Deposit Competition Beyond Rates [*]

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Abstract

We study competition and pass-through in the market for retail deposits. Federal funds rate increases are associated with only minimal increases in deposit rates. Leveraging new data on offers mailed by banks to households, we show that federal funds rate increases are strongly associated with changes in mail volumes, sign-up bonuses, and offers targeting new customers. These margins and not deposit rates are the primary ways in which interest rate changes affect the deposit market. We rationalize the use of these margins in a simple model with active and sleepy depositors. Our calibrated model and counterfactual analysis show that depositor heterogeneity and adverse selection, as opposed to market power, are the primary reasons why banks offer far less than the full present value of future rate spreads to depositors.

Keywords: Deposit Competition; Pass-through; Adverse Selection; Marketing

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1 Introduction

Major banks in the United States offer deposit rates of essentially zero on retail checking and savings accounts. When market interest rates, such as the federal funds rate (FFR), increase, these banks pass through little if any of the increase. This lack of pass-through has potentially important implications for competition in the banking market and for the conduct of monetary policy (Neumark and Sharpe, 1992; Drechsler et al., 2017; Wang et al., 2022).

However, banks can compete for depositors without offering higher interest rates. The most direct way to do this is by offering new depositors money to open an account.[1] Figure 1 shows two offers mailed by Chase and Citizens to households offering cash payments in exchange for opening checking and savings accounts. Both offers clearly emphasize sign-up bonuses (of $500 and $600, respectively), while reporting the interest rate (1bps and 2bps) in very small font. These ads make apparent a fact we confirm in data: almost all checking deposits and many savings deposits earn zero or close to zero interest. At the same time, many banks offer cash bonuses for opening new accounts and advertise these offers via direct mail. The marketing volume and size of these bonuses vary across banks and over time. That is, in the retail deposit market, the price through which competition occurs is the sign-up bonus, not the interest rate. Our contribution is to examine both empirically and theoretically this form of price competition and to study its implications for depositors behavior and banks franchise value.

In the first part of the paper, we document new facts on the pricing, advertising, and targeting strategies of large banks in the retail deposit market. To accomplish this, we obtained access to new data that allow us to measure these marketing efforts. We analyze images from about 2200 unique campaigns issued by 11 large banks in the period 2014-2023. The banks in our sample account for about 50% of the approximately $13 trillion of total deposits in domestic offices in the US as of 2019. We observe an estimated measure of total mail volume and dollar spent on campaigns, and we extract information on signup bonuses from images of the mailings using a large language model.

We confirm that for the largest banks in the US there is very limited pass-through from the federal funds rate to the interest rates on checking and savings accounts. The existing literature documents a non-zero (but far from complete) pass-through from the federal funds rate to banks' overall interest expense; this is driven by rates on certificates of deposits (CDs), wholesale deposit rates, and the like, as opposed to by checking and savings account rates.[2]

[1] Other ways for banks to compete include opening branches in new areas (Begenau and Stafford, 2023) and offering superior liquidity services (d'Avernas et al., 2023).

[2] We confirm a larger pass-through when looking at 12-months CDs in our data.

Figure 1: Sign-up bonuses and rates on mailed offers

Note: The figure shows the mailed offer for two banks in our sample. The left panel reports an offer from Chase, while the right panel reports an offer from Citizen bank. The red circles identify the location of the current interest rate on these accounts.

Second, we show that the median bonus for checking (saving) offers is \$300 (\$200). This offer is only a small fraction of the present value of sticky depositors with a life of ten or more years, but it is close to break-even in the event that the customer stays only for the minimum duration required to receive the bonus (usually 90 days). We also find that a 100-basis-points higher federal fund rate is associated to a \$16 (\$14) higher bonus for checking (savings) accounts, a 6% increase relative to their respective mean sign-up bonuses. We interpret bonus changes as a measure of the intensive margin of advertising effort.

Third, we study mail volumes, which we take as a measure of the extensive margin of advertising effort. The average bank in our sample sends more than 6 million mail offers for checking and saving accounts per month. The main volume has a very high correlation with the federal funds rate. That is, in a high rate environment, banks compete fiercely to attract new customers, while in a low rate environment they offer sign-up bonuses only to select prospects. We find that a 100-basis-points higher federal fund rate is associated to a 11% higher mail volumes for checking and savings accounts. The point estimate is larger for the most popular campaign for each bank and is driven by the banks paying the lowest rates on checking and savings accounts.

Fourth, we study banks' targeting of offers to different markets and demographic groups and how targeting varies with the federal funds rate. We define a target group as the interaction of state, income bins, and age bins and compute the Herfindahl-Hirschman Index (HHI) as a measure of bank campaign concentration in certain target groups. We find that a 100-basis-point higher federal funds rate is associated with a 3 percentage point lower HHI, which corresponds to a 12% decline relative to the average HHI. Combining the extensive margin and the targeting facts, our results point to banks expanding their customer acquisition unconditionally when rates are high, and contracting it conditionally (i.e., focusing on specific groups) when rates are low.

Finally, we leverage three surveys from the same data providers to study depositor switching behavior. Survey respondents who recently switched their checking or savings accounts are more likely to consider switching again in the future, and when switching savings products, respondents value the sign-up bonus more than the interest rate. We also find that a 100-basis-points higher federal fund rate is associated with a 10% higher switching rate for checking and savings accounts. While the point estimate is identified from only very limited time-series variation across survey waves, the direction of the relation is in line with work using detailed individual-level data on account openings and closures by Egan et al. (2025).

In the second part of the paper, we develop a model informed by our new facts to explain the use of these margins. Specifically, we develop a model to clarify why banks prefer to offer sign-up bonuses instead of high interest rates, why this bonus is sensitive to the level of rates, and why the extensive margin of advertising responds very strongly to the level of rates. Armed with such a model, we re-evaluate the degree of competition in the banking market for checking and savings accounts.

The key features of our model are as follows. First, we observe that deposits are "sticky", in the sense that many households do not change banks despite a financial incentive to do so. This creates the potential for bank franchise value. Second, we observe that bonus offers are well below this franchise value and are close to break-even in the event that the customer stays only for the minimum duration required to receive the bonus (usually 90 days), as well as that some depositors are more likely to keep switching banks over time. We interpret these findings as evidence of adverse selection; therefore, we model two types of depositors: "sleepy" and "active". Third, we observe that banks never offer very small (<$50) sign-up bonuses (and that changing bank accounts is a hassle), both of which are consistent with switching costs. We build a model that incorporates these key ingredients and use it to estimate (1) the degree to which bonuses are below the value of the deposit to the bank and (2) the degree to which this difference is driven by adverse selection vs. market power.

Under certain simplifying assumptions, we are able to analytically characterize the equilibrium in our model. Specifically, we construct an equilibrium in which bonuses are constant over time and banks vary their marketing efforts as interest rates change. This can be viewed as capturing our key empirical facts in a stylized way. The key driver of these results is that mailing more offers is a constant marginal cost method of acquiring new depositors whereas increasing the size of the bonus offer has an increasing marginal cost, because the higher bonus must be paid to depositors who would have switched to the bank for a lower bonus. This point in conjunction with the observation that banks' marketing efforts are strategic substitutes (when other banks market more, the value of a new depositor falls) delivers the result.

We calibrate this model using survey data on the rates at which people switch bank accounts in conjunction with our data on bonus levels and marketing volumes. We focus our calibration on savings accounts, since deposit accounts might have other sources of revenue (maintenance or overdraft fees) and costs (debit cards and ATM networks) that our model does not fully capture. The calibration delivers two key findings. First, the value of a deposit from a typical retail depositor is large as a fraction of the size of the deposit (more than 50%), as a consequence of the fact that banks can pay no interest on these deposits and that depositors switch bank accounts infrequently. Second, the value of a marginal depositor who is considering switching banks is far lower (about 6% of the deposit size), because these depositors are far more likely than the typical depositor to switch accounts again. Put another way, adverse selection (the difference in value between the representative depositor and the awake depositor) explains most of the gap between the value of a typical bank deposit and the bonus the bank offers; market power explains relatively little.

In the last part of the paper, we develop a quantitative version of our model and solve it without imposing the simplifying assumptions used to construct our analytic results. We confirm the findings of our calibration of the analytical model, in particular our conclusion that adverse selection accounts for the bulk of the difference between the value of a deposit from a typical depositor and the bonus. We find that adverse selection accounts for about 85% of the difference between the value of a representative depositor and the bonus paid by banks. This result holds in both high and low interest rate environments. In the baseline, sleepy types receive a $208 bonus when changing banks, which is 3%

of their value to the bank. In a counterfactual with all sleepy types (i.e. in the absence of adverse selection), sleepy types instead receive equilibrium bonuses of about $4700, which is approximately 75% of their value to the bank.

Related literature. Our paper contributes both empirically and theoretically to the large literature studying competition and pricing in banking more generally, and the deposit market more specifically.

First, a large body of literature has studied the role of banks' market power in the limited pass-through of monetary policy changes to both credit and deposit markets (Hannan and Berger, 1991; Neumark and Sharpe, 1992; Scharfstein and Sunderam, 2016; Drechsler, Savov, and Schnabl, 2017; Benetton and Fantino, 2021; Wang, Whited, Wu, and Xiao, 2022; Buchak, Matvos, Piskorski, and Seru, 2024). Furthermore, market power on the deposit side has been identified as the key source of banks' franchise value, which varies with the level of the federal fund rate, potentially leading to financial fragility and runs (Egan, Hortaçsu, and Matvos, 2017; Drechsler, Savov, and Schnabl, 2021; Egan, Lewellen, and Sunderam, 2022; Drechsler, Savov, Schnabl, and Wang, 2023; DeMarzo, Krishnamurthy, and Nagel, 2024). Some recent papers have proposed heterogeneity across markets and banks' business models and leveraged new microdata on depositors' behavior to explain and better measure the drivers of depositors' demand (e.g., convenience value) and its (low) average sensitivity to rates (Begenau and Stafford, 2023; d'Avernas, Eisfeldt, Huang, Stanton, and Wallace, 2023; Blickle, Li, Lu, and Ma, 2024; Kundu, Muir, and Zhang, 2024; Lu, Song, and Zeng, 2024; Argyle, Iverson, Kotter, Nadauld, and Palmer, 2025; Cirelli and Olafsson, 2025).

We complement this literature by exploiting supply-side data on bank offers and marketing effort, thus providing novel evidence on margins of deposit competition that have been largely overlooked in the literature.[3] Our model incorporates these new margins and shows how they can help disentangle the role of market power versus adverse selection for deposit pricing and banks' franchise value.

Second, a growing number of papers using tools from the industrial organization literature on differentiated products have been developing structural models to study bank deposit competition and financial fragility (Egan, Hortaçsu, and Matvos, 2017), monetary transmission through shadow banks (Xiao, 2020), the sources of bank value and synergies across markets in deposit taking and lending (Egan, Lewellen, and Sunderam, 2022; Aguirregabiria, Clark, and Wang, 2024), the impact of digital banking (Koont, 2023; Jiang, Yu, and Zhang, 2024), and the role of depositor sleepiness for bank's competition and stability (Egan, Hortaçsu, Kaplan, Sunderam, and Yao (2025), Lu and Wu (2025)).

The majority of these papers have focused on exogenous vertical and horizontal differentiation

[3]One exception is Chen et al. (2026), who study the narrative content of banks' television advertisement.

across banks as the source of market power and endogenous deposit rates as the key competition variable.[4] Our new facts and modeling approach contribute to this literature by adding targeted marketing and multi-dimensional pricing as important choice variables for bank competition, in line with some recent work studying the deposit (Honka, Hortaçsu, and Vitorino, 2017) and mortgage (Gurun, Matvos, and Seru, 2016; Benetton, Gavazza, and Surico, 2025) markets.

Third, the banking literature has extensively studied the role of asymmetric information and adverse selection on the asset side of banks' balance sheet (Dell'Ariccia and Marquez, 2004; Crawford, Pavanini, and Schivardi, 2018; Hertzberg, Liberman, and Paravisini, 2018; Nelson, 2018; Lester, Shourideh, Venkateswaran, and Zetlin-Jones, 2019; Matcham, 2024; Benetton and Buchak, 2024). In credit markets, increasing loan rates, decreasing downpayment requirements, and extending maturity attract borrowers that are more likely to default.

We show that adverse selection is an important friction also on the liability side of banks' balance sheet. Temporarily high deposit rates or sign-up bonuses attract depositors that are more likely to leave, and hence less valuable in present-value terms for banks. This force restrains competition between banks, and leads in equilibrium to banks' earning positive profits even in the perfectly competitive limit of our model (which, in other respects, is reminiscent of the shrouding model of Gabaix and Laibson (2006)). Our results show that equilibrium prices on banks' deposit side are jointly determined by market power and adverse selection, with implications for market structure (e.g., barriers to entry) and depositor welfare.

Overview. The result of the paper is organized as follows. Section 2 describes the new data we use and summary statistics. Section 3 presents the facts on deposit pricing and acquisition. Section 4 presents our model and a special case with an analytic solution. Section 5 presents the calibration and counterfactual analyses. Section 6 concludes.

2 Data and Summary Statistics

2.1 Data Sources

Our analysis exploits a rich database of offers for checking and saving accounts mailed by banks to households during the period 2014-2024. We complement our main database on mailed offers with additional data on banks' deposit rates, the federal fund rate, and households deposits.

[4] Koont (2023) also endogenize technology adoption and Jiang, Yu, and Zhang (2024) also endogenize branch choice. Egan, Hortaçsu, Kaplan, Sunderam, and Yao (2025) study dynamic competition with heterogeneous banks and sleepy depositors.

Comperemedia. The Comperemedia Direct Mail dataset collected by Mintel is a comprehensive database that tracks and analyzes marketing campaigns in the United States sent through direct mail (and more recently via email) by companies across various industries, with a particular emphasis on financial services, telecommunications, insurance, and other consumer-focused sectors. The database is based on a panel of more than 17,000 households (15,000 rolling and 2,500 lifecycle), which are paid to collect all direct mailers and send the originals to Mintel. The company invites each month about 6,000-6,500 panelists to participate, balancing them by age, income, region and household size, and targets a response rate of 50%. Similar data have been used in previous academic research on credit cards (Ru and Schoar, 2016; Han et al., 2018). We are the first to our knowledge to exploit this data for deposit account offers (checking and savings).

We obtained access to the Comperemedia Archives which is a repository of all Comperemedia Direct campaign data collected from the last 10 years. We obtained all the images captured from offers sent by the largest banks in the US between 2014 and 2024 that were categorized by Comperemedia as checking and/or savings. We run all the images through a large language model (LLM) to extract the relevant information on the offer: sign-up bonus, minimum deposit or balance requirement, minimum days. We also attempted to extract information on interest rate. However, in line with the evidence in Figure 1, interest rates are generally reported in very small font making it difficult to extract precise data. For this reason, we complement our rich information on contract offer with standard information on deposit rates from RateWatch.

The bonus offers sent to households are not standardized, and banks do not use a common vocabulary when describing the accounts they offer. For these reasons, building a structured dataset of comparable offers from the raw image data is a non-trivial endeavor. In Appendix A, we describe the classification schema we implement and the way in which we use the LLM to construct the dataset.

RateWatch - S&P Global Market Intelligence. For deposit rates we follow previous research and use data from RateWatch, which since 2018 has been acquired by S&P Global Market Intelligence. The S&P Capital IQ Pro platform provides a vast database containing deposit, loan and fee information for financial institutions across the US collected from over 96,000 branch locations. We focus on the same list of banks that we have data for based on the Comperemedia dataset and extract information on the two most popular deposit rate products in our sample: (i) interest checking with a minimum deposit of $2.500, and (ii) savings with a minimum deposit of $2.500.

FRED Economic Data. We obtained information on the federal fund rate (FFR) at the monthly level from the FRED, Federal Reserve Bank of St. Louis. The data can be downloaded here: `https:`

//fred.stlouisfed.org/series/FEDFUNDS.

Survey of Consumer Finances. We obtained information on households average deposits from the Survey of Consumer Finances. The data can be downloaded here: https://www.federalreserve.gov/econres/scfindex.htm.

2.2 Summary Statistics

Table 1 reports summary statistics on the main variables used in the analysis. Our main database is based on approximately 2200 unique campaigns issued by 11 lenders in the period 2014-2023. The lenders in our sample are the following: Bank of America, Capital One, Chase, Citibank, Citizens Bank, Discover, Fifth Third Bank, PNC, TD Bank, U.S. Bank, and Wells Fargo. Together, these banks account for about 50% of the about $13 trillions of total deposit in domestic offices in the US as of 2019.

Panel A of Table 1 reports these variables for the full sample. The average campaign is mailed to 1.1 million US households. The median campaign is sent to about 315 thousand households, but some popular ones are sent to more than 40 million households. The scale of these numbers emphasizes that physical mail is an important tool for banks.

Comperemedia classifies mail as acquisition and retention. The former are defined as offers of a new product or service to a consumer who is not a current customer of the issuing bank, while the latter are new product or service offered to a current customer of the issuing bank in addition to the current product or service, without replacing the current product or service. An example of a retention offer is a mail to a Chase credit card account holder offering a bonus for opening a Chase checking account. In our sample about 80% of the mail are classified as acquisition and the remaining ones as retention.

Panels B and C of Table 1 report variables for checking and saving offers, respectively. We treat these two product categories separately because they are typically associated with different types of bonus offers and may differ in terms of the revenues they create for banks. Checking accounts typically require a low or zero minimum deposit to receive the sign-up bonus, and instead require that the depositor set up direct deposit. Saving accounts generally require that the depositor maintain specified balance for a given amount of time (e.g. a $15,000 minimum deposit for at least 90 days).

Almost 90% of checking offers are for the acquisition of new customers, consistent with checking being an entry product for potentially future cross-selling. Indeed, according to the Survey of Consumer Finances in 2023, almost 95% of respondents have a checking account, while about 55% have

Table 1: SUMMARY STATISTICS

	Observations	Mean	Std. Dev.	Minimum	P5	Median	P95	Maximum
Panel A: Full sample								
Mail volumes (.000)	3,970	1,291.79	2,825.76	38.50	70.83	359.51	5,591.83	44,707.12
Mail dollar (.000)	3,970	479.59	921.61	18.48	36.72	162.09	2,110.27	12,875.65
Acquisition (%)	3,970	85.59	35.12	0.00	0.00	100.00	100.00	100.00
Retention (%)	3,970	14.21	34.92	0.00	0.00	0.00	100.00	100.00
Federal Fund Rate (%)	3,970	1.65	1.58	0.05	0.08	1.21	5.08	5.33
Panel B: Checking								
Acquisition (%)	2,775	90.52	29.30	0.00	0.00	100.00	100.00	100.00
Retention (%)	2,775	9.23	28.94	0.00	0.00	0.00	100.00	100.00
Incentive ($)	2,775	278.67	84.08	0.00	150.00	300.00	400.00	600.00
Minimum deposit ($)	2,775	965.77	1,481.71	0.00	0.00	500.00	4,000.00	15,000.00
Minimum balance days	1,178	90.38	8.81	30.00	90.00	90.00	90.00	182.00
Rate (%)	944	0.03	0.06	0.00	0.01	0.01	0.20	0.30
Panel C: Saving								
Acquisition (%)	2,345	84.48	36.22	0.00	0.00	100.00	100.00	100.00
Retention (%)	2,345	15.52	36.22	0.00	0.00	0.00	100.00	100.00
Incentive ($)	2,345	218.11	118.82	0.00	100.00	200.00	400.00	2,500.00
Minimum balance ($)	2,345	16,002.13	13,163.21	5,000.00	5,000.00	15,000.00	25,000.00	250,000.00
Incentive - 15K min balance ($)	2,345	230.22	111.13	0.00	100.00	200.00	500.00	600.00
Minimum balance days	1,977	82.96	20.32	10.00	60.00	90.00	90.00	365.00
Rate (%)	1,039	0.27	0.70	0.01	0.01	0.01	1.75	4.35

Note: The Table shows the main variable in our analysis. Panel A reports statistics for the full sample of offers, Panel B focuses on offers for checking accounts, and Panel C focuses on offers for saving accounts. Mail volumes is the research panel's adjusted direct mail volume sent by a company in a given period for a specific campaign projected to the entire national population. Acquisition is a dummy for offers of a new product or service to a consumer who is not a current customer. Retention is a dummy for offers a new product or service to a current customer of the issuing bank. Incentive is the value in dollars of the sign-up bonus. Rate is the average rate from Ratewatch for checking and saving products.

a savings account. The average incentive is approximately $280, which is very similar to the median incentive of $300. The median minimum deposit for checking account is $500, while the average is much higher at $1500, due to some large outliers. Some offers require a minimum number of days that the checking account should be kept open, with the median being 90 days. Finally, the average rate is three basis points and the median rate is one basis point, in line with checking accounts paying close to zero interest. Note that the sample size for the rate variable is smaller than the full sample size, reflecting the LLM's inability to discern the interest rate from the offer in many cases.

Turning to saving accounts Panel C of Table 1 shows that about 85% of saving offers are for acquisition of new customers. The average incentive is approximately $218, which is very similar to the median incentive of $200. The average minimum balance for checking account is about $16,000, which again is very similar to the median balance of $15,000. There is a strong relationship between the minimum balance and the incentive; many banks offer a tiered bonus structure with higher incentive payments for larger deposits. The ”Incentive - 15K min balance” variable reports the incentive payment per \$15k of minimum balance.

The mean interest rate on savings accounts is higher than on checking accounts, as expected. However, the median rate is still low at one basis point, while the average is about 27 basis points. This difference reflects heterogeneity in banks' strategies. Some of the banks in our sample (Capital One, Discover, Citizens) offer savings rates that co-move with the federal funds rate; the other banks in our sample offer savings rates that are essentially zero. In what follows, we will refer to the three aforementioned banks as ”high-rate” banks.

3 Facts on Deposit Pricing and Acquisition

In this section we present several facts derived from the data just described. Our goal is to document the main patterns with respect to deposit pricing and competition in the US in the period 2014-2024. We begin by looking at deposit rates, which have been the primary object of analysis in the existing literature. We then pay special attention to our new outcome variables: sign-up bonuses, marketing, and targeting.

We first present our facts as a simple unconditional correlation of different variables capturing banks' deposit pricing and acquisition with the federal fund rate. We then provide conditional correlation controlling for time-invariant differences across banks and other time-varying factors affecting deposit pricing. The purpose of banks fixed effects is to show that our facts hold within banks over time and are not driven by bank composition effects as the federal fund rate changes.[5] Using our monthly panel of campaigns over time, we estimate the following baseline regression:

$$y^k_{ilt} = \alpha^k FFR_t + \lambda^k X_{lt} + \gamma^k_l + \epsilon^k_{ilt}, \tag{1}$$

where k index the product (checking or saving), FFR_t is the federal fund rate (in percentage points), X_{lt} are lender and time (interacted) controls, and γ_l are lender fixed effects. The dependent variables y^k_{ilt} are: the interest rate (in basis points), sign-up bonus (in \$), total mail volumes and dollar spent (in log), and two measures of targeting which we discuss below. For rates and sign-up bonuses we estimate equation (1) separately for checking and saving offers. The key parameter of interest is α, which captures the correlation of various margins of banks' deposit pricing and competition with the

[5] The large banks we study operate for the entirety of the sample. However, there are certain bank-time pairs in which no mail is sent by that bank during that period, which generates the compositional effects our fixed effect specification is meant to address.

federal fund rate.

3.1 Deposit Rates

We begin by looking at deposit rates on checking and saving accounts and how banks adjust them as a function of the federal fund rate. Figure 2a shows the federal fund rate from 2014 to 2024 as well as the average rate on the two most popular checking and saving products using the RateWatch - S&P Global Market Intelligence data. The average rate on checking accounts is close to zero and remarkably stable over time despite the large changes in the federal fund rate. The average rate on saving accounts is on average higher than the rate on deposit account and shows more variability over time. Most notably, as the federal fund rate increases also the rate on saving account increases, but the changes in the latter are much smaller than the changes in the former in line with the existing literature on imperfect deposit rate pass-through (Neumark and Sharpe, 1992; Drechsler et al., 2017).

Columns (1) and (2) of Table 2 show the results of equation (1) using the checking and saving rates as dependent variables. Column (1) shows that within bank a 100-basis-point higher federal fund rate is associated with less than 1-basis-point higher rate on checking accounts. The point estimate in column (1) is statistically significant, but the magnitude is tiny at about 0.26 basis points. Turning to saving accounts, column (2) of Table 2 shows that a 100-basis-point higher federal fund rate is associated with a 13-basis-point higher rate on saving accounts. In line with the literature and the unconditional pattern from Figure 2, the within-bank pass-through from the federal fund rate to the deposit rate is higher for saving accounts than checking accounts, but also for saving accounts, the change in rates is only a small fraction of the change in the federal fund rate.[6]

In Appendix B we report additional analyses to test the robustness of our descriptive facts and explore some relevant dimensions of heterogeneity. The results are robust to excluding Chase, which is by far the largest advertiser in our sample, and the point estimates are only slightly larger when considering the cumulative effect of four months lags in the federal fund rate. Finally, we estimate (1) with an additional interaction between the federal fund rate and a dummy for banks with relatively higher rates throughout our sample period. We find that the larger correlation between the federal fund rate and the saving rate is driven by a specific set of banks, in line with the work by Kundu et al. (2024). When the federal fund rate is 100 basis points higher, the majority of large banks barely change their rate on saving accounts, while a few high-yield banks offer almost 50-basis-point higher

[6] We estimate the same specification for rates on 12-month certificate of deposits with an account size of $10,000 also using the RateWatch - S&P Global Market Intelligence data. We find a point estimate for the coefficient on the federal fund rate of about 30 basis points, again in line with the literature finding relatively larger but still imperfect pass-through for time deposits (Drechsler et al., 2017).

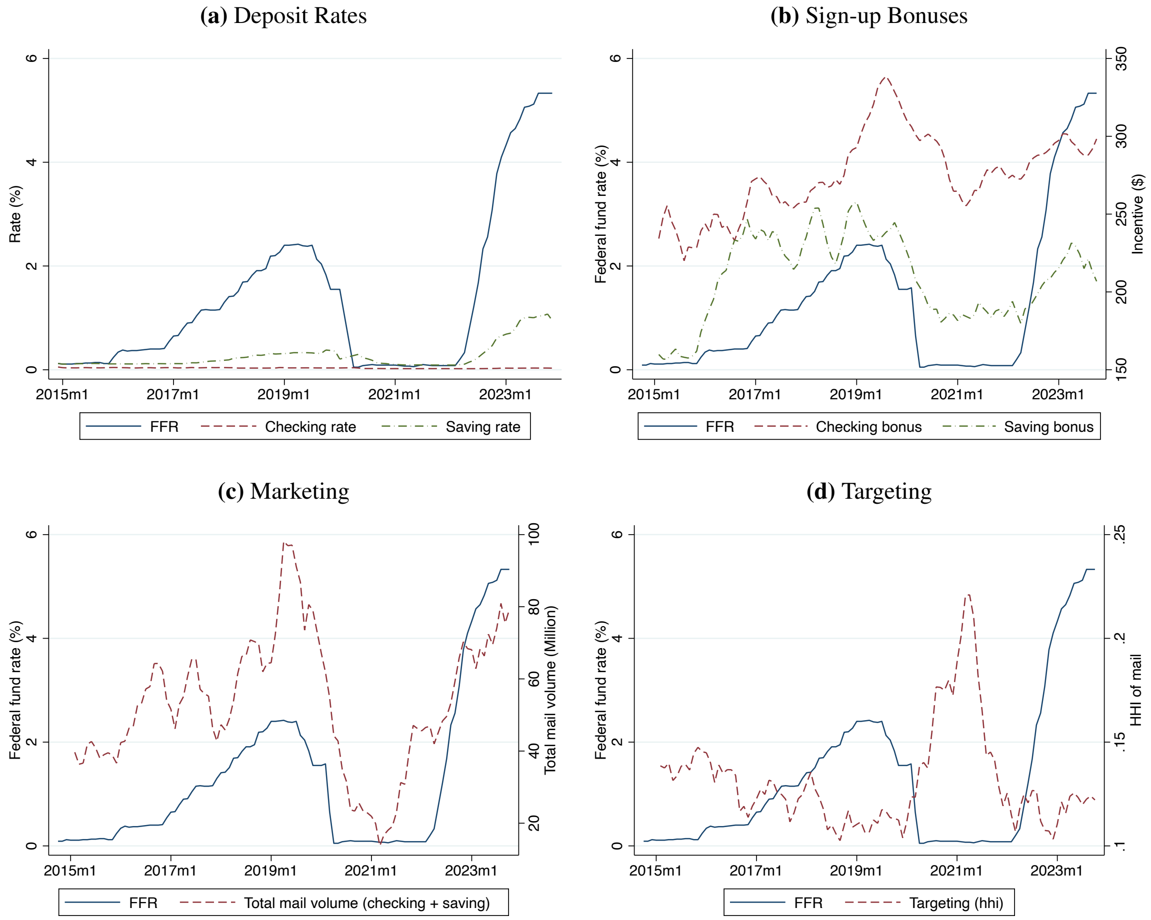

Figure 2: Pass-through on Checking and Savings Products - Rates, Mails, Bonuses and Targeting
Note: The figure plots the federal funds rate (2014–2024) alongside the following variables: (a) average checking and saving rates from RateWatch – S&P Global Market Intelligence; (b) total mail volume in millions of checking and saving offers using the Comperemedia data; (c) average sign-up bonus in dollars for checking and saving products using the Comperemedia data; (d) average Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins using the Comperemedia data.

saving rates. We summarize the findings on deposit rates, which we view as consistent with the existing literature, as follows.

Fact 1: Deposit rates. *The average rate on checking (saving) accounts is about 3 (27) basis points. A 100-basis-points higher federal fund rate is associated with a higher rate on checking account by less than 1 basis point and on savings accounts by about 13 basis points. The latter is driven by a few high-yield banks, while the majority of large banks do not change saving rates.*

Table 2: PASS-THROUGH ON CHECKING AND SAVINGS PRODUCTS - RATES, MAILS, BONUSES AND TARGETING

	Deposit Rates (bps)		Sign-up Bonus ($)		Marketing (log)		Targeting	
	Checking (1)	Saving (2)	Checking (3)	Saving (4)	Mail volume (5)	Dollar spent (6)	HHI (7)	Number of states (8)
Federal fund rate (%)	0.26***	12.55***	16.39***	13.40***	0.11***	0.11***	-0.02***	0.60***
	(0.05)	(1.45)	(1.51)	(2.18)	(0.02)	(0.02)	(0.00)	(0.13)
BANK F.E.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
BANK-TIME CONTROLS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
DEP. VAR. (MEAN)	3.27	26.84	276.92	214.51	13.43	12.64	0.25	10.24
DEP. VAR. (SD)	5.76	70.01	88.84	114.32	1.53	1.39	0.29	9.34
R^2	0.84	0.62	0.28	0.42	0.28	0.28	0.08	0.41
OBSERVATIONS	944	1,039	2,838	2,353	3,966	3,966	3,803	3,803

Note: The Table shows the results of equation (1). Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

3.2 Sign-up Bonuses

Next we look at banks' sign-up bonuses. As shown in Table 1 the median bonus for checking (saving) offers is $300 ($200). Is this a large or small bonus? We provide a preliminary answer to this question with a simple back of the envelope calculation, and return to this question more thoroughly with our quantitative model in Section 5. We focus on saving offers and compute the net present value with a deposit of $15,000 (the median from the Survey of Consumer Finance) for different baseline rates (1% and 4%) and different durations of the relationship (from 1 quarter to 20 years). Figure 3 shows that a $200 bonus is only a small fraction of sticky depositors present value. The present value net of the bonus of a depositor who stays with the bank for ten years – the length in Drechsler et al. (2023) and Egan et al. (2025) – is more than $1000 with a baseline rate of 1% and almost $5000 with a baseline rate of 4%. However, Figure 3 also reveals that banks lose money on depositors who take the sign-up bonus and leave in the subsequent quarters (recall from Table 1 that the minimum number of days to receive the typical bonus is 90 days).

We then look at bonuses over time as the federal fund rate changes. Since some banks do not mail bonus offers in each month, some of the time series variation is the result of banks moving in and out of the sample. To account for this compositional effect, we regress sign-up bonuses on bank and time fixed effects and then compute the average predicted sign-up bonuses using the estimated time fixed effects. Figure 2b reports the results. The average sign-up bonuses on both checking and saving accounts are positively correlated with the federal fund rate, but the correlation is less stark than the one of mail volumes shown in Figure 2c.

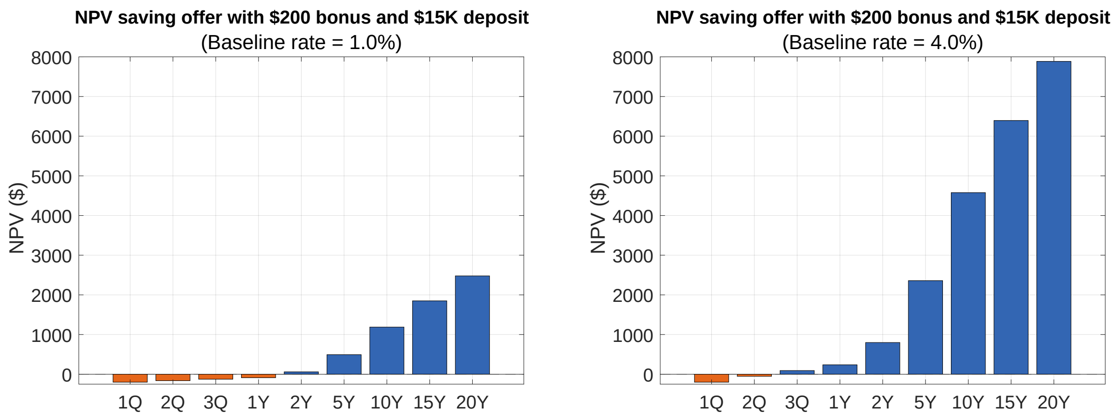

Figure 3: NPV saving offer with $200 bonus and $15K deposit

Note: The figure plots the present value net of the bonus of a deposit of $15,000 (the median from the Survey of Consumer Finance) for different baseline rates (1% and 4%) and different durations of the relationship (from 1 quarter to 20 years)

Columns (3) and (4) of Table 2 show the results of equation (1) using the average sign-up bonus for checking and saving offers as dependent variables. A 1-percentage-point increase in the FFR is associated with a $16 increase in sign-up bonuses for checking accounts. Given an average sign-up bonus of about $277, this corresponds to approximately a 6% increase relative to the mean. Turning to savings accounts, a 1-percentage-point increase in the federal fund rate is associated with a $13 increase in sign-up bonuses, which corresponds to a 6% increase relative to an average sign-up bonus of about $214.

In Appendix B we show that the point estimates are robust to excluding the Chase (Table A1), and slightly larger when focusing on the largest campaign within each bank-time pair (Table A2) and when considering the cumulative effect of four months lags in the federal fund rate (Table A3). Finally, we find that our baseline estimates in Table 2 are similar for checking accounts across all banks in our sample, while for savings accounts the increase in sign-up bonuses is driven by high-yield banks. Table A4 shows that when the federal fund rate is 100-basis-points higher, the majority of large banks increase sign-up bonuses for savings by only $6 (3% relative to the mean), while a few high-yield banks increase sign-up bonuses for savings by almost $30 (13% relative to the mean). We summarize these new findings on sign-up bonuses as follows.

Fact 2: Sign-up bonuses. *The average saving offer has a bonus of $200, which is only a small fraction of the present value of a sticky depositors with a life of ten or more years. A 100-basis-points higher federal fund rate is associated with higher sign-up bonuses for checking and savings account by $13-16, a 6% increase relative to their respective mean sign-up bonuses.*

3.3 Marketing via Direct Mail

Next we look at how banks adjust marketing effort in relation to the federal fund rate. We measure marketing effort using total mail volume and dollars spent on campaigns to attract new depositors for checking and savings accounts. The dollars spent measure is an estimate by Comperemedia of the cost of sending the mailings, and does not include the cost of paying the sign-up bonuses. Figure 2c shows that mail volume varies with the level of the federal fund rate. In 2015, when the federal fund rate is around zero, the total amount of mail sent by banks is about 40 million pieces of mail per month. As the federal fund rate rose to more than 2% in 2019, the total amount of monthly mail peaked at around 100 million pieces per month, before dropping below 30 million as the federal fund rate went back to zero in response to the COVID-19 pandemic. Following the recent monetary policy tightening in 2022 to fight inflationary pressure, mail volume started rising again, reaching 80 million by 2023.

Overall, the correlation between the federal fund rate and the mail volume is about 0.7. Figure A1a in Appendix B shows a very similar correlation between the federal fund rate and the amount of dollars spent monthly on mails for checking and saving accounts offers, which was below 10 million in 2021 and rose above 30 million by 2023. Figure A1b in Appendix B shows that the volume of mail offers for mortgages displays a very different pattern, increasing when rates are low (presumably because the of the incentive for refinancing). As a consequence of the divergence between the mail volumes of deposit offers and mortgage offers, bank-level measures of advertising expenditures are not a reliable guide to the amount of advertising at the product level.

Columns (5) and (6) of Table 2 show the results of equation (1) using the total mail volumes and dollars spent on mailings for checking and saving offers as dependent variables. Column (3) shows that within bank a 100-basis-point higher federal fund rate is associated with a significant and large increase in mail volume and dollar spent. The point estimate is about 11%.

In Appendix B we show that the point estimates drop to 9% when excluding the Chase, but remain large and statistically significant (Table A1), and increase to 17%, when focusing on the largest campaign within each bank-time pair (Table A2) . The point estimates are also slightly larger at about 14% when considering the cumulative effect of four months lags in the federal fund rate (Table A3). Finally, we find that our baseline estimates in Table 2 are driven by the largest banks (who pay essentially zero rates on checking and savings accounts). Table A4 shows that when the federal fund rate is 100-basis-points higher, the majority of large banks increase their direct mail volume by about 14% (relative to our average estimate of 12%), while a few high-yield banks do not change their marketing effort (i.e., the interaction term is negative and equal in magnitude to the direct effect). We summarize these new findings on marketing via direct mail as follows.

Fact 3: Marketing. *The average (median) bank in our sample sends about 6.5 (2) million mail offers for checking and saving accounts per month. A 100-basis-points higher federal fund rate is associated with a higher volume of mail offers for checking and saving accounts by 11-17%. The effect is driven by the banks that offer an interest rate close to zero on checking and savings accounts.*

3.4 Targeting

Finally, we look at how banks target their marketing effort at different levels of the federal fund rate. We construct two simple proxies for targeting. First, we look at the number of unique states each campaign is sent to. Second, we construct target groups as triplets of state, income, and age bins and compute for each campaign-month the volume of mails sent to each group. We then sum across groups to compute the Herfindahl-Hirschman Index (HHI) at the campaign-month level as a measure of concentration in certain target groups. A campaign with an HHI of one is sent to only a specific target group (e.g., young households with low income living in California).

