Strategic Complementarity and Persistence in Linear-Quadratic Mean Field Games

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Outline

- Linear Quadratic problem w idiosyncratic shocks & interaction term (game):
  1. *Agent's* state w/ *Aggregate* (average) state: $-\Theta_X$
  2. *Agent's* action w/ *Aggregate* (average) action: $-\Theta_A$

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- Eqbm of MFG described by a Riccati equation

- Eqm. is unique & exists if strategic complementarity remains below a threshold

- single agent Riccati eqn $\longleftrightarrow$ MFG Eqbm Riccati eqn w/different curvature
  1. *strategic complementarity in states decreases curvature own state:* $Q \to Q + \Theta_X$
  2. *strategic complementarity in actions decreases curvature on actions:* $\Gamma \to \Gamma + \Theta_A$
- Speed of convergence:
  1. *decreases with strategic complementarity in states:* $\Theta_X \to \lambda_j$
  2. *increases with strategic complementarity in actions:* $\Theta_A \to \lambda_j$

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- Speed of convergence:
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  2. *increases with strategic complementarity in actions:* $\Theta_A \to \lambda_j$
- Planner equivalent to Eqbm w/stronger interactions: $\Theta_X \to 2\Theta_X, \Theta_A \to 2\Theta_A$
- Add aggregate shocks, study identification: generalized missing intercept Θ_X, Θ_A

- Period return function:

$$F(x, X) + R(\alpha, \mathcal{A}) \equiv -\frac{1}{2}x'Qx - X'\Theta_X x - \frac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}'\Theta_{\mathcal{A}}\alpha \text{ where}$$

$$X \equiv \int x\, m(x)\, dx, \quad \mathcal{A} \equiv \int \alpha_*(x)\, m(x)\, dx$$

  - $x \in \mathbb{R}^n$: idiosyncratic state
  - $X \in \mathbb{R}^n$: aggregate state
  - $\alpha \in \mathbb{R}^k$: action, $\mathcal{A} \in \mathbb{R}^k$: aggregate action, using $\alpha_*(\cdot)$ optimal action of others
  - $m(\cdot)$: density over the states x
  - Σ, Q, Θ_X: are $n \times n$ symmetric matrices, $\Theta_{\mathcal{A}}$ is $k \times k$.

- The state x evolves as a the following diffusion:

$$dx = \mu(\alpha, x)\, dt + \Sigma^{1/2} dW \equiv (B\,\alpha - A\,x)\, dt + \Sigma^{1/2} dW$$

  - W: n dimensional standard Brownian Motion vector
  - B : $n \times k$ matrix, and Γ : $k \times k$ positive definite, $k \geq n$.
  - A: $n \times n$ matrix

Optimality given distribution: Hamilton-Jacobi-Bellman (HJB)

$$\rho u(x,t) = F(x, X(t)) + H(u_x(x,t), x, \mathcal{A}(t)) + \frac{1}{2}\sum_i \sum_j \sigma_{ij} u_{x_i x_j}(x,t) + u_t(x,t)$$

$$= -\tfrac{1}{2}x'Qx - X(t)'\Theta_X x + H(u_x(x,t), x, \mathcal{A}(t)) + \tfrac{1}{2} tr\left(\Sigma\, u_{xx}(x,t)\right) + u_t(x,t)$$

where the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is defined as:

$$H(p, x, \mathcal{A}) = \max_{\alpha \in \mathbb{R}^k} R(\alpha, \mathcal{A}) + p'\mu(\alpha, x) = \max_{\alpha \in \mathbb{R}^k} -\tfrac{1}{2}\alpha'\Gamma\alpha + p'\left(B\,\alpha - A\,x\right) - \mathcal{A}'\Theta\alpha$$

for any $p \in \mathbb{R}^n$.

Optimal action in Hamiltonian: $\alpha_*(p, \mathcal{A}) = \arg\max_\alpha R(\alpha, \mathcal{A}) + p'\mu(\alpha, x)$

Envelope: drift μ given by the derivative of the Hamiltonian:

$$\mu\left(\alpha_*(p, \mathcal{A}), x\right) = H_p(p, x, \mathcal{A})$$

Maintained Assumptions on F and R

1. positive discount factor $\rho \geq 0$,
2. the matrices Γ and Q are positive definite,
3. the matrix $B\Gamma^{-1}B'$ is invertible

- ☐ A 1-3 $\implies$ detectability & stabilizability $\implies$ finite $u(x,t) = \bar{u}(x)$ for $\Theta_X = \Theta_{\mathcal{A}} = 0$
- ☐ i.e. individual problem well defined if there is no dependence on others.
- ☐ when we introduce interactions, we assume Θ_X & $\Theta_{\mathcal{A}}$ are symmetric

Aggregation of optimal decision rules: Kolmogorov Forward (KFE)

Using that the optimal action is given by the derivative of the Hamiltonian:

$$m_t(x,t) = -div\Big(\underbrace{H_p(u_x(x,t), x, \mathcal{A}(t))}_{\text{optimal drift}} \, m(x,t) \Big) + \frac{1}{2} tr\left(\Sigma \, m_{xx}(x,t)\right)$$

with $m(x,0) = m_0(x)$

where $f = \{f^i(x)\}$ is as a vector field, $div(f)(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f^i(x)$.

Using the m and α_* we can define

$$X(t) = \int x \, m(x,t) \, dx \text{ for all } t \geq 0$$

$$\mathcal{A}(t) = \int \underbrace{\alpha_* \left(H_p \left(u_x \left(x,t \right) x, \mathcal{A} \left(t \right) \right), \mathcal{A} \left(t \right) \right)}_{\text{optimal action of agent with } x} m(x,t) \, dx \text{ for all } t \geq 0$$

Equilibrium Definition MFG given m_0

a value function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, a density $m : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, and paths $X : \mathbb{R}_+ \to \mathbb{R}^n, \mathcal{A} : \mathbb{R}_+ \to \mathbb{R}^k$, s.t:

$$\rho u(x,t) = -\tfrac{1}{2}x'Qx - X(t)'\Theta_X x + H(u_x(x,t), x, \mathcal{A}(t)) + \tfrac{1}{2}tr\left(\Sigma\, u_{xx}(x,t)\right) + u_t(x,t)$$

$$m_t(x,t) = -div\left(H_p(u_x(x,t), x, \mathcal{A}(t))\, m(x,t)\right) + \frac{1}{2}tr\left(\Sigma\, m_{xx}(x,t)\right)$$

for all $x \in \mathbb{R}^k$ and $t \geq 0$

$$H(p,x,\mathcal{A}) = \max_{\alpha \in \mathbb{R}^k} -\tfrac{1}{2}\alpha'\Gamma\alpha + p'\left(B\alpha - Ax\right) - \mathcal{A}'\Theta_{\mathcal{A}}\alpha \text{ for all } (p,x) \in \mathbb{R}^{2n}$$

$$X(t) = \int x\, m(x,t)\, dx \text{ for all } t \geq 0$$

$$\mathcal{A}(t) = \int \alpha_*\left(H_p\left(u_x\left(x,t\right), x, \mathcal{A}\left(t\right)\right), \mathcal{A}\left(t\right)\right) m(x,t)\, dx \text{ for all } t \geq 0$$

α_* is argmax of H, with $m(x,0) = m_0(x)$, & boundary conditions for m and u

Steady State

Definition of Strategic Complements/Substitutes

- Strategic complementarity or substitutability on **states**

  if $\frac{\partial^2}{\partial x \partial X} F(x, X) = -\Theta_X$ is negative definite has strategic substitutability

  if $\frac{\partial^2}{\partial x \partial X} F(x, X) = -\Theta_X$ is positive definite, has has strategic complementarity

- Strategic complementarity or substitutability on **actions**

  if $\frac{\partial^2}{\partial \alpha \partial \mathcal{A}} R(\alpha, \mathcal{A}) = -\Theta_{\mathcal{A}}$ is negative definite, has strategic substitutability

  if $\frac{\partial^2}{\partial \alpha \partial \mathcal{A}} R(\alpha, \mathcal{A}) = -\Theta_{\mathcal{A}}$ is positive definite, has strategic complementarity

Outline

Example I: Capital accumulation in industry equilibrium

- Each firm produces with DRTS function of capital, subject to exogenous shock
- Capital is accumulated subject to adjustment cost
- Price of investment good depend on total investment across firms
- Each firm is a monopolistic competitor selling its differentiated good
- Differentiated goods bought by sector with CES aggregator
- Exogenous constant interest rate

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1. Second order approximation around steady state
2. State is capital (deviation from steady state) of each firm
3. Strategic complementarity in states: decreasing returns + “demand externality"/CES
4. Strategic substitutability in action due upward slopping supply of investment

details

Example II: Multi-product Price Setting

- Constant marginal cost to produce each product
- Products aggregate w/CES demand
- Demand of store aggregate with Kimball, variable elasticity
- Price changes subject to Rotemberg cost, possible different across products
- Exogenous constant interest rate

Example II: Multi-product Price Setting

- Constant marginal cost to produce each product
- Products aggregate w/CES demand
- Demand of store aggregate with Kimball, variable elasticity
- Price changes subject to Rotemberg cost, possible different across products
- Exogenous constant interest rate

1. Second order approximation around flexible prices
2. State is price (deviation from flexible case) of n product of each firm
3. Strategic complementarity/substitutability depending on super-elasticity
4. Symmetry of cross derivatives Θ_X even though products are non-symmetric.

