's interim expected utility is

$$\begin{aligned}\pi^B(\sigma,b|B,S) \;&=\; \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b} f_y^B(y|\mu)\left(\mathbb{E}\left[v|\mu,\sigma\right]-y\right)dy\right. \\ &\quad\left.+\int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)\left(\mathbb{E}\left[v|\mu,\sigma\right]-b\right)dydx\right\} f_\mu(\mu|\sigma)d\mu.\end{aligned} \tag{38}$$

To gain insight into this equation, we contrast it with the corresponding equation from the CPV case:

$$\begin{aligned}\pi(v,b|B) \;&=\; \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b} f_y(y|\mu)\left(v-y\right)dy\right. \\ &\quad\left.+\int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}(x,y|\mu)\left(v-b\right)dydx\right\} f_\mu(\mu|v)d\mu.\end{aligned} \tag{39}$$

We first note that the buyer's expected utility in (38) now depends upon his signal and both strategies B and S, whereas in (39) it depends upon his private value v and the strategy B (with the dependence upon honest revelation by each seller suppressed in the notation). The first term in each of these two integrals represents the expected utility to the buyer when he trades at a price below his bid b and the second line represents the expected utility when his bid b sets the price. The integral in the CIV case is computed with respect to the conditional density $f_\mu(\mu|\sigma)$ as opposed to $f_\mu(\mu|v)$, which reflects the difference in his private information between the two cases. Finally, the focal buyer's private value v in (39) is replaced by $\mathbb{E}\left[v|\mu,\sigma\right]$ in (38), reflecting his uncertainty about his value at the time he bids in the CIV case.

Formula (38) implies the following first order condition for the focal buyer's optimal choice of a bid:

$$\pi_b^B(\sigma, b|B, S) = \int_{-\infty}^{\infty} \left\{ \left(\mathbb{E}\left[v|\mu, \sigma\right] - b\right) f_x^B(b|\mu) - \int_{-\infty}^{b} \int_{b}^{\infty} f_{xy}^B(x, y|\mu) dy dx \right\} f_\mu(\mu|\sigma) d\mu = 0. \quad (40)$$

We compare this to the first order condition in the CPV case:

$$\pi_b(v, b|B) = \int_{-\infty}^{\infty} \left\{ (v - b) f_x(b|\mu) - \int_{-\infty}^{b} \int_{b}^{\infty} f_{xy}(x, y|\mu) dy dx \right\} f_\mu(\mu|v) d\mu = 0. \quad (41)$$

The first term in each integral is the marginal gain from acquiring an item at price b while the second term is the marginal cost from raising the price at which he trades. The marginal gain is now $\mathbb{E}\left[v|\mu, \sigma\right] - b$.

Finally, we note that the densities are expanded in Appendix F in terms of the probabilities $M_{n,m}^B$, $K_{m,n}^B$, $L_{m,n}^B$. These have the same meaning as presented in the CPV case with two important differences in the formulas: (i) they now depend not only on B' and B^{-1} but also on (the unknown) S' and S^{-1}; (ii) instead of F, they are now defined in terms of the convolution $H = F \star G$ of the distributions F and G, which is the distribution of $\varepsilon + \delta$.

Notice that (40) implies that

$$\begin{aligned} b &= \frac{\int_{-\infty}^{\infty} \mathbb{E}\left[v|\mu, \sigma\right] f_x^B(b|\mu) f_\mu(\mu|\sigma) d\mu}{\int_{-\infty}^{\infty} f_x^B(b|\mu) f_\mu(\mu|\sigma) d\mu} - \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{b} \int_{b}^{\infty} f_{xy}^B(x, y|\mu) dy dx f_\mu(\mu|\sigma) d\mu}{\int_{-\infty}^{\infty} f_x^B(b|\mu) f_\mu(\mu|\sigma) d\mu} \Rightarrow \\ b &= \mathbb{E}[v|\sigma, x = b] - \frac{\Pr[x < b < y|\sigma]}{f_x^B(b|\sigma)}. \end{aligned} \quad (42)$$

As mentioned in Section 11.1 the event $\{x = b\}$ is when the focal buyer is *pivotal*, i.e., when he sets the price (and bidding infinitesimally less would lead to no trade). Hence, the first part of the left hand side of (42) is the updated expectation of his value given his signal and being pivotal; this is exactly the pricing-taking bid b_{PT} as defined before in (37). The second term reflects the buyer's *price impact* in the BBDA. Of course the focal buyer's bid is not going to be equal to b_{PT} since that satisfies (37), while the optimal bid satisfies (42). It is also clear then that the buyer in the CIV will underreport his "best estimate" of his value by a positive term similar to the one in the CPV case where his optimal bid was given by the fixed point equation:

$$b = v - \frac{\Pr[x < b < y|v]}{f_x(b|v)},$$

the main difference being the buyer's knowledge of his value v.

11.2.2 Seller's Expected Utility and First Order Condition

Reflecting the pricing rule of the BBDA, a seller's expected utility and marginal expected utility differ from a buyer's in that it does not have a term that corresponds to trade in which his ask sets the price. In the CIV case, however, a seller must account for the fact that the event of trading affects his estimate of his own cost.

Consider a focal seller with signal σ and ask a. Let x now denote the m^{th} order statistic and $f_x^S(x|\cdot)$ its conditional density for a sample of all m buyers using strategy B and $n-1$ sellers using strategy S. The seller's ex post utility is determined by the relative sizes of a and x:

- if $a < x$, then the seller trades at the price of x and receives $x - c$;
- if $a \geq x$, then the seller may or may not trade (based on a random draw) but irrespectively receives 0.

The ex ante utility of the seller is $(x - c)\chi_{a<x}$, and so his interim expected utility is,

$$\pi^S(\sigma, a; B, S) = \int_{-\infty}^{\infty} \left\{ \int_a^{\infty} (x - \mathbb{E}[c|\mu, \sigma])\, f_x^S(x|\mu) dx \right\} f_\mu(\mu|\sigma) d\mu. \tag{43}$$

Taking the derivative with respect to a one gets the first order condition

$$\pi_a^S(\sigma, a; B, S) = \int_{-\infty}^{\infty} \left\{ (a - \mathbb{E}[c|\mu, \sigma])\, f_x^S(a|\mu) \right\} f_\mu(\mu|\sigma) d\mu = 0. \tag{44}$$

The term $\mathbb{E}[c|\mu, \sigma]$ reflects the seller's uncertainty about his cost c. In the private values case, it reduces to c, which produces the optimal ask of $a = c$. While the term $\mathbb{E}[c|\mu, \sigma]$ does not depend upon the strategies B and S, they are implicit in the density $f_x^S(a|\mu)$. Because x is the price at which the focal seller sells when he trades, this density captures the effect of the event of trading upon his calculation of his marginal expected cost. Similarly as for a buyer from (44)

$$\begin{aligned} a &= \frac{\int_{-\infty}^{\infty} \mathbb{E}[c|\mu, \sigma]\, f_x^S(a|\mu) f_\mu(\mu|\sigma) d\mu}{\int_{-\infty}^{\infty} f_x^S(a|\mu) f_\mu(\mu|\sigma) d\mu} \Rightarrow \\ a &= \mathbb{E}[c|\sigma, x = a]. \end{aligned} \tag{45}$$

As mentioned in Section 11.1 the event $\{x = a\}$ is when the focal seller is *pivotal*, i.e., when she sets the price (and asking infinitesimally less would lead to trade). Hence, this is exactly the pricing-taking ask a_{PT} as defined in (36). Non-surprisingly this is the seller's optimal ask given the rules of the BBDA. As noted before, in (42) and (45) x refers to the m^{th} order statistic of (in general) different populations.

Invariance Property. Setting $b = B(\sigma)$ and $a = S(\sigma)$ in (40) and (44) produces a pair of differential equations in the equilibrium strategies B, S that are presented in Appendix F. Our goal in this section is to explore the robustness of the conclusions we reached in the private values case to the introduction of interdepence among values/costs. We therefore conjecture a similar form of offset equilibrium for the CIV model in which B and S have the form

$$\begin{aligned} B(\sigma_B) &= \sigma_B - \lambda_B, \\ S(\sigma_S) &= \sigma_S + \lambda_S, \end{aligned} \tag{46}$$

where λ_B, λ_S are constants that are not restricted to be positive.

Assuming that traders use offset strategies of the form (46), the following lemma states that the buyer's marginal utility $\pi_b^B(\sigma_B, b; B, S)$ and the seller's marginal utility $\pi_a^S(\sigma_S, a; B, S)$ do not depend upon the values of the signals σ_B and σ_S. As a consequence, by solving numerically for offsets λ_B and λ_S that equate both marginal utilities to zero for particular values of the signals σ_B and σ_S, we obtain offsets that solve the first order conditions at *all* σ_B and σ_S. The strategies (46) defined by these offsets therefore satisfy the first order conditions for equilibrium at all σ_B and σ_S, with only the sufficiency of the first order approach remaining to be verified in order to demonstrate equilibrium. The lemma therefore substantially simplifies the calculation of an equilibrium.

Lemma 4 *Assume that buyers and sellers use the offset strategies in (46).The marginal utilities of a focal buyer and focal seller in this case satisfy*

$$\begin{aligned} \pi_a^S(\sigma_S, \sigma_S + \lambda_S; B, S) &= \pi_a^S(\sigma_S + \rho_S, \sigma_S + \rho_S + \lambda_S; B, S), \\ \pi_b^B(\sigma_B, \sigma_B - \lambda_B; B, S) &= \pi_b^B(\sigma_B + \rho_B, \sigma_B + \rho_B - \lambda_B; B, S) \end{aligned}$$

for all $(\rho_B, \rho_S) \in \mathbb{R}^2$.

The lemma is proven in Appendix F.

11.3 Numerical Results

For our numerical results, we take $\varepsilon_i, \varepsilon_j \sim N(0, e)$, $\delta_i, \delta_j \sim N(0, d)$ for all i and j. Then,

$$\begin{aligned} f_\mu(\mu|\sigma) &= \frac{1}{\sqrt{e+d}} \phi\left(\frac{\mu - \sigma}{\sqrt{e+d}}\right), \\ \mathbb{E}\left[v|\mu, \sigma\right] &= \mathbb{E}\left[c|\mu, \sigma\right] = \frac{e\sigma + d\,\mu}{e+d}. \end{aligned}$$

For $d \to 0$ we get the CPV model studied in the previous sections.

We produce values of the buyers' and sellers' offset for different values of m and n in Tables 6 and 7. To verify that these are equilibrium strategies we plot the utility and marginal utility of the focal buyer as a function of his bid (for $\sigma = 0$), conditional on other traders following the offset strategy (λ_B, λ_S), see for example Fig. 10 (a) for $m = n = 8$, $e = 1$ and $d = 0.25$. The vertical line signifies λ_B. We observe that the focal buyer's utility is (uniquely) maximized at $b = \sigma - \lambda_B$ and hence his best response is to imitate the other buyers' strategy. Now, in Fig. 10 (b) we depict the counterpart for the focal seller. Again, the seller uniquely chooses to ask $a = \sigma + \lambda_S$. Therefore, the computed (λ_B, λ_S) constitute an offset equilibrium.

$m \backslash n$	2	4	8	16
2	$1.3404, 0.4124$	$0.8372, 0.4912$	$0.3642, 0.6508$	$-0.0361, 0.8546$
4	$1.2189, 0.1332$	$0.7036, 0.2172$	$0.2657, 0.3948$	$-0.1128, 0.6192$
8	$1.2084, -0.1712$	$0.7431, -0.0787$	$0.3417, 0.1091$	$-0.0212, 0.3494$
16	$1.3011, -0.4677$	$0.8853, -0.3756$	$0.5175, -0.1886$	$0.1754, 0.0614$

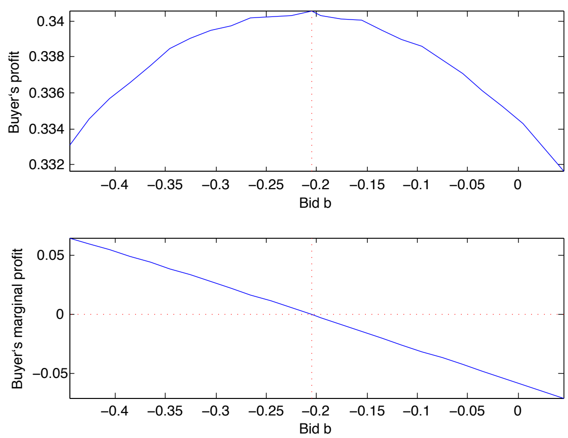
Table 6: Results for buyers' and sellers' offset λ_B, λ_S, for different values of m and n ($e = 1$, and $d = 1$).

Observing the values in Tables 6 and 7 recall that a buyer weighs two effects in choosing his offset: (i) the possibility of affecting price (price impact); (ii) the estimation of his value given his signal and the event that he trades (price taking). Holding n constant and increasing the number m of buyers, effect (i) diminishes as the likelihood that a buyer influences price goes to zero, while effect (ii) increases as a buyer who trades knows that more and more signals $(m - 1)$ are below his when he trades. As we go down a column in either of these two tables we therefore observe a buyer's offset first decreases due to effect (i) and then increase as effect (ii) dominates. Interestingly, a seller's offset monotonically increases for fixed m, as n increases. As n increases, a seller who trades knows that there are an increasing number of signals above his. The effect of increasing n is therefore to provide increasing evidence to the seller who trades that his signal is relatively low, causing him to increase his offset. A seller never affects the price at which he trades and so there is no effect similar to (i) in his selection of his offset.

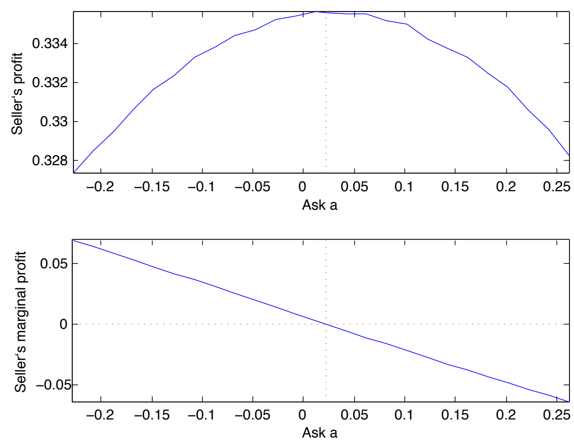
m	λ_B, λ_S
2	$1.7416, 0.4076$
4	$1.7164, 0.1532$
8	$1.7372, -0.1367$
16	$1.8075, -0.4257$
32	$1.9118, -0.6945$
64	$2.0336, -0.9383$
128	$2.1629, -1.1595$

Table 7: Results for buyers' and sellers' offset λ_B, λ_S, for $n = 1$ and different values of m ($e = 1$, and $d = 1$).

Moreover, Table 7 reflects exactly why a double auction with one seller and m buyers is *not* a standard one-sided auction. The "reservation price" of the seller, given by his offset in this case, is neither a given nor an unknown constant, but rather an equilibrium quantity, i.e., it depends on the buyers' optimal strategy. From a methodological perspective, as we mentioned, this implies that a buyer's signal in this case is not affiliated with the maximum, i.e., the m^{th} order statistic, of other buyers' optimal bids and the seller's "reservation price", as it is the case for example, in a standard first price auction. Nonetheless, offset equilibria exist and so our methodology can be used to compute equilibria of one-sided auctions with an endogenous reservation price.



(a) Buyer



(b) Seller

Figure 10: Focal trader's utility and marginal utility. Other sellers use the offset $\lambda_S = 0.0220$, and other buyers use the offset strategy $\lambda_B = 0.2053$, for $m = n = 8$, $e = 1$ and $d = .25$ ($\sigma = 0$).

