Testing Weak Factors in Asset Pricing

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Outline

Empirical Asset Pricing

- One of the most famous equations in AP is

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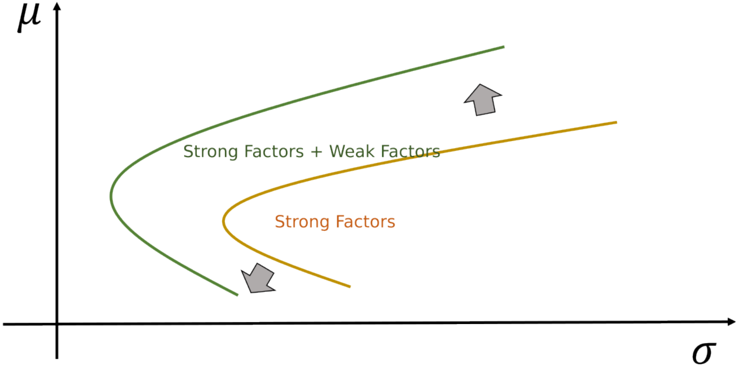
  - Standard empirical approach is the Two-Pass CSR method
  - Once you decide to take it seriously, lots of complexity arises in empirical application
- This paper considers the issue of weak factors
  - When risk exposures are close to zero for most assets
  - PCA will confound them with noise

Weak Factors and Investment

- Mean-Variance



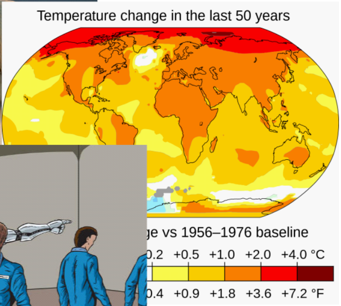
- Hence, as an investor, she will have a strong incentive to search for the weak factors!

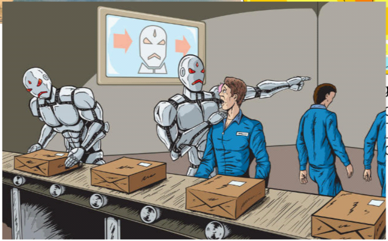
Weak Factors and Asset Pricing Test

- When some factors are weak, lots of distortion may happen
  - weak factors without premia may appear to be important
  - strong factors with significant premium may appear insignificant
- Especially, when the literature proposes hundreds of factors, we need some criteria

Furthermore, Rapidly Changing Economic Landscape

- We need to discern which factors are strong or weak
    - in a rapidly changing economic environment
- For example, paradigm shifts such as climate changes, new assets (cryptos and bitcoins), job destruction due to AI, all beg for a short-T method





Origins of Weak Factors

Weak factors may emerge...

- ...when constructing factors based on anomalies or when considering macroeconomic factors, as they are typically exposed to a smaller subset of the test assets under examination.
- ...under market incompleteness, as not all sources of risk are adequately spanned across all assets.
- ...when constructing a portfolio (call it alpha or SA portfolio) that is neutral to systematic risk but instead only exposed to unsystematic risk (see DelloPreite, Raponi, and Zaffaroni 2025).

Key Insights of Paper

- Back to the famous equation,

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- Taxonomy of empirical asset pricing econometrics

	Small T	Large T
Strong Factors	$B_{strong} \sim er$	$B_{strong} \gg er$
Weak Factors	$B_{weak} \ll er$	$B_{weak} \sim er$

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1. Traditionally, estimation errors in estimated beta are source of trouble
2. We flip it as a blessing to reveal whether a given factor is weak or not:
3. Key insight I: distinguish between behaviour of (sum of) B^2 from (sum of) $|B|$
4. Key insight II: power of test from estimating zero-beta rate (intercept) instead of risk premia (slope).

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Our Contribution

- We propose a novel test for weak factors under *fixed T* and *large N* setup that:
  - builds on the two-pass methodology - simple and intuitive.
  - handles conditional asset pricing models but robust to
    - misspecified conditional dynamics (semi-parametric)
    - omitted risk factors (PCA)
  - detect whether observed risk factors are *(locally)* weak or not

Big Picture: Cross-Sectional Asset Pricing - Large N

- Research agenda on estimating/testing/using AP models when using an unbalanced panel of returns/characteristics data for many (N) assets and limited (T) periods.
- Large N - Fixed T: APT, conditional asset pricing, robustness to misspecification, local risk factor, single stocks.
- Testing Beta Pricing Models using Large Cross-Sections (RFS, 2020).
- Factor Models for Conditional Asset Pricing (JPE forthcoming)
- Cross-Sectional Asset Pricing with Unsystematic Risk (under revision).
- Dissecting Anomalies for Conditional Asset Pricing (under revision).
- Statistical Arbitrage without Arbitrage.
- .....

