# Learning to Be Rich
## How Expectations Amplify Wealth Inequality*

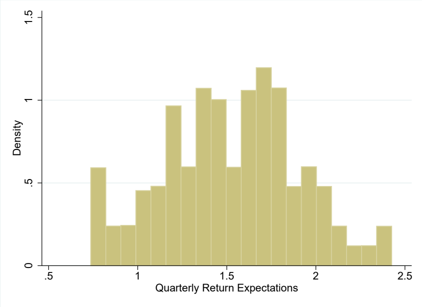
Pau Belda[1] Janko Heineken[2] Adrian Ifrim[3]

[1]Bank of England
[2]European Commission
[3]European Commission · JRC

*Only personal views.

# Motivation

* Explaining the extreme wealth concentration at the top remains challenging.
* Recent literature: refocus from **labor** markets to **asset** markets.
  * Portfolio choice and Asset Pricing have come to the forefront.
* Typical assumption: **beliefs** behind portfolios and asset prices are the same for everyone ("Communism of beliefs").
* **Microdata**: measured **expectations** look more like a full **distribution**.




Cross-section of expected stock returns in the RAND survey.

# Our questions

1. What are the main facts about the distribution of households' expected stock returns?

2. How can subjective beliefs influence top concentration?

3. Where do HE come from?

4. How can we solve HA models with HE? Are they even less tractable?

5. How much do HE influence Wealth Inequality?

# Our answers

1. What are the key facts of the distribution of households' expected stock returns?
   * Strong association between expectations and wealth.
2. How can subjective beliefs influence top concentration?
   * Optimism generates rare winners and thickens the Pareto tail.
3. Where do HE come from?
   * Learning from one's own portfolio experience.
4. How can we solve HA models with HE? Are they even less tractable?
   * No, they're simpler: prices become state variables. No need to forecast distributions.
5. How much do HE influence Wealth Inequality?
   * HE raises top 10 wealth share by $\approx +$ 10pp.

# Contributions to the literature

1. Measured expectations:
   * Characterize the joint distribution of wealth and expectations (Greenwood and Shleifer, 2014; Coibion and Gorodnichenko, 2015; Kohlhas and Walther, 2021).
   * Model the dominance of individual fixed effects (Giglio et al., 2021).

2. Wealth inequality:
   * Microfoundation of heterogeneous returns and portfolios (Gabaix et al., 2016; Fagereng et al., 2020; Hubmer et al., 2021; Xavier, 2021; Fernández-Villaverde and Levintal, 2024).
   * New amplification mechanism based on expectations.

3. Solution method for HA models with aggregate risk: Internal Rationality & Deep Learning. Make the models consistent with microdata... and easier!

Theories of wealth inequality

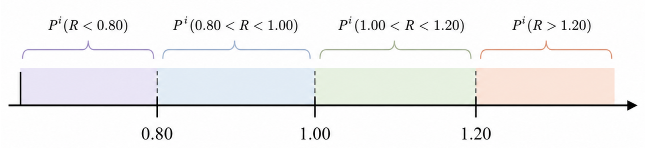
# 1. Facts about the distribution of expectations

# Measuring households' expectations

* Dataset: RAND - American Life Panel.
    * Representative US household survey.
    * 2008-2017, 61 waves.
    * Unbalanced panel $\approx$ 3,000 participants
        * We keep households with at least 10 observations ($\approx 2,300$ households).

* Only public panel data set that tracks wealth & stock return expectations.

* Questions yield a rough distribution of one year ahead (ex-dividend) returns for each agent:




Questions

# Deviations from FIRE in RAND

RAND dataset show the familiar deviations from FIRE...

1. Forecast errors predictability.
   * Extrapolation; over-reaction (consistent with e.g., Adam, Marcet & Beutel, 2017; Bordalo et al., 2020; Kohlhas and Walter, 2021).

     ⇒ Many papers address this. More relevant for cyclical fluctuations.

2. **Persistent disagreement**.
   * Individual Fixed Effects explain most of the panel variation ( Giglio et al., 2021).

     ⇒ Fewer papers discuss this. We conjecture that this is more relevant for the (long-run) wealth distribution.

Regressions

# Wealth and Expectations in RAND

| | **Average change in predicted probability (pp)** | | | |
|---|---|---|---|---|
| | $R < -20\%$ | $-20\% < R < 0\%$ | $0\% < R < 20\%$ | $R > 20\%$ |
| 2x wealth | 0.01 | $-1.37^{***}$ | $1.40^{***}$ | $0.15^{**}$ |
| Risky share +10 pp | 0.08 | $-1.66^{*}$ | $1.35^{***}$ | $0.33^{***}$ |
| Controls | Home ownership, retirement acct., housing wealth, income, age | | | |
| Observations | 12,025 | | | |
| Realized returns | Mean: 14.88% | Min: 1.38% | Max: 32.33% | |

*Note: Entries report average changes in predicted probabilities from separate fractional logit regressions.*

- Wealth is associated, on average, with more accurate return beliefs.
- The main source of greater accuracy is a lower probability assigned to moderately pessimistic returns.
- Wealth modestly increases the probability of over-optimistic beliefs.
- Portfolio composition exhibits the same pattern.

Logit model

# Summary of the RAND facts

1. Persistent disagreement (Giglio et al., 2021).
2. Higher wealth is associated with more accurate beliefs.
3. Higher wealth is associated with more optimistic beliefs.
4. Portfolios and beliefs show similar associations.

We want to reproduce these patterns in a model,
and understand how much they contribute to overall inequality.

# 2. A heterogeneous-agent model with subjective beliefs

# Structure

* Idiosyncratic wage risk and incomplete markets (à la Aiyagari-Bewley-Huggett).

* Aggregate risk (à la Krusell-Smith).

* Endowment economy (no production).

* Portfolio choice between safe and risky assets.

* Internal Rationality (à la Adam-Marcet) and imperfect information.