We now turn our attention to the symmetric case of equal numbers of buyers and sellers $m = n$. We fix $m = n = 1$ and consider different market sizes scaled by $\tau \in \mathbb{N}^*$ as before. So for different τ in Table 8 we report the terms corresponding to effects (i) and (ii) for buyers (Columns 2 & 3) and effect (i) for sellers (Column 4), which is the only effect for them. These of course correspond to the terms we identified in equations (42) and (45). In particular for buyers it is the sum of the price taking and the price impact terms that produces their equilibrium bid, while for sellers the price taking term is exactly their equilibrium ask, i.e.,

$$
\begin{aligned}
-\lambda_B &= \mathbb{E}[v|0, x = -\lambda_B] - \frac{\Pr[x < -\lambda_B < y|0]}{f_x^B(-\lambda_B|0)}, \\
+\lambda_S &= \mathbb{E}[c|0, x = +\lambda_S],
\end{aligned}
$$

The equilibrium (λ_B, λ_S) can be also read from the diagonal terms in Table 6. Now the buyer's price impact term does not depend on σ, this follows from similar arguments that led to the invariance result, see Lemma 4. On the other hand, as one would suspect, both the buyer's and the seller's

τ	$\mathbb{E}[v\|0, x = -\lambda_B]$	$\frac{\Pr[x<-\lambda_B<y\|0]}{f_x^B(-\lambda_B\|0)}$	$\mathbb{E}[c\|0, x = +\lambda_S]$
2	-0.4125	0.9279	0.4125
4	-0.2172	0.4864	0.2172
8	-0.1091	0.2326	0.1091
16	-0.0615	0.1139	0.0615

Table 8: Results for buyers' price taking term (Column 2), price impact term (Column 3), and seller's price taking term (Column 4) vs the market scale factor τ (Column 1) for $m = n = 1$, $e = 1$, and $d = 1$.

price taking terms do depend on the signal (recall that λ_B and λ_S are constants that do not depend on σ). However, for these offset equilibria the dependence is pretty trivial in the sense that[32]

$$\begin{aligned}\mathbb{E}[v|\sigma, x = \sigma - \lambda_B] &= \sigma + \mathbb{E}[v|0, x = -\lambda_B],\\ \mathbb{E}[c|\sigma, x = \sigma + \lambda_S] &= \sigma + \mathbb{E}[c|0, x = +\lambda_S].\end{aligned}$$

Also from the table it is pretty clear that[33]

$$\mathbb{E}[v|0, x = -\lambda_B] = -\mathbb{E}[c|0, x = +\lambda_S].$$

So in Table 8 we isolate the effect in the price taking term that comes from pure hedging rather than from the different value of a trader's signal. It is now notable that all terms decrease as we go down the rows of Table 8 at what appears to be a $O(1/\tau)$ rate. The price impact term of Column 3 captures the rate at which the strategic effect of buyers goes away, and hence the $O(1/\tau)$ rate is the same as in the CPV case. The fact that the price taking terms for both buyers and sellers also go away at a rate of $O(1/\tau)$ reflects the increasing meaningfulness of the price (the sample median in this symmetric case) as an estimate of the state μ as the sample size increases.[34] This may see to contradict our finding in Section 9 that the estimation error (of μ) goes away at a rate $\Theta(1/\sqrt{\tau})$ so a slower rate than $O(1/\tau)$. However, note that the price taking asks/bids are a hedge to *others'* estimate of μ and so they are essentially the best estimate of the market's estimate, and not of μ directly. Hence, intuitively we can see why the rate is the square of the estimation rate of μ (this is similar to our arguments in proving allocation efficiency, see Theorem 26).

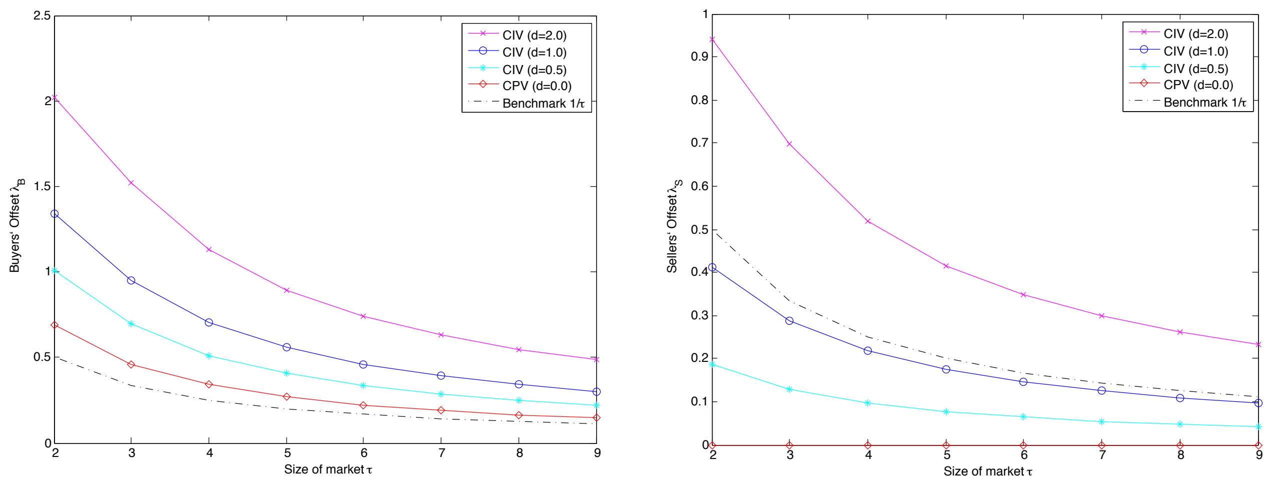
Finally, in Figures 11 (a) and (b) we depict the evolution of buyers' and sellers', respectively, offset strategy for size of the market τ for different values of variance of δ, d. For $d = 0$ a trader's signal is equal to her value and hence this is the CPV case. As was proved, and depicted in Fig.

[32] This is straightforward by observing the expressions for these terms provided in Appendix F.

[33] In these equations the x are *different* random variables.

[34] As τ increases, a seller who sells knows that an increasing number of signals are above his (τm) but this is counterweighted by the increasing number of signals that are lower ($\tau m - 1$). Similarly for a buyer.

11 (b), in that case sellers report their true values/signals and so the offset is zero for all values of τ, and buyers underbid strategically. For a specific value of τ we see that both buyers' and sellers' offset increase as the value of d increases. This follows from the fact that the more unsure traders are of their valuation over the asset the more they want to hedge against this uncertainty. Moreover, for a given value of d and τ, the buyers' offset is higher than the sellers' offset since for buyers the offsets has also the strategic component (price impact) besides the hedging component (price taking). As τ grows for all values of d both λ_B and λ_S converge to zero (since $m = n$ we have $\xi_q = 0$ and for $\sigma = 0$, $\lim_{\tau\to\infty} a_{\mathrm{PT}} = \lim_{\tau\to\infty} b_{\mathrm{PT}} = 0$) and according to the graphs this is at a rate of $1/\tau$ (benchmark line), that is λ_B and λ_S are $O(1/\tau)$ as was mentioned before.



(a) Buyers

(b) Sellers

Figure 11: Rate of Convergence to Truth Telling in the CPV & CIV case for different size markets ($m = n = 1$ and $e = 1$).

12 Conclusion

The double auction with its two sided incomplete information, correlated and, perhaps, interdependent values models important aspects of the centralized security and commodity markets that play a critical role in determining prices in market economies. In this paper we have analyzed small double auctions for both the CPV and CIV environments using a novel information structure. The bulk of our analysis is for the CPV environment using the BBDA. For it we prove, given that sellers follow their dominant strategies of reporting their costs honestly, that symmetric strategies exist that solve buyers' first order conditions. We then show that for both the normal and the Laplace distributions these solutions are in fact equilibria. We also prove that the rates of convergence to price-taking behavior and to ex post efficiency that are now well-established in the double auction literature hold in this model. We add to these convergence results a proof that the effect of strategic behavior upon the aggregation of information by the market is small relative to the error that is inherent in the market's finiteness. Finally, our asymptotic analysis of a buyer's decision problem produces a simple formula in the numbers of buyers and sellers that converges to a buyer's optimal bidding strategy as the market becomes larger.

Our model's information structure specifies that traders' ex ante beliefs concerning the market's underlying state μ is a diffuse, uniform prior on the real line. This, together with an additive, idiosyncratic value/cost term, implies that each trader's decision problem is invariant with respect to his own value/cost. Consequently an offset strategy is the plausible and focal equilibrium strategy for buyers. Computation of this offset strategy is easy for both small and large markets because it is a one dimensional problem, not a two dimensional problem as is usual in a game of incomplete information with a continuum of types. Checking that the computed offset strategy is an equilibrium strategy is also easy because it too is a one dimensional computation. This approach gives us results for small markets, e.g., it demonstrates that the rates of convergence hold even in such cases. It also distinguishes our work from much of the double auction literature, which typically addresses the outcome of a "sufficiently large" market. Together with the formula for approximate equilibria obtained through the asymptotic analysis, the computed equilibria provide insight into bidding behavior for a literature in which few equilibria had previously been derived.

After analyzing the BBDA's performance in the CPV environment we apply the same techniques to determine its performance in the CIV environment. The further complexity that follows from adding an additive noise term to each trader's private, value/cost signal prevents us from proving either existence of a solution to the buyer's first order condition or the rates at which equilibria approach allocational efficiency and informational accuracy. But the invariance of each trader's decision problem with respect to his own signal continues to hold. This, just as in the CPV case, makes both computation of an offset solution to the buyer's first order condition and checking that the solution is an equilibrium easy. The one complication is that, with interdependence, each buyer and each seller is subject to the winner's/loser's curse. Systematic numeric experiments where both the idiosyncratic value/cost term and the noise term are normally distributed establishes that for this case: (i) equilibria do exist for small markets as well as asymptotically for large markets, (ii) the rates of convergence for both allocational efficiency and informational efficiency remain unchanged from the CPV environment, and (iii) the realized market price converges to the REE price as the market size goes from small to large. The latter result provides new theoretical evidence that REE prices can be reasonably approximated by a Bayesian Nash equilibrium even in small markets for the CIV environment, which is exactly the environment for which REE prices are necessary for allocational efficiency.

A Some Proofs of Sections 4 and 5

Proof of Lemma 2. The invariance of $V'(v,b)$ allows us to set $v=0$ with no loss of generality. We also restrict attention to $b \le v = 0$. The functions $K^*_{n,m}(0,b|\mu)$, $L^*_{n,m}(0,b|\mu)$ and $M^*_{n,m}(0,b|\mu)$ that appear in formula (10) satisfy the bounds

$$K^*_{n,m}(0,b|\mu) \ge F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1},$$

$$L^*_{n,m}(0,b|\mu) \ge nF\left(-\mu\right)^{m-2} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1},$$

and

$$M^*_{n,m}(0,b|\mu) \le k_{n,m} F\left(-\mu\right)^{m-1} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1},$$

where $k_{n,m}$ in the last formula denotes

$$k_{n,m} = \sum_{\substack{i+j=m \\ 0 \le i \le m-1 \\ 1 \le j \le n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix}.$$

The bound on $K^*_{n,m}(0,b|\mu)$ is obtained by focusing on its $i=m-1$, $j=0$ term and the bound on $L^*_{n,m}(0,b|\mu)$ is obtained from its $i=m-2$, $j=1$ term. The bound on $M^*_{n,m}(0,b|\mu)$ is obtained by applying $b \le v = 0$, which implies that the monomial in its i^{th} term satisfies

$$\begin{aligned} & F\left(-\mu\right)^{i} F\left(b-\mu\right)^{j} \overline{F}\left(-\mu\right)^{m-1-i} \overline{F}\left(b-\mu\right)^{n-j} \\ \le\ & F\left(-\mu\right)^{i+j-1} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{m-1-i+n-j} \\ =\ & F\left(-\mu\right)^{m-1} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1}. \end{aligned}$$

Substitution into (10) and factoring the integrand in the numerator implies that $V'(0,b)$ is bounded above by

$$\frac{\int_{-\infty}^{\infty} \left[k_{n,m} F\left(b-\mu\right) + nbf(b-\mu)\right] F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu) d\mu}{-(m-1)nb \int_{-\infty}^{\infty} F\left(-\mu\right)^{m-2} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu}. \tag{47}$$

The integral in the numerator of (47) is now bounded separately over the intervals (-∞, b] and [b, ∞). The proof is completed by showing that: (i) the integral over (-∞, b] divided by the denominator of (47) converges to zero as $b \to -\infty$; (ii) the integral over [b, ∞) is negative for b sufficiently negative. These two steps are considered separately below.

Integral over $(-\infty, b]$**.** This integral is bounded above by

$$
\begin{aligned}
& \int_{-\infty}^{b} k_{n,m} F\left(b-\mu\right) F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu) d\mu \\
\leq \; & k_{n,m} \int_{-\infty}^{b} f(-\mu) d\mu \\
= \; & k_{n,m} \overline{F}(-b) \\
= \; & k_{n,m} F(b).
\end{aligned}
$$

The first line follows from $b \leq 0$, the second from

$$
F\left(b-\mu\right) F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1} \leq 1,
$$

and the last equality follows from the symmetry of F. The integral in the denominator of (47) satisfies

$$
\begin{aligned}
& \int_{-\infty}^{\infty} F\left(-\mu\right)^{m-2} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu \\
\geq \; & \int_{-\infty}^{0} F\left(-\mu\right)^{m-2} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu \\
\geq \; & F\left(b\right) \int_{-\infty}^{0} F\left(-\mu\right)^{m-2} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu,
\end{aligned}
$$

where the last inequality reflects $F\left(b-\mu\right) \geq F(b)$ for $\mu \leq 0$. Combining these two bounds implies

$$
\begin{aligned}
& \frac{\int_{-\infty}^{b} \left[k_{n,m} F\left(b-\mu\right) + nbf(b-\mu)\right] F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu) d\mu}{-(m-1)nb \int_{-\infty}^{\infty} F\left(-\mu\right)^{m-2} F\left(b-\mu\right) \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu} \\
\leq \; & \frac{k_{n,m} F(b)}{-(m-1)nbF\left(b\right) \int_{-\infty}^{0} F\left(-\mu\right)^{m-2} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu} \\
= \; & \frac{k_{n,m}}{-(m-1)nb \int_{-\infty}^{0} F\left(-\mu\right)^{m-2} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu)^2 d\mu}.
\end{aligned}
$$

The integrand in the denominator increases as b decreases and the ratio therefore goes to 0 as $b \to -\infty$.

Integral over $[b, \infty)$**.** This integral equals

$$
\int_{b}^{\infty} \left[k_{n,m} + nb \frac{f(b-\mu)}{F\left(b-\mu\right)}\right] F\left(b-\mu\right) F\left(-\mu\right)^{m-1} \overline{F}\left(b-\mu\right)^{n-1} f(-\mu) d\mu. \tag{48}
$$

We have $b-\mu \leq 0$ because $\mu \geq b$. Our assumptions on F implies the existence of a constant $K>0$ such that

$$\frac{f(x)}{F(x)} \geq K \text{ for } x \leq 0.$$

The integral (48) is therefore bounded above by

$$\int_b^\infty \left[k_{n,m}+nbK\right] F(b-\mu) F(-\mu)^{m-1} \overline{F}(b-\mu)^{n-1} f(-\mu) d\mu.$$

The integrand and hence the integral itself are negative for

$$k_{n,m}+nbK<0 \Leftrightarrow b<-\frac{k_{n,m}}{nK}.$$

■

Proof of Lemma 3. The assumption that $V'(v,b) \geq \delta$ means that attention can be restricted to v, b such that

$$v-b \leq L^*(F,\delta) \tag{49}$$

where $L^*(F,\delta)$ is the constant whose value is provided by Lemma 2. Select $L>L^*$ such that

$$\frac{F(-L)}{\left(F(-2L^*)\overline{F}(2L^*)\right)^\varphi}, \frac{\overline{F}(L)}{\left(F(-2L^*)\overline{F}(2L^*)\right)^\varphi}<\frac{1}{2}. \tag{50}$$

The denominator of each ratio on the left side of (50) is constant and so a sufficiently large L will satisfy the inequalities. The number L is selected with foresight for its use later in the proof.