Figure 2d shows that the HHI varies with the level of the federal fund rate. In 2015, when the federal fund rate is around zero, the HHI was around 0.15. As the federal fund rate rose to more than 2% in 2019, the HHI declined to 0.10, before increasing to more than 0.20 in 2021 as the federal fund rate declined in response to the COVID-19 pandemic. Following the recent monetary policy tightening in 2022, the HHI declined again to around 0.12.

We provide additional evidence of time-varying targeting in Appendix B. Figure A2a reports the fraction of households receiving any mail for checking or saving offers relative to the total number of respondents. During the year 2020, only 5-7% of households received an offer, while following the monetary policy tightening by 2023, one in five households received an offer for a checking or saving product. Figure A2b exploits the split of offers into acquisition (entirely new customer for the bank) vs retention (existing customer solicited for a new product), as discussed in Section 2. Mail volume for acquiring new customers for checking and saving products strongly comoves with the federal fund rate, while mail volume for retaining existing customers is acyclical. We interpret this as evidence that banks target specific depositors when rates are low (perhaps with cross-selling strategies), while they expand their reach when rates are high.

Columns (7) and (8) of Table 2 show the results of equation (1) using our main proxies for targeting as dependent variables. Column (7) shows that within bank a 100-basis-point higher federal fund rate is associated with a significant decline in the HHI. The point estimate is 2 percentage points, which corresponds to a 8% decline relative to the average HHI. Column (8) shows that within bank a 100-basis-point higher federal fund rate is also associated with a significant increase in the number of different states where banks send mail offers. The average campaign is sent to 10 different states, and when the federal fund rate increase by about 175 basis points the campaign is sent to a new state.

In Appendix B we show that the point estimate for the HHI is remarkably stable. The point estimate for the number of states drops when excluding the Chase, but remains large and statistically significant, and increases when focusing on the largest campaign within each bank-time pair. Finally, we find that our baseline estimates in Table 2 for both the HHI and the number of states are driven by the largest banks paying essentially zero rates on checking and savings accounts (Table A4). We summarize these new findings on targeting as follows.

Fact 4: Targeting. *The average within-campaign HHI in our sample is 0.25, which corresponds to a relatively high level of concentration. A 100-basis-points higher federal fund rate is associated with a 2-percentage-points lower concentration of mail offers, which corresponds to a 8% decline relative to the average. Large banks expand their customer acquisition unconditionally when rates are high, and contract their effort conditionally (i.e., focusing on specific groups) when rates are low.*

3.5 Depositors Switching

Our new facts so far focus on variables that are under the direct control of banks, such as rates and sign-up bonuses on checking and savings accounts, marketing volume, and targeting. We now look at depositors switching, which is the result of both banks' efforts (via sign-up bonuses, marketing, etc.) and depositors' actions. We capture the latter using three surveys from Comperemedia in 2021, 2022 and 2025 and the answer to the following question: *"For which of the following financial products have you switched your primary provider in the past 12 months?"*.[7] The survey are three repeated cross-sections, which prevent us from studying the behavior of the same depositor over time. However, the two most recent surveys also included a question about future intention: *For which of the following products would you consider switching to an account with a new primary provider?*.

Table 3 shows that survey respondents who, in the last 12 months, changed their primary provider, are more likely to consider switching again. This is true for both checking (column (1)) and saving (column (3)) accounts and the point estimates are large in magnitude. Columns (2) and (4) show

[7]Note that switching the primary provider for a checking or saving account does not require closing the existing accounts.

Table 3: SWITCHING OF CHECKING AND SAVINGS ACCOUNTS

	Checking		Saving		
	(1)	(2)	(3)	(4)	(5)
	Plan to switch	Plan to switch	Plan to switch	Plan to switch	Switched
Switched checking in past 12 months	0.43***	0.33***			
	(0.02)	(0.02)			
Switched saving in past 12 months			0.42***	0.35***	
			(0.02)	(0.02)	
Value interest					0.28
					(2.11)
Value bonus					6.00***
					(2.24)
YEAR F.E.	No	Yes	No	Yes	Yes
STATE F.E.	No	Yes	No	Yes	Yes
CONTROLS	No	Yes	No	Yes	Yes
DEP. VAR. (MEAN)	33.18	33.18	33.11	33.11	31.60
R^2	0.15	0.22	0.14	0.20	0.20
OBSERVATIONS	3,342	3,342	2,809	2,809	2,044

Note: The Table shows the results of equation (1). Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

that the correlation between past switching and future switching intentions is robust to controlling for observable demographics, suggesting that there are unobserved individual characteristics (to us and potentially to the bank) that might be driving switching behavior. Finally, column (5) of Table 3 shows that when switching saving products, respondents value the sign-up bonus more than the interest rate, whose coefficient is insignificant and small in magnitude.[8]

We then study the relation between the level of the federal fund rate and the fraction of households switching their primary checking or saving account provider. We compute two measures of switching. Our lower measure considers households who switched their primary checking and/or savings account without switching additional products. Our upper measure includes households switching checking or saving accounts regardless of whether the household switched any other products. Consider as an example a household that switches mortgage providers and checking account providers in the same month. Our lower measure excludes this household, while our upper measure includes it.

Figure 4 shows that as the federal fund rate increased from close to zero in 2021 to about 5% in 2023 the fraction of households switching their primary checking or saving account provider also

[8]The question from the survey is: *Which of the following would encourage you to switch to a new financial service provider? Please select all that apply.* The two options we focus on are: *Improved rewards program (e.g., higher rewards earned on credit cards, higher APY on savings)* and *Sign-up bonus or acquisition incentive*.

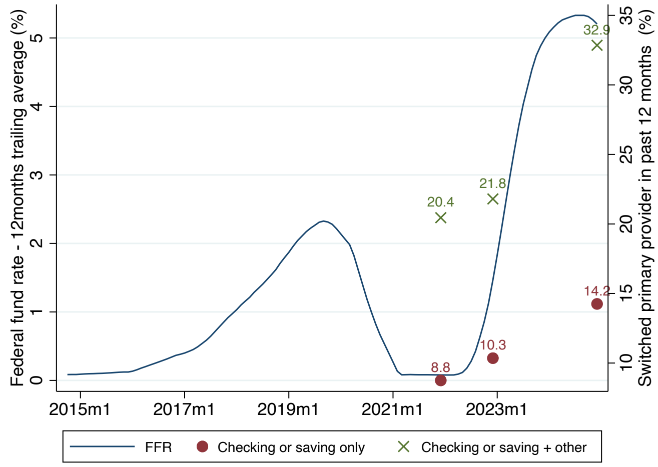

Figure 4: Switching of Checking and Savings Accounts

Note: The figure shows the 12-months trailing average of the federal fund rate and the fraction of respondents who switched their primary provider in the last 12 months. We report both the fraction that switched checking and saving accounts only and the fraction that switched checking and saving accounts plus any other product (e.g., mortgages, credit card). The Table shows the results of equation (1) estimated on our survey data. The dependent variables is the fraction of respondent who switched their primary provider in the past 12 months.

increased. Our time-series variation is limited to five data point.[9] For the sake of completeness, we nevertheless regress the average fraction of households switching their main checking or saving account provider on a 12-months trailing average of the federal fund rates. We find that a 100-basis-points higher federal fund rate is associated with one-percentage-point more switching when looking at checking and savings only, and 2.6-percentage-points when looking at checking and savings plus other products. In both cases the effect corresponds to about 10% of the respective mean. While the point estimate is identified from only very limited time-series variation across survey waves, the direction of the relation is in line with work using detailed individual-level data on account openings and closures by Egan et al. (2025).

We emphasize that our estimates in Figure 4 do not represent an elasticity of depositor demand to the deposit rate. Indeed we show that the deposit rate on checking and saving accounts barely moves with the federal fund rate (Fact 1). However, banks adjust other margins as the federal fund rate changes (Facts 2, 3 and 4). Hence the higher depositor switching rate at higher levels of the federal fund rate (Fact 5) might well be caused by higher interest rates without being caused by higher *deposit* rates. We summarize these new findings on switching as follows.

Fact 5: Switching. *Survey respondents who recently switched their checking or savings accounts are*

[9] Our data are from three surveys. The first two surveys were conducted in two different months and hence generate two data points each.

more likely to consider switching again in the future, and when switching savings products, they value higher sign-up bonuses over higher rates. A 100-basis-point higher federal fund rate is associated with 10% more switching for checking and savings accounts.

4 Model

In this section, we develop a model that speaks to the facts presented in the previous section. The primary purpose of this model is to explain why banks compete on the extensive margin (mailing more bonus offers) instead of on price margins (either by raising bonuses or by offering non-zero interest rates) as interest rates rise.

After developing the model, we will consider (1) its implications for the relationship between interest rates and franchise value and (2) the model's explanation for why bonuses are low relative to the net present value of net interest margins.

On the latter point, our model incorporates two possible explanations for this gap:

1. Banks could have a large amount of market power.

2. Bonus offers could be subject to adverse selection, in that depositors who respond to the bonus offer are far more likely to subsequently switch away from the bank than the average depositor.

The goal of our subsequent quantitative analysis is to determine the relative importance of these two possibilities.

We model competition within a single product/market. There are a continuum $j \in [0, 1]$ of banks and a continuum of depositors $i \in [0, 1]$. The depositors can be of one of two fixed types, which we will call type S ("sleepy") and A ("active"). There is a mass μ_S of sleepy depositors and a mass $\mu_A = 1 - \mu_S$ of active depositors. The type of a depositor is known to the depositor but not the bank, and we assume for simplicity that it is fixed over time.

Time is continuous. At each time t, each depositor will hold a fixed amount of deposits w at a single bank. The bank will earn an interest rate r_t by investing these deposits. For now, we will assume that banks pay no interest to depositors, and justify this assumption in an extension below. Banks can offer sign-up bonuses $b_{jt} \geq 0$ to a fraction $\alpha_{jt} \in [0, 1]$ of the population of depositors to induce them to switch their deposits. These bonuses are proportional to the amount deposited, and are not available to existing customers of the bank.

The cost to bank j of offering a bonus to a fraction α_{jt} of depositors is proportional to α_{jt}. The constant of proportionality depends on the bank's ability to target its offers. A bank that is unable to target its offers faces a flow cost $\chi\alpha_{j,t}dt$. A bank that is perfectly able to target its offers to the portion of the population that will consider the offer pays a flow cost $\chi\alpha_{j,t}\psi^* A(\cdot)dt$, where $A(\cdot)$ is the fraction of the population that will attend to the offer and ψ^* is a constant, both defined below. In our analytic results, we assume perfect targeting, which allows the model to generate sharper results at the expense of missing certain features of the data. In our quantitative analysis, we will consider the opposite case with no targeting, and find that results similar to those of our analytic model.

In each instant dt, a fraction $\psi_A dt$ and $\psi_S dt$ of active and sleepy depositors will consider switching banks (be "awake"). Each of these depositors will choose between their current bank j and one other bank j', drawn in an endogenous way from the set of banks (excluding the depositors' current bank). If the depositor received a bonus offer from bank j', they will take the bonus $b_{j't}$ into account. If they did not receive a bonus offer, they will treat that bank as-if they received a bonus offer of zero. Whether a particular depositor awakens is independent across time and depositor. The active depositors are awake more often than the sleepy depositors, $\psi_A > \psi_S$.

Depositors experience utility from consumption. As a simplification, we will assume depositors are ex-ante identical and that they consume the bonus payments they receive, so that the deposit wealth w_{it} remains constant, $w_{it} = w$ for all t. This simplification avoids the need to track a changing distribution of wealth across time. We will assume that the depositors' rate of time preference is equal to the interest rate, r_t, and that their utility is linear in their bonus income.[10] Implicitly, we have in mind that borrowers keep only the wealth required for transactions in deposit form, and that they are at least as impatient as the owners of the bank. This interpretation can also explain why the depositors' wealth is not sensitive to interest rates—depositors who are sensitive to interest rates have already minimized the share of their wealth in non-interest-bearing accounts.

We will study a Markov equilibrium with time-variation in the log short rate banks earn on their investment, r_t. Let $\mathbb{Q}$ denote the bank's risk-neutral measure. This measure, as opposed to the physical measure $\mathbb{P}$, will be the relevant measure in most of what follows. We assume that this rate is generated from a reflected diffusion factor model with K factors $f_t \in \mathcal{I} \subset \mathbb{R}^K$ that follow a continuous Markov process under $\mathbb{Q}$. That is, we assume

$$df_t = \Phi(f_t - \bar{f})dt + \Lambda \cdot dW_t + \gamma(f_t)dL_t,$$

where W_t is a K-dimensional Brownian motion under $\mathbb{Q}$, Λ is a $K \times K$ matrix of diffusion coeffi-

[10] Our results would extend readily to the case in which depositors are less patient than the bank. If depositors were instead more patient than the bank, there would be gains from paying them slowly over time via interest rates instead of upfront via bonuses.

cients, and Φ is a Hurwitz-stable matrix (so that a steady state distribution exists). The coefficients $\gamma(f_t)$ are an inward-pointing vector on $\partial\mathcal{I}$ and L_t is a non-decreasing process, increasing when $f_t \in \partial\mathcal{I}$ so that f_t remains in $\mathcal{I}$. The short rate $r_t = r(f_t)$ is a Lipschitz-continuous function of f that is bounded away from zero ($\inf_{f\in\mathcal{I}} r(f) > 0$). This structure allows for a flexible model of the term structure while avoiding the need for additional technical assumptions.

Let $Q_t \in \Delta(\{A, S\} \times [0, 1])$ be the density of active and sleepy households at each bank $j \in [0, 1]$ at time t, and let $Q_{Ajt} = Q_t(A, j)$ and $Q_{Sjt} = Q_t(S, j)$ be the population at bank j. The population density Q_t and the term structure factors f_t are the aggregate state variables of the model.

We will construct a symmetric Markov equilibrium in which the bonuses b_{jt} and offer volumes α_{jt} are functions of the factors f_t only and $Q_{Xt}(j) = \mu_X$ for $X \in \{A, S\}$. The key to this construction is that each bank is infinitesimal, and treats the population of potential depositors at other banks as fixed. This rules out strategic interactions that are in our view second-order.[11]

We will first describe the depositor's problem, and then the bank's problem.

4.1 The Depositor's Problem

Depositors awaken each period with probability ψ_X, where $X \in \{A, S\}$ is the depositor's type. When awake, they make their switching decision in two stages. First, they select a bank that is not their current bank to consider. Then, they decide whether or not to switch from their current bank to the selected bank.

In the equilibria we study, from the perspective of a individual depositor, all banks are identical apart from the bonuses they offer. For this reason, even though in principle the depositor's problem is forward-looking, nothing in the choices the depositor makes today affects their future opportunities. The depositor's problem can thus be re-cast as a static optimization. In the main text, we summarize the equations that determine depositor choice; their derivations can be found in Appendix Section C.

Suppose a borrower is considering switching from bank j to bank j' and has been offered a bonus of $b \cdot w$. If the borrower switches, she has to pay a switching cost $\delta \cdot w$ and will enjoy a random utility (in monetary units) of $\sigma\epsilon_{ij't}$ per unit wealth. If the borrower stays, they will instead enjoy the random utility $\sigma\epsilon_{ijt}$. The random utilities ϵ_{ijt} and $\epsilon_{ij't}$ are type-1 extreme value distributed and I.I.D. across agents, banks, and time. Let $D(b)$ denote the probability of switching to bank j' under these

[11]One example is the following: each sleepy depositor the bank attracts makes the future population it could attract slightly more likely to be active, which slightly reduces the value of attracting a sleepy depositor. We eliminate this effect by making the bank infinitesimal, so that it treats the population of depositors at other banks as exogenous.

circumstances. Given our assumptions,

$$D(b) = \frac{\exp(\sigma^{-1}(b-\delta))}{1+\exp(\sigma^{-1}(b-\delta))}. \tag{2}$$

Let us now consider how the depositor selects a bank j' to consider. Suppose that two banks have offered bonuses b and b'. We assume a functional form that implies that the relative likelihood of the depositor considering these two banks is $(1-D(b))^{-\sigma\theta}/(1-D(b'))^{-\sigma\theta}$.

The intuition behind this functional form is that borrowers are more likely to consider banks that offer them higher expected utility (again, see Appendix Section C for details). The parameter θ controls the extent to which this is the case. When $\theta = 0$, borrowers are equally likely to consider all banks. In this case, banks trying to attract new depositors must compete only against the depositor's current bank, which cannot offer a bonus but benefits from the switching cost. When $\theta = \sigma^{-1}$, the model is equivalent to a standard logit model with a continuum of choices, and features a constant elasticity. As θ becomes arbitrarily large, to be considered, a bank must offer more value (i.e. a higher bonus) than any other bank. The parameter θ can thus be understood as indexing the degree to which depositors consider a single bonus offer or multiple competing bonus offers.

Now suppose that one bank has offered a bonus of b, a share α^* of the other banks have offered a bonus b^*, and the rest have not sent bonus offers (which is equivalent to offering a bonus of zero). Under these circumstances, the PDF that defines the likelihood of the depositor considering the bank that offered a bonus of b is

$$A(b, \alpha^*, b^*) = \frac{(1-D(b))^{-\sigma\theta}}{\alpha^*(1-D(b^*))^{-\sigma\theta} + (1-\alpha^*)(1-D(0))^{-\sigma\theta}}. \tag{3}$$

If the bank offering b chooses to send that offer to a fraction α of potential depositors, the average switching density (i.e. the demand curve) will be

$$D(\alpha, b, \alpha^*, b^*) = \alpha D(b) A(b, \alpha^*, b^*) + (1-\alpha) D(0) A(0, \alpha^*, b^*). \tag{4}$$

This demand curve is the key determinant of bank behavior (i.e. the bank's choice of (α, b)), which is the main focus of our model. Note that sleepy and active households have the same demand curves conditional on being awake; they differ only in their probability of being awake.

4.2 A Bank's Problem

Now consider the problem of the bank in a Markov (in (f_t, Q_t)) equilibrium in which the policies of other banks are to offer bonuses $b^*(f_t, Q_t)$ to a randomly selected share $\alpha^*(f_t, Q_t)$ of the depositors. The density of type X at bank j evolves as

$$dQ_{Xjt} = \underbrace{\mu_X \psi_X D(\alpha, b, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))dt}_{\text{New Depositors}} - \underbrace{Q_{Xjt}\psi_X D^*(f_t, Q_t)dt}_{\text{Departing Depositors}}, \tag{5}$$

where b is the bonus set by bank j, α is share of potential depositors of bank j receiving offers, and

$$D^*(f_t, Q_t) = D(\alpha^*(f_t, Q_t), b^*(f_t, Q_t), \alpha^*(f_t, Q_t), b^*(f_t, Q_t)).$$

On the path of the symmetric equilibrium, this density will be constant and equal to μ_X.

We assume that the bank earns the interest rate r_t and has no ongoing costs associated with the deposit account. Implicitly, we treat the costs of branches, employees, mobile apps, etc... as sunk, and focus on the value of a marginal depositor. Despite this assumption, our model will feature two distinct notions of franchise value: the value of the current and future population and the value of a marginal depositor attracted from another bank. The latter is relevant for the bank's bonus decision, while the former determines the overall value of the bank.

The bank's problem is to choose processes for $\alpha_t \in [0, 1]$ and $b_t \in \mathbb{R}_+$ to maximize the net presentive value of net interest margins less customer acquisition costs. Let Θ be the set of admissible policies.[12] Let $\psi^* = \sum_{X\in\{A,S\}} \psi_X \mu_X$ be the population-average rate at which depositors awaken.

[12] I.e. let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathbb{Q})$ be the filtered probability space satisfying the usual conditions that describes the bank's information set, with the exogenous state variables (f_t, Q_t) and initial conditions (Q_{Aj0}, Q_{Sj0}) adapted to this filtration. The set of admissible controls is the set of processes for $\alpha_t \in [0, 1]$ and $b_t \in \mathbb{R}_+$ that are progressively measurable with respect to this filtration.

The problem of the bank in the Markov equilibrium is

$$
\begin{aligned}
V^B(f_0, Q_{Aj0}, Q_{Sj0}, Q_0) = \sup_{\{\alpha_t, b_t\}_{t\in\mathbb{R}_+}\in\Theta} & \underbrace{w\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} r(f_t)(Q_{Ajt} + Q_{Sjt})dt]}_{\text{NPV of Net Interest Margin}} \\
& \underbrace{-\chi\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds}\alpha_t(1-\omega+\omega\psi^* A(\bar{b}, \alpha^*(f_t, Q_t), b^*(f_t, Q_t)))dt]}_{\text{NPV of Bonus Marketing Costs}} \\
& \underbrace{-w\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds}\psi^*\alpha_t b_t D(b_t) A(b_t, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))dt]}_{\text{NPV of Bonus Payments}},
\end{aligned}
\tag{6}
$$

subject to (5).

Bonuses are paid only to depositors who have received bonus offers, and not to all new depositors.[13] The bank pays a cost of advertising that is proportional to the quantity of depositors it advertises bonuses to, α. The constant of proportionality depends on the scale parameter χ and bank's ability to target offers, as measured by the parameter $\omega \in [0, 1]$. When the bank cannot target offers, $\omega = 0$, and it must pay a cost per household mailed regardless of whether that household will consider the bank's offer. When $\omega = 1$, the bank is able to perfectly predict which households would consider a bonus offer of $\bar{b}$ (a fixed reference level), and mail only those households. Note that the cost is proportional to the attention paid given a bonus offer of $\bar{b}$, rather than the actual bonus offer. We make this assumption to avoid the implication that the bank should consider how changing its bonus offer affects its ability to target offers when choosing the optimal bonus level.[14]

Because of the linearity of the bank's problem, we can decompose the value function into the value of the existing depositors and the value of new depositors.

Lemma 1 *The value function satisfies*

$$
V^B(f_t, Q_{Ajt}, Q_{Sjt}, Q_t) = \underbrace{V_0^B(f_t, Q_t)}_{\textit{Value of New Depositors}} + \underbrace{Q_{Ajt}V_A^B(f_t, Q_t) + Q_{Sjt}V_S^B(f_t, Q_t)}_{\textit{Value of Existing Depositors}}, \tag{7}
$$

[13] This is consistent with the observation that many bonus offers in our dataset have coupon codes or other mechanisms to restrict the offer to the depositor who received the mailing. There are, however, some banks who make their offers available to all potential depositors, by e.g. advertising the offer on a poster outside the bank.

[14] In principle, such an effect could exist; a bank might know more about the marginal borrower who would consider a \$200 bonus offer than the one who would consider a \$400 bonus offer. Our view is that such effects are not likely to be quantitatively significant and omitting them simplifies our analysis.

where the marginal values $V_X^B(f_t, Q_t)$ satisfy

$$V_X^B(f_t, Q_t) = w\mathbb{E}_t^{\mathbb{Q}}[\int_t^\infty e^{-\int_t^s (r(f_u)+\psi_X D^*(f_u,Q_u))du} r(f_s)ds]. \tag{8}$$

and

$$\begin{aligned} V_0^B(f, Q_t) = & \sup_{\{\alpha,b\}\in\Theta} \\ & \underbrace{\sum_{X\in\{A,S\}} \psi_X\mu_X\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} D(\alpha_t, b_t, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))V_X^B(f_t, Q_t)dt|f_0 = f]}_{\textit{Value of Future New Depositors}} \\ & - \underbrace{\chi\psi^*\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds}\alpha_t A(\bar{b}, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))|f_0 = f]}_{\textit{Future Fixed Costs of Acquiring New Depositors}} \\ & - \underbrace{w\psi^*\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds}\alpha_t b_t D(1, b_t, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))dt|f_0 = f]}_{\textit{Future Bonus Payments to New Depositors}}. \end{aligned}$$

Proof. See Appendix Section D.1. ■

The intuition behind the marginal values $V_X^B(\cdot)$ is straightforward. Attracting a depositor allows the bank to earn the interest spread $r(f)$. This benefit is discounted with the effective discount rate $r(f) + \psi_X D^*(f, Q)$, which reflects the interest rate and the rate at which the depositor will leave. Active depositors leave at a higher rate than sleepy depositors ($\psi_A > \psi_S$), and hence are less valuable ($V_A^B(\cdot) < V_S^B(\cdot)$).

The weighted-average value of a marginal depositor (scaled by the size of the deposit) can be defined as

$$MV(f_t, Q_t) = \frac{\sum_{X\in\{A,S\}} \psi_X\mu_X V_X^B(f_t, Q_t)}{w\psi^*}. \tag{9}$$

Note that active types are over-represented in this weighted average, relative to their population share (i.e. the weights are $\psi_X\mu_X$, not μ_X). This over-representation of less-valuable active types among marginal depositors is how the model captures adverse selection.[15]

The optimal (but possibly negative) value of b conditional on offering a bonus, $b^+(MV)$, can be

[15]The population of awake depositors is adversely selected relative to the entire population, but there is no adverse selection within the population of awake depositors (active and sleepy types make identical decisions conditional on being awake).

derived from the first-order condition as

$$b^{+}(MV) = \underbrace{MV}_{\text{Marginal Value}} - \underbrace{\frac{D(1, b^{+}(MV), \cdot, \cdot)}{\frac{\partial}{\partial b} D'(1, b, \cdot, \cdot)|_{b=b^{+}(MV)}}}_{\text{Optimal Markdown = Inverse Elasticity}} \tag{10}$$

$$= MV - (\sigma^{-1}(1 - D(b^{+}(MV))) + \theta D(b^{+}(MV)))^{-1}.$$

The optimal bonus (when positive) is equal to the value of a marginal depositor (a weighted average of V_A^B and V_S^B) and the optimal markdown, which is the inverse of the demand elasticity.

The demand elasticity with respect the bonus offer is a weighted average of σ^{-1} and θ. When the bonus offered is less than the switching cost δ, $D(b) < \frac{1}{2}$, and the demand elasticity is driven primarily by the depositor's choice between the considered bank and their current bank, as captured by the elasticity σ^{-1}. When the bonus is greater than δ, $D(b) > \frac{1}{2}$, and the key elasticity is instead the comparison between the considered bank and other banks that might have been considered, as captured by the elasticity θ. In a limit in which both σ^{-1} and θ become infinitely large, the depositor will always consider only the largest bonus offer and switch if and only this offer exceeds the switching cost. This case captures the perfect competition limit, and the optimal bonus in this case converges to the marginal value.

Now consider the first-order condition for the offer volume α, given the optimal bonus b^*. Let α^* denote an optimal policy. If this optimal policy is interior ($\alpha^* \in (0,1)$), and all other banks employ the strategy $(\alpha^*(f_t, Q_t), b^*(f_t, Q_t))$, we must have

$$0 = \underbrace{(MV(f_t, Q_t) - b^*)(\frac{D(b^*)A(b^*, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))}{D(0)A(0, \alpha^*(f_t, Q_t), b^*(f_t, Q_t)))} - 1)}_{\text{Marginal Value of Additional New Depositors per Inframarginal Depositor}}$$

$$- \underbrace{b^*}_{\text{Cost of Bonuses to Inframarginal New Depositors}} - \underbrace{\frac{\chi}{w} \frac{1 - \omega + \omega \psi^* A(\bar{b}, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))}{\psi^* D(0) A(0, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))}}_{\text{Marginal Marketing Cost per Inframarginal Depositor}}.$$

If $\alpha^* = 1$ (resp. $= 0$), the right-hand side must be ≥ 0 (resp. $\leq$). Using the structure of the $A(\cdot)$ function from equation (3), we can simplify this as

$$0 = h(b^*, MV, A_0) = (MV - b^*)(\frac{D(b^*)(1 - D(b^*))^{-\sigma\theta}}{D(0)(1 - D(0))^{-\sigma\theta}} - 1) - b^* \tag{11}$$

$$- \chi\omega \frac{(1 - D(\bar{b}))^{-\sigma\theta}}{wD(0)(1 - D(0))^{-\sigma\theta}} - (1 - \omega)\frac{\chi}{w\psi^* D(0) A_0}.$$

The bank trades off the benefit of acquiring additional customers against the costs of making the offer. The benefit consists of the value of the new depositors less the bonus paid, $w(MV - b^*)$, multiplied by the number of additional depositors attracted by the bonus. The latter value relative to quantity of inframarginal new depositors can be viewed as an elasticity. If this elasticity and the one in (10) are small, the markdown is large, and attracting new customers is profitable ($MV - b^*$ is large) but difficult (because the elasticity is low). If the elasticities are instead large, the optimal markdown is small, but it is possible to attract a significant share of new customers by offering bonuses.

The function $h(b^*, MV, A_0)$ is the marginal profit from sending an additional bonus offer to a depositor per inframarginal new depositors (i.e. the quantity of depositors who would come to the bank even with no bonus offer). It depends on the strategies of the other banks in two ways. First, it depends on $(\alpha^*(f_t, Q_t), b^*(f_t, Q_t))$ in the current period through the competition for inframarginal depositors' consideration choices, $A_0 = A(0, \alpha^*(f_t, Q_t), b^*(f_t, Q_t))$. If the bank cannot target its offers ($\omega = 0$), when others banks market aggressively, each offer the bank sends is less effective because it is less likely to be considered. Targeting can solve this problem; when $\omega = 1$, the bank can avoid this form of competition, and the strategies of the other banks in the current period are irrelevant. However, the marginal value of a new depositor still depends on other banks' future strategies, $(\alpha^*(f_{t+s}, Q_{t+s}), b^*(f_{t+s}, Q_{t+s}))$ for $s > t$, because those strategies determine the rate at which a newly acquired depositor will leave the bank in the future. Our model thus features "dynamic strategic substitutability," in that the more a bank expects other banks to aggressively market in the future, the less incentive the bank has to market today.

The following lemma summarizes some basic properties of the functions $b^+(\cdot)$ and $h(\cdot)$ that describe the first-order conditions.

Lemma 2 *(Properties of b^+ and h) $b^+(MV)$ is the unique solution to* (10)*, is strictly increasing, and satisfies, for all A_0,*

$$\max(b^+(MV), 0) = \arg\max_{b \geq 0} h(b, MV, A_0).$$

Proof. See Appendix Section D.2. ■

In our analytic model, we will assume $\omega = 1$, which is to say the bank can perfectly target its offers. In our quantitative model, with $\omega < 1$, the cost of mailing per inframaginal new depositor depends on the number of inframarginal new depositors, which in turn depends on the marketing strategies of the other banks in the current period. Through the lens of (11), it is as-if the cost of marketing per inframarginal depositor rises when other banks compete more aggressively.

The following lemma summarizes the necessary first-order conditions.

Lemma 3 *(Necessity and Sufficiency of the FOCs) Given* (f_t, Q_t), (α, b) *is optimal in the bank's HJB equation given* $(\alpha^*(f_t, Q_t), b^*(f_t, Q_t), MV(f_t, Q_t))$ *if and only if:*

- *if* $\alpha > 0$, *then* $b = \max\{b^+(MV(f_t, Q_t)), 0\}$,
- $h(b, MV(f_t, Q_t), A(0, \alpha^*(f_t, Q_t), b^*(f_t, Q_t)) > 0$ *only if* $\alpha = 1$, *and*
- $h(b, MV(f_t, Q_t), A(0, \alpha^*(f_t, Q_t), b^*(f_t, Q_t)) < 0$ *only if* $\alpha = 0$.

Proof. See Appendix Section D.3. ■

Building on this, we define the equilibria of interest. Observe that the best response to $(\alpha^*(f_t), b^*(f_t))$ (a policy that does not depend on Q_t) is a policy that does not depend on Q_t (because $V_X^B(\cdot)$ does not depend on Q_t in this case). We define a symmetric Markov equilibrium as a pair of functions $\alpha^* : \mathbb{R}^K \to [0, 1]$ and $b^* : \mathbb{R}^K \to \mathbb{R}_+$ such that, if all other banks are assumed to play the strategy (α^*, b^*), each bank has a best response of (α^*, b^*).

Building on the result presented thus far, the following lemma summarizes conditions that are sufficient to characterize a symmetric Markov equilibrium.

Lemma 4 *(Sufficient Conditions for an Equilibrium) The functions* (α^*, b^*) *are a symmetric Markov equilibrium if, for all* $f \in \mathbb{R}^K$,

$$(\alpha^*(f), b^*(f)) \in \sup_{\alpha \in [0,1], b \geq 0} \alpha h(b, MV(f), A(0, \alpha^*(f), b^*(f)),$$

where

$$MV(f) = \mathbb{E}^{\mathbb{Q}}[\int_0^\infty \sum_{X \in \{A,S\}} \frac{\mu_X \psi_X}{\psi^*} e^{-\int_0^t (r(f_s) + \psi_X d^*(f_s) ds} r(f_t) dt | f_0 = f]$$

and

$$d^*(f) = D(\alpha^*(f), b^*(f), \alpha^*(f), b^*(f)).$$

Proof. See Appendix Section D.4. ■

This lemma can be thought of as stating: if the policies are optimal given the marginal depositor value and the marginal depositor value is consistent with those policies, then the policies are an equilibrium.

4.3 Analytic Results

We next provide some analytic results under special-case assumptions. The purpose of these results is to clarify the intuition of our model and our subsequent quantitative results. Our first assumption is that banks can perfectly target their bonus offers.

Assumption 1 *(Perfect Targeting)* $\omega = 1$.

This assumption is equivalent to the statement that the bank can target its offer only to those who might pay attention. The key consequence of this assumption is that the $h(b, MV, A_0)$ function does not in fact depend on A_0 or the parameter $\bar{b}$. That is, the net benefit to the bank of sending mail depends on the marginal value of acquiring a depositor but not on the volume of mail sent by other banks, because the bank is able to target its mail to those who will attend to its offers. We relax this assumption in our quantitative analysis.

Given this assumption, there is a minimum marginal value, MV_{min}, such that if marginal values fall below this level, bonuses will never be offered in equilibrium.

Lemma 5 *(Uniqueness and Positivity of MV_{min}) Under Assumption 1, There is a unique strictly positive value of MV, MV_{min}, with $b^+(MV_{min}) = b_{min} > 0$, that satisfies*

$$h(b^+(MV_{min}), MV_{min}, 1) = 0.$$

If $MV(f_t) < MV_{min}$ (resp. $>$), then $h(b^+(MV(f_t)), MV(f_t), 1) < 0$ (resp. $>$).

Proof. See Appendix Section D.5. ■

If $MV_{min} \geq 1$, bonuses will never be offered in equilibrium. Our second assumption rules out this possibility.

Assumption 2 *(Bonuses are Sometimes Positive)* $MV_{min} < 1$.

This assumption can be expressed in terms of the primitive parameters $(\sigma, \theta, \delta, \chi, w)$, in which case it is equivalent to the assumption that χ (the cost of sending offers) is not too large relative to the other parameters. This ensures it is profitable to send bonus offers if the marginal value of a new depositor is sufficiently large.

Note that, combining Lemmas 2 and 5, it follows that a bonus less than $b_{min} > 0$ will never be offered in equilibrium. Low bonuses imply low marginal values ($b^+(MV)$ is increasing in MV), and for sufficiently low marginal values the fixed costs do not justify sending offers.

We next define a constant λ, which is a transformation of MV_{min}. This constant will in equilibrium be related to the ratio of the departure rate to the interest rate, when interest rates are close to their long-run average values.

Lemma 6 *(Uniqueness and Positivity of λ) Under Assumption 2, there is a unique value of $\lambda > 0$ that satisfies*

$$MV_{min} = \sum_{x \in \{A,S\}} \frac{\mu_X \psi_X}{\psi^*} \frac{1}{1 + \lambda \psi_X}.$$

Proof. See Appendix Section D.6. ■

Our third assumption is that $K = 1$ (a one-factor term structure model), which simplifies the proof of our analytic results; we will consider a multi-factor term structure model in our quantitative analysis.

Assumption 3 *(One-Factor Term Structure) $K = 1$, $\mathcal{I} = (f_-, f_+)$ for some $f_+ > f_- > 0$, and $r(f) = f$.*

Our fourth and final assumption is that, at the long-run average level of interest rates, banks would want to send bonus offers if no other banks were sending bonus offers, but would not want to send bonus offers if all other banks were sending bonus offers. We will refer to the region of interest rates at which this situation arises as the region of "extensive margin competition," for reasons that will become clear below. The existence of such a region is a general feature of our model, in the sense that we expect this kind of situation to occur for some value of interest rates under most parameters.[16] The content of our assumption is that the relevant region in which this situation occurs contains the long-run interest rate. This assumption simplifies our proof by anchoring the region of the extensive margin competition in a known location; we do not impose it in our quantitative results. We also assume that the lowest possible rate is sufficiently low and the highest possible rate is sufficiently high, which ensures that the region of extensive margin competition is not the entire range of possible rates.

Assumption 4 *(Extensive Margin Competition at Long-Run Rates)*

$$\lambda f_- < D(0) < \lambda \bar{f} < D(b_{min}) < \lambda f_+.$$

Armed with these assumptions, we can characterize an equilibrium of our model. We first do so in the deterministic case ($\Lambda = 0$), in which interest rates gradually revert towards $\bar{f}$. This case provides "sharp" results, in that we can exactly characterize the region of extensive margin competition.

Theorem 1 *Under Assumptions 1-4, if $\Lambda = 0$, there is a symmetric Markov equilibrium in f_t in which:*

[16] We have not proven this, and we believe it may not hold when the interest rate volatility Λ is sufficiently high.

- *If* $r(f_t) < \lambda^{-1}D(0)$, $\alpha^*(f_t) = 0$.
- *If* $r(f_t) > \lambda^{-1}D(b_{min})$, $\alpha^*(f_t) = 1$ *and* $b^*(f_t) \geq b_{min}$.
- *If* $f_t \in [\lambda^{-1}D(0), \lambda^{-1}D(b_{min})]$, $b^*(f_t) = b_{min}$ *and* $\alpha^*(f_t) \in (0,1)$ *is the unique solution to*

$$D(\alpha^*(f_t), b_{min}, \alpha^*(f_t), b_{min}) = \lambda f_t.$$

Proof. See Appendix Section D.7. ■

The first result in this theorem says that if interest rates are low, then it is not worth offering bonuses, even if other banks are not offering bonuses and the departure rate conditional on a depositor being awake is $D(0)$. We will refer to this region as the "no competition" region. The second result says that if interest rates are high, it is always worth offering bonuses, even if all of the other banks are offering a bonus of at least b_{min} and hence the departure rate conditional on being awake is at least $D(b_{min})$. We refer to this region as the "intensive margin competition" region, because banks compete by varying the bonuses they offer with the level of rates. The third result says that if interest rates lie in an intermediate interval, bonuses will be constant and equal to b_{min}, banks will send offers to some but not all of the population, and the departure rate will vary linearly with the short-term interest rate. This is what we mean by the "extensive margin competition" region. This last result, in a stylized way, is what we find in our data.

To understand why this is an equilibrium, consider the problem of the bank when the interest rate is slightly above $\lambda^{-1}D(0)$. The bank finds it (weakly) profitable to send bonus offers. Now suppose it wants to attract some additional new depositors. Mailing offers to more borrowers (increasing α) has a constant marginal cost. In contrast, raising its bonus offer b has an increasing marginal cost– the higher bonus level must be paid not only to the additional depositors the bank attracts but also to all the depositors the bank would have attracted even at the lower bonus level. Put another way, mailing more offers is a linear cost technology for attracting new depositors, whereas raising bonuses is a convex cost technology for attracting new depositors. Starting from a point where the two marginal costs are equal, as the incentive to acquire more customers increases (i.e. rates rise), the bank will first exhaust its linear cost technology before turning to its convex cost technology.