# Household problem

* Savings-consumption + portfolio choice problem:

$$\max_{\{C_t^i, A_t^i, v_t^i\}_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \beta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \tag{1}$$

* Subject to the budget constraint

$$C_t^i + A_t^i = W_t^i + R_t^{A,i}(v_{t-1}^i) A_{t-1}^i \tag{2}$$

where assets are stocks and bonds

$$A_t^i = P_t S_t^i + B_t^i \tag{3}$$

and the return to wealth is

$$R_t^{A,i}(v_{t-1}^i) = v_{t-1}^i R_t^S + (1 - v_{t-1}^i) R_t^B; \quad v_{t-1}^i = \frac{P_{t-1} S_{t-1}^i}{A_{t-1}^i} \tag{4}$$

* Subject to asset holding limits

$$\begin{aligned} A_t^i &\geq 0 \\ 0 \leq v_t^i &\leq \bar{v} \end{aligned} \tag{5}$$

* Given initial endowments $S_0^i = 1$ and $B_0^i = 0$.

# Subjective probability measure $\mathcal{P}^i$

* Internal Rationality: agents maximize expected utility given a well-specified subjective probability measure $\mathcal{P}^i$.
* $\mathcal{P}^i$ is a primitive of the model (like e.g., $U(C_t^i)$).
* $\mathcal{P}^i$ is defined over payoff-relevant variables that are external to the household, including market outcomes such as prices.
  * Underlying probability space $(\mathbf{\Omega}, \mathcal{B}, \mathcal{P}^i)$ with a typical element $\omega \in \mathbf{\Omega}$ with $\omega = \{D_t, W_t^i, P_t, R_t^B\}_{t=0}^{\infty}$.
* Rational Expectations is recovered when agents have Full Information about the true laws of motion.
* We deviate from FIRE to model heterogeneous beliefs:
  * Full Information about income processes.
  * Imperfect market knowledge prevents agents from deducing the equilibrium stochastic process for asset prices.

# 3. Analytics: Subjective Beliefs and the Pareto tail

# An analytical version

* With CRRA preferences and an infinite horizon, impose:
  * Constant, positive, and non-pledgeable wages $\Rightarrow$ additive income floor needed for a stationary Kesten distribution.
  * No binding portfolio constraints $\Rightarrow$ Merton's portfolio rule.
  * i.i.d. log-N subjective returns and a constant safe rate $\Rightarrow$ Merton–Samuelson policies. Subjective model
  * Lognormal true returns $\Rightarrow$ expected log-growth formulas.
  * At the top, small wages relative to wealth $\Rightarrow$ Merton-Samuelson are asymptotically valid at the top.
* Approximate constant saving rate and risky share:

$$\kappa_i \simeq \left[\delta\,\mathbb{E}^{\mathcal{P}^i}\left[\left(R^{A,i}_{t+1}\right)^{1-\gamma_i}\right]\right]^{1/\gamma_i}, \qquad v_i \simeq \frac{\log\left(\mathbb{E}^{\mathcal{P}^i}_t[R^S_{t+1}]\right) - \log R^B}{\gamma_i \sigma_S^2}$$

* Wealth evolves as a Kesten process:

$$A^i_t = \kappa^i R^{A,i}_t A^i_{t-1} + \kappa^i \bar{W}^i$$

# Accurate beliefs maximize expected log wealth growth

* Subjective expected excess return: $\Delta^i \equiv \log\left(\mathbb{E}_t^{\mathcal{P}^i}[R_{t+1}^S]\right) - \log R^B$. Objective one is $\Delta^0$.
* Expected log growth of wealth is (approx):

$$\lambda^i \equiv \mathbb{E}\left[\log\left(\kappa_i R_t^{A,i}\right)\right] = \frac{1}{\gamma}\left[\log\delta + r_B + \frac{(\Delta^0)^2}{2\sigma^2} - \frac{(\Delta^i - \Delta^0)^2}{2\sigma^2}\right]$$

* Three immediate results:
  1. Accurate beliefs maximize objective expected log wealth growth (as e.g., Martin and Papadimitriou, 2022, and RAND data).
  2. Wealth growth is hump-shaped in optimism.
  3. Beliefs errors make the Kesten stationarity condition (i.e., $\lambda^i < 0$) looser.

Kesten stationarity

# Optimism can thicken the Pareto tail

* Under the usual Kesten–Goldie conditions, the stationary right tail is Pareto with exponent $\xi_i > 0$ satisfying (normalized by growth):

$$\mathbb{E}\left[\exp\left\{\xi_i(r_t^A - g_t - c^i)\right\}\right] = 1, \quad c^i \equiv -\log \kappa^i$$

* Approximate closed-form solution:

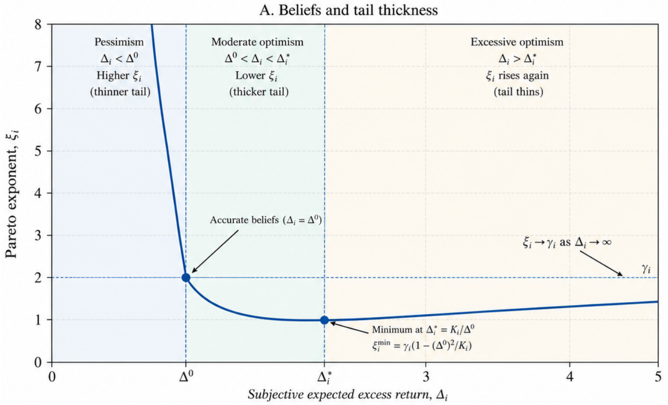
$$\xi_i = \gamma_i \frac{(\Delta^i)^2 - 2\Delta^0\Delta^i + 2\sigma^2\left(\gamma g - \log\delta - r_B\right)}{(\Delta^i)^2}$$

* The Pareto exponent is minimized at

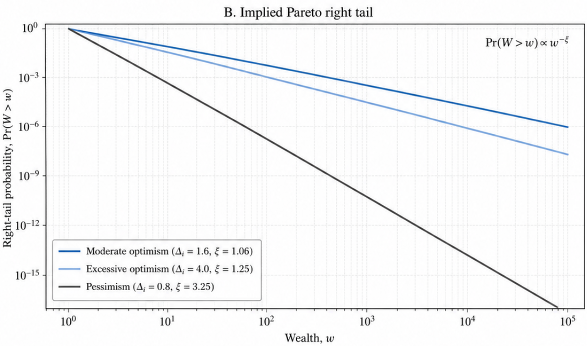
$$(\Delta^i)^\star = \frac{2\sigma^2\left(\gamma_i g - \log\delta - r_B\right)}{\Delta^0} > \Delta^0$$

(the inequality follows from assuming the accurate household is stationary).

