The Savings Wedge*

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Abstract

We derive a new criterion for evaluating over- or under-saving in over-lapping generations economies with both ex-ante and ex-post household heterogeneity. Our criterion, which we call the Savings Wedge, nests the traditional 'r vs. g' (Golden Rule) criterion as a special case, but accounts for how aggregate savings affects households' exposure to uninsurable risk and shifts the income distribution, and can be thought of as a risk and redistribution robust Golden Rule criterion. A second-order approximation of the Savings Wedge yields a tractable formula of measurable statistics. Using PSID data from 1978 to 2024, we measure the Savings Wedge in the United States and compare our measure to the traditional Golden Rule. The latter implies mild under-saving (10 percent) before 2005, then mild over-saving (6 percent). Relative to the Golden Rule, our criterion suggests substantially more under-saving, driven by both high levels of capital income risk and capital income inequality being greater than labor income inequality. We find the U.S. under-saves by 30 percent before 2005, with no over-saving in the later years.

Keywords: savings, capital accumulation, household heterogeneity, idiosyncratic risk, redistribution

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1 Introduction

What is the optimal amount of aggregate savings? That is, what does it mean for an economy to be over-saving or under-saving, and what would we need to know to make this determination? Knowing whether an economy is over or under-saving is crucial in designing optimal fiscal policy – in particular policies like debt levels and inter-generational transfers that shift the supply of aggregate savings. If we assume that households understand their own private value of savings, then the answer to this question depends on whether there is a wedge between households' private value and the 'social' value of savings.

The benchmark measure of whether an economy is over or under-saving is the *Golden Rule* capital stock [Phelps, 1961]: the level of savings at which investment is equal to the rate of return on capital.[1] Economies with savings rates above the Golden Rule capital stock are *dynamically inefficient*: Saving less would be a Pareto improvement, as it would make the current generation better off without making future generations worse off [Diamond, 1965]. Similarly, the so-called modified Golden Rule criterion allows for discounting the welfare of future generations and states that the optimal gap between the return to capital and investment should increase with this rate of discounting (Cass and Shell [1976]; Dixit [1976]; Atkinson and Sandmo [1980]). In particular, if the economy is growing at constant rate g, then the welfare of future generations can be discounted by (1+g), implying current households should sacrifice less consumption to benefit future (richer) generations. In this case, understanding whether we are saving above or below the Golden Rule level corresponds to the familiar "r versus g" question.[2] Golden Rule criteria capture the trade-off between consuming today and setting resources aside for the future, and have the advantage of being both easy to understand and measure in the data (e.g. Abel et al. [1989]; Barro [2023]).

In this paper, we revisit the question of how to measure over-or under-saving in the context of a model with rich household heterogeneity. In addition to over-lapping generations, our model features uninsurable idiosyncratic shocks to both labor and capital income (ex-post heterogeneity), and permanent labor skill and investment return types (ex-ante heterogeneity). Existing work has shown that models with rich heterogeneity can generate savings behavior that pushes an economy above or below the traditional Golden Rule capital stock.[3] However, both uninsurable risk and systematic inequality *change the welfare*

[1]Equivalently, is the marginal product of capital is equal to the depreciation rate? That is, "is the capital sector on net a spout or a sink?" (Abel et al. [1989]).

[2]This question has attracted renewed attention in recent years, see for instance Blanchard [2019] or Barro [2023].

[3]See for example classic studies like Aiyagari [1994] and Angeletos [2007], who show that ex-post heterogeneity (risk) can generate excessive pre-cautionary savings, and more recent work like Morrison [2026] for ex-ante heterogeneity.

consequences of savings, meaning that the classic Golden Rule may not be the appropriate criterion to *evaluate* over- or under-accumulation of saving in heterogeneous agent models (Dávila et al. [2012], Krueger et al. [2021]). In this paper, we aim to provide a parsimonious measure of whether an economy is saving the socially optimal amount – analogous to a Golden Rule criterion, but appropriate for heterogeneous agent models.

Specifically, we derive our measure of over- or under- saving directly by simply solving for the marginal value of additional savings from the point of view of a utilitarian planner who internalizes both the distributional consequences and risk externalities associated with greater capital not taken into account by atomistic households. We refer to our measure as the *Savings Wedge*. We show that in a simple over-lapping generations (OLG) model as in Diamond [1965], this wedge corresponds to the modified Golden Rule criterion and implies over (under) saving whenever r is less than (greater than) g.[4] However, in a model with rich heterogeneity, in addition to dynamic trade-offs, our wedge also captures the impact of savings on the *intra-generational* distribution of income, as well as the impact of savings and investment on households' *exposure to idiosyncratic risk*. Intuitively, a greater supply of savings (and therefore a higher capital stock), pushes down rates of return and pushes up wages. This both increases (decreases) aggregate exposure to idiosyncratic labor income (capital income) risk, and redistributes income from capital owners to wage earners. The savings wedge takes these forces into account and can therefore be thought of as a *risk and redistribution robust Golden Rule*.

We then show that our savings wedge can be approximated by a simple formula comprised of a set of measurable statistics. We show that – to second order – the savings wedge is approximately equal to the weighted sum of (i) the variance of both labor and capital productivity shocks, (ii) the distribution of permanent labor and investment income within a generation, and (iii) the distribution of permanent labor and capital income across generations. These 3 terms correspond to the 3 primary components of the savings wedge: uninsurable risk, intra-generational redistribution, and inter-generational redistribution. The weights are intuitive functions of key parameters of the economy like the capital share, the profit share, and the degree of risk aversion, as well as the Pareto weights assigned to types and generations by the planner. We show that our formula's basic structure is highly robust to models with increasing realism and complexity, and take a version of our wedge derived in a rich model to the data to see whether it alters the traditional r vs. g criterion, and why.

Using data from the Panel Study of Income Dynamics (PSID), we measure the savings wedge in the United States from 1978 to 1993 and again from 2007 to 2022.[5] We decompose

[4] This holds exactly in the case of log utility.

[5] These are the years in which data on capital income are available.

changes in the savings wedge over time into the contributions of risk and distributional effects, and compare how our savings wedge criterion compares to the more traditional r vs. g criterion. We also show the sensitivity of our measure of the savings wedge to the Pareto weights assigned to different subgroups and generations.

Our preliminary estimates show that, relative to the traditional Golden Rule criterion which implies that the U.S. was under-saving modestly for most of our sample and slightly over-saving as real rates of return fell in the later years, our criterion suggests a much higher level of under-saving, particularly in the earlier periods. This difference is driven by two forces that the Golden Rule criterion ignores. First, throughout our sample, capital income risk substantially exceeds labor income risk. Since more savings lowers equilibrium rates of return, it reduces households' exposure to capital income shocks while increasing their exposure to labor income shocks. When capital income risk dominates, this shift is welfare-improving, generating a positive risk channel that pushes the savings wedge above the Golden Rule.

Second, throughout our sample, capital income is far more concentrated than labor income: the richest households disproportionately earn their income through capital, while lower-income households — who receive the highest welfare weights under our egalitarian criterion — earn relatively more through labor. Because more capital raises wages and depresses returns, it redistributes income toward labor earners and therefore toward the most highly weighted households. This redistribution motive is absent from the Golden Rule, and its presence substantially amplifies the case for more saving relative to the traditional criterion. As r falls relative to g in the later period, – and therefore the weight on younger households falls – and markups compress the effective labor share, this redistribution motive weakens and eventually reverses, explaining why the gap between our criterion and the Golden Rule narrows toward the end of our sample.

Finally, we test the validity of our approximate savings wedge formula by solving a quantitative model calibrated to match the same steady-state moments used in the formula. The quantitative model also allows us to perform policy counter-factuals. [Quantitative results to be added.]

Framework and methodology. To highlight the key channels driving the savings wedge, we start with a simple overlapping generations (OLG) model with 2 generations, in which households work in a competitive labor market and invest in and operate their own firms as entrepreneurs. We allow for ex-ante heterogeneity in both labor market skill and entrepreneurial productivity. Households also face uninsurable shocks to their labor productivity and entrepreneurial productivity. We use the terms entrepreneurial productivity shocks and capital income shocks interchangeably. In the context of this simple model, we solve the problem of an utilitarian social planner who discounts future generations at a constant rate and can choose the level of savings of all households in the economy, taking the existing distribution of skills and the incompleteness of markets as given. We label the gap between the marginal value of additional savings from the planner's point of view and the marginal value from the point of view of individual households' (their Euler equations) as the *savings wedge.*

Our first result is that the savings wedge can be decomposed into 3 components. The first is a risk exposure channel. By lowering equilibrium rates of return and increasing wages, savings shifts the composition of all households' income away from investment income towards wage income, amplifying the effect of labor productivity shocks and dampening the effects of capital income shocks. The second component is a redistribution term: the sum of each permanent type's share of labor income relative to their share of capital income, weighted by their welfare weight.[6] This term captures the idea that higher wages and lower rates of return redistribute income from capital earners to wage earners. Finally, the third component is the expected welfare weight of future generations, and captures the traditional modified Golden Rule trade-off.

We then take a second order approximation of the wedge around the non-stochastic equilibrium. We show that for small shocks, the wedge is approximately equal to the weighted sum of (i) the variance of the investment income shocks less the variance of labor income shocks, (ii) each type's share of total labor income relative to their share of total capital income within the current generation, and (iii) the share of labor income of future generations. Intuitively, if the variance of capital income shocks is greater than that of labor income shocks, more savings improves welfare by lowering overall exposure to idiosyncratic risk through changes in factor prices. Furthermore, the greater the welfare weight of disproportionate labor earners – in this case those with high Pareto weights, low incomes, or the young – the higher the savings wedge.

Our formula also highlights key *interactions* between the risk and redistribution forces. As capital income inequality grows relative to labor income inequality, this amplifies the redistribution channel, as high-income households tend to be disproportionate capital income earners. However, this dampens the risk channel, as the welfare benefits of lowering exposure to idiosyncratic capital income risk are concentrated among relatively high-income households with lower welfare weights. We show that effect of rising capital income inequality on the savings wedge is therefore ambiguous.

We then solve a richer version of the simple model, adding multiple assets, monopolistic

[6]Here, a type's welfare weight is their population and Pareto weighted expected marginal utility.

firms, additional fiscal policy, and extend households' life-cycle to an arbitrary number of periods, and re-derive the savings wedge in this model. We show that the approximated version of the wedge retains the same structure as before – the weighted sum of the same set of measurable statistics – but the additional model features simply change each term's respective weights. This richer formula can be taken to the data and measured over time. We assume that the idiosyncratic income shocks are log-normally distributed, and use a fixed effect panel regression to estimate the variance of these shocks. At the same time, we estimate each households 'permanent' (capital or labor) income in a given year by estimating average income in a rolling symmetric window around that year. Households are then put into age-specific groups that reflect their labor and capital income percentile.[7] These estimates, along with values for statistics like firm markups, aggregate growth rates, and risk aversion taken from the literature, are plugged into the formula which is plotted over time.

1.1 Related Literature

Our savings wedge nests many forces already identified in the existing literature. For example, the impact of greater savings on aggregate exposure to idiosyncratic labor income risk is the key mechanism in Dávila et al. [2012] and Krueger et al. [2021]. We show that a symmetric logic applies to capital income risk as in Angeletos [2007]. The effect of savings on the distribution of income between households who earn mostly capital income and households who earn mostly labor income also features prominently in Dávila et al. [2012] and Morrison [2026]. Our formula shows what moments in the data determine the relative importance of each of these competing forces, but also reveals key *interactions* between these channels that are obscured in models that consider only 1 or 2 of them at a time.

We are also related to the literature measuring dynamic efficiency in an attempt to assess over or under-saving. Abel et al. [1989] ask if cash flows generated by capital exceed investment in every sector and in every period, implying the economy cannot be dynamically inefficient. Applying this criterion to the United States and six other OECD economies, they find that capital income consistently exceeds investment, and therefore conclude that these countries are not over-saving.

A more recent literature has revisited this question through the lens of the r versus g comparison directly. Blanchard [2019], in his AEA Presidential Address, documents that the US safe rate has been below the nominal growth rate for most of the postwar period and argues that this is more the historical norm than the exception. As a result, the welfare costs of reduced capital accumulation may be smaller than conventionally assumed. In

[7]For example, if a group might consist of all households in the 2nd quintile of labor income and the 3rd quintile of capital income among households in their age group.

contrast, Barro [2023] argues that the relevant rate for the dynamic efficiency condition is the average real return on equity rather than the risk-free rate: while $r^f \leq g$ is common in the data and does not by itself signal dynamic inefficiency, $E(r^e) > E(g)$ is both necessary for dynamic efficiency and consistent with the long-run evidence. Our paper complements this literature by showing that even when r and g are correctly measured, their gap is not a sufficient statistic for evaluating the social value of aggregate savings in a heterogeneous agent economy. Our savings wedge nests the modified Golden Rule as a special case but adds risk and redistribution terms that can independently push the economy toward under- or over-saving.

2 A Modern Heterogeneous Agent Framework

In this section, we solve a simple OLG model in which members of each household both supply labor exogenously to a competitive labor market and invest savings into their private firms as entrepreneurs. Households are grouped into permanent types that govern both their expected labor productivity and entrepreneurial ability. Households also face idiosyncratic uninsurable shocks to both their labor and entrepreneurial productivity. In this setting, we consider the problem of a utilitarian social planner with the ability to dictate savings rates. We define the gap between the marginal value of additional savings to the social planner and the value to individual private households as the savings wedge. We decompose the wedge into three distinct terms corresponding to risk, static redistribution, and dynamic redistribution, and show that the wedge can be approximated by the weighted sum of a set of measurable statistics.

2.1 The Economic Environment

Time is discrete and indexed by $t = 0, 1, \ldots, \infty$. We consider a one-sector production economy with labor and capital inputs. There are two overlapping generations, young y and old o. Households are heterogeneous in two key dimensions: (i) their permanent *ex ante* labor and entrepreneurial ability types, and (ii) their idiosyncratic *ex post* labor and entrepreneurial productivity.

Households. In each period, a new generation of households is born that lives for two periods. Households discount the future period at factor $\beta \in (0, 1)$. Preferences are defined over consumption $(c^y_{j,t}, c^o_{j,t+1})$ with a constant relative risk aversion utility function:

$$V_{j,t} = u(c^y_{j,t}) + \beta \mathbb{E}\left[u(c^o_{j,t+1})\right] \ , \quad \text{with} \quad u' > 0, u'' < 0, u''' > 0 \ .$$

Define the coefficients of relative risk aversion (η) and relative prudence (σ) in the following way (see Kimball [1990]):

$$\eta \equiv -\mathbb{E}\left[c^o_j\right] \frac{u''(\mathbb{E}\left[c^o_j\right])}{u'(\mathbb{E}\left[c^o_j\right])} > 0 \ ,$$
$$\sigma \equiv -\mathbb{E}\left[c^o_j\right] \frac{u'''(\mathbb{E}\left[c^o_j\right])}{u''(\mathbb{E}\left[c^o_j\right])} > 0 \ .$$

Households are grouped into a permanent type, $j \in J$, with corresponding equal discrete measure $\omega_j = \frac{1}{2J}$ where $\sum_j \omega_j = 1$. Households exogenously supply θ^y_j efficiency units of labor to a competitive labor market when young, and θ^o_j when old. When young, households can invest capital into their own private firm. However, households have no other savings vehicle.[8] Type-j households have expected entrepreneurial productivity ζ_j. The household budget constraints are

$$c^y_{j,t} = w_t \theta^y_j - k_{j,t+1} \ ,$$
$$c^o_{j,t+1} = (1 + r_{j,t+1}) k_{j,t+1} + \epsilon w_{t+1} \theta^o_j \ ,$$

where the general equilibrium wage rates are w_t and w_{t+1}, private firm investment is $k_{j,t+1}$ with corresponding return $r_{j,t+1}$, and ϵ denotes ex-post idiosyncratic labor productivity. We assume that ϵ takes only positive values and $\mathbb{E}\left[e\right] = 1$. The households choose $k_{j,t+1}$, $c^y_{j,t}$, and $c^o_{j,t+1}$ to maximize lifetime utility, $V_{j,t}$ subject to their budget constraints.

Technology. The private firm of household type j produces with technology:

$$y_{j,t} = A_t f(k_{j,t}, n^d_{j,t}) = A_t (\psi \zeta_j k_{j,t})^\alpha (n^d_{j,t})^{1-\alpha} \ .$$

Here ψ is an idiosyncratic entrepreneurial productivity shock and ζ_j is a type-j household's expected entrepreneurial productivity. Idiosyncratic capital risk, ψ, takes only positive values and $\mathbb{E}\left[\psi\right] = 1$. Importantly, ψ is realized at beginning of period $t + 1$ *after* capital $k_{j,t+1}$ is

[8] This assumption on market incompleteness is stark, and assumed only to emphasize the role of capital income risk. We relax this assumption in the following section.

installed but *before* employment $n^d_{j,t+1}$ is chosen. Profits are revenues net of labor costs:

$$\pi_{j,t} = A_t(\psi\zeta_j k_{j,t})^\alpha (n^d_{j,t})^{1-\alpha} - w_t n^d_{j,t} \, .$$

Consequently, the net return on private savings is $r_{j,t}(\psi, \zeta_j) = \pi_{j,t}/k_{j,t} - \delta$. Entrepreneurs choose their labor demand, $n^d_{j,t}$ to maximize profit, $\pi_{j,t}$ given their invested capital from the previous period, $k_{j,t}$ and the realization of their entrepreneurial productivity shock, ψ.

Technological Growth. We assume A_t grows at a constant exogenous growth rate, g.

Lasseiz-Faire Equilibrium. An equilibrium is a sequences of prices, $\{w_t\}_{t\geq 0}$ and allocations, $\{\{c^y_{j,t}, c^o_{j,t}, y_{j,t}, k_{j,t}, n^d_{j,t}\}_{j\in J}\}_{t\geq 0}$ such that labor demand $n^d_{j,t}$ (and therefore $y_{j,t}$) maximizes entrepreneur profit given $k_{j,t}$ and ψ and market wages, w_t, $c^y_{j,t}$, $c^o_{j,t}$ and $k_{j,t}$ maximizes households' lifetime utility subject to their budget constraints, and labor and goods markets clear.[9]

Because we have assumed a constant returns to scale production function, and because all entrepreneurs hire labor from a common labor market, individual rates of return can be expressed in terms of the marginal product of aggregate capital. Without heterogeneity in expected entrepreneurial productivity (type based returns), this formulation is exactly the one proposed in Angeletos [2007].

Lemma 1 *Private returns to savings are given by:*

$$r_{j,t} = \frac{\pi_{j,t}}{k_{j,t}} - \delta = \psi\zeta_j r_t - \delta \, , \quad \textit{with} \quad r_t = \alpha k_t^{\alpha-1} n_t^{1-\alpha} \, ,$$

where $k_t = \sum_j \omega_j k_{j,t}$ *is the aggregate capital stock and* $n_t = \sum_J \omega_j(\theta^y_j + \theta^o_j)$ *is the aggregate labor supply. For a proof, see Appendix X.*

Lemma 1 says that returns can be written as the product of entrepreneurs' idiosyncratic investment risk (luck), permanent investment type (group fixed effect), and the marginal product of aggregate capital (time fixed effect). In the absence of type-based returns, the expression in the Lemma 1 corresponds to the idiosyncratic rates of return in Angeletos [2007]. Here, returns depend on aggregate capital – and therefore on the investment decisions of all other entrepreneurs – because all firms hire from the same competitive labor market. Intuitively, by investing more, firms bid up the wage, curbing the returns of other firms.

[9]Note that capital is not traded.

2.2 The Social Value of Saving

In this section, we study the problem of a utilitarian social planner who has the power to mandate savings/investment in capital, $i_{j,t}$. For convenience, we denote the Lasseiz-Faire (LF) saving stock that type-j households would save in the absence of intervention as $\hat{i}_{j,t}$ and the mandated savings level $i_{j,t} \equiv \Gamma_{j,t}\hat{i}_{j,t}$. In this section, we evaluate whether our LF economy is over- or under-saving along the balanced growth path with the following two exercises. First, we characterize the *marginal* value of mandating additional saving to a utilitarian planner. That is, what is the marginal change in social welfare resulting from an increase in the savings mandate, $\Gamma_{j,t}$, evaluated at $\Gamma_{j,t} = 1$ for all j (no mandate). Second, we solve for the *optimal* savings mandate in order to quantify the degree of over- or under-saving. In other words, at what level of saving do the marginal welfare costs of distorting households' Euler equations equal the marginal welfare benefits of increasing or decreasing the capital stock.