This partial equilibrium logic explains why banks first respond on the extensive margin (mailing more offers) as rates rise, and only increase their bonus offers once they are sending offers to all potential depositors. If the departure rate of depositors were constant or an exogenous function of interest rates, the bank would instantly transition between sending no offers and sending offers to the entire population. However, because the departure rate endogenously depends on the extent to which other banks are sending offers, there is an additional general equilibrium consideration. When interest rates are above $\lambda^{-1}D(0)$ but below $\lambda^{-1}D(b_{min})$, if no other bank were sending offers ($\alpha^*(f_t) = 0$), an individual bank would strictly prefer to send offers to all depositors ($\alpha = 1$). Likewise, if all banks were sending offers ($\alpha^*(f_t) = 1$), an individual bank would strictly prefer not to send offers ($\alpha = 0$). In this sense, banks' actions are strategic substitutes. The only symmetric equilibrium in this case is if each bank sends offers to an intermediate share of the population.[17]

To sustain this equilibrium, the marginal value of a new depositor must lie exactly at the banks' indifference threshold, $MV(f_t) = MV_{min}$, and hence the bonus offer is $b(f_t) = b_{min}$. Lemma 6 shows that the marginal value will equal MV_{min} if $V_X^B(f_t) = (1 + \lambda\psi_X)^{-1}$. It is straightforward to show that this is consistent with the definition of the marginal values, (8), if $D^*(f_t, Q_t) = \lambda f_t$. That is, as interest rates rise, competition intensifies in response to rising rates, in a way that exactly offsets the increase in interest margins.

One striking aspect of the result is the observation that the degree of extensive margin competition ($\alpha^*(f_t)$) depends only on the level of the short rate and not the slope of the yield curve (as captured by the parameter Φ). Two versions of our model with different slopes of the yield curve (Φ) but all other parameters identical would have the same value of a marginal depositor, level of the bonus, and degree of extensive margin competition ($\alpha^*(f_t)$) in the intermediate region, and the boundaries of that region are also the same.[18] We confirm this finding in our multi-factor quantitative model, described in the next section.

Our model predicts that, in the intermediate region, the marginal value of a depositor is constant. This does not imply that franchise value is small, or even that it is constant, because the franchise value also includes the value of future depositors. The following corollary characterizes the portion of franchise value that comes from future depositors, under the assumption that interest rates always remain in the intermediate region.

Corollary 1 *Under Assumptions 1-4, if $\Lambda = 0$ and $f_t \in [\lambda^{-1}D(0), \lambda^{-1}D(b_{min})]$, the net present value of future new depositors, $V_0^B(f)$, is*

$$V_0^B(f_t) = w \int_t^\infty \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^s f_u du} \psi^* MV_{min} D(0) A(0, \alpha^*(f_s), b_{min})] ds$$

[17] By the same logic, there are also asymmetric equilibria in which α represents the share of banks sending offers rather than the share of population receiving offers.

[18] This is result can be contrasted with the result of models with fixed departure rates. With a fixed departure rate, the net present value of a depositor depends primarily on long-maturity discount rates (see remark 3 in Drechsler et al. (2023)).

Proof. See Appendix Section D.8. ■

The net present value of future depositors consists of the marginal value of those depositors less the costs of acquiring them, which includes both marketing costs and bonuses. In the intermediate region, the bank receives $\psi^* D(0)A(0, \alpha^*(f_t), b_{min})dt$ new depositors each instant "for free," even if it does not mail any bonus offers, and the value of these depositors is MV_{min}. The bank might choose to send out bonus offers, but the marginal benefit of doing so is exactly equal to the marginal cost, and as a result the value of future depositors can be computed as-if the bank did not send out bonus offers. It follows that if interest rates remain in the intermediate region forever, the value of future depositors is equal to the net present value of this flow.

An immediate consequence of this result is that, subject to the assumption of Corollary 1, the bank's overall franchise value is a *positive* duration asset. A persistent increase in interest rates both increases the discount rate and increases the volume of offers sent by other banks (α^*), which reduces the flow of "free" new depositors. In this equilibrium, the value of the bank's existing population of depositors has no interest rate sensitivity, and the value of the bank's future depositors decreases as interest rates increase.

We next show that equilibria in the stochastic case ($\Lambda > 0$) are qualitatively similar, provided that Λ is not too large. The formal results mirror the structure of Proposition 1, but are weaker in that we are silent about what happens at the "corners" when λf is close to $D(0)$ or $D(b_{min})$, and do not provide an explicit expression for α^*. The reason for these omissions is the presence of option-value-like effects near the transitions between no competition, extensive margin competition, and intensive margin competition. Nevertheless, we are able to show that the general structure of the equilibrium is similar to that of the deterministic case.

Theorem 2 *Under Assumptions 1-4, for any $\xi > 0$ such that*

$$\lambda^{-1}D(0) + \xi < r(\bar{f}) < \lambda^{-1}D(b_{min}) - \xi,$$

there exists a $\bar{\Lambda}_\xi > 0$ such that for all $\Lambda \in [0, \bar{\Lambda}_\xi)$, there is a symmetric Markov equilibrium in f_t in which

- *If $r(f_t) \le \lambda^{-1}D(0) - \xi$, $\alpha^*(f_t) = 0$.*
- *If $r(f_t) \ge \lambda^{-1}D(b_{min}) + \xi$, $\alpha^*(f_t) = 1$ and $b^*(f_t) \ge b_{min}$.*
- *If $r(f_t) \in (\lambda^{-1}D(0) + \xi, \lambda^{-1}D(b_{min}) - \xi)$, $b^*(f_t) = b_{min}$ and $\alpha^*(f_t) \in (0, 1)$.*

Proof. See Appendix Section D.7. ■

Discussion. It is useful to place these results in the context of the literature on deposit franchise value and its interest rate risk. With regards to the value of new depositors, our model is intermediate between Drechsler et al. (2023) (who assume the value of new depositors is zero) and DeMarzo et al. (2024) (who assume new and existing depositors are equally valuable); our quantification in Section 5 will be closer to Drechsler et al. (2023).

With regards to interest rate risk, the lack of rate sensitivity for the marginal value of a deposit is closer to the result of DeMarzo et al. (2024) than to Drechsler et al. (2023), but operates through a different mechanism. That mechanism can be thought of as what Drechsler et al. (2023) call "rate-driven outflows" (see also Xiao (2020)). In the intermediate rate region, such outflows arise endogenously through bank competition, as a result of banks sending more bonus offers in response to higher interest rates. Those authors instead emphasize an exogenous substitution between banks and money market funds.[19] These rate-driven outflows must be strong enough to cancel out the effect of rising rates, because otherwise banks would be tempted to advertise to more borrowers. If they did so, this would increase the quantity of rate-driven outflows, reducing the value of a new depositor (who might soon be stolen away by another bank). This reduction in value in turn reduces the incentive to advertise. Equilibrium is reached when the increase in rate-driven outflows induced by advertising exactly offsets the increase in value that comes from rising interest rates.

Finally, the literature on franchise value considers aspects of franchise value (e.g., costs, franchise value from lending) that our model does not address. Even if we restrict attention only to the deposit franchise value and exclude costs, it may well be the case that most of the deposit franchise value comes from high-balance corporate and individual accounts, as opposed to the mass-market retail accounts our model captures (the evidence of Argyle et al. (2025) points in this direction). Setting all of these issues aside, our intermediate equilibrium features a weak negative relationship between interest rates and deposit franchise value. The value of existing depositors is not interest rate sensitive and is likely to capture the lion's share of bank value. The value of future depositors is the only rate-sensitive component (it has positive duration), but it is a relatively small component of the overall value.

[19] Our baseline model does not include this kind of substitution. A straightforward extension of our model with exogenous departures to outside the banking system generates almost identical results (an intermediate equilibrium in which the total departure rate is linear in rates with coefficient λ and the marginal depositor values are constant), with a different mapping between the total departure rate and α.

4.4 Why Not Pay Interest?

Before turning to our calibration, we revisit the question of why banks offer bonuses instead of paying interest to their depositors. First, we should caveat that we view offers with temporarily high interest rates for new depositors as equivalent to bonus offers.[20] The question of this subsection is why banks do not offer non-trivial interest rates to their existing depositors, so as to prevent them from leaving.

We begin with the equilibrium described above, in which all banks are assumed not to pay interest, and consider the problem of a single bank who can both pay bonuses and pay interest.

We will allow the bank to commit, in one instant, to paying interest to its current depositors. The structure of this is as follows. At time t, the bank offers all of its current depositors an interest rate $\beta r(f_s)$ for all $s \in [t, t+\Delta)$, provided they remain at the bank. The time period length Δ is meant the capture the bank's limited ability to commit to offering interest.

We will assume this offer is not available to new depositors, as a simplification. As discussed above, offering some NPV of interest to new depositors is equivalent to a offering a bonus.[21]

If the depositor stays at the bank until time $t+\Delta$, the bank's offer is worth $\beta(1-e^{\int_t^{t+\Delta} r(f_s)ds})$ in NPV terms. Because the depositor might leave before earning interest all of this interest, the bank's offer is worth less than this amount. Let $\hat{D}(s, f, \alpha^*(f), b^*(f), \beta)$ be the departure rate in equilibrium given the offer β after s time has passed, noting that this potentially depends on the current f, because the depositor is receiving interest payments $\beta r(f)$.

If bank j contemplates offering interest at time zero, its problem is

$$
\begin{aligned}
&\hat{V}^B(f_0, Q_{Aj0}, Q_{Sj0}, Q_0) = \sup_{\beta \geq 0} V^B(f_0, Q_{Aj0}, Q_{Sj0}, Q_0) \\
&+ \sum_{X \in \{A,S\}} Q_{Xj0} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\Delta (r(f_s) + \psi_X \hat{D}(s, f_s, \alpha^*(f_s), b^*(f_s), \beta)) ds} V_X^B(f_\Delta) dt \right] \\
&- \sum_{X \in \{A,S\}} Q_{Xj0} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\Delta (r(f_s) + \psi_X \hat{D}(s, f_s, \alpha^*(f_s), b^*(f_s), 0)) ds} V_X^B(f_\Delta) dt \right] \\
&- w\beta \sum_{X \in \{A,S\}} Q_{Xj0} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\Delta e^{-\int_0^s (r(f_s) + \psi_X \hat{D}(s, f_s, \alpha^*(f_s), b^*(f_s), \beta)) ds} r(f_t) dt \right].
\end{aligned}
$$

[20]There are several examples of such offers in our data. However, they are far less common than offers a cash bonus. We speculate that a depositor with $10,000 to deposit might respond more strongly to a $200 cash bonus to a 4% higher annualized interest rate for six months, for behavioral or other reasons.

[21]We assume it is equally costly to advertise a bonus and a temporarily high interest rate. If the temporary rate is available only to depositors who receive an offer, it is equivalent to a bonus. A rate that is available to all new depositors is weakly dominated by a bonus, and strictly dominated if $\alpha_{jt} < 1$.

That is, the bank receives its usual equilibrium payoff, plus the benefit of a lower departure rate through time Δ, less the NPV of the promised interest payments it makes to depositors.

The following result provides a condition under which, for Δ sufficiently small, it will not be optimal to offer interest rates.

Proposition 1 *If* $\Delta^{-1} \geq \frac{\sum_{X\in\{A,S\}} \psi_X Q_{Xjt}}{\sum_{X\in\{A,S\}} Q_{Xjt}} \sigma^{-1}$, *it is optimal not to pay interest (*$\beta^* = 0$*).*

Proof. See Appendix Section D.9. ■

This expression is not the tightest possible bound; we employ it because it transparently captures the key forces. Specifically, the length of required commitment (Δ) to justify paying interest is decreasing in the elasticity (as captured by σ^{-1}) and in the attentiveness of the bank's current population. The role of elasticity in this expression is intuitive: the more sensitive depositors are to the interest rate offered, the shorter the length of commitment required to make offering them interest profitable.

The role of the population-weighted average of ψ_X highlights the key reason bonus payments to new depositors are superior to interest payments to existing depositors as a method of growing the deposit base. With bonus payments, some money is "wasted" on new depositors who would have switched to the bank even without a bonus. With interest rates for existing depositors, some money is "wasted" on existing depositors who would not have switched away from the bank even with zero interest. The former population is much smaller than the latter in the presence of inattention and/or switching costs. This can also be expressed in terms of markdowns and elasticities: the bank's current depositors are much less responsive to the offers of other banks than the bank's already awake potential depositors are to the bank's offers. Consequently, the optimal markdown on deposit rates is much larger than the optimal markdown on bonuses.

However, the strength of these effects can differ radically depending on the composition of the depositor base. A representative bank in our calibration will have a weighted-average value of ψ_X equal to $\psi^* \approx 0.1$ (quarterly). A bank with entirely active depositors, by contrast, would have a weighted-average value of one. Thus, a bank with entirely active depositors requires a commitment period that is less than one-tenth the commitment period required for a typical bank. We do not fully explore this kind of asymmetric equilibria in the present paper. However, we note that there are some pronounced differences in deposit rates between recent large entrants to the deposit market (e.g. Capital One, Discover, American Express, Ally) and more established competitors (Chase, Citibank, Wells Fargo, Bank of America). These differences might be explained by the newer entrants having a more active depositor base than their established competitors.

Lastly, we should note that our bound is derived by observing that the marginal value of a depositor considering leaving the bank must be less than w, which is to say that it implicitly considers the case in which the bank receives 100% of the value of a deposit (which would occur in our model only with infinite interest rates). In our quantitative analysis, we estimate the marginal value for a representative bank given typical interest rates as roughly one tenth of the deposit size. As a result, our bound is conservative by a factor of roughly ten for the typical bank, which is to say that the actual commitment period required to justify paying interest is substantially larger than the one given by our bound in most circumstances.

5 Quantitative Analysis

We next discuss the calibration of the equilibrium described in Proposition 1. Our calibration can be divided into two steps. To illustrate the role of adverse selection vs. market power, it is sufficient to calibrate a subset of the parameters of our model to match a relatively small number of moments. We will then consider what additional moments can tell us about the remainder of our parameters. We calibrate the model using our data on savings offers. We choose savings offers as opposed to checking offers because, consistent with the model, we believe that the primary source of income a bank derives from savings deposits is the interest it is able to earn on those deposits. For checking deposits, overdraft fees, debit interchange fees, and related types of income are potentially significant sources of revenue for the bank, even in a zero interest rate environment.

5.1 Simple Calibration

We will calibrate the model under the assumption that in 2024, interest rates were in the interior region described in Proposition 1, with $\alpha^*(\cdot) \in (0, 1)$ and $b = b_{min}$. This is consistent with our data, in particular Figure 2c, which shows that marketing volumes rose rapidly along with short-term interest rates, while the change in bonuses was more muted, as shown in Figure 2b. The average savings account bonus in our data at the end of 2023 is close to \$200. A common minimum balance required to earn the savings account bonus is \$15,000, which also happens to be close to the median savings account balance in the SCF. We set the wealth w to this value.

Table 4 Panel A lists the subset of moments we use to consider the question of adverse selection vs. market power, their associated data sources, and the interpretation of the moment in the context of the model. We use the 2024 Mintel data, combined with the average interest rate for 2024, to calibrate the model. The average Federal Funds rate for 2024, converted to a quarterly rate, was

1.26%. In the Mintel survey 2024, 32.9% of depositors reported switching their bank account in the last twelve months, which translates to a quarterly rate of 9.5%. We calibrate the half-life of a depositor relationship to sixteen years, consistent with the average length of a banking relationship reported in the SCF. This value will depend, of course, on the entire history of interest rates; when we translate this half-life in the context of the model, we assume that this half-life is consistent with the level of interest rates observed in 2024. Lastly, we assume the half-life of an active depositor is one quarter, consistent with the practice for savings accounts of requiring the balance to be maintained for that time period to receive the bonus. We test the robustness of our main result against different assumptions about these variables below.

Armed with these moments, we calibrate the key parameters of the model. Note that these moments are not sufficient to distinguish between inattention and switching costs. As a result, we will be able to identify e.g. $\psi_A D^*(f_{2024})$, but not ψ_A and $D^*(f_{2024})$ separately at this point. Despite this limitation, we will be able to compute marginal values, markdowns, and other moments of interest. Intuitively, the net present value of a depositor depends on the rate at which the depositor leaves for another bank, but not on whether that rate is governed primarily by sleepiness or switching costs.

Table 4 Panel B shows our calibrated parameters, $\psi_A D^*(f_{2024})$, μ_A, and $\psi_S D^*(f_{2024})$. The first of these ($\psi_A D^*(f_{2024})$) can be directly calculated from our half-life of active depositor assumption, moment 3 in Panel A of Table 4. The second parameter, the fraction of active depositor in the population (μ_A), must be a value that is slightly smaller than $9.5\%/(\psi_A D^*(f_{2024}))$. Intuitively, most of the 9.5% quarterly departure rate must come from active types, as very few sleepy types will depart in a single quarter. Given that active types depart at the rate $\psi_A D^*(f_{2024})$, their population must be close to the overall departure rate divided by the active departure rate. The last parameter, the departure rate of sleepy types ($\psi_S D^*(f_{2024})$), can be inferred from their population share and the population half-life of 64 quarters. Because almost all active types will have departed well before 64 quarters, the departure rate of the sleepy types must be roughly 1/64th of $ln(2)$, divided by their population share. The exact values listed in Table 4 are close to the ones given by these approximation arguments. The point of these approximations is to illustrate that any calibration featuring (1) a relatively high short-term departure rate and (2) a relatively long half-life will inevitably reach similar conclusions.

Given this calibration, we can compute the value of each type of depositor, the value of an awake depositor, and the value of a representative depositor. The difference between these last two is that active types are over-represented in the population of awake depositors; we interpret this difference in value as measure of adverse selection. We can also compute the difference between the value of an awake depositor and the bonus, which is to say the markdown. This is a measure of market power

Table 4: SIMPLE CALIBRATION

Panel A: Moments	Model Expression	Value	Source
1: Quarterly departure rate in 2024	$\psi^* D^*(f_{2024})$	9.5	Mintel Survey 2024
2: Half life of depositor at $f_t = f_{2024}$ (n, quarters)	$\frac{1}{2} = \mu_A \exp(-\psi_A D^*(f_{2024})n) + (1-\mu_A)\exp(-\psi_S D^*(f_{2024})n$	64	Survey of Consumer Finances
3: Half life of active depositor in 2024	$\frac{1}{2} = \exp(-\psi_A D^*(f_{2024}))$	1	Mintel Comperemedia

Panel B: Parameters	Model Expression	Value	Relevant Moments
Departure rate of active type	$\psi_A D^*(f_{2024})$	69%	3
Fraction of active type	μ_A	12.6%	1, 3
Departure rate of sleepy type	$\psi_S D^*(f_{2024})$	0.87%	1, 2, 3

Panel C: Derived values	Model Expression	Value
Value of sleepy type	$V_S^B(f_{2024})$	\$8863
Value of active type	$V_A^B(f_{2024})$	\$268
Value of awake depositor	$MV(f_{2024}) * w$	\$958
Value of representative depositor	$\mu_A V_A^B(f_{2024}) + \mu_S V_S^B(f_{2024})$	\$7779
Difference btw. rep. and awake	$\mu_A V_A^B(f_{2024}) + \mu_S V_S^B(f_{2024}) - MV(f_{2024}) * w$	\$6822
Markdown	$w(MV(f_{2024}) - b_{min})$	\$758
Implied demand Elasticity	$(\sigma^{-1}(1 - D(b_{min})) + \theta D(b_{min}))^{-1}$	19.8

Note: Panel A shows the moments used in the simple calibration and their source. Panel B shows the calibrated parameter value for the departure rate of active depositors, the departure rate of sleepy depositors and the fraction of active depositor in the population using the moments in Panel A . Panel C shows the value of sleepy depositors, the value of active depositors, the value of awake depositors, the value of the representative depositor, the markdown and the implied demand elasticity of awake depositors based on the estimated parameters.

and (scaled appropriately) the inverse of the demand elasticity.

The key to these computations is the result that, in the equilibrium described by Proposition 1, $\lambda^{-1}D^*(f_{2024}) = r(f_{2024})$, combined with the observation that $\lambda^{-1}D^*(f_{2024})$ and $\psi_X D^*(f_{2024})$ are together sufficient to compute $V_X^B(f_{2024})$ in the intermediate region. It follows that we are able to compute the marginal values of the active and sleepy types, and hence the marginal and average values of a depositor, without needing to separately identify the degree of inattention and scale of switching costs. We report these derived quantities in Panel C of Table 4.

We find that the value of a sleepy depositor is roughly \$8,800, which corresponds to about 60% of the size the median deposit (\$15,000). Intuitively, if the bank pays the depositor zero interest and the depositor never leaves, it is as-if the bank owned an infinite-maturity floating rate bond. The value of such a bond is always equal to its face value, which is to say that the bank captures the entire value of the deposit in this case. That the sleepy depositor will leave eventually reduces the value of the bank's claim, but given that the half-life is 16 years, it is not surprising that the bank's claim is most of the deposit's value.

In contrast, the value of an active depositor is about \$270, which is roughly 1.4 quarters worth of interest when the quarterly interest rate is 1.26% and the deposit is $15,000.[22] Active depositors are not worth very much to the bank, because they are likely leave and take the bonus offer of another bank shortly after arriving. Note, however, that their value exceeds the size of the bonus. In a perfectly competitive equilibrium with adverse selection (e.g. the "shrouding" model of Gabaix and Laibson (2006)), banks would lose money serving active types while earning profits on sleepy types, so that their profits overall were zero. With imperfect competition, the predicted sign of the profits of serving active depositors is ambiguous. Our calibration suggests small but positive profits, which is consistent with the observation that banks do not attempt to screen out active depositors and rarely refuse new deposits. Banks would prefer to attract sleepy rather than active depositors, but they will accept both types.

Given that banks profit from serving both types of depositors, it is not surprising that they enjoy substantial markdowns. Our estimate is that the marginal awake depositor is worth $958 to the bank, and hence the bank earns of a profit of $758 after paying the $200 bonus. Note, however, that this profit is small relative to the value of a representative depositor ($7779). Market power accounts for about 10% of the difference between the value of a representative depositor and the bonus paid by banks; about 90% is the result of adverse selection (the difference in value between the representative depositor and the average awake depositor).

The markdown we recover is consistent with a substantial demand elasticity. The elasticity of roughly 20 recovered from our calibration implies that if a bank offered an additional 1% of the deposit size as a bonus (i.e. if the bonus changed from $200 to $350), it would capture an additional 20% of *awake* depositors who consider switching. Put another way, consistent with introspection, the large national banks in our sample offer savings accounts that are close substitutes for the accounts offered by other banks. The problem, from the bank's perspective, is that few depositors consider switching, and those that do are far less valuable than the typical depositor.

Robustness. We show that our decomposition of the importance of adverse selection (the difference in value between the representative depositor and the average awake depositor) relative to market power in explaining the low level of bonuses is robust to changes in the value of the identifying moments used in Panel A of Table 4. Figure 5 reports the relative importance of adverse selection versus market power varying: (i) the average life of all deposits (baseline is 64 quarters); (ii) the average life

[22] Recall that in our equilibrium, this value is constant, meaning that an active depositor is worth e.g. roughly 3.6 quarters of interest at a 0.5% quarterly interest rate. That is, holding interest rates fixed, the bonus can be thought of as compensation for some time period's worth of interest. However, it is the length of that time period and not the value of the bonus that changes as interest rates change, which is why we prefer not to conceptualize the bonus as equivalent to temporarily receiving the market interest rate.

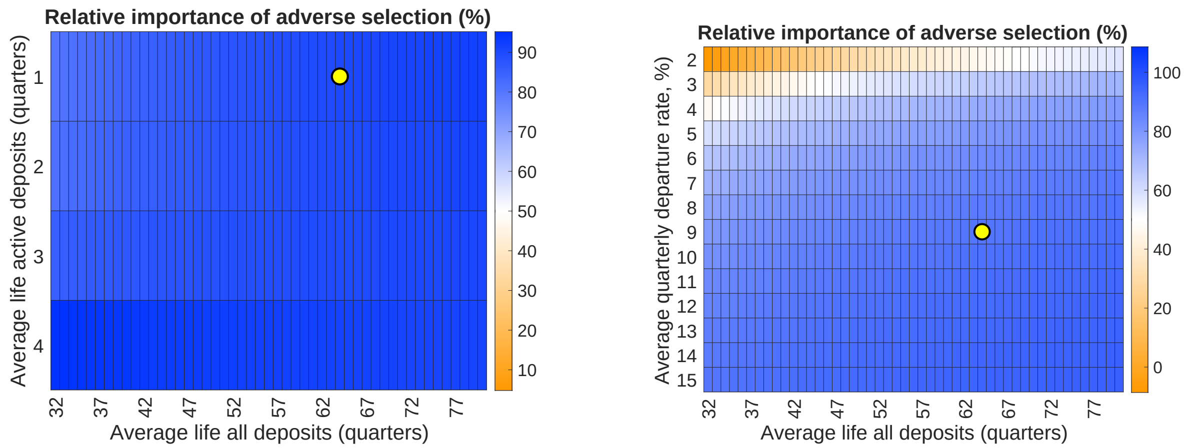

Figure 5: Relative Importance of Adverse Selection: Robustess

Note: The figure plots the relative imporance of adverse selection versus market power varying: (i) the average life of all deposits; (ii) the average life of active deposit; and (iii) the average quarterly departure rate. The yellow dot identifies our baseline estimate with: an average life of all deposits 64 quarters; an average life of active deposit of one quarter; and (iii) an average quarterly departure rate of 9.5%.

of active deposits (baseline is one quarter); and (iii) the average quarterly departure rate (baseline is 9.5%).

All else equal, as the average life of all deposit decreases, the relative importance of adverse selection also decreases. However, even for a short average life of a deposit account around eight years, like the one documented for savings accounts by Egan et al. (2025) (See their Figure 1), adverse selection still accounts for more than 70% of the difference between the value of a representative depositor and the bonus paid by banks. Increasing the average life of active depositors from one to four quarters does not significantly affect the relative importance of adverse selection. A shorter quarterly departure rate decreases the importance of adverse selection; however, for adverse selection to account for less than 50%, the departure rate needs to drop to less than 3%, which is less than a third of what is inferred from our data. Hence, adverse selection accounts for a large proportion of the difference between the value of a representative depositor and the bonus paid by banks across a large set of the moments space.

5.2 Full Calibration

Our simple calibration focused on the intermediate region of the equilibrium in Proposition 1. We now present the results of our full calibration, which relaxes the assumptions of perfect targeting and an absorbing interest rate interval. These changes allow the bonus to vary in the intermediate region and also make the intermediate region span a smaller range of interest rates (due to effects

Table 5: FULL CALIBRATION - MOMENTS

Moments	Data	Model
Average bonus in 2024 ($)	200	208
Departure rate in 2024 (%)	9.5	9.6
Avg. relationship length (quarters)	64.0	64.4
Average bonus in 2022 ($)	180	174
Mail prob. ratio 2024/2022	2.0	2.0

Note: The Table shows the value of the target moments in the data and in the model.

akin to option value). Our full calibration will also leverage additional moments to identify primitive parameters of our model. We simulate an AR(1) process for the interest rate with 30 states and solve for the equilibrium of the model at each of these states using value function iteration. We calibrate the parameters targeting some moments based on our new facts from Section 3. We recast the calibration as a mathematical program with equilibrium constraints, which has several advantages for forward-looking demand models where the Bellman equation must be solved repeatedly (Dubé et al., 2013).

Table 5 reports the moments used to calibrate the model. We choose to match the average bonus in a high interest rate environment ($200), the departure rate in a high interest rate environment (9.5%), the average length of the relation between a depositor and her bank from the survey of consumer finances (64 quarters), the average bonus in a low interest rate environment ($180), and the ratio of mail volume in a high versus a low interest rate environment (2). We interpret the last of these as the ratio of $\alpha^*(f)$ between the 2024 and 2022 interest rate environments. The comparison between the values observed in the data and those derived from the model shows that the model matches the targeted moments well.

Table 6 shows the parameters implied by our calibration. The fraction of active type is about 8.3%, which is slightly lower but in the same order of magnitude of the one from our simple calibration in Table 4. The probability that sleepy types wake up is low, at around 1%. We estimate a small switching cost conditional on being awake.[23] The advertising cost is about $400. This number should not be interpreted as the cost of sending one mail offer (which is about $0.5), but as the cost of reaching potentially interested depositors. Suppose, for example, that the bank needs to send 800 mails to have one potential customer actually look at one offer; then the cost is about $400. Note that we calibrate the model with $\omega = 0$; as a result, $\bar{b}$ is not a parameter of the model.

The two parameters listed at the bottom of Table 6 are not calibrated. Calibrating the first (θ) would require data from both high and low bonus environments; interest rates did not get high enough

[23]Note that "being awake" could be relabeled as "having low switching costs" in a model in which switching costs varied over time, so this result should not be interpreted as stating that switching costs are low for the entire population.

Table 6: FULL CALIBRATION - PARAMETERS

Parameters	Model Expression	Value
Calibrated		
Fraction of active type (%)	μ_A	8.33
Wake-up prob. sleepy type (%)	ψ_S	1.52
Switching cost ($)	δ	14
Taste shock scale ($)	σ	48
Advertising cost ($)	χ	387
Others		
Elasticity competing offers	θ	9.5
Advertising cost shock scale ($)	σ_χ	4.00

Note: The Table shows the value of the target moments in the data and in the model.

in our sample to generate substantial variation on bonus levels. The second parameter σ_χ is the standard deviation of bank-level shock to χ, which generates smooth policy functions. This can be viewed as a numerical device and we set it to a small value. We test the robustness of our estimates by varying these values.

In Table 7 we report several outcomes of interest in the baseline and alternative counterfactuals, in which we change the fraction of active depositors (μ_A). Panel A shows the high interest rate environment (4.5%), while panel B shows the low interest rate environment (1.5%). We discuss the main insights from the comparisons between (i) the high- and low-rate environments in the baseline; and (ii) the baseline and the all sleepy and all active counterfactuals in the high-rate environment (similar patterns apply for the same comparison in the low-rate environment).

The average bonus is higher when rates are higher because the marginal depositor value is higher. The difference in the bonus is only about $35, while the mailing probability with high rates is twice the mailing probability with low rates. Going from low to high rates generates two opposing forces. First, higher rates mean higher revenues for the bank due to the ability to pay (close to) zero rates on savings accounts. Second, higher rates mean more competition via higher mailing volumes, which increases the probability that existing depositors will leave. For example, active depositors have a probability of staying close to zero when rates are high, while they remain at their current bank with a 30% probability with low rates. Despite the higher endogenous competitive response, the first channel dominates, and the values of both active and sleepy depositors are higher with higher rates in the baseline. Interestingly, we find that the bank is losing money on active depositors (the bonus is greater than their value - $b > V^{active}$), which differs from our simple calibration in Table 4. Despite this difference, our estimate of the marginal depositor value to the bank is about $1000 in the high rate

Table 7: FULL CALIBRATION - BASELINE AND COUNTERFACTUALS

	Baseline $\mu_{active} = 0.087$	All active $\mu_{active} = 1$	More active $\mu_{active} = 0.15$	More sleepy $\mu_{active} = 0.03$	All sleepy $\mu_{active} = 0$
Panel A: High rate					
Bonus b	208	133	177	382	4706
Mail prob. p_b	1.00	0.00	0.76	1.00	1.00
Stay prob.^{active}	0.02	0.57	0.16	0.00	0.00
V^{active}	185	414	215	182	182
Stay prob.^{sleepy}	0.99	0.99	0.99	0.98	0.98
V^{sleepy}	6350	9302	7452	6190	6184
V^{avg}	5837	414	6367	5950	6184
V^{mrg}	1066	414	788	1784	6184
Market power (MP): $V^{mrg} - b$	858	281	611	1403	1478
Implied demand elasticity	14.6	32.8	19.2	9.6	9.5
Adverse selection (AS): $V^{avg} - V^{mrg}$	4771	0	5579	4166	0
Relative adverse selection	0.85	0.00	0.90	0.75	0.00
Panel B: Low rate					
Bonus b	174	54	154	246	3786
Mail prob. p_b	0.50	0.00	0.00	1.00	1.00
Stay prob.^{active}	0.29	0.57	0.57	0.01	0.00
V^{active}	78	148	143	55	55
Stay prob.^{sleepy}	0.99	0.99	0.99	0.98	0.98
V^{sleepy}	5725	8694	7222	5349	5336
V^{avg}	5254	148	6160	5137	5336
V^{mrg}	886	148	703	1467	5336
Market power (MP): $V^{mrg} - b$	711	93	549	1222	1550
Implied demand elasticity	19.7	101.5	24.9	11.8	9.5
Adverse selection (AS): $V^{avg} - V^{mrg}$	4369	0	5458	3670	0
Relative adverse selection	0.86	0.00	0.91	0.75	0.00

Note: The Table shows several outcomes of interest in the baseline and alternative counterfactuals, in which we change the fraction of active depositors (μ_A). Panel A shows the high interest rate environment, while panel B shows a low interest rate environment.

environment, which is close to the estimate from our simple calibration in Table 4 (\$958).[24] Hence, the bank earns a profit of \$860 after paying the \$208 bonus. Similarly to the simple calibration, market power accounts for about 15% of the difference between the value of a representative depositor and the bonus paid by banks; about 85% is the result of adverse selection. This result holds in the low interest rate environment as well.

We then compare our baseline to counterfactuals in which we exogenously vary the fraction of active types. Our main interest is to study the equilibrium effects of removing adverse selection, which in our context is captured by the difference between the average and the marginal depositor value. In an economy where all depositors are active, banks stop sending mail and offering bonuses.[25]

[24] Our full calibration implies lower values for active and sleepy depositors, but a lower fraction of the active type in the population.

[25] Note that in this counterfactual banks might start competing on rates which we rule out when implementing the

The shadow bonus (in an equilibrium with no mail, no bonuses are offered) declines to about \$130 as the value of the marginal depositor is now lower since all depositors are active. The value of the average depositor decreases by an even larger absolute and percentage amount from about \$6000 to only \$414.

Finally, we study an equilibrium in which all depositors are sleepy. This counterfactual is closer to the estimated baseline than the all-active counterfactual, given the high fraction of sleepy depositors in the baseline. The bonus experiences a very large increase as the value of the marginal depositor converges to the (high) value of the average depositor. In the baseline sleepy types only receive a \$208 bonus when changing banks, which is 3% of their value to the bank. Removing the adverse selection due to the presence of active types allows passive types to receive bonuses of about \$4700, which is approximately 75% of their value to the bank. However, even small population of active types dramatically reduces the size of the bonus. If we consider a "more sleepy" calibration of $\mu_A = 3\%$ instead of an "all sleepy" calibration of zero, the bonus level drops from \$4706 to \$382.

6 Conclusion

This paper studies competition in the deposit market. We provide novel descriptive evidence on how large US banks price discriminate between existing and new depositors using a combination of rates, bonuses, and targeted advertising. We also show, in line with the existing literature, very limited pass-through of market interest rates into deposit rates, but document a novel large pass-through to marketing, specifically the mail volume of sign-up bonuses.

The descriptive analysis motivates us to develop and estimate a dynamic equilibrium model with multi-dimensional customer acquisition and unobservable depositor heterogeneity. We use the estimated model to quantify the frictions affecting banks franchise value and the imperfect pass-through of market rates to depositor rates. Our model rationalizes the use of targeted bonuses rather than rates when a large fraction of existing customers are sleepy. Banks respond to rising rates by first increasing the volume of offers they send out, and increase the size of the bonus they offer only after exhausting their ability to increase offer volume. That banks respond first on the extensive margin is both consistent with our new data and implies that the value of a marginal depositor is constant as interest rates rise. Our quantitative estimates indicate that adverse selection, rather than market power, explains the majority of the difference between the value of a representative depositor and the

counterfactual. As we noted when discussing proposition 1 a bank with entirely active depositors would require a much lower commitment period to make offering them interest profitable.

bonus paid by banks. That is, adverse selection as opposed to market power is the main source of banks' franchise value.

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Online Appendix

In this online appendix we report additional figures and tables about results on the facts and structural model. Appendix A describe the classification schema we implement and the way in which we use the LLM to construct the dataset. Appendix B provides supplementary figures and tables for the data and setting. Appendix C provides additional details on the model in relation to the depositor problem. Appendix D provides proofs for the results given in the main text.

A Additional Detail on Data Construction

In this section we describe the classification schema we implement and the way in which we use Claude Sonnet 4.0 (a large language model) to construct the dataset. Our raw data consists of a list of campaign identifiers and the images in jpeg for each campaign. For each campaign, we upload the images and the prompt below using the Anthropic API. We prepend the system prompt (an Anthropic-written prompt used in the Claude chat application and included in our replication code) to our prompt, which we report below in italics. We run a script that repeats this process for each campaign in our dataset. We run this script twice and use as our dataset the observations that in both iterations give the same results.

You are a helpful research assistant. Your task is to analyze the provided images that show offers that banks send to consumers related to opening bank accounts.

CRITICAL: NUMERICAL ACCURACY REQUIREMENT
Before analyzing any offer, you must:

- *Read each monetary amount twice to verify accuracy.*
- *Pay special attention to distinguishing between similar numbers like $5,000, $15,000, and $50,000.*

Definitions

1. ***Type A Offer:*** *an offer that requires maintaining a minimum balance of $5,000 or more to receive the bonus payment.*
2. ***Type B Offer:*** *An offer that meets ALL of the following criteria:*
   a) *Is not a Type A offer.*
   b) *Requires direct deposit OR requires some amount of debit card purchases to receive a bonus payment.*
   c) *EITHER:*
      - *Does not have any minimum balance requirement to avoid account fees, OR*
      - *Has a minimum balance requirement to avoid account fees that is strictly less than $5,000.*

d) *EITHER:*

- *Does not have any minimum balance requirement to receive the bonus, OR*
- *Has a minimum balance requirement to receive the bonus that is strictly less than $5,000.*

3. ***Unclassified Offer:** an offer that does not meet the criteria for a Type A offer and does not meet the criteria for a Type B offer.*

After analyzing the offer, return a JSON that exactly follows the format below. All fields should be returned in the order listed below and no fields should be skipped. Return only one JSON and be mindful of lowercase and uppercase letters. For each entry:

1. *Return only lowercase letters for text entries.*
2. *Return all monetary values as numerical values in string form. Ex. Format $50,000 as '50000'.*
3. *Return all date values in the following format: YYYY-MM-DD.*

For an "empty JSON", return a JSON with the exact same format, but with every value in the key-value pair as "", except for `bank_name`*,* `campaign_id`*,* `unclassified_offer`*,* `suspected_not_promotion`*, and* `excluded_promotion_type`*.*

```
{"bank_name" : "", "campaign_id" : "", "type_a_offer" : "",
"type_b_offer" : "", "unclassified_offer" : "",
"type_a_incentive" : "", "type_b_incentive" : "",
"type_a_minimum_balance" : "", "type_b_minimum_deposit" : "",
"opening_day" : "", "balance days" : "",
"combined_incentive" : "", "targeted_offer" : "",
"special_rate" : "", "allows_assets" : "",
"suspected_not_promotion" : "",
"excluded_promotion_type" : "", "cap_one" : ""}
```

For the provided campaign ID, please calculate the bonus (if any) that the bank is offering for opening a Type A account, a Type B account, or both accounts. Please note that sometimes there is an additional "combined" bonus for opening both types of accounts. If there is only a bonus for opening both types of accounts, please list the incentive under the "combined" bonus.