# Moderate optimism lowers the Pareto exponent



# Moderate optimism increases the Pareto tail




* The tail depends on two forces:

$$\underbrace{\text{lower mean growth}}_{\text{thins the tail}} \quad vs. \quad \underbrace{\text{higher return variance}}_{\text{thickens the tail}}$$

* Accurate households perform best along the typical long-run history.
* Optimistic households overinvest in risk, lowering their typical growth but creating more extreme winners.
* The Pareto tail is determined by rare sequences of exceptionally high multiplicative returns rather than by average performance.

# General equilibrium brings damping forces

* More optimism raises risky demand and, conditional on $r_t^B$, the market-clearing stock price:

$$P_t(r_t^B) = \frac{\sum_i v_i(r_t^B)\kappa_i(r_t^B)\left(w_t^i + D_t s_{t-1}^i + b_{t-1}^i\right)}{1 - \sum_i v_i(r_t^B)\kappa_i(r_t^B) s_{t-1}^i}$$

$$\text{optimism} \uparrow \quad \Rightarrow \quad v_i \uparrow \quad \Rightarrow \quad P_t \uparrow \quad \Rightarrow \quad \mathbb{E}_t[r_{S,t+1}] \downarrow$$

* With bonds in zero net supply and common risk aversion,

$$r_t^B = \sum_i \frac{\kappa_i(r_t^B) y_t^i}{\sum_j \kappa_j(r_t^B) y_t^j} \log\left(\mathbb{E}_t^{\mathcal{P}^i}[R_{t+1}^S]\right) - \gamma \sigma_S^2$$

  so greater saving-weighted optimism also implies $r_t^B \uparrow$.
* Hence,

$$\Delta_t^0 \equiv \mathbb{E}_t[r_{S,t+1}] - r_t^B \downarrow$$

  damping the partial-equilibrium tail amplification.
* But there is no general ordering

$$\xi_i^{GE} > \xi_i^{PE}$$

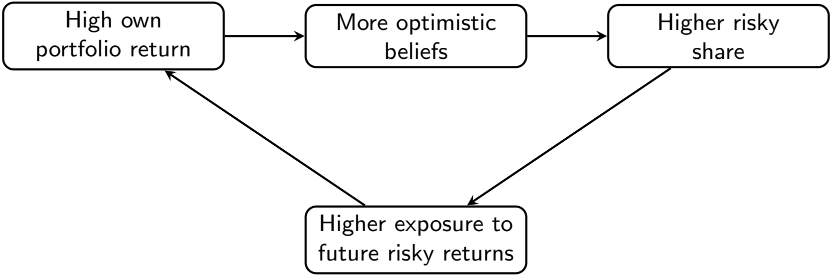
  GE also changes $v_i$, $\kappa_i$, and wealth-growth risk: the net effect is quantitative.

# 4. Where do Heteregeneous Expectations come from?

# Beyond reduced-form belief heterogeneity

* Analytical benchmark:
    * Impose persistent fixed effects in households' subjective models of prices.
* This can match the observed belief heterogeneity, but leaves its origin unexplained.
* Taken at face value, some households are simply endowed with optimism and become rich partly because of it.
    * A similar issue arises when persistent heterogeneity in discount factors or risk aversion is treated as primitive.
* Can we explain the origins of persistent belief differences?

# A mechanism




* Portfolio experience affects beliefs about future stock returns.
* Beliefs affect risky exposure.
* Risky exposure affects future portfolio experience.

This feedback loop is the mechanism behind *Learning to Be Rich*.

# Learning from own portfolio experience

* Household $i$ perceives an unobserved expected return state:

$$\underbrace{\log R^S_{t+1}}_{\text{stock return}} = \underbrace{Z^i_t}_{\text{latent expected log return}} + \varepsilon^{S,i}_{t+1}$$

* Portfolio performance is interpreted as a noisy signal $\mathcal{S}_t$ about this state:

$$\mathcal{S}^i_t = Z^i_t + \nu^i_t; \quad Z^i_t = Z^i_{t-1} + \varepsilon^i_t$$

where the experience signal is given by

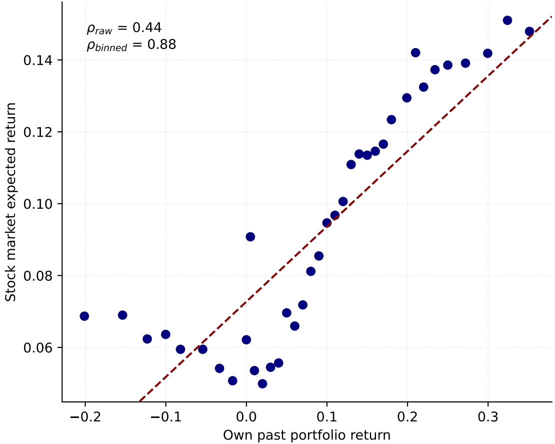
$$\mathcal{S}^i_t = \kappa_L + \mu_L \log R^{A,i}_t + (1 - \mu_L) \log R^S_t$$

* Let $\zeta^i_t \equiv \mathbb{E}^i_{t-1}[Z^i_t]$ denote household $i$'s estimate of the latent state.
* A steady-state Kalman filter gives the constant-gain updating rule

$$\zeta^i_{t+1} = (1 - g_L)\zeta^i_t + g_L \left[\kappa_L + \mu_L \log R^{A,i}_t + (1 - \mu_L) \log R^S_t\right]$$

# Evidence consistent with learning from own returns

* The UBS Gallup survey contains own-portfolio and market expected returns.
* Past own-portfolio returns predict expected market returns.




Regression

# 5. Quantitative model

# Internal Rationality simplifies the solution method

* In equilibrium, prices depend on the cross-sectional distribution:

$$(P_t, Q_t) = p(\Gamma_t, D_t)$$

* Under Rational Expectations, households know this pricing function. Hence, the distribution is part of their perceived state:

$$(\boldsymbol{x}_t^i)^{RE} = \left(W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, \Gamma_t\right).$$

  They must forecast a high-dimensional object in order to forecast prices.