Formula (10) implies that $V'(v,b) \geq \delta > 0$ only if

$$0 \leq \int_{-\infty}^\infty \left[M_{n,m}^* - nf(b-\mu)(v-b)K_{n,m}^*\right] f(v-\mu) d\mu. \tag{51}$$

This integral is now computed over three intervals:

$$\begin{aligned} 0 \quad \leq \quad & \int_{-\infty}^{b-L} M_{n,m}^* f(v-\mu) d\mu \\ & + \int_{b-L}^{v+L} \left[M_{n,m}^* - nf(b-\mu)(v-b)K_{n,m}^*\right] f(v-\mu) d\mu \\ & + \int_{v+L}^\infty M_{n,m}^* f(v-\mu) d\mu. \end{aligned}$$

The $nf(b-\mu)(v-b)K_{n,m}^*$ term has been dropped from the integrands of the first and third integrals, which can only increase their values. This implies that $v-b$ is at most

$$\frac{\int_{b-L}^{v+L} M_{n,m}^{*} f(v-\mu)d\mu}{\int_{b-L}^{v+L} nf(b-\mu)K_{n,m}^{*} f(v-\mu)d\mu} \tag{52}$$

$$+\frac{\int_{-\infty}^{b-L} M_{n,m}^{*} f(v-\mu)d\mu+\int_{v+L}^{\infty} M_{n,m}^{*} f(v-\mu)d\mu}{\int_{b-L}^{v+L} nf(b-\mu)K_{n,m}^{*} f(v-\mu)d\mu}. \tag{53}$$

The proof is completed below by showing that that (52) is $O\left(1/(n\wedge m)\right)$ and (53) is $O(2^{-n\wedge m})$. The bound we seek is thus determined over the states μ near v and b in $[b-L, v+L]$, with states μ in the tails $[-\infty, b-L]$ and $[v+L,\infty]$ shown to be relatively inconsequential.

Expression (52) is $O\left(1/(n\wedge m)\right)$. For $\mu\in[b-L, v+L]$, the bounds

$$\mu\geq b-L \Leftrightarrow b-\mu\leq L \tag{54}$$

and

$$\mu\leq v+L \Leftrightarrow -L\leq v-\mu$$

hold. The bound (49) then implies

$$v-\mu=v-b+b-\mu\leq L^{*}+L,$$

and

$$b-\mu=b-v+v-\mu\geq -L^{*}-L,$$

and so

$$v-\mu, b-\mu\in\left[-L^{*}-L, L+L^{*}\right]. \tag{55}$$

Turning to the numerator of (52), we have

$$\begin{aligned}\int_{b-L}^{v+L} M_{n,m}^{*} f(v-\mu)d\mu &= \int_{b-L}^{v+L} \frac{M_{n,m}^{*}}{nK_{n,m}^{*}} nK_{n,m}^{*} f(v-\mu)d\mu \\ &\leq \int_{b-L}^{v+L} \frac{M_{n,m}^{*}}{(n\wedge m)\, K_{n,m}^{*}} nK_{n,m}^{*} f(v-\mu)d\mu \\ &\leq \frac{2}{n\wedge m}\int_{b-L}^{v+L} \frac{F(v-\mu)\overline{F}\,(b-\mu)}{\overline{F}\,(v-\mu)} nK_{n,m}^{*} f(v-\mu)d\mu.\end{aligned} \tag{56}$$

The step from the first to the second line follows from the fact $M_{n,m}^{*}/nK_{n,m}^{*}$ is non-increasing in n

and in m when $m \geq 2$ (see Williams (1991, Lemma 4.2)). The inequality

$$\frac{M_{n\wedge m}}{K_{n\wedge m}} < 2\frac{F(v-\mu)\overline{F}\,(b-\mu)}{\overline{F}\,(v-\mu)}$$

follows from substitution into formulas (5.3) and (5.4) for $z(v_2, b)$ of Satterthwaite and Williams (1989) with $v-\mu$ replacing v_2, $b-\mu$ replacing b, and F replacing F_1 and F_2.

The fact that $v-\mu$ and $b-\mu$ lie in the compact interval (55) implies that

$$\frac{F(v-\mu)\overline{F}\,(b-\mu)}{\overline{F}\,(v-\mu)}$$

is bounded above and $f(b-\mu)$ is bounded below for $\mu \in [b-L, v+L]$. Applying these bounds and (56), there exists a constant F^* such that (52) is at most

$$\frac{F^*}{n\wedge m}\cdot\frac{\int_{b-L}^{v+L} nK^*_{n,m} f(v-\mu)d\mu}{\int_{b-L}^{v+L} nK^*_{n,m} f(v-\mu)d\mu} = \frac{F^*}{n\wedge m},$$

which completes the proof that (52) is $O\left(1/\left(n\wedge m\right)\right)$.

Expression (53) is $O(2^{-n\wedge m})$. Consider first the denominator of (53). Reduce the support of this integral from $[b-L, v+L]$ to $[b-L^*, v+L^*]$. Replacing L with L^* in the analysis (54)-(55) implies

$$v-\mu, b-\mu \in [-2L^*, 2L^*]$$

for $\mu \in [b-L^*, v+L^*]$. The fact that $v \geq b$ together with formula (5) for $K^*_{n,m}$ imply

$$\begin{aligned}
& nK^*_{n,m}(v,b,\mu) \\
&\geq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n\begin{pmatrix} m-1\\ i\end{pmatrix}\begin{pmatrix} n-1\\ j\end{pmatrix}\cdot F\left(b-\mu\right)^{m-1}\overline{F}\left(v-\mu\right)^{n-1} \\
&\geq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n\begin{pmatrix} m-1\\ i\end{pmatrix}\begin{pmatrix} n\\ j\end{pmatrix}\cdot F\left(-2L^*\right)^{m-1}\overline{F}\left(2L^*\right)^{n-1}.
\end{aligned}$$

Letting

$$f^* = \inf_{x\in[-2L^*,2L^*]} f(x),$$

the remaining term in the denominator satisfies

$$\begin{aligned}\int_{b-L^*}^{v+L^*} f(b-\mu)f(v-\mu)d\mu &\geq (f^*)^2 (v-b+2L^*) \\ &\geq (f^*)^2 (2L^*).\end{aligned}$$

The denominator of (53) is thus at least

$$\sum_{\substack{i+j=m-1\\0\leq i,j\leq m-1\\0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \cdot F(-2L^*)^{m-1}\, \overline{F}(2L^*)^{n-1} (f^*)^2 (2L^*). \tag{57}$$

Turning to the numerator of (53), notice that

$$\mu \leq b-L \Leftrightarrow L \leq b-\mu$$

and

$$\mu \geq v+L \Leftrightarrow v-\mu \leq -L.$$

Applying $b \leq v$, Formula (4) implies

$$M_{n,m}^* \leq \sum_{\substack{i+j=m\\0\leq i\leq m-1\\1\leq j\leq n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} F(v-\mu)^m\, \overline{F}(b-\mu)^{n-1}.$$

The change of index $j+1 \to j$ implies

$$\begin{aligned}\sum_{\substack{i+j=m\\0\leq i\leq m-1\\1\leq j\leq n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} &= \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j+1 \end{pmatrix} \\ &= \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} \frac{n}{j+1} \\ &\leq n \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix}.\end{aligned}$$

Therefore, for $\mu \in [-\infty, b-L]$,

$$M_{n,m}^{*} \leq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} \overline{F}(L)^{n-1}, \tag{58}$$

and for $\mu \in [v+L, \infty]$,

$$M_{n,m}^{*} \leq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} F(-L)^{m}. \tag{59}$$

We now combine the bounds (57), (58) and (59). Notice that

$$\sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix}$$

appears in each of the bounds (57), (58) and (59) and therefore cancels. The ratio (53) is therefore bounded above with:

$$\begin{aligned}
&\frac{\overline{F}(L)^{n-1}\int_{-\infty}^{b-L} f(v-\mu)d\mu + F(-L)^{m}\int_{v+L}^{\infty} f(v-\mu)d\mu}{F(-2L^{*})^{m-1}\,\overline{F}(2L^{*})^{n-1}\,(f^{*})^{2}\,(2L^{*})} \\
&\leq \frac{1}{(f^{*})^{2}(2L^{*})} \cdot \frac{\overline{F}(L)^{n} + F(-L)^{m}}{F(-2L^{*})^{m}\,\overline{F}(2L^{*})^{n}} \\
&\leq \frac{1}{(f^{*})^{2}(2L^{*})} \cdot \left[\frac{\overline{F}(L)^{n}}{\overline{F}(2L^{*})^{\varphi n}\,\overline{F}(2L^{*})^{\varphi n}} + \frac{F(-L)^{m}}{\overline{F}(2L^{*})^{\varphi m}\,\overline{F}(2L^{*})^{\varphi m}}\right] \\
&\leq \frac{1}{(f^{*})^{2}(2L^{*})} \cdot \left[\left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{m}\right] \\
&\leq \frac{1}{(f^{*})^{2}(L^{*})} \cdot \left(\frac{1}{2}\right)^{n\wedge m},
\end{aligned}$$

where the bound (50) is applied in the fourth line of this calculation. ■

Proof of Theorem 6. This proof adapts the argument in Rustichini, Satterthwaite, and Williams (1994, Thm. 3.2) to the model of this paper. The state μ is fixed. The denominator of relative inefficiency is the potential gains from trade. In each state μ, the gains from trade are at least $n \wedge m$ times the expected gains from trade $\lambda(F)$ between a single buyer and a sngle seller. Notice that the number $\lambda(F)$ is the same in each state μ. The expected gains from trade in state μ are therefore at least $(n \wedge m)\, \lambda(F)$.

The numerator of relative inefficiency in (26) is the expected value of the efficient trades given μ that fail to be made because of buyer misrepresentation. Given the lower bound $(n \wedge m)\, \lambda(F)$ on the denominator, it remains to be shown that the numerator is $O\left(1/\left(n \wedge m\right)\right)$. We first show that the value of a trade that inefficiently fails to occur is $O(1/\left(n \wedge m\right))$. The second step is then to show that the expected number of trades that inefficiently fail to occur is bounded above by a constant that does not depend on n or m.

Given a sample of m values and n costs, we first consider the value of a trade between a buyer with value v and a seller with cost c that occurs in the efficient allocation but does not occur in the equilibrium $< \widetilde{S}, B >$. Let p denote the market price in the BBDA determined by the sample and by $< \widetilde{S}, B >$. This buyer and this seller each fail to trade in the BBDA only if

$$B(v) \leq p \leq c. \tag{60}$$

Efficiency requires that both should trade only if v is among the largest n costs/values and c is among the m smallest costs/values. Consequently, trade between this buyer and this seller occurs in the efficient allocation only if

$$c \leq v. \tag{61}$$

Combining (60) and (61) implies

$$B(v) \leq c \leq v.$$

Theorem 5 then implies that the value $v - c$ of a trade that inefficiently fails to occur satisfies

$$v - c \leq v - B(v) \leq \frac{K}{n \wedge m}.$$

We suppress the dependence of K upon the distribution F and the bound φ on the relative size of n and m for notational simplicity in this proof.

The proof is completed by bounding above the expected number of trades that inefficiently fail to occur in the state μ by a constant determined by F that is independent of n, m and μ. The following notation is used: $t_{(m+1)}$ denotes the $(m+1)^{\text{st}}$ smallest cost/value in a sample of m buyers' values and m sellers' costs, $F_{t_{(m+1)}}\left(t_{(m+1)}\,|\mu\right)$ denotes the distribution of $t_{(m+1)}$ conditional on μ, and $L(t_{(m+1)}, \mu)$ is the expected number of trades that inefficiently fail to occur given the realization of $t_{(m+1)}$ in the state μ. The proof is completed by bounding the expected number of missed trades in the state μ,

$$\int_{-\infty}^{\infty} L(t_{(m+1)}, \mu) dF_{t_{(m+1)}}\left(t_{(m+1)}\,|\mu\right), \tag{62}$$

by a constant that does not depend on n, m or μ. The integral (62) is represented as the sum

$$\int_{-\infty}^{\mu+\gamma} L(t_{(m+1)}\,|\mu)dF_{t_{(m+1)}}\left(t_{(m+1)}\,|\mu\right) \tag{63}$$

$$+\int_{\mu+\gamma}^{\infty} L(t_{(m+1)}\,|\mu)dF_{t_{(m+1)}}\left(t_{(m+1)}\,|\mu\right) \tag{64}$$

where γ is a constant chosen with foresight to its use later in the proof so that

$$\overline{F}(\gamma) \le \frac{1}{2^{3\varphi}}. \tag{65}$$

The two integrals (63) and (64) are bounded above in separate arguments.

Bounding (63). Because buyers underbid by at most $K/(n \wedge m)$, the equilibrium market price (i.e., the $(m+1)$st bid/ask) necessarily lies in the interval

$$\left[t_{(m+1)} - \frac{K}{n \wedge m}, t_{(m+1)}\right].$$

The value of a buyer who inefficiently fails to trade must lie (i) at or below $t_{(m+1)}+K(F,\varphi)/\left(n \wedge m\right)$ so that his bid can fall below the market price, and (ii) at or above $t_{(m+1)}$ so that he should in fact trade. Given μ and $t_{(m+1)}$, the expected number of trades that inefficiently fail to occur is therefore at most equal to the expected number of buyers' values in the interval

$$\left[t_{(m+1)}, t_{(m+1)} + \frac{K(F,\varphi)}{n \wedge m}\right]. \tag{66}$$

This expected value will be bounded using an upper bound on the density of trader's cost/value in this interval given μ.

We now compute the bound on this density. Let x denote a trader's cost/value above $t_{(m+1)}$. Conditional on μ and $t_{(m+1)}$, there are $n-1$ traders' values above $t_{(m+1)}$ that are independently distributed with density

$$\frac{f(x-\mu)}{\overline{F}(t_{(m+1)}-\mu)}.$$

This satisfies the bound

$$\frac{f(x-\mu)}{\overline{F}(t_{(m+1)}-\mu)} \le \frac{f(x-\mu)}{\overline{F}(x-\mu)}$$

because $x \geq t_{(m+1)}$. In particular, for

$$x \in \left[t_{(m+1)}, t_{(m+1)} + \frac{K}{n \wedge m}\right] \tag{67}$$

with

$$t_{(m+1)} \in (-\infty, \mu + \gamma], \tag{68}$$

it is true that

$$x - \mu \leq t_{(m+1)} + \frac{K}{n \wedge m} - \mu \leq \gamma + \frac{K}{n \wedge m} \leq \gamma + K.$$

Define

$$\zeta(F) = \sup_{y \leq \gamma + K(F,\varphi)} \frac{f(y)}{\overline{F}(y)},$$

so that

$$\frac{f(x-\mu)}{\overline{F}(t_{(m+1)} - \mu)} \leq \zeta(F)$$

for x and $t_{(m+1)}$ satisfying (67) and (68).