Outline

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Model

- Conditional factor structure for asset $i = 1, \cdots, N$ at $t = 1, \cdots, T$:

$$R_{it} = \alpha_{it-1} + \underbrace{\boldsymbol{\beta}'_{fit-1}\boldsymbol{f}_t}_{\text{strong}} + \underbrace{\boldsymbol{\beta}'_{git-1}\boldsymbol{g}_t}_{\text{weak}} + e_{it},$$

where

$$\boldsymbol{\beta}_{fit-1} = (\beta_{f_1 it-1}, \cdots, \beta_{f_K it-1})', \ \ \boldsymbol{f}_t = (f_{1t}, \cdots, f_{Kt})$$
$$\boldsymbol{\beta}_{git-1} = (\beta_{g_1 it-1}, \cdots, \beta_{g_L it-1})', \ \ \boldsymbol{g}_t = (g_{1t}, \cdots, g_{Lt})$$

RGP+APT+Local Smoothness

- We use the conditional AP model as a locally unconditional AP model (smoothness assumption):

$$\mathbf{R}_t = \gamma_{zt-1}\mathbf{1}_N + \mathbf{B}_f\boldsymbol{\delta}_{ft} + \mathbf{B}_g\boldsymbol{\delta}_{gt} + \boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\delta}_{ft}$ and $\boldsymbol{\delta}_{gt}$ are *expost* risk premia:

$$\boldsymbol{\delta}_{ft} = \boldsymbol{\gamma}_{ft-1} + \boldsymbol{f}_t - E\left[\boldsymbol{f}_t|\mathcal{I}_{t-1}\right], \boldsymbol{\delta}_{gt} = \boldsymbol{\gamma}_{gt-1} + \boldsymbol{g}_t - E\left[\boldsymbol{g}_t|\mathcal{I}_{t-1}\right]$$

and γ_{zt-1} zero-beta rate.

Local Factor Strength

- For some $0 \leq \rho \leq 1$, the matrix $\mathbf{B}_g$ satisfies

$$\|\mathbf{B}_g\|^2 \asymp O\left(N^{\rho}\right), \ \|\mathbf{B}_g' 1_N\| \asymp o\left(N^{\frac{\rho+1}{2}}\right)$$

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  - The difference in the convergence speed plays a key role to learn ρ
    - Analogy to well-spread portfolio $\boldsymbol{w}$, $\boldsymbol{w}'\mathbf{1}_N = 1$ and $\boldsymbol{w}'\boldsymbol{w} \to 0$
- Our methodology works regardless of the form of weakness (sparse or uniform).

Outline

Target Equation

- First, we consider the case that there is no factor $\boldsymbol{f}$:
  - RGP

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_g \boldsymbol{g}_t + \boldsymbol{\epsilon}_t$$

Target Equation

- First, we consider the case that there is no factor $\boldsymbol{f}$:
  - RGP
    $$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_g \boldsymbol{g}_t + \boldsymbol{\epsilon}_t$$
  - Along with the pricing, $\mu = B \times \gamma$
    $$\mathbf{R}_t = \gamma_{zt-1} \mathbf{1}_N + \mathbf{B}_g \boldsymbol{\delta}_{gt} + \boldsymbol{\epsilon}_t,$$
    which gives the target equation:
    $$\overline{\mathbf{R}} = \overline{\gamma}_z \mathbf{1}_N + \mathbf{B}_g \overline{\boldsymbol{\delta}}_g + \overline{\boldsymbol{\epsilon}}$$
  - Note that we are interested in whether $\boldsymbol{g}$ is weak or not...not in estimating risk premia!