* Under Internal Rationality, households instead condition directly on observed prices and their subjective belief state:

$$(\boldsymbol{x}_t^i)^{IR} = \left(W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, P_t, Q_t, \zeta_t^i\right).$$

  The distribution determines equilibrium prices, but does not enter the household problem.

* This delivers a two-stage solution:

$$\underbrace{\mathcal{P}^i \implies h_i(\boldsymbol{x}_t^i)}_{\text{1. Solve subjective household problems}} \qquad \implies \qquad \underbrace{\sum_i \mu_i h_i \implies (P_t, Q_t) = p(\Gamma_t, D_t)}_{\text{2. Impose market clearing}}.$$

Two-Stage Deep Learning algorithm (2SDL) Details

# Calibration

* Obtain quantitative inputs and targets from RAND:
  1. Convert the RAND coarse probability distribution into expected returns, following Von Gaudecker and Wogrolly (2022). Details
  2. Match RAND to the 2016 SCF. RAND–SCF
* Calibration:
  * Dividend growth matches US stock-market data. Dividends
  * Labor income from Fernández-Villaverde and Levintal (2024). Wages
  * Permanent types determine patience, and risk aversion. Types
* SMM for common wage-dividend scale. Targets
* All portfolio choices, asset prices and income and wealth distributions are endogenous.
* We assess the model's stationary wealth distribution against SCF 2016 moments.
  * Focus on financial wealth (excluding housing and other nonfinancial assets).
* Results are preliminary.

# Aggregate variables

| Moment | Model | Data |
|---|---|---|
| *Household aggregates* | | |
| Capital-income share (%) | 19.78 | 27.00 |
| Wage-income share (%) | 80.22 | 73.00 |
| Wealth-to-income ratio (%) | 236.42 | 491.00 |
| Average risky asset share (%) | 25.35 | 30.50 |
| Risky assets / aggregate wealth (%) | 69.40 | 70.80 |
| Saving rate (%) | 5.53 | 6.70 |
| *Aggregate returns (annual, %)* | | |
| Mean stock return | 12.77 | 9.29 |
| Mean safe rate | 1.22 | 0.75 |
| Mean dividend growth | 2.77 | 2.84 |
| SD dividend growth | 6.32 | 6.40 |

Back

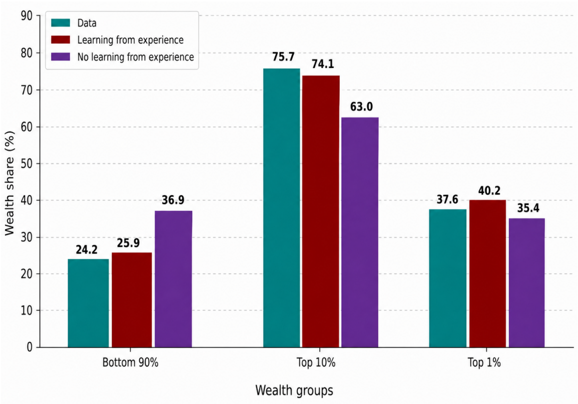
# Learning from own portfolio returns generate the belief gradient across wealth groups

| | Top 1% | | Next 9% | | Middle 40% | | Bottom 50% | |
|---|---|---|---|---|---|---|---|---|
| | **Model** | **Data** | **Model** | **Data** | **Model** | **Data** | **Model** | **Data** |
| Wealth share | 40.22 | 37.60 | 33.94 | 38.10 | 17.18 | 22.80 | 8.65 | 1.40 |
| Income share | 18.35 | 18.40 | 25.86 | 24.70 | 33.57 | 35.20 | 22.22 | 21.60 |
| Wage share | 9.58 | 12.30 | 23.80 | 21.60 | 39.55 | 38.50 | 27.07 | 27.50 |
| Risky asset share | 89.32 | 77.30 | 70.11 | 62.30 | 33.53 | 39.60 | 13.97 | 18.10 |
| Expected stock return | 8.60 | 7.88 | 8.14 | 6.78 | 6.72 | 5.39 | 5.09 | 4.77 |
| Portfolio return | 12.09 | 8.30 | 8.97 | 6.12 | 4.72 | 3.63 | 2.53 | 3.40 |

# The effect of Heterogeneous Expectations

| | Top 1% | | Next 9% | | Middle 40% | | Bottom 50% | |
|---|---|---|---|---|---|---|---|---|
| | **Hetero** | **Homo** | **Hetero** | **Homo** | **Hetero** | **Homo** | **Hetero** | **Homo** |
| Wealth share | 40.22 | 35.42 | 33.94 | 27.61 | 17.18 | 23.83 | 8.65 | 13.14 |
| Income share | 18.35 | 23.64 | 25.86 | 25.03 | 33.57 | 34.84 | 22.22 | 16.49 |
| Wage share | 9.58 | 11.88 | 23.80 | 23.41 | 39.55 | 41.06 | 27.07 | 23.64 |
| Risky asset share | 89.32 | 97.60 | 70.11 | 79.26 | 33.53 | 51.16 | 13.97 | 17.87 |
| Expected stock return | 8.60 | 9.07 | 8.14 | 9.07 | 6.72 | 9.07 | 5.09 | 9.07 |
| Portfolio return | 12.09 | 11.95 | 8.97 | 4.49 | 4.72 | 2.51 | 2.53 | 0.68 |

# Learning from own experience increase top inequality $\approx +10$pp

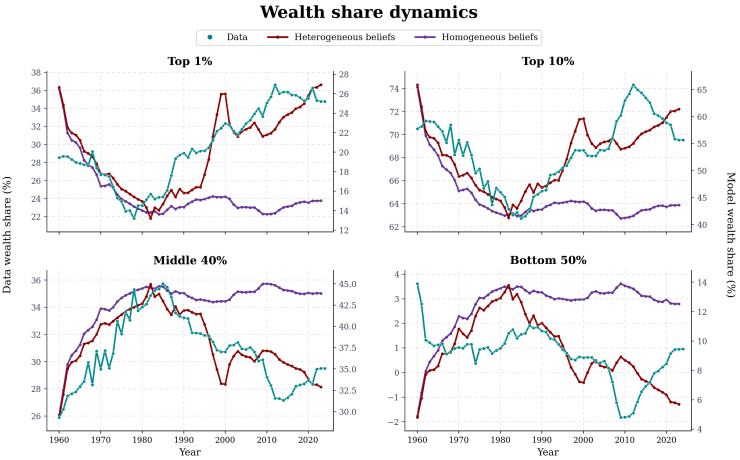


# Historical Transitional Dynamics

**Exercise:** feed the model the realized U.S. asset-price history and ask how heterogeneous beliefs reshape the evolution of portfolios and wealth inequality.