Our social planner assigns Pareto weights, λ_j to all type-j households and discounts future generations at constant rate, $\gamma \in [0, 1)$. After substituting households' budget constraints into their utility functions, social welfare can be written as:

$$\mathbb{SW} = \sum_{t=0}^{\infty} \gamma^t \sum_{j \in J} \omega_j \lambda_j \Bigg(u\Big(\theta_j^y w_t - k_{j,t+1})\Big) + \beta \mathbb{E}\Big[u\Big((1 - \delta + \psi \zeta_j r_{t+1}) k_{j,t+1} + \epsilon \theta_j^o w_{t+1} \Big) \Big] \Bigg) .$$

The marginal value of additional saving from type-j households born in period t, $\frac{d\mathbb{SW}}{di_{j,t}}$ is equal to the sum of the *direct* effect on households' private utility, $\mathcal{E}_{j,t}$ plus an *indirect effect* or *wedge*, μ_{t+1} resulting from changes in factor prices.

$$\frac{d\mathbb{SW}}{di_{j,t}} = \gamma^t \omega_j \lambda_j \mathcal{E}_{j,t} + \omega_j \zeta_j \mu_{t+1} \ , \tag{1}$$

where

$$\mathcal{E}_{j,t} = -u'(c_{j,t}) + \beta \mathbb{E}\Big[(1 + r_{t+1} \zeta_j \psi) u'(c_{ij,t+1}^o) \Big]$$

and

$$\mu_{t+1} = \Bigg(\sum_J \omega_j \lambda_j \beta \mathbb{E}\Bigg[u'(c_{jt+1}^o) \Bigg(\psi \zeta_j k_{j,t+1} \frac{\partial r_{t+1}}{\partial k_{t+1}} + \epsilon_i \theta_j^o \frac{\partial w_{t+1}}{\partial k_{t+1}} \Bigg) \Bigg] + \gamma \sum_J \omega_j \lambda_j u'(c_{j,t+1}^y) \theta_j^y \frac{\partial w_{t+1}}{\partial k_{t+1}} \Bigg)$$

The first term, $\mathcal{E}_{j,t}$ corresponds to a type-j households' Euler equation and captures the private marginal value of additional savings. In the absence of intervention – the Lasseiz-

Faire equilibrium where $\Gamma_{j,t} = 1$ for all types – because households are all optimizing, this *direct* effect of changing savings rates on household welfare is zero. The second term captures the *indirect* effect of the change savings on social welfare. Each type's weight, $\omega_j \zeta_j$ governs the contribution of a type-j households' savings to the aggregate capital stock in period t+1, while the aggregate savings wedge, $\gamma^\tau \mu_{t+1}$ captures the welfare impact of a change in the future capital stock. Because old households consume their capital in the final period of life, investment in period t only affects capital in period t+1. Individual households do not internalize these effects. When the savings wedge is positive, more saving and capital would improve social welfare and we say that the economy is under-saving. When this wedge is negative, the economy is over-saving. The following Proposition summarizes the channels through which these changes in wages and average rates of return impact welfare.

Proposition 1 *Define the labor income share and capital income share of type-j households respectively as:*

$$\chi_{jt}^h \equiv \frac{w_t \omega_j \theta_j^h n_j}{w_t n} \text{ for } h \in \{y, o\} \ , \quad \text{and} \quad \xi_{jt} \equiv \frac{r_t \omega_j \zeta_j k_{jt}}{r_t k_t} \ .$$

The savings wedge, μ_t is proportional to the following expression:

$$\mu_t \propto r_t \sum_J \lambda_j \mathbb{E}\Bigg[\beta u'(c_{j,t}^o)\Bigg(\underbrace{\chi_{jt}^o(\epsilon - 1) - \xi_{jt}(\psi - 1)}_{\textit{idiosyncratic risk}} + \underbrace{\chi_{jt}^o - \xi_{jt}}_{\textit{static redistribution}}\Bigg) + \underbrace{\gamma u'(c_{j,t+1}^y)\chi_{jt}^y}_{\textit{dynamic redistribution}}\Bigg] \ .$$

For a proof, see Appendix A.2.

Proposition 1 tells us that changing aggregate savings impacts welfare through three channels. The first term captures the effect of the change in factor prices on households' exposure to idiosyncratic risk. Intuitively, an increase in capital generates an increase in the market wage, and an off-setting decline in the aggregate rate of return. For a household whose share of total labor income is equal to their share of total capital income, this shift would have no effect on their expected income in the second period. However, these changes would shift households' exposure to their idiosyncratic labor and entrepreneurial productivity risk. In particular, more capital increases exposure to labor income shocks through higher wages and decreases exposure to entrepreneurial productivity (capital income) shocks. If these shocks are drawn from different distributions, this shift will change the variance and higher-order moments of total income.

The second term captures the effect of the change in factor prices on the distribution of expected income *within the current old generation*. Higher wages and lower average rates of return shifts income towards households who earn a greater share of their total income as a labor income and a smaller share as capital income. If the planner puts greater social welfare weights, $\nu_j \equiv \lambda_j u'(c^o_{j,t})$ on certain household types, then this static redistribution term is increasing in the labor income share relative to the capital income share of these households. Intuitively, if the planner assigns equal Pareto weights to all types – and therefore low-income households are assigned higher social welfare weights – then the static redistribution term will be positive when capital-income inequality is greater than labor income inequality, as low-income types earn a higher share of their household income as wages. If instead the planner sets the Pareto weights to the *Negishi weights* – the weights that rationtionalize the lasseiz-faire distribution of resources across types – then the generalized social welfare weights are uniform and the welfare impact of redistribution is zero.[10]

Finally, the dynamic redistribution term captures the impact of more capital on the wages and consumption of future generations. This term represents the gains from setting aside more resources for the future, and is straightforwardly increasing in the weight assigned to future generations by the planner, γ. Because we assume the economy is on a balanced growth path in which wages growth at constant rate, g, type-j households in the current young generation have marginal utility equal to the marginal utility of the current on in the previous generation, discounted by $(1+g)^\eta$. As the Proposition notes, In fact, in the Diamond [1965] model with no risk, ex-ante heterogeneity, or second period labor, this dynamic redistribution term corresponds directly to a modified Golden Rule criterion in a simple OLG model as in Diamond [1965]. This result is summarized in Corollary 1.

Corollary 1 *Suppose ϵ and ψ are constants, $J = 1$, and $\theta^o = 0$. Then, in the Lasseiz-Faire balanced growth path,*

$$\mu_{t+1} \geq 0 \ \textit{iff.} \ \frac{1}{\gamma}(1+g)^\eta \leq 1 + r_{t+1} = 1 + f_k - \delta \, .$$

For a proof, see Appendix A.2.1.

To see why this is true, note that in a Diamond [1965] economy in which the oldest generation does not work, without risk or permanent heterogeneity, the idiosyncratic risk term outlined in Proposition 1 is zero, while the static redistribution term is simply equal to 1, multiplied by $u'(c^o_{t+1})$. Note also that, along the Lasseiz-Faire balanced growth path (BGP), $u'(c^y_{t+1}) = (1+g)^{-\eta} u'(c^y_t) = \beta(1+r)u'(c^o_{t+1})(1+g)^{-\eta}$. Intuitively, in a given period, the old earn all of the capital income but no labor income. Mandating that the generation born in period t saves more will redistribute resources in period t+1 from period t+1 capital

[10]In particular, this would imply setting the Pareto weights such that $\frac{\lambda_j}{\lambda_{j'}} = \frac{u'(c^o_{j'})}{u'(c^o_j)}$.

income earners (the old) to period t+1 wage earners (the young). The lower the LF level of capital – equivalently the higher the interest rate r along the BGP – the greater the impact this additional capital will have on the wages of the future young. Because along the BGP the future generations have wages that are g percent higher than previous generations, the planner discounts the welfare gains of this redistribution by $\frac{\gamma}{(1+g)^{\eta}}$.

From Corollary 1 we see that, when household heterogeneity is removed, the savings wedge captures precisely the same dynamic trade-offs that generates the classic Golden Rule capital stock from Diamond [1965] and Cass and Shell [1976]. As γ approaches 1, whether μ_{t+1} is positive or negative – and therefore whether the economy is over or under-saving – collapses to a dynamic efficiency, "r versus g" criterion. In particular, if $\gamma = 1$ (no discounting of future generations) and we assume log utility, then $\mu_{t+1} > 0$ if and only if $r > g$. By comparing the savings wedge in Proposition 1 and Corollary 1, we can see that the savings wedge nests the classic Golden Rule criterion, but captures additional welfare impacts of savings generated by systematic labor and capital income inequality and idiosyncratic risk. For this reason, our savings wedge can be thought of as a *risk and redistribution robust Golden Rule.*

2.3 Discussion

Relationship to related literature. Proposition 1 shows how our framework helps unify the existing literature on the over- or under- accumulation of savings and capital. Dávila et al. [2012] study the normative properties of a model with infinitely lived ex-ante identical households and uninsurable idiosyncratic labor income shocks as in Aiyagari [1994]. As households accumulate shock realizations, their start of period wealth begins to diverge. To show the effect of this divergence, in Section 2.2 of their paper, the authors consider a 2-period version of their model with initial heterogeneity in wealth. They show that whether the economy is over- or under- saving depends on the strength of 2 opposing forces: a 'pecuniary externality' in which more capital and higher wages amplifies idiosyncratic risk, and a redistributive force that redistributed from the wealthy to the non-wealthy.

These forces are nested by our savings wedge, and correspond to a case in which $\gamma = 0$, $\psi = 1$ is a constant, and $\theta_j = 1$ for all j. That is, a case without weight on future generations, without capital income risk, and without ex-ante labor productivity heterogeneity, implying wealth inequality is more severe than labor income inequality by construction.

In addition to nesting the forces operating in Dávila et al. [2012], our framework – which allows for ex-ante return heterogneity – clarifies that what matters for the strength of the redistribution term is the degree of labor income inequality relative to *capital income* inequality rather than wealth inequality per se. This is an important distinction given that the degree of capital income inequality appears to be more severe than wealth inequality in the data [Gaillard et al., 2023].

Both Panousi and Reis [2021] and Park [2013] conduct normative policy analysis in a model with idiosyncratic investment income risk as in Angeletos [2007] The model studied in Angeletos [2007] and Panousi and Reis [2021] is nested by our savings wedge, and corresponds to a case in which ζ_j and θ_j are uniform, ϵ is a constant, and $\gamma = 0$. That is, a model without ex-ante heterogeneity, labor income risk, or weight on future generations.[11] Angeletos [2007] answers a *positive* question: does investment income risk generate more or less saving relative to an economy without it. The normative policy exercise conducted in Panousi and Reis [2021] is closer in spirit to our exercise – their optimal capital tax is meant to address the over or under accumulation of savings – but whether the optimal net tax is positive or negative depends both on whether more savings would be welfare improving, and on whether a higher tax would generate more or less saving.[12] In contrast, our framework makes clear that, all else equal, idiosyncratic income risk leads to under-saving.[13]

Finally, our framework nests the literature on over- and under- accumulation of savings in OLG models. As shown in Corollary 1, in the absence of idiosyncratic risk and ex-ante heterogeneity, our model nests the classic Golden Rule criterion emphasized in Diamond [1965]. Our model also nests the trade-offs present in Krueger et al. [2021] who study optimal capital taxation in a model with idiosyncratic labor income risk and over-lapping generations. Their benchmark model corresponds to a case in which $\gamma > 0$, but ϵ is a constant, and both θ_j and ζ_j are uniform.

One important takeaway from Proposition 1 is that whether an economy is over or under-accumulating savings depends crucially on both the nature of the income shocks households face and on their joint distribution. Krueger et al. [2021] consider an extension with both labor and capital income risk, but do not explore interactions between ex-ante heterogeneity and risk or explicitly link the relative strengths of labor or capital income risk to moments of the joint distribution of shocks. Furthermore, both Angeletos [2007] and Panousi and Reis [2021] study the effect of idiosyncratic capital income risk in the presence of *additive* income endowment risk. When aggregate capital increases, this dampens exposure to capital income risk as average rates of return fall, but the subsequent rise in wages does not amplify exposure to the endowment risk. In contrast, labor income risk in our model is *multiplicative*,

[11] It should be noted that both papers include a riskless asset. We show in the next Section that the presence of a riskless asset does not qualitatively change our results.

[12] That is, whether the income or substitution effects dominate.

[13] These papers have both a risky and riskless asset. In the following section, we re-derive our formula in the presence of a multiple riskless assets and show that it is qualitatively the same.

meaning that the welfare effects of the two types of risk push in opposite directions and could cancel one another out.

The advantage of our framework is that by unifying many studies int the existing literature into a single formula, we are able to both show how features of the economic environment amplify or dampen each competing channel and to uncover key interactions between ex-ante inequality and idiosyncratic risk. In particular, in Section 2.5 we examine how the effect of rising capital income inequality – relative to labor income inequality – on the savings wedge depends crucially on the relative severity of capital income risk compared to labor income risk. Furthermore, our measurable formula will allow us to estimate the degree of over- and under- saving empirically.[14]

Efficiency vs. Redistribution. A non-zero savings wedge tells us that the level of capital may both be *inefficient* and may generate a distribution of income that is *too unequal relative to the planner's preferences.* The presence of idiosyncratic risk is a source of *additional* static inefficiency, beyond dynamic inefficiency. By failing to internalize the effect of saving on factor prices, households impose a pecuniary externality on other households. In particular, when capital income risk is higher (lower) than labor income risk, savers fail to internalize that by saving more (less), they could reduce aggregate exposure to risk.

The second two terms capture the planner's pure redistribution motive. In other words, as long as $r > g$ along our BGP, it is possible to find Pareto weights $\{\{\lambda_j\}_{j\in J}, \gamma\}$ that *rationalize* any distribution of expected income such that both the static and dynamic redistribution terms are 0.[15] In the absence of idiosyncratic risk (when both ϵ and ψ are constants), the savings wedge simply captures the deviation of the Negishi weights that rationalize the equilibrium and the planner's preferred Pareto weights.

2.4 Measuring Over- or Under-Saving

In order to evaluate and solve for the degree of over- or under- saving empirically, we take a second-order approximation of the planner's first order condition (X) around the zero-risk equilibrium. Doing so transforms the expression for μ_{t+1} from Proposition 1 into a formula of measurable statistics and welfare weights that can be taken to the data. Furthermore, we can use our measurable statistic formula to examine what features of the economic environment amplify and dampen each component of the wedge. In Section XX, we re-derive the measurable saving wedge in the context of a more realistic model, and show that the

[14]Want this exercise to be in the same spirit as Abel Mankiw.
[15]The Negishi or market weights.

basic structure of our measurable formula is highly robust to additional realism. Before presenting the formula, we first must define the coefficients of relative risk aversion and relative prudence, as well as our generalized social welfare weights.[16]

Definition. Define the general social welfare weight for type-j households $\nu_j^o \equiv \lambda_j\, u'\big(\mathbb{E}[c_{j,t}^o]\big)$.

Proposition 2 *Suppose that* $\mathrm{Cov}(\psi, \epsilon) = 0$*. Denote aggregate output in period* t *by* Y_t*. Then the second-order approximation of the savings wedge around the zero-risk equilibrium is given by the following expression.*

$$\mu_t \approx (1-\alpha) r_t \beta \Big[\mu_I + \mu_R + \mu_D \Big] \,,$$

where the sub-components are given by:

$$\begin{aligned}
\mu_I &= \eta \sum_j \frac{\nu_j^o}{\omega_j} \frac{Y_t}{\mathbb{E}[c_{j,t}^o]} \times \Big(\alpha\, \xi_{jt}^2 \,\mathrm{Var}(\psi) - (1-\alpha)(\chi_{jt}^o)^2 \,\mathrm{Var}(\epsilon) \Big) \,, \\
\mu_R &= \sum_j \nu_j^o \big(\chi_{jt}^o - \xi_{jt}\big) \Big(1 + \tfrac{1}{2}\eta\sigma \,\frac{\mathrm{Var}(c_{j,t}^o)}{\mathbb{E}[c_{j,t}^o]^2} \Big) \,, \\
\mu_D &= \gamma \sum_j \nu_j^o \, \frac{1 + \mathbb{E}[r_{j,t}]}{(1+g)^\eta} \, \Xi_{j,t} \, \chi_{jt}^y \,,
\end{aligned}$$

with the risk-adjustment factor

$$\Xi_{j,t} = 1 + \tfrac{1}{2}\eta\sigma \,\frac{\mathrm{Var}(c_{j,t}^o)}{\mathbb{E}[c_{j,t}^o]^2} - \eta \,\frac{k_{j,t}\,\mathrm{Var}(r_{j,t})}{\big(1 + \mathbb{E}[r_{j,t}]\big)\,\mathbb{E}[c_{j,t}^o]} \,.$$

For a proof and complete statement with $\mathrm{Cov}(\psi, \epsilon) \neq 0$*, see Appendix A.2.2.*

The first term corresponds to the idiosyncratic risk channel in Proposition 1. As relative risk aversion increases, the impact of shifting exposure to idiosyncratic risk on the savings wedge is amplified. This term is increasing in the variance of the capital income shocks and decreasing in the variance of labor income shocks. As discussed in Proposition 1, by increasing wages and decreasing average returns, savings increases households' exposure to labor income risk and decreases exposure to capital income risk. Now however, our measurable formula tells us that what determines the relative strength of these two channels (to second order) is the *weighted* variance of the two shocks.

As the labor share, $1-\alpha$ grows, households earn a lower share of total income through capital and a greater share through labor, meaning that the welfare gains associated with

[16]Cite Saez and Stantcheva for this.

decreasing capital income risk exposure fall, while the welfare costs of increasing exposure to labor income risk increase. All else equal, an increase in the labor share implies a smaller savings wedge. The relative strength of the two income risks channels also depends on the joint distribution of capital and labor income. The planner puts a higher weight on the welfare of household types with a high social welfare weight, ν_j and a low share of total income, $\frac{\mathbb{E}[c^o_{j,t+1}]}{Y_{t+1}}$. The higher the capital income share, $\xi_{j,t}$ of highly weighted households, the greater the importance of the capital income risk channel. Similarly, the higher their share of total labor income, χ^o_{jt}, the greater the importance of the labor income risk channel.

The second and third term correspond to the static redistribution and dynamic redistribution terms in Proposition 1. Again, more savings and higher capital constitutes a redistribution towards households who earn relatively more of their income in the labor market rather than through capital investment. The higher the relative social welfare weight of those types, the greater the welfare impact of this redistribution term. Similarly, the greater the weight put on future generations (γ), the greater the welfare impact of savings.

From the formula we can see the importance of both relative risk aversion and relative prudence in determining the degree of over or under-saving. As relative risk aversion increases, the size of the idiosyncratic risk channel, μ_I increases. At the same time, the product of η and σ amplifies the contribution of the welfare of type-j households to the redistribution term. The product $\eta\sigma$ captures both how much households dislike risk (η) and how the curvature of their marginal utility responds to uncertainty (σ). Households facing more volatility relative to their expected consumption get proportionally larger welfare weights in the redistribution term, μ_R.

Characterizing the amount of over- or under- saving. In order to characterize how far the optimal savings level is from Lasseiz-Faire, we set our planner's first order conditions with respect to each type-j savings mandate, $\Gamma_{j,t}$ (1) equal to zero, and take second order approximation around the zero-risk steady state. Doing so yields a system of J equations of measurable statistics in J unknowns, $\Gamma^*_{j,t}$. However, under a simplified set of assumptions, the measurable savings wedge, μ_t is a constant (independent of $\Gamma_{j,t}$) and we can characterize the optimal savings mandate – and therefore the degree of over or under-saving – in closed form. The following Proposition summarizes this result.

Proposition 3 *Suppose the planner constrained to choose a single savings mandate Γ_t. Suppose further that there was full capital depreciation between periods ($\delta = 1$) and that $u(c_t) = ln(c_t)$. Then the measurable savings wedge, μ is a constant and the degree of over- or under-saving at the Lasseiz-Faire equilibrium relative to the optimal allocation, $\Gamma^*_t \equiv k^*_t/\hat{k}_t$*

is characterized by equation (2).

$$\alpha\left[\beta\sum_j \lambda_j \xi_{j,t+1}\left(\frac{\bar{c}^o_{j,t+1}}{Y_{t+1}}\right)^{-1}\Xi_{j,t} + (1-\alpha)\mu_{t+1}\right] = \sum_j \omega_j\lambda_j \frac{\Gamma^*_t}{1/\hat{\rho}_{j,t} - \Gamma^*_t} \tag{2}$$

Here μ_{t+1} is the measurable savings wedge and $\Xi_{j,t}$ is the risk-adjustment factor, both characterized in Proposition 2, and $\hat{\rho}_j$ is type-j households' Lasseiz-faire savings rate, $\hat{k}_{j,t}/w_t\theta^y_j$.