Some campaigns only advertise Type A offers, some only advertise Type B offers, and some advertise both; please keep this in mind.

For offers that require opening multiple accounts to receive any bonus, do not mark the offer as both Type A and Type B unless separate standalone bonuses are available for each account type.

Based on the offer type from the above criteria, mark the offer as "Yes" for `type_a_offer`*, "Yes" for* `type_b_offer`*, or "Yes" for both.*

Sometimes, banks will offer a schedule of different bonus sizes, depending on the balance maintained or deposit size. These can be present for Type A offers or Type B offers, or both. If a tiered offer exists, please choose only one of those offers to analyze, according to the following criteria:

1. *For a Type B offer under the above criteria, choose the largest bonus level that can be achieved while maintaining a minimum deposit closest to or equal to $2,500.*

2. *For a Type A offer under the above criteria, choose the offer with a minimum balance closest to $10,000. If a tie exists, choose the lower minimum balance value and analyze the corresponding offer. For example, if there are tiers with minimum balances of $5,000 and $15,000 and no tier with a minimum balance of $10,000, choose the tier with the minimum balance of $5,000.*

3. *Report an empty JSON for an unclassified offer.*

4. *For combined offers, choose the bonus offer with a minimum deposit closest to or equal to $2,500 for the Type B portion and the offer with a minimum balance closest to $10,000 for the Type A portion.*

If a campaign ID has one Type A offer and one Type B offer, those should be combined into one JSON.

Many of the Type A and Type B accounts require maintaining a certain balance for a certain amount of time to earn the bonus. For Type A accounts, note the minimum balance required to obtain the bonus (if applicable). For Type B accounts, note the minimum deposit required to obtain the bonus (if applicable). Make sure to check the fine print for balance and deposit levels. Where possible, please note the length of time required to earn the bonus.

Note that some offers do not have any bonus at all. You should report a bonus of zero for these offers. You should interpret language like "Not Specified" as an offer without any bonus.

Pay close attention to potentially confusing language like "Up to...". Banks sometimes advertise the largest bonus possible, but offer smaller bonuses for maintaining a smaller level of deposit. Please *choose the bonus levels corresponding to a minimum deposit equal to or closest to but under $2,500 and corresponding to a minimum balance equal to or closest to $10,000. Also, pay close attention to terms like "additional" if there are multiple images for a single campaign ID. You should report the total bonus, not the incremental bonus, for these offers.*

Other criteria

1. *Exclude "Citigold" and "Chase Private Client" or "Chase Sapphire Banking" offers.*
2. *Exclude campaigns that offer bonuses or incentives only in the form of rewards points, loyalty points, airline miles, or the like.*
3. *Exclude offers for children (ex. "debit card for kids") or students (ex. "Citibank Student Package").*
4. *Exclude offers for certificates of deposits (CDs).*

If any of these have the closest minimum deposit to $2,500 or closest minimum balance to $10,000 of a tiered offer, choose the next closest offer that fulfills the criteria above. Ensure that the bonus, deposit level, and balance level reported do not correspond to a Citigold, Chase Private Client, or Chase Sapphire Banking tier.

Please process the images given, and return a JSON structure according to the following:

- ***Bank Name/Primary Company***
- ***Campaign ID***
- ***Type A Offer:*** *Yes / No*
- ***Type B Offer:*** *Yes / No*
- ***Unclassified Offer:*** *Yes / No. If this is Yes, return an empty JSON.*
- ***Type A Incentive:*** *$ amount / Not specified / Not applicable / Others (e.g., Reward points, Ipad, etc.). Please only return a numerical value, 'not specified', 'not applicable', or 'others' for this field.*
- ***Type B Incentive:*** *$ amount / Not specified / Not applicable / Others (e.g., Reward points, Ipad, etc.). Please only return a numerical value, 'not specified', 'not applicable', or 'others' for this field.*

- ***Type A Minimum Balance:*** *$ amount / Not applicable / Not specified. If this requires multiple deposits over months, for example, $500 each month for 3 months, multiply those together to result in the total minimum balance (in this case, $1500).*

- ***Type B Minimum Deposit:*** *$ amount / Not specified. This refers to the amount that is required in direct deposits within a specific number of days to receive the bonus, not the deposit required to open the account.*

- ***Opening day:*** *Date / Not specified. Refers to the last day that a customer can sign up for an account that receives the promotion or bonus. This should not be the day the customer has to maintain the balance until, or the day they receive the bonus. This is often specified in the fine print.*

- ***Balance days:*** *Number of days / Not specified. Applies only to Type A accounts, and refers to the amount of days a customer must maintain the* `type_a_minimum_balance` *to receive the bonus or promotion. If an offer is for a Type A and Type B account, this field should reflect the balance days for the Type A offer. If the offer states language like "maintain this balance until (specific date)," subtract the last day the balance can be deposited by from the date given.*

- ***Combined incentive:*** *ONLY the incremental/additional $ amount offered for opening both accounts that is above and beyond the sum of the individual Type A and Type B bonuses. If the total combined bonus equals the sum of individual bonuses (Type A + Type B), enter '0'. If no combined offer exists, enter 'not applicable'. For example: if Type A offers $200, Type B offers $300, and the combined offer is $600 total, the combined incentive would be '100' (the extra $100 above the $500 sum).*

- ***Targeted Offer:*** *Yes / No. Yes if the offer contains a coupon code or language stating that the offer is only available to the person receiving the mailer. Please look for language similar to "Visit ___________ and input your unique offer code" or "For _____ cardholders" or "For existing customers". Additionally, if there is a whited-out section next to a coupon/promo code or in a URL, or if a reference code is provided, this is a targeted offer. This is also true if the URL cuts off after .com/.*

- ***Special Rate:*** *Yes / No. Yes if the offer mentions a specific special interest rate that is available for only a limited amount of time. This does not refer to language like "5X competitor interest rates" or "5X the national savings rate", or to APY's always associated with specific types* *of high-yield savings accounts. Please look specifically for an interest rate that is higher than normal for a limited time or with a specific minimum balance that is greater than $1 when the customer takes advantage of a promotion.*

- ***Allows Assets:*** *Yes / No. Yes if the offer explicitly states that the minimum balance needed to receive the bonus can be satisfied with securities or other investments.*
- ***Suspected Not Promotion:*** *Yes / No. Yes if the mailing in question is suspected not to be a promotional offer for a bank account, such as a bank statement, confirmation, or notification. If this is Yes, return an empty JSON.*
- ***Excluded Promotion Type:*** *Yes / No. Yes if the mailing in question is a promotional offer for miles, a private client offer, or another offer type that has been excluded. If this is Yes, return an empty JSON.*
- ***Cap One:*** *360 / 360 Performance / Other. For each Capital One account, identify if the account is a 360 Account, 360 Performance Account, or different type of account. If the account type is not specified, categorize it as "Other".*

Before completing your JSON response, systematically verify:

- *The minimum deposit amount required (read the number twice).*
- *The minimum balance maintenance amount required (read the number twice).*
- *That these two amounts are correctly distinguished from each other.*
- *The bonus amount matches what you've recorded.*
- *During verification, for each monetary amount, describe what you see digit by digit (e.g., 'I see: dollar sign, one, five, comma, zero, zero, zero = $15,000').*

MANDATORY VERIFICATION: *After completing your analysis, re-read all monetary amounts in the original image to confirm:*

- *Deposit amount is correctly transcribed.*
- *Balance maintenance amount is correctly transcribed.*
- *Bonus amount is correctly transcribed.*

- *No numbers have been confused with similar-looking amounts.*
- *Double-check all numbers if the ratio of the bonus to the minimum balance is 0.05 or greater for any type A offer.*

B Additional Results on Facts

In this section we report additional figures and tables in relation to the facts.

(a) Dollar spent on mail: Checking & Savings **(b)** Direct mail volume: Mortages

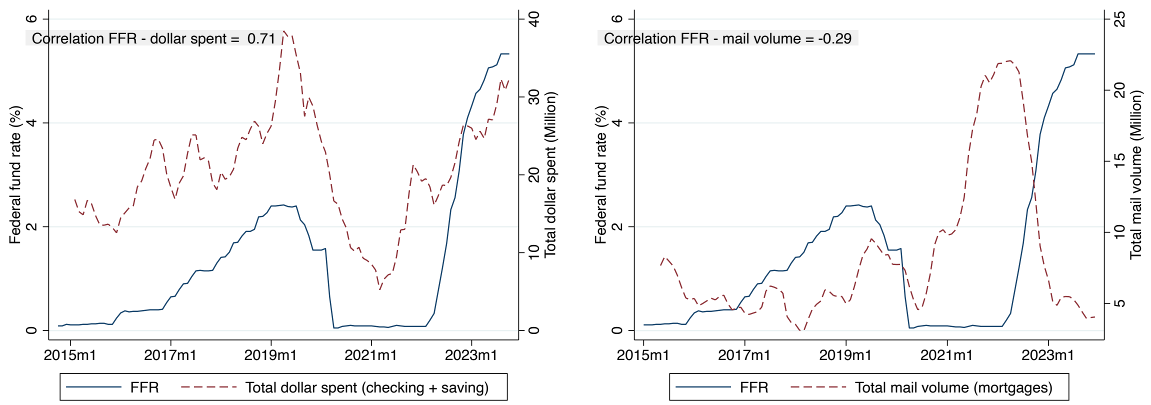

Figure A1: Pass-through on Checking and Savings Products - Marketing Extra

Note: The figure plots the federal funds rate (2014–2024) alongside the following variables: (a) total dollar spent in millions for checking and saving offers using the Comperemedia data; (b) total mail volume for mortgage offers.

(a) Fraction of household receiving mail **(b)** Acquisition VS Retention

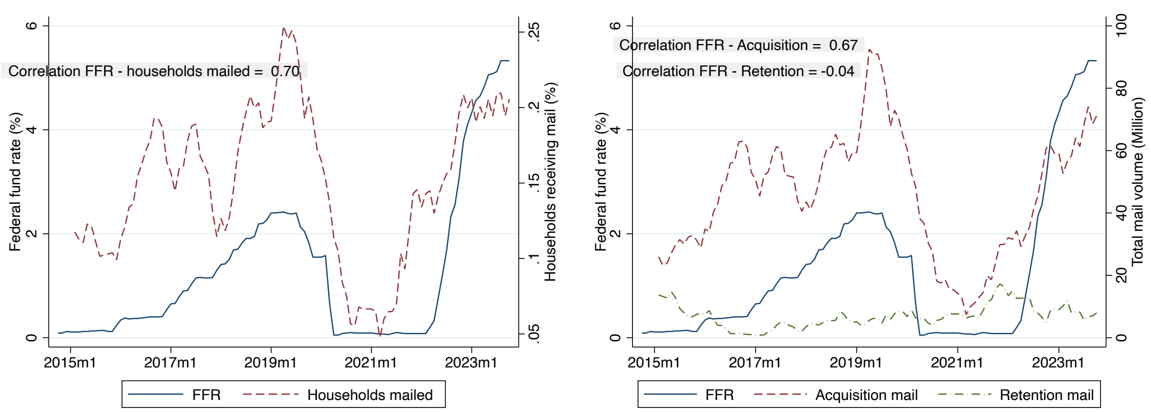

Figure A2: Pass-through on Checking and Savings Products - Targeting Extra

Note: The figure plots the federal funds rate (2014–2024) alongside the following variables: (a) total dollar spent in millions for checking and saving offers using the Comperemedia data; (b) total mail volume for mortgage offers.

Table A1: PASS-THROUGH ON CHECKING AND SAVINGS PRODUCTS - EXCLUDING CHASE

	Deposit Rates (bps)		Sign-up Bonus ($)		Marketing (log)		Targeting	
	Checking	Saving	Checking	Saving	Mail volume	Dollar spent	HHI	Number of states
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Federal fund rate (%)	0.30***	13.63***	17.22***	15.29***	0.09***	0.09***	-0.02***	0.28**
	(0.05)	(1.55)	(1.72)	(2.64)	(0.03)	(0.02)	(0.00)	(0.13)
BANK F.E.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
BANK-TIME CONTROLS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
DEP. VAR. (MEAN)	3.55	29.71	276.98	219.79	13.19	12.44	0.27	8.55
DEP. VAR. (SD)	6.06	73.24	95.66	128.59	1.36	1.28	0.30	8.02
R^2	0.84	0.63	0.32	0.43	0.15	0.20	0.06	0.29
OBSERVATIONS	839	935	1,849	1,389	2,767	2,767	2,620	2,620

Note: The Table shows the results of equation (1) excluding Chase. Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

Table A2: PASS-THROUGH ON CHECKING AND SAVINGS PRODUCTS - LARGEST CAMPAIGN

	Sign-up Bonus ($)		Marketing (log)		Targeting	
	Checking	Saving	Mail volume	Dollar spent	HHI	Number of states
	(1)	(2)	(3)	(4)	(5)	(6)
Federal fund rate (%)	17.33***	16.81***	0.16***	0.16***	-0.02***	0.72***
	(2.27)	(3.50)	(0.04)	(0.03)	(0.01)	(0.19)
BANK F.E.	Yes	Yes	Yes	Yes	Yes	Yes
BANK-TIME CONTROLS	Yes	Yes	Yes	Yes	Yes	Yes
DEP. VAR. (MEAN)	272.55	214.41	13.52	12.73	0.25	10.47
DEP. VAR. (SD)	90.29	120.36	1.58	1.45	0.29	9.73
R^2	0.28	0.45	0.30	0.31	0.09	0.47
OBSERVATIONS	600	373	802	802	786	786

Note: The Table shows the results of equation (1) focusing on the largest campaign by number of volume for each bank-month. Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

Table A3: PASS-THROUGH ON CHECKING AND SAVINGS PRODUCTS - CUMULATIVE EFFECT

	Deposit Rates (bps)		Sign-up Bonus ($)		Marketing (log)		Targeting	
	Checking (1)	Saving (2)	Checking (3)	Saving (4)	Mail volume (5)	Dollar spent (6)	HHI (7)	Number of states (8)
Cumulative federal fund rate (%)	0.30***	13.93***	19.75***	13.90***	0.13***	0.13***	-0.02***	0.66***
	(0.06)	(1.78)	(1.82)	(2.55)	(0.03)	(0.03)	(0.01)	(0.15)
BANK F.E.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
BANK-TIME CONTROLS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
DEP. VAR. (MEAN)	3.27	26.84	280.14	214.78	12.99	12.17	0.29	8.84
DEP. VAR. (SD)	5.76	70.01	82.54	108.04	1.38	1.26	0.31	8.42
R^2	.84	.62	.29	.42	.28	.28	.09	.41
OBSERVATIONS	944	1,039	2,838	2,353	3,966	3,966	3,803	3,803

Note: The Table shows the results of equation (1) modified to include four lags federal fund rate and report the cumulative effect. Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

Table A4: PASS-THROUGH ON CHECKING AND SAVINGS PRODUCTS - HETEROGENEITY

	Deposit Rates (bps)		Sign-up Bonus ($)		Marketing (log)		Targeting	
	Checking (1)	Saving (2)	Checking (3)	Saving (4)	Mail volume (5)	Dollar spent (6)	HHI (7)	Number of states (8)
Federal fund rate (%)	0.30***	0.07	15.82***	5.87**	0.13***	0.13***	-0.02***	0.76***
	(0.03)	(0.26)	(1.63)	(2.41)	(0.03)	(0.02)	(0.00)	(0.13)
Federal fund rate (%) × High-rate bank	-0.24	49.55***	4.27	27.01***	-0.12**	-0.11**	0.02*	-0.86**
	(0.22)	(2.73)	(4.88)	(4.26)	(0.06)	(0.05)	(0.01)	(0.36)
BANK F.E.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
BANK-TIME CONTROLS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
DEP. VAR. (MEAN)	3.27	26.84	276.92	214.51	13.43	12.64	0.25	10.24
DEP. VAR. (SD)	5.76	70.01	88.84	114.32	1.53	1.39	0.29	9.34
R^2	0.84	0.86	0.28	0.44	0.28	0.28	0.09	0.41
OBSERVATIONS	944	1,039	2,838	2,353	3,966	3,966	3,803	3,803

Note: The Table shows the results of equation (1) with an additional interaction for high-rate banks (Capital One, Discover, Citizens). Deposit rates is the average deposit rate in basis points for checking and saving products from Ratewatch. Mail volume and dollar spent are the total across all banks in our sample of estimated direct mail volume and dollar spent from Comperemedia. Sign-up bonus is the average sign-up bonus in dollar for checking and saving offer from Comperedia. HHI is the Herfindahl-Hirschman Index (HHI) at the campaign-month based on the volume of mails sent to target groups defined as triplets of state, income, and age bins. Robust standard errors in parenthesis. *, **, and *** denote significance at the 10%, 5% and 1% levels, respectively.

C Derivations of the Depositors' Choice Probabilities

Let $V^X(f, Q, j)$ be the depositor's value function given the aggregate state (f, Q) if the depositor currently uses bank j. Now imagine a depositor who is considering switching to bank j' and has been offered bonus b (scaled by the deposit size w). If the depositor switches to bank j', they will receive bw but pay a switching cost δw and experience non-pecuniary utility $w\sigma\epsilon_{ij't}$. If the depositor stays at their current bank j, they will not receive a bonus or pay a switching cost, and will experience non-pecuniary utility $w\sigma\epsilon_{ijt}$.

The expected value of considering a switch to bank j' at time t, if that bank has offered the depositor a bonus b, is

$$\begin{aligned}\bar{V}^X(f_t, Q_t, j, j', b) = \int \max_{j'' \in \{j, j'\}} \{ &\mathbf{1}\{j'' \neq j\}(b - \delta)w + \sigma w \epsilon_{i,j'',t} \\ &+ V^X(f_t, Q_t, j'')\} dg(\epsilon_{ijt}, \epsilon_{i,j',t}),\end{aligned}$$

where the expectation is taken over possible realizations of the non-pecuniary utility shocks, whose joint probability measure is $g(\cdot)$.

We assume that depositors are more likely to consider switching to bank j' when the value of doing so is higher. Let $\mathbf{B}$ be the set of functions from $[0, 1]$ to $\mathbb{R}_+$. The function $\mathbf{b}_i \in \mathbf{B}$ describes the set of bonus offers received by household i from the banks. Given these offers, we define the density for depositor i of considering bank j' (relative to the Lebesgue measure over the set of potential banks) as $A(\cdot)$. This can depend in principle on the current bank j, aggregate state (f, Q), and entire set of bonus offers to depositor i,

$$A(j'; \mathbf{b}_i, j, f, Q) = \frac{e^{\theta \bar{V}^X(f, Q, j, j', \mathbf{b}_i(j'))}}{\int_0^1 e^{\theta \bar{V}^X(f, Q, j, j'', \mathbf{b}_i(j''))} dj''}. \tag{12}$$

$V^X(\cdot)$ is the depositor's value function. If the borrower does not awaken, they receive zero flow payoff (the banks pay no interest). If they awaken, they consider bank j' with probability $A(j'; \cdot)$ and then decide between that bank and their current bank. Avoiding the details of how (f_t, Q_t) evolve over time, we can write the depositor's HJB equation as

$$\begin{aligned}r(f_t)V^X(f_t, Q_t, j) = &\lim_{h \downarrow 0} h^{-1} \mathbb{E}^{\mathbb{P}}[V^X(f_{t+h}, Q_{t+h}, j) - V^X(f_t, Q_t, j)] \\ &+ \psi_X \mathbb{E}^{\mathbb{P}}[\int_0^1 \bar{V}^X(f_t, Q_t, j, j', \mathbf{b}_i(j')) A(j'; \mathbf{b}_i, j, f_t, Q_t) dj'],\end{aligned}$$

where the expectation in the second line is taken over the set of offers $\mathbf{b}_i$ the depositor will receive.

The functional form of $A(\cdot)$, (12), can be justified as arising from the optimal allocation of attention to bank offers subject to a constraint on mutual information, in the style of Sims (2010) and Matějka and McKay (2015). Specifically, the Bellman equation we study is equivalent to the problem of maximizing utility over the set of all possible attention strategies subject to a constraint on mutual information and given an exchangeable prior over banks (i.e. one in which banks are symmetric). That is,

$$\begin{aligned} r(f_t)V^X(f_t,Q_t,j) &= \sup_{p\in\mathcal{P}}\lim_{h\downarrow 0} h^{-1}\mathbb{E}^{\mathbb{P}}[V^X(f_{t+h},Q_{t+h},j)-V^X(f_t,Q_t,j)] \\ &+\psi_X\mathbb{E}^{\mathbb{P}}[\int_0^1 \bar{V}^X(f_t,Q_t,j,j',\mathbf{b}_i(j'))p(j';\mathbf{b}_i)dj'] \\ &-\psi_X\theta^{-1}\mathbb{E}^{\mathbb{P}}[\int_0^1 p(j';\mathbf{b_i})\ln(p(j';\mathbf{b_i}))dj'] \end{aligned}$$

where $\mathcal{P}$ is the set of functions from $\mathbf{B}$ to probability densities on $[0,1]$ satisfying, for any j', j'', $\mathbb{E}^{\mathbb{P}}[p(j';\mathbf{b}_i)] = \mathbb{E}^{\mathbb{P}}[p(j'';\mathbf{b}_i)]$ (here, the expectations are taken over possible realizations of $\mathbf{b}_i$), and θ^{-1} is the cost of attending to banks' offers, paid only when awake. The attention policy A described above is the optimal p in this problem. We can thus understand the $\theta\to\infty$ limit as one in which it is costless for the depositor to consider only the bank with the best offer, and the $\theta\to 0$ as one in which the depositor randomly searches for a new bank without knowing which banks are offering bonuses. Intermediate cases can be understood as ones in which depositors have partial but imperfect awareness of the terms banks are offering, and consider the bank that is offering the best deal in expectation under the depositor's beliefs.[26]

We can simplify the depositor's problem by observing that all banks are identical after the bonus has been paid. This yields the functional form for $D(b)$ in the main text.

Moreover, under our assumptions, the value function in the consideration stage is the "inclusive value," which up to a constant can be written as

$$\bar{V}^X(f_t,Q_t,j,j',b) = -\sigma\ln(1-D(b)).$$

This, combined with our definition of $A(\cdot)$, yields the functional form given in the main text. The

[26]We could have described both choice stages as arising from type-1 extreme value taste shocks or both as arising from rational inattention. There is nothing in behavior that can distinguish between unobserved preference shocks and inattention. Our choice to describe the selection of which bank to consider as "inattention" and the choice to switch to the considered bank or stay at the current bank as "preferences" suits our priors but is not essential.

demand curve is then derived by considering a single bank that offers a bonus of b if a fraction α^* of other banks offer a bonus of b^*, and the remainder offer no bonus.[27]

[27] There are some subtleties here with respect to the rational inattention microfoundation of A if this contemplated configuration of bonuses lies outside the support of the depositor's prior (see Ravid (2020)). We view these issues as outside the scope of the present paper.

D Proofs

D.1 Proof of Lemma 1

Using (5),

$$d(Q_{Xjt}e^{\int_0^t \psi_X D^*(f_s,Q_s)ds}) = \mu_X \psi_X e^{\int_0^t \psi_X D^*(f_s,Q_s)ds} D(\alpha_t, b_t, \alpha^*(f_t,Q_t), b^*(f_t,Q_t))dt$$

from which it follows that

$$\begin{aligned} Q_{Xjt} = \mu_X \psi_X \int_0^t & e^{-\int_s^t \psi_X D^*(f_l,Q_l)dl} D(\alpha_s, b_s, \alpha^*(f_s,Q_s), b^*(f_s,Q_s))ds \\ &+ e^{-\int_0^t \psi_X D^*(f_s,Q_s)ds} Q_{Xj0}. \end{aligned}$$

Plugging this into the sequence problem of the bank, the NPV of the net interest margin is

$$\begin{aligned} w\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} r(f_t)(Q_{Ajt}+Q_{Sjt})dt] = {} & wQ_{Aj0}\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t (r(f_s)+\psi_A D^*(f_s,Q_s))ds} r(f_t))dt] \\ &+ wQ_{Sj0}\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t (r(f_s)+\psi_S D^*(f_s,Q_s))ds} r(f_t))dt] \\ +w\sum_{X\in\{A,S\}} \mu_X\psi_X \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} r(f_t) \int_0^t & e^{-\int_s^t \psi_X D^*(f_l,Q_l)dl} D(\alpha_s, b_s, \alpha^*(f_s,Q_s), b^*(f_s,Q_s))dsdt]. \end{aligned}$$

The last line of this expression can be rewritten using the definition of $V_X^B(\cdot)$ via

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} r(f_t) \int_0^t e^{-\int_s^t \psi_X D^*(f_l,Q_l)dl} D(\alpha_s, b_s, \alpha^*(f_s,Q_s), b^*(f_s,Q_s))dsdt] = \\ \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^s r(f_l)dl} D(\alpha_s, b_s, \alpha^*(f_s,Q_s), b^*(f_s,Q_s)) \int_s^\infty e^{-\int_s^t (r(f_l)+\psi_X D^*(f_l,Q_l))dl} r(f_t)dtds] = \\ \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^s r(f_l)dl} D(\alpha_s, b_s, \alpha^*(f_s,Q_s), b^*(f_s,Q_s)) V_X^B(f_s,Q_s)ds] \end{aligned}$$

The result follows from the observation that this expression and the cost components of the sequence problem (the NPV of bonus payments and marketing costs) do not depend on (Q_{Aj0}, Q_{Sj0}).

D.2 Proof of Lemma 2

Define the function $FOC : \mathbb{R} \to \mathbb{R}$ by

$$FOC(b) = b + \sigma(1 + (\sigma\theta - 1)D(b))^{-1},$$

noting that the first-order condition is $MV = FOC(b)$. Observe that this function is continuously differentiable, with

$$FOC'(b) = 1 - (\sigma\theta - 1)\frac{D(b)(1 - D(b))}{(1 + (\sigma\theta - 1)D(b))^2}$$
$$= \frac{1 + (1 + D(b))(\sigma\theta - 1)D(b) + (\sigma\theta - 1)^2 D(b)^2}{(1 + (\sigma\theta - 1)D(b))^2}$$

If $\sigma\theta \geq 1$, $FOC'(b) > 0$ is immediate from the second line by $D(b) \in (0, 1)$. If $\sigma\theta \in (0, 1)$, $FOC'(b) > 0$ is immediate from the first line. Moreover, $\lim_{b\to\infty} FOC(b) = \infty$ and $\lim_{b\to-\infty} FOC(b) = -\infty$. It follows by the inverse function theorem that the inverse function, $b^+(MV)$, is defined for all $MV \in \mathbb{R}$ and is strictly increasing.

Observe that

$$\frac{\partial h(b, MV, A_0)}{\partial b} \propto MV\frac{\partial}{\partial b}(D(b)(1 - D(b))^{-\sigma\theta}) - \frac{\partial}{\partial b}b(D(b)(1 - D(b))^{-\sigma\theta}) \propto MV - FOC(b).$$

It follows that $\max\{b^+(MV), 0\}$ is the unique maximizer of $\alpha h(b, MV, A_0)$ on $b \geq 0$ given any $\alpha > 0$ and $MV \in \mathbb{R}$.

D.3 Proof of Lemma 3

First observe that the policy-dependent portion of the bank's HJB equation, under the linearity conjecture, is proportional to $\alpha h(b, MV, A_0)$. By Lemma 2, $b = max(b^+(MV(f_t, Q_t), 0)$ is necessary and sufficient to establish the optimality of b given $\alpha > 0$.

The necessity of $h(b, MV, A_0) > 0$ only if $\alpha = 1$ and $h(b, MV, A_0) < 0$ only if $\alpha = 0$ are immediate via the first-order condition for α. We prove sufficiency by contradiction. Suppose, given (MV, A_0), there is some alternative policy $(\tilde{\alpha}, \tilde{b})$ that is strictly superior to the policy (α, b) satisfying the conditions of Lemma 1. If $h(\tilde{b}, MV, A_0) > 0$, we must have $h(b, MV, A_0) \geq h(\tilde{b}, MV, A_0) > 0$, as b is a maximizer by the argument above. This implies $\alpha = 1$, and therefore $\alpha h(b, MV, A_0) \geq \tilde{\alpha} h(\tilde{b}, MV, A_0)$, contradicting strict optimality. If $h(\tilde{b}, MV, A_0) \leq 0$, then $0 \geq \tilde{\alpha} h(\tilde{b}, MV, A_0) > \alpha h(b, MV, A_0)$ requires $\alpha > 0$ and $h(b, MV, A_0) < 0$, contradicting the conditions of Lemma 1. It follows that the conditions of Lemma 1 are necessary and sufficient to establish the optimality of (α, b).

D.4 Proof of Lemma 4

Recall the definition $MV(f_t) = \sum_{X\in\{A,S\}} \frac{\psi_X \mu_X}{w\psi^*} V_X^B(f_t)$ and observe that

$$\sum_{X\in\{A,S\}} \frac{\psi_X \mu_X}{w} D(\alpha_t, b_t, \alpha^*(f_t), b^*(f_t)) V_X^B(f_t) - \chi \alpha_t \psi^* A(\bar{b}, \alpha^*(f_t), b^*(f_t)) - \psi^* \alpha b_t D(1, b_t, \alpha^*(f_t), b^*(f_t)) =$$

$$\psi^* D(0) A(0, \alpha^*(f_t), b^*(f_t))(MV(f_t) + \alpha_t h(b_t, MV(f_t), A(0, \alpha^*(f_t), b^*(f_t))).$$

We can therefore rewrite $V_0^B(f)$ (defined in Lemma 1) as

$$V_0^B(f) = \sup_{\{\alpha,b\}\in\Theta} w\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t r(f_s)ds} \psi^* D(0) A(0, \alpha^*(f_t), b^*(f_t))$$
$$\times (MV(f_t) + \alpha_t h(b_t, MV(f_t), A(0, \alpha^*(f_t), b^*(f_t)))dt | f_0 = f].$$

It follows immediately by the strict positivity of the interest rate that if (α, b) satisfy the conditions of Lemma 3 given the function $MV(f)$ for all $f \in \mathbb{R}^K$, then they are an optimal policy in the bank's sequence problem given the (α^*, b^*) used to define the functions $V_X^B(f)$ and hence $MV(f)$, as all other policies must generate weakly lower values of $\alpha_t h(b_t, MV(f_t), A(0, \alpha^*(f_t), b^*(f_t)))$.

D.5 Proof of Lemma 5

Observe that under the assumption of $\omega = 1$, $h(b, MV, A_0)$ does not depend on A_0. In what follows, we omit the A_0 argument to emphasize this.

Observe by the envelope theorem that

$$\frac{\partial}{\partial MV} \max_{b\in\mathbb{R}} h(b, MV) = \frac{D(b^+(MV))(1 - D(b^+(MV)))^{-\theta\sigma}}{D(0)(1 - D(0))^{-\theta\sigma}} - 1.$$

This is strictly positive wherever $b^+(MV) > 0$.

Define $MV_0 = \sigma(1 + (\sigma\theta - 1)D(0)))^{-1}$, and observe that $b^+(MV_0) = 0$. We have

$$h(b^+(MV_0), MV_0) = -\frac{\chi}{w} \frac{(1 - D(\bar{b}))^{-\sigma\theta}}{D(0)(1 - D(0))^{-\sigma\theta}} < 0.$$

Now observe that, for any arbitrary $b > 0$, $h(b^+(MV), MV) \geq h(b, MV)$ and $\lim_{MV\to\infty} h(b, MV) = \infty$, from which it follows that there exists a value of MV such that $h(b^+(MV), MV) > 0$, and hence by continuity some $MV_{min} > MV_0 > 0$ such that $h(b^+(MV), MV) = 0$. Defining $b_{min} = b^+(MV_{min})$, the uniqueness and positivity of b_{min} follow from Lemma 2. The claims about

$h(\cdot) < 0$ (resp. > 0) follow from $\frac{\partial}{\partial MV} \max_{b \in \mathbb{R}} h(b, MV) > 0$ in the neighborhood of MV_{min}.

D.6 Proof of Lemma 6

We can rewrite this equation as

$$(1 + \lambda\psi_A)(1 + \lambda\psi_S)MV_{min} - \frac{\mu_S\psi_S}{\psi^*}(1 + \lambda\psi_A) - \frac{\mu_A\psi_A}{\psi^*}(1 + \lambda\psi_S) = 0,$$

which can be further refined to

$$c_2\lambda^2 + c_1\lambda + c_0 = 0,$$

where

$$\begin{aligned} c_2 &= \psi_A\psi_S MV_{min}, \\ c_1 &= (\psi_A + \psi_S)MV_{min} - \frac{\psi_A\psi_S}{\psi^*}, \\ c_0 &= MV_{min} - 1. \end{aligned}$$

Noting that, by $c_0 < 0$ and $c_2 > 0$,

$$c_1^2 - 4c_2c_0 > c_1^2 > 0,$$

it follows that this quadratic equation has two solutions,

$$\lambda = \frac{-c_1 + \sqrt{c_1^2 - 4c_2c_0}}{2c_2} > 0$$

and

$$\frac{-c_1 - \sqrt{c_1^2 - 4c_2c_0}}{2c_2} < 0,$$

from which the claim follows.

D.7 Proof of Theorems 1 and 2

We construct a Markov-in-f equilibrium. Define the function $d : [MV_{min}, 1] \to \mathbb{R}_+$ by

$$d(m) = D(b^+(m)),$$

where $D(\cdot)$ and b^+ are defined as in the main text, and let $D_{min} = d(MV_{min})$. Note that $d(\cdot)$ is strictly increasing and differentiable.

By Lemma 5, the departure rate $D^*(f)$ satisfies the sufficient conditions for the existence of an equilibrium described in Lemma 4 if, for all $f \in \mathcal{I}$, $D^*(f) \in \Gamma(m(f))$, where

$$\Gamma(m) = \begin{cases} \{D(0)\}, & m < MV_{min}, \\ [D(0), D_{min}], & m = MV_{min}, \\ \{d(m)\}, & m > MV_{min}. \end{cases} \tag{13}$$

(This follows from $h(b^+(m), m, 1) = 0$ if and only if $m = MV_{min}$, with single-crossing, plus $D^*(\alpha, b_{min}, \alpha, b_{min}) = \alpha D_{min} + (1 - \alpha D(0).)$

It is also without loss of generality to normalize $w = 1$. Thus, given some $\Lambda > 0$, it is sufficient to construct a set of real-valued measurable functions on $\mathcal{I}$, $(V_{\Lambda,S}, V_{\Lambda,A}, m_\Lambda, D^*_\Lambda)$, such that:

1. The functions $V_{\Lambda,X}$ satisfy (8) given $D^*(f, Q) = D^*_\Lambda(f)$ with $w = 1$,

2. m_Λ satisfies (9) given $V_{\Lambda,X}$ and $w = 1$, and

3. $D^*_\Lambda(f) \in \Gamma(m_\Lambda(f))$ for all $f \in \mathcal{I}$.

We prove two constructive results. The first concerns the zero volatility limit.

Theorem 3 *Under Assumptions 1-4, if $\Lambda = 0$ there exists an equilibrium $(V_{0,S}, V_{0,A}, m_0, D^*_0)$ such that:*

1. *If $f < \lambda^{-1} D(0)$, $D^*_0(f) = D(0)$,*

2. *If $f > \lambda^{-1} D_{min}$, $D^*_0(f) = d(m_0(f)) > D_{min}$,*

3. *If $f \in [\lambda^{-1} D(0), \lambda^{-1} D_{min}]$, $D^*_0(f) = \lambda f$.*

Proof. See Section E. ■

This proves Theorem 1.

Our second result shows that, away from the frozen limit, an equilibrium exists that resembles the frozen limit.

Theorem 4 *Under Assumptions 1-4, for any $\xi > 0$ such that*

$$\lambda^{-1} D(0) + \xi < \bar{f} < \lambda^{-1} D_{min} - \xi,$$

there exists a $\bar{\Lambda}_\xi > 0$ such that, for all $\Lambda \in [0, \bar{\Lambda}_\xi)$, there exists an equilibrium $(V_{\Lambda,S}, V_{\Lambda,A}, m_\Lambda, D^*_\Lambda)$ such that:

1. *If $f \leq \lambda^{-1}D(0) - \xi$, $D^*_\Lambda(f) = D(0)$,*

2. *If $f \geq \lambda^{-1}D_{min} + \xi$, $D^*_\Lambda(f) = d(m_\Lambda(f)) > D_{min}$,*

3. *If $f \in [\lambda^{-1}D(0) + \xi, \lambda^{-1}D_{min} - \xi]$, $D^*_\Lambda(f) \in (D(0), D_{min})$.*

Proof. See Section E. ■

This proves Theorem 2.

D.8 Proof of Corollary 1

As in the proof of Lemma 4, observe that $V_0^B(f)$ as defined in Lemma 1 can be written as

$$V_0^B(f) = \sup_{\{\alpha,b\}\in\Theta} \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t f_s ds}\psi^* D(0)A(0,\alpha^*(f_t),b^*(f_t))$$
$$\times(MV(f_t) + \alpha_t h(b_t, MV(f_t), A(0,\alpha^*(f_t),b^*(f_t)))dt|f_0 = f].$$

The result follows from Proposition 1 under that observation that in the absorbing region we have $b^*(f_t) = b_{min}$ and $MV(f_t) = MV_{min}$, implying $h(\cdot) = 0$.

D.9 Proof of Proposition 1

Define

$$v_X(s,\beta,\Delta) = \mathbf{1}\{s < \Delta\}w\mathbb{E}_s^{\mathbb{Q}}[\int_s^\Delta e^{-\int_s^t (r(f_u)+\psi_X \hat{D}(u,f_u,\alpha^*(f_u),b^*(f_u),\beta))du} r(f_t)dt]$$

as the value paid by the bank to the depositor as interest.