The expected number of buyers' values in the interval (66) is therefore at most

$$1 + ((n-1) \wedge m)\, \zeta(F) \cdot \frac{K}{n \wedge m} \leq 1 + \zeta(F) \cdot K,$$

where the "1" counts the trader's cost/value equal to $t_{(m+1)}$ and $\zeta(F) \cdot K / (n \wedge m)$ bounds above the probability that the cost/value of any of the other $n-1$ traders above $t_{(m+1)}$ lies in the interval (66). Because there are m buyers, the term $(n-1) \wedge m$ bounds above the number of buyers' values that are above $t_{(m+1)}$. The desired bound on (63) is now computed as follows:

$$\begin{aligned} \int_{-\infty}^{\mu+\gamma} L(t_{(m+1)}, \mu) dF_{t_{(m+1)}}\left(t_{(m+1)} \,|\mu\right) &\leq \int_{-\infty}^{\mu+\gamma} \left(1 + \zeta(F) \cdot K\right) dF_{t_{(m+1)}}\left(t_{(m+1)} \,|\mu\right) \\ &\leq 1 + \zeta(F) \cdot K. \end{aligned}$$

Bounding (64). We focus on the low probability that $t_{(m+1)} \geq \mu + \gamma$. Specifically,

$$\begin{aligned} \int_{\mu+\gamma}^{\infty} L(t_{(m+1)}, \mu) dF_{t_{(m+1)}}\left(t_{(m+1)} \,|\mu\right) &\leq \int_{\mu+\gamma}^{\infty} (n \wedge m)\, dF_{t_{(m+1)}}\left(t_{(m+1)} \,|\mu\right) \\ &= n \wedge m \cdot \left(1 - F_{t_{(m+1)}}(\mu + \gamma \,|\mu)\right). \end{aligned} \tag{69}$$

The event "$t_{(m+1)} \geq \mu+\gamma$" requires that n costs/values be above $\mu+\gamma$. The term $1-\Phi(\mu + \gamma \,|\mu)$ is therefore the probability given μ that at least n in a sample of $n+m$ costs/values are greater than or equal to $\mu+\gamma$. This is calculated by summing over the number i of costs/values below $\mu+\gamma$ and then reducing as follows:

$$\begin{aligned}
1-F_{t_{(m+1)}}(\mu+\gamma\,|\mu) &= \sum_{i=0}^{m}\binom{n+m}{i}F(\gamma)^i\overline{F}(\gamma)^{n+m-i} \\
&= \overline{F}(\gamma)^n\sum_{i=0}^{m}\binom{n+m}{i}F(\gamma)^i\overline{F}(\gamma)^{m-i} \\
&\leq \overline{F}(\gamma)^n\sum_{i=0}^{n+m}\binom{n+m}{i}1^i1^{n+m-i} \\
&\leq \frac{1}{2^{3\varphi n}}\cdot(1+1)^{n+m} \\
&= \frac{1}{2^{2\varphi n-m}} \\
&\leq \frac{1}{2^{\varphi n}}.
\end{aligned}$$

The first inequality drops $F(\gamma)^i\overline{F}(\gamma)^{m-i}<1$ from the i^{th} term, the second inequality increases the sum by adding terms from $i=m+1$ to $i=n+m$, and the final inequality applies the binomial formula and the bound (65).

Substitution into (69) implies

$$\int_{\mu+\gamma}^{\infty} L(t_{(m+1)},\mu)dF_{t_{(m+1)}}\left(t_{(m+1)}\,|\mu\right)\leq\frac{n\wedge m}{2^{\varphi n}}<1,$$

which completes the proof. ■

B Bilateral Results

Proof of Theorem 4. Assumptions A1-A5 and A5′, as well as the structure of our model, have the following implications:

I1: As mentioned, the diffuse prior on μ implies that $f_\mu(\mu|v)=f_v(v|\mu)$ (similar for costs) for all $(v,\mu)\in\mathbb{R}^2$.

I2: As also mentioned, the additive structure of values implies than $f_v(v|\mu)=f(v-\mu)$ (similar for costs) for all $(v,\mu)\in\mathbb{R}^2$.

I3: A3 and A4 imply that $f(x)/\bar{F}(x)$ is increasing for all $x\in\mathbb{R}$.

I4: I2, I3 & A5′ imply that $f_v(v|\mu)$ is MLRP for all $(v,\mu)\in\mathbb{R}^2$.

I5: I4 implies that the elements of the random vector

$$(\mu, v_1, v_2, \ldots, v_m, c_1, \ldots, c_n),$$

are affiliated for any $m, n \geq 1$, i.e., the density of the vector satisfies the MTP$_2$ property, see Tong (1990, Page 75).

I6: I5 implies that

$$\frac{\partial}{\partial v}\frac{F_{c|v}(b|v)}{f_{c|v}(b|v)} \quad \leq \quad 0,$$

for all $(v, c) \in \mathbb{R}^2$, see Milgrom and Weber (1982, Lemma 1), where $F_{c|v}$ ($f_{c|v}$) is the distribution (density) of the seller's cost c given the buyer's value v.

I7: The diffuse prior of μ and A4 imply that

$$\frac{F_{c|v}(b|v)}{f_{c|v}(b|v)} = \frac{F_{c|v}(v-c|0)}{f_{c|v}(v-c|0)},$$

for all $(v, c) \in \mathbb{R}^2$.[35]

[35] Proof that $f_{c|v}(c|v) = f_{c|v}(v-c|0)$:

$$\begin{aligned}
f_{c|v}(c|v) &= \int_{-\infty}^{\infty} f(c-\mu)f(v-\mu)d\mu \quad \text{(from the diffuse prior)} \\
&= \int_{-\infty}^{\infty} f(c-v-\tilde{\mu})f(0-\tilde{\mu})d\tilde{\mu} \quad (\tilde{\mu} = \mu - v) \\
&= \int_{-\infty}^{\infty} f(v-c+\tilde{\mu})f(\tilde{\mu})d\tilde{\mu} \quad (f(x) = f(-x)) \\
&= \int_{-\infty}^{\infty} f(v-c-\mu)f(-\mu)d\mu \quad (\mu = -\tilde{\mu}) \\
&= f_{c|v}(v-c|0).
\end{aligned}$$

Proof that $F_{c|v}(c|v) = \overline{F}_{c|v}(v-c|0)$:

$$\begin{aligned}
F_{c|v}(c|v) &= \int_{-\infty}^{c} f_{c|v}(x|v)dx \\
&= \int_{-\infty}^{c} f_{c|v}(v-x|0)dx \quad \text{(from above)} \\
&= \int_{v-c}^{\infty} f_{c|v}(y|0)dy \quad (y = v - x) \\
&= \overline{F}_{c|v}(v-c|0).
\end{aligned}$$

I8: I6 and I7 imply that

$$\frac{\partial}{\partial c}\frac{F_{c|v}(b|v)}{f_{c|v}(b|v)} \;\; \geq \;\; 0,$$

for all $(v,c) \in \mathbb{R}^2$, so notice that in our case the part of assumption (A2) in Kadan (2007) that we need follows from affiliation and need not be assumed separately.

I9: A3, A5, I6, I7, and I8, see Kadan (2007, Theorem 1), imply that (18) has a unique offset solution that is the unique equilibrium strategy of this bilateral game

$$B(v) = v - \lambda,$$

where λ is a positive constant.[36]

■

Proof of Theorem 7. The example concerns the standard Cauchy distribution, i.e., the Cauchy distribution with location parameter 0 and scale parameter 1. The buyer's beliefs conditional on his value v is therefore that c is Cauchy distributed with location parameter v and scale parameter 2. Using the formula for the cumulative of the Cauchy distribution, we have

$$\pi(v,b) = (v-b)\left(\frac{1}{\pi}\arctan\left(\frac{b-v}{2}\right)+\frac{1}{2}\right),$$

and his marginal expected utility is

$$\pi_b(v,b) = -\left(\frac{1}{\pi}\arctan\left(\frac{b-v}{2}\right)+\frac{1}{2}\right) + (v-b)\frac{1}{\pi}\left(\frac{2}{4+(b-v)^2}\right).$$

It is sufficient to consider the case of $v=0$ and $b \leq 0$, which simplifies the notation to

$$\begin{aligned}\pi_b(0,b) &= -\left(\frac{1}{\pi}\arctan\left(\frac{b}{2}\right)+\frac{1}{2}\right) - \frac{b}{\pi}\left(\frac{2}{4+b^2}\right) \\ &= -\frac{1}{\pi}\left(\arctan\left(\frac{b}{2}\right)+\frac{\pi}{2}+\frac{2b}{4+b^2}\right). \end{aligned} \tag{70}$$

At $b=0$ we have

$$\pi_b(0,0) = -\frac{1}{2} < 0,$$

[36]The fact that the solution is an offset follows directly from the form of the first order condition

$$v - b \;\; = \;\; \frac{\overline{F}_{c|v}(v-b|0)}{f_{c|v}(v-b|0)}.$$

The positiveness of the offset just follows from rationality of the buyer.

and

$$\lim_{b \to -\infty} \pi_b(0, b) = 0.$$

The remainder of the argument shows that

$$\pi_{bb}(0, b) < 0$$

for $b < 0$, where $\pi_{bb} = \partial^2\pi/\partial b^2$, which implies that $\pi_b(0, b)$ decreases from its limiting value of 0 at $b = -\infty$ to its value of $-1/2$ at $b = 0$. This implies

$$\pi_b(0, b) < 0$$

for all $b < 0$. The buyer's expected utility thus strictly increases as b becomes more and more negative.

Working from (70), the second derivative is

$$\begin{aligned}
\pi_{bb}(0, b) &= -\frac{1}{\pi}\left(\frac{2}{4 + b^2} + \frac{8 - 2b^2}{(4 + b^2)^2}\right) \\
&= -\frac{1}{\pi\,(4 + b^2)^2}\left(8 + 2b^2 + 8 - 2b^2\right) \\
&= -\frac{16}{\pi\,(4 + b^2)^2} < 0,
\end{aligned}$$

which completes the proof. ■

C An Example of Failure of Affiliation

The point of this section is to inquire on whether order statistics of bids/asks are affiliated with the value (signal) of a focal buyer. Consider $m \geq 2$ buyers and $n \geq 1$ sellers, where buyer one is the focal buyer. Conditional on μ their values/costs $(v_1, \ldots, v_m, c_1, \ldots, c_n)$ are independent and identically distributed. Hence the density of $(\mu, v_1, \ldots, v_m, c_1, \ldots, c_n)$ is,

$$f_{\mu,v_1,\ldots,v_m,c_1,\ldots,c_n}(x_0, \ldots, x_{m+n}) = g(x_0)\Pi_{i=1}^{m+n} f(x_i - x_0),$$

where let g the prior density of μ and f is, as before, the density of the idiosyncratic component ε. Affiliation requires that

$$\partial^2 \ln f_{\mu,v_1,\ldots,v_m,c_1,\ldots,c_n}(x_0, \ldots, x_{m+n})/\partial x_i \partial x_j \geq 0,$$

for all $i \neq j$ and $i, j \in \{0, \ldots, m + n\}$. For f satisfying assumptions A1-A5 & A5$'$ (so that the marginals of values/costs given μ satisfy the MLRP) this is true.

Now, let $\tilde{s}_{(i)}$ be the i^{th} order statistic of $(v_2, \ldots, v_m, c_1, \ldots, c_n)$, then the density of $(\mu, v_1, \tilde{s}_{(1)}, \ldots, \tilde{s}_{(m+n-1)})$ is

$$\begin{aligned} f_{\mu,v_1,\tilde{s}_{(1)},\ldots,\tilde{s}_{(m+n-1)}}(x_0, \ldots, x_{m+n}) &= g(x_0)(m+n-1)!\Pi_{i=1}^{m+n} f(x_i - x_0) \\ &= g(x_0)f(x_1 - x_0)(m+n-1)!\Pi_{i=2}^{m+n} f(x_i - x_0), \end{aligned}$$

for $x_2 \leq \ldots \leq x_{m+n}$. Again the assumptions on f guarantee that $f_{\mu,v_1,\tilde{s}_{(1)},\ldots,\tilde{s}_{(m+n-1)}}$ is affiliated.

Similarly conditional on μ the elements of the vector $(v_1, b_2, \ldots, b_m, c_1, \ldots, c_n)$, where $b_i = v_i - \lambda$, $i = \{2, \ldots, m\}$, $\lambda \in \mathbb{R}$, are independent but not identically distributed, in particular,

$$f_{\mu,v_1,b_2,\ldots,b_m,c_1,\ldots,c_n}(x_0, \ldots, x_{m+n}) = g(x_0)f(x_1 - x_0)\Pi_{i=2}^{m} f(x_i + \lambda - x_0)\Pi_{j=m+1}^{m+n} f(x_j - x_0),$$

this is not an issue however, since the assumptions on f still guarantee affiliation.

Now, let $s_{(i)}$ be the i^{th} order statistic of $(b_2, \ldots, b_m, c_1, \ldots, c_n)$, then the density of $(\mu, v_1, s_{(1)}, \ldots, s_{(m+n-1)})$ David and Nagaraja (2003, Page 98) is

$$f_{\mu,v_1,s_{(1)},\ldots,s_{(m+n-1)}}(x_0, \ldots, x_{m+n}) = g(x_0)f(x_1 - x_0)\text{Permanent}(A),$$

for $x_2 \leq \ldots \leq x_{m+n}$, and

$$A = \begin{bmatrix} f(x_2 + \lambda - x_0) & \ldots & f(x_2 + \lambda - x_0) & f(x_2 - x_0) & \ldots & f(x_2 - x_0) \\ f(x_3 + \lambda - x_0) & \ldots & f(x_3 + \lambda - x_0) & f(x_3 - x_0) & \ldots & f(x_3 - x_0) \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ f(x_{m+n} + \lambda - x_0) & \ldots & f(x_{m+n} + \lambda - x_0) & f(x_{m+n} - x_0) & \ldots & f(x_{m+n} - x_0) \end{bmatrix},$$

where in each row $i \in \{1, \ldots, m+n-1\}$ there are $m-1$, $f(x_{i+1}+\lambda-x_0)$ terms, and n, $f(x_{i+1}-x_0)$ terms. The permanent of a matrix is like the determinant but where all the signs are positive. We can write this particular permanent as

$$\text{Permanent}(A) = (m-1)!n! \sum_{i=1}^{\binom{m+n-1}{m-1}} \left[\Pi_{j=1}^{m-1} f\left(x_{\pi_{ij}} + \lambda - x_0\right) \Pi_{j=1}^{n} f\left(x_{\pi_{i-j}} - x_0\right)\right],$$

where π_i is the i^{th} selection (without loss of generality, arbitrarily ordered) of $m-1$ elements out of the $m+n-1$ in $\{2, \ldots, m+n\}$, and π_{ij} is the j^{th} element (without loss of generality, also arbitrarily ordered) of the i^{th} selection. So $i \in \left\{1, \ldots, \binom{m+n-1}{m-1}\right\}$, and $j \in \{1, \ldots, m-1\}$. Furthermore, π_{i-} is every element not picked in the i^{th} selection and π_{i-j} is the j^{th} element (out of n) of π_{i^-}.[37]

Now even though f satisfies A1-A5 and A5$'$ we need to inquire more about the terms in the square brackets. Consider the partial $\partial^2 \ln f_{\mu,v_1,s_{(1)},\ldots,s_{(m+n-1)}}(x_0,\ldots,x_{m+n})/\partial x_k \partial x_l$, for any $m+n \geq l > k > 1$. We have

$$\frac{\partial^2}{\partial x_k \partial x_l} \ln f_{\mu,v_1,s_{(1)},\ldots,s_{(m+n-1)}}(x_0,\ldots,x_{m+n}) =$$

$$\frac{\partial^2}{\partial x_k \partial x_l} \ln \left\{ \sum_{i=1}^{\binom{m+n-1}{m-1}} \left[\Pi_{j=1}^{m-1} f\left(x_{\pi_{ij}} + \lambda - x_0\right) \Pi_{j=1}^{n} f\left(x_{\pi_{i^-j}} - x_0\right) \right] \right\}. \tag{71}$$

Proof of Theorem 3. We will provide an example of failure of affiliation in the case of the the standard normal, i.e., for $f = \phi$. In this case the term in the square brackets of (71) becomes

$$\Pi_{j=1}^{m-1} f\left(x_{\pi_{ij}} + \lambda - x_0\right) \Pi_{j=1}^{n} f\left(x_{\pi_{i^-j}} - x_0\right) =$$

$$\frac{1}{\sqrt{2\pi}^{m+n-1}} \exp\left\{ -\frac{1}{2} \left[\sum_{j=2}^{m+n} (x_j - x_0)^2 + (m-1)\lambda^2 + 2\lambda \sum_{j=1}^{m-1} \left(x_{\pi_{ij}} - x_0\right) \right] \right\} =$$

$$\frac{1}{\sqrt{2\pi}^{m+n-1}} \exp\left\{ -\frac{1}{2} \left[\sum_{j=2}^{m+n} (x_j - x_0)^2 + (m-1)\lambda(\lambda - 2x_0) + 2\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}} \right] \right\} =$$

$$\frac{1}{\sqrt{2\pi}^{m+n-1}} \exp\left\{ -\frac{1}{2} \left[\sum_{j=2}^{m+n} (x_j - x_0)^2 + (m-1)\lambda(\lambda - 2x_0) \right] \right\} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}} \right]$$

for each $i \in \left\{1,\ldots,\binom{m+n-1}{m-1}\right\}$, and so after summing over i and taking the logarithm the relevant term is

$$\ln \left\{ \sum_{i=1}^{\binom{m+n-1}{m-1}} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}} \right] \right\}.$$

[37]For example if $m-1 = n = 3$, a particular selection, let π_1, of 3 out of 6 elements is $\{1,2,3\}$, and so $\pi_1 = \{1,2,3\}$, $\pi_{11} = 1$, and $\pi_{1^-} = \{4,5,6\}$, $\pi_{1^-2} = 5$.