1. Historical inputs, 1960–2024: observed stock prices, dividends, and calibrated price moments for the period.
2. Start from the observed initial wealth distribution.
3. Subjective expected capital gains match the RAND gradient.
4. Model output: endogenous portfolio shares, wealth shares, capital-income shares, and inequality statistics over 1960–2024.

# Historical Transitional Dynamics




Table

# 6. Conclusions

# Main results

1. Key facts from microdata:
   * Persistent disagreement.
   * Strong association between return beliefs, wealth, and risky holdings.

2. Accurate beliefs maximize objective expected log growth, but moderate optimism can thicken the Pareto tail.

3. Internal Rationality makes HA models with aggregate risk
   * Consistent with the microdata.
   * Easier to solve.

4. Learning from portfolio experience amplifies ex-ante differences, helping to explain concentration at the very top.

# Appendix

# What explains Wealth Inequality?

* **Labour income inequality:** insufficient to explain wealth inequality at the top De Nardi and Fella (2017)
* **Bequests:** De Nardi (2004)
* **Preference heterogeneity**: discount rate (Krusell and Smith (1998)), risk-aversion (Fernandez-Villaverde and Levintal (2024))
* **Earning risk:** helps in matching bottom 60% De Nardi et al. (2016)
* **Heterogeneous capital/wealth returns:** Gabaix et al. (2016), Hubmer et al. (2020) , Xavier (2021)
* **Entrepreneurship:** Quadrini (1999, 2000), Cagetti and De Nardi (2006)
* **Decline in tax progressivity** Hubmer et al. (2020)

Back

# RAND Expectations Questions

1. *By next year at this time, what are the chances that mutual fund shares invested in blue chip stocks like those in the Dow Jones Industrial Average will be* **worth more** *than they are today?*

2. *By next year at this time, what are the chances that mutual fund shares invested in blue-chip stocks like those in the Dow Jones Industrial Average will have* **increased** *in value by* **more than 20 percent** *compared to what they are worth today?*

3. *By next year at this time, what are the chances that mutual fund shares invested in blue-chip stocks like those in the Dow Jones Industrial Average will have* **fallen** *in value by* **more than 20 percent** *compared to what they are worth today?*

A typical answer would be 1. 65%, 2. 15%, 10%.
From it, we can back out:
$P(1 < R < 1.2) = 65\% - 15\% = 50\%$;
$P(0.8 < R < 1) = (100\% - 65\%) - 10\% = 25\%$.
Internally inconsistent (negative probabilities) observations are dropped. Back

# Features of measured expectations

1. Extrapolation: $\underbrace{R_{t+k} - \mathbb{E}_t^{\mathcal{S}^i}(R_{t+k})}_{\text{Individual forecast error}} = \alpha^i + b \underbrace{R_t}_{\text{Current market return}} + \varepsilon_t^i$
2. Reaction to news: $R_{t+k} - \mathbb{E}_t^{\mathcal{S}^i}(R_{t+k}) = \alpha^i + d \underbrace{\left(\bar{\mathbb{E}}_t^{\mathcal{S}}(R_{t+k}) - \bar{\mathbb{E}}_{t-1}^{\mathcal{S}}(R_{t+k})\right)}_{\text{Average Forecast revision}} + \varepsilon_t^i$

   FIRE null hypothesis: $b = d = 0$.
3. Persistent disagreement: $\mathbb{E}_t^{\mathcal{S}^i}(R_{t+k}) = \alpha^i + \mu_t + v_t^i$

| | **1. Forecast Error Predictability** | | |
|---|---|---|---|
| *Individual Forecast Error* | (1) | (2) | (3) |
| $R_t$ | -0.13*** | - | -0.14*** |
| $\bar{\mathbb{E}}_t^{\mathcal{S}}(R_{t+k}) - \bar{\mathbb{E}}_{t-1}^{\mathcal{S}}(R_{t+k})$ | - | -0.50*** | -0.51*** |
| Obs. | 64749 | 64749 | 64749 |
| | **2. Persistent Disagreement** | | |
| *Fixed Effects* | $\mu_t$ | $\alpha^i$ | $\mu_t,\ \alpha^i$ |
| $R^2$ | 0.01 | 0.50 | 0.52 |

Back

# Regression specification

For each return bin $j$, estimate a separate fractional logit:

$$p_{ijt} = \Lambda\left(\alpha_j + X_{it}'\beta_j + \delta_{jt}\right)$$

* Dependent variables: subjective probabilities $p \in (0, 1)$ assigned to the four bins.
* For each observation, we compute the change in predicted probability under the counterfactual:

$$\widehat{\Delta p}_{ijt} = \widehat{p}_{ijt}(X_{it}^{CF}) - \widehat{p}_{ijt}(X_{it}).$$

* We then report the average across observations:

$$\widehat{\Delta p}_j = \frac{1}{N} \sum_{i,t} \widehat{\Delta p}_{ijt}$$

Back

# Subjective model for returns

* Remember a key empirical pattern: persistent disagreement.
* We first treat persistent belief differences as given to isolate their consequences.
* Subjective model of returns:

$$R_t^S = \zeta^i \varepsilon_t^{P,i}, \quad \ln \varepsilon_t^{P,i} \sim i.i.\mathcal{N}(-\frac{1}{2}\sigma_P^2, \sigma_P^2) \tag{6}$$

* $\zeta^i$ captures persistent heterogeneity in subjective return beliefs.
* We endogenize $\zeta^i$ later!
* The subjective one-period bond return is i.i.d. and homogeneous.