The value of a depositor of type X at this bank at time s is

$$V^X(s,f_s,\beta) = \beta v_X(s,\beta,\Delta)$$
$$+\psi_X \mathbb{E}_s^{\mathbb{Q}}[\int_s^\infty e^{-\int_s^t (r(f_u)+\psi_X)du}\bar{V}^X(t,\beta,f_t)dt]$$

where

$$\bar{V}^X(t,f,\beta) = \alpha^*(f)A(b^*(f),\alpha^*(f),b^*(f))w\mathbb{E}[\max\{b^*(f)-\delta+\sigma_{ijt}+V^X(t,f,0),V^X(t,f,\beta)\}]$$
$$+ (1-\alpha^*(f))A(0,\alpha^*(f),b^*(f))w\mathbb{E}[\max\{-\delta+\sigma_{ijt}+V^X(t,f,0),V^X(t,f,\beta)\}].$$

Taking expectations,

$$\bar{V}^X(t,f,0) = V^X(t,f,0) + w\alpha^*(f)A(b^*(f),\alpha^*(f),b^*(f))\sigma\ln(1-D(b^*(f)))$$
$$+ w(1-\alpha^*(f))A(0,\alpha^*(f),b^*(f))\sigma\ln(1-D(0))$$

and

$$\bar{V}^X(t,f,\beta) = \bar{V}^X(t,f,0) + (1-\hat{D}(t,f,\alpha^*(f),b^*(f),\beta))(V^X(t,f,\beta)-V^X(t,f,0)).$$

Subtracting off the value without interest,

$$V^X(s,f_s,\beta) - V^X(s,f_s,0) = \beta v_X(s,\beta,\Delta)$$
$$+ \psi_X\mathbb{E}_s^{\mathbb{Q}}[\int_s^\infty e^{-\int_s^t(r(f_u)+\psi_X)du}(\bar{V}^X(t,f_t,\beta)-\bar{V}^X(t,f_t,0))dt].$$

From this,

$$V^X(s,f_s,\beta) - V^X(s,f_s,0) = \beta v_X(s,\beta,\Delta)$$
$$+ \psi_X\mathbb{E}_s^{\mathbb{Q}}[\int_s^\infty e^{-\int_s^t(r(f_u)+\psi_X)du}(1-D(t,f_t,\alpha^*(f_t),b^*(f_t),\beta))(V^X(f_t,\beta)-V^X(f_t,0))dt].$$

Letting $\mathcal{L}$ be the generator associated with f, the corresponding PDE is

$$\begin{aligned}\mathcal{L}^f(V^X(s,f_s,\beta)-V^X(s,f_s,0)) = &-\beta w r(f_s)\\ &-\psi_X(1-D(t,f_s,\alpha^*(f_s),b^*(f_s),\beta))(V^X(s,f_s,\beta)-V^X(s,f_s,0))\\ &+(r(f_s)+\psi_X)(V^X(s,f_s,\beta)-V^X(s,f_s,0))\\ &+\frac{\partial}{\partial s}(V^X(s,f_s,\beta)-V^X(s,f_s,0)).\end{aligned}$$

which is solved by

$$V^X(s,f_s,\beta) - V^X(s,f_s,0) = \beta v_X(s,f_s,\beta).$$

Define

$$\begin{aligned} D(\alpha,b,v) &= \alpha A(b,\alpha,b)D(b-v) \\ &\quad + (1-\alpha)A(0,\alpha,b)D(-v) \end{aligned}$$

and it follows that

$$\hat{D}(s,f_s,\alpha^*(f_s),b^*(f_s),\beta)) = D(\alpha^*(f_u),b^*(f_u),\beta w^{-1}v_X(s,f_s,\beta)).$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial v}D(\alpha,b,v) &= -\sigma^{-1}\alpha A(b,\alpha,b)D(b-v)(1-D(b-v)) + \\ &\quad - \sigma^{-1}\alpha A(0,\alpha,b)D(-v)(1-D(-v)) \\ &\leq -\sigma^{-1}D(\alpha,b,v). \end{aligned}$$

By $D(\alpha,b,v) \leq D(\alpha,b,0)$, it follows that

$$\begin{aligned} \hat{D}(s,f_s,\alpha^*(f_s),b^*(f_s),\beta)) &\geq D^*(f_s) - \frac{\beta}{w\sigma}D^*(f_s)v_X(s,f_s,\beta) \\ &\geq D^*(f_s) - \frac{\beta}{w\sigma}D^*(f_s)v_X(0,f_s,\beta). \end{aligned}$$

The β-relevant portion of the banks objective function is therefore

$$\begin{aligned} &\sum_{X\in\{A,S\}} Q_{Xj0}(\mathbb{E}^{\mathbb{Q}}[e^{-\int_s^{\Delta}(r(f_u)+\psi_X\hat{D}(u,f_u,\alpha^*(f_u),b^*(f_u),\beta))du}V_X^B(f_\Delta)] - \beta v_X(0,f_s,\beta)) \leq \\ &\sum_{X\in\{A,S\}} Q_{Xj0}(\mathbb{E}^{\mathbb{Q}}[e^{-\int_s^{\Delta}(r(f_u)+\psi_X D^*(f_u)(1-\frac{\beta}{w\sigma}v_X(0,f_s,\beta))du}V_X^B(f_\Delta)] - \beta v_X(0,f_s,\beta)). \end{aligned}$$

Differentiating this upper bound with respect β, and using

$$\mathbb{E}^{\mathbb{Q}}[e^{-\int_s^{\Delta}(r(f_u)+\psi_X D^*(f_u)(1-\frac{\beta}{w\sigma}v_X(0,f_s,\beta))du}V_X^B(f_\Delta)] \leq w,$$

and $D^*(f_u) < 1$,

$$\frac{\partial}{\partial\beta}\sum_{X\in\{A,S\}} Q_{Xj0}(\mathbb{E}^{\mathbb{Q}}[e^{-\int_s^{\Delta}(r(f_u)+\psi_X D^*(f_u)(1-\frac{\beta}{w\sigma}v_X(0,f_s,\beta))du}V_X^B(f_\Delta)] - \beta v_X(0,f_s,\beta)) <$$

$$\frac{\partial}{\partial\beta}(\beta v_X(0,f_s,\beta)))\sum_{X\in\{A,S\}} Q_{Xj0}(\psi_X\sigma^{-1}\Delta - 1),$$

and therefore

$$\Delta^{-1} \leq \sigma^{-1}\frac{\sum_{X\in\{A,S\}} Q_{Xj0}\psi_X}{\sum_{X\in\{A,S\}} Q_{Xj0}}$$

is sufficient for the objective to be decreasing in β and hence for $\beta = 0$ to be optimal.

E Proof of Theorems 3 and 4

We first prove an existence result with a property for m that is related to single-crossing. The construction uses penalized functions to establish the single-crossing property and a regularization approach to identify the limit of the differential inclusion. Define $\mathcal{I}_+ = (\bar{f}, f_+)$, $\mathcal{I}_- = (f_-, \bar{f})$,

$$B(a) = (\psi_A + \psi_S)MV_{min} - \frac{\psi_A \psi_S}{\psi^*} a,$$

and

$$R(f, D, a) = f(1 - MV_{min}) - DB(a).$$

Define the set

$$\begin{aligned} \mathcal{A} = \{a \in [0,1] : D(0)B(a) \leq \bar{f}(1 - MV_{min}) \leq D_{min}B(a)\} \\ \cap \{a \in [0,1] : 0 \leq \hat{V}_S(a, MV_{min}) \leq 1\} \\ \cap \{a \in [0,1] : 0 \leq \hat{V}_A(a, MV_{min}) \leq 1\}. \end{aligned}$$

We introduce some algebraic relations that are useful in what follows, and show that the set $\mathcal{A}$ is non-empty.

Lemma 7 *The constant $\hat{a}$ is strictly positive, $\hat{V}_X(\hat{a}, MV_{min}) = (1 + \lambda\psi_X)^{-1}$,*

$$\hat{a} = 1 - \psi^* MV_{min} \lambda,$$

and

$$\frac{(1 - MV_{min})}{B(\hat{a})} = \lambda.$$

By Assumption 4, $\hat{a} \in int(\mathcal{A})$.

Proof. See Section G.1. ■

Define the constant

$$\bar{D} = \max\{d(1), \frac{2f_+(1 - MV_{min})}{\psi_S MV_{min}}\},$$

which will serve as an upper bound on the departure rate.

Next, given any $\bar{a} \in \mathcal{A}$, we construct a fixed point.

Proposition 2 *(Existence) For any $\bar{a} \in \mathcal{A}$ and $\Lambda > 0$, there exist measurable functions $a_{\Lambda,\bar{a}} : \mathcal{I} \to [0,1]$, $m_{\Lambda,\bar{a}} : \mathcal{I} \to [0,1]$, and $D^*_{\Lambda,\bar{a}} : \mathcal{I} \to [0, \bar{D}]$ with $a_{\Lambda,\bar{a}}$ and $m_{\Lambda,\bar{a}}$ in $\mathcal{C}^{1,1}(\mathcal{I}_+ \cup \mathcal{I}_-) \cap \mathcal{C}^0(\mathcal{I})$ such* *that, a.e. on $\mathcal{I}$,*

$$\mathcal{L}a_{\Lambda,\bar{a}}(f) = fa_{\Lambda,\bar{a}}(f) - f + \psi^* D^*_{\Lambda,\bar{a}}(f) m_{\Lambda,\bar{a}}(f),$$

$$\mathcal{L}m_{\Lambda,\bar{a}}(f) = (f + (\psi_A + \psi_S) D^*_{\Lambda,\bar{a}}(f)) m_{\Lambda,\bar{a}}(f) - f - \frac{\psi_A \psi_S}{\psi^*} D^*_{\Lambda,\bar{a}}(f) a_{\Lambda,\bar{a}}(f),$$

$$D^*_{\Lambda,\bar{a}}(f) = \begin{cases} d(m_{\Lambda,\bar{a}}(f)) & m_{\Lambda,\bar{a}}(f) > MV_{min}, \\ D(0) & m_{\Lambda,\bar{a}}(f) < MV_{min}, \\ \frac{f(1-MV_{min})}{B(a_{\Lambda,\bar{a}}(f))} & m_{\Lambda,\bar{a}}(f) = MV_{min}, \end{cases}$$

with $m_{\Lambda,\bar{a}}(\bar{f}) = MV_{min}$, $a_{\Lambda,\bar{a}}(\bar{f}) = \bar{a}$, $a'_{\Lambda,\bar{a}}(f_-) = a'_{\Lambda,\bar{a}}(f_+) = 0$, $m'_{\Lambda,\bar{a}}(f_-) = m'_{\Lambda,\bar{a}}(f_+) = 0$, and

$$B(a_{\Lambda,\bar{a}}(f)) > 0 \textit{ a.e. on } \{m_{\Lambda,\bar{a}}(f) = MV_{min}\}$$

Moreover, on $f \in \mathcal{I}_+$, $m_{\Lambda,\bar{a}}(f) \geq MV_{min}$ and, a.e. where $m_{\Lambda,\bar{a}}(f) = MV_{min}$,

$$R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) \leq 0.$$

Likewise, on $f \in \mathcal{I}_-$, $m_{\Lambda,\bar{a}}(f) \leq MV_{min}$ and, a.e. where $m_{\Lambda,\bar{a}}(f) = MV_{min}$,

$$R(f, D(0), a_{\Lambda,\bar{a}}(f)) \geq 0.$$

Proof. See Section F. ■

Given any family of non-empty compact sets $\mathcal{A}^*_\Lambda \subseteq \mathcal{A}$, define, for $f \neq \bar{f}$,

$$m^{*+}(f) = \lim \sup_{\Lambda \downarrow 0, f' \to f} \sup_{\bar{a} \in \mathcal{A}^*_\Lambda} m_{\Lambda,\bar{a}}(f'),$$

and

$$m_{*-}(f) = \lim \inf_{\Lambda \downarrow 0, f' \to f} \inf_{\bar{a} \in \mathcal{A}^*_\Lambda} m_{\Lambda,\bar{a}}(f'),$$

and define a^{*+} and a_{*-} from $a_{\Lambda,\bar{a}}$ in the same fashion,

$$a^{*+}(f) = \lim \sup_{\Lambda \downarrow 0, f' \to f} \sup_{\bar{a} \in \mathcal{A}^*_\Lambda} a_{\Lambda,\bar{a}}(f'),$$

and

$$a_{*-}(f) = \lim \inf_{\Lambda \downarrow 0, f' \to f} \inf_{\bar{a} \in \mathcal{A}^*_\Lambda} a_{\Lambda,\bar{a}}(f').$$

We leave the values of these functions at $\bar{f}$ undefined for now. Note that these functions depend on the definition of $\mathcal{A}^*_\Lambda$; we suppress this dependence to simplify notation.

We construct the equilibrium outward from $\bar{f}$. First, we show that the average values are near $\hat{a}$ in the vicinity of $\bar{f}$ (in a one-sided sense).

Lemma 8 *Under Assumption 4, with $\mathcal{A}^*_\Lambda = \mathcal{A}$ for all Λ,*

$$\limsup_{f \downarrow \bar{f}} a^{*+}(f) \leq \hat{a}, \qquad \liminf_{f \uparrow \bar{f}} a_{*-}(f) \geq \hat{a}.$$

Proof. See Section G.2. ■

The following lemma shows that the zero-volatility limit has a contact interval.

Lemma 9 *Under Assumption 4, with $\mathcal{A}^*_\Lambda = \mathcal{A}$ for all Λ, there exists a $\delta > 0$ such that, for all $f \in (\bar{f}, \bar{f} + \delta]$,*

$$m^{*+}(f) = MV_{min}, \qquad R(f, D_{min}, a^{*+}(f)) < 0,$$

and for all $f \in [\bar{f} - \delta, \bar{f})$,

$$m_{*-}(f) = MV_{min}, \qquad R(f, D(0), a_{*-}(f)) > 0.$$

Proof. See Section G.3. ■

We next show that if contact occurs on an open set away from $\bar{f}$, then contact must occur for small but non-zero volatility at some point. Note that this lemma holds for any family of non-empty compact sets $\mathcal{A}^*_\Lambda \subseteq \mathcal{A}$.

Lemma 10 *If for some $f_+ > f_2 > f_1 > \bar{f}$, $m^{*+}(f) = MV_{min}$ and $R(f, D_{min}, a^{*+}(f)) < 0$ for all $f \in [f_1, f_2]$, then there exists a $\bar{\Lambda}(f_1, f_2)$ such that, for all $\Lambda \in (0, \bar{\Lambda}(f_1, f_2))$ and $\bar{a} \in \mathcal{A}^*$, there exists an $f \in (f_1, f_2)$ with*

$$m_{\Lambda,\bar{a}}(f) = MV_{min}.$$

Likewise, if for some $f_- < f_1 < f_2 < \bar{f}$, $m_{*-}(f) = MV_{min}$ and $R(f, D(0), a_{*-}(f)) > 0$ for all $f \in [f_1, f_2]$, then there exists a $\bar{\Lambda}(f_1, f_2)$ such that, for all $\Lambda \in (0, \bar{\Lambda}(f_1, f_2))$ and $\bar{a} \in \mathcal{A}^*_\Lambda$, there exists an $f \in (f_1, f_2)$ with

$$m_{\Lambda,\bar{a}}(f) = MV_{min}.$$

Proof. See Section G.4. ■

Using this result, we show there is a contact interval with non-zero but sufficiently small volatility.

Lemma 11 *Under Assumption 4, there exists a* $\delta' > 0$ *and* $\bar{\Lambda} > 0$ *such that for all* $f \in [\bar{f} - \delta', \bar{f} + \delta']$, $\bar{a} \in \mathcal{A}$, *and* $\Lambda \in (0, \bar{\Lambda})$, $m_{\Lambda,\bar{a}}(f) = MV_{min}$.

Proof. See Section G.5. ■

Define $\bar{\Lambda}$ as in Lemma 11.

Lemma 12 *Under Assumption 4, there exists a* $\bar{\Lambda}' > 0$ *such that, for all* $\Lambda \in (0, \bar{\Lambda}')$, *there exists an* $\bar{a}_\Lambda \in \mathcal{A}$ *and* $(a_{\Lambda,\bar{a}_\Lambda}, m_{\Lambda,\bar{a}_\Lambda}, D^*_{\Lambda,\bar{a}_\Lambda})$ *from Proposition 2 such that*

$$\lim_{f \uparrow \bar{f}} a'_{\Lambda,\bar{a}_\Lambda}(f) = \lim_{f \downarrow \bar{f}} a'_{\Lambda,\bar{a}_\Lambda}(f).$$

Proof. See Section F. ■

In what follows, assume $\Lambda \in (0, \bar{\Lambda}')$ and let

$$a_\Lambda = a_{\Lambda,\bar{a}_\Lambda}, \qquad m_\Lambda = m_{\Lambda,\bar{a}_\Lambda}, \qquad D^*_\Lambda = D^*_{\Lambda,\bar{a}_\Lambda}$$

where $\bar{a}_\Lambda$ is defined as in Lemma 12. For all of the lemmas that follow, let $\mathcal{A}^*_\Lambda = \{\bar{a}_\Lambda\}$ as defined from Lemma 12 for some $\Lambda \in (0, \bar{\Lambda}')$.

Using this lemma, we can show that smooth pasting holds at $\bar{f}$, which allows us to verify an equilibrium exists.

Lemma 13 *For all* $\Lambda \in (0, \bar{\Lambda}')$, *the functions*

$$V_{\Lambda,X}(f) = \hat{V}_X(a_\Lambda(f), m_\Lambda(f))$$

satisfy (8).

Proof. See Section G.6. ■

Lemma 13 almost proves the equilibrium existence portion of Theorem 4. What remains to be shown is that $D^*_\Lambda(f) \in \Gamma(m_\Lambda(f))$ as well as the claimed properties of the equilibrium and the existence result for the $\Lambda = 0$ case. We proceed by showing that there is an interval on which the average value is constant in the zero-volatility limit.

From m_Λ, define the upper and lower half-relaxed limits

$$m^*(f) = \lim \sup_{\Lambda \downarrow 0, f' \to f} m_\Lambda(f'),$$

and

$$m_*(f) = \lim \inf_{\Lambda \downarrow 0, f' \to f} m_\Lambda(f'),$$

and define a^* and a_* from a_Λ in the same fashion.

Lemma 14 *Under Assumption 4, there exists a $\delta'' > 0$, such that, for all $f \in [\bar{f} - \delta'', \bar{f} + \delta'']$,*

$$a^*(f) = a_*(f) = \hat{a}.$$

Proof. See Section G.7 ■

We next prove a uniform Lipschitz bound away from $\bar{f}$. The proof method follows that of Lemma 2.4 of Stynes and Stynes (2018). Let

$$\delta^* = \min\{\delta', \delta'', \lambda^{-1} D_{min} - \bar{f}, \bar{f} - \lambda^{-1} D(0)\},$$

where δ' and δ'' are defined in Lemmas 11 and 14.

Lemma 15 *Under Assumption 4, there exists a $\bar{\Lambda} > 0$ such that, on the set $K = [f_-, \bar{f} - \delta^*] \cup [\bar{f} + \delta^*, f_+]$, m_Λ and a_Λ are Lipschitz-continuous on K, with Lipschitz constants that are uniform on $\Lambda \in (0, \bar{\Lambda})$.*

Proof. See Section G.8. ■

We now extend our results concerning the contact region in the zero-volatility case to the full interval.

Lemma 16 *Under Assumption 4, for all $f \in [\lambda^{-1} D(0), \lambda^{-1} D_{min}]$,*

$$a^*(f) = a_*(f) = \hat{a}$$

and

$$m^*(f) = m_*(f) = MV_{min}.$$

Proof. See Section G.9. ■

From Lemmas 15 and 16, it follows that every sequence $\Lambda \downarrow 0$ has a subsequence on which m_Λ and a_Λ converge compactly on $\mathcal{I}$. We show that each of these limits satisfies the conditions of Theorem 3.

Lemma 17 *Under Assumption 4, for any sequence $\Lambda_n \downarrow 0$, every compact-open subsequential limit $(m_{\Lambda_n}, a_{\Lambda_n}) \to (m_0, a_0)$ satisfies $m_0(f) > MV_{min}$ if and only if $f > \lambda^{-1} D_{min}$ and $m_0(f) < MV_{min}$ if and only if $f < \lambda^{-1} D(0)$. Moreover, $a_0(f) \geq \hat{a}$ for $f \geq \bar{f}$ and $a_0(f) \leq \hat{a}$ for $f \leq \bar{f}$.*

Proof. See Section G.10. ■

We next show that each of these limits constitutes an equilibrium with zero volatility.

Lemma 18 *Under Assumption 4, given any sequence $(\Lambda_n, m_{\Lambda_n}, a_{\Lambda_n}) \to (0, m_0, a_0)$, there exists a $D_0^* : \mathcal{I} \to [0,1]$ with $D_0^*(f) \in \Gamma(m_0(f))$ for all $f \in \mathcal{I}$, such that the functions*

$$V_{0,X}(f) = \hat{V}_X(a_0(f), m_0(f))$$

satisfy (8) given D_0^*.

Proof. See Section G.11. ■

Lemmas 17 and 18 prove Theorem 3 (that m_0 is consistent with the $V_{0,X}$ is automatic given the definition of the $\hat{V}_X$ functions). To finish the proof, we prove that a version of D_Λ^* satisfies $D_\Lambda^*(f) \in \Gamma(m_\Lambda(f))$ and then show the required properties in the $\Lambda > 0$ case, which in conjunction with Lemma 13 proves Theorem 4. Note that selecting a representative from the class D_Λ^* does not affect the conclusions of Lemma 13.

Lemma 19 *Under Assumption 4, for any $\xi > 0$ such that*

$$\lambda^{-1} D(0) + \xi < \bar{f} < \lambda^{-1} D_{min} - \xi,$$

there exists a $\bar{\Lambda}_\xi > 0$ such that, for all $\Lambda \in [0, \bar{\Lambda}_\xi)$, there is a version of D_Λ^* that satisfies

1. *If $f \leq \lambda^{-1} D(0) - \xi$, $D_\Lambda^*(f) = D(0)$,*

2. *If $f \geq \lambda^{-1} D_{min} + \xi$, $D_\Lambda^*(f) = d(m_\Lambda(f)) > D_{min}$,*

3. *If $f \in [\lambda^{-1} D(0) + \xi, \lambda^{-1} D_{min} - \xi]$, $D_\Lambda^*(f) \in (D(0), D_{min})$,*

with $D_\Lambda^*(f) \in \Gamma(m_\Lambda(f))$ for all $f \in \mathcal{I}$.

Proof. See Section G.12. ■

F Proof of Proposition 2 and Lemma 12

This section proves Proposition 2 and Lemma 12 by constructing a fixed point with the desired properties. We start by defining the relevant objects for the fixed point.

Define

$$\bar{d}_0(f,a) = \max\{D(0), \frac{f(1-MV_{min})}{\max\{D_{min}^{-1}f(1-MV_{min}), B(a)\}}\}$$

and let $\rho_\epsilon : \mathbb{R} \to [0,1]$ be a smooth function such that $\rho_\epsilon(x) = 0$ on $x \leq 0$ and $\rho_\epsilon(x) = 1$ for $x \geq \epsilon$.

Let V_S^+ and V_A^+ be $\mathcal{C}^{1,1}$ functions on $\mathcal{I}_+ \to [0,1]$, let V_S^- and V_A^- be $\mathcal{C}^{1,1}$ functions on $\mathcal{I}_- \to [0,1]$, and let $\mathcal{V}_4$ be the space of quadruples of such functions equipped with the topology induced by the $\mathcal{C}^1$ norm. Given any $v \in \mathcal{V}_4$ and $\bar{a} \in \mathcal{A}$, define

$$m(f;v) = \sum_{X\in\{A,S\}} \frac{\mu_X\psi_X}{\psi^*}(\mathbf{1}\{f > \bar{f}\}V_X^+(f) + \mathbf{1}\{f < \bar{f}\}V_X^-(f) + \mathbf{1}\{f = \bar{f}\}MV_{min})$$

and

$$a(f;v) = \sum_{X\in\{A,S\}} \mu_X(\mathbf{1}\{f > \bar{f}\}V_X^+(f) + \mathbf{1}\{f < \bar{f}\}V_X^-(f) + \mathbf{1}\{f = \bar{f}\}\bar{a}).$$

For any $\epsilon > 0$, define

$$D_\epsilon^+(f;v) =$$
$$\begin{cases} (1-\rho_\epsilon(m(f;v) - MV_{min}))\bar{d}_0(f, a(f;v)) + \rho_\epsilon(m(f;v) - MV_{min})d(m(f;v)) & m(f;v) \geq MV_{min}, \\ (1-\rho_\epsilon(MV_{min} - m(f;v)))\bar{d}_0(f, a(f;v)) & m(f;v) < MV_{min}. \end{cases}$$

Likewise, define

$$D_\epsilon^-(f;v) =$$
$$\begin{cases} (1-\rho_\epsilon(MV_{min} - m(f;v)))\bar{d}_0(f, a(f;v)) + \rho_\epsilon(MV_{min} - m(f;v))D(0) & m(f;v) < MV_{min}, \\ (1-\rho_\epsilon(m(f;v) - MV_{min}))\bar{d}_0(f, a(f;v)) + \rho_\epsilon(m(f;v) - MV_{min})\bar{D} & m(f;v) \geq MV_{min}. \end{cases}$$

Let $T_\epsilon(v, \bar{a}) = (\hat{V}_S^+, \hat{V}_A^+, \hat{V}_S^-, \hat{V}_A^-)$ be a map from $\mathcal{V}_4 \times \mathcal{A}$ to $\mathcal{V}_4$ constructed as follows:

- $\hat{V}_X^+$ is the (a.e.) solution to

$$\mathcal{L}\hat{V}_X^+(f) = (f + \psi_X D_\epsilon^+(f;v))\hat{V}_X^+(f) - f$$

with boundary conditions $\hat{V}_X^+(\bar{f}) = \hat{V}_X(\bar{a}, MV_{min})$ and $\hat{V}_X^{+\prime}(f_+) = 0$,

- $\hat{V}_X^-$ is the (a.e.) solution to

$$\mathcal{L}\hat{V}_X^-(f) = (f + \psi_X D_\epsilon^-(f; v))\hat{V}_X^-(f) - f$$

with boundary conditions $\hat{V}_X^-(\bar{f}) = \hat{V}_X(\bar{a}, MV_{min})$ and $\hat{V}_X^{-\prime}(f_-) = 0$,

Note that D_ϵ^+ is zero if m is ϵ below MV_{min}, and likewise that D_ϵ^- is large (equal to $\bar{D}$) if m is ϵ above MV_{min}.

We first observe that the range of $T_\epsilon(v, \bar{a})$ lies within $\mathcal{V}_4$. For $T_\epsilon(v, \bar{a}) = (\hat{V}_S^+, \hat{V}_A^+, \hat{V}_S^-, \hat{V}_A^-)$, the existence of the $(\hat{V}_S^+, \hat{V}_A^+, \hat{V}_S^-, \hat{V}_A^-)$ and the uniqueness of the solutions to their differential equations given the boundary conditions are standard. Moreover, these functions are uniformly (in $\bar{a}$ and ϵ) bounded in the $\mathcal{C}^{1,1}$-norm on the closures of $\mathcal{I}_+/\mathcal{I}_-$ by standard regularity results Gilbarg and Trudinger (1998).

That these functions are each bounded above by one and below by zero follows from the Feynman-Kac representation. Specifically, for $f > \bar{f}$,

$$\begin{aligned}\hat{V}_X^+(f) = \mathbb{E}^{\mathbb{Q}}[\int_0^\tau e^{-\int_0^t (f_s + \psi_X D_\epsilon^+(f_s; v))ds} f_t dt | f_0 = f] \\ + \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^\tau (f_s + \psi_X D_\epsilon^+(f_s; v))ds} \hat{V}_X(\bar{a}, MV_{min}) | f_0 = f].\end{aligned}$$

That $\hat{V}_X^+(f) \in [0, 1]$ follows from the definition of $\mathcal{A}$, which ensures $\hat{V}_X(\bar{a}, MV_{min}) \in [0, 1]$, and $D_\epsilon^+(\cdot) \geq 0$. The same argument applies for $f < \bar{f}$ and $\hat{V}_X^-$.

The continuity of the $(\hat{V}_S^+, \hat{V}_A^+, \hat{V}_S^-, \hat{V}_A^-)$ functions in v follows from standard stability results. The range of $T(\cdot)$ is relatively compact in $\mathcal{V}_4$ by the uniform $C^{1,1}$ bound on the $\hat{V}$ functions and the Arzela-Ascoli theorem, given the assumed topologies. The closed convex hull of this range in $\mathcal{V}_4$ is therefore convex and compact (see Theorem 5.35 of Aliprantis and Border (2006)). It follows by the Brouwer-Schauder-Tychonoff fixed point theorem, applied to the closed convex hull of the range, that given any $\bar{a} \in \mathcal{A}$, a fixed point exists: $T_\epsilon(v_\epsilon^*, \bar{a}) = v_\epsilon^*$.

By the compactness of the range of $T_\epsilon(\cdot)$, taking the limit $\epsilon \downarrow 0$, there exists a subsequence on which $v_\epsilon^* = (V_{\epsilon,S}^{*+}, V_{\epsilon,A}^{*+}, V_{\epsilon,S}^{*-}, V_{\epsilon,A}^{*-}) \to v^* = (V_S^{*+}, V_A^{*+}, V_S^{*-}, V_A^{*-})$. Defining $D_\epsilon^{*+} = D_\epsilon^+(\cdot; v_\epsilon^*)$ and $D_\epsilon^{*-} = D_\epsilon^-(\cdot; v_\epsilon^*)$, and observing that these functions are uniformly bounded in L^∞, there exists a further subsequence on which $D_\epsilon^{*+} \to D^{*+}$ and $D_\epsilon^{*-} \to D^{*-}$ in the weak* sense. It follows, by $V_{\epsilon,X}^{*+} \to V_X^{*+}$ uniformly and $D_\epsilon^{*+} \to D^{*+}$ in the L^∞/weak* sense that $D_\epsilon^{*+} V_{\epsilon,X}^{*+} \to D^{*+} V_X^{*+}$ and

$D_\epsilon^{*-}V_{\epsilon,X}^{*-} \to D^{*-}V_X^{*-}$ in a distributional sense, from which it follows that in a distributional sense,

$$\mathcal{L}V_X^{*+}(f) = (f + \psi_X D^{*+}(f))V_X^{*+}(f) - f,$$
$$\mathcal{L}V_X^{*-}(f) = (f + \psi_X D^{*-}(f))V_X^{*-}(f) - f,$$

with (a.e.)

$$D^{*+}(f) \in \begin{cases} \{d(m(f;v^*))\} & m(f;v^*) > MV_{min}, \\ [0, D_{min}] & m(f;v^*) = MV_{min}, \\ \{0\} & m(f;v^*) < MV_{min}, \end{cases}$$

and (a.e.)

$$D^{*-}(f) \in \begin{cases} \{\bar{D}\} & m(f;v^*) > MV_{min}, \\ [D(0), \bar{D}] & m(f;v^*) = MV_{min}, \\ \{D(0)\} & m(f;v^*) < MV_{min}. \end{cases}$$

That $D^{*-}(f) \in [D(0), \bar{D}]$ a.e. follows from $D_\epsilon^{*-}(f) \in [D(0), \bar{D}]$ via standard test function arguments. That $D^{*+}(f) \le D_{min}$ a.e. on $m(f;v^*) = MV_{min}$ follows from $D_\epsilon^{*+}(f) \le \max\{D_{min}, d(m(f;v_\epsilon^*))\}$ and the uniform convergence of that bound on the set $m(f;v^*) = MV_{min}$ to D_{min}. That $D^{*+}(f) = d(m(f;v^*))$ a.e. on $m(f;v^*) > MV_{min}$ follows from the uniform convergence of $m(\cdot;v_\epsilon^*)$ in a neighborhood of f and $D_\epsilon^{*+} = d(m(\cdot;v_\epsilon^*))$ in that neighborhood for sufficiently small ϵ. The $(D^{*-}, m(\cdot;v^*) > MV_{min})$ case and the $m(\cdot;v^*) < MV_{min}$ cases follow by the same argument.

Defining $m_{\Lambda,\bar{a}}$ and $a_{\Lambda,\bar{a}}$ by $m_{\Lambda,\bar{a}}(f) = m(f;v^*)$ and $a_{\Lambda,\bar{a}}(f) = a(f;v^*)$, and

$$D_{\Lambda,\bar{a}}^*(f) = \begin{cases} D^{*+}(f) & f > \bar{f}, \\ D^{*-}(f) & f < \bar{f}, \\ \bar{d}_0(f, a_{\Lambda,\bar{a}}(f)) & f = \bar{f}, \end{cases}$$

it follows that $m_{\Lambda,\bar{a}}$ and $a_{\Lambda,\bar{a}}$ are in $\mathcal{C}^{1,1}(\mathcal{I}_- \cup \mathcal{I}_+) \cap C^0(\mathcal{I})$ and that, a.e. on $\mathcal{I}_- \cup \mathcal{I}_+$,

$$\mathcal{L}a_{\Lambda,\bar{a}}(f) = f a_{\Lambda,\bar{a}}(f) - f + \psi^* D_{\Lambda,\bar{a}}^*(f) m_{\Lambda,\bar{a}}(f),$$
$$\mathcal{L}m_{\Lambda,\bar{a}}(f) = (f + (\psi_A + \psi_S)D_{\Lambda,\bar{a}}^*(f))m_{\Lambda,\bar{a}}(f) - f - \frac{\psi_A\psi_S}{\psi^*} D_{\Lambda,\bar{a}}^*(f) a_{\Lambda,\bar{a}}(f),$$

and that the Neumann boundary conditions are satisfied.

We next prove that $m_{\Lambda,\bar{a}}(f) \geq MV_{min}$ for all $f > \bar{f}$. Suppose not: then either an interior minimum below MV_{min} exists or a minimum below MV_{min} exists at f_+ (by $m_{\Lambda,\bar{a}}(\bar{f}) = MV_{min}$). Let f' be the location of this minimum (noting that $m'_{\Lambda,\bar{a}}(f') = 0$ either because f' is an interior minimum or by the Neumann boundary condition). By continuity, there exists an open connected subset of $\mathcal{I}_+$ containing f' (if f' is interior) or whose upper endpoint is f' (if $f' = f_+$) on which $m_{\Lambda,\bar{a}}(f) < MV_{min}$. Almost everywhere on this set, by $D^{*+}(f) = 0$,

$$\mathcal{L}m_{\Lambda,\bar{a}}(f) = fm_{\Lambda,\bar{a}}(f) - f.$$

The continuity of the right-hand side ensures that $m''_{\Lambda,\bar{a}}$ is equal a.e. to a continuous function (by $m_{\Lambda,\bar{a}} \in \mathcal{C}^{1,1}$) and hence continuous, from which it follows that the ODE holds classically on this set. By $fm_{\Lambda,\bar{a}}(f) - f < f(MV_{min} - 1) < 0$, $m_{\Lambda,\bar{a}}$ cannot be constant on this set. It follows by the maximum principle that an interior minimum cannot exist, and ruling out f' interior. By the Hopf boundary point lemma, if the upper boundary of the set is $f' = f_+$, this point is a strict minimum (recalling that an interior minimum cannot exist), and thus $m'_{\Lambda,\bar{a}}(f_+) < 0$, contradicting the Neumann boundary condition. It follows that $m_{\Lambda,\bar{a}}(f) \geq MV_{min}$ for all $f > \bar{f}$.

The lower side is similar but slightly more complicated. We prove that $m_{\Lambda,\bar{a}}(f) \leq MV_{min}$ for all $f < \bar{f}$. Suppose not: then either an interior maximum above MV_{min} exists or a maximum above MV_{min} exists at f_- (by $m_{\Lambda,\bar{a}}(\bar{f}) = MV_{min}$), and let f' be the location of this maximum.

Using

$$\begin{aligned}
(\psi_A + \psi_S)m_{\Lambda,\bar{a}}(f') - \frac{\psi_A\psi_S}{\psi^*}a_{\Lambda,\bar{a}}(f') &= \psi_S m_{\Lambda,\bar{a}}(f') + \frac{\psi_A}{\psi^*}(\psi^* m_{\Lambda,\bar{a}}(f') - \psi_S a_{\Lambda,\bar{a}}(f')) \\
&= \psi_S m_{\Lambda,\bar{a}}(f') + \frac{\psi_A}{\psi^*}(\psi^* - \psi_S)V_A^{*-}(f') \\
&> \psi_S MV_{min},
\end{aligned}$$

it follows by continuity that there exists an open connected subset of $\mathcal{I}_-$ containing f' (if f' is interior) or whose lower endpoint is f' (if $f' = f_-$) on which $m_{\Lambda,\bar{a}}(f) > MV_{min}$ and

$$(\psi_A + \psi_S)m_{\Lambda,\bar{a}}(f) - \frac{\psi_A\psi_S}{\psi^*}a_{\Lambda,\bar{a}}(f) > \psi_S MV_{min}.$$

A.e. on this set, by $D^{*-}(f) = \bar{D}$,

$$\mathcal{L}m_{\Lambda,\bar{a}}(f) = \bar{D}[(\psi_A + \psi_S)m_{\Lambda,\bar{a}}(f) - \frac{\psi_A\psi_S}{\psi^*}a_{\Lambda,\bar{a}}(f)] \\ + fm_{\Lambda,\bar{a}}(f) - f.$$

The continuity of the right-hand side ensures that $m''_{\Lambda,\bar{a}}$ is equal a.e. to a continuous function (by $m_{\Lambda,\bar{a}} \in \mathcal{C}^{1,1}$) and hence continuous, from which it follows that the ODE holds classically on this set. By

$$\mathcal{L}m_{\Lambda,\bar{a}}(f) > fMV_{min} - f + \bar{D}\psi_S MV_{min} > 0,$$

$m_{\Lambda,\bar{a}}$ cannot be constant on this set (due to the strictness of the inequality). It follows by the maximum principle that an interior maximum cannot exist, and ruling out f' interior. If the lower boundary of the set is $f' = f_-$, this point is a strict maximum (recalling that an interior maximum cannot exist), and by the Hopf boundary lemma $m'_{\Lambda,\bar{a}}(f_-) < 0$, contradicting the Neumann boundary condition. It follows that $m_{\Lambda,\bar{a}}(f) \le MV_{min}$ for all $f < \bar{f}$.

By Stampacchia's lemma,[28] on the contact set ($m_{\Lambda,\bar{a}}(f) = MV_{min}$) away from $\bar{f}$, we must have (a.e.)

$$0 = (f + (\psi_A + \psi_S)D^*_{\Lambda,\bar{a}}(f))MV_{min} - f - \frac{\psi_A\psi_S}{\psi^*}D^*_{\Lambda,\bar{a}}(f)a_{\Lambda,\bar{a}}(f),$$

from which it follows that (a.e. on the contact set) $B(a_{\Lambda,\bar{a}}(f)) > 0$ and

$$D^*_{\Lambda,\bar{a}}(f) = \frac{f(1 - MV_{min})}{B(a_{\Lambda,\bar{a}}(f))}.$$

By $D^*_{\Lambda,\bar{a}}(f) \le D_{min}$ on $\{f > \bar{f} : m_{\Lambda,\bar{a}}(f) = MV_{min}\}$, it follows that

$$R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) \le 0$$

on this set. Likewise, by $D^*_{\Lambda,\bar{a}}(f) \ge D(0)$ on $\{f < \bar{f} : m_{\Lambda,\bar{a}}(f) = MV_{min}\}$, it follows that

$$R(f, D(0), a_{\Lambda,\bar{a}}(f)) \ge 0$$

on this set. These inequalities hold a.e.