Outline

Construction and Analysis of Eqbm of MFG

1. Maximization problem given path of $\{X(t)\}$.
2. Dynamics of $X(t)$ using aggregation and optimal controls
3. Characterize the equilibrium as a dynamical system in 2n variables,
4. Rewrite solution of dynamics of game for $X(t)$ as matrix Riccati equation
5. Necessary and Sufficient Conditions for existence solution as function of $\Theta_X, \Theta_{\mathcal{A}}$.
6. Comparative statics w.r.t. Θ_X and $\Theta_{\mathcal{A}}$.

Quadratic Hamiltonian: straight-forward linear algebra

H is quadratic in $(p, x, \mathcal{A})$ and H_p linear in (x, p).

$$H(p, x, \mathcal{A}) = \max_{\alpha \in \mathbb{R}^k} -\frac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}'\Theta_{\mathcal{A}}\alpha + p'(B\alpha - Ax)$$

for $p \in \mathbb{R}^n$. Computation gives

$$\alpha_*(p, \mathcal{A}) = \Gamma^{-1}\left(B'p - \Theta_{\mathcal{A}}\mathcal{A}\right) \qquad \text{linear in } (p, \mathcal{A})$$

$$H(p, x, \mathcal{A}) = \frac{1}{2}\left(B'p - \Theta_{\mathcal{A}}\mathcal{A}\right)'\Gamma^{-1}\left(B'p - \Theta_{\mathcal{A}}\mathcal{A}\right) - \frac{1}{2}\left(p'Ax + x'A'p\right) \qquad \text{quadratic in } (p, x, \mathcal{A})$$

$$H_p(p, x, \mathcal{A}) = \left(p'B - \mathcal{A}'\Theta_{\mathcal{A}}\right)\Gamma^{-1}B' - x'A' \qquad \text{linear in } (x, p)$$

Time varying LQ $\Longrightarrow$ Quadratic Value Function in x

- $u(\cdot, t)$ is quadratic in x w/coefficients that are functions of time

$$u(x,t) = \beta_0(t) + \beta_1'(t)x + \frac{1}{2}x'\beta_2(t)x \quad \text{for all } t \geq 0$$

<u>scalar</u>:	$\beta_0(t)$ for all $t \geq 0$,
<u>$n-$dim vector</u>:	$\beta_1(t)$ for all $t \geq 0$,
<u>$n \times n$ matrix</u>:	$\beta_2(t)$ symmetric matrix for all $t \geq 0$

- Straightforward computations

$$\begin{aligned}
u_x(x,t) &= \beta_1'(t) + x'\beta_2(t) \\
H(u_x(x,t), x, \mathcal{A}) &= \frac{1}{2}\left(\beta_1'(t)B - \mathcal{A}'\Theta_{\mathcal{A}}\right)\Gamma^{-1}\left(B'\beta_1(t) - \Theta_{\mathcal{A}}\mathcal{A}\right) \\
&\quad + \frac{1}{2}x'\beta_2(t)B\Gamma^{-1}B'\beta_2(t)x + \left(\beta_1'(t)B - \mathcal{A}'\Theta_{\mathcal{A}}\right)\Gamma^{-1}B'\beta_2(t)x \\
&\quad - \left(\beta_1'(t) + x'\beta_2(t)\right)Ax \\
H_p(u_x(x,t), x, \mathcal{A}) &= \left(\left(\beta_1'(t) + x'\beta_2(t)\right)B - \mathcal{A}'\Theta_{\mathcal{A}}\right)\Gamma^{-1}B' - x'A'
\end{aligned}$$

- *Replace them back into the HJB*, impose hold for all x, t.

Optimal decisions given $X \implies$ system of ODEs for β's

Recall: $u(x,t) = \beta_0(t) + \beta_1'(t)x + \frac{1}{2}x'\beta_2(t)x$ for all $t \geq 0$

- Form of u + HBJ $\implies$ system of ODEs given paths for $X, \mathcal{A}$

$$\dot\beta_0(t) = \rho\beta_0(t) - \frac{1}{2}\left(B'\beta_1(t) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right)'\Gamma^{-1}\left(B'\beta_1(t) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right) - \frac{1}{2}tr\left(\Sigma\,\beta_2(t)\right)$$

$$\dot\beta_1(t) = \left(\rho I_n + A'\right)\beta_1(t) + \Theta_X X(t) - \beta_2(t)B\Gamma^{-1}B'\beta_1(t) + \beta_2(t)B\Gamma^{-1}\Theta_{\mathcal{A}}\mathcal{A}(t)$$

$$\dot\beta_2(t) = Q - \beta_2(t)B\Gamma^{-1}B'\beta_2(t) + \beta_2(t)A + A'\beta_2(t) + \rho\beta_2(t)$$

- System is recursive, solve for $\beta_2 \to \beta_1 \to \beta_0$ Given paths $X(t), \mathcal{A}(t)$
- $u(x,t)$ is concave in x for each $t \implies \beta_2(t)$ is negative definite
- Autonomous ODE $\beta_2(t)$, independent of $\Theta_X, \Theta_{\mathcal{A}} \implies \beta_2(t) = \bar\beta_2$ standard Riccati.

Aggregation of Optimal decision rules given $\beta_1 \implies$ equation for $\mathcal{A}$

- First order condition for α_*:

$$\alpha_*(p, \mathcal{A}) = \Gamma^{-1}\left(B'p - \Theta_{\mathcal{A}}\mathcal{A}\right) \text{ with } p = \beta_1(t) + \bar{\beta}_2 x$$

- Replacing quadratic value function on β's

$$\alpha^*(x, t) = \Gamma^{-1}\left(B'\left(\beta_1(t) + \bar{\beta}_2 x\right) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right)$$

- Integrating w.r.t. to $m(x, t)$:

$$\mathcal{A}(t) = \Gamma^{-1}\left(B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right)$$

- Solving for $\mathcal{A}$, trivial fixed point:

$$\mathcal{A}(t) = \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$$

Aggregation HJB, obtain first o.d.e. for (β_1, X)

- We have obtained an o.d.e. for β_1 depending on forcing $X(t)$ and $\mathcal{A}(t)$

$$\dot{\beta}_1(t) = \left(\rho I_n + A' - \bar{\beta}_2 B \Gamma^{-1} B'\right) \beta_1(t) + \Theta_X X(t) + \bar{\beta}_2 B \Gamma^{-1} \Theta_{\mathcal{A}} \mathcal{A}(t)$$

- Replace optimal $\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$ obtained just before

- We obtain a constant coefficient o.d.e. for β_1 <u>without aggregate action $\mathcal{A}(t)$:</u>

$$\begin{aligned}\dot{\beta}_1(t) = &\left(\rho I_n + A' - \bar{\beta}_2 B (\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right) \beta_1(t) \\ &+ \left[\bar{\beta}_2 \left(B \Gamma^{-1} B - B' (\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right) \bar{\beta}_2 + \Theta_X\right] X(t)\end{aligned}$$

details are here

Aggregation KFE, obtain second o.d.e. for (β_1, X)

- Individual law of motion $dx(t) = [B\alpha(x,t) - Ax(t)] + \Sigma^{1/2} dW$
- Definition of $X(t)$ aggregate of x using $m(x,t)$
- Definition of $\mathcal{A}(t)$ aggregate of $\alpha^*(x,t)$ using $m(x,t)$
- Individual law of motion $\dot{X}(t) = B\mathcal{A}(t) - AX(t)$
- Replace optimal $\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$ obtained just before.
- Obtained an expression without optimal action, constant coefficient o.d.e.:

$$\dot{X}(t) = B\,(\Gamma + \Theta_{\mathcal{A}})^{-1} B' \beta_1(t) + \left[B\,(\Gamma + \Theta_{\mathcal{A}})^{-1} B' \bar{\beta}_2 - A\right] X(t)$$

Rigorous proof in back .

Recursive Dynamical System: $\beta_2 \to (\beta_1, X) \to \mathcal{A} \to \beta_0$

$$0 = Q - \bar{\beta}_2 B \Gamma^{-1} B' \bar{\beta}_2 + \bar{\beta}_2 A + A' \bar{\beta}_2 + \rho \bar{\beta}_2$$

$$\dot{\beta}_1(t) = \left(\rho I_n + A' - \bar{\beta}_2 B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \beta_1(t) + \left(\Theta_X + \bar{\beta}_2 \left(B\Gamma^{-1}B' - B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \bar{\beta}_2\right) X(t)$$

$$\dot{X}(t) = B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \beta_1(t) + \left(B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \bar{\beta}_2 - A\right) X(t)$$

$$\mathcal{A}(t) = \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$$

$$\dot{\beta}_0(t) = \rho \beta_0(t) - \frac{1}{2} tr\left(\Sigma \bar{\beta}_2\right) - \frac{1}{2}\left(B'\beta_1(t) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right)' \Gamma^{-1} \left(B'\beta_1(t) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right)$$

- given initial condition $X(0)$
- (β_1, X) eigenvalues real part less than $\rho/2$ so that β_0 finite.