Intuitively, a uniform savings mandate and full capital depreciation ensures that the income ratios inside the savings wedge remain constant at their Lasseiz-Faire values as Γ_t shifts away from 1. The planner increases Γ_t until the marginal social benefit of increasing aggregate investment – the savings wedge – is equal to the sum of the marginal costs of distorting households' Euler equations.

2.5 Increasing Capital Income Inequality Relative to Labor Income Inequality

The measurable formula presented in Proposition 2 can help us understand how shifts in the economic environment over time change the relative magnitude of each of the components of the savings wedge. One notable change in the United States in recent decades has been the increase in the severity of *capital income inequality relative to labor income inequality.* We can use our formula to ask how this shift affects the degree of over- or under-saving.[17]

Consider the effect of a small increase in capital income inequality: a redistribution of capital income shares away from household types with lower-than-average shares of capital income towards households with a higher-than-average share, keeping aggregate capital and the distribution of labor productivity unchanged. Below, we show that the impact of such a change on the savings wedge depends on (1) the magnitude of capital income risk relative to labor income risk, and (2) the current degree of *relative* income inequality; here we define relative income inequality as the covariance between households' total income (and therefore their consumption) and their share of total capital income relative to their share of total labor income. This term captures the severity of capital income inequality relative to labor income inequality – when this term is high, 'rich' households have a disproportionately higher share of total capital income. [18]

We first consider the effect of an increase in capital income inequality on the idiosyncratic

[17]One notatble study of the rise of capital income inequality relative to labor income inequality in the United States since xxx is... (cite)

[18]A zero covariance (no relative inequality) implies that $\frac{\zeta_j k_{j,t+1}}{k_{t+1}} = \theta_j$ for all j. A high correlation implies that households with high over-all income are disproportionate capital income earners.

risk exposure term, μ_I. Intuitively, this term is the welfare weighted-sum of households' net exposure to risk. Receiving a greater share of capital income simultaneously lowers a household's welfare weight (by increasing their expected consumption), but also concentrates exposure to capital income risk relative to labor income risk (a concentration effect). Which of these two forces dominates determines the impact of an increase in capital income inequality on μ_I.

In the absence of *relative* inequality, each type's share of capital income is the same as their share of labor income. As a result, all households have the same net-exposure to idiosyncratic risk: a change in factor prices would increase the variance of all type's income in a proportional way. Given this even exposure to risk across types, the overall importance of the idiosyncratic risk channel is determined by the severity of *total income* inequality. When total income inequality is higher – that is, the variance of consumption is greater – the consequences of changing households' exposure to risk are amplified. Through a *reweighting* effect, an increase in capital income inequality increases the degree of total consumption inequality, and amplifies the benefits (costs) of greater capital when the variance of ψ is greater (smaller) than the variance of ϵ.

If at the same time there was already substantial *consumption* inequality before the change, more capital income inequality would unambiguously make μ_I more negative, as it would further concentrate net risk exposure among households with lower incomes – and therefore higher welfare weights. Intuitively, the benefits of reducing the exposure to capital income risk are concentrated among households whom the planner now 'cares less about'. Whenever $\eta < 2/\alpha - 1$, the absolute magnitude of this concentration effect is greater than the reweighting effect. These results are summarized in Proposition 4.

Proposition 4 *Suppose $\lambda_j = 1$ for $j \in J$, $\eta < (2/\alpha - 1)$, and $Var(\mathbb{E}[c^o_{j,t+1}]) > 0$. Define $\mathcal{X}^H \subset J$ as the set of all types with higher than average capital income. Consider a small change in the distribution of capital income, $\{\Delta_j\}_{j\in J}$ such that $\sum_j \Delta_j = 0$ and $\Delta_j > 0$ for all $j \in \mathcal{X}^H$ and $\Delta_j < 0$ for all $j \notin \mathcal{X}^H$.*

If $Cov\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}\theta_j}, \mathbb{E}[c^o_{j,t+1}]\right) = 0$, then the total change in the idiosyncratic risk term, $d\mu_I < 0$. For a proof, see Appendix A.3.

Intuitively, Proposition 4 tells us that for an *egalitarian planner* facing *no relative inequality*, as long as the concentration effect outweighs the reweighting effect, an increase in capital income inequality always decreases the idiosyncratic risk channel. If however, relative inequality were high – and therefore capital income risk was already heavily concentrated among a small subset of households – further concentrating this risk would be costly to the planner.[19] For sufficiently high levels of capital income risk, this would imply that the planner would put *more* welfare weight on these high-risk-exposure types, and therefore assign greater importance to reducing capital income risk. As a result, the effect on μ_I of greater capital income inequality would be ambigous.

We can also use our measurable formula to examine the impact of the increase in capital income inequality described above on the redistribution term, μ_R. Assuming an egalitarian planners facing some overall income inequality, the effect of increasing capital income inequality relative to labor income inequality on the redistribution channel is summarized in the following proposition.

Proposition 5 *Suppose $\lambda_j = 1$ for $j \in J$ and the $Var(\mathbb{E}[c^o_{j,t+1}]) > 0$. Define $\mathcal{X}^H \subset J$ as the set of all types with higher than average income. Consider a small change in the distribution of capital income, $\{\Delta_j\}_{j\in J}$ such that $\sum_j \Delta_j = 0$ and $\Delta_j > 0$ for all $j \in \mathcal{X}^H$ and $\Delta_j < 0$ for all $j \notin \mathcal{X}^H$.*

If any of the following conditions are satisfied:

(i) $Cov\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}\theta_j}, \mathbb{E}[c^o_{j,t+1}]\right) = 0$,

(ii) $Cov\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}\theta_j}, \mathbb{E}[c^o_{j,t+1}]\right) > 0$ *and* $Var(\psi) < \frac{1-\alpha}{\alpha} Var(\epsilon)$,

(iii) $Cov\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}\theta_j}, \mathbb{E}[c^o_{j,t+1}]\right) < 0$ *and* $Var(\psi) > \frac{1-\alpha}{\alpha} Var(\epsilon)$

then the change in the redistribution term, $d\mu_R > 0$.

For a proof, see Appendix A.3.

Proposition 5 provides sufficient conditions under which increasing capital income inequality increases the redistributive effect of aggregate savings. In the absence of relative inequality, all types have the same net exposure to idiosyncratic risk and therefore an increase in capital income inequality relative to labor income inequality unambiguously amplifies the redistribution effect. When relative inequality is positive – the rich are disproportionate capital income earners – but labor income risk is more severe than capital income risk, lower income households have higher net risk exposure. The same is true when relative inequality is negative and capital income risk is more severe.

[19] That is, the increase in expected income and consumption from an increasing capital share is not worth the increased variance of income.

In all three cases, households with lower expected income also have income that is at least as variable as that of higher income households. As a result, the planner buts unambiguously higher weight on these households. Greater capital income inequality then implies that aggregate savings redistributes a greater amount from low-welfare weight types towards high welfare weight types, increasing μ_R.

Note that the conditions outlined in Proposition 5 are sufficient but not necessary. Even if high-income households have greater net risk exposure – as would be the case if capital income risk were more severe and high income households were disproportionate capital income earners – it may still be the case that increasing capital income inequality relative to labor income inequality amplifies the redistribution channel. However, for sufficiently high levels of capital income risk, further concentrating capital income among high income types may actually dampen μ_R, as the high variance of these households' income would lead the planner to weigh these households more.

3 A More General Model

In this Section, we re-derive the savings wedge and the solve for the degree of over- or under-saving in a more general setting with additional savings instruments, a general life-cycle of arbitrary length, fiscal policy, and firm market power. Doing so gives us a measurable formula whose components have closer analogs in the data. We show that locally, at the Lasseiz-Faire equilibrium, the savings wedge is driven by the same three forces: idiosyncratic risk, static redistribution, and dynamic redistribution, and that the measurable savings wedge is simply a re-weighted version of the formula presented in Proposition 2.

However, with a multi-period life cycle and multiple assets, we no longer have a single notion of *the amount* over- or under-saving. Recall that in the previous section, we defined the degree of over or under-saving, Γ_t as the ratio between the level of investment along the Lassiez-Faire BGP and the planner's optimal capital: $\Gamma_t \equiv i_t^*/\hat{i}_t$. In the more general setting, we could in principle consider the amount by which households should be saving more at different ages or in different assets. That is,

To begin, we will solve for the simplest measure of over- or under- saving; one that keeps households' Lasseiz-Faire portfolio allocations and relative savings levels over their life-cycles intact. That is, mirroring Proposition 3, we will first consider the problem of a planner who is able to scale households investment in each asset, at each period of life, by a single constant, Γ_t. Doing so provides a simple, one-dimensional notion of the optimal aggregate lifetime saving rate, keeping households' portfolio allocations across assets and over their life-cycle unchanged.

We then solve for the degree of over- or under- saving by age, type, and asset.

3.1 The Economic Environment

Households. In each period, a new generation of households is born that lives for H periods. Preferences are defined over consumption at each age:

$$V_{j,t} = \sum_{h=0}^{A} \beta^h u(c^h_{j,t+h}) \ , \quad \text{with} \quad u' > 0, u'' < 0, u''' > 0 \ .$$

Households at age h exogenously supply θ^h_j efficiency units of labor to a competitive labor market. In addition to investing capital, $k^h_{j,t+h}$ into their own private firm, households can invest capital, $k^{ch}_{j,t+h}$ into a public corporate sector through a financial intermediary and can save and borrow in a one period bond, $b^h_{j,t+h}$. Households are subject to a linear labor tax, $\tau_{\ell,t}$ on their labor income and a capital tax, $\tau_{k,t}$ on their capital income. Households' period budget constraint at age h is now given by:

$$\begin{aligned} c^h_{j,t+h} = (1-\tau_{\ell,t+h})\theta^h_j \epsilon w_{t+h} - b^{h+1}_{j,t+h+1} - k^{ch}_{j,t+h+1} - k^h_{j,t+h+1} + (1-\tau_{k,t+h})((1+r^b_{t+h})b^h_{j,t+h} \\ + (1+r^c_{t+h})k^{ch}_{j,t+h} + (1+r_{j,t+h})k^h_{j,t+h}) \end{aligned} \tag{3}$$

Final Good Firm. A competitive final good firm aggregates goods from intermediate goods firms who produce differentiated varieties $i \in [0,1]$ with the following CES production function.

$$y_t = \left(\int_I y_{it}^{\frac{\rho-1}{\rho}} di \right)^{\frac{\rho}{\rho-1}} \ .$$

Intermediate Goods Firms. Let A_t be aggregate productivity. Both private firms run by entrepreneurs and the corporate sector produce intermediate goods to sell to the final goods aggregator using both labor and capital. Entrepreneurs investing $k_{j,t}$ into their private business and earn the private stochastic return $r_{j,t}$, which we derive below. Entrepreneurs producing variety-i operate the following production function with productivity shock, ψ_{ijt}.

$$y_{ij,t} = A_t(\psi\zeta_j)^\alpha (k_{ij,t})^\alpha (n^d_{ij,t})^{1-\alpha} \ .$$

A competitive intermediary pools capital invested into the corporate sector one period in advance and invests it in corporate firms. These firms produce variety i using labor and capital according to the following production function:

$$y_{i,t}^c = A_t (k_{i,t}^c)^\alpha (n_t^d)^{1-\alpha} \ .$$

After corporate firms produce, they return the un-depreciated capital to the competitive intermediary who transfers the un-depreciated capital to households along with a share of corporate firm profits proportional to their investment, $k_{j,t}^c$.

Technology growth. As before, we assume that aggregate productivity grows at constant exogenous rate, g.

Government. The government uses tax revenue to pay interest on its debt, B_t and to fund government spending, G_t, i.e.:

$$G_t + r_t^b B_{t-1} = \tau_{kt}(b_t r_t^b + \sum_J \omega_j k_{j,t} r_{j,t} + k_t^c r_t^c) + \tau_{\ell t} w_t + B_t \ . \tag{4}$$

Definition. We define aggregate capital, k_t as follows:

$$k_t = \sum_j \omega_j \zeta_j k_{j,t} + k_t^c \ .$$

Lemma 2 (Returns and Wages) *Private returns to savings are given by:*

$$r_{j,t} = \frac{\pi_{j,t}}{k_{j,t}} - \delta = \psi \zeta_j r_t - \delta, \quad \textit{where } r_t = \alpha k_t^{\alpha-1} n_t^{1-\alpha} \ .$$

Corporate net returns, $r_t^c = r_t - \delta$ and real market wages, $w_t = \frac{\rho-1}{\rho}(1-\alpha)k_t^\alpha n_t^{-\alpha}$ For a proof, see Appendix A.4.1

From Lemma 2 we can see that monopolistic competition drives a wedge between wages and the aggregate marginal product of capital. Note however that because households manage their private firms, the markup does not appear in the rate of return on capital. Corporate capital and bonds both return the same risk-free rate of return.

Laissez-faire equilibrium. Given government policies $\{\tau_{\ell,t}, \tau_{k,t}, G_t, B_t\}_{t\geq 0}$, a laissez-faire equilibrium is a sequence of prices $\{w_t, r_t, r_t^c, r_t^b\}_{t\geq 0}$, along with household allocations $\{c_{j,t}^h, k_{j,t}^h, k_{j,t}^{ch}, b_{j,t}^h\}_{j\in J, h\in H, t\geq 0}$, and firm allocations $\{n_{j,t}^{d,h}, n_t^c, k_t^c\}_{j\in J, h\in H, t\geq 0}$ such that: (i) taking prices and policies as given, households choose consumption and savings to maximize lifetime utility subject to their budget constraints (3), with optimal asset holdings characterized by the Euler equations (13)–(15); (ii) firms choose labor to maximize profits, as characterized in Lemma 2; (iii) returns satisfy the no-arbitrage condition (16); (iv) the government budget (4) balances; and (v) the labor, bond, corporate-capital, and goods markets clear, (17)–(20).

Mandated equilibrium. Given the laissez-faire policies augmented with the savings mandate Γ_t, a mandated equilibrium is defined as above, except that the household Euler equations (13)–(15) are replaced by the savings mandate

$$\{k^h_{j,t+1},\, k^{ch}_{j,t+1},\, b^h_{j,t+1}\} = \Gamma_t\, \{\hat{k}^h_{j,t+1},\, \hat{k}^{ch}_{j,t+1},\, \hat{b}^h_{j,t+1}\} \quad \text{for all } j, h, \tag{5}$$

where hats denote laissez-faire values and consumption is determined residually by the budget constraints. Conditions (ii)–(v) are unchanged.

3.2 A Measurable Savings Wedge

As before, we consider the marginal value of additional saving to our egalitarian social planner and again define our savings wedge as the gap between households' private value and the social value of savings. The planner chooses the sequence of net investments $\{i^h_{j,t}\}_{j\in J,\,0\leq h<H,\,t\geq 0}$, where $i^h_{j,t} \equiv k^{h+1}_{j,t+1} - (1-\delta)k^h_{j,t}$ denotes the new saving an age-h household of type j adds on top of the principal it rolls forward within its life, to maximize

$$SW = \sum_{t=0}^{\infty} \gamma^t \sum_{j\in J} \omega_j \lambda_j \sum_{h=0}^{H} \beta^h\, \mathbb{E}\big[u(c^h_{j,t+h})\big], \tag{6}$$

subject to the household budget constraints, the laws of motion for household capital, and the aggregation and pricing conditions listed in Appendix A. Households are born with no capital, and the age-H stock is consumed at death, so a marginal unit of saving persists in the aggregate capital stock only for its owner's remaining lifetime. The marginal value of additional net investment by age-h households of type j at time t is then

$$\frac{dSW}{di^h_{j,t}} = \gamma^{t-h}\beta^h\, \omega_j\lambda_j \sum_{s=0}^{H-1-h} \big(\beta(1-\delta)\big)^s\, \widehat{\mathcal{E}}^{h+s}_{j,t+s} \;+\; \gamma^t\, \omega_j\zeta_j \sum_{s=0}^{H-1-h} \big((1-\delta)\gamma\big)^s\, \mu_{t+1+s}, \tag{7}$$

where

$$\widehat{\mathcal{E}}^a_{j,m} \equiv -\,u'(c^a_{j,m}) + \beta\, \mathbb{E}\big[\big(1-\delta+\psi\zeta_j r_{m+1}\big)\, u'(c^{a+1}_{j,m+1})\big] \tag{8}$$

is the age-a Euler residual.

Proposition 6 *Assume the economy is currently on the balanced growth path. Define the auxiliary welfare weights defining* $\nu_{j,t} \equiv \lambda_j u'(\mathbb{E}[c^H_{j,t}])$ *and* $\hat{\nu}_{j,t} \equiv \nu_{j,t}\left(1 + \frac{1}{2}\eta\sigma\frac{Var(c^H_{j,t})}{\mathbb{E}[c^H_{j,t}]^2}\right)$*, and denote the inverse of the monopolists' markup,* $\mathcal{M} \equiv \frac{\rho-1}{\rho}$*. Let* $\bar{\xi}^h_{jt}$ *be age-h, type-j households' share of total asset income, while* ξ^h_{jt} *is their risky capital income as a share of total asset income. Then the second-order approximation of the savings wedge around the zero-risk equilibrium is given by the following expression:*

$$\mu \propto r\bigg(\mu_I + \mu_R\bigg)\,,$$

where

$$\begin{aligned}\mu_{It} = \eta \sum_j \sum_{h=0}^{H} \frac{\nu_{j,t}}{\omega_j}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h} \frac{y_t}{\mathbb{E}[c^h_{j,t}]}\Bigg[&\alpha(1-\tau_{kt})^2\big(\xi^h_{jt}\big)^2\, Var(\psi) \\ &- (1-\alpha)\mathcal{M}^2(1-\tau_{\ell t})^2\big(\chi^h_{jt}\big)^2\, Var(\epsilon) \\ &+ \mathcal{M}(1-2\alpha)(1-\tau_{\ell t})(1-\tau_{kt})\,\chi^h_{jt}\,\xi^h_{jt}\, Cov(\epsilon,\psi)\Bigg],\end{aligned}$$

$$\mu_{Rt} = \sum_j \sum_{h=0}^{H} \hat{\nu}_{j,t}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}\Big[\mathcal{M}(1-\tau_{\ell t})\,\chi^h_{jt} - (1-\tau_{kt})\,\bar{\xi}^h_{jt}\Big].$$

For a proof, see Appendix A.2.2.

From Proposition 6, we see that the measurable savings wedge maintains the same basic structure as in the simple model – an idiosyncratic risk term and a redistribution term – but with richer weights. The idiosyncratic risk term now weights the variance of capital income shocks by the welfare-weighted sum of *after-tax* capital income shares for all types *across generations*. Similarly, the variance of labor income is weighted by the sum across types and generations of after-tax shares of labor income, *discounted by the markup.* Intuitively, because monopolistic pricing generates a wedge between the market wage and worker's marginal product, dampening the impact of addition capital on wages. We also allow for a non-zero covariance between households' labor and capital income.

The redistribution term is now the welfare-weighted sum across types and generations of households' after-tax labor income share, discounted by the markup, less their total after tax capital income share. Households' capital income share includes fixed income and public capital income alongside private capital income. Note that we have consolidated the static and dynamic redistribution terms into a single term accounting for both intra- and intergenerational redistribution.

Corollary 2 *Suppose there is no idiosyncratic risk, taxes, markups, and no ex-ante heterogeneity within generations. Then the measurable savings wedge simplifies to:*

$$\mu \propto r \cdot \mu_R,$$

where

$$\mu_R = \sum_h \left(\frac{\gamma(1+r)}{(1+g)^\eta} \right)^{H-h} [\Theta_h - Z_h], \tag{9}$$

and where $\Theta_h \equiv \sum_j \omega_j \theta_j^h$ *and* $Z_h \equiv \sum_j \frac{\omega_j \zeta_j k_j^h}{k}$ *denote the total population-weighted labor and capital income shares of generation* h*, respectively.*

For a proof, see Appendix A.6.1

Equation (9) is reminiscent of Corollary 1, which shows that without risk or ex-ante heterogeneity, assuming that the young work and the old live off capital income, the savings wedge collapses to an r vs. g criterion. Here, if we restricted $h \in (0, 1)$ and allocate all labor income to the young and all capital income to the old, we recover this criterion.

In the following section, we take the measurable formula in Proposition 6 as well as the simplified formula from equation (9) to the data to see how the savings wedge has evolved over time, to account for the forces driving these changes, and to compare our criterion to the traditional r vs. g measure.

