Higher-Order Moment Inequality Restrictions for SVARs[1]

P. Andrade, F. Ferroni and L. Melosi

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[1] *The views expressed in this paper are those of the authors and are not necessarily reflective of views at the Federal Reserve Bank of Boston, the Federal Reserve Bank of Chicago, or the Federal Reserve System.*

Outline

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  - ⋆ Large changes might not be so infrequent: $\epsilon > 3\sigma$ might occur more often that just once every 700 obs (Gaussian)
- A wealth of information for identifying macro-financial shocks may rest on their higher-order moments.

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- Introduce inequality restrictions on higher-order moments (HOM) of the structural shock and combine them with more standard restrictions.

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     - ⋆ HOM inequality restrictions only set identify a shock of interest: ideally combined with other restrictions derived from economics.
- Strike a balance between using HOM restrictions to improve identification and being robust to misspecification and sample bias pervasive when estimating HOM.

What we do: Applications

- Controlled experiment:
  - excess kurtosis inequality restrictions improve the identification of monetary policy shocks in NK models (e.g. SW model and estimated shocks; Uhlig restrictions).
- US monetary policy shocks:
  - document that MP shock proxies are leptokurtic; use this restriction to study monetary policy transmission in the U.S.,
  - compared to sign restrictions only: when adding HOM restrictions clear negative impact on output; smaller impact on prices.
- Sovereign spread shocks in the Euro Area
  - restriction on both skewness and excess kurtosis,
  - compared to other identifications: stronger recessionary effects and sizable pass-through from sovereign to corporate spread.
- Geopolitical Risk (Iacoviello-Caldara).
  - restriction on both skewness and excess kurtosis,
  - larger macroeconomic impact of geopolitical risk shocks.

Literature Review

- Point identification through non-Gaussianity
  - Specific non-gaussian distribution for VAR errors or matching empirical innovations moments: Lanne, Meitz and Saikkonen (2017), Gourieroux, Monfort and Renne (2017, 2019) , Lanne, Liu and Luoto (2022).
  - Specific non-Gaussian distribution for structural shocks: Brunnermeier et al. (2021), Jarocinsky (2022).
  - Our setup allows for more flexible distributional assumption for structural shocks.

- Identified-set refinements through non-Gaussianity:
  - Drautzburg and Wright (2023) discard rotations violating statistical independence of shocks.
  - Hoesch, Lee and Mesters (2024) develop a method to refine the identified set when the independent shocks features weak deviations from non-Gaussianity.
  - We leverage on the non-Gaussianity of structural shocks but do not impose independence.

- Sign restrictions and weak identification:
  - Kilian and Murphy (2012), Arias, Rubio-Ramirez and Waggoner (2018), Wolf (2020, 2022) show that imposing sign restrictions alone is often too weak to provide adequate identification of structural shocks.
  - Additional constraints: Antolin-Diaz and Rubio-Ramirez (2018); Arias, Caldara and Rubio-Ramirez (2018).
  - We use higher order moment restrictions.

Outline

Static example

1/ ι empirical innovations (observed); ν structural shocks (unobserved). Linear mapping:

$$\iota = \begin{pmatrix} \iota_1 \\ \iota_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_o & -\sin\theta_o \\ \sin\theta_o & \cos\theta_o \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = A_o \nu$$

where θ_o is the 'true' unknown angle of rotation with $\theta_o \in (-\pi/2, \pi/2)$ and $\theta_o \neq 0$ (otherwise trivial).

$E(\nu_1^2) = (\nu_2^2) = 1$ and $E(\nu_1 \nu_2) = 0$.

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2/ 3^{rd} moments of structural shocks: $E(\nu_1^3) = 0$ and $E(\nu_2^3) = 1$, and and cross second and third moments are zero, i.e. $E(\nu_1 \nu_2^2) = E(\nu_1^2 \nu_2) = 0$,

$$E(\nu\nu' \otimes \nu') = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\Rightarrow$ 3^{rd} moments of empirical innovations: $E(\iota_1^3) = -\sin^3\theta_o$, $E(\iota_2^3) = \cos^3\theta_o$,

$E(\iota_1^2 \iota_2) = \sin^2\theta_o \cos\theta_o$, $E(\iota_1 \iota_2^2) = -\sin\theta_o \cos^2\theta_o$.