This concludes the proof of Proposition 2. We next prove Lemma 12, taking Lemma 11 as given

[28] See, specifically, Theorem 7.7 of Gilbarg and Trudinger (1998).

for and using $\delta' > 0$ and $\bar{\Lambda} > 0$ as defined in that lemma. Define

$$g(a) = (a - \hat{a}) + \psi^* MV_{min}(1 - MV_{min})(B(a)^{-1} - B(\hat{a})^{-1}),$$

observing that $g(a)$ is strictly increasing wherever $B(a) > 0$, with $g(a) > 0$ (resp. < 0) if and only if $a > \hat{a}$ (resp. $a < \hat{a}$). Let $\bar{a}_L$ and $\bar{a}_H$ be the minimal and maximal elements of $\mathcal{A}$, with $g(\bar{a}_L) < 0 = g(\hat{a}) < g(\bar{a}_H)$. Define

$$\bar{\Lambda}' = \min\{(f_-|g(\bar{a}_L)|)^{\frac{1}{2}}\delta', (f_-|g(\bar{a}_H)|)^{\frac{1}{2}}\delta', \bar{\Lambda}\}$$

and assume $\Lambda \in (0, \bar{\Lambda}')$ in what follows.

Define

$$\Delta(v) = \lim_{f \downarrow \bar{f}} a'(f; v) - \lim_{f \uparrow \bar{f}} a'(f; v).$$

Note that Δ is continuous in v given the topology induced by the $\mathcal{C}^1$-norm. Observe that the set $\mathcal{A}$ is convex and compact. For arbitrary constant $\gamma > 0$, define

$$T_a(v, \bar{a}) = \arg\min_{a \in \mathcal{A}} |a - \bar{a} - \gamma\Delta(v)|.$$

By standard arguments, $T_a(v, \bar{a})$ is continuous and has a compact range. By essentially the same fixed point arguments as above, for any $\epsilon > 0$ there exists a $(v^*_\epsilon, \bar{a}_\epsilon) \in \mathcal{V}_4 \times \mathcal{A}$ such that

$$(T_\epsilon(v^*_\epsilon, \bar{a}_\epsilon), T_a(v^*_\epsilon, \bar{a}_\epsilon)) = (v^*_\epsilon, \bar{a}_\epsilon).$$

By the same compactness argument as above, there exists a subsequential limit as $\epsilon \downarrow 0$, $(v^*, \bar{a}_0)$, satisfying the conditions of Proposition 2 and with

$$\bar{a}_0 = \arg\min_{a \in \mathcal{A}} |a - \bar{a}_0 - \gamma\Delta(v^*)|.$$

If $\bar{a}_0$ is interior, then $\Delta(v^*) = 0$ concludes the proof of Lemma 12. If $\bar{a}_0$ is the maximal element of $\mathcal{A}$, $\Delta(v^*) \geq 0$, and if it is the minimal element, $\Delta(v^*) \leq 0$.

By $m_{\Lambda,\bar{a}_0} = MV_{min}$ on $[\bar{f} - \delta', \bar{f} + \delta']$ (Lemma 11), we know that a.e. on this set,

$$D^*_{\Lambda,\bar{a}_0}(f) = \frac{f(1 - MV_{min})}{B(a_{\Lambda,\bar{a}_0}(f))} \leq \bar{D},$$

from which it follows that $B(a_{\Lambda,\bar{a}_0}(f)) \geq (\bar{D})^{-1} f_-(1 - MV_{min}) > 0$. By the continuity of $a_{\Lambda,\bar{a}_0}$, this bound holds everywhere on this set. It follows that, a.e.,

$$\mathcal{L}a_{\Lambda,\bar{a}_0}(f) = f a_{\Lambda,\bar{a}_0}(f) - f + \psi^* MV_{min} \frac{f(1 - MV_{min})}{B(a_{\Lambda,\bar{a}_0}(f))}.$$

By $a_{\Lambda,\bar{a}_0} \in \mathcal{C}^{1,1}$ away from $\bar{f}$ and the aforementioned bound, the right-hand side is continuous, and $a'_{\Lambda,\bar{a}_0}$ is continuous. Thus $a''_{\Lambda,\bar{a}_0}$ is a.e. equal to a continuous function and this equation holds classically on the relevant set excluding $\bar{f}$.

Note that $\hat{a}$ is in the interior of $\mathcal{A}$ by Lemma 7, with

$$f\hat{a} - f + \psi^* MV_{min} \frac{f(1 - MV_{min})}{B(\hat{a})} = 0,$$

and therefore

$$\mathcal{L}(a_{\Lambda,\bar{a}_0}(f) - \hat{a}) = f g(a_{\Lambda,\bar{a}_0}(f))$$

Define

$$h(f) = e^{\Lambda^{-2}\Phi(f-\bar{f})^2} a'_{\Lambda,\bar{a}_0}(f)$$

and observe that (for $f \neq \bar{f}$ in a neighborhood of $\bar{f}$),

$$\begin{aligned}
h'(f) &= \frac{2}{\Lambda^2}(\Phi(f-\bar{f})h(f) + \frac{\Lambda^2}{2} a''_{\Lambda,\bar{a}_0}(f) e^{\Lambda^{-2}\Phi(f-\bar{f})^2}) \\
&= \frac{2}{\Lambda^2}\Phi(f-\bar{f})(h(f) - a'_{\Lambda,\bar{a}_0}(f) e^{\Lambda^{-2}\Phi(f-\bar{f})^2}) + \frac{2}{\Lambda^2} e^{\Lambda^{-2}\Phi(f-\bar{f})^2} f g(a_{\Lambda,\bar{a}_0}(f)) \\
&= \frac{2}{\Lambda^2} e^{\Lambda^{-2}\Phi(f-\bar{f})^2} f g(a_{\Lambda,\bar{a}_0}(f)).
\end{aligned}$$

If $\bar{a}_0$ is maximal, then $\bar{a}_0 > \hat{a}$, and by continuity this inequality holds in a neighborhood of $\bar{f}$. In this case,

$$\lim_{f \to \bar{f}} \frac{2}{\Lambda^2} e^{\Lambda^{-2}\Phi(f-\bar{f})^2} f g(a_{\Lambda,\bar{a}_0}(f)) = \frac{2}{\Lambda^2} \bar{f} g(\bar{a}_0) > 0,$$

and it follows that $h'(f) > 0$ on a neighborhood of $\bar{f}$. We prove by contradiction that $\Delta(v^*) < 0$. Suppose not ($\Delta(v^*) \geq 0$). In this case, either $\lim_{\epsilon \downarrow 0} h(\bar{f} + \epsilon) \geq 0$ or $\lim_{\epsilon \downarrow 0} h(\bar{f} - \epsilon) \leq 0$ (or both).

If $\lim_{\epsilon \downarrow 0} h(\bar{f} + \epsilon) \geq 0$, then for all $f \in (\bar{f}, \bar{f} + \delta']$,

$$h(f) \geq \int_{\bar{f}}^{f} h'(f') df'$$

and

$$a_{\Lambda,\bar{a}_0}(f) = \bar{a}_0 + \int_{\bar{f}}^{f} e^{-\Lambda^{-2}\Phi(f'-\bar{f})^2} h(f')df'.$$

Let $\tilde{f}$ be the smallest $f \in (\bar{f}, \bar{f}+\delta']$ for which $a_{\Lambda,\bar{a}_0}(f) \leq \hat{a}$, if such a point exists. For all $f \in (\bar{f}, \tilde{f})$, $h'(f) > 0$, and it follows that $a_{\Lambda,\bar{a}_0}(\tilde{f}) > \bar{a}_0$ and hence that such a point cannot exist. Therefore, $a_{\Lambda,\bar{a}_0}(f) \geq \hat{a}$ on this set, implying $h'(f) \geq 0$, from which $a_{\Lambda,\bar{a}_0}(f) \geq \bar{a}_0$ follows, and

$$a_{\Lambda,\bar{a}_0}(f) \geq \bar{a}_0 + \int_{\bar{f}}^{f} e^{-\Lambda^{-2}\Phi(f'-\bar{f})^2} \int_{\bar{f}}^{f'} \frac{2}{\Lambda^2} e^{\Lambda^{-2}\Phi(f''-\bar{f})^2} f'' g(\bar{a}_0) df'' df'.$$

We can bound this (using $e^{\Lambda^{-2}\Phi(f''-\bar{f})^2} e^{-\Lambda^{-2}\Phi(f'-\bar{f})^2} \geq 1$) as

$$a_{\Lambda,\bar{a}_0}(f) \geq \bar{a}_0 + \frac{f_{-}g(\bar{a}_0)}{\Lambda^2}(f-\bar{f})^2.$$

Evaluating at $f = \bar{f} + \delta'$ yields (by $a_{\Lambda,\bar{a}_0} \in [0,1]$)

$$a_{\Lambda,\bar{a}_0}(\bar{f}+\delta') \geq \bar{a}_0 + \frac{f_{-}g(\bar{a}_0)}{\Lambda^2}(\delta')^2 > 1,$$

a contradiction.

If $\lim_{\epsilon\downarrow 0} h(\bar{f}-\epsilon) \leq 0$, then for all $f \in [\bar{f}-\delta', \bar{f})$,

$$0 \geq h(f) + \int_{f}^{\bar{f}} h'(f')df',$$

and

$$a_{\Lambda,\bar{a}_0}(f) + \int_{f}^{\bar{f}} e^{-\Lambda^{-2}\Phi(f'-\bar{f})^2} h(f')df' = \bar{a}_0.$$

Now let $\tilde{f}$ be the largest $f \in [\bar{f}-\delta', \bar{f})$ for which $a_{\Lambda,\bar{a}_0}(f) \leq \hat{a}$, if such a point exists. For all $f \in (\tilde{f}, \bar{f})$, $h'(f) > 0$, $h(f) < 0$, and it follows that $a_{\Lambda,\bar{a}_0}(\tilde{f}) > \bar{a}_0$ and hence that such a point cannot exist. Therefore, $a_{\Lambda,\bar{a}_0}(f) \geq \hat{a}$ on this set, implying $h'(f) \geq 0$, from which $a_{\Lambda,\bar{a}_0}(f) \geq \bar{a}_0$ follows, and

$$a_{\Lambda,\bar{a}_0}(f) \geq \bar{a}_0 + \int_{f}^{\bar{f}} e^{-\Lambda^{-2}\Phi(f'-\bar{f})^2} \int_{f'}^{\bar{f}} \frac{2}{\Lambda^2} e^{\Lambda^{-2}\Phi(f''-\bar{f})^2} f'' g(\bar{a}_0) df'' df'$$

which again leads to

$$a_{\Lambda,\bar{a}_0}(\bar{f}-\delta') \geq \bar{a}_0 + \frac{f_{-}g(\bar{a}_0)}{\Lambda^2}(\delta')^2 > 1,$$

a contradiction. We conclude that $\Delta(v^*) < 0$, contradicting $\Delta(v^*) \geq 0$. Therefore, $\bar{a}_0$ cannot be maximal.

The case in which $\bar{a}_0$ is minimal is ruled out by a symmetric argument: if $\bar{a}_0 < \hat{a}$, $g(\bar{a}_0) < 0$, and supposing $\Delta \leq 0$, then either $\lim_{\epsilon \downarrow 0} h(\bar{f} - \epsilon) \geq 0$ or $\lim_{\epsilon \downarrow 0} h(\bar{f} + \epsilon) \leq 0$, either of which yields $a < \bar{a}_0$, with $a < 0$ for sufficiently small Λ. Therefore, $\bar{a}_0$ is interior and $\Delta(v^*) = 0$, which is the result.

G Proofs of Supporting Lemmas

G.1 Proof of Lemma 7

Define $\tilde{a}$ as the unique solution to

$$1 - \tilde{a} = \frac{\psi^* MV_{min}(1 - MV_{min})}{(\psi_A + \psi_S)MV_{min} - \frac{\psi_A \psi_S}{\psi^*}\tilde{a}}.$$

on $(\psi_A + \psi_S)MV_{min} > \frac{\psi_A \psi_S}{\psi^*}\tilde{a}$ (the left-hand side is decreasing and the right-hand side is increasing on the relevant domain, and the limits diverge). Recall that

$$MV_{min} = \sum_{X \in \{A,S\}} \frac{\mu_X \psi_X}{\psi^*} \frac{1}{1 + \lambda \psi_X}.$$

We guess and verify that

$$\tilde{a} = \hat{a} = \sum_{X \in \{A,S\}} \frac{\mu_X}{1 + \lambda \psi_X}.$$

Plugging this in,

$$1 - \sum_{X \in \{A,S\}} \frac{\mu_X}{1 + \lambda \psi_X} = \frac{\psi^* MV_{min} \sum_{X \in \{A,S\}} \frac{\mu_X \psi_X}{\psi^*}(1 - \frac{1}{1+\lambda\psi_X})}{\sum_{X \in \{A,S\}} \mu_X (\frac{\psi_X(\psi_A + \psi_S)}{\psi^*} - \frac{\psi_A \psi_S}{\psi^*})\frac{1}{1+\lambda\psi_X}}$$

which simplifies as

$$1 - \sum_{X \in \{A,S\}} \frac{\mu_X}{1 + \lambda \psi_X} = \psi^* MV_{min} \frac{\sum_{X \in \{A,S\}} \mu_X \frac{\lambda \psi_X^2}{1+\lambda\psi_X}}{\sum_{X \in \{A,S\}} \mu_X \frac{\psi_X^2}{1+\lambda\psi_X}} = \psi^* MV_{min} \lambda.$$

Note that $\sum_{X\in\{A,S\}}\mu_X\frac{\psi_X^2}{1+\lambda\psi_X}>0$, verifying $(\psi_A+\psi_S)MV_{min}>\frac{\psi_A\psi_S}{\psi^*}\hat{a}$ for our conjecture. This result immediately implies

$$\frac{(1-MV_{min})}{B(\hat{a})}=\frac{1-\hat{a}}{\psi^* MV_{min}}=\lambda.$$

To verify the conjecture,

$$\psi^* MV_{min}\lambda=\sum_{X\in\{A,S\}}\mu_X\frac{\lambda\psi_X}{1+\lambda\psi_X}=\sum_{X\in\{A,S\}}\mu_X(1-\frac{1}{1+\lambda\psi_X})=1-\hat{a}.$$

The claims for $\hat{V}_X$ then follow.

That $\hat{a}$ is interior given Assumption 4 follows from

$$\hat{V}_X(\hat{a},MV_{min})=\frac{1}{1+\lambda\psi_X}\in(0,1)$$

and

$$D(0)B(\hat{a})=\lambda^{-1}D(0)(1-MV_{min})<\bar{f}(1-MV_{min})\leq D_{min}B(\hat{a})=\lambda^{-1}D_{min}(1-MV_{min}).$$

G.2 Proof of Lemma 8

By Proposition 2, a.e. on $\mathcal{I}_+$, $m_{\Lambda,\bar{a}}(f)\geq MV_{min}$. Set $\theta\in(0,(\bar{D})^{-1})$ and define

$$d^+(f,a)=\min\{D_{min},\frac{f(1-MV_{min})}{\max\{\theta f(1-MV_{min}),B(a)\}}\}.$$

If $m_{\Lambda,\bar{a}}(f)>MV_{min}$, then

$$D^*_{\Lambda,\bar{a}}(f)=d(m_{\Lambda,\bar{a}}(f))\geq D_{min}\geq d^+(f,a_{\Lambda,\bar{a}}(f)).$$

If $m_{\Lambda,\bar{a}}(f)=MV_{min}$, then $B(a_{\Lambda,\bar{a}}(f))>0$, and (a.e.)

$$D^*_{\Lambda,\bar{a}}(f)=\frac{f(1-MV_{min})}{B(a_{\Lambda,\bar{a}}(f))}\geq\frac{f(1-MV_{min})}{\max\{\theta f(1-MV_{min}),B(a_{\Lambda,\bar{a}}(f))\}}\geq d^+(f,a_{\Lambda,\bar{a}}(f)).$$

Therefore, a.e., $D^*_{\Lambda,\bar{a}}(f)\geq d^+(f,a_{\Lambda,\bar{a}}(f))$.

It follows that on the interval $\mathcal{I}_+$, $a_{\Lambda,\bar{a}}$ is an a.e. (and hence viscosity) subsolution of $F^+_\Lambda(f,a,p,X)=0$, where

$$F^+_\Lambda(f,a,p,X)=fa-f+\psi^* MV_{min}d^+(f,a)-\Phi p(f-\bar{f})-\frac{\Lambda^2}{2}X.$$

This equation is proper and elliptic in the sense of Crandall et al. (1992). By the usual viscosity arguments, $a^{*+}(f)$ is a subsolution $F_0^+(f, a, p) = 0$,

$$F_0^+(f, a, p) = fa - f + \psi^* MV_{min} d^+(f, a) - p\Phi(f - \bar{f}).$$

Extend the definition of a^{*+} to $\bar{f}$ by

$$a^{*+}(\bar{f}) = \limsup_{f \downarrow \bar{f}} a^{*+}(f),$$

which ensures a^{*+} is u.s.c. on $[\bar{f}, \bar{f} + \delta']$.

For any $\xi > 0$ and $\delta' \in (0, f_+ - \bar{f})$, let

$$f_\xi \in \arg \max_{f \in [\bar{f}, \bar{f} + \delta']} a^{*+}(f) - \frac{1}{2\xi}(f - (\bar{f} + \frac{1}{2}\delta'))^2.$$

which exists by the u.s.c. property of a^{*+}. Note that

$$a^{*+}(\bar{f}) - \frac{1}{2\xi}(\bar{f} - (\bar{f} + \frac{1}{2}\delta'))^2 = a^{*+}(\bar{f}) - \frac{(\delta')^2}{8\xi}.$$

and

$$a^{*+}(\bar{f} + \delta') - \frac{1}{2\xi}(\bar{f} + \delta' - (\bar{f} + \frac{1}{2}\delta'))^2 = a^{*+}(\bar{f} + \delta') - \frac{(\delta')^2}{8\xi}.$$

It follows by $a^{*+}(\cdot) \in [0, 1]$ that for ξ small enough that $\frac{(\delta')^2}{8\xi} > 1$, f_ξ is interior. By

$$a^{*+}(f_\xi) - \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2 \geq a^{*+}(\bar{f} + \frac{1}{2}\delta')$$

it follows that

$$\frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2 \leq 1.$$

Let $\delta'_n \to 0$ be a sequence of values such that

$$\lim_{n \to \infty} a^{*+}(\bar{f} + \frac{1}{2}\delta'_n) = a^{*+}(\bar{f})$$

and set $\xi_n = \frac{(\delta'_n)^2}{16}$. We have

$$\lim_{n\to\infty}(f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n))^2 \leq \lim_{n\to\infty} 2\xi_n = 0$$

and therefore

$$\lim_{n\to\infty} f_{\xi_n} = \bar{f}.$$

We have

$$\lim\sup_{n\to\infty}[\frac{1}{2\xi_n}(f_{\xi_n} - (\bar{f}+\frac{1}{2}\delta'_n))^2 + a^{*+}(\bar{f}+\frac{1}{2}\delta'_n)] \leq \lim\sup_{n\to\infty} a^{*+}(f_{\xi_n}) \leq \lim\sup_{n\to\infty} a^{*+}(\bar{f}+\frac{1}{2}\delta'_n),$$

and therefore

$$0 \leq \lim\sup_{n\to\infty} \frac{1}{2\xi_n}(f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n))^2 \leq 0.$$

This is equivalent to

$$\lim\sup_{n\to\infty} 8(\frac{f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n)}{\delta'_n})^2 = 0$$

and therefore

$$\lim_{n\to\infty}(\frac{f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n)}{\delta'_n}) = 0.$$

Define the test function

$$g(f) = a^{*+}(f_\xi) - \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2 + \frac{1}{2\xi}(f - (\bar{f} + \frac{1}{2}\delta'))^2,$$

which satisfies $g(f_\xi) = a^{*+}(f_\xi)$ and, for all $f \in [\bar{f}, \bar{f} + \delta']$,

$$\begin{aligned} g(f) - a^{*+}(f) &= \frac{1}{2\xi}(f - (\bar{f} + \frac{1}{2}\delta'))^2 - \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2 + a^{*+}(f_\xi) - a^{*+}(f) \\ &\geq 0. \end{aligned}$$

Therefore, $g(f) \geq a^{*+}(f)$ on $f \in (\bar{f}, \bar{f} + \delta')$ with equality at some interior f_ξ. It follows by the viscosity subsolution property that

$$F_0^+(f_\xi, g(f_\xi), g'(f_\xi)) \leq 0,$$

which simplifies as

$$F_0^+(f_\xi, g(f_\xi), g'(f_\xi)) = f_\xi a^{*+}(f_\xi) - f_\xi + \psi^* MV_{min} d^+(f_\xi, a^{*+}(f_\xi)).$$
$$- \frac{1}{\xi}\Phi(f_\xi - (\bar{f} + \frac{1}{2}\delta'))(f_\xi - \bar{f}) \leq 0.$$

The quantity $f_\xi a + \psi^* MV_{min} d^+(f_\xi, a)$ is strictly increasing in a, and therefore by

$$a^{*+}(f_\xi) \geq a^{*+}(\bar{f} + \frac{1}{2}\delta') + \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2$$

we have

$$\begin{aligned}
f_\xi(a^{*+}(\bar{f} + \frac{1}{2}\delta') + \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2) - f_\xi + \\
\psi^* MV_{min} d^+(f_\xi, a^{*+}(\bar{f} + \frac{1}{2}\delta') + \frac{1}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2) + \\
-\Phi\frac{\delta'}{2\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta')) - \Phi\frac{1}{\xi}(f_\xi - (\bar{f} + \frac{1}{2}\delta'))^2 \leq \\
f_\xi a^{*+}(f_\xi) - f_\xi + \psi^* MV_{min} d^+(f_\xi, a^{*+}(f_\xi)) + \\
-\frac{1}{\xi}\Phi(f_\xi - (\bar{f} + \frac{1}{2}\delta'))(f_\xi - \bar{f}) \leq 0.
\end{aligned}$$

Using $\Phi < 0$, this simplifies in the case of $(\delta', \xi) = (\delta'_n, \xi_n)$ as

$$\begin{aligned}
f_{\xi_n}(a^{*+}(\bar{f} + \frac{1}{2}\delta'_n) + \frac{1}{2\xi_n}(f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n))^2) - f_{\xi_n} + \\
\psi^* MV_{min} d^+(f_{\xi_n}, a^{*+}(\bar{f} + \frac{1}{2}\delta'_n) + \frac{1}{2\xi_n}(f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n))^2) + \\
-8\Phi(\frac{f_{\xi_n} - (\bar{f} + \frac{1}{2}\delta'_n)}{\delta'_n}) \leq 0.
\end{aligned}$$

Taking the limit superior as $n \to \infty$,

$$\bar{f} a^{*+}(\bar{f}) - \bar{f} + \psi^* MV_{min} d^+(\bar{f}, a^{*+}(\bar{f})) \leq 0.$$

By Assumption 4 and Lemma 7,

$$D(0) < \bar{f}\frac{(1 - MV_{min})}{B(\hat{a})} < D_{min} < \bar{D},$$

and therefore $d^+(\bar{f}, \hat{a}) = \bar{f}\frac{(1-MV_{min})}{B(\hat{a})}$ and

$$\bar{f}\hat{a} - \bar{f} + \psi^* MV_{min} d^+(\bar{f}, \hat{a}) = 0.$$

It follows that by the fact that $fa - f + \psi^* MV_{min} d^+(f, a)$ is strictly increasing in a that $a^{*+}(\bar{f}) \leq \hat{a}$.

We now repeat the argument for the other side. By Proposition 2, a.e. on $\mathcal{I}_-$, $m_{\Lambda,\bar{a}}(f) \leq MV_{min}$. Define

$$d^-(f, a) = \max\{D(0), \frac{f(1 - MV_{min})}{\max\{\theta f(1 - MV_{min}), B(a)\}}\}.$$

If $m_{\Lambda,\bar{a}}(f) < MV_{min}$, then

$$D^*_{\Lambda,\bar{a}}(f) = D(0) \leq d^-(f, a_{\Lambda,\bar{a}}(f)).$$

If $m_{\Lambda,\bar{a}}(f) = MV_{min}$, then $B(a_{\Lambda,\bar{a}}(f)) > 0$,

$$D^*_{\Lambda,\bar{a}}(f) = \frac{f(1 - MV_{min})}{B(a_{\Lambda,\bar{a}}(f))} \leq \bar{D} < \theta^{-1},$$

and therefore

$$D^*_{\Lambda,\bar{a}}(f) = \frac{f(1 - MV_{min})}{\max\{\theta f(1 - MV_{min}), B(a_{\Lambda,\bar{a}}(f))\}} \leq d^-(f, a_{\Lambda,\bar{a}}(f)).$$

It follows that on the interval $\mathcal{I}_-$, $a_{\Lambda,\bar{a}}$ is an a.e. (and hence viscosity) supersolution of $F^-_\Lambda(f, a, p, X) = 0$, where

$$F^-_\Lambda(f, a, p, X) = fa - f + \psi^* MV_{min} d^-(f, a) - \Phi p(f - \bar{f}) - \frac{\Lambda^2}{2} X.$$

This equation is proper and elliptic in the sense of Crandall et al. (1992). By the usual viscosity arguments, $a_{*-}(f)$ is a supersolution $F^-_0(f, a, p) = 0$,

$$F^-_0(f, a, p) = fa - f + \psi^* MV_{min} d^-(f, a) - p\Phi(f - \bar{f}).$$

Extend the definition of a_{*-} to $\bar{f}$ by

$$a_{*-}(\bar{f}) = \liminf_{f \uparrow \bar{f}} a_{*-}(f),$$

which ensures a_{*-} is l.s.c. on $[\bar{f} - \delta', \bar{f}]$.

For any $\xi > 0$ and $\delta' \in (0, \bar{f} - f_-)$, let

$$f_\xi^- \in \arg \min_{f \in [\bar{f} - \delta', \bar{f}]} a_{*-}(f) + \frac{1}{2\xi}(f - (\bar{f} - \frac{1}{2}\delta'))^2.$$

Note that a minimizer exists by the l.s.c. property of a_{*-}. By essentially the same argument as above, for ξ small enough that $\frac{(\delta')^2}{8\xi} > 1$, f_ξ^- is interior and

$$\frac{1}{2\xi}(f_\xi^- - (\bar{f} - \frac{1}{2}\delta'))^2 \leq 1.$$

Now let $\delta'_n \to 0$ be a sequence of values such that

$$\lim_{n \to \infty} a_{*-}(\bar{f} - \frac{1}{2}\delta'_n) = a_{*-}(\bar{f})$$

and set $\xi_n = \frac{(\delta'_n)^2}{16}$. We have

$$\lim_{n \to \infty} (f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n))^2 \leq \lim_{n \to \infty} 2\xi_n = 0$$

and therefore

$$\lim_{n \to \infty} f_{\xi_n}^- = \bar{f}.$$

Again by the same argument as above,

$$\lim_{n \to \infty} (\frac{f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n)}{\delta'_n}) = 0.$$

Define the test function

$$g(f) = a_{*-}(f_\xi^-) + \frac{1}{2\xi}(f_\xi^- - (\bar{f} - \frac{1}{2}\delta'))^2 - \frac{1}{2\xi}(f - (\bar{f} - \frac{1}{2}\delta'))^2,$$

which satisfies $g(f_\xi^-) = a_{*-}(f_\xi^-)$ and, for all $f \in [\bar{f} - \delta', \bar{f}]$,

$$\begin{aligned} g(f) - a_{*-}(f) &= -\frac{1}{2\xi}(f - (\bar{f} - \frac{1}{2}\delta'))^2 + \frac{1}{2\xi}(f_\xi^- - (\bar{f} - \frac{1}{2}\delta'))^2 + a_{*-}(f_\xi^-) - a_{*-}(f) \\ &\leq 0. \end{aligned}$$

Therefore, $g(f) \leq a_{*-}(f)$ on $f \in (\bar{f} - \delta', \bar{f})$ with equality at some interior f_ξ^-. It follows by the viscosity supersolution property that

$$F_0^-(f_\xi^-, g(f_\xi^-), g'(f_\xi^-)) \geq 0,$$

which simplifies as

$$\begin{aligned}F_0^-(f_\xi^-, g(f_\xi^-), g'(f_\xi^-)) = f_\xi^- a_{*-}(f_\xi^-) - f_\xi^- + \psi^* MV_{min} d^-(f_\xi^-, a_{*-}(f_\xi^-)).\\ + \frac{1}{\xi}\Phi(f_\xi^- - (\bar{f} - \frac{1}{2}\delta'))(f_\xi^- - \bar{f}) \geq 0.\end{aligned}$$

The quantity $fa + \psi^* MV_{min} d^-(f,a)$ is strictly increasing in a, and therefore by

$$a_{*-}(f_\xi^-) \leq a_{*-}(\bar{f} - \frac{1}{2}\delta') + \frac{1}{2\xi}(f_\xi^- - (\bar{f} - \frac{1}{2}\delta'))^2$$

this simplifies in the case of $(\delta', \xi) = (\delta'_n, \xi_n)$ as

$$\begin{aligned}f_{\xi_n}^-(a_{*-}(\bar{f} - \frac{1}{2}\delta'_n) + \frac{1}{2\xi_n}(f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n))^2) - f_{\xi_n}^- +\\ \psi^* MV_{min} d^-(f_{\xi_n}^-, a_{*-}(\bar{f} - \frac{1}{2}\delta'_n) + \frac{1}{2\xi_n}(f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n))^2) +\\ + \frac{1}{\xi_n}\Phi(f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n))^2 - 8\Phi(\frac{f_{\xi_n}^- - (\bar{f} - \frac{1}{2}\delta'_n)}{\delta'_n}) \geq 0.\end{aligned}$$

Using $\Phi < 0$ and our earlier limit estimates, taking the limit inferior as $n \to \infty$,

$$\bar{f} a_{*-}(\bar{f}) - \bar{f} + \psi^* MV_{min} d^-(\bar{f}, a_{*-}(\bar{f})) \geq 0.$$

By the same argument as above, $d^-(\bar{f}, \hat{a}) = \bar{f}\frac{(1-MV_{min})}{B(\hat{a})}$ and

$$\bar{f}\hat{a} - \bar{f} + \psi^* MV_{min} d^-(\bar{f}, \hat{a}) = 0.$$

It follows that by the fact that $fa - f + \psi^* MV_{min} d^-(f,a)$ is strictly increasing in a that $a_{*-}(\bar{f}) \geq \hat{a}$.

G.3 Proof of Lemma 9

By Assumption 4 and Lemma 7, $R(\bar{f}, D_{min}, \hat{a}) < 0$ and $R(\bar{f}, D(0), \hat{a}) > 0$. It follows by Lemma 8 and the continuity of $R(\cdot)$ and its monotonicity in a that for some $\eta > 0$ there exists a

$\delta^+ \in (0, \lambda^{-1} D_{min} - \bar{f})$ such that

$$R(f, D_{min}, a^{*+}(f)) < -\eta < 0$$

for all $f \in (\bar{f}, \bar{f} + \delta^+]$, and likewise a $\delta^- \in (0, \bar{f} - \lambda^{-1} D(0))$ such that

$$R(f, D(0), a_{*-}(f)) > \eta > 0$$

for all $f \in [\bar{f} - \delta^-, \bar{f})$. Choose $\delta = \min\{\delta^+, \delta^-\}$.

By Proposition 2, on $\mathcal{I}_+ \cup \mathcal{I}_-$, $m_{\Lambda,\bar{a}}$ satisfies

$$\mathcal{L} m_{\Lambda,\bar{a}}(f) = (f + (\psi_A + \psi_S) D^*_{\Lambda,\bar{a}}(f))(m_{\Lambda,\bar{a}}(f) - MV_{min}) - f(1 - MV_{min}) + D^*_{\Lambda,\bar{a}}(f) B(a_{\Lambda,\bar{a}}(f)).$$

Consider first $\mathcal{I}_+$, on which $m_{\Lambda,\bar{a}}(f) \geq MV_{min}$. If $m_{\Lambda,\bar{a}}(f) = MV_{min}$, then

$$D^*_{\Lambda,\bar{a}}(f) B(a_{\Lambda,\bar{a}}(f)) = f(1 - MV_{min}),$$

and if $m_{\Lambda,\bar{a}}(f) > MV_{min}$, then $D^*_{\Lambda,\bar{a}}(f) = d(m_{\Lambda,\bar{a}}(f)) \geq D_{min}$ and

$$D^*_{\Lambda,\bar{a}}(f) B(a_{\Lambda,\bar{a}}(f)) = d(m_{\Lambda,\bar{a}}(f)) B(a_{\Lambda,\bar{a}}(f)).$$

Therefore,

$$\begin{aligned} \mathcal{L} m_{\Lambda,\bar{a}}(f) \geq\ & (\bar{f} + (\psi_A + \psi_S) D_{min})(m_{\Lambda,\bar{a}}(f) - MV_{min}) \\ & - \max\{0, R(f, d(m_{\Lambda,\bar{a}}(f)), a_{\Lambda,\bar{a}}(f))\}. \end{aligned}$$

Note that the right-hand side of this equation is decreasing in $a_{\Lambda,\bar{a}}(f)$. By the usual viscosity arguments, m^{*+} is a subsolution of $G_0^+(f, m, a^{*+}(f), p) = 0$,

$$\begin{aligned} G_0^+(f, m, a, p) = & (\bar{f} + (\psi_A + \psi_S) D_{min})(m - MV_{min}) \\ & - \max\{0, R(f, d(m), a)\} \\ & - \Phi(f - \bar{f}) p. \end{aligned}$$

By standard viscosity arguments for first-order equations (Crandall et al. (1992)), using the strict positivity of the discount rate and the boundedness of m^{*+},

$$m^{*+}(f) - MV_{min} \leq$$
$$\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-(\bar{f}+(\psi_A+\psi_S)D_{min})t}\max\{0, R(f_t, d(m^{*+}(f_t)), a^{*+}(f_t))\}|f_0 = f, \Lambda = 0].$$

Now observe that $f_t \in (\bar{f}, f]$ for all $t \geq 0$. It follows that for $f \in (\bar{f}, \bar{f}+\delta]$,

$$R(f_t, D_{min}, a^{*+}(f_t)) < 0,$$

which implies $B(a^{*+}(f_t)) > 0$. Given this, $R(f_t, D, a^{*+}(f_t))$ is decreasing in D, and therefore for all $t \geq 0$,

$$R(f_t, d(m^{*+}(f_t)), a^{*+}(f_t)) < 0,$$

from which it follows that $m^{*+}(f) \leq MV_{min}$, and equality holds by $m^{*+}(f) \geq m_{*-}(f) \geq MV_{min}$ on $\mathcal{I}_+$.

Now consider $\mathcal{I}_-$, on which $m_{\Lambda,\bar{a}}(f) \leq MV_{min}$. If $m_{\Lambda,\bar{a}}(f) = MV_{min}$, then

$$D^*_{\Lambda,\bar{a}}(f)B(a_{\Lambda,\bar{a}}(f)) = f(1 - MV_{min}),$$

and if $m_{\Lambda,\bar{a}}(f) < MV_{min}$, then $D^*_{\Lambda,\bar{a}}(f) = D(0)$ and

$$D^*_{\Lambda,\bar{a}}(f)B(a_{\Lambda,\bar{a}}(f)) = D(0)B(a_{\Lambda,\bar{a}}(f)).$$

Therefore,

$$\begin{aligned}\mathcal{L}m_{\Lambda,\bar{a}}(f) \leq (f_- &+ (\psi_A+\psi_S)D(0))(m_{\Lambda,\bar{a}}(f) - MV_{min}) \\ &- \min\{0, R(f, D(0), a_{\Lambda,\bar{a}}(f))\}.\end{aligned}$$

Note that the right-hand side of this equation is decreasing in $a_{\Lambda,\bar{a}}(f)$. By the usual viscosity arguments, m_{*-} is a supersolution of $G_0^-(f, m, a_{*-}(f), p) = 0$,

$$\begin{aligned}G_0^-(f, m, a, p) = &(f_- + (\psi_A+\psi_S)D(0))(m - MV_{min}) \\ &- \min\{0, R(f, D(0), a)\} \\ &- \Phi(f - \bar{f})p.\end{aligned}$$

By standard viscosity arguments for first-order equations (Crandall et al. (1992)), using the strict positivity of the discount rate and the boundedness of m_{*-},

$$m_{*-}(f) - MV_{min} \geq$$
$$\mathbb{E}^{\mathbb{Q}}[\int_0^{\infty} e^{-(f_- + (\psi_A+\psi_S)D(0))t} \min\{0, R(f_t, D(0), a_{*-}(f_t))\} | f_0 = f, \Lambda = 0].$$

Now observe that $f_t \in [f, \bar{f})$ for all $t \geq 0$. It follows that for $f \in [\bar{f} - \delta, \bar{f})$,

$$R(f_t, D(0), a_{*-}(f_t)) > 0,$$

from which it follows that $m_{*-}(f) \geq MV_{min}$, and equality holds by $MV_{min} \geq m^{*+}(f) \geq m_{*-}(f)$ on $\mathcal{I}_-$.

G.4 Proof of Lemma 10

We prove the upper side ($f_2 > f_1 > \bar{f}$) by contradiction: suppose there exists a sequence $(\Lambda_n, \bar{a}_n)$ with $\Lambda_n \downarrow 0$ and $\bar{a}_n \in \mathcal{A}^*_{\Lambda_n}$ such that, for all $f \in (f_1, f_2)$,

$$m_{\Lambda_n, \bar{a}_n}(f) > MV_{min}.$$

Define η by

$$\begin{aligned} -\eta &= \limsup_{n\to\infty} \sup_{f \in [f_1, f_2]} R(f, d(m_{\Lambda_n,\bar{a}_n}(f)), a_{\Lambda_n,\bar{a}_n}(f)) \\ &\leq \sup_{f\in[f_1,f_2]} R(f, D_{min}, a^{*+}(f)) < 0, \end{aligned}$$

noting that $m_{\Lambda_n,\bar{a}_n} \geq MV_{min}$ converges uniformly on $[f_1, f_2]$ to MV_{min} and that a^{*+} is u.s.c. on $[f_1, f_2]$.

We must have, by Proposition 2, a.e. on $[f_1, f_2]$,

$$\begin{aligned} \frac{\Lambda_n^2}{2} m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f - \bar{f}) m'_{\Lambda_n,\bar{a}_n}(f) &= (f + (\psi_A + \psi_S) d(m_{\Lambda_n,\bar{a}_n}(f))) m_{\Lambda_n,\bar{a}_n}(f) \\ &\quad - f - \frac{\psi_S \psi_A}{\psi^*} d(m_{\Lambda_n,\bar{a}_n}(f)) a_{\Lambda_n,\bar{a}_n}(f). \end{aligned} \tag{14}$$

By $m_{\Lambda_n,\bar{a}_n}$ and $a_{\Lambda_n,\bar{a}_n}$ in $\mathcal{C}^{1,1}$, $m''_{\Lambda_n,\bar{a}_n}$ is equal a.e. to a continuous function, and hence this equation holds classically on this interval.

Let

$$h_\Lambda(f) = \mathbb{E}^{\mathbb{Q}}[\int_0^\tau \frac{1}{2}\eta dt | f_0 = f] = \frac{1}{2}\eta \mathbb{E}^{\mathbb{Q}}[\tau | f_0 = f]$$

where $\tau = \inf\{t \geq 0 : f_t \notin (f_1, f_2)\}$.