Let

$$H\left(x_k, x_l; X_{kl^-}\right) = \sum_{i=1}^{\binom{m+n-1}{m-1}} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}}\right],$$

where $X_{kl^-} = \{x_i : i \neq k,\, i \neq l,\, i \in \{2, \ldots, m+n\}\}$. Then the partial derivative of interest is

$$\begin{aligned}\frac{\partial^2}{\partial x_k \partial x_l} \ln f_{\mu, v_1, s_{(1)}, \ldots, s_{(m+n-1)}}(x_0, \ldots, x_{m+n}) &= \frac{\partial^2}{\partial x_k \partial x_l} \ln \left\{H\left(x_k, x_l; X_{kl^-}\right)\right\} \\ &= \frac{\frac{\partial^2 H}{\partial x_k \partial x_l} H - \frac{\partial H}{\partial x_k} \frac{\partial H}{\partial x_l}}{R^2}\end{aligned}$$

So we look into the sign of the numerator

$$\frac{\partial^2 H}{\partial x_k \partial x_l} H - \frac{\partial H}{\partial x_k} \frac{\partial H}{\partial x_l} =$$

$$\lambda^2 \sum_{i \in \left\{1, \ldots, \binom{m+n-1}{m-1}\right\} : (k,l) \in \pi_i} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}}\right] \sum_{i \in \left\{1, \ldots, \binom{m+n-1}{m-1}\right\}} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}}\right]$$

$$-\lambda^2 \sum_{i \in \left\{1, \ldots, \binom{m+n-1}{m-1}\right\} : k \in \pi_i} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}}\right] \sum_{i \in \left\{1, \ldots, \binom{m+n-1}{m-1}\right\} : l \in \pi_i} \exp\left[-\lambda \sum_{j=1}^{m-1} x_{\pi_{ij}}\right].$$

After some (tedious) algebra we get that

$$\frac{\partial^2 H}{\partial x_k \partial x_l} H - \frac{\partial H}{\partial x_k} \frac{\partial H}{\partial x_l} = -\lambda^2 \exp[-\lambda(x_k + x_l)] \tilde{H}\left(X_{kl-}\right),$$

where $\tilde{H}\left(X_{kl-}\right)$ (available upon request) is a sum of exponentials involving sums of elements of X_{kl-}. So we just showed that the partial of interest is negative for any $\lambda \neq 0$, and so we have a violation of affiliation.[38] ■

D Approximate Formula

Proof of Theorem 10. For a fixed number m of buyers and n of sellers, we consider a sequence of markets with τm buyers and τn sellers for $\tau \in \mathbb{N}^*$. For every τ we pick a focal buyer, buyer τm (without loss of generality), with value $v_{\tau m} = v$.[39]

For notational brevity let x, y be the $(\tau m)^{\text{th}}$, $(\tau m+1)^{\text{st}}$, respectively, order statistic of the population of $\tau(m+n)-1$ traders, i.e., the whole population excluding the focal buyer. Denote the length $w \equiv y - x$ with density and distribution $f_w(\cdot|x,\tau)$ and $F_w(\cdot|x,\tau)$, respectively, indexed by the order statistic x and by the size of the market τ.[40] Also let $\bar{F}_w(\cdot|x,\tau) = 1 - F_w(\cdot|x,\tau)$ be the right-hand distribution of F_w; therefore, conditional on v and τ,

$$\begin{aligned}
\Pr[x < b < y|v,\,\tau] &= \Pr[0 < b-x < y-x|v,\tau] \\
&= \Pr[0 < b-x < w|v,\tau] \\
&= \Pr[0 < b-x \;\&\; b-x < w|v,\tau] \\
&= \int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw}(x,w|v,\tau,\mu)\,dwdx \\
&= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw\mu}(x,w,\mu-v,\tau)\,dwdxd\mu \qquad (72)\\
&= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw}(x,w|\mu,v,\tau)\,f(\mu-v)\,dwdxd\mu \\
&= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw}(x,w|\mu,\tau)\,dwdxf(\mu-v)\,d\mu \\
&= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{y=b}^{\infty} f_w(w|x,\mu,\tau)\,f_x(x|\mu,\tau)\,dwdxf_\mu(\mu-v)\,d\mu \\
&= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b} \bar{F}_w(b-x|x,\mu,\tau)\,f_x(x|\mu,\tau)\,dxf(\mu-v)\,d\mu.
\end{aligned}$$

[38]Example for $m = n = 3$, $k = 4$, $l = 5$. Using the formula above we get that in this particular case,

$$\frac{\partial^2 H}{\partial x_4 \partial x_5}H - \frac{\partial H}{\partial x_4}\frac{\partial H}{\partial x_5} =$$

$$-\lambda^2 e^{-(x_4+x_5)\lambda}\left(e^{-2(x_2+x_3)\lambda} + e^{-2(x_2+x_6)\lambda} + e^{-2(x_3+x_6)\lambda} + e^{-(2x_2+x_3+x_6)\lambda} + e^{-(x_2+2x_3+x_6)\lambda} + e^{-(x_2+x_3+2x_6)\lambda}\right)$$

$$< 0.$$

[39]Recall that the conditional density of μ given a buyer's value v is $f(\mu - v)$ with distribution $F(\mu - v)$, for all $(\mu, v) \in \mathbb{R}^2$.

[40]Observe that w is independent of μ and, therefore, of v.

We are interested in the q^{th} quantile where

$$q = m/(m+n).$$

Hence, as before, $F(\xi_q) = q$ so that $F(\xi_q + \mu - \mu) = q$. We write

$$\begin{aligned}
\bar{F}_w\left(b-x|x,\mu,\tau\right) &= \Pr\left[2\left[\tau\left(m+n\right)-1\right]f\left(\xi_q\right)w > 2\left[\tau\left(m+n\right)-1\right]f\left(\xi_q\right)\left(b-x\right)|x,\mu,\tau\right] \\
&= \Pr\left[\widehat{w} > 2\left[\tau(m+n)-1\right]f\left(\xi_q\right)\left(b-x\right)|x,\mu,\tau\right],
\end{aligned}$$

where $\widehat{w} \equiv 2\left[\tau\left(m+n\right)-1\right]f\left(\xi_q\right)w$. We have proven that conditional on μ, x is asymptotically normal, see Theorem 13 (presented in Appendix E). We also have that $\widehat{w}$ is χ^2 with two degrees of freedom and independent of x, see Siddiqui (1960).[41] Therefore, letting asymptotic distributions be denoted by tildes,

$$\begin{aligned}
\tilde{\bar{F}}_{\widehat{w}}\left(t|x,\mu,\tau\right) &= \exp\left(-\frac{t}{2}\right), t\in\mathbb{R}^+, \\
\tilde{f}_x\left(t|\mu,\tau\right) &= \frac{1}{\sqrt{2\pi\Sigma^2}}\exp\left(-\frac{\left(t-\xi_q-\mu\right)^2}{2\Sigma^2}\right), t\in\mathbb{R}, \\
\text{where } \Sigma^2 &= \frac{mn/(m+n)^2}{\tau(m+n)-1}\frac{1}{f^2\left(\xi_q\right)} = \frac{\Lambda}{\tau}, \text{ for,} \\
\Lambda &\equiv \frac{mn/(m+n)^2}{(m+n)-1/\tau}\frac{1}{f^2\left(\xi_q\right)}.
\end{aligned} \tag{73}$$

Then the inner integral on the last line of (72) can be asymptotically written as

$$\begin{aligned}
&\int_{x=-\infty}^{b}\tilde{\bar{F}}_{\widehat{w}}\left(2\left[\tau(m+n)-1\right]f\left(\xi_q\right)\left(b-x\right)|x,\mu,\tau\right)f_x\left(x|\mu,\tau\right)dx \\
= &\frac{1}{2\left[(m+n)-1/\tau\right]f\left(\xi_q\right)}\int_{t=0}^{\infty}\tilde{\bar{F}}_{\widehat{w}}\left(\tau\, t|x,\mu,\tau\right)\tilde{f}_x\left(b-\frac{t}{2\left[(m+n)-1/\tau\right]f\left(\xi_q\right)}\middle|\,\mu,\tau\right)dt \\
= &K\int_{t=0}^{\infty}\tilde{\bar{F}}_{\widehat{w}}\left(\tau\, t|x,\mu,\tau\right)\tilde{f}_x\left(b-K\,t|\mu,\tau\right)dt \\
= &K\sqrt{\frac{\tau}{2\pi\Lambda}}\int_{t=0}^{\infty}\left[\exp\left(-\frac{t}{2}\right)\exp\left(-\frac{\left(b-Kt-\xi_q-\mu\right)^2}{2\Lambda}\right)\right]^{\tau}dt,
\end{aligned}$$

where we change the variable of integration to $t = (b-x)/K$, and

$$K \equiv \frac{1}{2\left[(m+n)-1/\tau\right]f\left(\xi_q\right)}. \tag{74}$$

[41] This result applies even in the presence of two populations because our result on the asymptotic normality of x with two populations can be applied within the proof in Siddiqui (1960).

The integral above is equal to

$$\exp\left(-\frac{\tau}{8}\frac{4K(b-\xi_q-\mu)-\Lambda}{K^2}\right)\Phi\left(\frac{\sqrt{\tau}}{2}\frac{2K(b-\xi_q-\mu)-\Lambda}{\sqrt{\Lambda}K}\right),$$

where Φ is the distribution function of the standard normal. To get a simpler expression, though, we approximate the integral by, see (Fibich and Gavious 2010, Lemma 2),[42]

$$\begin{aligned}
& K\sqrt{\frac{\tau}{2\pi\Lambda}}\int_{t=0}^{\infty}\left[\exp\left(-\frac{t}{2}\right)\exp\left(-\frac{(b-Kt-\xi_q-\mu)^2}{2\Lambda}\right)\right]^{\tau}dt \\
= \quad & K\sqrt{\frac{\tau}{2\pi\Lambda}}\frac{1}{\tau}2\frac{\exp\left[-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\tau}}\right]}{1-\frac{2K}{\Lambda}(b-\xi_q-\mu)}\left[1+O\left(\frac{1}{\tau}\right)\right] \\
= \quad & \frac{2K}{\tau}\frac{1}{1-\frac{2K}{\Lambda}(b-\xi_q-\mu)}\tilde{f}_x\left(b|\mu,\tau\right)\left[1+O\left(\frac{1}{\tau}\right)\right] \\
= \quad & \left[\frac{2K}{\tau}\frac{1}{1-\frac{2K}{\Lambda}(b-\xi_q-\mu)}+O\left(\frac{1}{\tau^2}\right)\right]\tilde{f}_x\left(b|\mu,\tau\right).
\end{aligned}$$

Plugging this back into (72) gives,

$$\begin{aligned}
\Pr[x<b<y|v,\,\tau] \quad = \quad & \int_{\mu=-\infty}^{\infty}\frac{2K}{\tau}\frac{1}{1-\frac{2K}{\Lambda}(b-\xi_q-\mu)}\tilde{f}_x\left(b|\mu,\tau\right)f(\mu-v)d\mu \\
& +O\left(\frac{1}{\tau^2}\right)\tilde{f}_x(b|v,\tau) \\
= \quad & \frac{2K}{\tau}\int_{\alpha=-\infty}^{\infty}\frac{1}{1-\frac{2K}{\Lambda}\alpha}\frac{1}{\sqrt{2\pi\frac{\Lambda}{\tau}}}\exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\tau}}\right)f(\lambda+\xi_q+\alpha)d\alpha \\
& +O\left(\frac{1}{\tau^2}\right)\tilde{f}_x(b|v,\tau).
\end{aligned}$$

where we change the variable of integration to $\alpha=b-\xi_q-\mu=b-(\mu+\xi_q)$ and $\lambda=v-b$.

Up to here we have not specified the distribution of ε. However, to be able to compute the above integral in closed form and get a simple approximate formula for the offset we are going to restrict attention to the case of the standard normal distribution.