Back

# Kesten stationarity

* At high wealth, the additive wage term becomes negligible, so stationarity is governed by

$$\lambda_i \equiv \mathbb{E}^0\left[\log\left(\kappa_i R_{p,t}^{0,i}\right)\right] = \log\kappa_i + r_B^0 + v_i(r_S^0 - r_B^0) - \frac{1}{2}v_i^2\sigma_0^2 < 0.$$

* Equivalently,

$$\underbrace{-\log\kappa_i}_{\text{consumption drag}} + \underbrace{\frac{1}{2}v_i^2\sigma_0^2}_{\text{volatility drag}} > \underbrace{r_B^0 + v_i(r_S^0 - r_B^0)}_{\text{return component}}.$$

  High expected returns need not imply explosive wealth.
* Under correct beliefs, this becomes a restriction on patience:

$$\lambda_i^{\text{acc}} < 0 \quad \Longleftrightarrow \quad \delta < \exp\left\{-r_B - \frac{(\Delta^0)^2}{2\sigma^2}\right\}.$$

* With finitely many permanent types, fixed population weights $\mu_i$, and the usual Kesten regularity conditions, if $\mathbb{E}^0\left[\log(\kappa_i R_{p,t}^{0,i})\right] < 0 \quad \forall i$, each type has an invariant wealth distribution $F_i$, and the stationary cross-sectional distribution is

$$F(a) = \sum_i \pi_i F_i(a).$$

* In a finite economy, the joint wealth vector and wealth shares are stationary as well. Positive additive income rules out an absorbing monopoly state; no separate relative-growth condition is required.

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# Evidence consistent with learning from own returns

$$\ln \mathbb{E}_t^i(R_{t+1}^m) = \alpha_m + \gamma_m \ln R_{t-1}^i + \delta_m \ln \mathbb{E}_t^i(R_{t+1}^i) + X_t^{i\prime} \lambda_m + u_{m,t}^i$$

$$H_0 : \gamma_m = 0 \qquad \text{vs.} \qquad H_1 : \gamma_m > 0$$

| | (1) | (2) | (3) | (4) |
|---|---|---|---|---|
| | $\ln \mathbb{E}_t^i(R_{t+1}^m)$ | | | |
| $\ln(R_{t-1}^i)$ | 0.1304*** | 0.1206*** | 0.3834*** | 0.3852*** |
| | (0.025) | (0.025) | (0.007) | (0.008) |
| $\ln \mathbb{E}_t^i(R_{t+1}^i)$ | 0.6183*** | 0.6152*** | 0.6166*** | 0.6148*** |
| | (0.007) | (0.007) | (0.007) | (0.008) |
| Controls | – | ✓ | – | ✓ |
| Restriction $\delta_m = 1 - \gamma_m$ | – | – | ✓ | ✓ |
| $R^2$ | 0.438 | 0.441 | 0.201 | 0.203 |
| Obs. | 26,607 | 26,607 | 26,607 | 26,607 |

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# Recursive Formulation

* Specify the aggregate and individual state variables.
* Recall the budget constraint

$$C_t^i + A_t^i = W_t^i + \left(v_{t-1}^i R_t^S + (1 - v_{t-1}^i) R_t^B\right) A_{t-1}^i \tag{7}$$

* Household i position at the beginning of each period

$$\boldsymbol{x}_t^i = (W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, P_t, Q_t) \tag{8}$$

with $R_{t+1}^B = 1/Q_t$.
* In equilibrium, stock demand functions $h^s(\boldsymbol{x}_t^i)$.
* Asset markets equilibrium:

$$\begin{aligned} \sum_i \mu^i h^s(W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, P_t, Q_t) = 1 \\ \sum_i \mu^i h^b(W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, P_t, Q_t) = 0 \end{aligned} \tag{9}$$

* Solving that equation for prices

$$(P_t, Q_t) = p(\bar{\Gamma}_t, D_t) \tag{10}$$

* Conventionally, $\bar{\Gamma}$ is the cross-sectional distribution over stocks, bonds and wages:

$$\bar{\Gamma}_t \equiv \bar{\Gamma}(s, b, w) = \sum_{i=1}^{I} \mu^i \mathbf{1}\left\{A_{t-1}^i \leq a, v_{t-1}^i \leq v, W_t^i \leq w\right\} \tag{11}$$

# Recursive Formulation: RE vs. IR

* RE: agents know $(P_t, Q_t) = p(\bar{\Gamma}_t, D_t)$.
* Vector of states becomes

$$(\boldsymbol{x}_t^i)^{RE} = (W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, p(\bar{\Gamma}_t, D_t)) \tag{12}$$

* Problematic: deal with a highly-dimensional object.
* Under Internal Rationality (with imperfect info):

$$(\boldsymbol{x}_t^i)^{IR} = (W_t^i, A_{t-1}^i, v_{t-1}^i, D_t, P_t, Q_t, \zeta_t^i) \tag{13}$$

* Don't need to keep track of the distribution to solve households' problems!
* Prices still depend on the cross-sectional distribution

$$(P_t, Q_t) = p(\Gamma_t, D_t) \tag{14}$$

$$\Gamma_t \equiv \Gamma(s, b, w, \zeta) = \sum_{i=1}^{I} \mu^i \mathbf{1}\left\{A_{t-1}^i \leq a, v_{t-1}^i \leq v, W_t^i \leq w, \zeta_t^i \leq \zeta\right\} \tag{15}$$

# Recursive Competitive Equilibrium under IR

Given $\{\mathcal{P}^i, S^i_{-1}, B^i_{-1}\}_i$, a RCE is given by

* The policy functions $\boldsymbol{h}_i$ for all $i$:

$$\begin{pmatrix} C^i_t \\ v^i_t \end{pmatrix} = h_i((\boldsymbol{x}^i_t)^{IR}) \tag{16}$$

* The pricing function p:

$$(P_t, Q_t) = p(\Gamma_t, D_t) \tag{17}$$

Such that

1. Optimality conditions, budget constraints and asset holdings limits of each household are satisfied.
2. Markets clear.

# Recursive Competitive Equilibrium under IR: two-stage approach

Under Internal Rationality, computing the equilibrium can be separated into two steps

1. Solve the individual problems conditional on simulated price paths drawn from $\mathcal{P}^i$.

2. Use optimal demands to compute the market-clearing prices.
   * Step 2 recovers the implied equilibrium pricing function $p(\Gamma, \cdot)$.