4 Measurement

In this section, we measure the savings wedge over time using a combination of existing estimates from the literature and our own estimates.

A central empirical question in measuring the savings wedge is the appropriate measure of the return to capital, r. We consider two alternative measures. The first is the (annualized) real yield on 10-year US Treasury bills as in Blanchard [2019]. In particular, we use the annualized 10-year real treasury bond rate as our baseline measure of the real rate, r.[21] The second is the average real rate of return on equity. As documented in Barro [2024], these three measures can diverge substantially — particularly in recent decades as the bond rate has fallen while average returns to capital have remained relatively stable — implying that

[21]The measure of the annualized 10-year treasury bond rate come from Shiller [2026] and can be accessed at shillerdata.com.

Table 1: Statistics from the Literature

Statistic	Description	Value	Source
η^{RPR}	Coeff. relative prudence	$1+\eta^{RRA}$	Kimball [1990]
η^{RRA}	Coeff. relative risk aversion	1–5	Chetty [2006], Barsky et al. [1997]
$\mathcal{M}$	Inverse markup		De Loecker et al. [2020]
τ_k	Capital income tax rate	0.4	Mendoza et al. [1994]
τ_ℓ	Labor income tax rate	0.3	Mendoza et al. [1994]
r	Return to capital		Shiller [2026][20]
α	Capital share	.33	
g	Growth rate		BEA NIPA

the dynamic efficiency assessment is sensitive to which concept of r one adopts. We present results under both measures and discuss the sensitivity of the savings wedge to this choice.

Our data come from the Panel Survey of Income Dynamics (PSID), a longitudinal panel survey of several thousand US households and their descendants starting in 1968. The data include information on household labor income as well as household holdings of and income from various types of assets. We include all households in our sample, with no additional sample selection criteria. For each variable of interest, we use the PSID computed household aggregate if it exists, or otherwise construct the variable as the sum of head and spouse sub-measures. As the PSID questionnaire varies over time, certain measures of income and wealth are not available for the entire duration of the panel. In particular, wealth information is only available in 1984, 1989, 1994, and 1999 onward, expenditure information is only available from 1999 onward, and asset income is missing between 1995 and 2003. All measures are transformed into 2017 USD using the PCE price index and we apply no additional truncation or winsorization.

Household labor income is constructed as the sum of head and spouse labor income, excluding business income. Asset income is constructed as the sum of head and spouse farm, business, rent, interest, dividend, room and board, trust and royalty, market gardening, and other asset income. We define total capital income as aggregate asset income, and the *risky* component as business income specifically. Transfer income is constructed as the sum of social security income and all transfers excluding social security. Wealth is taken directly from household net wealth. Consumption expenditure is constructed as total expenditure less flow payments for mortgages and property taxes.

For the years from 2003 onward, both total capital income and its business income component are directly observable in the PSID, allowing us to compute each age-type j group's share of business income relative to total capital income. For earlier years in the sample, the PSID does not separately identify business income. As a result, we cannot estimate the variance of idiosyncratic business income shocks. In these years therefore, we impute risky capital income *shares* for each age-type j group by multiplying the group's total capital income by its average business income share estimated from the later sample and fix the value for the variance of ψ at it's initial value in the later sample. Table 2 presents the sampling-weight adjusted summary stats for the full sample, as well as the first and last year labor and asset income are jointly available.

Table 2: PSID Summary Statistics

	Full Sample	1976	2023
HH Labor Income	47,981.55	37,959.98	58,733.16
HH Asset Income	5,503.73	4,017.68	4,838.18
HH Asset Income (Risky)	2,734.81	–	1,839.77
HH Transfer Income	9,050.05	5,194.83	13,801.45
HH Consumption	31,087.75	–	35,395.88
HH Wealth	361,859.21	–	473,435.31
Head — Age	48.84	45.80	53.39
Head — Male	0.70	0.72	0.68
Head — Years of Schooling	12.90	11.67	13.73
HH — Married	0.50	–	0.43
HH — Number of Children	0.68	0.92	0.49
Observations	306,002	5,861	9,152

4.1 Empirical Strategy

Our first objective is to isolate the 'permanent' component of a households' labor and capital income in order to group them by permanent income types.[22] In order to do this, we follow Straub (2019) and construct symmetric moving averages for each individual household for each type of income. In particular, for individual i, income type y, age h, time t, and window W, we define:

$$\bar{y}^{ih}_{t,W} = \sum_{\tau=t-\frac{W-1}{2}}^{t+\frac{W-1}{2}} y^{i,h+\tau}_{\tau} \tag{10}$$

We construct these symmetric averages using both $W = 5$ and $W = 9$ for years in which both labor and asset income are jointly present: 1976 to 1993 and 2005–2023. The measure

[22]This corresponds to ex-ante types, j, in our model.

is only constructed for observations in which data are available for the entire window. Note that since the PSID switched from annual to biennial in 1999, the same window length contains fewer observations in the later period than the earlier one. We refer to $\bar{y}^{ih}_{t,W}$ as a household's permanent labor income when y is labor income, and their permanent capital income when y is capital income.

We then use these symmetric averages to group households into permanent income groups depending on which percentile their permanent capital and labor income falls into for households of their age in their year. In particular, we calculate quintiles of both types of income for a particular year for each 10 year age group between ages 25 and 65. An income group for households in a particular age group in a particular year is a permanent labor income quintile–permanent capital income quintile pair, with the bottom 2 capital income quintiles grouped together. Because households who are older than 65 are less likely to work, we maintain the same 4 capital income groups for the 65 to 74-year-old group and the 75 and older age-group, but collapse the labor income groups into 3 groups: those with no labor income, and those with above or below median labor earnings for over-65-year-olds in that year.

Once households are assigned to permanent income groups for their age and year, we calculate several key statistics for each group: in particular, we calculate each group's (i) population share, (ii) share of total labor income, (iii) 'risky' capital income as a share of total capital income, and (iv) total income shares. These shares correspond to the statistics in Table 1 where we use total income shares as a proxy for consumption shares.[23,24]

Figure 1 plots the evolution of labor income shares over time for each permanent income quintile, while Figure 2 plots the evolution of capital income shares. From the figures, it is clear that both substantial labor income and capital income inequality are present in the data, and that the share of the highest quintiles rises in the later years of our sample. However, there is substantially more capital income inequality as measure by the top quintiles share of total income. In the period between 1978 and 1993, the top quintile controlled around 80 percent of total capital income, rising to 90 percent in the years after 2005. Because households' labor and capital income quintiles are highly correlated, this implies that lower income households will tend to be disproportionate wage earners while higher income households will be net capital income earners. This then implies that lower (higher) income households will contribute negatively (positively) to the idiosyncratic risk term, but positively (negatively) to the redistribution term. Therefore, whether these terms are positive

[23]This exercise is analogous to the one performed in Kekre and Lenel [2025].

[24]For the years after 2005, data on consumption is directly available and we calculate consumption shares for each group.

Figure 1: Labor income shares by permanent labor income quintile

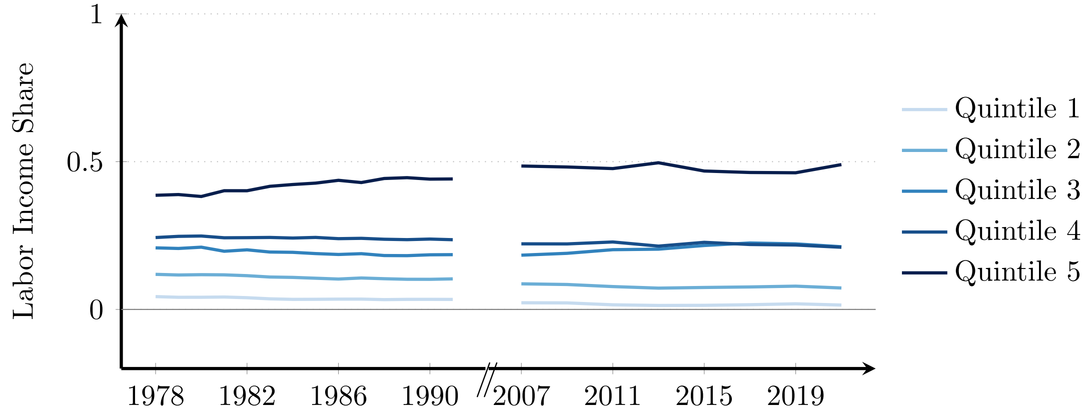
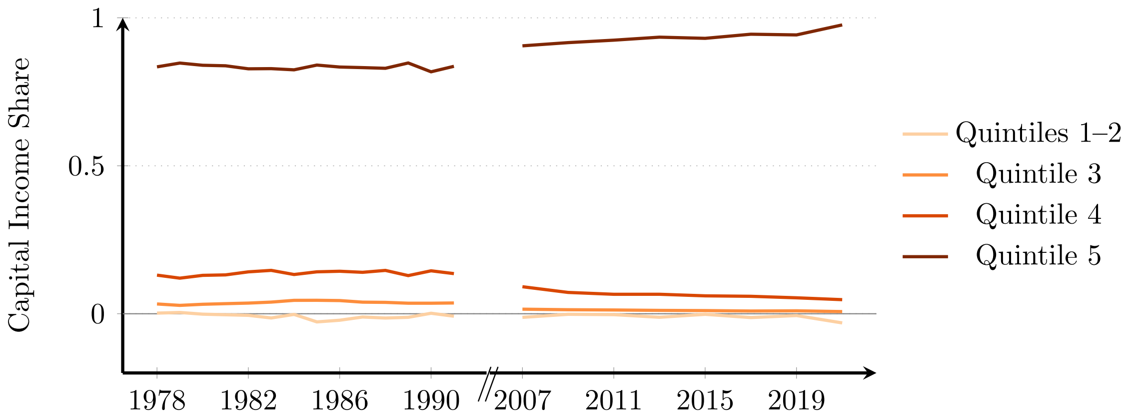


Figure 2: Capital income shares by permanent capital income quintile.



or negative depends on the welfare weights assigned to each permanent income group, ν_j and $\hat{\nu}_j^h$.

Our second objective is to estimate the variances of the idiosyncratic shocks ψ and ϵ. To do this, we run separate panel regressions of log risky capital income and log labor income on individual fixed effects, year fixed effects, and an age trend. The age trend coefficient is allowed to vary by permanent income type j, where type is assigned based on the group a household belongs to at the beginning of the sample window in which they are observed. Formally, for household i of permanent type j and age h in year t, we estimate:

$$\log y_{it} = \alpha_i + \gamma_t + \beta_j \cdot h_{it} + \varepsilon_{it}, \tag{11}$$

where α_i are individual fixed effects, γ_t are year fixed effects, and β_j is a type-specific age trend. The residuals ε_{it} correspond to $\log \psi_{it}$ and $\log \epsilon_{it}$ for capital and labor income respectively. Because the log transformation is particularly sensitive to values near zero, which are likely to reflect measurement error rather than true income realizations, we restrict the regression sample to households with strictly positive income above a minimum threshold.[25] Assuming log-normality of the underlying shocks, we can recover estimates of Var(ψ) and Var(ϵ) from the variance of the estimated residuals via the standard log-normal relationship $\text{Var}(x) = e^{\mu+\sigma^2/2}(e^{\sigma^2} - 1)$, where σ^2 is the variance of the log residuals.

Because a period is 10-years in our general model, for each variable we calculate a symmetric 10-year rolling average to use in the formula.[26]

4.2 Preliminary Estimates

We begin by estimating the baseline wedge for a social planner who sets egalitarian welfare weights, $\nu_j = \frac{1}{\mathbb{E}[c_j]^2}$ where $\mathbb{E}[c_j]$ is the expected consumption of type-j households over all ages over all shock realizations. We refer to these weights as egalitarian, as they correspond to the case in which $u(c) = ln(c)$ and $\lambda_j = 1$ for all $j \in J$. These weights are then normalize so that they sum to 1.

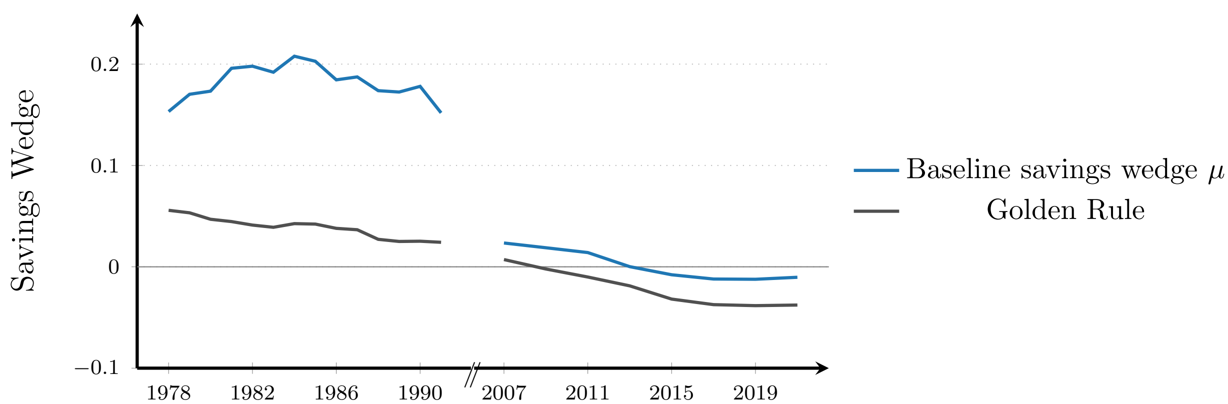


Figure 3: Baseline savings wedge μ and the Golden Rule benchmark.

Figure 3 plots the savings wedge over time against the wedge implied by the Golden Rule, derived in Corollary 1 in the economy of Diamond [1965]. From the figure, we can see that our baseline savings wedge implies substantially more under-saving than a traditional r vs. g comparison, particularly in the early years of the sample. This is driven by two forces. First, capital income risk is nearly twice as severe as labor income risk, pushing the *'idiosyncratic risk'* term, μ_I positive. Second, a redistribution term that disproportionately weights low-income types and the young generations, both of whom tend to have less capital income relative to labor income. Low income types have higher general social welfare weights

[25] Specifically, we drop households with annual labor or capital income below $1,000 (in 2017 USD) before taking logs.

[26] For example, capital income shares for group j in 1985 are the average shares for that group between 1980 and 1990. Rates of return and growth rates are 10-year rates.

due to their high marginal utility.

Young generations have higher implied weights in the earlier years, because *because* the return to capital is higher than the growth rate. Recall from the discussion in Section 2.2 that, as r declines relative to g, younger generations' welfare is discounted more heavily than older generations. Intuitively, growth pushes up future consumption and therefore pushes down the marginal utility, and thus the welfare weight of the young. When the marginal product of capital is especially high (high r) relative to the growth rate however, additional savings will translate into *substantially* higher wages in the future, pushing up the implied weight of the young. As r falls relative to g, this logic reverses. Because the young tend to have higher ratios of labor income shares to capital income shares relative to the old, rising r relative to g implies less welfare weight on wage earners.

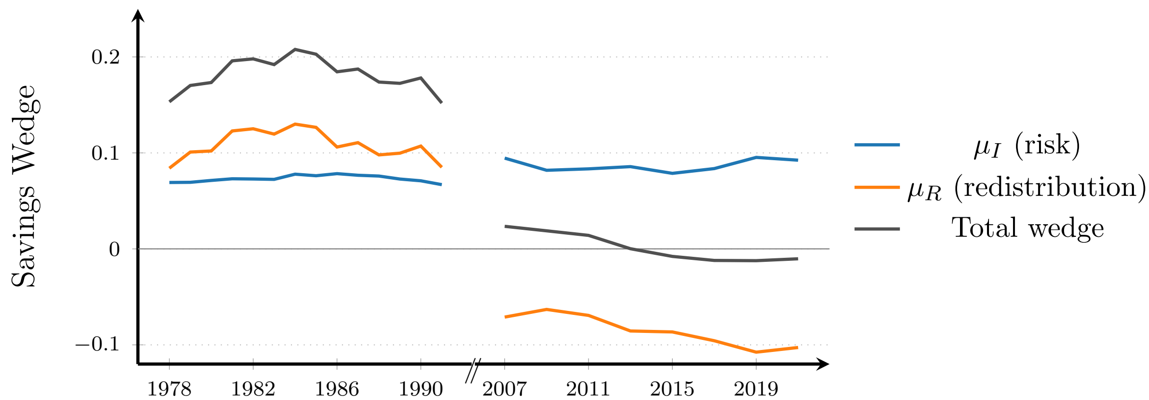


Figure 4: Decomposition of the savings wedge into the idiosyncratic-risk (μ_I) and redistribution (μ_R) channels, with the total.

To illustrate this point, we decompose the baseline wedge into the risk and redistribution sub-components. Figure 4 shows that the idiosyncratic risk term, μ_I is positive throughout our sample, implying that the contribution of the reduction in idiosyncratic capital income risk dominates the contribution of exacerbating labor income risk. This may seem surprising, as with egalitarian weights, the welfare of disproportionate labor income earners is given more weight than capital income earners. Despite this, the degree of capital income risk is substantially higher than labor income risk throughout the sample, generating a positive contribution of additional aggregate capital to idiosyncratic risk exposure.

Figure 4 also shows a large positive redistribution term in the early years of the sample, that turns negative in the later years. The decrease in the redistribution term can be attributed to the decline in r vs. g during this period.

Next, we illustrate the sensitivity of the savings wedge to our choice of Pareto weights, both within and across generations. Figure 5 plots the baseline savings wedge against the

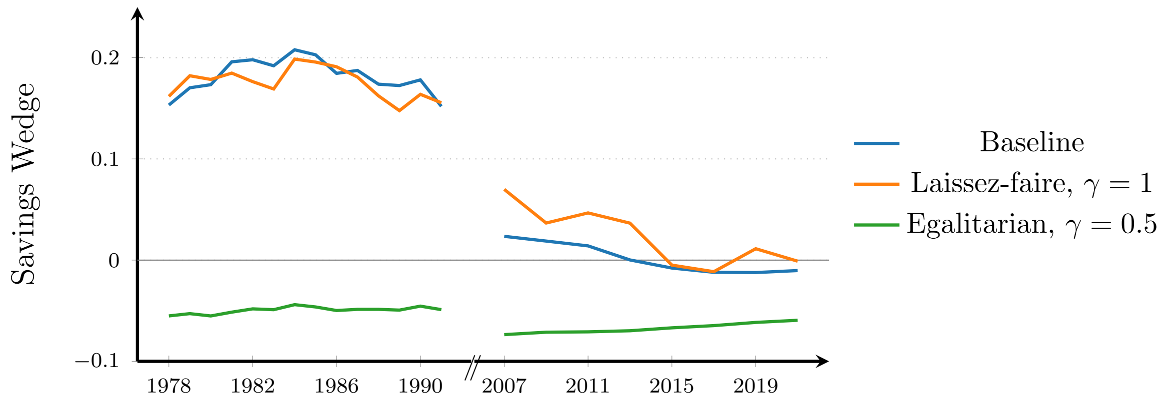


Figure 5: Baseline savings wedge under alternative welfare weights: lasseiz-faire ($\gamma = 1$) and egalitarian ($\gamma = 0.5$).

wedge calculated with no generational discounting but with *Lasseiz-faire* weights which are set to equal the inverse of the (risk adjusted) social welfare weights, so that the redistribution term is equal to zero by construction. The Figure also plots the savings wedge calculated at egalitarian Pareto weights but with heavy generational discounting.

From the figure, it is clear that shifting the Pareto weight put on future generations has a substantial impact on the savings wedge. When generations are weighted equally, the wedge suggests substantial *under-saving*, while when young generations are heavily discounted, the wedge suggests slight *over-saving*. Intuitively, these results are being driven by the redistribution effect. The young are disproportionate labor-income earners in our PSID data, benefiting from the higher wages associated with greater capital. Conversely, the old are disproportionate capital income earners, and are negatively impacted when their existing savings earn lower returns as aggregate capital increases.

Under the Lasseiz-faire weights, all permanent income types are given the same welfare weight, dampening the redistribution term, as only the welfare benefit of inter-generational redistribution remain in the term. However, because the welfare of 'rich' high-capital income households are no longer so heavily discounted, the welfare benefits of reducing capital income risk are amplified. As a result, the increase in μ_I off-sets the elimination of μ_R, generating a similarly-sized savings wedge.