Eqbm of MFG as state-costate $2n$ dynamical system

$$\begin{bmatrix} \dot{\beta}_1 \\ \dot{X} \end{bmatrix} \equiv \mathcal{H} \begin{bmatrix} \beta_1 \\ X \end{bmatrix} \text{ with the } 2n \times 2n \text{ matrix } \mathcal{H}$$

$$\mathcal{H} \equiv \begin{bmatrix} \rho I_n + A' - \bar{\beta}_2 B \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' & \Theta'_X + \bar{\beta}_2 \left(B\Gamma^{-1} B' - B \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' \right) \bar{\beta}_2 \\ B \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' & -A + B \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' \bar{\beta}_2 \end{bmatrix}$$

- $\mathcal{H} - \frac{\rho}{2} I_{2n}$ is a MFG Hamiltonian (symplectic) matrix
- $\mathcal{H}$ depends on $\bar{\beta}_2$ which is independent of $\bar{X}$ and $\Theta_X, \Theta_{\mathcal{A}}$.
- There is an $\underline{\text{initial condition for } X(0)}$, there is $\underline{\text{NO initial condition for } \beta_1(0)}$
- Solve for saddle path for $\beta_1(t) = S\,X(t)$, solve for matrix S
- Unique Steady state is $\bar{X} = \bar{\beta}_1 = 0$ (generically)

MFG Eqbm as a Saddle Path for $\beta_1(t) = S\,X(t)$ matrix S

$$S\,B\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1}B'\,S = S\left(A - B\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1}B'\bar{\beta}_2\right) + \left(A' - \bar{\beta}_2 B\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1}B'\right)S$$
$$+\rho S + \Theta_X + \bar{\beta}_2\left(B\Gamma^{-1}B' - B\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1}B'\right)\bar{\beta}_2$$

implies the following solution:

$$\dot{X} = \left[\underbrace{B\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1}B'S + B\left[\left(\Theta_{\mathcal{A}}+\Gamma\right)^{-1} - \Gamma^{-1}\right]B'\bar{\beta}_2}_{\text{effect of interactions: } J} + \underbrace{B\Gamma^{-1}B'\bar{\beta}_2 - A}_{\text{dynamic w/o interactons: } G}\right]X$$

J depends on $\Theta_X, \Theta_{\mathcal{A}}$, but G does NOT.

Solution S must have Re eig$(J+G) < \frac{\rho}{2}$

$J = S = 0$ if $\Theta_X = \Theta_{\mathcal{A}} = 0$

Complicated analysis, $\bar{\beta}_2$ is itself solution of Riccati eqn. $Q, A, B\Gamma^{-1}B', \rho$.

Eqm MFG solves $P = S + \bar{\beta}_2$

- Eqbm solved by Auxiliar Riccati equation P:

$$P\,B\,(\Theta_{\mathcal{A}} + \Gamma)^{-1}\,B'\,P = (Q + \Theta_X) + \rho P + PA + A'P$$

$$\dot{X} \equiv [J + G]X = \Big[B\,(\Theta_{\mathcal{A}} + \Gamma)^{-1}\,B'\,P - A\Big]X$$

- Uncoupled Riccati equation $\bar{\beta}_2$: $\Theta_X = \Theta_{\mathcal{A}} = 0$ single agent + aggregation

$$\bar{\beta}_2\,B\Gamma^{-1}B'\,\bar{\beta}_2 = Q + \rho\bar{\beta}_2 + \bar{\beta}_2 A + A'\bar{\beta}_2$$

$$\dot{X} \equiv G\,X = \Big[B\Gamma^{-1}B'\,\bar{\beta}_2 - A\Big]X$$

- Single Agent to Ebqm MFG: $Q \to Q + \Theta_{\mathcal{A}}$ and $\Gamma \to \Gamma + \Theta_{\mathcal{A}}$.
- Strategic Complementarity on states $(\Theta_X \prec 0)$: less curvature on own state
- Strategic Complementarity on actions $(\Theta_{\mathcal{A}} \prec 0)$: less curvature on own action

Existence, Uniqueness of MFG Equilibrium

Multiplicity: There exists at most one MFG Eqbm.

Necessity: If there is an MFG Eqbm, then
$Q + \Theta_X + \left(A' + \frac{\rho}{2} I_n\right) \left(B \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B'\right)^{-1} \left(A + \frac{\rho}{2} I_n\right)$ must be positive semi-definite.

Sufficiency: If $Q + \Theta_X$ & $\Gamma + \Theta_{\mathcal{A}}$ are positive definite, there exists a unique MFG Eqbm.

1. One dim. $n = 1$, there exist unique MFG Eqbm $\iff$ $q + \theta_X + \left(\frac{\rho}{2} + a\right)^2 \left(\frac{\gamma + \theta_{\mathcal{A}}}{b^2}\right) > 0$
   eigenvalues $\lambda = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2} + a\right)^2 + b^2 \, \frac{q + \theta_X}{\gamma + \theta_{\mathcal{A}}}}$
2. $n = 2$ example: Necessary hold, Sufficient doesn't, there is NO MFG Eqbm.
3. If $A = a\, I_n$ necessary and sufficient (back to it later)

$$\left(a + \frac{\rho}{2}\right)^2 + eig_j \left(\left(Q + \Theta_X\right)^{\frac{1}{2}} B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \left(Q + \Theta_X\right)^{\frac{1}{2}}\right) > 0$$

Comparative Statics with respect to $\Theta_X, \Theta_{\mathcal{A}}$ and “Persistence" I

useful for $n \geq 2$:

Eqbm dynamics: $\frac{d}{dt}X(t) = (J+G)X(t) \equiv [B(\Theta_{\mathcal{A}}+\Gamma)^{-1}B'P - A]X(t)$

$\Omega_0 \subset \mathbb{R}^n$ and $\Omega_t = \{X(t) \in \mathbb{R}^n : \dot{X}(s) = (J+G)X(s), 0 \leq s \leq t,$ with $X(0) \in \Omega_0\}$
$V_0 = vol(\Omega_0)$ and $V_t = vol(\Omega_t), 0 \leq s \leq t$

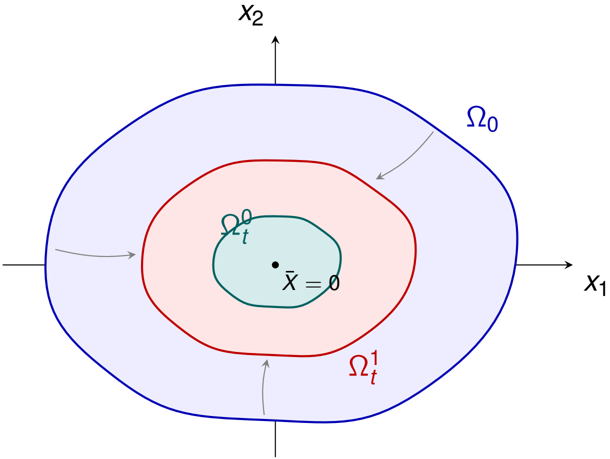
In words: Ω_0 set of initial conditions, Ω_t set of values after t.

$vol(\cdot)$ is area for $n = 2$, volume for $n = 3$, Lebesgue measure in general.

- more strat. compl. in **states** $(\Theta_X \downarrow) \implies vol(\Omega_t) \uparrow$ i.e. volume shrink less
- more strat. compl. in **actions** $(\Theta_{\mathcal{A}} \downarrow) \implies vol(\Omega_t) \downarrow$ i.e. volume shrink more

details

Volume Contraction and Persistence



Ω_0: initial conditions, $\bar{X} = 0 \in \Omega_0$

Ω_t^i: image of Ω_0 under eqbm flow $\dot{X} = (J + G)\, X$ at time t, for couplings $(\Theta^i{}_X, \Theta^i{}_{\mathcal{A}})$:

Ω_t^1: $\Theta_X \downarrow, \Theta_{\mathcal{A}} \uparrow$
more compl. states, more subst. actions
$\Rightarrow$ volume shrinks less: persistence

Ω_t^0: $\Theta_X \uparrow, \Theta_{\mathcal{A}} \downarrow$
more subst. states, more compl. actions
$\Rightarrow$ volume shrinks more

$$vol\,(\Omega_t) = e^{t\, tr(J+G)}\, vol\,(\Omega_0)$$

Comparative Statics with respect to $\Theta_X, \Theta_{\mathcal{A}}$ and “Persistence" II