Statistical identification

- First, the mapping between third moments of structural shocks and empirical innovations is given by

$$E(\iota\iota' \otimes \iota')E(\iota\iota' \otimes \iota')' = A_o E(\nu\nu' \otimes \nu')E(\nu\nu' \otimes \nu')' A_o' = A_o \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A_o'.$$

- We have that

$$E(\iota_t\iota_t' \otimes \iota_t')E(\iota_t\iota_t' \otimes \iota_t')' = \begin{pmatrix} \sin^2\theta_o & -\sin^2\theta_o\cos\theta_o \\ -\sin\theta_o\cos\theta_o & \cos^2\theta_o \end{pmatrix}. \quad (1)$$

- The characteristic polynomial of (1) is $(\sin^2\theta_o - \lambda)(\cos^2\theta_o - \lambda) - \sin^2\theta_o\cos^2\theta_o$ and the associated eigenvalues are zero and one respectively.

- The first structural shock third moment equals zero and the second structural shock third moment equals one. The eigenvector associated with the non-zero eigenvalue is $\begin{pmatrix} -\sin\theta_o & \cos\theta_o \end{pmatrix}'$.

Inequality restriction on HOM

- Let $\breve{\nu} = A'\iota$ and let A be a generic rotation with angle θ, i.e. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

- With II order moments, $\mathcal{I}_{ii} \equiv \{\theta | \ -\pi/2 < \theta < \pi/2\}$.

- Introduce the HOM inequality restriction of asymmetry, we have

$$\begin{aligned} & E(\breve{\nu}_2^3) > 0, \\ & E(-\sin\theta\iota_1 + \cos\theta\iota_2)^3 > 0, \\ & (\sin\theta\sin\theta_o + \cos\theta\cos\theta_o)^3 > 0, \\ & (\text{sign}(\cos\theta_o)\cos(\theta-\theta_o))^3 > 0. \end{aligned}$$

- Solution: $\mathcal{I}_{hm} \equiv \{\theta | \ \max\{-\pi/2, \theta_o - \pi/2\} < \theta < \min\{\pi/2, \theta_o + \pi/2\}\} \subset \mathcal{I}_{ii}$ $\theta_o \neq 0$.

Inequality restriction on HOM

- With HOM inequality restrictions
  - No need to rely on sample estimates of $E(\iota\iota' \otimes \iota')$;
  - Robust non-parametric methods to compute $E(\breve{\nu}_2^3)$ or $E(\breve{\nu}_2^4)$;
  - No assumptions about statistical independence or cross-HOM zero restrictions.
- Statistical identification (eigenvalue/eigenvector decomposition) is appropriate when

  (a) good sample estimates of HOM, i.e. $E(\iota\iota' \otimes \iota')$;

  (b) shocks are independent or (a weaker condition) the cross-third and all cross-fourth moments are zero, i.e. $E(\nu_1\nu_2^2) = (\nu_1^2\nu_2) = 0$.

  . cross-HOM relations

Introducing dynamics

- Let a $VAR(p)$ be:

$$y_t = \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + \Phi_0 + u_t.$$

- Reduced form errors, empirical innovations and structural shocks, (no distributional assumption on ν).

$$u_t = \Sigma^{1/2} \iota_t = \Sigma^{1/2} \, A_o \, \nu_t$$

- The Bayesian inference builds on the work by Petrova (JoE, 2022); it exploits asymptotic normality of the Quasi Maximum Likelihood (QML) estimator of reduced form parameters.

- Asympt. inference about Φ does not depend on the error term distribution: $\rightarrow$ valid inference on u_t.
Asympt. inference about Σ depends on fourth moments of the errors: $\rightarrow$ invalid inference on ι_t.
(assume no skewness for exposition simplicity)

- Asymptotic valid inference for Σ can be performed by drawing from the asymptotic normal distribution centered in the consistent estimator of Σ, i.e. the QML estimator $\hat{\Sigma}$, and with covariance matrix equal difference between the fourth mom and the 'squared' second moments.