[42]This lemma (with trivial modifications) states that for real function $R\in\mathcal{C}^2$, with $r=R'>0$, and $R(\infty)=0$ then

$$\int_0^{\infty}R^n(t)dt=-\frac{1}{n}\frac{R^{n+1}(0)}{r(0)}\left[1+O\left(\frac{1}{n}\right)\right].$$

Standard normal distribution case. Let the idiosyncratic component ε be standard normal, i.e., $F = \Phi$ $(f = \phi)$, so that

$$f(\mu - v) = f(v - \mu) = \phi(v - \mu) = \exp(-(v - \mu)^2/2)/\sqrt{2\pi}.$$

To calculate the integral above explicitly we will take a series expansion of $\left[1 - \frac{2K}{\Lambda}\alpha\right]^{-1}$ around zero, i.e., write

$$\frac{1}{1 - \frac{2K}{\Lambda}\alpha} = 1 + \frac{2K}{\Lambda}\alpha + \left[\frac{2K}{\Lambda}\alpha\right]^2 + O\left(\left[\frac{2K}{\Lambda}\alpha\right]^3\right).$$

Substituting and computing the integrals bearing in mind that for any symmetric regular equilibrium strategy λ is $O\left(1/\tau\right)$ yields:

$$\begin{aligned}
\Pr[x < b < y | v, \tau] &= \frac{2K}{\tau}\left(\frac{1}{1 + \frac{\Lambda}{\tau}} + O\left(\frac{1}{\tau}\right)\right)\left[\frac{1}{\sqrt{2\pi\left(1 + \frac{\Lambda}{\tau}\right)}}\exp\left(-\frac{\lambda^2}{2\left(1 + \frac{\Lambda}{\tau}\right)}\right)\right] + \\
&\quad + O\left(\frac{1}{\tau^2}\right)\tilde{f}_x(b|v, \tau) \\
&= \left[\frac{2K}{\tau}\frac{1}{1 + \frac{\Lambda}{\tau}} + O\left(\frac{1}{\tau^2}\right)\right]\tilde{f}_x(b|v, \tau),
\end{aligned}$$

where the second line follows from observing that the term in the square brackets in the first line is exactly equal to $\tilde{f}_x(b|v, \tau)$, i.e., the asymptotic distribution of the $(\tau m)^{\text{th}}$ order statistic given v. Given this the *asymptotic* first order condition of the focal buyer is:

$$(v - b)\tilde{f}_x(b|v, \tau) - \left(\frac{2K}{\tau + \Lambda} + O\left(\frac{1}{\tau^2}\right)\right)\tilde{f}_x(b|v, \tau) \quad = \quad 0,$$

which yields a unique solution for the offset $v - b$,

$$\lambda^* \quad = \quad \frac{2K}{\tau + \Lambda} + O\left(\frac{1}{\tau^2}\right) = \frac{1}{(m + n)\phi(\xi_q)}\frac{1}{\tau + C(m, n)} + O\left(\frac{1}{\tau^2}\right),$$

where

$$C(m, n) \quad = \quad \frac{mn - (m + n)^2\phi^2(\xi_q)}{(m + n)^3\phi^2(\xi_q)}.$$

Using the fact that $1/\varphi < m/n < \varphi$ we get the following approximate formula for the buyers' optimal offset strategy,[43]

$$\lambda_{\text{approx}} \quad = \quad \frac{1}{(m + n)\phi(\xi_q)}\frac{1}{\tau},$$

[43]What we mean is that $\lambda_{\text{approx}} \doteq \lambda^*$ in the sense that $\lim_{\tau \to \infty} \lambda^*/\lambda_{\text{approx}} = 1$.

for arbitrary values of m, n and τ. ■

E Asymptotic Distributions of Order Statistics

E.1 Asymptotic normality of the q^{th} quantile of the whole sample excluding an focal buyer

For a fixed number m of buyers and n of sellers, we consider a sequence of markets with τm buyers and τn sellers for $\tau \in \mathbb{N}^*$. For every τ we pick an focal buyer, whose value (without loss of generality) is $v_{\tau m}$ and look at the vector of bids of the $\tau m - 1$ remaining buyers $\{b_1, \ldots, b_{\tau m-1}\}$ and the vector of asks (or costs since sellers bid truthfully) $\{c_1, \ldots, c_{\tau n}\}$ of the τn sellers. Buyers play a symmetric regular equilibrium strategy $b_i = B_\tau(v_i)$ for all i (where we are explicitly on the dependence on τ) and from Theorem 5 and Remark 12,

$$\lim_{\tau\to\infty} \sup_{x\in\mathbb{R}} \left\{ \sqrt{\tau} \left| B_\tau^{-1}(x) - x \right| \right\} \quad = \quad 0,$$

which also implies that

$$\lim_{\tau\to\infty} \sup_{x\in\mathbb{R}} \left| B_\tau^{-1}(x) - x \right| \quad = \quad 0.$$

For notational brevity let

$$\lambda(x;\tau) \quad \equiv \quad B_\tau^{-1}(x) - x,$$

of course $\lambda(x;\tau) > 0$, for all $x \in \mathbb{R}$ and $\tau \in \mathbb{N}_+$. However, since we are not constraining attention to an offset strategy $\lambda(x;\tau)$ need not be a constant for all x given a τ.

Also, define the empirical distribution of the sample of bids and asks of the $\tau(m+n)-1$ traders remaining after we exclude the focal buyer,

$$\begin{aligned}
\tilde{F}_{\tau(m+n)-1}(x) \quad &\equiv \quad \frac{1}{\tau(m+n)-1} \sum_{i=1}^{\tau m-1} \mathbb{I}\{b_i \le x\} + \frac{1}{\tau(m+n)-1} \sum_{i=1}^{\tau n} \mathbb{I}\{c_i \le x\} \\
&= \quad \frac{1}{\tau(m+n)-1} \sum_{i=1}^{\tau m-1} \mathbb{I}\{v_i \le x + \lambda(x;\tau)\} + \frac{1}{\tau(m+n)-1} \sum_{i=1}^{\tau n} \mathbb{I}\{c_i \le x\},
\end{aligned}$$

which allows us to define the q^{th} quantile from this population

$$\tilde{\xi}_{q[\tau(m+n)-1]} \quad \equiv \quad \inf\{y : \tilde{F}_{\tau(m+n)-1}(y) \le q\},$$

where $\mathbb{I}\{\cdot\}$ is the indicator function.

Slightly abusing the notation in the main text for purposes of this section we will denote the distribution and density of values and consts conditional on μ as F_μ and f_μ, respectively. Of course conditional on μ values and costs are i.i.d. with $F_\mu(x) = F(x-\mu)$ and $f_\mu(x) = f(x-\mu)$, for all $(x,\mu) \in \mathbb{R}^2$. As before $\bar{F}_\mu = 1 - F_\mu$. Also by definition

$$F_\mu(x) \quad = \quad \lim_{\tau\to\infty} \frac{1}{\tau\, m} \sum_{i=1}^{\tau m} \mathbb{I}\{v_i \leq x\}.$$

The q^{th} quantile of the idiosyncratic component's distribution is

$$\xi_q \quad \equiv \quad \inf\{y : F(y) \leq q\} = F^{-1}(q),$$

so that

$$\xi_q + \mu \quad = \quad F_\mu^{-1}(q).$$

The following result establishes the asymptotic relationship between $\tilde{\xi}_{q[\tau(m+n)-1]}$ and ξ_q *conditional* on μ.

Theorem 11 *For all $t \in \mathbb{R}$, and $0 < q < 1$,*

$$\lim_{\tau\to\infty} \Pr\left(\frac{\sqrt{\tau(m+n)-1}\left(\tilde{\xi}_{q[\tau(m+n)-1]} - \xi_q - \mu\right)}{\sqrt{q(1-q)}/f\left(\xi_q\right)} \leq t\right) \quad = \quad \Phi(t),$$

where $\Phi(\cdot)$ is the standard normal distribution function. So

$$\tilde{\xi}_{q[\tau(m+n)-1]} \quad \sim \quad AN\left(\xi_q + \mu, \frac{q(1-q)}{(\tau(m+n)-1)f^2(\xi_q)}\right).$$

Proof. This proof is based on the proof of Serfling (1980, Theorem 2.3.3 A).

Let $A > 0$ be a normalizing constant to be specified later, and put

$$\begin{aligned}
G_{\tau(m+n)-1}(t) \quad &\equiv \quad \Pr\left(\frac{\sqrt{\tau(m+n)-1}\left(\tilde{\xi}_{q[\tau(m+n)-1]} - \xi_q - \mu\right)}{A} \leq t\right) \\
&= \quad \Pr\left(\tilde{\xi}_{q[\tau(m+n)-1]} \leq \xi_q + \mu + t\, A \sqrt{\tau(m+n)-1}^{-1}\right) \\
&= \quad \Pr\left(\tilde{F}_{\tau(m+n)-1}\left(\xi_q + \mu + t\, A\sqrt{\tau(m+n)-1}^{-1}\right) \geq q\right),
\end{aligned}$$

where the last line follows from Serfling (1980, Lemma 1.1.4 (iii)). Let

$$\Delta \equiv \xi_q + \mu + t\,A\sqrt{\tau(m+n)-1}^{-1},$$

then $\tilde{F}_{\tau(m+n)-1}(\Delta)$ is a random variable, with mean, and variance[44]

$$\begin{aligned}
\mathbb{E}[\tilde{F}_{\tau(m+n)-1}(\Delta)] &= \frac{\tau m-1}{\tau(m+n)-1}F_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{\tau(m+n)-1}F_\mu(\Delta), \\
Var[\tilde{F}_{\tau(m+n)-1}(\Delta)] &= \frac{\tau m-1}{(\tau(m+n)-1)^2}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) \\
&\quad + \frac{\tau n}{(\tau(m+n)-1)^2}F_\mu(\Delta)\bar{F}_\mu(\Delta).
\end{aligned}$$

So

$$\begin{aligned}
G_{\tau(m+n)-1}(t) &= \Pr\left(\tilde{F}_{\tau(m+n)-1}\left(\xi_q+\mu+tA\sqrt{\tau(m+n)-1}^{-1}\right)\geq q\right) \\
&= \Pr\left(\frac{\tilde{F}_{\tau(m+n)-1}(\Delta)-\mathbb{E}[\tilde{F}_{\tau(m+n)-1}(\Delta)]}{\sqrt{Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]}} \geq \frac{p-\mathbb{E}[\tilde{F}_{\tau(m+n)-1}(\Delta)]}{\sqrt{Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]}}\right) \\
&= \Pr\left(\tilde{F}^*_{\tau(m+n)-1}(\Delta)\geq c(\Delta)\right),
\end{aligned}$$

with straightforward definitions for $\tilde{F}^*_{\tau(m+n)-1}(\Delta)$ and $c(\Delta)$. By invoking the Lindeberg-Feller Central Limit Theorem (see Serfling (1980, Theorem 1.9.2 A)) for $t=0$ we can get that,

$$\lim_{\tau\to\infty}\Pr[\sqrt{\tau(m+n)-1}(\tilde{\xi}_{q[\tau(m+n)-1]}-\xi_q)\geq 0] = \Phi(0)=\frac{1}{2}.$$

Using the Berry-Esseen Theorem (see Serfling (1980, Theorem 1.9.5) for i.i.d. random variables and Batirov, Mavenich, and Nagaev (1977)) for i.ni.d. random variables as it is the case here, we have

$$\sup_{x\in\mathbb{R}}\left|\Pr\left(\tilde{F}^*_{\tau(m+n)-1}(\Delta)\leq x\right)-\Phi(x)\right| \leq K\frac{\beta(\Delta)}{\left[(\tau(m+n)-1)^2\,Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]\right]^3},$$

where K is a universal constant, and

$$\beta(x) \equiv \sum_{i=1}^{\tau m-1}\mathbb{E}\left[\left|\mathbb{I}\{v_i\leq x+\lambda(x;\tau)\}-F_\mu\left(x+\lambda(x;\tau)\right)\right|^3\right]+\sum_{i=1}^{\tau n}\mathbb{E}\left[\left|\mathbb{I}\{c_i\leq x\}-F_\mu(x)\right|^3\right].$$

[44]The dependence of Δ on μ, τ, and t will be suppressed for notational brevity.

It can be shown that

$$
\begin{aligned}
\mathbb{E}\left[\left|\mathbb{I}\{v_i \leq x+\lambda(x;\tau)\} - F_\mu\left(x+\lambda(x;\tau)\right)\right|^3\right] &= R(x;\tau)\,\bar{R}(x;\tau)\left[\bar{R}^2(x;\tau)+R^2(x;\tau)\right], \\
\mathbb{E}\left[\left|\mathbb{I}\{c_i \leq x\} - F_\mu(x)\right|^3\right] &= F_\mu(x)\,\bar{F}_\mu(x)\left[\bar{F}_\mu^2(x)+F_\mu^2(x)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
R(x;\tau) &\equiv F_\mu\left(x+\lambda(x;\tau)\right), \\
\bar{R}(x;\tau) &\equiv \bar{F}_\mu\left(x+\lambda(x;\tau)\right) = 1 - F_\mu\left(x+\lambda(\Delta;\tau)\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
\left|\Pr\left(\tilde{F}^*_{\tau(m+n)-1}(\Delta) \geq c(\Delta)\right) - \Phi(t)\right| \leq\; & K\frac{\beta(\Delta)}{\left[(\tau(m+n)-1)^2\,Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]\right]^3} \\
& + \left|\Phi(t) - \Phi\left(-c(\Delta)\right)\right|. \qquad (75)
\end{aligned}
$$

We need to show that

$$
\lim_{\tau\to\infty} \frac{\beta(\Delta)}{\left[(\tau(m+n)-1)^2\,Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]\right]^3} = 0. \qquad (76)
$$

We have

$$
\begin{aligned}
& \frac{\beta(\Delta)}{\left[(\tau(m+n)-1)^2\,Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]\right]^3} = \\
& \frac{(\tau m-1)\,R(\Delta;\tau)\,\bar{R}(\Delta;\tau)\left[\bar{R}^2(\Delta;\tau)+R^2(x;\tau)\right] + \tau n\,F_\mu(\Delta)\,\bar{F}_\mu(\Delta)\left[\bar{F}^2(\Delta)+F^2(\Delta)\right]}{\left[(\tau m-1)\,R(\Delta;\tau)\bar{R}(\Delta;\tau)+\tau n\,F_\mu(\Delta)\bar{F}_\mu(\Delta)\right]^3} = , \\
& \frac{(m-1/\tau)\,R(\Delta;\tau)\,\bar{R}(\Delta;\tau)\left[\bar{R}^2(\Delta;\tau)+R^2(x;\tau)\right] + n\,F_\mu(\Delta)\,\bar{F}_\mu(\Delta)\left[\bar{F}^2(\Delta)+F^2(\Delta)\right]}{\left[(\tau^{2/3}\,m-\tau^{-1/3})\,R(\Delta;\tau)\bar{R}(\Delta;\tau)+\tau^{2/3}\,n\,F_\mu(\Delta)\bar{F}_\mu(\Delta)\right]^3}.
\end{aligned}
$$

Taking the limit of $\tau\to\infty$, bearing in mind that

$$
\begin{aligned}
\lim_{\tau\to\infty}\lambda(\Delta;\tau) &= 0, \\
\lim_{\tau\to\infty}\Delta &= \xi_q+\mu, \\
\lim_{\tau\to\infty}F_\mu(\Delta) &= \lim_{\tau\to\infty}R(\Delta;\tau) = q, \\
\lim_{\tau\to\infty}\bar{F}_\mu(\Delta) &= \lim_{\tau\to\infty}\bar{R}(\Delta;\tau) = 1-q,
\end{aligned}
$$

yields 0, which proves (76). To complete the proof we have to find appropriate constant A such that

$$\lim_{\tau\to\infty} c(\Delta) \quad = \quad -t. \tag{77}$$

We have

$$\begin{aligned}
c(\Delta) &= \frac{q - \mathbb{E}[\tilde{F}_{\tau(m+n)-1}(\Delta)]}{\sqrt{Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]}} \\
&= -\frac{\mathbb{E}[\tilde{F}_{\tau(m+n)-1}(\Delta)] - F_\mu(\xi_q+\mu)}{\sqrt{Var[\tilde{F}_{\tau(m+n)-1}(\Delta)]}} \\
&= -\frac{\frac{\tau m-1}{\tau(m+n)-1}F_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{\tau(m+n)-1}F_\mu(\Delta) - F_\mu(\xi_q+\mu)}{\sqrt{\frac{\tau m-1}{(\tau(m+n)-1)^2}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{(\tau(m+n)-1)^2}F_\mu(\Delta)\bar{F}_\mu(\Delta)}} \\
&= -\frac{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu}{\sqrt{\frac{\tau m-1}{(\tau(m+n)-1)^2}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{(\tau(m+n)-1)^2}F_\mu(\Delta)\bar{F}_\mu(\Delta)}} \\
&\quad \times \frac{\frac{\tau m-1}{\tau(m+n)-1}F_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{\tau(m+n)-1}F_\mu(\Delta) - F_\mu(\xi_q+\mu)}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\frac{\tau(m+n)-1}{\sqrt{\tau}}\left(\Delta+\lambda(\Delta;\tau)-\xi_q-\mu\right)}{\sqrt{\frac{\tau m-1}{\tau}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) + n\,F_\mu(\Delta)\bar{F}_\mu(\Delta)}} \\
&\quad \times \frac{\frac{\tau m-1}{\tau(m+n)-1}F_\mu(\Delta+\lambda(\Delta;\tau)) + \frac{\tau n}{\tau(m+n)-1}F_\mu(\Delta) - F_\mu(\xi_q+\mu)}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu} \\
&= -\frac{\frac{\tau(m+n)-1}{\sqrt{\tau}}\left(\Delta+\lambda(\Delta;\tau)-\xi_q-\mu\right)}{\sqrt{\frac{\tau m-1}{\tau}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) + n\,F_\mu(\Delta)\bar{F}_\mu(\Delta)}} \\
&\quad \times \left\{\frac{\tau m-1}{\tau(m+n)-1}\frac{F_\mu(\Delta+\lambda(\Delta;\tau)-\xi_q-\mu+\xi_q+\mu) - F_\mu(\xi_q+\mu)}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu}\right. \\
&\quad \left. + \frac{\tau n}{\tau(m+n)-1}\frac{F_\mu(\Delta-\xi_q-\mu+\xi_q+\mu) - F_\mu(\xi_q+\mu)}{\Delta-\xi_q-\mu} \times \frac{\Delta-\xi_q-\mu}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu}\right\}.
\end{aligned}$$