□ $A = a\, I_n$ case: every eigenvalue of $\lambda_i = eig_i(J+G)$, satisfies

$$\lambda_i = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+a\right)^2 + eig_j\left((Q+\Theta_X)^{\frac{1}{2}}\, B(\Gamma+\Theta_{\mathcal{A}})^{-1} B'\, (Q+\Theta_X)^{\frac{1}{2}}\right)}$$

□ Scalar case: $\lambda = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+a\right)^2 + b\frac{(q+\theta_X)}{(\gamma+\theta_{\mathcal{A}})} b}$

Implications:

- more strat. compl. in **states** $(\Theta_X \downarrow) \implies$ every $\lambda_j \uparrow$: more persistent
- more strat. compl. in **actions** $(\Theta_{\mathcal{A}} \downarrow) \implies$ every $\lambda_j \downarrow$: less persistent

details

Outline

Utilitarian Planner

- Sum of agent utilities becomes

$$\max_{\alpha(x,t),\text{ all } x,t} \int_0^\infty e^{-\rho t}\left[\mathcal{U}(m(t)) - \frac{1}{2}\int \alpha'(x,t)\,\Gamma\,\alpha(x,t)\,m(x,t)\,dx\right]dt$$
$$- \int_0^\infty e^{-\rho t}\left(\int \alpha(x,t)\,m(x)dx\right)'\Theta_{\mathcal{A}}\left(\int \alpha(x,t)\,m(x)dx\right)dt$$

  where the period utility of the planner is the sum of the one by agents

$$\mathcal{U}(m) \equiv \int F(x,m)m(x)dx = -\frac{1}{2}\int x'Qx\,m(x)dx - \left(\int x\,m(x)dx\right)'\Theta_X\left(\int x\,m(x)dx\right)$$

- Planner controls action of each agent $\alpha(x,t)$, i.e. for all x, t.
- Subject to KFE for m_t:

$$m_t(x,t) = -div\left((B\alpha(x,t) - Ax)m(x,t)\right) + \frac{1}{2}tr\left(\Sigma\, m_{xx}(x,t)\right)\ \text{ all } x, t$$

Coupled PDEs described solution of Planner's problem

- Solution of planner's problem: coupled p.d.e.s for all $t \geq 0$ and $x \in \mathbb{R}^n$:

  $$\rho\lambda(x,t) = -\frac{1}{2}x'Qx - X(t)'(\Theta_X' + \Theta_X)x + H(\lambda_x(x,t), x, \mathcal{A}(t)) + \frac{1}{2}tr\,(\Sigma\lambda_{xx}(x,t)) + \lambda_t(x,t)$$

  $$m_t(x,t) = -div\,(H_p(\lambda_x(x,t), x, \mathcal{A}(t))\, m(x,t)) + \frac{1}{2}tr\,(\Sigma m_{xx}(x,t))$$

  where $\lambda(x,t)$ is Lagrange multiplier i planner's problem m_t at each (x,t), and

  where Hamiltonian H defined with $\Theta_{\mathcal{A}}' + \Theta_{\mathcal{A}}$

- Eqbm $u(x,t)$ w/ $(2\,\Theta_X\,,\,2\,\Theta_{\mathcal{A}}) \longleftrightarrow$ Planner $\lambda(x,t)$ w/ $(\Theta_X\,,\,\Theta_{\mathcal{A}})$

  $\Longrightarrow$ Strat. compl. states/actions: planner more/less persistence than Eqbm

Outline

Aggregate shock: Jump of size $\Upsilon \in \mathbb{R}^n_+$

- $N^+(t)$ and $N^-(t)$ independent Poisson counting processes with intensity $\kappa/2 > 0$

$$dx(t) = (B\alpha(t) - Ax(t))\, dt + \Sigma^{1/2} dW(t) + \Upsilon\, d\mathcal{J}(t) \text{ where } \mathcal{J}(t) = N^+(t) - N^-(t)$$

- Agents assume (later verified) stochastic process for $X(t)$ w/Jumps and $\mathcal{A}(t)$

$$dX(t) = M\, X(t)\, dt + \Upsilon d\mathcal{J}(t) \;\; , \;\; \mathcal{A}(t) = CX(t)$$

- Value function v has arguments (x, X), instead of (x, t). Takes M, C as given:

$$\rho v(x, X) = -\tfrac{1}{2}x'Qx - X'\Theta_X x + H(v_x(x, X), x, CX) + \tfrac{1}{2}tr\,(\Sigma\, v_{xx}(x, X))$$
$$+ v_X(x, X)'M\, X + \kappa\left[\tfrac{1}{2}v(x + \Upsilon, X + \Upsilon) + \tfrac{1}{2}v(x - \Upsilon, X - \Upsilon) - v(x, X)\right]$$

- We show that in Equilibrium the **fixed point** for M,C we get

$$dX(t) = M\, X(t) dt + \Upsilon\, d\mathcal{J}(t) \;\text{ with }\; M = B\,(\Theta_{\mathcal{A}} + \Gamma)^{-1} B'P - A$$

- Unsurprising, due to Certainty Equivalence

go

Outline

Recursive Representation

- Rewrite the state as two independent process (z, X) with $\boxed{z \equiv x - X}$

$$dX = \left[B\left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P - A\right] X\,dt + \Upsilon\,d\mathcal{J} \qquad \leftarrow \text{ rep. agent l.o.m.}$$

$$dz = \left[B\Gamma^{-1} B' \bar{\beta}_2 - A\right] z\,dt + \Sigma^{1/2} dW \qquad \leftarrow \text{ detrended l.o.m.}$$

$$\alpha \equiv \alpha^*(x, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P\,X + \Gamma^{-1} B' \bar{\beta}_2\,z \qquad \leftarrow \text{ any agent action}$$

$$\mathcal{A} \equiv \alpha^*(X, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P\,X \qquad \leftarrow \text{ rep. agent action}$$

- Study Identification, recall:
  - $\bar{\beta}_2$ solution of LQ with no coupling with others, i.e. $\Theta_X = \Theta_{\mathcal{A}} = 0$.
  - P solution of MFG equilibrium, crucially depend on $\Theta_X, \Theta_{\mathcal{A}}$.

$$dX = \left[B\left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P - A \right] X \, dt + \Upsilon \, d\mathcal{J} \qquad \leftarrow \text{ rep. agent l.o.m.}$$

$$dz = \left[B\Gamma^{-1} B' \bar{\beta}_2 - A \right] z \, dt + \Sigma^{1/2} dW \qquad \leftarrow \text{ detrended l.o.m.}$$

$$\alpha \equiv \alpha^*(x, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P \, X + \Gamma^{-1} B' \bar{\beta}_2 \, z \qquad \leftarrow \text{ any agent action}$$

$$\mathcal{A} \equiv \alpha^*(X, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P \, X \qquad \leftarrow \text{ rep. agent action}$$

Identification: Assume a data set is generated by a model. A data set identifies a parameter vector if it is consistent with at most one such vector.

1. Path $\{z(t) = x(t) - X(t), \alpha(t) - \mathcal{A}(t)\}$, or **net of time effects**, has NO info. on $\Theta_X, \Theta_{\mathcal{A}}$

$$dX = \left[B\left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P - A \right] X\, dt + \Upsilon\, d\mathcal{J} \qquad \leftarrow \text{ rep. agent l.o.m.}$$

$$dz = \left[B\Gamma^{-1} B' \bar{\beta}_2 - A \right] z\, dt + \Sigma^{1/2} dW \qquad \leftarrow \text{ detrended l.o.m.}$$

$$\alpha \equiv \alpha^*(x, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P\, X + \Gamma^{-1} B' \bar{\beta}_2\, z \qquad \leftarrow \text{ any agent action}$$

$$\mathcal{A} \equiv \alpha^*(X, X) = \left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P\, X \qquad \leftarrow \text{ rep. agent action}$$

Identification: Assume a data set is generated by a model. A data set identifies a parameter vector if it is consistent with at most one such vector.

1. Path $\{z(t) = x(t) - X(t), \alpha(t) - \mathcal{A}(t)\}$, or **net of time effects**, has NO info. on $\Theta_X, \Theta_{\mathcal{A}}$
2. Assume $\Theta_{\mathcal{A}}, \Gamma, \rho, Q, A, B$ are known. Time series of $\{\ X(t)\}$ identifies Θ_X.

$$dX = \left[B \left(\Theta_{\mathcal{A}} + \Gamma \right)^{-1} B' P - A \right] X \, dt + \Upsilon \, d\mathcal{J} \qquad \leftarrow \text{ rep. agent l.o.m.}$$

$$dz = \left[B \Gamma^{-1} B' \bar{\beta}_2 - A \right] z \, dt + \Sigma^{1/2} dW \qquad \leftarrow \text{ detrended l.o.m.}$$

$$\alpha \equiv \alpha^*(x, X) = \left(\Theta_{\mathcal{A}} + \Gamma \right)^{-1} B' P \, X + \Gamma^{-1} B' \bar{\beta}_2 \, z \qquad \leftarrow \text{ any agent action}$$

$$\mathcal{A} \equiv \alpha^*(X, X) = \left(\Theta_{\mathcal{A}} + \Gamma \right)^{-1} B' P \, X \qquad \leftarrow \text{ rep. agent action}$$

Identification: Assume a data set is generated by a model. A data set identifies a parameter vector if it is consistent with at most one such vector.