FamaMcBeth Two-Pass

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_{g0} = \mathbf{B}_g + \epsilon \mathcal{P}_g,$$

where $\mathbf{R} = (\mathbf{R}_1, \cdots, \mathbf{R}_T)'$, $G = (\boldsymbol{g}_1, \cdots, \boldsymbol{g}_T)'$, $\mathcal{J}_T = I_T - \frac{1}{T}\mathbf{1}_T\mathbf{1}_T', \mathcal{P}_g = \mathcal{J}_T G \left(G'\mathcal{J}_T G\right)^{-1}$

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- Second-pass cross-sectional OLS gives:

$$\widehat{\boldsymbol{\Gamma}}_{g0} = \begin{bmatrix} \widehat{\gamma}_{0g0} \\ \widehat{\boldsymbol{\delta}}_{g0} \end{bmatrix} = \left(\widehat{\mathbf{X}}_{g0}'\widehat{\mathbf{X}}_{g0}\right)^{-1}\widehat{\mathbf{X}}_{g0}'\overline{\mathbf{R}}$$

$$\asymp \begin{bmatrix} \overline{\gamma}_z \\ 0_L \end{bmatrix} + \begin{bmatrix} O\left(\frac{\mathbf{B}_g' 1_N}{N}\right) \\ O\left(\frac{\mathbf{B}_g'\mathbf{B}_g}{N}\right) \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right),$$

where

$$\widehat{\mathbf{X}}_{g0} = \left[1_N \ \hat{\mathbf{B}}_{g0}\right]$$

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Properties of FMB 1 - Risk Premia (Slope)

Theorem 1. *Under some Assumptions, the two-pass estimator $\widehat{\boldsymbol{\delta}}_{g0}$ in $\widehat{\boldsymbol{\Gamma}}_{g0} = \left[\widehat{\gamma}_{z0}\ \widehat{\boldsymbol{\delta}}'_{g0}\right]'$ behaves as follows:*

	$\widehat{\boldsymbol{\delta}}_{g0} \to_p$	$\sqrt{N}\widehat{\boldsymbol{\delta}}_{g0} \to_d$
$\rho < \frac{1}{2}$		$\mathcal{N}\left(\mathbf{0}_L, \frac{\kappa_4 + Ts_4}{T^2 s_2^2} G' \mathcal{J}_T G\right)$
$\rho = \frac{1}{2}$	$\mathbf{0}_L$	$\mathcal{N}\left(\mathbf{0}_L, \frac{\kappa_4 + Ts_4}{T^2 s_2^2} G' \mathcal{J}_T G\right) + O_p(1)$
$\frac{1}{2} < \rho < 1$		$\pm\infty$
$\rho = 1$	$\widehat{\boldsymbol{\delta}}_{g0} \nrightarrow_p \overline{\boldsymbol{\delta}}_g$	

where $s_2 = \lim_N \frac{1}{N}\sum_i \epsilon_{it}^2$, $\kappa_4 = \left(\lim_N \frac{1}{N}\sum_i \epsilon_{it}^4 - 3s_4\right)$ *and* $s_4 = \lim_N \frac{1}{N}\sum_i E\left[\epsilon_{it}^2\right]^2$

Asymptotically valid SE can be constructed.

Relationship to Standard OLS

Theorem 2. *Under the assumption that residuals are normal i.i.d, the OLS statistics* R^2_{g0} *and t-stats and F-stat on* $\widehat{\boldsymbol{\delta}}_{g0}$ *behaves as follows:*

	$R^2_{\mathrm{g0}} \to_p$	$t_{g0,k} \to_p$	$F_{g0} \to_p$
$\rho < \frac{1}{2}$		$\mathcal{N}(0,1)$	$\frac{\chi^2_L}{L}$
$\rho = \frac{1}{2}$	0	$\mathcal{N}(0,1) + O_p(1)$	$\frac{\chi^2_L}{L} + O_p(1)$
$\frac{1}{2} < \rho < 1$		$\pm\infty$	∞
$\rho = 1$	$(0,1)$		

Properties of FMB 2 - Zero-Beta Rate (Intercept)

Theorem 3. *Under some Assumptions, the two-pass estimator* $\widehat{\gamma}_{z0}$ *in* $\widehat{\boldsymbol{\Gamma}}_{g0} = \left[\widehat{\gamma}_{z0}\ \widehat{\boldsymbol{\delta}}'_{g0}\right]'$ *behaves as follows:*

	$\widehat{\gamma}_{z0} \to_p$	$\sqrt{N}\left(\widehat{\gamma}_{0g0} - \overline{\gamma}_0\right) \to_d$
$\rho = 0$	$\overline{\gamma}_z$	$\mathcal{N}\left(0, \frac{s_2}{T}\right)$
$0 < \rho < 1$	$\overline{\gamma}_z$	$\pm\infty$
$\rho = 1$	$\widehat{\gamma}_{z0} \nrightarrow_p \overline{\gamma}_z$	$\pm\infty$