Now, note that

$$
\begin{aligned}
\lim_{\tau\to\infty} \Delta + \lambda(\Delta;\tau) - \xi_q - \mu &= \lim_{\tau\to\infty} t\,A\sqrt{\tau(m+n)-1}^{-1} + \lambda(\Delta;\tau) = 0,\\
\lim_{\tau\to\infty} \Delta - \xi_q - \mu &= \lim_{\tau\to\infty} t\,A\sqrt{\tau(m+n)-1}^{-1} = 0,
\end{aligned}
$$

so that,

$$
\begin{aligned}
\lim_{\tau\to\infty} \frac{\tau\,m-1}{\tau(m+n)-1}\frac{F_\mu(\Delta+\lambda(\Delta;\tau)-\xi_q-\mu+\xi_q+\mu)-F_\mu(\xi_q+\mu)}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu} &= \frac{m}{m+n}F'(\xi_q) = \frac{m}{m+n}f(\xi_q),\\
\lim_{\tau\to\infty} \frac{\tau n}{\tau(m+n)-1}\frac{F_\mu(\Delta-\xi_q-\mu+\xi_q+\mu)-F_\mu(\xi_q+\mu)}{\Delta-\xi_q-\mu} &= \frac{n}{m+n}F'(\xi_q) = \frac{n}{m+n}f(\xi_q).
\end{aligned}
$$

Also,

$$
\lim_{\tau\to\infty} \sqrt{\frac{\tau\,m-1}{\tau}F_\mu(\Delta+\lambda(\Delta;\tau))\bar{F}_\mu(\Delta+\lambda(\Delta;\tau)) + n\,F_\mu(\Delta)\bar{F}_\mu(\Delta)} = \sqrt{(m+n)q(1-q)}.
$$

Furthermore recall that $\lambda(\Delta;\tau)$ is $O(1/\tau)$ to get

$$
\begin{aligned}
\lim_{\tau\to\infty} \frac{\tau(m+n)-1}{\sqrt{\tau}}\left(\Delta+\lambda(\Delta;\tau)-\xi_q-\mu\right) &= \lim_{\tau\to\infty} \frac{\tau(m+n)-1}{\sqrt{\tau}}\Big(t\,A\sqrt{\tau(m+n)-1}^{-1} + O(1/\tau)\Big)\\
&= \lim_{\tau\to\infty}\left(\sqrt{\frac{\tau(m+n)-1}{\tau}}t\,A + O\left(\frac{1}{\sqrt{\tau}}\right)\right)\\
&= \sqrt{m+n}\,t\,A,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\tau\to\infty} \frac{\Delta-\xi_q-\mu}{\Delta+\lambda(\Delta;\tau)-\xi_q-\mu} &= \lim_{\tau\to\infty} \frac{t\,A\sqrt{\tau(m+n)-1}^{-1}}{t\,A\sqrt{\tau(m+n)-1}^{-1}+\lambda(\Delta;\tau)}\\
&= \lim_{\tau\to\infty} \frac{t\,A}{t\,A + O\left(\frac{1}{\sqrt{\tau}}\right)}\\
&= 1.
\end{aligned}
$$

Putting all these together we get,

$$
\lim_{\tau\to\infty} c(\Delta) = -\frac{t\,A}{\sqrt{q(1-q)}}f(\xi_q),
$$

so to get (77) we have to pick,

$$
A = \frac{\sqrt{q(1-q)}}{f(\xi_q)}.
$$

Putting (76) and (77) we get from (75) that,

$$\lim_{\tau\to\infty} \left| \Pr\left(\tilde{F}^{*}_{\tau(m+n)-1}(\Delta) \geq c(\Delta) \right) - \Phi(t) \right| = 0,$$

which establishes the desired result. ■

E.2 Asymptotic normality of the q^{th} quantile of the whole sample

Now if we have the vector of all buyers' bids and seller's asks/costs we can similarly define the empirical distribution of the whole sample of $\tau(m+n)$ bids and asks,

$$\begin{aligned} \tilde{F}_{\tau(m+n)}(x) &\equiv \frac{1}{\tau(m+n)} \sum_{i=1}^{\tau m} \mathbb{I}\{b_i \leq x\} + \frac{1}{\tau(m+n)} \sum_{i=1}^{\tau n} \mathbb{I}\{c_i \leq x\} \\ &= \frac{1}{\tau(m+n)} \sum_{i=1}^{\tau m} \mathbb{I}\{v_i \leq x + \lambda(x;\tau)\} + \frac{1}{\tau(m+n)} \sum_{i=1}^{\tau n} \mathbb{I}\{c_i \leq x\}, \end{aligned}$$

and its q^{th} quantile

$$\tilde{\xi}_{q[\tau(m+n)]} \equiv \inf\{y : \tilde{F}_{\tau(m+n)}(y) \leq q\}.$$

We present the following Theorem that establishes the asymptotic relationship between $\tilde{\xi}_{q[\tau(m+n)]}$ and ξ_q *conditional* on μ. The proof is identical with that of Theorem 11, with the terms $\tau\, m - 1$, and $\tau(m+n) - 1$ replaced by $\tau\, m$ and $\tau(m+n)$, respectively.

Theorem 12 *For all $t \in \mathbb{R}$, and $0 < q < 1$,*

$$\lim_{\tau\to\infty} \Pr\left(\frac{\sqrt{\tau(m+n)}\left(\tilde{\xi}_{q[\tau(m+n)]} - \xi_q - \mu\right)}{\sqrt{q(1-q)}/f\left(\xi_q\right)} \leq t \right) = \Phi(t),$$

where $\Phi(\cdot)$ is the standard normal distribution function. So

$$\tilde{\xi}_{q[\tau(m+n)]} \sim AN\left(\xi_q + \mu, \frac{q(1-q)}{\tau(m+n)\, f^2(\xi_q)}\right).$$

E.3 Relating quantiles to order statistics

We are interested in the

$$q = \frac{m}{m+n}$$

quantile. Let $s_{\tau m:\tau(m+n)-1}$, $s_{\tau m+1:\tau(m+n)-1}$ be the $(taum)^{\text{th}}$, $(\tau m+1)^{\text{st}}$ order statistic of the sample of $\tau(m+n)-1$ traders which excludes the focal buyer; this is of particular importance because these are the order statistics the focal buyer uses to make his decision of how much to bid. Similarly let $s_{\tau m+1:\tau(m+n)}$ be the $(\tau m+1)^{\text{st}}$ order statistic of the sample of all $\tau(m+n)$ traders; this is of particular importance because it is the market price in the BBDA.

Notice that,

$$\begin{aligned}
\frac{\tau m}{\tau(m+n)-1} &= \frac{m}{m+n}+\frac{m}{(m+n)(\tau(m+n)-1)} = \frac{m}{m+n}+o\left(\frac{1}{\tau^{1-\epsilon}}\right),\ \tau\to\infty,\\
\frac{\tau m+1}{\tau(m+n)-1} &= \frac{m}{m+n}+\frac{2\,m+n}{(m+n)(\tau(m+n)-1)} = \frac{m}{m+n}+o\left(\frac{1}{\tau^{1-\epsilon}}\right),\ \tau\to\infty,\\
\frac{\tau m+1}{\tau(m+n)} &= \frac{m}{m+n}+\frac{1}{\tau(m+n)} = \frac{m}{m+n}+o\left(\frac{1}{\tau^{1-\epsilon}}\right),\ \tau\to\infty,
\end{aligned}$$

for any small $\epsilon>0$. Then we can invoke Serfling (1980, Theorem and Corollary 2.5.2) to get the asymptotic relationships between $s_{\tau m:\tau(m+n)-1}$, $s_{\tau m+1:\tau(m+n)-1}$, $s_{\tau m+1:\tau(m+n)}$ and ξ_q *conditional* on μ.

Theorem 13 *We have*

$$\begin{aligned}
&(i) && s_{\tau m:\tau(m+n)-1}\sim AN\left(\xi_q+\mu,\frac{m\,n/(m+n)^2}{[\tau(m+n)-1]\,f^2(\xi_q)}\right),\\
&(ii) && s_{\tau m+1:\tau(m+n)-1}\sim AN\left(\xi_q+\mu,\frac{m\,n/(m+n)^2}{[\tau(m+n)-1]\,f^2(\xi_q)}\right),\\
&(iii) && s_{\tau m+1:\tau(m+n)}\sim AN\left(\xi_q+\mu,\frac{m\,n/(m+n)^2}{\tau(m+n)\,f^2(\xi_q)}\right).
\end{aligned}$$

F Correlated Interdependent Values: First Order Approach

F.1 Buyer

Pick a focal buyer with signal σ_B and unknown value v. Let x, y be the m^{th} and $(m+1)^{\text{st}}$ order statistic of the other traders' bids/asks. The selected buyer's expected utility when he bids b is

$$\begin{aligned}
\pi^B(\sigma_B,b;B,S) &= \int\!\!\int\!\!\int v\chi_{x<y<b}f^B_{vxy}(v,x,y|\sigma_B)dvdxdy-\int\!\!\int\!\!\int y\chi_{x<y<b}f^B_{vxy}(v,x,y|\sigma_B)dvdxdy\\
&+\int\!\!\int\!\!\int v\chi_{x\le b\le y}f^B_{vxy}(v,x,y|\sigma_B)dvdxdy-\int\!\!\int\!\!\int b\chi_{x\le b\le y}f^B_{vxy}(v,x,y|\sigma_B)dvdxdy,
\end{aligned}$$

where,

$$
\begin{aligned}
f^B_{vxy}(v,x,y|\sigma_B) &= \int_{-\infty}^{\infty} f^B_{vxy\mu}(v,x,y,\mu|\sigma_B)d\mu = \int_{-\infty}^{\infty} f^B_{xy\mu}(x,y,\mu|v,\sigma_B)f_v(v|\sigma_B)d\mu \\
&= \int_{-\infty}^{\infty} f^B_{xy}(x,y|\mu,v,\sigma_B)f_\mu(\mu|v,\sigma_B)f_v(v|\sigma_B)d\mu = \int_{-\infty}^{\infty} f^B_{xy}(x,y|\mu)f_\mu(\mu|v)f_v(v|\sigma_B)d\mu \\
&= \int_{-\infty}^{\infty} f^B_{xy}(x,y|\mu)f(v-\mu)g(\sigma_B-v)d\mu.
\end{aligned}
$$

Notice that from the *diffuse prior* on μ, we have

$$f_\mu(\mu|\sigma_B) = f_\sigma(\sigma_B|\mu) = \int_{-\infty}^{\infty} f(v-\mu)g(\sigma_B-v)dv.$$

Let

$$h(\sigma_B-\mu) = \int_{-\infty}^{\infty} f(v-\mu)g(\sigma_B-v)dv,$$

i.e.,

$$h(s) = (f\star g)(s) = \int_{-\infty}^{\infty} f(t)\,g(s-t)dt,$$

where $\star$ is the standard convolution operation, and of course $H(s) = \int_{-\infty}^{s} h(t)dt$. Also let,

$$V(\mu,\sigma_B) = \int_{-\infty}^{\infty} v f(v-\mu)g(\sigma_B-v)dv,$$

so that

$$\frac{V(\mu,\sigma_B)}{f_\mu(\mu|\sigma_B)} = \mathbb{E}[v|\mu,\sigma_B].$$

Rewriting the expected utility for the exceptional buyer we have,

$$\begin{aligned}
\pi_B(\sigma_B, b; B, S) &= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b}\int_{x}^{b} f_{xy}^B(x,y|\mu)\left(V(\mu,\sigma_B) - y f_\mu(\mu|\sigma_B)\right) dydx\right. \\
&\quad \left. + \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)\left(V(\mu,\sigma_B) - b f_\mu(\mu|\sigma_B)\right) dydx\right\} d\mu \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b}\int_{x}^{b} f_{xy}^B(x,y|\mu)\left(\frac{V(\mu,\sigma_B)}{f_\mu(\mu|\sigma_B)} - y\right) dydx\right. \\
&\quad \left. + \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)\left(\frac{V(\mu,\sigma_B)}{f_\mu(\mu|\sigma_B)} - b\right) dxdy\right\} f_\mu(\mu|\sigma_B)d\mu \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b}\int_{x}^{b} f_{xy}^B(x,y|\mu)\left(\mathbb{E}[v|\mu,\sigma_B] - y\right) dydx\right. \\
&\quad \left. + \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)\left(\mathbb{E}[v|\mu,\sigma_B] - b\right) dydx\right\} f_\mu(\mu|\sigma_B)d\mu \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{b} f_{y}^B(y|\mu)\left(\mathbb{E}[v|\mu,\sigma_B] - y\right) dy\right. \\
&\quad \left. + \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)\left(\mathbb{E}[v|\mu,\sigma_B] - b\right) dydx\right\} f_\mu(\mu|\sigma_B)d\mu.
\end{aligned}$$

Taking the derivative with respect to b produces the first order condition for the buyers in the CIV case,

$$\int_{-\infty}^{\infty}\left\{\left(\mathbb{E}[v|\mu,\sigma_B] - b\right) f_x^B(b|\mu) - \int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)dydx\right\} f_\mu(\mu|\sigma_B)d\mu = 0,$$

which $b = B(\sigma_B)$ satisfies in equilibrium. Following similar notation with that of the previous sections let $\int_{-\infty}^{b}\int_{b}^{\infty} f_{xy}^B(x,y|\mu)dydx = M_{m,n}^B(b;\sigma_B,\mu,B,S)$, and we will express $f_x^B(b|\mu)$ in terms of the probabilities $K_{m,n}^B(b;\sigma_B,\mu,B,S)$ and $L_{m,n}^B(b;\sigma_B,\mu,B,S)$.