1. Path $\{z(t) = x(t) - X(t), \alpha(t) - \mathcal{A}(t)\}$, or **net of time effects**, has NO info. on $\Theta_X, \Theta_{\mathcal{A}}$
2. Assume $\Theta_{\mathcal{A}}, \Gamma, \rho, Q, A, B$ are known. Time series of $\{ X(t)\}$ identifies Θ_X.
3. Assume $\Theta_{\mathcal{A}}, \Gamma, \rho$ known. Time series $\{X(t), z(t), \alpha(t), \mathcal{A}(t)\}$ identifies $\{Q, \Theta_X, A, B\}$

Proofs are constructive $\rightarrow$ algorithm for estimation. $\Theta_{\mathcal{A}}$ & Θ_X not separately identified.

Outline

Coupling with higher moments

- Assume $x \in \mathbb{R}$ or $n = 1$. Denote each moment of x using $m(\cdot, t)$ at time t by

$$M_r(t) \equiv \int x^r \, m(x,t) dx \text{ for all } t \geq 0 \text{ and } r = 1, 2, \ldots, \bar{r}$$

- In this case, the period return function is given by

$$F(x, M_1(t), \ldots, M_{\bar{r}}(t)) + R(\alpha) \equiv -\tfrac{1}{2} x \, q \, x - \left(\theta_0 + \sum_{r=1}^{\bar{r}} M_r(t) \theta_r \right) x - \tfrac{1}{2} \alpha' \, \Gamma \, \alpha$$

- Strategic complementarity: If m_a is stochastically higher than m_b, then $F(x, M^a) - F(x, M^b)$ is weakly increasing in $x \implies \theta_r = 0$ for r even.

Equilibrium

An equilibrium for the MFG given m_0 is a value function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, a density $m : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\bar{r}$ moments $M_r : \mathbb{R}_+ \to \mathbb{R}$ for $r = 1, \ldots, \bar{r}$ satisfying:

$$\rho u(x,t) = -\tfrac{1}{2} x\, q\, x - \left(\theta_0 + \sum_{r=1}^{\bar{r}} M_r(t)\theta_r\right) x + H(u_x(x,t), x) + \tfrac{1}{2}\sigma^2 u_{xx}(x,t) + u_t(x,t)$$

$$m_t(x,t) = -\left(H_p(u_x(x,t), x)\, m(x,t)\right)_x + \tfrac{1}{2}\sigma^2\, m_{xx}(x,t)$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$ where

$$H(p,x) = \max_{\alpha \in \mathbb{R}^{\ltimes}} -\tfrac{1}{2}\,\alpha'\,\Gamma\alpha + p\,(B\alpha - ax) \text{ for all } (p,x) \in \mathbb{R}^2$$

$$M_r(t) = \int x^r\, m(x,t)\, dx \text{ for all } t \geq 0 \text{ and } r = 1, \ldots, \bar{r}$$

with $m(x,0) = m_0(x)$ and appropriate boundary conditions for m and u.

Quadratic value function and o.d.e. for β_1

$$\dot{\beta}_2 = q - \beta_2\gamma^{-1}\beta_2 + \beta_2 a + a\beta_2 + \rho\beta_2 \implies \beta_2 = \bar{\beta}_2$$

$$\dot{\beta}_1 = (\rho + a)\,\beta_1 + \theta_0 + \sum_{r=1}^{\bar{r}} \theta_r M_r - \bar{\beta}_2\gamma_B^{-1}\beta_1$$

$$\dot{\beta}_0 = \rho\beta_0 - \tfrac{1}{2}\beta_1'\gamma_B^{-1}\beta_1 - \tfrac{1}{2}\sigma^2\bar{\beta}_2$$

- define scalar $\gamma_B \equiv 1/(B\Gamma^{-1}B')$
- same argument and value for $\beta_2(t) = \bar{\beta}_2$
- once $\beta_1(t)$ is solved check that $\beta_0(t)$ finite.
- β_1 o.d.e. linear in β_1 and moments $M_r(t)$.

Aggregation of optimal rules $\implies$ o.d.e for all moments

Kolmogorov forward equation implies the following $\bar{r}$ dimensional system of linear o.d.e.s:

$$\dot{M}_r(t) = r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) M_r(t) + r\gamma^{-1} \underbrace{\beta_1(t)\, M_{r-1}(t)}_{\text{non-linear}} + \underbrace{r(r-1)\frac{\sigma^2}{2}}_{\text{depends on } \sigma} M_{r-2}(t)$$

for $r = 1, \ldots, \bar{r}$ and $t \geq 0$, for given path β_1 and initial conditions $M_r(0)$ for $r = 1, \ldots, \bar{r}$.

- If $\bar{r} = 1$ same as before corresponds to $X = M_1$: linear in β_1 and independent of σ^2
- If $\bar{r} > 1$, non-linear, and involves all moments.

$\bar{r}$ States and one Co-state, form an $(\bar{r}+1)$ dynamical system for Eqbm:

An equilibrium is solution for:

$$\dot{\beta}_1(t) = \left(\rho + a - \bar{\beta}_2 \gamma_B^{-1}\right) \beta_1(t) + \theta_0 + \sum_{r=1}^{\bar{r}} \theta_r M_r(t)$$

$$\dot{M}_r(t) = r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) M_r(t) + r\gamma_B^{-1}\beta_1(t) M_{r-1}(t) + \frac{r(r-1)\sigma^2}{2} M_{r-2}(t)$$

for $r = 1, 2, \ldots, \bar{r}$ that keeps $\beta_0(t)$ finite.

- Recall, we interpret x as steady state deviation, so it must have $\bar{M}_1 = 0$.
- $\bar{M}_1 = 0$ in equation for $r = 1$ requires $\bar{\beta}_1 = 0$.
- Study steady state system of equations, potentially several.
- Focus on steady state consistent with $\int x \bar{m}(x) dx = 0$ ss-analysis .
- Symmetric s,s. distribution of deviations: all zero odd moments $\bar{M}_r = 0$ for odd r.

Local Analysis Around Steady State

Linearize system for $(\beta_1, M_1, \ldots, M_{\bar{r}})$ around steady state. Define deviations:

$$\hat{\beta}_1(t) = \beta_1(t) - \bar{\beta}_1 \text{ and } \hat{M}_r(t) = M_r(t) - \bar{M}_r, \text{ for all } t \geq 0 \text{ and } r = 1, \ldots, \bar{r}$$

The linearized system of o.d.e's becomes:

$$\frac{d}{dt}\hat{\beta}_1(t) = \left(\rho + a - \bar{\beta}_2\gamma_B^{-1}\right)\hat{\beta}_1(t) + \sum_{r=1}^{\bar{r}} \theta_r \hat{M}_r(t)$$

$$\frac{d}{dt}\hat{M}_r(t) = r\gamma_B^{-1}\bar{M}_{r-1}\hat{\beta}_1(t) + r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right)\hat{M}_r(t) + \frac{r(r-1)\sigma^2}{2}\hat{M}_{r-2}(t)$$

or stacking the system into a vector of steady state deviations around zero mean:

$$\dot{Y}(t) = \widetilde{\mathcal{H}}\, Y(t) \text{ where } Y \equiv \left(\hat{\beta}_1, \hat{M}_1, \hat{M}_2, \ldots, \hat{M}_{\bar{r}}\right)'$$

where Y is a $\bar{r}+1$ column vector and $\widetilde{\mathcal{H}}$ is an $(\bar{r}+1) \times (\bar{r}+1)$ real matrix

Speed of Convergence: one co-state and $\bar{r}$ states (all moments)

WLOG $\bar{r}$ is even.

Let $\{\tilde{\lambda}_r\}_{r=1}^{\bar{r}+1}$ eigenvalues of the matrix $\widetilde{\mathcal{H}}$, linearization of Y of deviations around zero mean distribution:

$$\tilde{\lambda}_1 = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+a\right)^2 + \frac{q+\bar{\theta}_X}{\gamma_B}} < 0 \, , \; \tilde{\lambda}_{\bar{r}+1} = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}+a\right)^2 + \frac{q+\bar{\theta}_X}{\gamma_B}} > 0 \, , \text{ and}$$

$$\tilde{\lambda}_r = r\left(\frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+a\right)^2 + \frac{q}{\gamma_B}}\right) < 0 \text{ for } r = 2, 3, \ldots, \bar{r} \text{ where}$$

$$\bar{\theta}_X \equiv \sum_{r=1,3,5,\ldots}^{\bar{r}-1} r \underbrace{\bar{M}_{r-1}}_{>0} \theta_r = \text{ linear combination of odd couplings on even moments}$$

path of <u>even moment deviations</u> $\hat{M}_r(t)$ for $r = 2, 4, \ldots, \bar{r}$ is autonomous.