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where $s_2 = \lim_N \frac{1}{N} \sum_i \epsilon_{it}^2$

- *Given that we do not observe* $\overline{\gamma}_z$ *(except R is an excess return), the asymptotic distribution is not directly useful*
- *Hence, we propose a new test using* $\sqrt{N}\frac{\mathbf{1}'_N \widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g$
- *No need for this when working with excess returns.*
- *Asymptotically valid SE can be constructed.*

Properties of FMB 2: Feasible Version

Theorem 4. *Under some Assumptions,* $\sqrt{N}\frac{\mathbf{1}_N'\widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g$ *behaves as follows:*

	$\sqrt{N}\frac{\mathbf{1}_N'\widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g \to_d$
$\rho = 0$	$\mathcal{N}\left(0, s_2\overline{\boldsymbol{\delta}}_g'\left(G'\mathcal{J}_T G\right)^{-1}\overline{\boldsymbol{\delta}}_g\right)$
$0 < \rho < 1$	$\pm\infty$
$\rho = 1$	$\pm\infty$

- *Furthermore, we observe all the elements for the asymptotic variance except* $s_2 = \lim_N \frac{1}{N}\sum_i \epsilon_{it}^2$*!*

Asymptotically valid SE can be constructed.

Summary of Our Tests of Factors Strength

- We propose two tests: (i) coefficients on the noisy betas (risk premia) and (ii) zero-beta rate

<table>
<tr><th></th><th>$\sqrt{N}\widehat{\boldsymbol{\delta}}_{g0}$</th><th>$\sqrt{N}\widehat{\gamma}_{0g0}$</th></tr>
<tr><td>$\rho = 0$</td><td rowspan="2">null</td><td>null</td></tr>
<tr><td>$0 < \rho < \frac{1}{2}$</td><td rowspan="2">Alternative</td></tr>
<tr><td>$\rho \geq \frac{1}{2}$</td><td>Alternative</td></tr>
</table>

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Modified FMB Two-Pass with Observed Strong Factors

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_f = \mathbf{B}_f + \epsilon\mathcal{P}_f, \widehat{\mathbf{B}}_g = \mathbf{B}_g + \epsilon\mathcal{P}_{g_\perp},$$

where $\mathcal{P}_f = \mathcal{J}_T F \left(F' \mathcal{J}_T F\right)^{-1},\ \mathcal{P}_{g\perp} = \mathcal{J}_T G_\perp \left(G'_\perp \mathcal{J}_T G_\perp\right)^{-1}$

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- Second-pass cross-sectional OLS gives:

$$\widehat{\Gamma}_g = \left[\begin{array}{c} \widehat{\gamma}_z \\ \widehat{\boldsymbol{\delta}}_g \end{array} \right] = \left(\widehat{\mathbf{X}}_g'\widehat{\mathbf{X}}_g\right)^{-1}\widehat{\mathbf{X}}_g'\left(\overline{\mathbf{R}} - \widehat{\mathbf{B}}_f\overline{\boldsymbol{\delta}}_f\right),$$

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  where

$$\widehat{\mathbf{X}}_g = \left[1_N \ \widehat{\mathbf{B}}_g\right]$$

- If we include $\widehat{\mathbf{B}}_f$ in the second pass regressor
  - It is well known that the estimator is biased due to estimation error
  - The bias-correction such as Shanken (1992) does not work

Slight Modification of Tests

- Two tests have similar properties

	$\sqrt{N}\widehat{\boldsymbol{\delta}}_g \to_d$	$\frac{\mathbf{1}_N' \widehat{\mathbf{B}}_g}{\sqrt{N}} \overline{\boldsymbol{\delta}}_g \to_d$
$\rho = 0$	$\mathcal{N}(\mathbf{0}_L, V_1)$	$\mathcal{N}(0, V_2)$
$\rho < \frac{1}{2}$	$\mathcal{N}(\mathbf{0}_L, V_1)$	
$\rho = \frac{1}{2}$	$\mathcal{N}(\mathbf{0}_L, V_1) + O_p(1)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	

where

$$V_1 = \frac{s_4}{s_2^2}\mathbf{l}'\mathbf{l} G_\perp' G_\perp + \frac{\kappa_4}{s_2^2} G_\perp' \operatorname{diag}(\mathbf{l} \odot \mathbf{l})\, G_\perp$$

$$\mathbf{l} = \frac{1_T}{T} - \mathcal{P}_f \overline{\boldsymbol{\delta}}_f$$

$$V_2 = s_2 \overline{\boldsymbol{\delta}}_g' \left(G_\perp' G_\perp\right)^{-1} \overline{\boldsymbol{\delta}}_g$$