We first prove a lower bound on $h_\Lambda(\frac{1}{2}(f_1 + f_2))$. Let $T = \frac{1}{4}\frac{f_2 - f_1}{|\Phi| \max_{f\in[f_1,f_2]} |f - \bar{f}|} > 0$. Observe that starting from $f = \frac{1}{2}(f_1 + f_2)$, exit can occur before T only if

$$\Lambda \sup_{t\in[0,T]} |W_t - W_0| \geq \frac{1}{4}(f_2 - f_1),$$

by $|\Phi(f' - \bar{f})| \leq |\Phi| \max_{f\in[f_1,f_2]} |f - \bar{f}|$ for $f' \in (f_1, f_2)$. By the reflection principle, the probability of this is

$$\begin{aligned}
&Pr\{\sup_{t\in[0,T]} |W_t - W_0| \geq \frac{f_2 - f_1}{4\Lambda}\} \\
\leq &Pr\{\sup_{t\in[0,T]} W_t - W_0 \geq \frac{f_2 - f_1}{4\Lambda}\} + Pr\{\sup_{t\in[0,T]} W_0 - W_t \geq \frac{f_2 - f_1}{4\Lambda}\} \\
= &2Pr\{W_T - W_0 \geq \frac{f_2 - f_1}{4\Lambda}\} + 2Pr\{W_0 - W_T \geq \frac{f_2 - f_1}{4\Lambda}\} \\
= &2Pr\{|W_T - W_0| \geq \frac{f_2 - f_1}{4\Lambda}\}.
\end{aligned}$$

By Chebyshev's inequality,

$$2Pr\{|W_T - W_0| \geq \frac{f_2 - f_1}{4\Lambda}\} \leq \frac{32T\Lambda^2}{(f_2 - f_1)^2}.$$

For any $\Lambda \leq \frac{1}{8}(f_2 - f_1)T^{-\frac{1}{2}}$,

$$Pr\{\sup_{t\in[0,T]} |W_t - W_0| \geq \frac{f_2 - f_1}{4\Lambda}\} \leq \frac{1}{2},$$

which yields

$$h_\Lambda(\frac{1}{2}(f_1 + f_2)) \geq \frac{1}{4}\eta T.$$

By assumption there exists an $\bar{n}(f_1, f_2)$ such that for all $n > \bar{n}(f_1, f_2)$,

$$\Lambda_n < \frac{1}{8}(f_2 - f_1)T^{-\frac{1}{2}},$$

$$\sup_{f\in[f_1,f_2]} |m_{\Lambda_n,\bar{a}_n}(f) - MV_{min}| < \frac{1}{8}\eta T,$$

and

$$\sup_{f\in[f_1,f_2]} R(f, d(m_{\Lambda_n,\bar{a}_n}(f)), a_{\Lambda_n,\bar{a}_n}(f)) < -\frac{1}{2}\eta.$$

We can rewrite (14) as

$$\begin{aligned}\frac{\Lambda_n^2}{2}m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f-\bar{f})m'_{\Lambda_n,\bar{a}_n}(f) = (f + (\psi_A+\psi_S)d(m_{\Lambda_n,\bar{a}_n}(f)))(m_{\Lambda_n,\bar{a}_n}(f) - MV_{min})\\ - R(f, d(m_{\Lambda_n,\bar{a}_n}(f)), a_{\Lambda_n,\bar{a}_n}(f)),\end{aligned}$$

and for all $f \in (f_1, f_2)$, by $m_{\Lambda_n,\bar{a}_n}(f) \geq MV_{min}$ on $\mathcal{I}_+$,

$$\frac{\Lambda_n^2}{2}m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f-\bar{f})m'_{\Lambda_n,\bar{a}_n}(f) > \frac{1}{2}\eta > 0. \tag{15}$$

Now observe that on $f \in (f_1, f_2)$,

$$\frac{\Lambda^2}{2}h''_{\Lambda}(f) + \Phi(f-\bar{f})h'_{\Lambda}(f) + \frac{1}{2}\eta = 0$$

and therefore

$$\frac{\Lambda_n^2}{2}(m''_{\Lambda_n,\bar{a}_n}(f) + h''_{\Lambda_n}(f)) + \Phi(f-\bar{f})(m'_{\Lambda_n,\bar{a}_n}(f) + h'_{\Lambda_n}(f)) > 0.$$

By the strictness of the inequality, $m_{\Lambda_n,\bar{a}_n}(f) + h_{\Lambda_n}(f)$ is not constant on the interior of the interval. By the maximum principle, $m_{\Lambda_n,\bar{a}_n}(f) + h_{\Lambda_n}(f)$ cannot attain an interior maximum on $f \in (f_1, f_2)$. It follows that on this interval,

$$m_{\Lambda_n,\bar{a}_n}(f) + h_{\Lambda_n}(f) \leq \max\{m_{\Lambda_n,\bar{a}_n}(f_1), m_{\Lambda_n,\bar{a}_n}(f_2)\} \leq MV_{min} + \frac{1}{8}\eta T.$$

Applying this to $f = \frac{1}{2}(f_1+f_2)$,

$$m_{\Lambda_n,\bar{a}_n}(\frac{1}{2}(f_1+f_2)) \leq MV_{min} + \frac{1}{8}\eta T - h_{\Lambda_n}(\frac{1}{2}(f_1+f_2)),$$

which contradicts $m_{\Lambda_n,\bar{a}_n}(\frac{1}{2}(f_1+f_2)) \geq MV_{min}$.

We next prove the lower side $f_1 < f_2 < \bar{f}$ by contradiction. Suppose there exists a sequence $(\Lambda_n, \bar{a}_n)$ with $\Lambda_n \downarrow 0$ and $\bar{a}_n \in \mathcal{A}^*_{\Lambda_n}$ such that, for all $f \in (f_1, f_2)$,

$$m_{\Lambda_n,\bar{a}_n}(f) < MV_{min}.$$

Define η by

$$\eta = \liminf_{n\to\infty} \inf_{f\in[f_1,f_2]} R(f, D(0), a_{\Lambda_n,\bar{a}_n}(f))$$
$$\geq \inf_{f\in[f_1,f_2]} R(f, D(0), a_{*-}(f)) > 0,$$

noting that $m_{\Lambda_n,\bar{a}_n} \leq MV_{min}$ converges uniformly on $[f_1, f_2]$ to MV_{min} and a_{*-} is l.s.c. on $[f_1, f_2]$. Define h_Λ from η as above, noting that the value of η is different on the lower side, and define T as above.

We must have, by Proposition 2, a.e. on $[f_1, f_2]$,

$$\frac{\Lambda_n^2}{2} m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f-\bar{f}) m'_{\Lambda_n,\bar{a}_n}(f) = (f + (\psi_A + \psi_S)D(0)) m_{\Lambda_n,\bar{a}_n}(f) - f - \frac{\psi_S\psi_A}{\psi^*} D(0) a_{\Lambda_n,\bar{a}_n}(f),$$

and by the same argument as above this must hold classically. By assumption there exists an $\bar{n}(f_1, f_2)$ such that for all $n > \bar{n}(f_1, f_2)$,

$$\Lambda_n < \frac{1}{8}(f_2 - f_1)T^{-\frac{1}{2}},$$

$$\sup_{f\in[f_1,f_2]} |m_{\Lambda_n,\bar{a}_n}(f) - MV_{min}| < \frac{1}{8}\eta T,$$

and

$$\inf_{f\in[f_1,f_2]} R(f, D(0), a_{\Lambda_n,\bar{a}_n}(f)) > \frac{1}{2}\eta.$$

Noting that, for all $f \in \mathcal{I}_-$ with $m_{\Lambda_n,\bar{a}_n}(f) < MV_{min}$,

$$\frac{\Lambda_n^2}{2} m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f-\bar{f}) m'_{\Lambda_n,\bar{a}_n}(f) = (f + (\psi_A + \psi_S)D(0))(m_{\Lambda_n,\bar{a}_n}(f) - MV_{min})$$
$$- R(f, D(0), a_{\Lambda_n,\bar{a}_n}(f)),$$

for all $f \in (f_1, f_2)$, by $m_{\Lambda_n,\bar{a}_n}(f) \leq MV_{min}$ on $\mathcal{I}_-$,

$$\frac{\Lambda_n^2}{2} m''_{\Lambda_n,\bar{a}_n}(f) + \Phi(f-\bar{f}) m'_{\Lambda_n,\bar{a}_n}(f) < -\frac{1}{2}\eta < 0$$

and therefore

$$\frac{\Lambda_n^2}{2}(m''_{\Lambda_n,\bar{a}_n}(f) - h''_{\Lambda_n}(f)) + \Phi(f-\bar{f})(m'_{\Lambda_n,\bar{a}_n}(f) - h'_{\Lambda_n}(f)) < 0.$$

By the strictness of the inequality, $m_{\Lambda_n,\bar{a}_n}(f) - h_{\Lambda_n}(f)$ is not constant on the interior of the interval.

By the minimum principle, $m_{\Lambda_n,\bar{a}_n}(f) - h_{\Lambda_n}(f)$ cannot attain an interior minimum on $f \in (f_1, f_2)$. It follows that

$$m_{\Lambda_n,\bar{a}_n}(f) - h_{\Lambda_n}(f) \geq \min\{m_{\Lambda_n,\bar{a}_n}(f_1), m_{\Lambda_n,\bar{a}_n}(f_2)\} \geq MV_{min} - \frac{1}{8}\eta T.$$

Applying this to $f = \frac{1}{2}(f_1 + f_2)$,

$$m_{\Lambda_n,\bar{a}_n}(\frac{1}{2}(f_1 + f_2)) \geq MV_{min} - \frac{1}{8}\eta T + h_{\Lambda_n}(\frac{1}{2}(f_1 + f_2)),$$

which contradicts $m_{\Lambda_n,\bar{a}_n}(\frac{1}{2}(f_1 + f_2)) \leq MV_{min}$.

G.5 Proof of Lemma 11

Observe, by $a_{\Lambda,\bar{a}}(\bar{f}) = \bar{a} \in \mathcal{A}$, that $R(\bar{f}, D_{min}, a_{\Lambda,\bar{a}}(\bar{f})) \leq 0$ for all $\Lambda > 0$ and $\bar{a} \in \mathcal{A}$.

By Lemma 9, there exists a $\delta > 0$ such that $R(f, D_{min}, a^{*+}(f)) < 0$ for all $f \in (\bar{f}, \bar{f} + \delta]$.

Define $\tilde{a}(f)$ as the solution to $R(f, D_{min}, \tilde{a}(f)) = 0$, noting that this function is linear with slope $\beta < 0$. By Assumption 4 and Lemma 7, $R(\bar{f}, D_{min}, \hat{a}) < 0$. It follows that for sufficiently small $\delta'_+ \in (0, \delta)$ with $\bar{f} + \delta'_+ < \lambda^{-1} D_{min}$,

$$R(\bar{f} + \delta'_+, D_{min}, \hat{a} + \frac{\Phi\beta\delta'_+}{\bar{f}}) \leq 0.$$

By Lemma 9 and standard half-relaxed limit arguments, there exists a $\Lambda^+(\delta'_+) > 0$ such that for all $\Lambda \in (0, \Lambda^+(\delta'_+))$ and $\bar{a} \in \mathcal{A}$,

$$R(\bar{f} + \delta'_+, D_{min}, a_{\Lambda,\bar{a}}(\bar{f} + \delta'_+)) < 0.$$

We prove by contradiction that for all $\Lambda \in (0, \Lambda^+(\delta'_+))$ and $\bar{a} \in \mathcal{A}$,

$$\max_{f \in [\bar{f}, \bar{f} + \delta'_+]} R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) \leq 0.$$

Suppose not: then for some $\Lambda \in (0, \Lambda^+(\delta'_+))$ and $\bar{a} \in \mathcal{A}$, define the non-empty set

$$J = \{f \in [\bar{f}, \bar{f} + \delta'_+] : R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) > 0\}.$$

By the monotonicity of $R(\cdot)$ in a, this implies $a_{\Lambda,\bar{a}}(f) > \tilde{a}(f)$ a.e. on J.

For a.e. $f \in J$, $B(a_{\Lambda,\bar{a}}(f))D_{min} < f(1 - MV_{min})$ and we must have $D^*_{\Lambda,\bar{a}}(f) > D_{min}$. It follows that a.e. on J,

$$\mathcal{L}a_{\Lambda,\bar{a}}(f) \geq f a_{\Lambda,\bar{a}}(f) - f + \psi^* MV_{min} D_{min}.$$

By $f \leq \bar{f} + \delta'_+ < \lambda^{-1} D_{min}$,

$$\mathcal{L}a_{\Lambda,\bar{a}}(f) \geq f(a_{\Lambda,\bar{a}}(f) - \hat{a}) + f(\hat{a} - 1 + \psi^* MV_{min} \lambda)$$

and the second term is zero by Lemma 7.

Let f' be a maximum of $w(f) = a_{\Lambda,\bar{a}}(f) - \tilde{a}(f)$ on $f \in [\bar{f}, \bar{f} + \delta'_+]$, which exists by continuity and must be interior by the endpoint bounds above and an element of J by construction. At such a maximum, the function

$$g(f) = w(f') \geq w(f)$$

is a function that is weakly above $w(\cdot)$ and touches at f'. Observing that

$$\mathcal{L}w(f) \geq f(w(f) + \tilde{a}(f) - \hat{a}) - \Phi(f - \bar{f})\beta$$

in an a.e. sense with continuous coefficients, $w(\cdot)$ is a subsolution is the viscosity sense. The subsolution property for $w(\cdot)$ applied to the test function g requires

$$\Phi(f' - \bar{f})\beta \geq f'(a_{\Lambda,\bar{a}}(f') - \hat{a}).$$

Using the bounds on f',

$$a_{\Lambda,\bar{a}}(f') \leq \hat{a} + \frac{\Phi\beta\delta'_+}{\bar{f}}.$$

Because $R(f, D_{min}, a)$ is increasing in f and a,

$$R(f', D_{min}, a_{\Lambda,\bar{a}}(f')) \leq R(\bar{f} + \delta'_+, D_{min}, \hat{a} + \frac{\Phi\beta\delta'_+}{\bar{f}}) \leq 0,$$

a contradiction. We conclude that $R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) \leq 0$ for all $f \in [\bar{f}, \bar{f} + \delta'_+]$ and $\Lambda \in (0, \Lambda^+(\delta'_+))$. This implies $B(a_{\Lambda,\bar{a}}(f)) \geq 0$, and therefore

$$R(f, d(m_{\Lambda,\bar{a}}(f)), a_{\Lambda,\bar{a}}(f)) \leq R(f, D_{min}, a_{\Lambda,\bar{a}}(f)) \leq 0.$$

Recall that $\delta'_+ < \delta$, where δ is defined in the statement of Lemma 9. By Lemmas 9 and 10, for some $\Lambda_1^+ > 0$ and all $\Lambda \in (0, \Lambda_1^+)$ and $\bar{a} \in \mathcal{A}$, there exists an $f_H \in (\bar{f} + \frac{1}{2}\delta'_+, \bar{f} + \delta'_+)$ such that $m_{\Lambda,\bar{a}}(f_H) = MV_{min}$.

Recall by Proposition 2 that, a.e. on $f \in \mathcal{I}_+$ with $m_{\Lambda,\bar{a}}(f) > MV_{min}$,

$$\begin{aligned}\mathcal{L}m_{\Lambda,\bar{a}}(f) &= (f + (\psi_A + \psi_S)d(m_{\Lambda,\bar{a}}(f)))(m_{\Lambda,\bar{a}}(f) - MV_{min}) \\ &\quad - R(f, d(m_{\Lambda,\bar{a}}(f)), a_{\Lambda,\bar{a}}(f)),\end{aligned}$$

which implies (by $m_{\Lambda,\bar{a}}(f) \geq MV_{min}$ on $\mathcal{I}_+$) that, a.e. on $f \in (\bar{f}, f_H)$,

$$\mathcal{L}m_{\Lambda,\bar{a}}(f) \geq 0.$$

Applying Dynkin's formula up to the first hitting time of $m_{\Lambda,\bar{a}}(f_t) = MV_{min}$, for all $f \in (\bar{f}, f_H)$ (and hence all $f \in (\bar{f}, \bar{f} + \frac{1}{2}\delta'_+])$,

$$m_{\Lambda,\bar{a}}(f) \leq \mathbb{E}^{\mathbb{Q}}[m_{\Lambda,\bar{a}}(\tau)|f_0 = f]$$

where $\tau = \inf\{t \geq 0 : f_t \notin (\bar{f}, f_H) \text{ or } m_{\Lambda,\bar{a}}(f_t) = MV_{min}\}$. By construction, $m_{\Lambda,\bar{a}}(\tau) = MV_{min}$, which implies $m_{\Lambda,\bar{a}}(f) \leq MV_{min}$, and hence equality must hold. This proves the upper side result for $\delta' \leq \frac{1}{2}\delta'_+$ and $\bar{\Lambda} \leq \min\{\Lambda^+(\delta'_+), \Lambda_1^+\}$.

The other side (for $f \in (\bar{f} - \delta', \bar{f})$) follows by a symmetric argument.

G.6 Proof of Lemma 13

By Proposition 2 and Lemma 12, $a_\Lambda \in \mathcal{C}^{1,1}(\mathcal{I})$. By Lemma 11, $m_\Lambda(f) = MV_{min}$ for $f \in [\bar{f} - \delta', \bar{f} + \delta']$ and hence by Proposition 2 that $m_\Lambda \in \mathcal{C}^{1,1}(\mathcal{I})$ (the constant interval around $\bar{f}$ bridges the two $\mathcal{C}^{1,1}$ regions of Proposition 2).

From this, the functions $V_{\Lambda,X}(f) = \hat{V}_X(a_\Lambda(f), m_\Lambda(f))$ are also in $\mathcal{C}^{1,1}(\mathcal{I})$, and satisfy (a.e. on $\mathcal{I}$)

$$\mathcal{L}(V_{\Lambda,X}(f)) = -f + (f + \psi_X D^*_\Lambda(f))V_{\Lambda,X}(f).$$

By the Ito-Krylov formula, for any fixed horizon T,

$$V_{\Lambda,X}(f) = \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\int_0^t (f_s+\psi_X D^*_\Lambda(f_s))ds} f_t dt | f_0 = f] \\ + \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T (f_s+\psi_X D^*_\Lambda(f_s))ds} V_{\Lambda,X}(f_T) | f_0 = f].$$

By the boundedness of a_Λ and m_Λ (Proposition 2) and the strict positivity of the discounting term, taking the limit as $T \to \infty$,

$$V_{\Lambda,X}(f) = \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-\int_0^t (f_s+\psi_X D^*_\Lambda(f_s))ds} f_t dt | f_0 = f]$$

which is the result.

G.7 Proof of 14

Applying Lemma 11, for $\delta' > 0$, there exists a $\bar{\Lambda}$ such that for all $\Lambda \in (0, \bar{\Lambda})$,

$$m_\Lambda(f) = MV_{min}$$

for all $f \in [\bar{f} - \delta', \bar{f} + \delta']$.

By Proposition 2, a.e. on this interval, $D^*_\Lambda(f) = \frac{f(1-MV_{min})}{B(a_\Lambda(f))}$ and $B(a_\Lambda(f)) \geq (\bar{D})^{-1} f(1 - MV_{min}) > 0$. By the continuity of a_Λ, the lower bound applies to the entire interval, with $B(a_\Lambda(f))$ bounded strictly away from zero. On this interval, a_Λ is an a.e. solution of $F_\Lambda(f, a, p, X) = 0$, where

$$F_\Lambda(f, a, p, X) = fa - f + \psi^* MV_{min} \frac{f(1 - MV_{min})}{B(a)} - \Phi p(f - \bar{f}) - \frac{\Lambda^2}{2} X.$$

This equation is proper and elliptic in the sense of Crandall et al. (1992), with continuous coefficients. By the usual viscosity arguments, $a^*(f)$ is a subsolution and $a_*(f)$ is a supersolution of $F_0(f, a, p) = 0$,

$$F_0(f, a, p) = fa - f + \psi^* MV_{min} \frac{f(1 - MV_{min})}{B(a)} - p\Phi(f - \bar{f}).$$

For any $\xi > 0$, let

$$f_\xi \in \arg \max_{f \in [\bar{f} - \delta', \bar{f} + \delta']} a^*(f) - \frac{1}{2\xi}(f - \bar{f})^2$$

which exists by the u.s.c. property of a^*. For $2\xi < (\delta')^2$, f_ξ is guaranteed to be interior, and

$\lim_{\xi\downarrow 0} f_\xi = \bar{f}$. Moreover, by

$$a^*(f_\xi) - \frac{1}{2\xi}(f_\xi - \bar{f})^2 \geq a^*(\bar{f})$$

and the u.s.c. property of a^*,

$$0 \leq \limsup_{\xi\downarrow 0} \frac{1}{2\xi}(f_\xi - \bar{f})^2 \leq \limsup_{\xi\downarrow 0} a^*(f_\xi) - a^*(\bar{f}) \leq 0,$$

and therefore

$$\lim_{\xi\downarrow 0} a^*(\bar{f}) + \frac{1}{2\xi}(f_\xi - \bar{f})^2 = a^*(\bar{f}).$$

Define the test function $g(f) = a^*(f_\xi) - \frac{1}{2\xi}(f_\xi - \bar{f})^2 + \frac{1}{2\xi}(f - \bar{f})^2$ satisfies $g(f_\xi) = a^*(f_\xi)$ and, for all $f \in [\bar{f} - \delta', \bar{f} + \delta']$,

$$\begin{aligned} g(f) - a^*(f) &= \frac{1}{2\xi}(f - \bar{f})^2 - \frac{1}{2\xi}(f_\xi - \bar{f})^2 + a^*(f_\xi) - a^*(f) \\ &\geq 0. \end{aligned}$$

Therefore, $g(f) \geq a^*(f)$ on $f \in [\bar{f} - \delta', \bar{f} + \delta']$ with equality at f_ξ. It follows by the viscosity subsolution property that

$$F_0(f_\xi, g(f_\xi), g'(f_\xi)) \leq 0,$$

which simplifies as

$$F_0(f_\xi, g(f_\xi), g'(f_\xi)) = f_\xi a^*(f_\xi) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a^*(f_\xi))} - \frac{1}{\xi}\Phi(f_\xi - \bar{f})^2 \leq 0.$$

The quantity $fa + \psi^* MV_{min} \frac{f(1-MV_{min})}{B(a)}$ is strictly increasing in a, and therefore by

$$a^*(f_\xi) \geq a^*(\bar{f}) + \frac{1}{2\xi}(f_\xi - \bar{f})^2,$$

and $\Phi < 0$,

$$\begin{aligned} f_\xi(a^*(\bar{f}) + \frac{1}{2\xi}(f_\xi - \bar{f})^2) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a^*(\bar{f}) + \frac{1}{2\xi}(f_\xi - \bar{f})^2)} \leq \\ f_\xi a^*(f_\xi) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a^*(f_\xi))} - \frac{1}{\xi}\Phi(f_\xi - \bar{f})^2 \leq 0. \end{aligned}$$

Taking the limit as $\xi \to 0$,

$$\bar{f}a^*(\bar{f}) - \bar{f} + \psi^* MV_{min} \frac{\bar{f}(1 - MV_{min})}{B(a^*(\bar{f}))} \leq 0,$$

from which it follows by Lemma 7 that $a^*(\bar{f}) \leq \hat{a}$.

We repeat the argument on the l.s.c./supersolution side. Let

$$f_\xi \in \arg \min_{f \in [\bar{f}-\delta', \bar{f}+\delta']} a_*(f) + \frac{1}{2\xi}(f - \bar{f})^2$$

and observe that $\lim_{\xi \downarrow 0} f_\xi = \bar{f}$ and

$$a_*(f_\xi) + \frac{1}{2\xi}(f_\xi - \bar{f})^2 \leq a_*(\bar{f}).$$

By the l.s.c. property of a_*,

$$0 \geq \liminf_{\xi \downarrow 0} -\frac{1}{2\xi}(f_\xi - \bar{f})^2 \geq \liminf_{\xi \downarrow 0} a_*(f_\xi) - a_*(\bar{f}) \geq 0,$$

and hence

$$\lim_{\xi \downarrow 0} a_*(\bar{f}) - \frac{1}{2\xi}(f_\xi - \bar{f})^2 = a_*(\bar{f}).$$

Define the test function $g(f) = a_*(f_\xi) + \frac{1}{2\xi}(f_\xi - \bar{f})^2 - \frac{1}{2\xi}(f - \bar{f})^2$ satisfies $g(f_\xi) = a_*(f_\xi)$ and, for all $f \in [\bar{f} - \delta', \bar{f} + \delta']$,

$$\begin{aligned} g(f) - a_*(f) &= -\frac{1}{2\xi}(f - \bar{f})^2 + \frac{1}{2\xi}(f_\xi - \bar{f})^2 + a_*(f_\xi) - a_*(f) \\ &\leq 0. \end{aligned}$$

Therefore, $g(f) \leq a_*(f)$ on $f \in [\bar{f} - \delta', \bar{f} + \delta']$ with equality at f_ξ. It follows by the viscosity supersolution property that

$$F_0(f_\xi, g(f_\xi), g'(f_\xi)) \geq 0,$$

which simplifies as

$$F_0(f_\xi, g(f_\xi), g'(f_\xi)) = f_\xi a_*(f_\xi) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a_*(f_\xi))} + \frac{1}{\xi}\Phi(f_\xi - \bar{f})^2 \geq 0.$$

Therefore, by

$$a_*(f_\xi) \le a_*(\bar{f}) - \frac{1}{2\xi}(f_\xi - \bar{f})^2,$$

and $\Phi < 0$,

$$f_\xi(a_*(\bar{f}) - \frac{1}{2\xi}(f_\xi - \bar{f})^2) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a_*(\bar{f}) - \frac{1}{2\xi}(f_\xi - \bar{f})^2)} \ge$$
$$f_\xi a_*(f_\xi) - f_\xi + \psi^* MV_{min} \frac{f_\xi(1 - MV_{min})}{B(a_*(f_\xi))} + \frac{1}{\xi}\Phi(f_\xi - \bar{f})^2 \ge 0.$$

Taking the limit as $\xi \to 0$,

$$\bar{f} a_*(\bar{f}) - \bar{f} + \psi^* MV_{min} \frac{\bar{f}(1 - MV_{min})}{B(a_*(\bar{f}))} \ge 0,$$

from which it follows that $a_*(\bar{f}) \ge \hat{a}$. By $\hat{a} \le a_*(\bar{f}) \le a^*(\bar{f}) \le \hat{a}$, equality follows.

We next prove that this conclusion extends to the interval $(\bar{f} - \delta', \bar{f} + \delta')$. By $m_\Lambda(f) = MV_{min}$ on this interval and Proposition 2, a.e. on the interior,

$$\mathcal{L} a_\Lambda(f) = f a_\Lambda(f) - f + \psi^* MV_{min} \frac{f(1 - MV_{min})}{B(a_\Lambda(f))}.$$

Therefore (by $\mathcal{L}\hat{a} = 0$ and $f\hat{a} - f + \psi^* MV_{min} \frac{f(1-MV_{min})}{B(\hat{a})} = 0$),

$$\mathcal{L}(a_\Lambda(f) - \hat{a}) = \beta(f)(a_\Lambda(f) - \hat{a})$$

where

$$\beta(f) = \begin{cases} f + \psi^* MV_{min} \frac{f(1-MV_{min})}{a_\Lambda(f) - \hat{a}} (B(a_\Lambda(f))^{-1} - B(\hat{a})^{-1}) & a_\Lambda(f) \ne \hat{a} \\ f + \psi^* MV_{min} f(1 - MV_{min}) \frac{\partial}{\partial a} B(a)^{-1}|_{a=\hat{a}} & a_\Lambda(f) = \hat{a}. \end{cases}$$

Note that $B(a_\Lambda(f)) \ge (\bar{D})^{-1} f(1 - MV_{min}) > 0$ a.e. on $f \in [\bar{f} - \delta', \bar{f} + \delta']$ by Proposition 2. By the continuity of a_Λ, $B(a_\Lambda(f))$ is bounded below away from zero and bounded above. It follows that $\beta(f)$ is bounded above and below by a positive constant.

By Ito-Krylov,

$$a_\Lambda(f) - \hat{a} = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^\tau \beta(f_s)ds}(a_\Lambda(f_\tau) - \hat{a})|f_0 = f]$$

where $\tau = \inf\{t \ge 0 : f_t \notin (\bar{f} - \delta', \bar{f}) \cup (\bar{f}, \bar{f} + \delta')\}$. By the boundedness of a_Λ and $e^{-\int_0^\tau \beta(f_s)ds} \in$

$(0,1]$,

$$a_\Lambda(f) - \hat{a} \le (a_\Lambda(\bar{f}) - \hat{a})\mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{f_\tau = \bar{f}\}e^{-\int_0^\tau \beta(f_s)ds}|f_0 = f] \\ + (1-\hat{a})\mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{f_\tau \neq \bar{f}\}|f_0 = f],$$

and

$$a_\Lambda(f) - \hat{a} \ge (a_\Lambda(\bar{f}) - \hat{a})\mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{f_\tau = \bar{f}\}e^{-\int_0^\tau \beta(f_s)ds}|f_0 = f] \\ - \hat{a}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{f_\tau \neq \bar{f}\}|f_0 = f].$$

Choose any $\delta'' \in (0, \delta')$. Given $\Phi < 0$ (reversion towards $\bar{f}$), $\lim_{\Lambda \downarrow 0} Pr\{f_\tau \neq \bar{f}|f_0 = f\} = 0$ uniformly on $f \in [\bar{f} - \delta'', \bar{f} + \delta'']$ (by the one-dimensional scale function formula, the probability of reaching $\bar{f} \pm \delta'$ before $\bar{f}$ vanishes exponentially as $\Lambda \downarrow 0$). By the result above, it follows that $a^*(f) \le \hat{a}$ and $a_*(f) \ge \hat{a}$, and therefore

$$a^*(f) = a_*(f) = \hat{a}.$$

G.8 Proof of Lemma 15

On the interval $I = [\bar{f} + \delta^*, f_+]$, by Proposition 2, in an a.e. sense, for $f \in I$,

$$\mathcal{L}m_\Lambda(f) = F(f),$$

where

$$F(f) = (f + (\psi_A + \psi_S)D_\Lambda^*(f))m_\Lambda(f) - f - \frac{\psi_S\psi_A}{\psi^*}D_\Lambda^*(f)a_\Lambda(f).$$

The definition of $\mathcal{L}$ is

$$\mathcal{L}m_\Lambda(f) = \Phi(f - \bar{f})m'_\Lambda(f) + \frac{\Lambda^2}{2}m''_\Lambda(f).$$

Define $H(f, f_+) = -\int_f^{f_+} \Phi(f' - \bar{f})df'$, and define

$$\tilde{m}(f) = m'_\Lambda(f)e^{\frac{2}{\Lambda^2}H(f,f_+)}.$$

It follows that

$$\frac{2}{\Lambda^2}F(f)e^{\frac{2}{\Lambda^2}H(f,f_+)} = \tilde{m}'(f).$$

Therefore,

$$\tilde{m}(f_+) - \tilde{m}(f) = \int_f^{f_+} \frac{2}{\Lambda^2} F(f') e^{\frac{2}{\Lambda^2} H(f', f_+)} df'.$$

We can rewrite this as

$$\begin{aligned} m'_\Lambda(f) &= m'_\Lambda(f_+) e^{-\frac{2}{\Lambda^2} H(f, f_+)} - \int_f^{f_+} \frac{2}{\Lambda^2} F(f') e^{\frac{2}{\Lambda^2}(H(f', f_+) - H(f, f_+))} df' \\ &= -\int_f^{f_+} \frac{2}{\Lambda^2} F(f') e^{\frac{2}{\Lambda^2}(\int_f^{f'} \Phi(f'' - \bar{f}) df'')} df' \end{aligned}$$

using the Neumann boundary condition $m'_\Lambda(f_+) = 0$. It follows that

$$|m'_\Lambda(f)| \leq ||F||_{\infty, I} \int_f^{f_+} \frac{2}{\Lambda^2} e^{\frac{2}{\Lambda^2}(\int_f^{f'} \Phi(f'' - \bar{f}) df'')} df'.$$

where $|| \cdot ||_{\infty, I}$ denotes the L^∞ norm on I. Using $\Phi < 0$,

$$\begin{aligned} \int_f^{f_+} \frac{2}{\Lambda^2} e^{\frac{2}{\Lambda^2}(\int_f^{f'} \Phi(f'' - \bar{f}) df'')} df' &\leq \\ \int_f^{f_+} \frac{2}{\Lambda^2} e^{\frac{2}{\Lambda^2}(f' - f)\Phi\delta^*} df' &= \\ \frac{1 - e^{\frac{2}{\Lambda^2}(f_+ - f)\Phi\delta^*}}{-\Phi\delta^*} &\leq \frac{1}{-\Phi\delta^*}. \end{aligned}$$

It follows that for all $f \in [\bar{f} + \delta^*, f_+]$,

$$|m'_\Lambda(f)| \leq \frac{||F||_{\infty, I}}{-\Phi\delta^*}.$$

By $||F||_{\infty, I}$ bounded uniformly in Λ (which follows by the boundedness of f on I and Proposition 2), m_Λ is Lipschitz-continuous on $[\bar{f} + \delta^*, f_+]$, with a Lipschitz constant that does not depend on Λ. This argument applies, with the direction of drift reversed, to the interval $[f_-, \bar{f} - \delta^*]$. By the boundedness of m_Λ and $(\bar{f} + \delta^*) - (\bar{f} - \delta^*) = 2\delta^* > 0$, m_Λ is Lipschitz-continuous across the "gap." It follows that m_Λ is uniformly (in Λ) Lipschitz-continuous on the union of the two sets.

The argument for a proceeds in the same fashion, using

$$\mathcal{L} a_\Lambda(f) = G(f),$$

where

$$G(f) = fa_\Lambda(f) - f + \psi^* D^*_\Lambda(f) m_\Lambda(f),$$

noting that $||G||_{\infty,I}$ is also bounded uniformly in Λ and that a_Λ is bounded and satisfies the same Neumann boundary conditions.

G.9 Proof of Lemma 16

Define the set

$$B = \{f' \in (\lambda^{-1}D(0), \bar{f}] : \forall f \in (f', \bar{f}],\ m_*(f) = MV_{min}\ and\ a^*(f) = a_*(f) = \hat{a}\}.$$

Note that $m_*(f) = MV_{min}$ implies $m^*(f) = MV_{min}$ on B by $m_\Lambda(f) \leq MV_{min}$ for $f \leq \bar{f}$ from Proposition 2.

By Lemmas 9 and 14, $\inf B \leq \bar{f} - \delta^*$. Suppose that $f^* = \inf B > \lambda^{-1}D(0)$. We will prove this cannot be the case.

By $a^*(f) = a_*(f) = \hat{a}$ and $m_*(f) = m^*(f) = MV_{min}$ on $(f^*, \bar{f}]$, convergence from (m_Λ, a_Λ) to $(MV_{min}, \hat{a})$ is uniform on any compact subset of $(f^*, \bar{f}]$.

For any $f < \bar{f}$, by the usual viscosity arguments and using $\psi^* m_\Lambda(f) D^*_\Lambda(f) \geq 0$ (a.e.),

$$\mathcal{L}a^*(f) \geq fa^*(f) - f.$$

Fix some $\xi \in (0, \bar{f} - f^*)$ and let $T(f) = -\Phi^{-1}\ln(\frac{f-\bar{f}}{f^*+\xi-\bar{f}})$, so that $e^{\Phi T(f)}(f - \bar{f}) = f^* + \xi - \bar{f}$. By standard viscosity arguments for first-order equations,

$$a^*(f) - 1 \leq \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(a^*(f^* + \xi) - 1)|f_0 = f, \Lambda = 0],$$

which yields

$$a^*(f) \leq 1 - (\frac{f - \bar{f}}{f^* + \xi - \bar{f}})^{\bar{f}\Phi^{-1}} + (\frac{f - \bar{f}}{f^* + \xi - \bar{f}})^{\bar{f}\Phi^{-1}}\hat{a}.$$

Using $\psi^* m_\Lambda(f) D^*_\Lambda(f) \leq \psi^* MV_{min}\bar{D}$ a.e. on $f < \bar{f}$ (either by $m_\Lambda(f) < MV_{min}$ and $D^*_\Lambda(f) = D(0) \leq \bar{D}$ or by $m_\Lambda(f) = MV_{min}$ and $D^*_\Lambda(f) \leq \bar{D}$), a.e.,

$$\mathcal{L}a_*(f) \leq fa_*(f) - f + \psi^* MV_{min}\bar{D}.$$

From this,

$$a_*(f) - 1 \geq \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(a_*(f^* + \xi) - 1)|f_0 = f, \Lambda = 0]$$
$$- \psi^* MV_{min}\bar{D}\mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds}dt|f_0 = f, \Lambda = 0].$$

Using $f_s > f > 0$ for all $s \geq 0$ given $\Lambda = 0$,

$$a_*(f) - 1 \geq -\psi^* \bar{D} MV_{min} T(f)$$
$$+ e^{-T(f)f}(\hat{a} - 1),$$

which is

$$a_*(f) \geq \psi^* \bar{D} MV_{min} \Phi^{-1} \ln(\frac{f - \bar{f}}{f^* + \xi - \bar{f}})$$
$$+ 1 - (\frac{f - \bar{f}}{f^* + \xi - \bar{f}})^{f\Phi^{-1}} + (\frac{f - \bar{f}}{f^* + \xi - \bar{f}})^{f\Phi^{-1}}\hat{a}.$$

Choosing $f = f^* - \xi$, it follows that

$$\liminf_{\xi \to 0} a_*(f^* - \xi) \geq \hat{a}$$

and

$$\limsup_{\xi \to 0} a^*(f^* - \xi) \leq \hat{a}.$$

By Lemma 15, a_Λ is Lipschitz-continuous, uniformly in Λ, on $[f_-, \bar{f} - \delta^*]$, and hence a^* and a_* are Lipschitz continuous on the same. By $f^* \in (f_-, \bar{f} - \delta^*]$, it follows that

$$\liminf_{\xi \to 0} a_*(f^* - \xi) = a_*(f^*), \ \limsup_{\xi \to 0} a^*(f^* - \xi) = a^*(f^*),$$

and therefore $\hat{a} \leq a_*(f^*) \leq a^*(f^*) \leq \hat{a}$. Note that these results apply even if $f^* = \lambda^{-1}D(0)$.

For $f^* > \lambda^{-1}D(0)$, it follows by $R(f^*, D(0), \hat{a}) > 0$ and $R(f, D(0), a)$ increasing in a that

$$\liminf_{\xi \to 0} R(f^* - \xi, D(0), a_*(f^* - \xi)) > 0.$$

Thus, for some $\delta > 0$ and $\eta > 0$, $R(f, D(0), a_*(f)) > \eta$ for all $f \in [f^* - \delta, f^*)$. This conclusion extends to all $f \in [f^* - \delta, \bar{f}]$ by $a_*(f) = \hat{a}$ for $f \in [f^*, \bar{f}]$ and $f > \lambda^{-1}D(0)$.