1. Thee coefficients $\{\theta\}_{r=1}^{\bar{r}}$ only enter in two eigenvalues: $\tilde{\lambda}_1$ and $\tilde{\lambda}_{\bar{r}+1}$.
2. Only the coefficients on the odd moments enter into the eigenvalues $\tilde{\lambda}_{\bar{r}+1}$.
3. The interpretation of $\bar{\theta}$ is the same as in the benchmark scalar case: as any θ_r increases (for odd r), then the convergence to the steady state is faster.
4. There is a value $\bar{\theta}^*$ s.t. if $\bar{\theta} < \bar{\theta}^*$, then there is no local equilibrium
5. If there is strategic complementarity, i.e. if $\bar{\theta} < 0$, then $\tilde{\lambda}_1 > \tilde{\lambda}_r$ for all $r = 2, 3, \ldots, \bar{r}$. That is, with strategic complementarity $\tilde{\lambda}_1$ it is the dominant eigenvalue.
6. If $\sigma^2 = 0$, then $\bar{\theta} = \theta_1$. If all $\theta_r < 0\,(> 0)$ for r odd, then $\bar{\theta}$ is decreasing in σ^2. If there is strategic complementarity, then convergence to equilibrium is slower for higher σ^2
7. For small σ^2 expansion gives: $\bar{\theta} = \theta_1 + \theta_3 \frac{\sigma^2 3}{2(a - \gamma_B^{-1} \bar{\beta}_2)} + o(\sigma^2)$

Outline

- LQ problem w/ idiosncratic shocks & interactions (game) in $-\Theta_X$ state & $\Theta_{\mathcal{A}}$ actions
- Eqbm of MFG described by a Riccati equation
- Eqm. is unique & exists if strategic complementarity remains below a threshold
- single agent Riccati eqn $\longleftrightarrow$ MFG Eqbm Riccati eqn w/different curvature
  1. *strategic complementarity in states decreases curvature own state:* $Q \to Q + \Theta_X$
  2. *strategic complementarity in actions decreases curvature on actions:* $\Gamma \to \Gamma + \Theta_A$
- Speed of convergence:
  1. *decreases with strategic complementarity in states:* $\Theta_X \to \lambda_j$
  2. *increases with strategic complementarity in actions:* $\Theta_A \to \lambda_j$
- Planner equivalent to Eqbm w/stronger interactions: $\Theta_X \to 2\Theta_X, \Theta_A \to 2\Theta_A$

  Planner magnifies persistence/convergence relative to Eqbm
- Add aggregate shocks, study identification: generalized missing intercept Θ_X, Θ_A
- Extend to on higher odd moments.

Appendix

Simple example:

- Monopolistic competitive firms, with production function $y(k) = k^{\nu}$
- Relative Price: $\left[\frac{y(k)}{Y(t)}\right]^{-\eta}$ where $Y(t) = \left[\int y(k)^{1-1/\eta}\, m(k,t)\, dk\right]^{1/(1-1/\eta)}$
- Law of motion of capital $dk = i - \delta k + k\sigma dW$, (investment in numeraire)
- Price of investment good in terms of numerair $P(I)$ where $I(t) = \int i^*(k,t) m(k,t) dk$
- Convex adjustment cost function $\psi(i - \delta k) \geq 0$, (paid in numeraire)

$$\text{H.J.B. : } \rho u(k,t) = \max_i \left[\frac{y(k)}{Y(t)}\right]^{-\eta} y(k) - i\, P(I(t)) - \psi(i - \delta k)$$
$$+ u_k(k,t)(i - \delta k) + u_{kk}(k,t) k^2 \sigma^2 + u_t(k,t)$$

- 2^{nd} order approx: $-\Theta_X = Q\, \frac{(1-\nu)\eta}{(1-\nu)\eta + \nu} > 0$: **<u>Strat. Compl. in states</u>**
- 2^{nd} order approx: $-\Theta_{\mathcal{A}} = \frac{P'(\delta \bar{k})}{P(\bar{k})} < 0$: **<u>Strat. Subst. in actions</u>**

back

Assumptions on $m_0(\cdot)$ and $m(\cdot, t)$

- Assume that $m_0(x)$ has finite second moments
- Look for solutions where $m(\cdot, t)$ satisfies for

$$
\begin{aligned}
1 &= \int m(x,t)\,dx \text{ for all } t \geq 0 \\
0 &= \lim_{x_i \to +\infty} m(x,t) = \lim_{x_i \to -\infty} m(x,t) = \lim_{x_i \to +\infty} m_{x_j}(x,t) = \lim_{x_i \to -\infty} m_{x_j}(x,t) \\
&= \lim_{x_i \to +\infty} x_i\, m(x,t) = \lim_{x_i \to -\infty} x_i\, m(x,t)
\end{aligned}
$$

Assumptions

In the case of $\Theta_X = \Theta_{\mathcal{A}} = 0$, single-agent LQ problem has solution in L_ρ

1. positive discount factor $\rho \geq 0$,
2. the matrix Γ is positive definite, and Σ, Q are positive semi-definite.
3. detectability of Q and A: there exists a matrix F for which $\mathrm{Re}\ \lambda\,(FC - A) < \frac{\rho}{2}$ where $Q = C'C$.
4. stabilizability of B and A: there exists a matrix L for which $\mathrm{Re}\ \lambda\left(B\Gamma^{-1/2}L - A\right) < \frac{\rho}{2}$

where $\mathrm{Re}\ \lambda(M)$: real part of the eigenvalues of matrix M.

Back

Literature Review on LQ MFG

- Lasry-Lions (2007), started literature MFG
- Caines (2009), early summary LQ MFG
- Bardi (2012), Bardi and Pruli (2014), Ergocdic and Discounted case
- Bensoussan et all (2016), comprehensive treatment LQ MG, w/monotonicity.
- Tchuendom (2018), common noise, non-linear coupling of first moment.
- Huan and Yang, (2021) planner problem. back

Steady state HBJ for $\bar{u}$ and $\bar{m}$

- Denote steady state values by $\bar{\beta}_i$ and $\bar{X}$, not functions of time.
- Solve, recursively, first for $\bar{\beta}_2$, then $\bar{\beta}_1, \bar{X}$ and then $\bar{\beta}_0$.

$$\text{from } \dot{\beta}_2 = 0 : \bar{\beta}_2 B\Gamma^{-1} B' \bar{\beta}_2 = Q + \bar{\beta}_2 A + A' \bar{\beta}_2 + \rho \bar{\beta}_2$$

$$\text{from } \dot{\beta}_1 = 0 : \Theta'_X \bar{X} = \left(I\rho + A' - \bar{\beta}_2 B\Gamma^{-1} B' \right) \bar{\beta}_1$$

$$\text{from } \dot{X} = 0 : B\Gamma^{-1} B' \bar{\beta}_1 = \left(B\Gamma^{-1} B' \bar{\beta}_2 - A \right) \bar{X}$$

$$\text{from } \dot{\beta}_0 = 0 : \rho \bar{\beta}_0 = \tfrac{1}{2} \bar{\beta}'_1 B\Gamma^{-1} B' \bar{\beta}_1 + tr \left(\Sigma \, \bar{\beta}_2 \right)$$

- $\bar{\beta}_2$ is independent of steady state $\bar{X}$
- Steady State of the Game $\bar{\beta}_1 = \bar{X} = 0$, unique generically

  $\bar{X} \neq 0$ requires $0 = det \left(\Theta_X + Q + A' \left(B\Gamma^{-1} B' \right)^{-1} A + \rho \left(B\Gamma^{-1} B' \right)^{-1} A \right)$. back

Utilitarian Planner vs non-atomic Potential Game, skip

- Differs from the Potential of the static game $\mathcal{F}$. To see this differentiate wrt m:

$$\frac{\partial \mathcal{U}(m,x)}{\partial m} = -\frac{1}{2}x'Qx - x'\Theta_X X - X'\Theta_X x = -\frac{1}{2}x'Qx - X'\left(\Theta_X' + \Theta_X\right)x \neq F(x,m)$$

- Two differences so that $\mathcal{F}(x,X) \neq \frac{\partial \mathcal{U}(m,x)}{\partial m}$
  - For (static) potential requires Θ_X to be symmetric
  - In $\mathcal{F}$, a $1/2$ factor multiplies $X'\Theta_X X$: difference between eq. path and planner's solution

back

Analysis of the Scalar Case $n = 1$

- Let $Q = q > 0, A = a, \Gamma = \gamma, \Theta_{\mathcal{A}} = \theta_{\mathcal{A}}, \Theta_X = \theta_X$. The eigenvalues are:

$$\lambda_{2,1} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2} + a\right)^2 + b^2 \frac{q + \theta_X}{\gamma + \theta_{\mathcal{A}}}}$$

- Define thresholds $-\theta^*$ and $-\theta^{**}$ for strategic complementarity $-\theta$ as:

$$0 < -\theta^* \equiv a(a + \rho)\frac{(\gamma + \theta_{\mathcal{A}})}{b^2} + q < -\theta^{**} \equiv \frac{(\gamma + \theta_{\mathcal{A}})}{b^2}\left(\frac{\rho}{2} + a\right)^2 + q$$