New solution algorithm: 2-Stage Deep Learning (2SDL).

# 2-Stage Deep Learning algorithm

1. **Stage 1: Solve household policies under subjective beliefs** Graph
   * Train a policy network for the saving rate and risky portfolio share:
   $$(\kappa_t^i, v_t^i) = \mathcal{N}_\theta\left(\boldsymbol{x}_t^i\right)$$
   * Initialize the network at the analytical Merton–Samuelson policy; the network learns a nonlinear correction around it.
   * Generate training states by simulating histories under households' subjective measure, without imposing market clearing.
   * Enrich the simulated histories with off-path collocation states, placing additional mass near the saving constraint.
   * Minimize the Euler–KKT loss: Fischer–Burmeister complementarity residuals for $A_t^i \geq 0$ and $0 \leq v_t^i \leq \bar{v}$, with a penalty for infeasible current or future resources.

2. **Stage 2: General Equilibrium**
   * Simulate thousands of households with agent-specific exogenous realizations (wages, belief anchors).
   * Use the trained policy network inside the full GE simulation.
   * At each date, solve equilibrium prices from market clearing.
   * Update household states and repeat along the simulated GE path.

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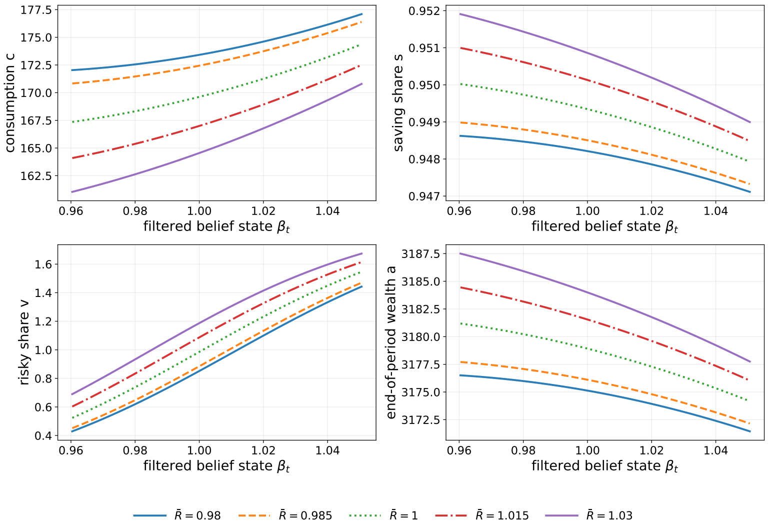
# A look into the policy functions

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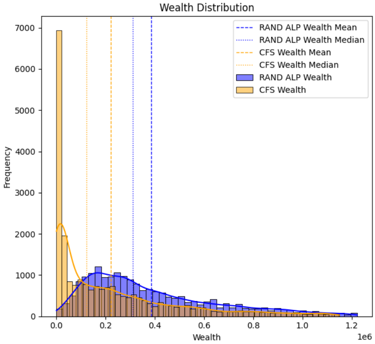
# Constructing RAND–SCF calibration targets

* Why RAND alone is not enough for quantitative calibration
  * Coarse probability distribution $\neq$ expected returns.
  * Limited asset coverage, completely omitting directly-held financial assets.
* Convert raw data into expected returns (like Von Gaudecker and Wogrolly, 2022).
* Use RAND to impute beliefs in the Survey of Consumer Finances (2016).
  * Use common observables: retirement account, equity share in the retirement account, home value, ownership states, wage income, and age to predict expected returns.
  * Interpretation: this is a statistical bridge for calibration targets, not a separate source of observed expectations in the SCF.

Details

# Imputed Expectations and Portfolios

* The validity of our matching approach hinges on RAND being sufficiently representative of stock market participants.
* Wealth distribution in the common categories look similar enough for the quantitative exercise.

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# Imputing Expectations

$$Y_t = \alpha + (\boldsymbol{Z}_t^i)'\beta + \epsilon_t^i, \text{ for } Y_t = \{\mathbb{E}_t^{\mathcal{S}^i}[R_{t+1}], R_{t+1} - \mathbb{E}_t^{\mathcal{S}^i}[R_{t+1}]\}$$

| Predictor | Expected Returns | Forecast Error |
|---|---|---|
| Constant | $-8.17$ | $6.3^{***}$ |
| Owns home? | $-2.1$ | $0.7$ |
| Has Retirement Account? | $1.66$ | $-0.27$ |
| Retirement Account (value) | $0.4^{***}$ | $-0.1^{***}$ |
| Stock Share in RA | $0.04^{***}$ | $-0.01^{***}$ |
| Home Value | $0.07$ | $-0.02$ |
| Wage Income | $0.5$ | $-0.21$ |
| Age | $0.002$ | $-0.01$ |

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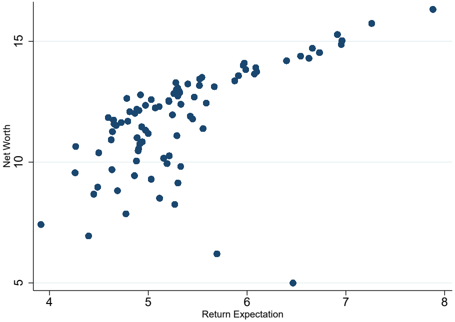
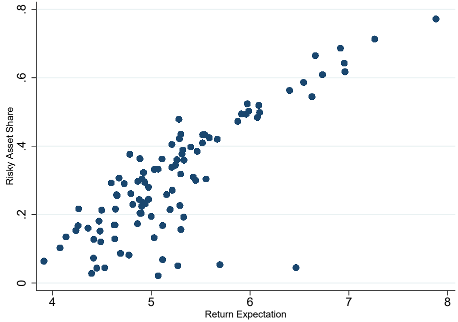
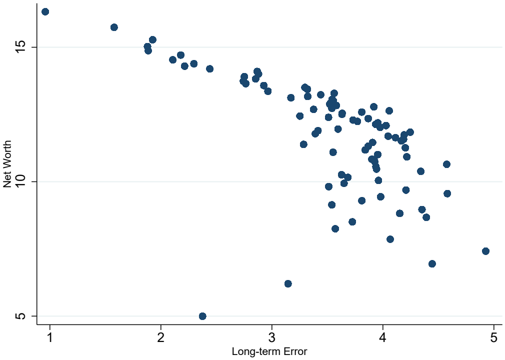
# Imputed RAND–SCF: expected returns and wealth