A.e. on $\mathcal{I}_-$, $D^*_\Lambda(f) \geq D(0)$ with equality if $m_\Lambda(f) < MV_{min}$, and

$$D^*_\Lambda(f)B(a_\Lambda(f)) \leq \max\{f(1 - MV_{min}), D(0)B(a_\Lambda(f))\}.$$

Therefore (a.e. on $\mathcal{I}_-$)

$$\begin{aligned}\mathcal{L}m_\Lambda(f) \leq {} & (f + (\psi_A + \psi_S)D(0))(m_\Lambda(f) - MV_{min}) \\ & + \max\{0, -R(f, D(0), a_\Lambda(f))\}.\end{aligned} \tag{16}$$

Using the usual viscosity results for the half-relaxed limits, m_* is a supersolution of $G(f, m, a_*(f), p) = 0$,

$$\begin{aligned}G(f, m, a, p) = {} & (f + (\psi_A + \psi_S)D(0))(m - MV_{min}) \\ & + \max\{0, -R(f, D(0), a)\} - \Phi(f - \bar{f})p.\end{aligned}$$

For any $f \in [f^* - \delta, f^* + \xi]$, $R(f, D(0), a_*(f)) \geq 0$, and therefore by standard viscosity arguments for first-order equations,

$$m_*(f) - MV_{min} \geq \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(m_*(f^* + \xi) - MV_{min})|f_0 = f, \Lambda = 0],$$

from which it follows that $m_*(f) \geq MV_{min}$ and hence (by $m_*(f) \leq m^*(f) \leq MV_{min}$) equality holds, with uniform convergence on $[f^* - \delta, f^*]$.

The interval $[f^* - \delta, f^* - \frac{1}{2}\delta]$ thus satisfies the conditions of Lemma 10 (with the half-relaxed limits formed along the selected family $(\Lambda, \bar{a}_\Lambda)$ from Lemma 12), and hence there exists a $\bar{\Lambda}$ such that for all $\Lambda \in (0, \bar{\Lambda})$, there exists an $f_\Lambda \in [f^* - \delta, f^* - \frac{1}{2}\delta]$ such that $m_\Lambda(f_\Lambda) = MV_{min}$ and such that, for all $f \in [f^* - \delta, \bar{f}]$, $R(f, D(0), a_\Lambda(f)) > 0$. (The latter follows from compactness, $R(f, D(0), a_*(f)) > \eta > 0$, that $R(\cdot, a)$ is increasing in a, and the l.s.c. property of a_*). It follows by (16) and $m_\Lambda \in \mathcal{C}^{1,1}(\mathcal{I}_-)$ that

$$\mathcal{L}m_\Lambda(f) \leq 0$$

a.e. on $f \in (f_\Lambda, \bar{f})$, and hence by the weak minimum principle for a.e. solutions (see Chapter 9 of Gilbarg and Trudinger (1998)), an interior minimum cannot exist. By $m_\Lambda(f_\Lambda) = m_\Lambda(\bar{f}) = MV_{min}$, it follows that $m_\Lambda(f) = MV_{min}$ for all $f \in [f^* - \frac{1}{2}\delta, \bar{f}]$ and $\Lambda \in (0, \bar{\Lambda})$. By Proposition 2,

$D^*_\Lambda(f) = f(1 - MV_{min})B(a_\Lambda(f))^{-1}$ a.e. on this interval, with (a.e.)

$$B(a_\Lambda(f)) \geq (\bar{D})^{-1}(f^* - \frac{1}{2}\delta)(1 - MV_{min}) > 0.$$

Note that this bound holds on the entire interval by the continuity of a_Λ, and applies to a^* and a_* as well.

From this, a.e.,

$$\mathcal{L}a_\Lambda(f) = f a_\Lambda(f) - f + \psi^* MV_{min}\frac{f(1 - MV_{min})}{B(a_\Lambda(f))}.$$

It follows that a^* is a subsolution and a_* is a supersolution of $F(f, a, p) = 0$,

$$F(f, a, p) = fa - f + \psi^* MV_{min}\frac{f(1 - MV_{min})}{B(a)} - \Phi p(f - \bar{f}).$$

Note by the above that $a^*(f^* + \xi) = a_*(f^* + \xi) = \hat{a}$. It follows that by standard first-order viscosity arguments that

$$\begin{aligned} a^*(f) - 1 \leq &-\psi^* MV_{min}\mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds}\frac{f_t(1 - MV_{min})}{B(a^*(f_t))}dt|f_0 = f, \Lambda = 0] \\ &+ \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(\hat{a} - 1)|f_0 = f, \Lambda = 0] \end{aligned}$$

and

$$\begin{aligned} a_*(f) - 1 \geq &-\psi^* MV_{min}\mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds}\frac{f_t(1 - MV_{min})}{B(a_*(f_t))}dt|f_0 = f, \Lambda = 0] \\ &+ \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(\hat{a} - 1)|f_0 = f, \Lambda = 0], \end{aligned}$$

noting that

$$B(a_*(f)) \geq B(a^*(f)) \geq (\bar{D})^{-1}(f^* - \frac{1}{2}\delta)(1 - MV_{min}).$$

By $a^*(f) \geq a_*(f)$ and $B(a)$ strictly decreasing in a, $B(a^*(f)) \leq B(a_*(f))$, and therefore

$$0 \geq \mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds} f_t(B(a_*(f_t))^{-1} - B(a^*(f_t))^{-1})dt|f_0 = f, \Lambda = 0] \geq 0.$$

From this, $a^*(f) = a_*(f)$ a.e. on $f \in [f^* - \frac{1}{2}\delta, f^*]$. By

$$\mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds} f_t(B(a_*(f_t))^{-1} - B(a^*(f_t))^{-1})dt | f_0 = f, \Lambda = 0] \geq$$
$$a^*(f) - a_*(f) \geq 0,$$

this conclusion holds pointwise. Therefore,

$$\begin{aligned} a^*(f) - 1 = -\psi^* MV_{min} \mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds} \frac{f_t(1 - MV_{min})}{B(a^*(f_t))} dt | f_0 = f, \Lambda = 0] \\ + \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(f)} f_s ds}(\hat{a} - 1) | f_0 = f, \Lambda = 0], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} a^*(f) - \hat{a} &= \psi^* MV_{min}(1 - MV_{min}) \\ &\times \mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} e^{-\int_0^t f_s ds} f_t(B(\hat{a})^{-1} - B(a^*(f_t))^{-1}) dt | f_0 = f, \Lambda = 0]. \end{aligned}$$

Define $a_L = \inf_{f \in [f^* - \frac{1}{2}\delta, f^*]} a^*(f)$, $a_H = \sup_{f \in [f^* - \frac{1}{2}\delta, f^*]} a^*(f)$, and let $\beta > 0$ be the Lipschitz constant of $B(a)^{-1}$ on $[a_L, a_H]$. We have, using $f_t \in (0, \bar{f})$,

$$|a^*(f) - \hat{a}| \leq \psi^* MV_{min}(1 - MV_{min}) \bar{f} \beta \mathbb{E}^{\mathbb{Q}}[\int_0^{T(f)} |a^*(f_t) - \hat{a}| dt | f_0 = f, \Lambda = 0].$$

Define the measure μ on $([f^* - \frac{1}{2}\delta, f^* + \xi], \mathcal{B}([f^* - \frac{1}{2}\delta, f^* + \xi]))$, where $\mathcal{B}(\cdot)$ is the Borel sigma-algebra, by

$$\mu(A) = \psi^* MV_{min}(1 - MV_{min}) \bar{f} \beta \mathbb{E}^{\mathbb{Q}}[\int_0^{T(f^* - \frac{1}{2}\delta)} \mathbf{1}\{f_t \in A\} dt | f_0 = f^* - \frac{1}{2}\delta]$$

for $A \in \mathcal{B}([f^* - \frac{1}{2}\delta, f^* + \xi])$. It follows that

$$|a^*(f) - \hat{a}| \leq \int_f^{f^* + \xi} |a^*(f') - \hat{a}| d\mu(f').$$

and that

$$\mu([f^* - \frac{1}{2}\delta, f]) = \psi^* MV_{min}(1 - MV_{min}) \bar{f} \beta \Phi^{-1} \ln(\frac{f - \bar{f}}{f^* - \frac{1}{2}\delta - \bar{f}}),$$

which is continuous in f. By the integral version of Grönwall's inequality,

$$|a^*(f) - \hat{a}| \leq 0,$$

and hence $a^*(f) = a_*(f) = \hat{a}$.

We conclude that $f^* > \lambda^{-1}D(0)$ cannot be an infimum, and therefore $f^* = \lambda^{-1}D(0)$. By $a^*(f) = a_*(f) = \hat{a}$ on $f \in (\lambda^{-1}D(0), \bar{f}]$, $R(f, D(0), a_*(f)) \geq 0$ for all $f \in (\lambda^{-1}D(0), \bar{f}]$, and thus

$$\begin{aligned}&m_*(\lambda^{-1}D(0)) - MV_{min} \geq \\ &\mathbb{E}^{\mathbb{Q}}[e^{-\int_0^{T(\lambda^{-1}D(0))} f_s ds}(m_*(f^* + \xi) - MV_{min})|f_0 = \lambda^{-1}D(0), \Lambda = 0],\end{aligned}$$

from which it follows that $m_*(\lambda^{-1}D(0)) = MV_{min}$. As shown above, $a^*(f^*) = a_*(f^*) = \hat{a}$, and $f^* = \lambda^{-1}D(0)$.

The upper side proof follows by a symmetric argument.

G.10 Proof of Lemma 17

Define

$$V_{0,X}(f) = \hat{V}_X(a_0(f), m_0(f))$$

from the subsequential limit. By Lemma 15 the functions (a_0, m_0) (and hence the $V_{0,X}$) are Lipschitz-continuous on $f \in [\lambda^{-1}D_{min}, f_+] \cup [f_-, \lambda^{-1}D(0)]$. By Lemma 16 and Lemma 7, $V_{0,X}$ is constant and equal to $(1 + \lambda\psi_X)^{-1}$ on $f \in [\lambda^{-1}D(0), \lambda^{-1}D_{min}]$. It follows that the $V_{0,X}$ are Lipschitz-continuous and $V_{\Lambda,X}$ converges to $V_{0,X}$ in the compact-open topology.

By Lemma 13, the $V_{\Lambda,X}$ functions are $\mathcal{C}^{1,1}$ and satisfy

$$\mathcal{L}V_{\Lambda,X}(f) = -f + (f + \psi_X D^*_\Lambda(f))V_{\Lambda,X}(f).$$

We first consider $f > \lambda^{-1}D_{min}$. By Proposition 2, either $D^*_\Lambda(f) = d(m_\Lambda(f))$ (if $m_\Lambda(f) > MV_{min}$) or $D^*_\Lambda(f) \leq D_{min} = d(m_\Lambda(f))$ (if $m_\Lambda(f) = MV_{min}$), and therefore $D^*_\Lambda(f) \leq d(m_\Lambda(f))$. It follows by $V_{\Lambda,X}(f) > 0$ (from Lemma 13 and $f > 0$) that

$$\mathcal{L}V_{\Lambda,X}(f) \leq -f + (f + \psi_X d(m_\Lambda(f)))V_{\Lambda,X}(f),$$

with equality if $m_\Lambda(f) > MV_{min}$.

By the convergence of $V_{\Lambda,X}$ to $V_{0,X}$ and of m_Λ to m_0, standard viscosity arguments, and the a.e. differentiability of $V_{0,X}$, a.e. on $\mathcal{I}$,

$$\Phi(f-\bar{f})V'_{0,X}(f) \leq -f + (f+\psi_X d(m_0(f)))V_{0,X}(f). \tag{17}$$

If $m_0(f) > MV_{min}$ (noting by continuity that this holds in a neighborhood of f), then we must have $m_\Lambda(f) > MV_{min}$ for sufficiently small Λ in the neighborhood of f, from which it follows that equality holds in (17) for such f.

Define

$$T(f) = \Phi^{-1}\ln(\frac{\lambda^{-1}D_{min}-\bar{f}}{f-\bar{f}})$$

using $V_{0,X}(\lambda^{-1}D_{min}) = (1+\lambda\psi_X)^{-1}$ and integrating,

$$V_{0,X}(f') - \frac{1}{1+\psi_X\lambda} \geq \\ \frac{\psi_X}{1+\psi_X\lambda}\mathbb{E}^{\mathbb{Q}}[\int_0^{T(f')} e^{-\int_0^t (f_s+\psi_X d(m_0(f_s)))ds}(\lambda f_t - d(m_0(f_t)))dt | f_0 = f', \Lambda = 0]. \tag{18}$$

If $m_0(f) > MV_{min}$ on some set (f_1, f_2), then for T such that $e^{\Phi T}(f_2-\bar{f}) \geq f_1 - \bar{f}$, equality holds on this set,

$$V_{0,X}(f_2) - \frac{1}{1+\psi_X\lambda} = \frac{\psi_X}{1+\psi_X\lambda}\mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\int_0^t (f_s+\psi_X d(m_0(f_s)))ds}(\lambda f_t - d(m_0(f_t)))dt | f_0 = f_2, \Lambda = 0] \\ + \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T (f_s+\psi_X d(m_0(f_s)))ds}(V_{0,X}(f_T) - \frac{1}{1+\psi_X\lambda}) | f_0 = f_2, \Lambda = 0],$$

from which differentiability follows (i.e. (17) holds with equality). Note that the derivative is continuous at such points.

Suppose that for some $f_H > \lambda^{-1}D_{min}$, $m_0(f_H) = MV_{min}$. It follows by $m_0(f_H) = MV_{min}$ that for at least one $X \in \{A, S\}$, $V_{0,X}(f_H) \leq \frac{1}{1+\psi_X\lambda}$. Given that the integral in (18) is weakly negative, either the integrand is strictly negative on a positive measure set or it is non-negative a.e. In the latter case, for the integral to be weakly negative, the integrand must be zero a.e., and by continuity, we would have $\lambda f_H = d(m_0(f_H)) = D_{min}$, a contradiction. Therefore, the integrand must be strictly negative on a positive measure set. It follows that on this set, $d(m_0(f)) > \lambda f > D_{min}$.

We will prove that a point on $f > \lambda^{-1}D_{min}$ with $d(m_0(f)) > \lambda f$ cannot exist. By the preceding argument, this proves $m_0(f) \neq MV_{min}$ on $f > \lambda^{-1}D_{min}$, and hence that $m_0(f) > MV_{min}$ (by $m_0(f) \geq MV_{min}$ on $\mathcal{I}_+$ from Proposition 2).

Let $f'_H \in (\lambda^{-1}D_{min}, f_+)$ be a point such that $d(m_0(f'_H)) > \lambda f'_H$. Define

$$f_1 = \sup\{f \in [\lambda^{-1}D_{min}, f'_H] : d(m_0(f)) \leq \lambda f\},$$

noting that $f_1 < f'_H$. Note also that, by continuity, $d(m_0(f_1)) = \lambda f_1$, and that $d(m_0(f)) \geq \lambda f$ for f in a neighborhood of f_1 with $f > f_1$. Recall that $d(\cdot)$ is smooth and has a strictly positive derivative.

If $f_1 > \lambda^{-1}D_{min}$, by the continuity of the derivative, at f_1 we must have, for at least $X_1 \in \{A, S\}$,

$$V'_{0,X_1}(f_1) \geq \frac{\lambda}{d'(d^{-1}(\lambda f_1))}.$$

If $f_1 = \lambda^{-1}D_{min}$, this statement holds for the right derivative, taking the limit of $V'_{0,X_1}(f_1)$ from above and observing that this limit exists by (17) holding with equality where $m_0(f) > MV_{min}$.

Either of these cases require

$$V_{0,X_1}(f_1) - \frac{1}{1+\psi_{X_1}\lambda} \leq \frac{\lambda}{d'(d^{-1}(\lambda f_1))}\frac{\Phi(f_1 - \bar{f})}{f_1(1+\psi_{X_1}\lambda)}.$$

Define

$$\xi = \inf_{f\in[\lambda^{-1}D_{min}, f'_H], X\in\{A,S\}, m\in[MV_{min},1]} \frac{\lambda}{d'(m)}\frac{-\Phi(f-\bar{f})}{f(1+\psi_X\lambda)}$$

and observe that

$$V_{0,X_1}(f_1) - \frac{1}{1+\psi_{X_1}\lambda} \leq -\xi < 0.$$

If $f_1 = \lambda^{-1}D_{min}$, this generates a contradiction with Lemma 16. Assume in what follows that $f_1 > \lambda^{-1}D_{min}$.

Let γ be the maximum of one and the two Lipschitz constants associated with $V_{0,X}$ on the set $f \in [\lambda^{-1}D_{min}, f'_H]$. It follows that

$$V_{0,X_1}(\max\{f_1 - \frac{\xi}{2\gamma}, \lambda^{-1}D_{min}\}) - \frac{1}{1+\psi_{X_1}\lambda} - \frac{\xi}{2\gamma}\gamma \leq V_{0,X_1}(f_1) - \frac{1}{1+\psi_{X_1}\lambda},$$

and therefore

$$V_{0,X_1}(\max\{f_1 - \frac{\xi}{2\gamma}, \lambda^{-1}D_{min}\}) - \frac{1}{1+\psi_{X_1}\lambda} < 0.$$

Using the lower bound as before, it follows that if $f_1 - \frac{\xi}{2\gamma} > \lambda^{-1}D_{min}$, there is a positive measure subset of $[\lambda^{-1}D_{min}, f_1 - \frac{\xi}{2\gamma}]$ on which

$$\lambda f < d(m_0(f)).$$

Repeating the argument if $f_1 - \frac{\xi}{2\gamma} > \lambda^{-1}D_{min}$, there must be some value of $f \in (\lambda^{-1}D_{min}, f_1 - \frac{\xi}{2\gamma}]$ such that $d(m_0(f)) > \lambda f$. Call this f_1'. We can thus define

$$f_2 = \sup\{f \in [\lambda^{-1}D_{min}, f_1'] : d(m_0(f)) \leq \lambda f\},$$

and conclude that for some $X_2 \in \{A, S\}$,

$$V_{0,X_2}(\max\{\lambda^{-1}D_{min}, f_2 - \frac{\xi}{2\gamma}\}) - \frac{1}{1 + \psi_{X_2}\lambda} < 0.$$

Repeating this argument at most $n = \lceil \frac{2\gamma}{\xi}(f_H' - \lambda^{-1}D_{min}) \rceil$ times in total, until we reach $\lambda^{-1}D_{min}$, we conclude that $V_{0,X_n}(\lambda^{-1}D_{min}) < \frac{1}{1+\psi_{X_n}\lambda}$, but this contradicts Lemma 16. Therefore, we cannot have a point $f > \lambda^{-1}D_{min}$ with $d(m_0(f)) > \lambda f$, and as argued above this conclusion rules out $m_0(f') = MV_{min}$ for any $f' > \lambda^{-1}D_{min}$. From (18), it follows that

$$V_{0,X}(f) \geq \frac{1}{1 + \psi_X \lambda}$$

for all $f > \lambda^{-1}D_{min}$, and hence that $a_0(f) \geq \hat{a}$ on this set.

Now consider the case of $f < \lambda^{-1}D(0)$. By Proposition 2, a.e. on $f \leq \lambda^{-1}D(0) < \bar{f}$,

$$D^*_\Lambda(f) \geq D(0),$$

with equality wherever $m_\Lambda(f) < MV_{min}$. Define

$$T^-(f) = \Phi^{-1}\ln(\frac{\bar{f} - \lambda^{-1}D(0)}{\bar{f} - f}).$$

By

$$\mathcal{L}V_{\Lambda,X}(f) \geq -f + (f + \psi_X D(0))V_{\Lambda,X}(f),$$

we have

$$\Phi(f - \bar{f})V_{0,X}'(f) \geq -f + (f + \psi_X D(0))V_{0,X}(f).$$

It follows by $V_{0,X}(\lambda^{-1}D(0)) = \frac{1}{1+\psi_X\lambda}$ (from Lemma 16) that

$$V_{0,X}(f') - \frac{1}{1+\psi_X\lambda} \leq$$
$$\frac{\psi_X}{1+\psi_X\lambda}\mathbb{E}^{\mathbb{Q}}[\int_0^{T^-(f')} e^{-\int_0^t (f_s+\psi_X D(0))ds}(\lambda f_t - D(0))dt|f_0 = f', \Lambda = 0].$$

By $\lambda f_t < D(0)$ on this set, $V_{0,X}(f') < \frac{1}{1+\psi_X\lambda}$, and therefore $m_0(f') < MV_{min}$ and $a_0(f') < \hat{a}$.

Combining these results with Lemma 16 proves the lemma.

G.11 Proof of Lemma 18

By Lemmas 17, $m_0(f) > MV_{min}$ if and only if $f > \lambda^{-1}D_{min}$ and $m_0(f) < MV_{min}$ if and only if $f < \lambda^{-1}D(0)$. Define

$$D_0^*(f) = \begin{cases} d(m_0(f)) & f > \lambda^{-1}D_{min}, \\ \lambda f & f \in [\lambda^{-1}D(0), \lambda^{-1}D_{min}], \\ D(0) & f < \lambda^{-1}D(0). \end{cases}$$

Note by construction that $D_0^*(f) \in \Gamma(m_0(f))$.

Define

$$V_{0,X}(f) = \hat{V}_X(a_0(f), m_0(f))$$

from the subsequential limit. By Lemma 15 the functions (a_0, m_0) (and hence the $V_{0,X}$) are Lipschitz-continuous on $f \in [\lambda^{-1}D_{min}, f_+] \cup [f_-, \lambda^{-1}D(0)]$. By Lemma 16 and Lemma 7, $V_{0,X}$ is constant and equal to $(1+\lambda\psi_X)^{-1}$ on $f \in [\lambda^{-1}D(0), \lambda^{-1}D_{min}]$. It follows that the $V_{0,X}$ are Lipschitz-continuous and (from the convergence of (a_Λ, m_Λ) to (a_0, m_0)) that $V_{\Lambda,X}$ converges to $V_{0,X}$ in the compact-open topology.

By Lemma 13, the $V_{\Lambda,X}$ functions are $\mathcal{C}^{1,1}$ and satisfy

$$\mathcal{L}V_{\Lambda,X}(f) = -f + (f + \psi_X D_\Lambda^*(f))V_{\Lambda,X}(f).$$

For any $f > \lambda^{-1}D_{min}$, by $m_0(f) > MV_{min}$, for sufficiently small Λ, $D_\Lambda^*(f) = d(m_\Lambda(f))$. By the convergence of $V_{\Lambda,X}$ to $V_{0,X}$ and of m_Λ to m_0, standard viscosity arguments, and the a.e. differentiability of $V_{0,X}$, a.e. on $f \in (\lambda^{-1}D_{min}, f_+)$,

$$\Phi(f-\bar{f})V'_{0,X}(f) = -f + (f + \psi_X d(m_0(f)))V_{0,X}(f).$$

Likewise, for any $f < \lambda^{-1}D(0)$, by $m_0(f) < MV_{min}$, for sufficiently small Λ, $D^*_\Lambda(f) = D(0)$. By the convergence of $V_{\Lambda,X}$ to $V_{0,X}$ and of m_Λ to m_0, standard viscosity arguments, and the a.e. differentiability of $V_{0,X}$, a.e. on $f \in (f_-, \lambda^{-1}D(0))$,

$$\Phi(f-\bar{f})V'_{0,X}(f) = -f + (f + \psi_X D(0))V_{0,X}(f).$$

By Lemma 16, on $(\lambda^{-1}D(0), \lambda^{-1}D_{min})$, $V_{0,X}(f) = (1+\psi_X\lambda)^{-1}$, and therefore

$$\Phi(f-\bar{f})V'_{0,X}(f) = -f + (f + \psi_X \lambda f)V_{0,X}(f).$$

It follows that $V_{0,X}$ satisfies, a.e. on $\mathcal{I}$,

$$\Phi(f-\bar{f})V'_{0,X}(f) = -f + (f + \psi_X D^*_0(f))V_{0,X}(f).$$

By this and Lipschitz-continuity, for any $T > 0$,

$$\begin{aligned} V_{0,X}(f) = \mathbb{E}^{\mathbb{Q}}[&\int_0^T e^{-\int_0^t (f_s + \psi_X D^*_0(f_s))ds} f_t dt | f_0 = f, \Lambda = 0] \\ &+ \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T (f_s + \psi_X D^*_0(f_s))ds} V_{0,X}(f_T) | f_0 = f, \Lambda = 0]. \end{aligned}$$

Verification follows taking the $T \to \infty$ limit, using the boundedness of $V_{0,X}$ and the boundedness away from zero of the discount rate.

G.12 Proof of Lemma 19

The $\Lambda = 0$ case follows from Lemma 18. Assume in what follows that $\Lambda > 0$.

Consider first the interval $I_- = [f_-, \lambda^{-1}D(0) - \xi]$, and suppose there exists a sequence $(\Lambda_n > 0, f_n \in I_-)$ with $\Lambda_n \downarrow 0$ such that $m_{\Lambda_n}(f_n) \geq MV_{min}$. Passing to a convergent subsequence, $f_n \to f_\infty \in I_-$. By Lemma 15 and the Arzelà-Ascoli theorem, there is a further subsequence on which $(a_{\Lambda_n}, m_{\Lambda_n}) \to (a_0, m_0)$ uniformly. For this further subsequence, $m_0(f_\infty) \geq MV_{min}$, contradicting Lemma 17. Therefore, there exists a Λ^- such that for all $\Lambda \in (0, \Lambda^-)$ and $f \in I_-$, $m_\Lambda(f) < MV_{min}$ and hence (a.e.) $D^*_\Lambda(f) = D(0)$.

Consider next the interval $I_+ = [\lambda^{-1}D_{min} + \xi, f_+]$, and suppose there exists a sequence $(\Lambda_n > 0, f_n \in I_+)$ with $\Lambda_n \downarrow 0$ such that $m_{\Lambda_n}(f_n) \leq MV_{min}$. Passing to a convergent subsequence, $f_n \to f_\infty \in I_+$. By Lemma 15 and the Arzelà-Ascoli theorem, there is a further subsequence on which $(a_{\Lambda_n}, m_{\Lambda_n}) \to (a_0, m_0)$ uniformly. For this further subsequence, $m_0(f_\infty) \leq MV_{min}$, contradicting Lemma 17. Therefore, there exists a Λ^+ such that for all $\Lambda \in (0, \Lambda^+)$ and $f \in I_+$, $m_\Lambda(f) > MV_{min}$ and hence (a.e.) $D^*_\Lambda(f) = d(m_\Lambda(f)) > D_{min}$.

Lastly, consider the interval $I_0 = [\lambda^{-1}D(0) + \frac{1}{2}\xi, \lambda^{-1}D_{min} - \frac{1}{2}\xi]$. By Lemma 16, $m^*(f) = m_*(f) = MV_{min}$ and $a^*(f) = a_*(f) = \hat{a}$ on I_0. It follows that (a_Λ, m_Λ) converges uniformly to $(\hat{a}, MV_{min})$ on this interval, and by Assumption 4 and Lemma 7,

$$\frac{f(1 - MV_{min})}{B(\hat{a})} = f\lambda \in [D(0) + \lambda\frac{1}{2}\xi, D_{min} - \lambda\frac{1}{2}\xi] \tag{19}$$

for $f \in I_0$.

It follows that the conditions of Lemma 10 are satisfied on the intervals $[\lambda^{-1}D(0)+\frac{1}{2}\xi, \lambda^{-1}D(0)+\xi]$ and $[\lambda^{-1}D_{min} - \xi, \lambda^{-1}D_{min} - \frac{1}{2}\xi]$ (with the half-relaxed limits formed along the selected family $(\Lambda, \bar{a}_\Lambda)$ from Lemma 12). Therefore, there exists a Λ^0 such that, for all $\Lambda \in (0, \Lambda^0)$,

1. there exists an $f_H \in [\lambda^{-1}D_{min} - \xi, \lambda^{-1}D_{min} - \frac{1}{2}\xi]$ with $m_\Lambda(f_H) = MV_{min}$,
2. for all $f \in [\bar{f}, \lambda^{-1}D_{min} - \frac{1}{2}\xi]$, $R(f, D_{min}, a_\Lambda(f)) \leq 0$,
3. there exists an $f_L \in [\lambda^{-1}D(0) + \frac{1}{2}\xi, \lambda^{-1}D(0) + \xi]$ with $m_\Lambda(f_L) = MV_{min}$,
4. for all $f \in [\lambda^{-1}D(0) + \frac{1}{2}\xi, \bar{f}]$, $R(f, D(0), a_\Lambda(f)) \geq 0$,
5. for all $f \in [\lambda^{-1}D(0) + \frac{1}{2}\xi, \lambda^{-1}D_{min} - \frac{1}{2}\xi]$, $D(0) < \frac{f(1-MV_{min})}{B(a_\Lambda(f))} < D_{min}$.

Note that items 2, 4, and 5 are a consequence of the uniform convergence discussed above and the strict margins in (19), and items 1 and 3 are consequences of Lemma 10.

Assume $\Lambda \in (0, \Lambda^0)$. On $\mathcal{I}_-$, $D^*_\Lambda(f) \geq D(0)$ a.e., with equality (a.e.) if $m_\Lambda(f) < MV_{min}$, and (also a.e.)

$$D^*_\Lambda(f)B(a_\Lambda(f)) \leq \max\{f(1 - MV_{min}), D(0)B(a_\Lambda(f))\}.$$

Therefore (a.e. on $\mathcal{I}_-$),

$$\begin{aligned}\mathcal{L}m_\Lambda(f) \leq (f + (\psi_A + \psi_S)D(0))(m_\Lambda(f) - MV_{min})\\ + \max\{0, -R(f, D(0), a_\Lambda(f))\}.\end{aligned}$$

It follows on $f \in [f_L, \bar{f}]$ that $\mathcal{L}m_\Lambda(f) \leq 0$ (by point 4 in the above list), and hence that $m_\Lambda(f) \geq MV_{min}$ (by the weak minimum principle for a.e. solutions, $m_\Lambda(\bar{f}) = MV_{min}$, and point 3 in the above list), and by $m_\Lambda(f) \leq MV_{min}$ on $\mathcal{I}_-$ equality holds.

Likewise, on $\mathcal{I}_+$, $D^*_\Lambda(f) = d(m_\Lambda(f))$ a.e. on $m_\Lambda(f) > MV_{min}$, and (a.e.)

$$D^*_\Lambda(f)B(a_\Lambda(f)) \geq \min\{f(1 - MV_{min}), d(m_\Lambda(f))B(a_\Lambda(f))\}.$$

Therefore (a.e. on $\mathcal{I}_+$),

$$\begin{aligned}\mathcal{L}m_\Lambda(f) \geq\ & (f + (\psi_A + \psi_S)d(m_\Lambda(f)))(m_\Lambda(f) - MV_{min}) \\ & + \min\{0, -R(f, d(m_\Lambda(f)), a_\Lambda(f))\}.\end{aligned}$$

By point 2 in the above list, on $f \in [\bar{f}, f_H]$, $R(f, D_{min}, a_\Lambda(f)) \leq 0$, which requires $B(a_\Lambda(f)) > 0$, and therefore (by $d(m_\Lambda(f)) \geq D_{min}$),

$$R(f, d(m_\Lambda(f)), a_\Lambda(f)) \leq R(f, D_{min}, a_\Lambda(f)),$$

which yields

$$\min\{0, -R(f, d(m_\Lambda(f)), a_\Lambda(f))\} \geq \min\{0, -R(f, D_{min}, a_\Lambda(f))\}.$$

It follows that on $f \in [\bar{f}, f_H]$, $\mathcal{L}m_\Lambda(f) \geq 0$ (by point 2 in the above list), and hence that $m_\Lambda(f) \leq MV_{min}$ (by the maximum principle, $m_\Lambda(\bar{f}) = MV_{min}$, and point 1 in the above list), and by $m_\Lambda(f) \geq MV_{min}$ on $\mathcal{I}_+$ equality holds.

Therefore, $m_\Lambda(f) = MV_{min}$ on $f \in [f_L, f_H] \supset [\lambda^{-1}D(0) + \xi, \lambda^{-1}D_{min} - \xi]$. Moreover, on this set, by point 5 in the above list, a.e.,

$$D^*_\Lambda(f) = \frac{f(1 - MV_{min})}{B(a_\Lambda(f))} \in (D(0), D_{min})$$

Choosing $\bar{\Lambda}_{\xi,1} = \min\{\Lambda^+, \Lambda^-, \Lambda^0\}$ proves the characterization result of the lemma in an a.e. sense.

What remains is to prove that $D^*_\Lambda(f) \in \Gamma(m_\Lambda(f))$ for all $f \in \mathcal{I}$. By Proposition 2, this conclusion is immediate a.e. on $\{m_\Lambda(f) > MV_{min}\}$ and $\{m_\Lambda(f) < MV_{min}\}$. By the above results, the contact set $\{m_\Lambda(f) = MV_{min}\}$ is a subset of $[\lambda^{-1}D(0) - \xi, \lambda^{-1}D_{min} + \xi]$.

By Proposition 2, on $f > \bar{f}$, if $m_\Lambda(f) = MV_{min}$ then $R(f, D_{min}, a_\Lambda(f)) \leq 0$, from which it follows that (a.e.)

$$D^*_\Lambda(f) = \frac{f(1 - MV_{min})}{B(a_\Lambda(f))} \le D_{min}.$$

By continuity $R(\bar{f}, D_{min}, a_\Lambda(\bar{f})) \le 0$ as well. Let $\mathcal{S}_\Lambda$ be the set $\{m_\Lambda(f) = MV_{min}\} \cap \{B(a_\Lambda(f)) > 0\}$, recalling that $B(a_\Lambda(f)) > 0$ a.e. on $\{m_\Lambda(f) = MV_{min}\}$ by Proposition 2.

We claim that there exists a $\bar{\Lambda}_{\xi,2} > 0$ such that, for all $\Lambda \in (0, \bar{\Lambda}_{\xi,2})$,

$$\inf_{f \in [\bar{f}, \lambda^{-1} D_{min} + \xi] \cap \mathcal{S}_\Lambda} \frac{f(1 - MV_{min})}{B(a_\Lambda(f))} \ge D(0).$$

Suppose not; then there exists a sequence (Λ_n, f_n) with $\Lambda_n \downarrow 0$ and $f_n \in [\bar{f}, \lambda^{-1} D_{min} + \xi]$ with $m_{\Lambda_n}(f_n) = MV_{min}$ and $B(a_{\Lambda_n}(f_n)) > 0$ such that

$$\frac{f_n(1 - MV_{min})}{B(a_{\Lambda_n}(f_n))} < D(0). \tag{20}$$

Passing to a convergent subsequence, $f_n \to f_\infty \in [\bar{f}, \lambda^{-1} D_{min} + \xi]$. By Lemma 15 and the Arzelà-Ascoli theorem, there is a further subsequence on which $(a_{\Lambda_n}, m_{\Lambda_n}) \to (a_0, m_0)$ uniformly. For this further subsequence, by Lemma 17, $a_0(f) \ge \hat{a}$ for $f \ge \bar{f}$. From (20), $\lim_{n\to\infty} B(a_{\Lambda_n}(f_n)) = B(a_0(f_\infty)) \ge f_\infty(1 - MV_{min}) D(0)^{-1} > 0$, from which it follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{f_n(1 - MV_{min})}{B(a_{\Lambda_n}(f_n))} &= \frac{f_\infty(1 - MV_{min})}{B(a_0(f_\infty))} \\
&\ge \frac{f_\infty(1 - MV_{min})}{B(\hat{a})} \\
&> \frac{\lambda^{-1} D(0)(1 - MV_{min})}{B(\hat{a})} = D(0),
\end{aligned}$$

a contradiction. We conclude that for all $\Lambda \in (0, \bar{\Lambda}_{\xi,2})$, $D^*_\Lambda(f) \ge D(0)$ a.e. on $\{m_\Lambda(f) = MV_{min}\} \cap \{f \ge \bar{f}\}$.

We next consider the other side. By Proposition 2, on $f < \bar{f}$, if $m_\Lambda(f) = MV_{min}$ then $R(f, D(0), a_\Lambda(f)) \ge 0$, from which it follows that (a.e.)

$$D^*_\Lambda(f) = \frac{f(1 - MV_{min})}{B(a_\Lambda(f))} \ge D(0).$$

By continuity $R(\bar{f}, D(0), a_\Lambda(\bar{f})) \ge 0$ as well. We claim that there exists a $\bar{\Lambda}_{\xi,3} > 0$ such that, for all $\Lambda \in (0, \bar{\Lambda}_{\xi,3})$,

$$\sup_{f \in [\lambda^{-1} D(0) - \xi, \bar{f}] \cap \mathcal{S}_\Lambda} \frac{f(1 - MV_{min})}{B(a_\Lambda(f))} \le D_{min}.$$

Suppose not; then there exists a sequence (Λ_n, f_n) with $\Lambda_n \downarrow 0$ and $f_n \in [\lambda^{-1}D(0) - \xi, \bar{f}]$ with $m_{\Lambda_n}(f_n) = MV_{min}$ and $B(a_{\Lambda_n}(f_n)) > 0$ such that

$$\frac{f_n(1 - MV_{min})}{B(a_{\Lambda_n}(f_n))} > D_{min}.$$

Passing to a convergent subsequence, $f_n \to f_\infty \in [\lambda^{-1}D(0) - \xi, \bar{f}]$. By Lemma 15 and the Arzelà-Ascoli theorem, there is a further subsequence on which $(a_{\Lambda_n}, m_{\Lambda_n}) \to (a_0, m_0)$ uniformly. For this further subsequence, by Lemma 17, $a_0(f) \leq \hat{a}$ for $f \leq \bar{f}$, and therefore $\lim_{n\to\infty} B(a_{\Lambda_n}(f_n)) = B(a_0(f_\infty)) \geq B(\hat{a}) > 0$ and

$$\begin{aligned}
\lim_{n\to\infty} \frac{f_n(1 - MV_{min})}{B(a_{\Lambda_n}(f_n))} &= \frac{f_\infty(1 - MV_{min})}{B(a_0(f_\infty))} \\
&\leq \frac{f_\infty(1 - MV_{min})}{B(\hat{a})} \\
&< \frac{\lambda^{-1}D_{min}(1 - MV_{min})}{B(\hat{a})} = D_{min},
\end{aligned}$$

a contradiction. We conclude that for all $\Lambda \in (0, \bar{\Lambda}_{\xi,3})$, $D^*_\Lambda(f) \leq D_{min}$ a.e. on $\{m_\Lambda(f) = MV_{min}\} \cap \{f \leq \bar{f}\}$.

Choosing $\bar{\Lambda}_\xi = \min\{\bar{\Lambda}_{\xi,1}, \bar{\Lambda}_{\xi,2}, \bar{\Lambda}_{\xi,3}\}$ proves the results in an a.e. sense (specifically, for the L^∞-equivalence class D^*_Λ). Choosing a representative of this equivalence class for which the conditions hold pointwise proves the result.