- Whenever the equilibrium exists, it is unique. Furthermore
    1. $-\infty < -\theta < -\theta^*$: unique saddle path stable solution, $X(t) \to 0$ at the rate $\lambda_1 < 0$
    2. $-\theta = -\theta^*$: unique solution with $\lambda_1 = 0$, where $X(t) = X(0)$ all t
    3. $-\theta^* < -\theta < -\theta^{**}$: unique solution where “slowly" $X(t) \to \pm\infty$ at the rate $0 < \lambda_1 < \rho/2$
    4. $-\theta \geq -\theta^{**}$: there is no equilibrium of the MFG

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Cross-Sectional Distribution: stochastic process

- $m(x,t)$ follows a stochastic process for each x:

$$dm(x,t) = \left[-div\left(H_p(v_x(x,X(t)),x,CX)\, m(x,t) \right) + \frac{1}{2} tr\left(\Sigma\, m_{xx}(x,t) \right) \right] dt$$
$$+ \left(m(x-\Upsilon,t) - m(x,t) \right) dN^+(t) + \left(m(x+\Upsilon,t) - m(x,t) \right) dN^-(t)$$

- We define $X(t)$ as before, but note it is an stochastic process

$$X(t) = \int x\, m(x,t) dx \quad \text{and} \quad CX = \int \alpha_*(H_p(v(x,X),x,CX))\, m(x,t) dx$$

- We show that in Equilibrium the **fixed point** gives

$$dX(t) = M\, X(t) dt + \Upsilon\, d\mathcal{J}(t) \quad \text{with} \quad M = B\left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P - A$$

- where: P solves the same algebraic Riccati equation as in the case $\Upsilon = 0$
- In words, speed of convergence matrix $J + G$ becomes drift M of stoch. process

Details of Aggregate Shock case

- Guess value function: $v(x, X) = S_0 + X' S_1 x + \frac{1}{2} x' S_2 x + \frac{1}{2} X' S_3 X$ where S_0 is a scalar and S_1, S_2, S_3 are $n \times n$ matrices and S_2 and S_3 symmetric.
- S_0 iscalar and S_1, S_2, S_3 are $n \times n$ matrices are functions of M
- Use KFE, Hamiltonian to show that:

$$dX(t) = M\, X(t) dt + \Delta\, d\mathcal{J}(t) \text{ where } M = (\Theta_{\mathcal{A}} + \Gamma)^{-1} [S_1 + S_2] - A$$

- We show that fixed point: $M = B\,(\Theta_{\mathcal{A}} + \Gamma)^{-1} P - A$ is same P as in model with $\Upsilon = 0$

System of equations on matrices S, given M, C. –NEED TO PROPERLY ADD C

$$\begin{aligned}
\rho S_0 &= \tfrac{1}{2} tr\left(\Sigma\, S_2\right) + \kappa \Delta' \left(S_1 + S_2 + S_3\right)\Delta \\
\rho S_1 &= -\Theta_X + S_1' B\Gamma^{-1} B' S_2 - S_1' A + M' S_1' \\
\rho S_2 &= -Q + S_2 B\Gamma^{-1} B' S_2 - \left(S_2 A + A' S_2\right) \\
\tfrac{\rho}{2} S_3 &= \tfrac{1}{2} S_1' B\Gamma^{-1} B' S_1 + S_3 M
\end{aligned}$$

- Given M, solve recursively: S_2 (Riccati equation) $\rightarrow S_1 \rightarrow S_3 \rightarrow S_0$
- The solution of this system is unique (for any given M).
- Equilibrium: Let $P \equiv S_1 + S_2$. Using M and Θ_X symmetric solution follows:

$$P\,B\left(\Theta_{\mathcal{A}} + \Gamma\right)^{-1} B' P = \left(Q + \Theta_X'\right) + \rho P + PA + A'P$$

- Same algebraic Riccati equation as in the case $\Delta = 0$

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- Introduce **forcing variables:** part of the state not controlled SKIP
- Split state $x = [x^{(1)}, x^{(2)}]'$ with $n = n_1 + n_2$, with n_2 exogenous forcing variables
- Assume $x^{(2)}$ independent of $x^{(1)}$ and not controllable:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{n_1 \times n_2} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ 0_{n_2 \times k} \end{bmatrix}$$

for arbitrary $\tilde{q}$ and $\tilde{\theta}$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \tilde{q} I_{n_2} \end{bmatrix} \text{ and } \Theta_X = \begin{bmatrix} \Theta_{X,11} & \Theta_{X,12} \\ \Theta_{X,21} & \tilde{\theta} I_{n_2} \end{bmatrix}$$

- Assume $\text{Real}(\text{eig}(A_{22})) + \frac{\rho}{2} < 0$, so they are not de-stabilizing.
- Allow assumptions only on endogenous part, e.g.: $Q_{11} + \Theta_{X,11}$ positive definite
- Riccati equation for P_{11} based upon $\Theta_{X,11} + Q_{11}, \Gamma, A_{11}, B_1$.

Analysis of Steady States

1. Solve for $\bar{M} = (\bar{M}_1, \ldots, M_{\bar{r}})$ as function of $\bar{\beta}_1$. Given $\bar{\beta}_1$ this is s linear system with recursive structure given by:

$$\bar{M}_r = \frac{1}{a - \gamma_B^{-1}\bar{\beta}_2}\left[\gamma_B^{-1}\bar{\beta}_1 \bar{M}_{r-1} + \frac{(r-1)\sigma^2}{2}\bar{M}_{r-2}\right] \text{ for } r = 1, 2 \ldots, \bar{r}$$

with $\bar{M}_0 = 1$. We let $\{\mathcal{M}_r(\bar{\beta}_1)\}_{r=1}^{\bar{r}}$ the unique solution of this linear equations.

2. For the second step we replace this solution into the equation for $\bar{\beta}_1$, defining

$$\mathcal{B}(\bar{\beta}_1; \theta_0) \equiv \frac{1}{\bar{\beta}_2\gamma_B^{-1} - a - \rho}\left[\theta_0 + \sum_{r=1}^{\bar{r}} \theta_r \mathcal{M}_r(\bar{\beta}_1)\right]$$

So, a steady state is a fixed point of the non-linear (polynomical) $\bar{\beta}_1 = \mathcal{B}(\bar{\beta}_1; \theta_0)$.

3. In general there are multiple steady states.

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Consistency requires $\bar{M}_1 = 0 \implies \bar{\beta}_1 = 0 \implies \theta_0 = \bar{\theta}_0$.

There is a unique steady state with $\bar{\beta}_1 = 0$ and zero odd moments, i.e. with $\bar{M}_r = 0$ for $r = 1, 3, \ldots, \bar{r}$, if θ_0 is set as follows:

$$\bar{\theta}_0 = - \sum_{r=2,4,6\ldots}^{\bar{r}} \theta_r \bar{M}_r \tag{1}$$

$$\bar{M}_r = 0 \text{ for } r = 1, 3, 5, \ldots, \bar{r} - 1 \tag{2}$$

$$\bar{M}_r = (r-1)!! \left(\frac{\sigma^2}{2\left(a - \gamma_B^{-1} \bar{\beta}_2\right)} \right)^{\frac{r}{2}} \text{ for } r = 2, 4, 6, \ldots, \bar{r} \tag{3}$$

where !! is the double factorial.

- $\bar{M}_r$ are the moments of a normal distribution with zero mean.
- same type of stationary state as in benchmark case with linear coupling back .

Aggregation of Optimal decision rules given $\beta_1 \implies$ system of ODEs for X

In equilibrium, vector X solves the following o.d.e. given a path of $\beta_1(t)$ and $\beta_2(t)$, and using previous result $\beta_2(t) = \bar{\beta}$ becomes constant coefficient ODE on $(\beta_1(t), X(t))$:

$$\dot{X}(t) = B\Gamma^{-1}B'\beta_1(t) + \left(B\Gamma^{-1}B'\bar{\beta}_2 - A\right) X(t) - B\Gamma^{-1}\Theta_{\mathcal{A}}\mathcal{A}(t) \text{ for all } t \geq 0$$

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Proof uses time derivative of def. X, KFE , form of $u(x,t)$ and envelope for H:

$$\dot{X}(t) = \int x\, m_t(x,t) dx$$

$$m_t(x,t) = -div\left(H_p(u_x(x,t), x, \mathcal{A})\, m(x,t)\right) + \frac{1}{2} tr\left(\Sigma\, m_{xx}(x,t)\right)$$

$$H_p(u_x(x,t), x) = \left[\left(\beta_1'(t) + x'\bar{\beta}_2\right) B - \mathcal{A}'\Theta_{\mathcal{A}}\right]\Gamma^{-1}B' - x'A'$$

Recall that div (in one dimension) is just derivative w.r.t x

Proof is replacing and integrating by parts.