- Furthermore, we can make the tests feasible using consistent estimators for components in the asymptotic variance

Outline

PCA (Most Useful Setting)

- We borrow the idea of Giglio and Xiu (2021) and span the (strong) factors' space by PCA.
- Following the local PCA methodology of Zaffaroni (2025), we obtain the systematic factors up to rotation (when T fixed and $N \to \infty$)

$$F_* - F\tilde{H} \to_p 0_{T \times K}$$

Modified FMB Two-Pass with PCA Factors

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_{f_*} = \mathbf{B}_{f_*} + \epsilon_* \mathcal{P}_{f_*},$$
$$\widehat{\mathbf{B}}_{g_*} = \mathbf{B}_{g_*} + \epsilon_* \mathcal{P}_{g_{*\perp}},$$

where $\mathcal{P}_{f_*} = \mathcal{J}_T F_* \left(F_*' \mathcal{J}_T F_*\right)^{-1}$, $\mathcal{P}_{g_{*\perp}} = \mathcal{J}_T G_{*\perp} \left(G_{*\perp}' \mathcal{J}_T G_{*\perp}\right)^{-1}$

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- Second-pass cross-sectional OLS gives:

$$\widehat{\boldsymbol{\Gamma}}_{g_*} = \begin{bmatrix} \widehat{\gamma}_{z_*} \\ \widehat{\boldsymbol{\delta}}_{g_*} \end{bmatrix} = \left(\widehat{\mathbf{X}}_{g_*}' \widehat{\mathbf{X}}_{g_*}\right)^{-1} \widehat{\mathbf{X}}_{g_*}' \left(\overline{\mathbf{R}} - \widehat{\mathbf{B}}_{f_*} \overline{\boldsymbol{\delta}}_{f_*}\right),$$

where

$$\widehat{\mathbf{X}}_{g_*} = \left[1_N \; \widehat{\mathbf{B}}_{g_*}\right]$$

Slight Modification of Tests

- Two tests have similar properties

	$\sqrt{N}\widehat{\boldsymbol{\delta}}_{g_*} \to_d$	$\sqrt{N}\widehat{\gamma}_{g0_*} \to_d$
$\rho = 0$	$\mathcal{N}\left(\mathbf{0}_L, V_{1*}\right)$	$\mathcal{N}\left(0, V_{2*}\right)$
$\rho < \frac{1}{2}$	$\mathcal{N}\left(\mathbf{0}_L, V_{1*}\right)$	$\pm\infty$
$\rho = \frac{1}{2}$	$\mathcal{N}\left(\mathbf{0}_L, V_{1*}\right) + O_p(1)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	$\pm\infty$

where

$$V_{1*} = \frac{s_4}{s_2^2}\mathbf{l}'\mathbf{l}G_\perp' G_\perp + \frac{\kappa_4}{s_2^2}G_\perp' \operatorname{diag}\left(\mathbf{l} \odot \mathbf{l}\right) G_\perp$$

$$V_{2*} = c_* + s_2\overline{\boldsymbol{\delta}}_g'\left(G_\perp' G_\perp\right)^{-1}\overline{\boldsymbol{\delta}}_g$$

- Furthermore, we can make the tests feasible using consistent estimator of the corresponding asymptotic variances

Outline

Simulation Design I (Conventional MC)