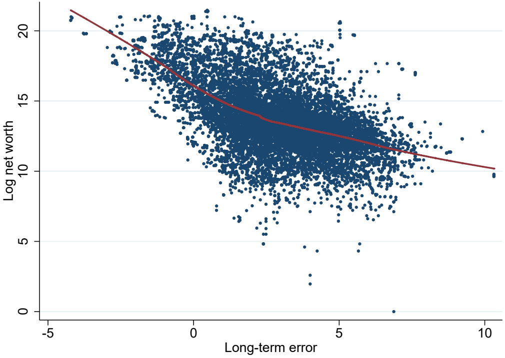
# Imputed RAND–SCF: expected returns and risky shares



# Imputed RAND–SCF: wealth and forecast errors



# Imputed RAND–SCF: wealth and forecast errors



# Aggregate Processes and Asset Returns

* Dividend growth is lognormal:

$$g_{t+1} \equiv \frac{D_{t+1}}{D_t} = \bar{g}_D \exp\left(-\frac{1}{2}\sigma_D^2 + \sigma_D \varepsilon_{t+1}^D\right), \qquad \varepsilon_{t+1}^D \sim \mathcal{N}(0,1).$$

* Calibration targets: annual mean dividend growth 2.84% and annual volatility 6.4%.
* The risky return and safe return realized in the GE simulation are

$$R_{t+1}^S = g_{t+1}\frac{pd_{t+1}+1}{pd_t}, \qquad R_{t+1}^B = \frac{1}{Q_t}.$$

* Aggregate return anchors: mean real stock return 9.29% and mean real safe rate 0.75% per year.

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# Idiosyncratic Wages: Two-State Process

* Wage persistence follows Fernández-Villaverde and Levintal, 2024; the two wage levels are disciplined by 2016 SCF wage-income shares.
* Every household faces the same transitory two-state wage process:

$$w_t^i = \bar{w}\,\varepsilon(z_t^i), \qquad z_t^i \in \{L, H\}.$$

  There is no permanent household-specific wage type.
* Annual wage-state transition matrix:

$$P^A = \begin{pmatrix} 0.9972 & 0.0028 \\ 0.0252 & 0.9748 \end{pmatrix}, \qquad \pi = (0.9, 0.1).$$

* The quarterly model uses the exact stochastic fourth root:

$$P^Q \simeq \begin{pmatrix} 0.999293 & 0.000707 \\ 0.006367 & 0.993633 \end{pmatrix}.$$

* Mean-one wage levels implied by the SCF shares are

$$\varepsilon_L = \frac{0.661}{0.9} = 0.7344, \qquad \varepsilon_H = \frac{0.339}{0.1} = 3.39.$$

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# Six Preference States

* Each household occupies one of six preference states:

$$\tau_t^i \in \{1, \ldots, 6\}, \qquad \tau_t^i \longmapsto \left(\delta_t^i,\, \gamma_t^i\right).$$

* The wage state is a separate transitory state:

$$z_t^i \in \{L, H\}, \qquad z_{t+1}^i \sim P^Q(z_t^i, \cdot).$$

  Every household faces the same two-state wage process.

* Beliefs are not assigned by type. All households start from the same prior, and expected stock returns subsequently evolve through learning from investment experience.

* Hence the full idiosyncratic state includes the preference state, the transitory wage state, and the household-specific belief state:

$$\left(\tau_t^i, z_t^i, \bar{R}_t^i\right).$$

* The six-state support captures heterogeneity in patience and risk aversion, while wages and beliefs evolve separately.

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# Six-State Preference Support

| State | Mass | $\delta$ | $\gamma$ |
|---|---|---|---|
| 1 | 39.1% | 0.952 | 13.00 |
| 2 | 12.2% | 0.973 | 2.60 |
| 3 | 22.0% | 0.956 | 5.50 |
| 4 | 16.7% | 0.951 | 9.50 |
| 5 | 9.0% | 0.954 | 7.50 |
| 6 | 1.0% | 0.975 | 1.45 |

* The discrete state determines patience and risk aversion—not wages or beliefs.
* Every preference state faces the same low and high wage states and the same wage-state transition matrix.
* Expected stock returns evolve endogenously and separately through learning from experience.

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# Stage-2 Targets and Diagnostics

* Aggregate targets:

| | | | |
|---|---|---|---|
| Capital-income share | 27.0% | Wealth-income ratio | 491% |
| Mean risky share | 30.5% | Wealth-weighted risky share | 70.8% |
| Saving rate | 6.7% | Mean stock return | 9.29% |
| Mean safe rate | 0.75% | | |

* Distributional targets by wealth group: wealth, income, wages, capital-income shares, risky shares, expected returns, and portfolio returns.
* Forecast errors are left as diagnostics.

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# Historical transitional dynamics: Distributions

| | Model: Heterogeneous Beliefs | | | | Model: Homogeneous Beliefs | | | |
|---|---|---|---|---|---|---|---|---|
| **By wealth percentile** | **Top 1%** | **Next 9%** | **Mid 40%** | **Bot 50%** | **Top 1%** | **Next 9%** | **Mid 40%** | **Bot 50%** |
| Wealth | 22.18 | 32.88 | 36.68 | 8.25 | 14.74 | 28.51 | 43.76 | 13.00 |
| Wages | 7.08 | 18.35 | 37.03 | 37.55 | 6.63 | 18.35 | 37.14 | 37.88 |
| Risky asset share | 86.79 | 74.68 | 60.52 | 41.94 | 65.48 | 60.56 | 55.08 | 50.84 |
| Expected return | 8.35 | 6.63 | 5.42 | 4.40 | 5.04 | 5.04 | 5.04 | 5.04 |
| Forecast error | -1.07 | 0.65 | 1.86 | 2.88 | 2.25 | 2.25 | 2.25 | 2.25 |
| Portfolio return | 7.46 | 6.43 | 5.14 | 2.16 | 5.39 | 5.05 | 4.60 | 2.26 |

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