Law of motion of X given aggregation of $\mathcal{A}$

- Use aggregation of foc of actions obtained above

$$\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$$

- Use aggregation of states from KFE and derivative of Hamiltonian:

$$\dot{X}(t) = B\Gamma^{-1} B' \beta_1(t) + \left(B\Gamma^{-1} B' \bar{\beta}_2 - A\right) X(t) - B\Gamma^{-1} \Theta_{\mathcal{A}} \mathcal{A}(t)$$

- Obtain Law of motion:

$$\begin{aligned}
\dot{X}(t) &= B\Gamma^{-1} B' \beta_1(t) + \left(B\Gamma^{-1} B' \bar{\beta}_2 - A\right) X(t) \\
&\quad - B\Gamma^{-1} \Theta_{\mathcal{A}} (\Gamma + \Theta_{\mathcal{A}})^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right) \\
&= B\Gamma^{-1} \left[I_n - \Theta_{\mathcal{A}} (\Gamma + \Theta_{\mathcal{A}})^{-1}\right] B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right) - AX(t) \\
&= B (\Gamma + \Theta_{\mathcal{A}})^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right) - AX(t)
\end{aligned}$$

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After replacing we get:

$$\begin{aligned}
\dot{\beta}_1(t) &= \left(\rho I_n + A' - \bar{\beta}_2 B\Gamma^{-1}B'\right)\beta_1(t) + \Theta_X X(t) \\
&+ \bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right) \\
&= \left(\rho I_n + A' - \bar{\beta}_2 B\Gamma^{-1}B' + \bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right)\beta_1(t) \\
&+ \left[\bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\bar{\beta}_2 + \Theta_X\right] X(t) \\
&= \left(\rho I_n + A' - \bar{\beta}_2 B\left(\Gamma^{-1} - \Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}\right) B'\right)\beta_1(t) \\
&+ \left[\bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\bar{\beta}_2 + \Theta_X\right] X(t) \\
&= \left(\rho I_n + A' - \bar{\beta}_2 B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right)\beta_1(t) \\
&+ \left[\bar{\beta}_2\left(B\Gamma^{-1}B' - B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right)\bar{\beta}_2 + \Theta_X\right] X(t)
\end{aligned}$$

The last equality uses that

$$\Gamma^{-1} - \Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} = \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}$$

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Eigenvalues of $J + G$ when $A = a\, I_n$

Notation: $\tilde{Q} \equiv Q + \Theta_X \succ 0$, $D \equiv B\,(\Gamma + \Theta_{\mathcal{A}})^{-1}\, B' \succ 0$, P minimal solution, $J + G = DP - aI_n$.

Proposition. If $A = a\, I_n$, the eigenvalues of $J + G$ are real and given by

$$eig_j\,(J+G) = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2} + a\right)^2 + eig_j\left(\tilde{Q}^{\frac{1}{2}}\, D\, \tilde{Q}^{\frac{1}{2}}\right)}\,, \quad j = 1, \ldots, n$$

Proof. *Step 1.* With $A = aI_n$ the Riccati equation collapses to $P\,D\,P = \tilde{Q} + (\rho + 2a)\,P$.

Step 2. Let $T \equiv D^{1/2} P\, D^{1/2}$: symmetric and, by congruence with $P \prec 0$, negative definite.

Multiplying the Riccati by $D^{1/2}$ on both sides:

$$T^2 - (\rho + 2a)\, T - C = 0\,, \qquad C \equiv D^{1/2} \tilde{Q}\, D^{1/2} \succ 0$$

Step 3. $C = T^2 - (\rho + 2a) T$ commutes with T; both symmetric $\implies$ simultaneously diagonalizable. On each common eigenvector, $t_j^2 - (\rho + 2a) t_j - c_j = 0$, so with $c_j \equiv \lambda_j(C) > 0$:

$$t_j = \left(\tfrac{\rho}{2} + a\right) \pm \sqrt{\left(\tfrac{\rho}{2} + a\right)^2 + c_j}\,; \qquad T \prec 0 \implies \text{negative root for every } j$$

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Eigenvalues of $J+G$ when $A = a\,I_n$, continued

Step 4. $J+G = DP - aI_n$ is *similar* to $T - aI_n$:

$$D^{-1/2}\,(DP - aI_n)\,D^{1/2} = D^{1/2}PD^{1/2} - aI_n = T - aI_n$$

so $\lambda_j(J+G) = t_j - a = \frac{\rho}{2} - \sqrt{(\frac{\rho}{2}+a)^2 + c_j}$. Finally $\lambda_j(D^{1/2}\tilde{Q}D^{1/2}) = \lambda_j(\tilde{Q}^{1/2}D\,\tilde{Q}^{1/2})$, both similar to $D\tilde{Q}$. ■

Corollary (mode-by-mode comparative statics). By Weyl monotonicity, each $\lambda_j(\tilde{Q}^{1/2}D\tilde{Q}^{1/2})$ is monotone in $\tilde{Q}$ and in D (Loewner order), so *every* eigenvalue of $J+G$ satisfies:

- more strat. compl. in **states** ($\Theta_X \downarrow \Rightarrow \tilde{Q}\downarrow$) $\implies$ every $\lambda_j \uparrow$: all modes more persistent
- more strat. compl. in **actions** ($\Theta_{\mathcal{A}} \downarrow \Rightarrow D\uparrow$, inverse is order-reversing) $\implies$ every $\lambda_j \downarrow$: all modes less persistent

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Trace of $J + G$: general A, general n

Proposition. Assume $\tilde{Q} \succ 0$, $\Gamma + \Theta_{\mathcal{A}} \succ 0$ (so $P \prec 0$), eqbm exists. Along any smooth path $\tau \mapsto (\tilde{Q}(\tau), D(\tau))$:

$$\frac{d}{d\tau} tr\,(J+G) \;=\; -tr\left(W\dot{\tilde{Q}}\right) \;+\; tr\left(\dot{D}\,\underbrace{(P + PWP)}_{\prec 0}\right), \qquad W \succeq 0$$

Hence $tr(J+G) = \sum_j Re\,\lambda_j(J+G)$: strat. compl. in states ($\dot{\tilde{Q}} \preceq 0$) raises it, strat. compl. in actions ($\dot{D} \succeq 0$) lowers it.

Proof. Write $\mathcal{G} \equiv J + G = DP - A$ and $\tilde{\mathcal{G}} \equiv \mathcal{G} - \frac{\rho}{2} I_n$, *stable* in eqbm: $Re\,\lambda(\tilde{\mathcal{G}}) < 0$.

Step 1 (perturbation of P). Differentiate $PDP = \tilde{Q} + \rho P + PA + A'P$ along the path, group terms:

$$\dot{P}\,\tilde{\mathcal{G}} + \tilde{\mathcal{G}}'\,\dot{P} = \dot{\tilde{Q}} - P\dot{D}P \quad\Longrightarrow\quad \dot{P} = -\int_0^\infty e^{\tilde{\mathcal{G}}'t}\left(\dot{\tilde{Q}} - P\dot{D}P\right)e^{\tilde{\mathcal{G}}t}\,dt$$

Note: $\dot{\tilde{Q}} \preceq 0$, $\dot{D} \succeq 0 \implies \dot{P} \succeq 0$: re-derives, in one line, monotonicity of P in *both* couplings (Wimmer 1985).

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Trace of $J + G$, continued: Gramian and key lemma

Step 2 (trace formula). Let $W \equiv \int_0^\infty e^{\tilde{\mathcal{G}}t} D\, e^{\tilde{\mathcal{G}}'t} dt \succeq 0$, the closed-loop Gramian: $\tilde{\mathcal{G}} W + W \tilde{\mathcal{G}}' = -D$. By Step 1 and cyclicity of the trace:

$$\frac{d}{d\tau} tr(\mathcal{G}) = tr(\dot{D}P) + tr(D\dot{P}) = -tr(W\dot{\tilde{Q}}) + tr\big(\dot{D}(P + PWP)\big)$$

Step 3 (key lemma: $P + PWP \prec 0$). Let $V \equiv -P^{-1} \succ 0$. Multiply the Riccati by P^{-1} on left and right, substitute V; then use $\tilde{\mathcal{G}} V = -D - AV - \frac{\rho}{2} V$:

$$\tilde{\mathcal{G}} V + V \tilde{\mathcal{G}}' = -D - V\tilde{Q}V \qquad \text{while} \qquad \tilde{\mathcal{G}} W + W \tilde{\mathcal{G}}' = -D$$

Subtract: $\tilde{\mathcal{G}}(V - W) + (V - W)\tilde{\mathcal{G}}' = -V\tilde{Q}V \prec 0$; stability of $\tilde{\mathcal{G}}$ gives $V - W \succ 0$. By congruence:

$$P + PWP = P\left(P^{-1} + W\right)P = -P\,(V - W)\,P \prec 0 \qquad \blacksquare$$

- $P + PWP \prec 0$: *direct* effect of $\Theta_{\mathcal{A}}$ (via D) dominates *indirect* effect (via P); $n = 1$: $\partial\lambda/\partial D = -\tilde{q}/(2s) < 0$.
- Trace strictly $\downarrow \implies$ volume contraction rate of flow.

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