1. Calibration: MacKinlay and Pastor (2000)

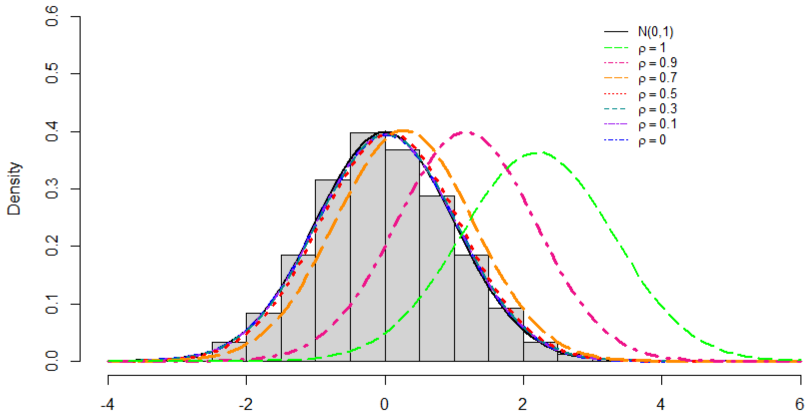
$$R_{it} = 0 + \beta_{fi} \boldsymbol{f}_t + \beta_{gi} \boldsymbol{g}_t + e_{it}$$

2. We consider a single strong factor and a single weak factor, $N = 3000,\ T = 24$
3. We focus on the distribution of the following two tests

	test 1: $\frac{1}{\sqrt{\widehat{AsyVar}}} \left(\sqrt{N} \widehat{\boldsymbol{\delta}}_g \right)$	test 2: $\frac{1}{\sqrt{\widehat{AsyVar}}} \left(\frac{\mathbf{1}_N' \widehat{\mathbf{B}}_g}{\sqrt{N}} \overline{\boldsymbol{\delta}}_g \right)$
$\rho = 0$	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$
$\rho < \frac{1}{2}$		
$\rho = \frac{1}{2}$	$\mathcal{N}(0,1) + O_p(1)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	

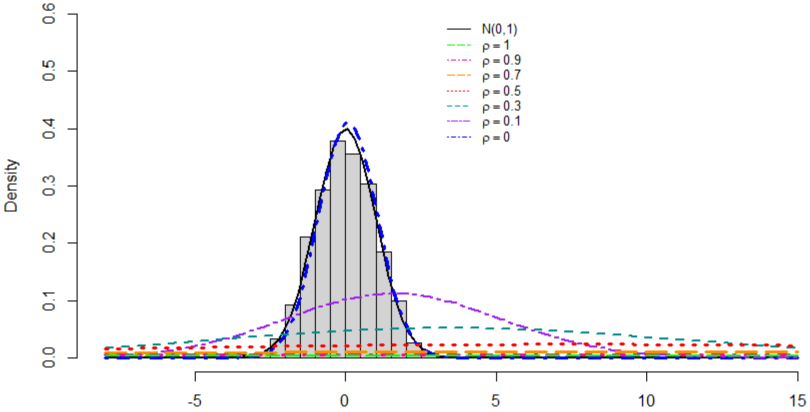
Test 1: under the null $\rho < \frac{1}{2}$ + DGP with $\rho \in [0, 1]$

- 3000 repetitions



Test 2: under the null $\rho = 0$ + DGP with $\rho \in [0, 1]$

- 3000 repetitions



Simulation Design II (Cool MC - SPCA)

- Giglio, Xiu, and Zhang (2024) introduced the Supervised PCA (SPCA) to estimate the risk premia of observed weak (semi-strong) factors.
- Their two-step methodology:(i) identify the subset of assets where candidate risk factor is strong; (ii) two-pass estimation of risk premia over the subset of assets.
- We demonstrate how our methodology (KRZ) can be used sequentially with SPCA:
  1. Use KRZ to test whether a given candidate risk factor g_t is weak, semi-strong, or strong.
  2. Use PCA (Giglio and Xiu 2020) or SPCA (Giglio, Xiu, and Zhang 2024) to estimate g_t risk premia.

KRZ and SPCA in the presence of a WEAK factor

Panel A: $s_0 = 1, T = 12$

		KRZ PCA test	SPCA + KRZ PCA test			KRZ on Non Selected
N	Scenario	% Reject	% Reject	N_{SPCA}	% (mode)	% Reject
200, N_0=20	G orth F	3.2	96.8	20	64.6	1.8
	G = F + z	11.0	80.0	20	59.8	1.0
	G ∼ F (cor = 0.99)	1.6	1.8	70	30.8	9.0
500, N_0=50	G orth F	8.8	96.2	50	90.8	1.5
	G = F + z	7.4	88.8	50	32.4	1.4
	G ∼ F (cor = 0.99)	1.2	1.8	400	16.2	10.3
1000, N_0=100	G orth F	11.2	100.0	100	87.2	0.8
	G = F + z	9.4	98.8	100	17.4	0.4
	G ∼ F (cor = 0.99)	0.8	5.6	150	15.8	2.4