4.3 Interpretation of the Wedge

Figure 6 translates the savings wedge into an optimal saving mandate, Γ^*, using Proposition 3. Whereas the wedge μ measures the sign and marginal magnitude of the distortion at the current allocation, Γ^* answers a different question: by what factor should the economy scale its saving to reach the optimal allocation? A value of $\Gamma^* = 1$ indicates that the laissez-faire allocation is optimal, $\Gamma^* > 1$ that the planner would mandate additional saving, and $\Gamma^* < 1$ that the planner would mandate less. Because Γ^* is anchored to the household Euler equation at $\Gamma = 1$, the wedge is the only object that moves the mandate away from one: when $\mu = 0$ the private and social returns to saving coincide and $\Gamma^* = 1$ exactly.

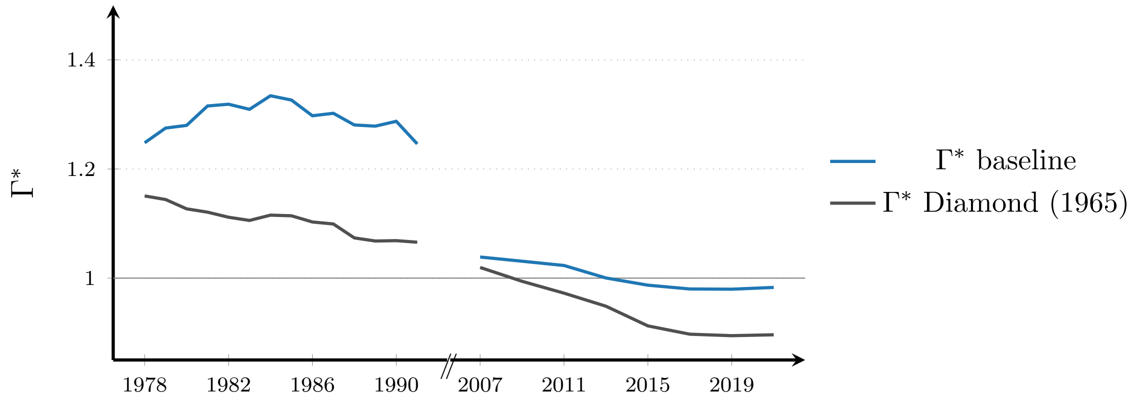


Figure 6: Optimal saving mandate Γ^*: baseline vs. Diamond (1965)

Quantitatively, the two criteria diverge sharply. The Diamond benchmark implies average under-saving of roughly 10 percent in the early sample, reversing to over-saving of about 6 percent in the later sample. Our baseline criterion instead implies nearly 30 percent under-saving in the early sample, and essentially no over-saving in the later periods. These magnitudes are large relative to observed saving: the average saving rate in our sample is low, around 10 percent, so that a mandated 30 percent increase corresponds to a rise of only about 3 percentage points in the saving rate.

4.4 Capital-skill complementarity

To be added.

5 Conclusion

This paper derives a new criterion for evaluating over- or under-saving in economies with rich household heterogeneity. Our Savings Wedge nests the classic Golden Rule as a special case but extends it to account for how aggregate savings affects households' exposure to idiosyncratic risk and the distribution of income. A tractable second-order approximation renders the wedge measurable from standard household panel data, decomposing it into a risk channel, a redistribution channel, and a dynamic efficiency channel. Applied to PSID data, our estimates suggest that the U.S. has been substantially under-saving relative to our criterion for most of the sample period — a conclusion that the traditional Golden Rule criterion would largely miss — driven by the dominance of capital income risk over labor income risk and the higher levels of capital income inequality relative to labor income inequality.

Our findings also illuminate how structural changes in the U.S. economy over recent decades have reshaped the forces driving the savings wedge. The secular rise in markups since the 1980s has compressed the effective labor share, weakening the redistribution motive for additional saving, while rising capital income concentration has reinforced it. These forces have increasingly offset one another, gradually eroding the wedge toward the end of our sample. More broadly, our results suggest that evaluations of optimal aggregate savings based on Golden Rule criteria may substantially understate the social value of saving in economies characterized by high inequality and uninsurable idiosyncratic risk, and that the welfare consequences of savings policy depend critically on the joint distribution of labor and capital income across households.

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A Appendix

A.1 Proof of Lemma X

The individual entrepreneurs profit is:

$$\pi_{j,t} = A_t(\psi\zeta_j k_{j,t})^\alpha (n^d_{j,t})^{1-\alpha} - n^d_{j,t} w$$

Labor demand is:

$$n^d_{j,t} = \left(\frac{1-\alpha}{w_t}\right)^{\frac{1}{\alpha}} \psi\zeta_j k_{j,t} A_t^{\frac{1}{\alpha}}$$

Aggregate labor is given by:

$$\ell_t = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} A_t^{\frac{1}{\alpha}} \sum_J \omega_j \zeta_j k_{j,t} \int \psi di = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} A_t^{\frac{1}{\alpha}} k_t$$

Then we can derive the equilibrium wage:

$$\ell_t^\alpha = (1-\alpha) A_t w_t^{-1} (k_t)^\alpha$$

Then wages are given by:

$$w_t = A_t(1-\alpha)\ell_t^{-\alpha} k_t^\alpha$$

Then an individual investors returns are given by:

$$\frac{\pi_{j,t}}{k_{j,t}} = A_t(\psi\zeta_j k_{j,t})^\alpha (n^d_{j,t})^{1-\alpha} - n^d_{j,t} w_t = A_t \psi\zeta_j \alpha k_t^\alpha \ell_t^{1-\alpha} = \psi\zeta_j r_t$$

A.2 Proof Proposition 1

Social welfare is given by:

$$\mathbb{SW} = \sum_{t=0}^{\infty} \gamma^t \sum_{j\in J} \omega_j \lambda_j \left(u\left(\theta_j^y w_t - k_{j,t+1}\right) + \beta \mathbb{E}\left[u\left(k_{j,t+1}(1+\psi\zeta_j r_{t+1}) + \epsilon\theta_j^o w_{t+1} \right)\right]\right).$$

The derivative of social welfare with respect to $k_{j,t+1}$ is given by the following expression:

$$\frac{d\mathbb{SW}}{dk_{j,t+1}} = \omega_j \lambda_j \left(-u'(c^y_{j,t}) + \beta \mathbb{E}\left[(1+r_{j,t}) u'(c^o_{j,t+1})\right]\right) + \zeta_j \omega_j \mu_{t+1}$$

Where the term μ_{t+1} is:

$$\mu_{t+1} \equiv \beta \sum_j \omega_j \lambda_j \mathbb{E}\left[u'(c^o_{j,t+1})\left(\psi \zeta_j k_{j,t} \frac{\partial r_{t+1}}{\partial k_{t+1}} + \epsilon \theta^o_j \frac{\partial w_{t+1}}{\partial k_{t+1}}\right)\right] + \gamma \sum_j \omega_j \lambda_j u'(c^y_{j,t+1}) \frac{\partial w_{t+1}}{\partial k_{t+1}} \theta^y_j$$

With a slight abuse of notation, we denote the aggregate marginal product of capital, $r_{t+1} = A_{t+1} \alpha k^{\alpha-1}_{t+1} n^{1-\alpha}_{t+1} = \alpha \frac{Y_{t+1}}{k_{t+1}}$ (constant along the balanced growth path) and $w_{t+1} = A_{t+1}(1-\alpha) k^{\alpha}_{t+1} n^{-\alpha}_{t+1} = (1-\alpha)\frac{Y_{t+1}}{n_t}$ (which grows at constant rate g), we obtain

$$\mu_{t+1} \equiv (1-\alpha) r_{t+1} \left\{ \beta \sum_j \omega_j \lambda_j \mathbb{E}\left[u'(c^o_{j,t+1})\left(\epsilon \theta^o_j - \psi \zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right)\right] + \gamma \sum_j \omega_j \lambda_j u'(c^y_{j,t+1}) \theta^y_j \right\}$$

Adding an intelligent zero inside the first term gives us

$$\begin{aligned} \mu_{t+1} \equiv (1-\alpha) r_{t+1} \Bigg\{ & \beta \sum_j \omega_j \lambda_j \mathbb{E}\left[u'(c^o_{j,t+1})\left((\epsilon - 1)\theta^o_j - (\psi - 1)\zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right)\right] \\ & + \beta \sum_j \omega_j \lambda_j \mathbb{E}\left[u'(c^o_{j,t+1})\left(\theta^o_j - \zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right)\right] \\ & + \gamma \sum_j \omega_j \lambda_j u'(c^y_{j,t+1}) \theta^y_j \Bigg\} \end{aligned}$$

Substituting in the definitions of the labor and capital income shares gives the expression in the Proposition.

A.2.1 Proof of Corollary 1.

Assume as stated in the Proposition that second period labor is 0, and that all ex-ante heterogeneity and risk is eliminated. Next, we note that along the balanced growth path, wages w_t grow at constant rate g, implying that the present value of resources for the current young $w_{t+1} = (1+g)w_t$, the resources available to the current old. Because preferences and rates of return remain constant along the BGP:

$$
\begin{aligned}
u'(c^y_{j,t}) = \beta(1+r)u'(c^o_{j,t+1}) \implies c^o_{j,t+1} = (\beta(1+r))^{1/\eta} c^y_{j,t} \\
c^y_{j,t} + \frac{c^o_{j,t+1}}{1+r} = w_t\theta_j \implies c^y_{j,t}\left(1 + \frac{(\beta(1+r))^{1/\eta}}{1+r}\right) = w_t\theta_j \\
\implies c^y_{j,t} = \frac{w_t\theta_j}{\Phi(r)}, \quad \Phi(r) \equiv 1 + \frac{(\beta(1+r))^{1/\eta}}{1+r} \\
w_{t+1} = (1+g)w_t \implies c^y_{j,t+1} = (1+g)c^y_{j,t} \\
\implies u'(c^y_{j,t+1}) = \left(c^y_{j,t+1}\right)^{-\eta} = (1+g)^{-\eta}\left(c^y_{j,t}\right)^{-\eta} = (1+g)^{-\eta}u'(c^y_{j,t})
\end{aligned}
$$

Using the households' Euler equation to sub-in for $u'(c^y_{j,t}) = (1+r)\beta u'(c^o_{j,t+1})$, we have that the final term in the wedge is given by:

$$
\gamma\beta\sum \lambda_j u'(c^o_{j,t+1})\theta^y_j\left(\frac{1+r}{(1+g)^\eta}\right)
$$

By construction, then $\theta^y = 1$. Then the expression from Proposition 1 becomes:

$$
\mu_{t+1} \propto r_{t+1}\left(-u'(c^o_t)\beta + \frac{\gamma}{(1+g)^\eta}u'(c^y_t)\right)
$$

Plugging in the Euler equation, evaluating the above at log utility gives the result.:

$$
\mu_{t+1} \propto r_{t+1}u'(c^y_t)\left(\frac{-1}{1+r_{t+1}} + \frac{\gamma}{1+g}\right)
$$

A.2.2 Measurable Savings Wedge

Before we proceed, let us state the following auxiliary result.

Result 1 *Consider the expectation of the product of two functionals of random variables, i.e., $\mathbb{E}[f(x)g(z)]$. Assuming $g(\mu_z) = 0$, to second order, we have*

$$
\begin{aligned}
\mathbb{E}[f(x)g(z)] \approx \mathbb{E}\big[& f(\mu_x)g(\mu_z) + f'(\mu_x)g(\mu_z)(x-\mu_x) + f(\mu_x)g'(\mu_z)(z-\mu_z) \\
& + \frac{1}{2}f''(\mu_x)g(\mu_z)(x-\mu_x)^2 + f'(\mu_x)g'(\mu_z)(x-\mu_x)(z-\mu_z) + \frac{1}{2}f(\mu_x)g''(\mu_z)(z-\mu_z)^2\big] \\
= & f'(\mu_x)g'(\mu_z)Cov(x,z) \ ,
\end{aligned}
$$

where the last line follows from $g(z) = z$ with $g'(\mu_z) = 1$, $g''(\mu_z) = 0$ and $g(\mu_z) = 0$.

First Term: Using the second period household budget constraint, i.e., $c^o_{j,t+1} = (1-\delta)k_{j,t+1} + \psi\zeta_j r_{t+1}k_{j,t+1} + \epsilon\theta^o_j w_{t+1}$, with $\mathbb{E}\left[c^o_{j,t+1}\right] = (1-\delta+\zeta_j r_{t+1})k_{j,t+1} + \theta^o_j w_{t+1}$, $f(c^o_{j,t+1}) \equiv u'(c^o_{j,t+1})$, and $z \equiv (\epsilon-1)\theta^o_j - (\psi-1)\zeta_j \frac{k_{j,t+1}}{k_{t+1}}$ we obtain:

$$
\begin{aligned}
\mathbb{E}\left[f(c^o_{j,t+1})z\right] &\approx u''(\mathbb{E}\left[c^o_{j,t+1}\right])\mathrm{Cov}\left((1-\delta+\psi\zeta_j r_{t+1})k_{j,t+1} + \epsilon\theta^o_j w_{t+1}, (\epsilon-1)\theta^o_j - (\psi-1)\zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right) \\
&= u''(\mathbb{E}\left[c^o_{j,t+1}\right])\mathrm{Cov}\left(\psi\zeta_j r_{t+1}k_{j,t+1} + \epsilon\theta^o_j w_{t+1}, \epsilon\theta^o_j - \psi\zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right) ,
\end{aligned}
$$

where we can write

$$
\begin{aligned}
&\mathrm{Cov}\left(\psi\zeta_j r_{t+1}k_{j,t+1} + \epsilon\theta^o_j w_{t+1}, \epsilon\theta^o_j - \psi\zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right) \\
&= \left[(\theta^o_j)^2 w_{t+1}\right]\mathrm{Var}(\epsilon) - \left[(\zeta_j k_{j,t+1})^2 \frac{r_{t+1}}{k_{t+1}}\right]\mathrm{Var}(\psi) + \left[\theta^o_j\zeta_j r_{t+1}k_{j,t+1} - \theta^o_j w_{t+1}\zeta_j \frac{k_{j,t+1}}{k_{t+1}}\right]\mathrm{Cov}(\epsilon,\psi) \\
&= Y_{t+1}\left\{(1-\alpha)(\theta^o_j)^2\mathrm{Var}(\epsilon) - \alpha\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\right)^2\mathrm{Var}(\psi) + (2\alpha-1)\theta^o_j\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\mathrm{Cov}(\epsilon,\psi)\right\}
\end{aligned}
$$

Second Term: To second order, we obtain

$$
\mathbb{E}\left[u'(c^o_{j,t+1})\left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right)\right] \approx \left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right)\left\{u'(\mathbb{E}\left[c^o_{j,t+1}\right]) + \frac{1}{2}u'''(\mathbb{E}\left[c^o_{j,t+1}\right])\mathrm{Var}(c^o_{j,t+1})\right\}
$$

which can be simplified to

$$
\begin{aligned}
\mathbb{E}\left[u'(c^o_{j,t+1})\left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right)\right] &\approx u'(\mathbb{E}\left[c^o_{j,t+1}\right])\left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right)\left\{1 + \frac{1}{2}\frac{u'''(\mathbb{E}\left[c^o_{j,t+1}\right])}{u'(\mathbb{E}\left[c^o_{j,t+1}\right])}\mathrm{Var}(c^o_{j,t+1})\right\} \\
&= u'(\mathbb{E}\left[c^o_{j,t+1}\right])\left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right)\left\{1 + \frac{1}{2}\eta\sigma\frac{\mathrm{Var}(c^o_{j,t+1})}{\mathbb{E}\left[c^o_{j,t+1}\right]^2}\right\} ,
\end{aligned}
$$

where we have defined the coefficients of relative risk aversion (η) and relative prudence (σ) as follows (see Kimball 1990, ECMA):

$$
\begin{aligned}
\eta &\equiv -\mathbb{E}\left[c^o_{j,t+1}\right]\frac{u''(\mathbb{E}\left[c^o_{j,t+1}\right])}{u'(\mathbb{E}\left[c^o_{j,t+1}\right])} > 0 , \\
\sigma &\equiv -\mathbb{E}\left[c^o_{j,t+1}\right]\frac{u'''(\mathbb{E}\left[c^o_{j,t+1}\right])}{u''(\mathbb{E}\left[c^o_{j,t+1}\right])} > 0 .
\end{aligned}
$$

Moreover, we can compute the variance of consumption according to

$$
\begin{aligned}
\mathrm{Var}(c^o_{j,t+1}) &= \mathrm{Var}((1-\delta+\psi\zeta_j r_{t+1})k_{j,t+1} + \epsilon\theta^o_j w_{t+1}) \\
&= Y^2_{t+1}\left\{\alpha^2\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\right)^2 \mathrm{Var}(\psi) + (1-\alpha)^2(\theta^o_j)^2\mathrm{Var}(\epsilon) + 2\alpha(1-\alpha)\theta^o_j\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\mathrm{Cov}(\epsilon,\psi)\right\}
\end{aligned}
$$

Substituting in all approximated terms, the *measurable* savings wedge is to second order given by:

$$
\mu_{t+1} \approx (1-\alpha)r_{t+1}\beta\Big[\mu_{PE} + \mu_{RE} + \mu_{FH}\Big],
$$

where the sub-components are given by:

$$
\begin{aligned}
\mu_{PE} \approx\; & \sum_j \omega_j \lambda'_j(\mathbb{E}\left[c^o_{j,t+1}\right])\frac{Y_{t+1}}{\mathbb{E}\left[c^o_{j,t+1}\right]} \\
& \times\left\{\alpha\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\right)^2\mathrm{Var}(\psi) - (1-\alpha)(\theta^o_j)^2\mathrm{Var}(\epsilon) + (1-2\alpha)\theta^o_j\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\mathrm{Cov}(\epsilon,\psi)\right\} \\
\mu_{RE} \approx\; & \sum_j \omega_j\lambda_j u'(\mathbb{E}\left[c^o_{j,t+1}\right])\left(\theta^o_j - \zeta_j\frac{k_{j,t+1}}{k_{t+1}}\right) \\
& \times\left\{1 + \frac{1}{2}\eta\sigma\left(\frac{Y_{t+1}}{\mathbb{E}\left[c^o_{j,t+1}\right]}\right)^2\left[\alpha^2\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\right)^2\mathrm{Var}(\psi) + (1-\alpha)^2(\theta^o_j)^2\mathrm{Var}(\epsilon) + 2\alpha(1-\alpha)\theta^o_j\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\mathrm{Cov}(\epsilon,\psi)\right]\right\} \\
\mu_{FH} =\; & \sum_j \lambda_j u'(c^y_{j,t+1})\theta^y_j\gamma
\end{aligned}
$$

Using the results from Section A.2.1, along the Lasseiz-Faire BGP, we have that $c^y_{j,t+1} = (1+g)c^y_{j,t}$ and therefore that $u'(c^y_{j,t+1}) = (1+g)^{-\eta}u'(c^y_{j,t})$. Using the households' Euler equations, we can therefore write the term $u'(c^y_{j,t+1}) = (1+g)^{-\eta}\beta\mathbb{E}[(1-\delta+r_{t+1}\zeta_j\psi)u'(c^o_{j,t+1})]$.