Deriving $f_x^B(b|\mu)$ and $M_{m,n}^B(b;\mu,\sigma_B,B,S)$. Given μ all other traders' signals are $h(\sigma-\mu)$.[45] Therefore

1. $\mathbb{P}[B(\sigma) \leq B(\sigma_B)|\mu] = \mathbb{P}[\sigma \leq \sigma_B|\mu] = H\left(\sigma_B - \mu\right).$
2. $\mathbb{P}[S(\sigma) \leq B(\sigma_B)|\mu] = \mathbb{P}[\sigma \leq S^{-1}\circ B(\sigma_B)|\mu] = H\left(S^{-1}\circ B(\sigma_B) - \mu\right).$
3. $\mathbb{P}[B(\sigma_B) \leq B(\sigma) \leq B(\sigma_B) + \Delta b|\mu] = \mathbb{P}[\sigma_B \leq \sigma \leq B^{-1}(B(\sigma_B) + \Delta b)|\mu]$. We look for Δx such

[45] For other traders' signals we use a generic σ since subscripting is superfluous.

that,

$$\begin{aligned}\sigma_B + \Delta x &= B^{-1}(B(\sigma_B) + \Delta b) \Rightarrow \\ B(\sigma_B + \Delta x) &= B(\sigma_B) + \Delta b.\end{aligned}$$

Now for smooth functions $B(\cdot)$ and small Δx we have $B(\sigma_B + \Delta x) \approx B(\sigma_B) + \Delta x\, B'(\sigma_B)$ and so,

$$\begin{aligned}B(\sigma_B) + \Delta x\, B'(\sigma_B) &= B(\sigma_B) + \Delta b \Rightarrow \\ \Delta x &= \Delta b / B'(\sigma_B).\end{aligned}$$

So,

$$\begin{aligned}\mathbb{P}[B(\sigma_B) \leq B(\sigma) \leq B(\sigma_B) + \Delta b|\mu]/\Delta b &= \mathbb{P}[\sigma_B \leq \sigma \leq \sigma_B + \Delta x|\mu]/B'(\sigma_B) \\ &= h\left(\sigma_B - \mu\right)/B'(\sigma_B).\end{aligned}$$

4. $\mathbb{P}[B(\sigma_B) \leq S(\sigma) \leq B(\sigma_B) + \Delta b|\mu] = \mathbb{P}[S^{-1} \circ B(\sigma_B) \leq \sigma \leq S^{-1}(B(\sigma_B) + \Delta b)|\mu]$. We look for Δx such that,

$$\begin{aligned}S^{-1} \circ B(\sigma_B) + \Delta x &= S^{-1}(B(\sigma_B) + \Delta b) \Rightarrow \\ S(S^{-1} \circ B(\sigma_B) + \Delta x) &= B(\sigma_B) + \Delta b.\end{aligned}$$

Now for smooth functions $S(\cdot)$ and small Δx we have $S(S^{-1} \circ B(\sigma_B) + \Delta x) \approx B(\sigma_B) + \Delta x\, S' \circ S^{-1} \circ B(\sigma_B)$ and so,

$$\begin{aligned}B(\sigma_B) + \Delta x\, S' \circ S^{-1} \circ B(\sigma_B) &= B(\sigma_B) + \Delta b \Rightarrow \\ \Delta x &= \Delta b / S' \circ S^{-1} \circ B(\sigma_B),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[B(\sigma_B) \leq S(\sigma) \leq B(\sigma_B) + \Delta b|\mu]/\Delta b &= \frac{\mathbb{P}[S^{-1} \circ B(\sigma_B) \leq \sigma \leq \sigma_B + \Delta x|\mu]}{S' \circ S^{-1} \circ B(\sigma_B)} \\ &= \frac{h\left(S^{-1} \circ B(\sigma_B) - \mu\right)}{S' \circ S^{-1} \circ B(\sigma_B)}.\end{aligned}$$

Then let $L^B_{m,n}(B(\sigma_B); \sigma_B, \mu, B, S)$ be the probability that from the population of the $m + n - 1$ remaining traders, another buyer's ask is the m^{th} order statistic, conditional on σ_B, μ and strategies

B, S. Then

$$L^B_{m,n}(B(\sigma_B);\sigma_B,\mu,B,S) = \sum_{\substack{i+j=m-1\\0\le i\le m-2\\0\le j\le n}} \begin{pmatrix} m-2 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} H\left(\sigma_B-\mu\right)^i H\left(S^{-1}\circ B(\sigma_B)-\mu\right)^j \cdot \bar{H}\left(\sigma_B-\mu\right)^{m-2-i} \bar{H}\left(S^{-1}\circ B(\sigma_B)-\mu\right)^{n-j}. \tag{78}$$

Similarly for a seller making the m^{th} order ask,

$$K^B_{m,n}(B(\sigma');\sigma,\mu,B,S) = \sum_{\substack{i+j=m-1\\0\le i\le m-1\\0\le j\le n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} H\left(\sigma_B-\mu\right)^i H\left(S^{-1}\circ B(\sigma_B)-\mu\right)^j \cdot \bar{H}\left(\sigma_B-\mu\right)^{m-1-i} \bar{H}\left(S^{-1}\circ B(\sigma_B)-\mu\right)^{n-1-j}. \tag{79}$$

Hence,

$$f^B_x(B(\sigma_B)|\mu) = (m-1)\frac{h\left(\sigma_B-\mu\right)}{B'(\sigma_B)}L^B_{m,n}(B(\sigma_B);\sigma_B,\mu,B,S) + n\frac{h\left(S^{-1}\circ B(\sigma_B)-\mu\right)}{S'\circ S^{-1}\circ B(\sigma_B)}K^B_{m,n}(B(\sigma_B);\sigma_B,\mu,B,S).$$

Also,

$$M^B_{m,n}(B(\sigma_B);\sigma_B,\mu,B,S) = \sum_{\substack{i+j=m\\0\le i\le m-1\\0\le j\le n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} H\left(\sigma_B-\mu\right)^i H\left(S^{-1}\circ B(\sigma_B)-\mu\right)^j \cdot \bar{H}\left(\sigma_B-\mu\right)^{m-1-i} \bar{H}\left(S^{-1}\circ B(\sigma_B)-\mu\right)^{n-j}. \tag{80}$$

F.2 Seller

Pick now a focal seller with signal σ_S, and cost c; let x, y be the m^{th}, $(m+1)^{\text{st}}$ statistic of the other traders' bids/asks (different from the ones for the focal buyer). The ex-post utility of the seller when he asks a is $(x-c)\chi_{a<x}$, and so his interim expected utility is,

$$\pi^S(\sigma_S,a;B,S) = \int\int\int (x-c)\chi_{a<x} f^S_{cxy}(c,x,y|\sigma_S)dcdxdy.$$

Following similar derivations as before and using symmetry one can arrive at the following,

$$\pi^S(\sigma_S,a;B,S) = \int_{-\infty}^{\infty}\left\{\int_a^{\infty}\left(x-\mathbb{E}[c|\mu,\sigma_S]\right)f^S_x(x|\mu)dx\right\}f_\mu(\mu|\sigma_S)d\mu.$$

Taking the derivative with respect to a one gets the first order condition for the sellers in the CIV case,

$$\int_{-\infty}^{\infty} (a - \mathbb{E}[c|\mu, \sigma_S])\, f_x^S(a|\mu) f_\mu(\mu|\sigma_S) d\mu = 0,$$

and $a = S(\sigma_S)$ in equilibrium. We will express $f_x^S(a|\mu)$ in terms of the probabilities $K_{m,n}^S(a; \sigma_S, \mu, B, S)$ and $L_{m,n}^S(a; \sigma_S, \mu, B, S)$.

Deriving $f_x^S(a|\mu)$. Similarly with before,

$$\begin{aligned}
\mathbb{P}[S(\sigma) \le S(\sigma_S)] &= H(\sigma_S - \mu), \\
\mathbb{P}[B(\sigma) \le S(\sigma_S)|\mu] &= H\left(B^{-1} \circ S(\sigma_S) - \mu\right), \\
\mathbb{P}[S(\sigma_S) \le S(\sigma) \le S(\sigma_S) + \Delta b|\mu]/\Delta b &= \frac{h(\sigma_S - \mu)}{S'(\sigma_S)}, \\
\mathbb{P}[S(\sigma_S) \le B(\sigma) \le S(\sigma_S) + \Delta b|\mu]/\Delta b &= \frac{h\left(B^{-1} \circ S(\sigma_S) - \mu\right)}{B' \circ B^{-1} \circ S(\sigma_S)}.
\end{aligned}$$

Let $L_{m,n}^S(S(\sigma_S); \sigma_S, \mu, B, S)$ be the probability that from the population of the $m + n - 1$ remaining traders, a buyer's ask is the m^{th} order statistic, conditional on σ_S, μ and strategies B, S. Then

$$\begin{aligned}
L_{m,n}^S(S(\sigma_S); \sigma_S, \mu, B, S) = & \sum_{\substack{i+j=m-1 \\ 0 \le i \le m-1 \\ 0 \le j \le n-1}} \binom{m-1}{i} \binom{n-1}{j} H\left(B^{-1} \circ S(\sigma_S) - \mu\right)^i H(\sigma_S - \mu)^j \\
& \cdot \bar{H}\left(B^{-1} \circ S(\sigma_S) - \mu\right)^{m-1-i} \bar{H}(\sigma_S - \mu)^{n-1-j}.
\end{aligned}$$

Similarly for another seller making the m^{th} order ask,

$$\begin{aligned}
K_{m,n}^S(S(\sigma_S); \sigma_S, \mu, B, S) = & \sum_{\substack{i+j=m-1 \\ 0 \le i \le m \\ 0 \le j \le n-2}} \binom{m}{i} \binom{n-2}{j} H\left(B^{-1} \circ S(\sigma_S) - \mu\right)^i H(\sigma_S - \mu)^j \\
& \cdot \bar{H}\left(B^{-1} \circ S(\sigma_S) - \mu\right)^{m-i} \bar{H}(\sigma_S - \mu)^{n-2-j}.
\end{aligned}$$

Hence,

$$f_x^S(S(\sigma_S)|\mu) = m\frac{h\left(B^{-1}\circ S(\sigma_S)-\mu\right)}{B'\circ B^{-1}\circ S(\sigma_S)}L^S_{m,n}(S(\sigma_S);\sigma_S,\mu,B,S)$$
$$+(n-1)\frac{h\left(\sigma_S-\mu\right)}{S'(\sigma_S)}K^S_{m,n}(S(\sigma_S);\sigma_S,\mu,B,S).$$

F.3 Invariance Property

Proof of Lemma 4. We demonstrate this fact for the buyer's marginal utility. A similar argument applies for the seller. The buyer's marginal utility reduces in the case of the offset strategies $B(\sigma_B)=\sigma_B-\lambda_B$ and $S\left(\sigma_S\right)=\sigma_S+\lambda_S$ to

$$\int_{-\infty}^{\infty}\left(\mathbb{E}[v|\mu,\sigma_B]-\sigma_B+\lambda_B\right)(m-1)h\left(\sigma_B-\mu\right)L^B_{m,n}(\sigma_B-\lambda_B;\sigma_B,\mu,B,S)f_\mu(\mu|\sigma_B)d\mu$$
$$+\int_{-\infty}^{\infty}\left(\mathbb{E}[v|\mu,\sigma_B]-\sigma_B+\lambda_B\right)nh\left(\sigma_B-\lambda_B-\lambda_S-\mu\right)K^B_{m,n}(\sigma_B-\lambda_B;\sigma_B,\mu,B,S)f_\mu(\mu|\sigma_B)d\mu$$
$$-\int_{-\infty}^{\infty}M^B_{m,n}(\sigma_B-\lambda_B;\sigma_B,\mu,B,S)f_\mu(\mu|\sigma_B)d\mu.$$

Inspection of formulas (78)-(80) for $L^B_{m,n}$, $K^B_{m,n}$ and $M^B_{m,n}$ show that in this case they depend only on the quantities $(\sigma_B-\mu,\sigma_B-\lambda_B-\lambda_S-\mu)$. We express this by slightly abusing notation and writing

$$\int_{-\infty}^{\infty}\left(\mathbb{E}[v|\mu,\sigma_B]-\sigma_B+\lambda_B\right)(m-1)h\left(\sigma_B-\mu\right)L^B_{m,n}(\sigma_B-\mu,\sigma_B-\lambda_B-\lambda_S-\mu)h\left(\sigma_B-\mu\right)d\mu$$
$$+\int_{-\infty}^{\infty}\left(\mathbb{E}[v|\mu,\sigma_B]-\sigma_B+\lambda_B\right)nh\left(\sigma_B-\lambda_B-\lambda_S-\mu\right)K^B_{m,n}(\sigma_B-\mu,\sigma_B-\lambda_B-\lambda_S-\mu)h\left(\sigma_B-\mu\right)d\mu$$
$$-\int_{-\infty}^{\infty}M^B_{m,n}(\sigma_B-\mu,\sigma_B-\lambda_B-\lambda_S-\mu)h\left(\sigma_B-\mu\right)d\mu. \tag{81}$$

where we have also substituted

$$f_\mu(\mu|\sigma_B)=h\left(\sigma_B-\mu\right).$$

Furthermore note that

$$\mathbb{E}[v|\mu+\rho_B,\sigma_B+\rho_B]-\sigma_B+\rho_B+\lambda_B = \frac{\int_{-\infty}^{\infty}vf\left(v-\mu-\rho_B\right)g\left(\sigma_B+\rho_B-v\right)dv}{\int_{-\infty}^{\infty}f\left(v-\mu-\rho_B\right)g\left(\sigma_B+\rho_B-v\right)dv}-\sigma_B+\rho_B+\lambda_B$$
$$=\frac{\int_{-\infty}^{\infty}\left(v-\rho_B\right)f\left(v-\mu\right)g\left(\sigma_B-v\right)dv}{\int_{-\infty}^{\infty}f\left(v-\mu\right)g\left(\sigma_B-v\right)dv}-\sigma_B+\rho_B+\lambda_B$$
$$=\mathbb{E}[v|\mu,\sigma_B]-\sigma_B+\lambda_B. \tag{82}$$

We are now ready to replace σ_B with $\sigma_B + \rho_B$ in (81) and show that the value of this integral does not change:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \{(\mathbb{E}[v|\mu, \sigma_B + \rho_B] - \sigma_B + \rho_B + \lambda_B)\,(m-1)h\,(\sigma_B + \rho_B - \mu) \\
& \times L^B_{m,n}(\sigma_B + \rho_B - \mu, \sigma_B + \rho_B - \lambda_B - \lambda_S - \mu)\}h\,(\sigma_B + \rho_B - \mu)\,d\mu \\
+ & \int_{-\infty}^{\infty} \{(\mathbb{E}[v|\mu, \sigma_B + \rho_B] - \sigma_B + \rho_B + \lambda_B)\,nh\,(\sigma_B + \rho_B - \lambda_B - \lambda_S - \mu) \\
& \times K^B_{m,n}(\sigma_B + \rho_B - \mu, \sigma_B + \rho_B - \lambda_B - \lambda_S - \mu)\}h\,(\sigma_B + \rho_B - \mu)\,d\mu \\
- & \int_{-\infty}^{\infty} M^B_{m,n}(\sigma_B + \rho_B - \mu, \sigma_B + \rho_B - \lambda_B - \lambda_S - \mu)h\,(\sigma_B + \rho_B - \mu)\,d\mu.
\end{aligned}$$

Applying the linear change of variable $\mu \to \mu + \rho_B$ produces

$$\begin{aligned}
& \int_{-\infty}^{\infty} \{(\mathbb{E}[v|\mu + \rho_B, \sigma_B + \rho_B] - \sigma_B + \rho_B + \lambda_B)\,(m-1)h\,(\sigma_B - \mu) \\
& \times L^B_{m,n}(\sigma_B - \mu, \sigma_B - \lambda_B - \lambda_S - \mu)\big\}\,h\,(\sigma_B - \mu)\,d\mu \\
+ & \int_{-\infty}^{\infty} \{(\mathbb{E}[v|\mu + \rho_B, \sigma_B + \rho_B] - \sigma_B + \rho_B + \lambda_B)\,nh\,(\sigma_B - \lambda_B - \lambda_S - \mu) \\
& \times K^B_{m,n}(\sigma_B - \mu, \sigma_B - \lambda_B - \lambda_S - \mu)\big\}\,h\,(\sigma_B - \mu)\,d\mu \\
- & \int_{-\infty}^{\infty} M^B_{m,n}(\sigma_B - \mu, \sigma_B - \lambda_B - \lambda_S - \mu)h\,(\sigma_B - \mu)\,d\mu.
\end{aligned}$$

This reduces to (81) by substituting (82). ■

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