Panel B: $s_0 = 0.5, T = 12$

		KRZ PCA test	SPCA + KRZ PCA test			KRZ on Non Selected
N	Scenario	% Reject	% Reject	N_{SPCA}	% (mode)	% Reject
200, N_0=20	G orth F	1.2	84.0	70	35.8	2.6
	G = F + z	0.0	86.6	20	48.0	0.4
	G ∼ F (cor = 0.99)	1.2	0.6	120	32.8	9.8
500, N_0=50	G orth F	1.6	81.0	50	35.6	0.7
	G = F +z	0.8	97.2	100	42.8	0.4
	G ∼ F (cor = 0.99)	2.0	2.4	150	17.0	8.4
1000, N_0=100	G orth F	1.8	99.0	100	31.8	1.0
	G = F + z	0.4	100.0	150	32.2	7.4
	G ∼ F (cor = 0.99)	0.0	10.8	150	14.6	19.4

KRZ and SPCA in the presence of a STRONG Factor

Panel A: $s = 1$, $T = 12$

		KRZ PCA test	SPCA + KRZ PCA test			KRZ on Non Selected
N	Scenario	% Reject	% Reject	N_{SPCA}	% (mode)	% Reject
200	G orth F	95.2	86.0	20	63.6	1.4
	G = F + v	100.0	50.2	70	48.8	6.6
	G ∼ F (cor = 0.99)	1.0	1.6	70	37.0	5.8
500, N_0=50	G orth F	99.6	94.2	50	66.8	1.0
	G = F + v	100.0	85.8	100	34.2	1.0
	G ∼ F (cor = 0.99)	2.0	4.8	100	22.6	3.6
1000, N_0=100	G orth F	99.2	97.8	150	26.2	0.6
	G = F + v	100.0	98.4	150	36.8	3.2
	G ∼ F (cor = 0.99)	10.6	7.6	150	27.8	7.2

Panel B: $s = 0.5$, $T = 12$

		KRZ PCA test	SPCA + KRZ PCA test			KRZ on Non Selected
N	Scenario	% Reject	% Reject	N_{SPCA}	% (mode)	% Reject
200	G orth F	66.6	97.4	20	53.8	0.8
	G = F + v	49.2	57.0	70	44.2	9.1
	G ∼ F (cor = 0.99)	0.2	1.2	70	33.8	8.0
500, N_0=50	G orth F	78.6	99.4	50	86.6	2.2
	G = F + v	100.0	90.4	50	31.4	15.6
	G ∼ F (cor = 0.99)	4.4	1.8	400	21.8	10.8
1000, N_0=100	G orth F	44.4	100.0	100	65.6	0.4
	G = F + v	97.2	99.0	150	20.4	11.0
	G ∼ F (cor = 0.99)	0.4	4.6	150	17.0	2.8

Outline

Outline Empirics

- We focus on the test 2 (zero-beta rate) on the null $\rho = 0$ over 1968-2022
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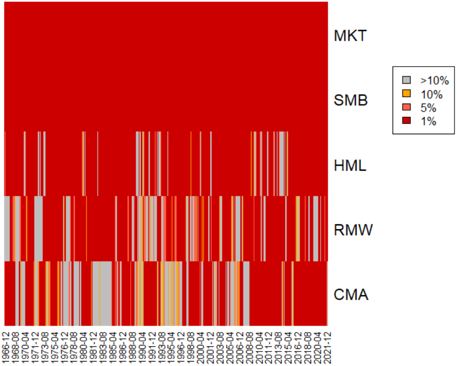
- Q4 ....

Any Strong Factors in FF5?

- HeatMap (Strong Red - ... - Weak Grey)

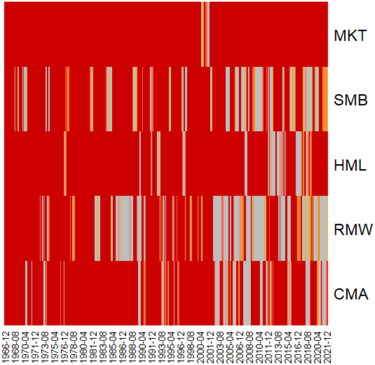


- Null on F and G

F	G
No Strong Factor	MKT
MKT	SMB, HML
FF3	RMW, CMA

Stong-Weak of FF5 in Utility Industry

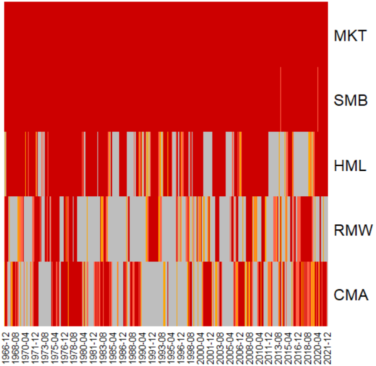
- HeatMap (Strong Red - ... - Weak Grey)