Take a second order approximation of $\mathbb{E}[((1-\delta)+r_{t+1}\zeta_j\psi)u'(c^o_{j,t+1})]$. Split the gross return into its deterministic and risky components, $(1-\delta)+r_{t+1}\psi\zeta_j = \big((1-\delta)+r_{t+1}\zeta_j\big) + r_{t+1}\zeta_j(\psi-1)$:

$$
\begin{aligned}
\mathbb{E}\big[\big((1-\delta)+r_{t+1}\psi\zeta_j\big)u'(c^o_{j,t+1})\big] &= \big((1-\delta)+r_{t+1}\zeta_j\big)\,\mathbb{E}\big[u'(c^o_{j,t+1})\big] \\
&\quad + r_{t+1}\zeta_j\,\mathbb{E}\big[(\psi-1)u'(c^o_{j,t+1})\big].
\end{aligned}
$$

The two second-order results

$$\begin{aligned}
\mathbb{E}\big[u'(c^o_{j,t+1})\big] &\approx u'(\bar c^o_{j,t+1})\Big[1+\tfrac{1}{2}\eta\sigma V_j\Big],\\
\mathbb{E}\big[(\psi-1)u'(c^o_{j,t+1})\big] &\approx u''(\bar c^o_{j,t+1})\,\mathrm{Cov}(c^o_{j,t+1},\psi)\\
&= -\eta\,\frac{u'(\bar c^o_{j,t+1})}{\bar c^o_{j,t+1}}\,\mathrm{Cov}(c^o_{j,t+1},\psi)
\end{aligned}$$

then, factoring the deterministic gross return out front,

$$\mathbb{E}\big[\big((1-\delta)+r_{t+1}\psi\zeta_j\big)u'(c^o_{j,t+1})\big] \approx \big((1-\delta)+r_{t+1}\zeta_j\big)\,u'(\bar c^o_{j,t+1})\,\tilde{\Xi}_{j,t+1},$$

where

$$\tilde{\Xi}_{j,t+1} \equiv 1+\tfrac{1}{2}\eta\sigma V_j - \eta\,\frac{r_{t+1}\zeta_j\,\mathrm{Cov}(c^o_{j,t+1},\psi)}{\big((1-\delta)+r_{t+1}\zeta_j\big)\,\bar c^o_{j,t+1}},$$

and

$$\begin{aligned}
\bar c^o_{j,t+1} &= \big((1-\delta)+r_{t+1}\zeta_j\big)k_{j,t+1}+\theta^o_j w_{t+1}, \qquad V_j \equiv \frac{\mathrm{Var}(c^o_{j,t+1})}{(\bar c^o_{j,t+1})^2},\\
\mathrm{Cov}(c^o_{j,t+1},\psi) &= r_{t+1}\zeta_j k_{j,t+1}\mathrm{Var}(\psi)+\theta^o_j w_{t+1}\mathrm{Cov}(\epsilon,\psi),\\
\mathrm{Var}(c^o_{j,t+1}) &= (r_{t+1}\zeta_j k_{j,t+1})^2\mathrm{Var}(\psi)+(\theta^o_j w_{t+1})^2\mathrm{Var}(\epsilon)\\
&\quad + 2\,r_{t+1}\zeta_j k_{j,t+1}\,\theta^o_j w_{t+1}\,\mathrm{Cov}(\epsilon,\psi).
\end{aligned}$$

Substituting the balanced-growth Euler:

$$u'(c^y_{j,t+1}) = (1+g)^{-\eta}\beta\,\mathbb{E}\big[((1-\delta)+r_{t+1}\zeta_j\psi)u'(c^o_{j,t+1})\big]$$

into μ_{FH}, applying the factored second-order result above for $\mathbb{E}[((1-\delta)+r_{t+1}\zeta_j\psi)u'(c^o_{j,t+1})]$, and pulling the β it carries to the front, all three channels share the common factor $(1-\alpha)r_{t+1}\beta$. Define the type-j labor and capital income shares

$$\chi^h_{jt} \equiv \frac{w_t\omega_j\theta^h_j n_j}{w_t n}\quad (h\in\{y,o\}), \qquad \xi_{jt} \equiv \frac{r_t\omega_j\zeta_j k_{jt}}{r_t k_t},$$

which absorb the population weight ω_j from the welfare sum. The shares enter the cross-type aggregation only; the per-type objects keep their bare loadings,

$$\begin{aligned}\bar{c}^o_{j,t+1} &\equiv \mathbb{E}[c^o_{j,t+1}] = \big((1-\delta)+r_{t+1}\zeta_j\big)k_{j,t+1} + \theta^o_j w_{t+1},\\ \tilde{\Xi}_{j,t+1} &\equiv 1 + \tfrac{1}{2}\eta\sigma V_j - \eta\,\frac{r_{t+1}\zeta_j\,\mathrm{Cov}(c^o_{j,t+1},\psi)}{\big((1-\delta)+r_{t+1}\zeta_j\big)\bar{c}^o_{j,t+1}},\end{aligned}$$

with the per-type consumption-risk ratio

$$\begin{aligned}V_j \equiv \frac{\mathrm{Var}(c^o_{j,t+1})}{(\bar{c}^o_{j,t+1})^2} = \Big(\frac{Y_{t+1}}{\bar{c}^o_{j,t+1}}\Big)^2\Big[&\alpha^2\Big(\tfrac{\zeta_j k_{j,t+1}}{k_{t+1}}\Big)^2\mathrm{Var}(\psi) + (1-\alpha)^2(\theta^o_j)^2\mathrm{Var}(\epsilon)\\ &+ 2\alpha(1-\alpha)\theta^o_j\tfrac{\zeta_j k_{j,t+1}}{k_{t+1}}\mathrm{Cov}(\epsilon,\psi)\Big].\end{aligned}$$

Then the *measurable* savings wedge is, to second order,

$$\mu_{t+1} \approx (1-\alpha)r_{t+1}\beta\big[\mu_{PE}+\mu_{RE}+\mu_{FH}\big],$$

where

$$\begin{aligned}\mu_{PE} &\approx \sum_j \frac{\lambda_j}{\omega_j}\,\eta\,u'(\bar{c}^o_{j,t+1})\frac{Y_{t+1}}{\bar{c}^o_{j,t+1}}\\ &\quad\times\Big\{\alpha\,\xi^2_{jt}\,\mathrm{Var}(\psi) - (1-\alpha)(\chi^o_{jt})^2\mathrm{Var}(\epsilon) + (1-2\alpha)\chi^o_{jt}\xi_{jt}\,\mathrm{Cov}(\epsilon,\psi)\Big\},\\ \mu_{RE} &\approx \sum_j \lambda_j\,u'(\bar{c}^o_{j,t+1})\big(\chi^o_{jt}-\xi_{jt}\big)\Big\{1+\tfrac{1}{2}\eta\sigma V_j\Big\},\\ \mu_{FH} &\approx \frac{\gamma}{(1+g)^\eta}\sum_j \lambda_j\,\big((1-\delta)+r_{t+1}\zeta_j\big)u'(\bar{c}^o_{j,t+1})\,\tilde{\Xi}_{j,t+1}\,\chi^y_{jt}.\end{aligned}$$

Setting the Covariance(ψ,ϵ)=0 gives the expression in the Proposition.

A.3 Interaction of the risk and redistribution effects.

Suppose that the Cov(ψ,ϵ)=0. Then the second-order approximation of the savings wedge around the zero-risk equilibrium is given by the following expression.

$$\mu_{t+1} \approx (1-\alpha)r_{t+1}\Bigg[\mu_I+\mu_R+\mu_D\Bigg]\ ,$$

where the sub-components are given by:

$$\mu_I = \beta\eta \sum_j \omega_j \nu_j \frac{Y_{t+1}}{\mathbb{E}\left[c^o_{j,t+1}\right]} \times \left(\alpha \left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}} \right)^2 \mathrm{Var}(\psi) - (1-\alpha)(\theta^o_j)^2 \mathrm{Var}(\epsilon) \right)$$

$$\mu_R = \beta \sum_j \omega_j \nu_j \left(\theta^o_j - \zeta_j \frac{k_{j,t+1}}{k_{t+1}} \right) \left(1 + \frac{1}{2}\eta\sigma \frac{\mathrm{Var}(c^o_{j,t+1})}{\mathbb{E}\left[c^o_{j,t+1}\right]^2} \right)$$

$$\mu_D = \gamma \sum_j \omega_j \nu^y_j \theta^y_j$$

Definition. Define the two auxiliary welfare weights:

$$\tilde{\nu}_j \equiv \nu_j \frac{Y_{t+1}}{\omega_j \mathbb{E}[c^o_{j,t+1}]}$$

$$\hat{\nu}_j \equiv \nu_j \left(1 + \frac{1}{2}\eta\sigma \frac{\mathrm{Var}(c^o_{j,t+1})}{\mathbb{E}\left[c^o_{j,t+1}\right]^2} \right)$$

Note that μ_I can be written in the following way:

$$\mu_I = \beta\eta \sum_j \tilde{\nu}_j \left(\alpha \left(\frac{\omega_j \zeta_j k_{j,t+1}}{k_{t+1}} \right)^2 Var(\psi) - (1-\alpha)(\omega_j \theta^o_j)^2 Var(\epsilon) \right) = \beta\eta \sum_j \tilde{\nu}_j \Omega_j$$

where Ω_j is defined as:

$$\Omega_j \equiv \left(\alpha \left(\frac{\omega_j \zeta_j k_{j,t+1}}{k_{t+1}} \right)^2 Var(\psi) - (1-\alpha)(\omega_j \theta^o_j)^2 Var(\epsilon) \right)$$

Similarly, μ_R can similarly be written as:

$$\mu_R = \beta \sum_j \hat{\nu}_j \omega_j \left(\theta^o_j - \frac{\zeta_j k_{j,t+1}}{k_{t+1}} \right)$$

A.3.1 Derivative of μ_I and μ_R to changes in the capital income distribution

Consider a set of infinitesimal changes to the distribution of capital income, $\zeta_j k_j + \Delta_j$ where $\sum_j \Delta_j = 0$, leaving off time subscripts for simplicity. We will characterize the distribution of changes, $\{\Delta_j\}_{j \in J}$ below.

For notational simplicity, we denote $c_j \equiv \mathbb{E}[c^o_{j,t+1}]$

Solve for $d\mu_I$. Consider small changes Δ_j to capital holdings $\zeta_j k_j$ such that $\sum_{j=1}^{J} \omega_j \Delta_j = 0$ (preserving total capital K). The first-order change in μ_I is:

$$\Delta\mu_I = \sum_{j=1}^{J} \Delta_j \cdot \frac{\partial \mu_I}{\partial(\zeta_j k_j)} = \sum_j \left(\Omega_j \Delta\tilde{v}_j + \tilde{v}_j \Delta\Omega_j \right)$$

Since $\tilde{\nu}_j = \lambda_j \frac{u'(c_j)Y_{t+1}}{\omega_j c_j}$, we have:

$$\begin{aligned}
\Delta\tilde{\nu}_j &= \frac{\lambda_j Y_{t+1}}{\omega_j} \left[\frac{u''(c_j)\Delta c_j \cdot c_j - u'(c_j)\Delta c_j}{c_j^2} \right] \\
&= \frac{\lambda_j Y_{t+1} \Delta c_j}{\omega_j c_j} \left[u''(c_j) - \frac{u'(c_j)}{c_j} \right] \\
&= \frac{\lambda_j Y_{t+1} r \Delta_j}{\omega_j c_j} \left[u''(c_j) - \frac{u'(c_j)}{c_j} \right]
\end{aligned}$$

The change in Ω_j is:

$$\Delta\Omega_j = \alpha Var(\psi) \frac{\omega_j^2}{K^2} 2\zeta_j k_j$$

The change in μ_I can be decomposed into a re-weighting effect and a concentration effect:

$$\Delta\mu_I = \Delta\mu_I^{\text{reweight}} + \Delta\mu_I^{\text{concentr}}$$

Where the reweighting effect is given by:

$$\Delta\mu_I^{\text{reweight}} = \sum_{j=1}^{J} \Delta_j \cdot \frac{\lambda_j Y_{t+1} r \Omega_j}{\omega_j c_j} \left(u''(c_j) - \frac{u'(c_j)}{c_j} \right)$$

And the concentration effect is (assuming uniform ω_j):

$$\Delta\mu_I^{\text{concentr}} = \sum_{j=1}^{J} \Delta_j \cdot \frac{2\alpha Var(\psi) \tilde{\nu}_j \omega_j^2 \zeta_j k_j}{K^2} = 2\alpha Var(\psi) \frac{\omega}{K} \sum_{j=1}^{J} \Delta_j \frac{\tilde{\nu}_j \omega_j \zeta_j k_j}{K}$$

Note that when relative inequality is 0 but absolute inequality is not, we have that:

$$Var\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}} \frac{1}{\theta_j} \right) = 0 \text{ and}$$
$$Var\left(c_{j,t+1}^o \right) > 0$$

This implies that $\frac{\zeta_j k_j}{k} = \theta_j$ and therefore that $\frac{\Omega_j}{c_j^2}$ is the same for all household types (intuitively, households capital income share = their labor share = their consumption share). Therefore we can write the the reweighting term as:

$$\Delta\mu_I^{\text{reweight}} = \frac{\Omega}{c^2}\sum_{j=1}^{J}\Delta_j \cdot \frac{ru'(c_j)}{\omega_j}(-\eta - 1)$$

Because $\Delta_j > 0$ for those with high $\zeta_j k_j$ and therefore high c_j and low $u'(c_j)$, the reweighting term is unambiguously negative when $\alpha Var(\psi) < (1-\alpha)Var(\epsilon)$. Intuitively, when labor income risk is more severe than capital income risk, the idiosyncratic risk term is negative, as higher wages and lower average returns amplify risk exposure. When households with a high welfare weight are given an even smaller share of capital income, this amplifies their net exposure, making the impact on welfare through idiosyncratic risk even more severe. If capital income risk is more severe, then further concentrating risk among high income households and removing exposure from low income households improves welfare and increases μ_I.

The sign of the concentration term in this case is unambiguously negative, as $\Delta\mu_I^{concentr}$ is given by:

$$\Delta\mu_I^{concentr} \propto \frac{Y_{t+1}}{\mathbb{E}[c^o_{j,t+1}]}\frac{\omega_j\zeta_j k_{j,t+1}}{k_{t+1}}\sum_{j=1}^{J}\Delta_j u'(\mathbb{E}[c^o_{j,t+1}]) < 0$$

Where here we again use the fact that in the case with no relative inequality, all households have the same ratio between their capital income share and their consumption share.

Let $\Omega = \alpha Var(\psi) - (1-\alpha)Var(\epsilon)$. Then the total change in μ_I can be written as:

$$\Delta\mu_I = \sum_{j=1}^{J}\Delta_j\omega_j u'(\theta_j Y)\left[-\frac{r\Omega(\eta+1)}{Y} + \frac{2\alpha\text{Var}(\psi)}{K}\right]$$

Whenever Ω is negative (labor risk dominates), this change is unambiguously negative. Whenever the following condition is met:

$$\left(\frac{2}{K} - \frac{(1+\eta)r}{Y}\right) = K\Bigg(2-(1+\eta)\alpha\Bigg) \geq 0$$

Then the term is unambiguously negative as well. For $\alpha \approx \frac{1}{3}$ and $\eta \leq 2$ this condition is satisfied.

Solve for $d\mu_R$. We can write μ_R as:

$$\mu_R = \beta\Bigg(\text{mean}(\hat{\nu}_j)\text{mean}\Bigg(\omega_j\theta_j^o - \frac{\omega_j\zeta_j k_{j,t+1}}{k_{t+1}}\Bigg) + Cov\Bigg(\hat{\nu}_j, \Bigg(\omega_j\theta_j^o - \frac{\omega_j\zeta_j k_{j,t+1}}{k_{t+1}}\Bigg)\Bigg)\Bigg)$$
$$= \beta Cov\Bigg(\hat{\nu}_j, \Bigg(\omega_j\theta_j^o - \frac{\omega_j\zeta_j k_{j,t+1}}{k_{t+1}}\Bigg)\Bigg)$$

Note that here, we're using the fact that the average difference between the share of labor income and the share of capital income is by construction equal to 0. Then we need to calculate the difference in μ_I and the difference in μ_R after a given change in the capital income distribution.

This term is easier to sign. If we can establish conditions under which $\hat{\nu}_j$ is higher for already low income households, then giving more capital income to high income households straightforwardly increases the above Covariance.

Expanding out the $\hat{\nu}_j$ term (again assuming 0 Covariance):

$$\hat{\nu}_j = \nu_j\left(1 + \frac{1}{2}\eta\sigma\left(\frac{Y_{t+1}}{\mathbb{E}\left[c^o_{j,t+1}\right]}\right)^2\left[\alpha^2\left(\frac{\zeta_j k_{j,t+1}}{k_{t+1}}\right)^2\text{Var}(\psi) + (1-\alpha)^2(\theta_j^o)^2\text{Var}(\epsilon)\right]\right)$$

If as in the text, we assume no relative inequality, then:

$$\frac{\zeta_j k_{j,t+1}}{k_{t+1}} = \theta_j \quad \forall\, j \in J$$

In this case, $\hat{\nu}_j \propto \nu_j$ and high-income households have unambiguously higher $\hat{\nu}_j$. In this case, the change in the capital income distribution $\{\Delta_j\}_{j\in J}$ assumed above would unambiguously increase μ_R by increasing $Cov\Bigg(\hat{\nu}_j, \Bigg(\omega_j\theta_j^o - \frac{\omega_j\zeta_j k_{j,t+1}}{k_{t+1}}\Bigg)\Bigg)$.

A.4 General Model Results.

A.4.1 Proof of Lemma 2

The final good firm's problem is a standard problem whose solution implies that firm n's demand and the aggregate price level are given by the following expressions.

$$y_{nt} = \left(\frac{p_{nt}}{P_t}\right)^{-\rho} y_t \qquad P_t = \left(\int p_{nt}^{1-\rho} di\right)^{\frac{1}{1-\rho}} \tag{12}$$

Private entrepreneurs. Private businesses earn the private stochastic return $r_{j,t}$, which we derive below. Entrepreneurs operate the following production function with productivity shock, ψ. We define the nominal wage, $W_t = w_t P_t$.

$$y_{j,t} = (\psi\nu_j)^\alpha (k_{j,t})^\alpha (\ell^d_{j,t})^{1-\alpha}$$

For a given capital stock, $k_{j,t}$, entrepreneurs profit in period t is given by:

$$\pi_{j,t} = p_{j,t}(y_{j,t})y_{j,t} - W_t\ell^d_{j,t} + P_t k_{j,t}(1-\delta)$$

Solving for labor demand as a function of ψ, the *nominal* wage W_t, and $k_{j,t}$:

$$\left(\frac{\partial p_{j,t}}{\partial y_{j,t}} y_{j,t} + p_{j,t}\right)\frac{\partial y_{j,t}}{\partial \ell^d_{j,t}} - W_t = 0$$

Plugging in the production function and the firm's demand function and dividing by P_t gives:

$$\left(\mathcal{M}\right)\frac{p_{j,t}}{P_t}\psi^\alpha \nu_j^\alpha (1-\alpha)(k_{j,t})^\alpha (\ell^d_{j,t})^{-\alpha} = w_t$$
$$\left(\left(\mathcal{M}\right)(1-\alpha)w_t^{-1}\right)^{\frac{1}{\alpha}} \psi\nu_j k_{j,t} = \ell^d_{j,t}$$

Where we assume that all firms set the same price in equilibrium (verify this!)[27]. Using the labor market clearing condition to solve for the equilibrium wage:

$$\ell_t = \sum_J \omega_j \ell^d_{j,t} + \ell^c_t$$
$$\rightarrow w_t = \mu(1-\alpha)\left(k^c_t + \sum_J \omega_j \nu_j k_{j,t}\right)^\alpha \ell_t^{-\alpha}$$

Subbing the expression for labor demand in, we can write (real) returns as:

$$r_{j,t} = \frac{\pi_{j,t}}{k_{j,t}} = \psi\nu_j\left(\left(\frac{\frac{(\rho-1)}{\rho}(1-\alpha)}{w_t}\right)^{\frac{1-\alpha}{\alpha}} - w_t\left(\frac{\mathcal{M}(1-\alpha)}{w_t}\right)^{\frac{1}{\alpha}}\right) + 1 - \delta$$
$$r_{j,t} = \psi\nu_j\left(\frac{1}{k_t^\alpha \ell_t^{-\alpha}}\right)^{\frac{1-\alpha}{\alpha}} - (1-\alpha)k_t^\alpha \ell^{1-\alpha}\left(\frac{1}{k_t^\alpha \ell_t^{-\alpha}}\right)^{\frac{1}{\alpha}} + 1 - \delta$$
$$r_{j,t} = \alpha\psi\nu_j k_t^{\alpha-1}\ell_t^{1-\alpha} + 1 - \delta$$

[27] Or just assume that prices have to be set in expectation

Here, we define aggregate capital, k_t as in the text:

$$k_t = \left(\sum \omega_j \int_I \nu_j k^e_{ijt} + k^c_{ijt} \right)$$

Corporate sector. A financial intermediary pools together equity from a measure of corporate firms and offers the return r^c_t. The rate of return on capital invested in the corporate sector, inclusive of profits, is:

$$r^c_t = \alpha k_t^{\alpha-1} \ell_t^{1-\alpha} + 1 - \delta$$

Deriving the bond interest rate. We can use (i) the bond market clearing condition, (ii) households' bond Euler equations, (iii) and the no-arbitrage condition between capital and bonds to derive an implicit function of the bond market.

A.5 Lasseiz-Faire equilibrium conditions.

Equilibrium conditions. The household and market-clearing conditions referenced in the equilibrium definitions are as follows; the firm optimality conditions are derived in the proof of Lemma 2.

Household Euler equations. Interior savings choices at age h in the bond, corporate account, and private firm satisfy

$$u'(c^h_{j,t}) = \beta(1 - \tau_{k,t+1})(1 + r^b_{t+1})\, \mathbb{E}\big[u'(c^{h+1}_{j,t+1})\big], \tag{13}$$

$$u'(c^h_{j,t}) = \beta(1 - \tau_{k,t+1})(1 + r^c_{t+1})\, \mathbb{E}\big[u'(c^{h+1}_{j,t+1})\big], \tag{14}$$

$$u'(c^h_{j,t}) = \beta(1 - \tau_{k,t+1})\, \mathbb{E}\big[(1 + r_{j,t+1})\, u'(c^{h+1}_{j,t+1})\big]. \tag{15}$$

No arbitrage. The riskless assets earn a common return; the private firm carries a risk premium consistent with the Euler equations above:

$$r^b_t = r^c_t \equiv r^f_t, \qquad \mathbb{E}[r_{j,t+1}] > r^f_{t+1}. \tag{16}$$

Market clearing. Define aggregate effective capital and labor as

$$k_t = \sum_j \omega_j \sum_h \zeta_j k^h_{j,t} + k^c_t, \qquad n_t = \sum_j \omega_j \sum_h \theta^h_j.$$

The labor, bond, corporate-capital, and goods markets clear:

$$\sum_j \omega_j \sum_h n_{j,t}^{d,h} + n_t^c = n_t, \tag{17}$$

$$\sum_j \omega_j \sum_h b_{j,t}^h = B_t, \tag{18}$$

$$\sum_j \omega_j \sum_h k_{j,t}^{ch} = k_t^c, \tag{19}$$

$$Y_t = \sum_j \omega_j \sum_h c_{j,t}^h + \big(k_{t+1} - (1-\delta)k_t\big) + G_t. \tag{20}$$

A.6 Deriving the general savings wedge.

Now the social welfare function is given by:

$$SW = \sum_{t=0}^{\infty} \gamma^t \sum_{j \in J} \omega_j \lambda_j \left(\sum_{h=0}^{H} \beta^h \mathbb{E}[u(c_{j,t+h}^h)] \right)$$

subject to:

$$\begin{aligned}
c_{j,t+h}^h &= (1-\tau_{\ell,t+h})\theta_j^h \epsilon w_{t+h} - b_{j,t+h+1}^{h+1} - k_{j,t+h+1}^{c,h+1} - k_{j,t+h+1}^{h+1} \\
&\quad + \big(1 + (1-\tau_{k,t+h}) r_{t+h}^b\big) b_{j,t+h}^h + \big(1 + (1-\tau_{k,t+h}) r_{t+h}^c\big) k_{j,t+h}^{ch} \\
&\quad + \big(1 + (1-\tau_{k,t+h}) r_{j,t+h}\big) k_{j,t+h}^h \\
r_t^c &= \alpha k_t^{\alpha-1} \ell_t^{1-\alpha} - \delta, \quad r_{j,t} = \alpha \psi \zeta_j k_t^{\alpha-1} \ell_t^{1-\alpha} - \delta, \quad w_t = \mathcal{M}(1-\alpha) k_t^{\alpha} \ell_t^{-\alpha}.
\end{aligned}$$

We assume that $B_t = B$ and that a uniform lump-sum tax, G_t absorbs any changes in revenue.

The total marginal value of additional saving from a type-j household at age h has the same structure for each asset: a *direct* Euler term, which vanishes at the laissez-faire optimum, plus an *indirect wedge* term operating through the asset's contribution to the aggregate capital stock. Because we scale only period-t saving flows, the perturbation moves the period-t capital stock alone, so a single contemporaneous wedge μ_t enters. Taking the three asset margins separately, each holding the other two fixed:

$$\begin{aligned}
\frac{dSW}{dk^h_{j,t}} &= \gamma^{t-h}\omega_j\lambda_j\beta^{h-1}\Big[-u'(c^{h-1}_{j,t-1}) + \beta\,\mathbb{E}\big[\big(1+(1-\tau_{kt})r_{j,t}\big)u'(c^h_{j,t})\big]\Big] + \omega_j\zeta_j\,\mu_t,\\
\frac{dSW}{dk^{ch}_{j,t}} &= \gamma^{t-h}\omega_j\lambda_j\beta^{h-1}\Big[-u'(c^{h-1}_{j,t-1}) + \beta\big(1+(1-\tau_{kt})r^c_t\big)\mathbb{E}\big[u'(c^h_{j,t})\big]\Big] + \omega_j\,\mu_t,\\
\frac{dSW}{db^h_{j,t}} &= \gamma^{t-h}\omega_j\lambda_j\beta^{h-1}\Big[-u'(c^{h-1}_{j,t-1}) + \beta\big(1+(1-\tau_{kt})r^b_t\big)\mathbb{E}\big[u'(c^h_{j,t})\big]\Big].
\end{aligned}$$

The private and corporate margins each enter aggregate capital $k_t = \sum_j \omega_j \sum_h \zeta_j k^h_{j,t} + k^c_t$, with weights $\omega_j\zeta_j$ and ω_j, and so carry the wedge; the bond does not enter k_t, so its margin is the Euler alone, with no wedge attached.
We evaluate these expressions at the laissez-faire equilibrium, where every Euler vanishes, implying that $\frac{dSW}{dk^h_{j,t}} = \omega_j\zeta_j\mu_t$, $\frac{dSW}{dk^{ch}_{j,t}} = \omega_j\mu_t$, and $\frac{dSW}{db^h_{j,t}} = 0$. Then if dk_t can be written as

$$dk_t = \sum_j\sum_h \omega_j\big(\zeta_j dk^h_{j,t} + dk^{hc}_{j,t}\big),$$

we have that $\frac{dSW}{dk_t} = \mu_t$, where μ_t is given by (folding the newborn generation in as the $h=0$ term, with $k^0_{j,t} = k^{c0}_{j,t} = b^0_{j,t} = 0$):

$$\begin{aligned}
\mu_t = \sum_{h=0}^{H}\sum_j \gamma^{t-h}\beta^h\lambda_j\omega_j\mathbb{E}\Bigg[u'(c^h_{j,t})\Bigg(&(1-\tau_{\ell,t})\theta^h_j\epsilon\tfrac{dw_t}{dk_t}\\
&+(1-\tau_{kt})\Big(b^h_{j,t}\tfrac{dr^b_t}{dk_t} + k^{ch}_{j,t}\tfrac{dr^c_t}{dk_t} + \zeta_j k^h_{j,t}\psi\tfrac{dr_t}{dk_t}\Big)\Bigg)\Bigg]
\end{aligned}$$

Recall that $r_t = A_t\alpha k_t^{\alpha-1}\ell_t^{1-\alpha}$ and $w_t = A_t\mathcal{M}(1-\alpha)k_t^{\alpha}\ell_t^{-\alpha}$. Therefore, $\frac{dw_t}{dk_t} = \mathcal{M}(1-\alpha)r_t\ell_t^{-1}$ and $\frac{dr_t}{dk_t} = (\alpha-1)r_tk_t^{-1}$; by Lemma 2 and no arbitrage, $\frac{dr^c_t}{dk_t} = \frac{dr^b_t}{dk_t} = \frac{dr_t}{dk_t}$. Factoring out $(1-\alpha)r_t$, the labor terms retain $\mathcal{M}/\ell_t$ and the capital terms $-1/k_t$. Following the same procedure as before, we can decompose μ_t into risk and redistribution terms by adding an intelligent zero.

$$\begin{aligned}
\mu_t = (1-\alpha)r_t\Bigg(&\sum_j\omega_j\lambda_j\sum_{h=0}^{H}\gamma^{t-h}\beta^h\mathbb{E}\Big[u'(c^h_{j,t})\Big((\epsilon-1)\mathcal{M}(1-\tau_{\ell t})\tfrac{\theta^h_j}{\ell_t} - (1-\tau_{kt})(\psi-1)\zeta_j\tfrac{k^h_{j,t}}{k_t}\Big)\Big]\\
&+\sum_j\omega_j\lambda_j\sum_{h=0}^{H}\gamma^{t-h}\beta^h\mathbb{E}\big[u'(c^h_{j,t})\big]\Big(\mathcal{M}(1-\tau_{\ell t})\tfrac{\theta^h_j}{\ell_t} - \big(\zeta_j\tfrac{k^h_{j,t}}{k_t} + \tfrac{k^{ch}_{j,t}}{k_t} + \tfrac{b^h_{j,t}}{k_t}\big)(1-\tau_{kt})\Big)\Bigg)
\end{aligned}$$

Because households hold the riskless corporate account (and bond) in interior amounts, the riskless Euler equation holds with equality at every age. Denote the after-tax riskless rate $r_t^f \equiv (1-\tau_{kt})(r_t - \delta)$. Taking unconditional expectations of the Euler equation and using balanced growth, $\mathbb{E}[u'(c_{j,t+1}^{h+1})] = (1+g)^{-\eta}\mathbb{E}[u'(c_{j,t}^{h+1})]$,

$$\mathbb{E}[u'(c_{j,t}^h)] = \beta(1+r^f)\,\mathbb{E}[u'(c_{j,t+1}^{h+1})] = \frac{\beta(1+r^f)}{(1+g)^\eta}\,\mathbb{E}[u'(c_{j,t}^{h+1})] \implies \mathbb{E}[u'(c_{j,t}^h)] = \left(\frac{\beta(1+r^f)}{(1+g)^\eta}\right)^{H-h}\mathbb{E}[u'(c_{j,t}^H)].$$

This referencing is exact: because the return is riskless, it factors out of the expectation, and no risk adjustment is required. Combining the chain with the cohort weight γ^{t-h} and the utility discount β^h, and using $\gamma^{t-h} = \gamma^{t-H}\gamma^{H-h}$, the weight on the age-$h$ term factors as

$$\gamma^{t-h}\beta^h\left(\frac{\beta(1+r^f)}{(1+g)^\eta}\right)^{H-h} = \underbrace{\gamma^{t-H}\beta^H}_{\text{common}}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}.$$

The common factor $\gamma^{t-H}\beta^H$ is independent of j and h and is absorbed into the constant of proportionality. The γ^{H-h} inside the bracket is pure cohort discounting — the age-h cohort was born $H-h$ periods after the oldest living cohort — while $(1+r^f)/(1+g)^\eta$ is pure Euler referencing. The chain applies to $\mathbb{E}[u'(c_{j,t}^h)]$ as a standalone unconditional mean; it therefore substitutes directly into the redistribution term, where $\mathbb{E}[u'(c_{j,t}^h)]$ multiplies non-stochastic income shares, but *not* into the risk term, where $u'(c_{j,t}^h)$ sits inside an expectation against the household's own shocks. Substituting into the redistribution term only, and folding the newborn wage term into the $h=0$ element of the sums (newborns hold no assets, $k_{j,t}^0 = k_{j,t}^{c0} = b_{j,t}^0 = 0$):

$$\begin{aligned}\mu_t = (1-\alpha)\,r_t\Bigg(&\sum_j \omega_j\lambda_j\sum_{h=0}^{H}\gamma^{t-h}\beta^h\,\mathbb{E}\Big[u'(c_{j,t}^h)\Big((\epsilon-1)\mathcal{M}(1-\tau_{\ell t})\tfrac{\theta_j^h}{\ell_t} - (1-\tau_{kt})(\psi-1)\zeta_j\tfrac{k_{j,t}^h}{k_t}\Big)\Big]\\
&+\gamma^{t-H}\beta^H\sum_j\omega_j\lambda_j\sum_{h=0}^{H}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}\mathbb{E}\big[u'(c_{j,t}^H)\big]\\
&\qquad\times\Big(\mathcal{M}(1-\tau_{\ell t})\tfrac{\theta_j^h}{\ell_t} - \big(\zeta_j\tfrac{k_{j,t}^h}{k_t} + \tfrac{k_{j,t}^{ch}}{k_t} + \tfrac{b_{j,t}^h}{k_t}\big)(1-\tau_{kt})\Big)\Bigg)\end{aligned}$$

The risk term retains its raw weight $\gamma^{t-h}\beta^h$ and age-h marginal utility; the chain will enter it below, only after the second-order approximation has converted the random $u'(c_{j,t}^h)$ into a deterministic coefficient.

Recall that if we define $z \equiv (\epsilon-1)\mathcal{M}(1-\tau_{\ell t})\frac{\theta_j^h}{\ell_t} - (1-\tau_{kt})(\psi-1)\zeta_j\frac{k_{j,t}^h}{k_t}$, then to second order

$$\mathbb{E}\big[u'(c_{j,t}^h)\, z\big] \approx u''(\mathbb{E}[c_{j,t}^h])\,\mathrm{Cov}\big(c_{j,t}^h, z\big) = -\eta\,\frac{u'(\mathbb{E}[c_{j,t}^h])}{\mathbb{E}[c_{j,t}^h]}\,\mathrm{Cov}\big(c_{j,t}^h, z\big).$$

To keep track of orders, write $\epsilon = 1+\varsigma e$ and $\psi = 1+\varsigma p$ with $\mathbb{E}[e]=\mathbb{E}[p]=0$. The zero-risk steady state corresponds to $\varsigma = 0$, and all shock variances and covariances are $O(\varsigma^2)$. Taking a second order approximation of $\mathbb{E}[u'(c_{j,t}^h)]$ around the zero-risk steady state gives:

$$\mathbb{E}[u'(c_{j,t}^h)] = u'(\mathbb{E}[c_{j,t}^h]) + \tfrac{1}{2}u'''(\mathbb{E}[c_{j,t}^h])\,\mathrm{Var}(c_{j,t}^h) + O(\varsigma^3),$$

Re-arranging to solve for $u'(\mathbb{E}[c_{j,t}^h])$:

$$u'(\mathbb{E}[c_{j,t}^h]) = \mathbb{E}[u'(c_{j,t}^h)] - \tfrac{1}{2}u'''(\mathbb{E}[c_{j,t}^h])\,\mathrm{Var}(c_{j,t}^h) + O(\varsigma^3),$$

Note that the second term is $O(\varsigma^2)$. Subbing this in to the risk term, the correction multiplies $\mathrm{Cov}(c_{j,t}^h, z) = O(\varsigma^2)$, making it $O(\varsigma^4)$ — beyond second order:

$$\mathbb{E}\big[u'(c_{j,t}^h)\, z\big] = -\eta\,\frac{\mathrm{Cov}(c_{j,t}^h, z)}{\mathbb{E}[c_{j,t}^h]}\,\mathbb{E}[u'(c_{j,t}^h)]\ \underbrace{+\,\eta\,\frac{\mathrm{Cov}(c_{j,t}^h, z)}{\mathbb{E}[c_{j,t}^h]}\,\tfrac{1}{2}u'''(\mathbb{E}[c_{j,t}^h])\,\mathrm{Var}(c_{j,t}^h)}_{O(\varsigma^4)}\ + O(\varsigma^3).$$

Now $\mathbb{E}[u'(c_{j,t}^h)]$ is a constant rather than a random variable inside the expectation, so we can sub in the riskless Euler chain just as in the redistribution term:

$$\mathbb{E}[u'(c_{j,t}^h)] = \left(\frac{\beta(1+r^f)}{(1+g)^\eta}\right)^{H-h}\mathbb{E}[u'(c_{j,t}^H)];$$

Replacing $\mathbb{E}[u'(c_{j,t}^H)]$ with $u'(\mathbb{E}[c_{j,t}^H])$ adds another $O(\varsigma^4)$ term. Note that the $1/\mathbb{E}[c_{j,t}^h]$ term is an age-h object and stays at age h. Then we have that:

$$\begin{aligned}\mathrm{Cov}\big(c_{j,t}^h, z\big) = y_t\Bigg[&\mathcal{M}^2(1-\alpha)(1-\tau_{\ell t})^2\Big(\tfrac{\theta_j^h}{\ell_t}\Big)^2\mathrm{Var}(\epsilon) - \alpha(1-\tau_{kt})^2\Big(\tfrac{\zeta_j k_{j,t}^h}{k_t}\Big)^2\mathrm{Var}(\psi)\\ &+\mathcal{M}(2\alpha-1)(1-\tau_{\ell t})(1-\tau_{kt})\tfrac{\theta_j^h}{\ell_t}\tfrac{\zeta_j k_{j,t}^h}{k_t}\,\mathrm{Cov}(\epsilon,\psi)\Bigg].\end{aligned}$$

For the redistribution term, the chain is exact in $\mathbb{E}[u']$, so we approximate only the terminal expectation: $\mathbb{E}[u'(c^H_{j,t})] \approx u'(\mathbb{E}[c^H_{j,t}])\big(1+\frac{1}{2}\eta\sigma\frac{\mathrm{Var}(c^H_{j,t})}{\mathbb{E}[c^H_{j,t}]^2}\big)$. Define the welfare weights

$$\nu_{j,t} \equiv \lambda_j u'(\mathbb{E}[c^H_{j,t}]), \qquad \hat{\nu}_{j,t} \equiv \nu_{j,t}\left(1+\tfrac{1}{2}\eta\sigma\frac{\mathrm{Var}(c^H_{j,t})}{\mathbb{E}[c^H_{j,t}]^2}\right),$$

and define the type-j age-h labor income share, risky capital income share, and total capital income share:

$$\chi^h_{jt} \equiv \frac{w_t\omega_j\theta^h_j}{w_t\ell_t}, \qquad \xi^h_{jt} \equiv \frac{r_t\omega_j\zeta_j k^h_{j,t}}{r_t k_t}, \qquad \bar{\xi}^h_{jt} \equiv \frac{r_t\omega_j\big(\zeta_j k^h_{j,t}+k^{ch}_{j,t}+b^h_{j,t}\big)}{r_t k_t}.$$

The shares each absorb one population weight ω_j. The redistribution term is linear in the shares, so its explicit ω_j is absorbed; the risk term is degree-two in the shares, so its weight retains a $1/\omega_j$. Subbing these approximations in to the expression for μ_t above, and absorbing the common factor $\gamma^{t-H}\beta^H$ into the constant of proportionality, the measurable savings wedge is given by:

$$\mu_t \propto (1-\alpha)\, r_t\,\big(\mu_{It}+\mu_{Rt}\big),$$

where

$$\begin{aligned}\mu_{It} = \eta\sum_j\sum_{h=0}^{H}\frac{\nu_{j,t}}{\omega_j}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}\frac{y_t}{\mathbb{E}[c^h_{j,t}]}\Bigg[&\alpha(1-\tau_{kt})^2\big(\xi^h_{jt}\big)^2\mathrm{Var}(\psi)\\ &-(1-\alpha)\mathcal{M}^2(1-\tau_{\ell t})^2\big(\chi^h_{jt}\big)^2\mathrm{Var}(\epsilon)\\ &+\mathcal{M}(1-2\alpha)(1-\tau_{\ell t})(1-\tau_{kt})\,\chi^h_{jt}\,\xi^h_{jt}\,\mathrm{Cov}(\epsilon,\psi)\Bigg],\end{aligned}$$

$$\mu_{Rt} = \sum_j\sum_{h=0}^{H}\hat{\nu}_{j,t}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}\Big[\mathcal{M}(1-\tau_{\ell t})\,\chi^h_{jt}-(1-\tau_{kt})\,\bar{\xi}^h_{jt}\Big].$$

A.6.1 Proof of Corollary 2

Suppose there is no idiosyncratic risk ($\epsilon=\psi=1$), no taxes ($\tau_{\ell t}=\tau_{kt}=0$), no markups ($\mathcal{M}=1$), and no ex-ante heterogeneity within generations ($\theta^h_j=\theta^h$ and $\zeta_j=\zeta$ for all j).

With $\mathrm{Var}(\psi)=\mathrm{Var}(\epsilon)=\mathrm{Cov}(\epsilon,\psi)=0$, the risk term vanishes, $\mu_{It}=0$, and $\hat{\nu}_{j,t}=\nu_{j,t}$. With no taxes, the after-tax riskless rate is $r^f=r-\delta$ and $\mathcal{M}=1$ removes the markup from the labor share. With no risk, all three assets pay the common return $1+r^f$, so only households' total asset position $a^h_{j,t}\equiv\zeta k^h_{j,t}+k^{ch}_{j,t}+b^h_{j,t}$ is pinned down, and the redistribution bracket depends on assets only through $\bar{\xi}^h_{jt} = \omega_j a^h_{j,t}/k_t$.