- SMB tends to be weaker in the Utility industry
  - 20% of tests in Uitlity vs 0% of tests in CRSP

Stong-Weak of FF5 in Consumer Nondurables

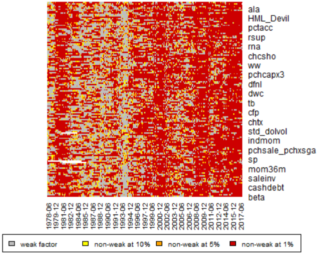
- HeatMap (Strong Red - ... - Weak Grey)



- HML tends to be weaker in the Consumer Nondurables industry
    - 27% of tests in Consumer Non-durables vs 10% of tests in CRSP

Factor Zoo with Strong (Subset of) FF5

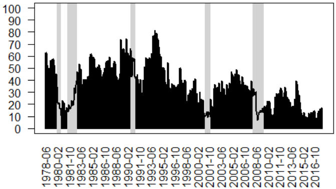
- FF5 as Strong Factors for each period



Business Cycle and Weakness of Factors

- Business Cycle and % of Weak factors in factor zoo

$$\% \text{ of weak factors} = a - \underbrace{10.6}_{t=19.69} * \text{NBER recession dummy} + e$$



Post-Publication Effect

- What happens to the weakness of a given factor post publication
  - We regress [the dummy on $|t| > 1.96$ from our test] on [the post-publication dummy]

    $$\text{Strong Dummy using our test} = a + \underbrace{0.19}_{t=45.92} * \text{Post Publication dummy} + e$$

  - Nice contrast with the results that the average returns tend to be lower post publication (McLean and Pontiff, 2016)
    - Public information => Pervasive & Fair price

Locally Strong - Weak Factors

- Literature suggests novel methods for how to handle (unconditional) weak factors, where weak factors are defined over a large T.
- Our local method (small T) can provide novel insights on
  - existence of local weakness
  - economic significance of weakness

Economic Significance of Weak Factors

- Data: Chen-Zimmerman (2022)'s 768 portfolios from 1963:07 to 2023:12
  - Large T PCA risk premia

PC	mean	std	annual SR
1	0.20	1.55	0.46
2	0.03	0.28	0.36
3	0.03	0.23	0.41
4	0.05	0.13	1.25
5	0.03	0.10	1.10
6	0.01	0.10	0.26
7	0.01	0.09	0.21

  - We consider the fourth/fifth long-term PCs as weak factors
- Our design
  - Set first three (unconditional) PCs as $\mathbf{F}$
  - Test whether fourth/fifth (unconditional) PCs $\mathbf{G}$ are locally weak

Economic Significance of Weak Factors

- Sorting on the t-stat of the slope (risk premium), one gets

PCA	Quintile	δ_g	g_{t+1}
4	low $\lvert t\text{-stat}\rvert$	0.035	0.037
		0.042	0.053
		0.048	0.054
		0.053	0.036
	high $\lvert t\text{-stat}\rvert$	0.065	0.060
5	low $\lvert t\text{-stat}\rvert$	0.026	0.011
		0.024	0.032
		0.025	0.023
		0.039	0.048
	high $\lvert t\text{-stat}\rvert$	0.050	0.047

- Unconditionally-weak factors are priced higher when they are locally-strong, i.e., when rejecting the null of locally weak.
- Unconditionally-weak factors offer higher return next period when they are locally-strong, i.e., when rejecting the null of locally weak.

Outline

Conclusion

- Novel methodology to test (local) weakness of factors
- Suitable for conditional asset pricing and robust to misspecified dynamics and risk factors.
  - Asymptotic theory (large N fixed T)
  - Simulation evidence I and II (complementarity to SPCA)
- Empirical findings
  - unconditionally weak factors tend to be stronger during recession and post publication
  - unconditionally weak factors tend to have economic significance (risk premium) when locally strong