With no ex-ante heterogeneity, all type-j households within a generation hold the same allocation, so $\mathbb{E}[c^H_{j,t}] = c^H_t$ and $\nu_{j,t} = \lambda_j u'(c^H_t)$. Then the j-sum in μ_{Rt} collapses:

$$\sum_j \nu_{j,t}\Big[\chi^h_{jt} - \bar{\xi}^h_{jt}\Big] = u'(c^H_t)\sum_j \lambda_j\omega_j\left[\frac{\theta^h}{\ell_t} - \frac{a^h_t}{k_t}\right] = u'(c^H_t)\Big(\sum_j \lambda_j\omega_j\Big)\Big[\Theta_h - Z_h\Big],$$

where

$$\Theta_h \equiv \frac{\sum_j \omega_j \theta^h_j}{\ell_t}, \qquad Z_h \equiv \frac{\sum_j \omega_j\big(\zeta_j k^h_{j,t} + k^{ch}_{j,t} + b^h_{j,t}\big)}{k_t}$$

denote generation h's share of total labor income and of total asset income, respectively. The constant $u'(c^H_t)\sum_j \lambda_j\omega_j$ is independent of h and is absorbed into the constant of proportionality. Then we have that:

$$\mu_t \propto r_t \sum_{h=0}^{H}\left(\frac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h}\Big[\Theta_h - Z_h\Big].$$

B The optimal mandate in the general model

The policy instrument

The planner sets a proportional mandate $\Gamma_t > 0$ scaling generation-t saving at every age and in every asset relative to laissez-faire,

$$\big\{k^h_{j,t+1}, k^{ch}_{j,t+1}, b^h_{j,t+1}\big\}(\Gamma_t) = \Gamma_t\big\{\hat{k}^h_{j,t+1}, \hat{k}^{ch}_{j,t+1}, \hat{b}^h_{j,t+1}\big\} \quad \text{for all } j, h, \qquad k_{t+1}(\Gamma_t) = \Gamma_t \hat{k}_{t+1},$$

with $\Gamma_t = 1$ laissez-faire. Mandated saving is held in the household's own firm, the corporate sector, and the bond, and returned with its realized return; there are no transfers across types and no distorting tax beyond the existing τ_ℓ, τ_k. Because every type and age scales by the common Γ_t, the instrument acts on the planner's net investment $i^h_{j,t} \equiv k^{h+1}_{j,t+1} - (1-\delta)k^h_{j,t}$; at $\delta = 1$ households are born with no capital and consume the whole stock at death, so $i^h_{j,t} = k^{h+1}_{j,t+1}$ and $di^h_{j,t}/d\Gamma_t = \hat{k}^{h+1}_{j,t+1}$, with no cross-period coupling through the rolled-forward principal.

The planner's problem

Utilitarian, with Pareto weights λ_j across types and geometric weights γ across generations:

$$\max_{\{\Gamma_t\}} SW, \qquad SW = \sum_{t=0}^{\infty} \gamma^t \sum_{j \in J} \omega_j \lambda_j \sum_{h=0}^{H} \beta^h \, \mathbb{E}\big[u(c_{j,t+h}^h)\big].$$

First-order condition

The mandate scales all three asset positions of generation t at every age, so $dx_{j,t+1}^{a,h+1}/d\Gamma_t = \hat{x}_{j,t+1}^{a,h+1}$ for each asset $a \in \{k, c, b\}$, writing $x^k = k$, $x^c = k^c$, $x^b = b$ for the private, corporate, and bond holdings. Aggregating the text's net-investment conditions across assets and ages—each instrument collapsing at $\delta = 1$ to a single within-life step—and collecting own-quantity terms into the asset Euler residuals and price terms into the wedge,

$$\frac{dSW}{d\Gamma_t} = \gamma^t \Bigg[\sum_j \omega_j \lambda_j \sum_{a \in \{k,c,b\}} \sum_{h=0}^{H-1} \left(\tfrac{\beta}{\gamma}\right)^h \hat{x}_{j,t+1}^{a,h+1} \, \widehat{\mathcal{E}}_{j,t}^{a,h} \; + \; \hat{k}_{t+1} \, \mu_{t+1} \Bigg],$$

where the age-h Euler residual on asset a is

$$\widehat{\mathcal{E}}_{j,t}^{a,h} \equiv -\, u'(c_{j,t}^h) + \beta \, \mathbb{E}\big[R_{j,t+1}^a \, u'(c_{j,t+1}^{h+1})\big],$$

with after-tax gross returns

$$R_{j,t+1}^k = (1-\tau_k)(1-\delta+\psi\zeta_j r_{t+1}), \quad R_{t+1}^c = (1-\tau_k)(1-\delta+r_{t+1}), \quad R_{t+1}^b = (1-\tau_k)(1+r_{t+1}^b),$$

the first risky, the latter two safe. Bonds are in zero net supply, so the scaled positions cancel in aggregate capital $\hat{k}_{t+1}$ and enter only through their Euler residuals; private and corporate capital both contribute to $\hat{k}_{t+1}$. Setting $dSW/d\Gamma_t = 0$ gives the policy condition

$$\sum_j \omega_j \lambda_j \sum_{a \in \{k,c,b\}} \sum_{h=0}^{H-1} \left(\tfrac{\beta}{\gamma}\right)^h \hat{x}_{j,t+1}^{a,h+1} \, \widehat{\mathcal{E}}_{j,t}^{a,h} \; + \; \hat{k}_{t+1} \, \mu_{t+1} = 0. \tag{21}$$

This is the general-model counterpart of the simple-model condition $\sum_j \omega_j \lambda_j k_j^{LF} \mathcal{E}_j + k^{LF} \mu = 0$: the single saving level k_j^{LF} is replaced by the within-life, across-asset sum $\sum_a \sum_h (\beta/\gamma)^h \hat{x}_{j,t+1}^{a,h+1}$, each age weighted by its planner discount relative to the cohort, and the forward multiplier $1/[1-\gamma(1-\delta)]$ equals one at $\delta = 1$.

Second-order approximation of the Euler term

The saving household at age h reaps at age $h+1$. The safe assets (c, b) carry deterministic returns, so $\widehat{\mathcal{E}}^{c,h}$ and $\widehat{\mathcal{E}}^{b,h}$ expand trivially; only the risky private margin $\widehat{\mathcal{E}}^{k,h}$ carries risk, through the term $\mathbb{E}[\psi\, u'(c^{h+1}_{j,t+1})]$. With $\bar{c}^{h+1}_{j,t+1} \equiv \mathbb{E}[c^{h+1}_{j,t+1}] = \zeta_j r_{t+1} k^{h+1}_{j,t+1} + \theta^{h+1}_j w_{t+1}$ and $\psi = 1 + (\psi - 1)$, the same expansion as the simple model gives

$$\widehat{\mathcal{E}}^{k,h}_{j,t} \approx -u'(c^h_{j,t}) + \beta\zeta_j r_{t+1}\, u'(\bar{c}^{h+1}_{j,t+1})\, \Xi^{h+1}_{j,t+1},$$

$$\Xi^{h+1}_{j,t+1} \equiv 1 + \tfrac{1}{2}\eta^{RRA}\eta^{RPR}\frac{\mathrm{Var}(c^{h+1}_{j,t+1})}{(\bar{c}^{h+1}_{j,t+1})^2} - \eta^{RRA}\,\frac{\zeta_j r_{t+1} k^{h+1}_{j,t+1}\,\mathrm{Var}(\psi)}{\bar{c}^{h+1}_{j,t+1}},$$

with $u'' = -\eta^{RRA}u'/\bar{c}$ and $u''' = \eta^{RRA}\eta^{RPR}u'/\bar{c}^2$. The young (age-$h$) marginal cost $u'(c^h_{j,t})$ is deterministic and stays in levels.

Lemma 3 *With full depreciation, $\delta = 1$, the consumption-risk bracket $\Xi^{h+1}_{j,t+1}$ is independent of the mandate Γ.*

Proof. At $\delta = 1$ there is no undepreciated principal, so old-age consumption at each age is capital plus labor income, $\bar{c}^{h+1}_j = \zeta_j r\, k^{h+1}_j + \theta^{h+1}_j w$. Under the proportional mandate $k^{h+1}_j = \Gamma\hat{k}^{h+1}_j$ and the Cobb–Douglas scaling $r = r^{LF}\Gamma^{\alpha-1}$, $w = w^{LF}\Gamma^{\alpha}$, both components scale as output,

$$\zeta_j r\, k^{h+1}_j = a_j\Gamma^{\alpha}, \quad \theta^{h+1}_j w = b_j\Gamma^{\alpha}, \quad \text{so} \quad \bar{c}^{h+1}_j = (a_j + b_j)\Gamma^{\alpha},$$

with $a_j \equiv \zeta_j r^{LF}\hat{k}^{h+1}_j$ and $b_j \equiv \theta^{h+1}_j w^{LF}$. The Γ-linear principal term that survives at $\delta < 1$ is absent, so every ingredient of Ξ is a ratio of equal powers of Γ:

$$\pi_j = \frac{\zeta_j r k^{h+1}_j}{\bar{c}^{h+1}_j} = \frac{a_j}{a_j + b_j}, \qquad V_j = \frac{\mathrm{Var}(c^{h+1}_j)}{(\bar{c}^{h+1}_j)^2}$$
$$= \pi_j^2\mathrm{Var}(\psi) + (1-\pi_j)^2\mathrm{Var}(\epsilon) + 2\pi_j(1-\pi_j)\mathrm{Cov}(\psi, \epsilon),$$

both constant in Γ. Hence $\Xi^{h+1}_{j,t+1} = 1 + \frac{1}{2}\eta^{RRA}\eta^{RPR}V_j - \eta^{RRA}\,\pi_j\mathrm{Var}(\psi)$ is constant in Γ. ■

The wedge in powers of Γ

The wedge μ_{t+1} is the object of Proposition 6. To isolate its Γ-dependence, evaluate it at the mandated allocation $k = \Gamma\hat{k}$. Under $\delta = 1$ every level is a power of Γ,

$$r = r^{LF}\Gamma^{\alpha-1}, \quad w = w^{LF}\Gamma^{\alpha}, \quad Y = Y^{LF}\Gamma^{\alpha}, \quad \bar{c}^h_j = \bar{c}^{h,LF}_j\Gamma^{\alpha}, \quad u'(\bar{c}^h_j) = (\bar{c}^{h,LF}_j)^{-\sigma}\Gamma^{-\alpha\sigma},$$

while the shares S_j^h, χ_j^h, the ratios $\bar{c}_j^h/Y$, the brackets Ξ_j^{h+1} (by the Lemma), and the shock moments are scale-free.

The lifecycle weights require care, because what the wedge truly weights is the age-h old-age marginal utility $u'(\bar{c}_{j,t+1}^h)$ at the mandated allocation. The Proposition 6 weight $\nu_{j,t}\big(\gamma(1+r^f)/(1+g)^{\eta^{RRA}}\big)^{H-h}$ rewrites that object through the household's Euler equation,

$$u'(\bar{c}_j^h) = \frac{\gamma(1+r^f)}{(1+g)^{\eta^{RRA}}}\, u'(\bar{c}_j^{h+1}),$$

which holds only at laissez-faire; under the mandate the household is forced off its saving margin, so this intertemporal identity fails and the chain $(1+r(\Gamma))^{H-h}$ would assert an MRS the distorted household does not equate. The correct object is the explicit marginal utility,

$$u'(\bar{c}_j^h) = (\bar{c}_j^{h,LF})^{-\sigma}\Gamma^{-\alpha\sigma}, \qquad u'(\bar{c}_j^{h,LF}) = \nu_{j,t}\left(\tfrac{\gamma(1+r^{f,LF})}{(1+g)^{\eta^{RRA}}}\right)^{H-h},$$

which scales by the common power $\Gamma^{-\alpha\sigma}$ at every age; the lifecycle weights are its valid laissez-faire instance, and so enter at their laissez-faire values and are invariant to Γ.

The price prefactor $r \propto \Gamma^{\alpha-1}$ together with the common $\Gamma^{-\alpha\sigma}$ combine to a single age-independent power. Pulling the prefactor $(1-\alpha)r$ out of the Proposition 6 wedge, $\mu_{t+1} = (1-\alpha)r_{t+1}(\mu_I + \mu_R)$, and defining the normalized wedge $\tilde{\mu} \equiv Y^{LF}(\mu_I+\mu_R)|_{\Gamma=1}$,

$$\mu_{t+1} = (1-\alpha)\, r^{LF}\, (Y^{LF})^{-1}\, \Gamma^{\alpha(1-\sigma)-1}\, \tilde{\mu}, \qquad \tilde{\mu} \equiv Y^{LF}\, (\mu_I+\mu_R)\big|_{\Gamma=1}, \tag{22}$$

where the Y^{LF} cancels the $Y^{-\sigma}$ in each $\nu_j = \lambda_j u'(\bar{c}_j^H)$, leaving $\tilde{\mu}$ a pure-share object in the laissez-faire lifecycle weights, income shares, ratios $\bar{c}_j^{h,LF}/Y^{LF}$, and brackets $\Xi_j^{h+1,LF}$. Since $\hat{k}_{t+1}\, r^{LF} = \alpha Y^{LF}$, the wedge term in the first-order condition collapses to

$$\frac{dSW}{d\Gamma_t} = \gamma^t\Bigg[\sum_j \omega_j\lambda_j \sum_{a\in\{k,c,b\}} \sum_{h=0}^{H-1} \left(\tfrac{\beta}{\gamma}\right)^h \hat{x}_{j,t+1}^{a,h+1}\, \widehat{\mathcal{E}}_{j,t}^{a,h} \;+\; \alpha(1-\alpha)\, \Gamma^{\alpha(1-\sigma)-1}\, \tilde{\mu}\Bigg].$$

The first-order condition in Γ

Let $\bar{y}_{j,t}^h$ denote the predetermined period-t resources of an age-h household—labor income plus gross after-tax asset returns on the stocks it brought in, all set by $\Gamma_{t-1} = 1$—and $\hat{s}_j^h$ its laissez-faire saving:

$$\bar{y}_{j,t}^h \equiv (1-\tau_\ell)\theta_j^h \epsilon\, w_t + (1-\tau_k)\big[(1+r_t^b)\hat{b}_{j,t}^h + (1+r_t^c)\hat{k}_{j,t}^{ch} + (1+r_{j,t})\hat{k}_{j,t}^h\big], \qquad \hat{s}_j^h \equiv \sum_a \hat{x}_{j,t+1}^{a,h+1}.$$

The budget constraint gives $c_{j,t}^h = \bar{y}_{j,t}^h - \Gamma \hat{s}_j^h$. Under CRRA, with $\hat{R}_j^a \equiv R_j^a\,\Gamma^{1-\alpha}$ the Γ-stripped after-tax return,

$$\widehat{\mathcal{E}}_{j,t}^{a,h} = -\big(\bar{y}_{j,t}^h - \Gamma \hat{s}_j^h\big)^{-\sigma} + \Gamma^{\alpha(1-\sigma)-1}\,\beta\,\hat{R}_j^a\,(\bar{c}_j^{h+1,LF})^{-\sigma}\,\Xi_j^{a,h+1}.$$

Summing over assets, $\sum_a \hat{x}_j^{a,h+1} = \hat{s}_j^h$, condition (21) becomes

$$\mathcal{C}(\Gamma) = \Gamma^{\alpha(1-\sigma)-1}\big[\beta\mathcal{R} + \alpha(1-\alpha)\,\tilde{\mu}\big], \tag{23}$$

$$\mathcal{C}(\Gamma) \equiv \sum_j \omega_j \lambda_j \sum_{h=0}^{H-1} \left(\frac{\beta}{\gamma}\right)^h \hat{s}_j^h\,\big(\bar{y}_{j,t}^h - \Gamma \hat{s}_j^h\big)^{-\sigma},$$
$$\mathcal{R} \equiv \sum_j \omega_j \lambda_j \sum_{h=0}^{H-1} \left(\frac{\beta}{\gamma}\right)^h (\bar{c}_j^{h+1,LF})^{-\sigma} \sum_a \hat{x}_j^{a,h+1}\,\hat{R}_j^a\,\Xi_j^{a,h+1}.$$

At $\Gamma = 1$, $\widehat{\mathcal{E}}_{j,t}^{a,h} = 0$ and

$$\left.\frac{dSW}{d\Gamma_t}\right|_{\Gamma=1} = \gamma^t\,\alpha(1-\alpha)\,\tilde{\mu}.$$

Log ($\sigma = 1$): $\Gamma^{\alpha(1-\sigma)-1} = \Gamma^{-1}$; multiplying (23) by Γ, with $\rho_j^h \equiv \hat{s}_j^h / \bar{y}_{j,t}^h$,

$$\sum_j \omega_j \lambda_j \sum_{h=0}^{H-1} \left(\frac{\beta}{\gamma}\right)^h \frac{\Gamma \rho_j^h}{1 - \Gamma \rho_j^h} = \beta\mathcal{R} + \alpha(1-\alpha)\,\tilde{\mu}, \qquad \Gamma^* \in \Big(0,\ \min_{j,h} \tfrac{1}{\rho_j^h}\Big).$$

Normalization of $\mathcal{R}$

Let aggregate capital income be $\sum_{j'} \omega_{j'} \sum_a \hat{x}_{j'}^{a,h+1} \hat{R}_{j'}^a = \alpha Y^{LF}$ (markup-free). Define the total asset-income share, the income-share-weighted bracket, and the consumption share of an age-h, type-j household:

$$\bar{\xi}_j^{h+1} \equiv \frac{\sum_a \hat{x}_j^{a,h+1} \hat{R}_j^a}{\sum_{j'} \omega_{j'} \sum_a \hat{x}_{j'}^{a,h+1} \hat{R}_{j'}^a}, \qquad \overline{\Xi}_j^{h+1} \equiv \frac{\sum_a \hat{x}_j^{a,h+1} \hat{R}_j^a\,\Xi_j^{a,h+1}}{\sum_a \hat{x}_j^{a,h+1} \hat{R}_j^a}, \qquad \hat{c}_j^{h+1} \equiv \frac{\bar{c}_j^{h+1,LF}}{Y^{LF}}.$$

Then the asset block of $\mathcal{R}$ is

$$\sum_a \hat{x}_j^{a,h+1} \hat{R}_j^a\,\Xi_j^{a,h+1} = \Big(\sum_a \hat{x}_j^{a,h+1} \hat{R}_j^a\Big)\overline{\Xi}_j^{h+1} = \alpha Y^{LF}\,\bar{\xi}_j^{h+1}\,\overline{\Xi}_j^{h+1},$$

and $(\bar{c}_j^{h+1,LF})^{-1} = (\hat{c}_j^{h+1} Y^{LF})^{-1}$, so the Y^{LF} factors cancel:

$$\mathcal{R} = \alpha \sum_j \omega_j \lambda_j \sum_{h=0}^{H-1} \left(\tfrac{\beta}{\gamma}\right)^h \frac{\bar{\xi}_j^{h+1} \Xi_j^{h+1}}{\hat{c}_j^{h+1}}.$$

Normalization of $\tilde{\mu}$

The factor Y^{LF} in $\tilde{\mu} \equiv Y^{LF}(\mu_I + \mu_R)|_{\Gamma=1}$ cancels the $Y^{-\sigma}$ in each auxiliary weight; the explicit $Y/\mathbb{E}[c_j^h]$ in μ_I is already the share $(\hat{c}_j^h)^{-1}$. Define the share-form weights

$$\mathsf{n}_j \equiv \lambda_j \big(\hat{c}_j^H\big)^{-\sigma}, \qquad \hat{\mathsf{n}}_j \equiv \lambda_j \big(\hat{c}_j^H\big)^{-\sigma} \Big(1 + \tfrac{1}{2}\sigma(\sigma+1)V_j^H\Big), \qquad \hat{c}_j^h \equiv \frac{\bar{c}_j^{h,LF}}{Y^{LF}},$$

so that $Y^{LF}\nu_j = \mathsf{n}_j$ and $Y^{LF}\hat{\nu}_j = \hat{\mathsf{n}}_j$. Then $\tilde{\mu} = \tilde{\mu}_I + \tilde{\mu}_R$ is a pure-share object,

$$\begin{aligned}
\tilde{\mu}_I = \eta \sum_j \sum_{h=0}^{H} \frac{\mathsf{n}_j}{\omega_j} \left(\tfrac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h} \frac{1}{\hat{c}_j^h} \Big[& \alpha(1-\tau_k)^2 (\xi_j^h)^2 \mathrm{Var}(\psi) \\
& - (1-\alpha)\mathcal{M}^2 (1-\tau_\ell)^2 (\chi_j^h)^2 \mathrm{Var}(\epsilon) + \mathcal{M}(1-2\alpha)(1-\tau_\ell)(1-\tau_k)\chi_j^h \xi_j^h \mathrm{Cov}(\epsilon,\psi) \Big], \\
\tilde{\mu}_R = \sum_j \sum_{h=0}^{H} \hat{\mathsf{n}}_j \left(\tfrac{\gamma(1+r^f)}{(1+g)^\eta}\right)^{H-h} \Big[& \mathcal{M}(1-\tau_\ell)\chi_j^h - (1-\tau_k)\bar{\xi}_j^h \Big].
\end{aligned}$$