Identification

- Let $\Sigma^{(j)}$ and $\Phi^{(j)}$ be the j^{th} draw. Draw $\breve{\Omega}$ from a uniform distribution with the Rubio-Ramirez el al. (RESTUD, 2010) algorithm
  I. compute the impulse response function and check if the sign (or any other economic) restrictions are verified,
  II. compute the implied structural shocks

$$\breve{\nu}_t^{(j)} = \breve{\Omega}' \left(\Sigma^{(j)}\right)^{-1/2} (y_t - \Phi_1^{(j)} y_{t-1} - \ldots - \Phi_p^{(j)} y_{t-p} - \Phi_0^{(j)}),$$

  II. compute $\mathcal{S}(\breve{\nu}_{n,t}^{(j)})$ and/or $\mathcal{K}(\breve{\nu}_{n,t}^{(j)})$ and check if the higher-order moment inequality restrictions are satisfied.

  If both [I] and [III] are satisfied, keep the draw $\Omega^{(j)} = \breve{\Omega}$. Else repeat [I], [II] and [III]. ▸ Gibbs Sampler

- Fourth and third sample moments:

$$\mathcal{S}(x) = \frac{\bar{x} - F^{-1}(0.5)}{\text{std}(x)}, \quad \mathcal{K}(x) = \frac{F^{-1}(0.975) - F^{-1}(0.025)}{F^{-1}(0.75) - F^{-1}(0.25)} - 2.9,$$

  where $F^{-1}(\alpha)$ is the α-percentile of the empirical distribution of x.

- These restrictions are modular, i.e. can be combined with sign (and zero), magnitude, narrative ... any restrictions that generate set-identification.

Outline

What are the effects of monetary policy on output?

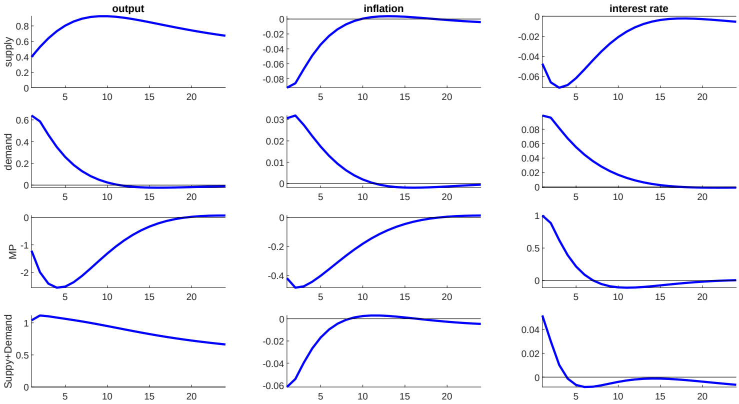
- Uhlig (JME, 2005) is after ‘real effect of MP’. Imposes sign restrictions on inflation and interest rate (moving in opposite directions) and is *agnostic* about output.
- Results from this agnostic identification procedure by Uhlig (JME, 2005) point at no clear effect on output. Many positive trajectories after a MP tightening.
- Wolf (AEJMacro, 2020) shows that in the NK models this occurs because supply and demand shocks tend to *masquerade* or disguise as monetary policy shocks when only sign restrictions on inflation and interest rate are imposed.
- Identification can be improved with instruments or restrictions on the monetary policy rule coefficients (e.g. Arias, Caldara and Rubio-Ramirez (JME, 2019)). Multiple shocks identification (Fry and Pagan (JEL, 2011)).
- We suggest to use higher-order moment to improve identification.

Smets and Wouters (2007) model

- The Smets and Wouters (2007) (SW) model is perhaps the most well-known example of an empirically successful New-Keynesian business cycle model.
- We use this model as a realistic laboratory to show how the higher order moments can sharpen identification.
- Consider the SW posterior mode parameterization and the smoothed estimates of the shocks using postwar US data on output, consumption, investment, real wages, inflation, interest rate and hours worked as in their original work.
- Shocks: technology (supply), risk premium (demand) and MP
- Observables: output, inflation and interest rate

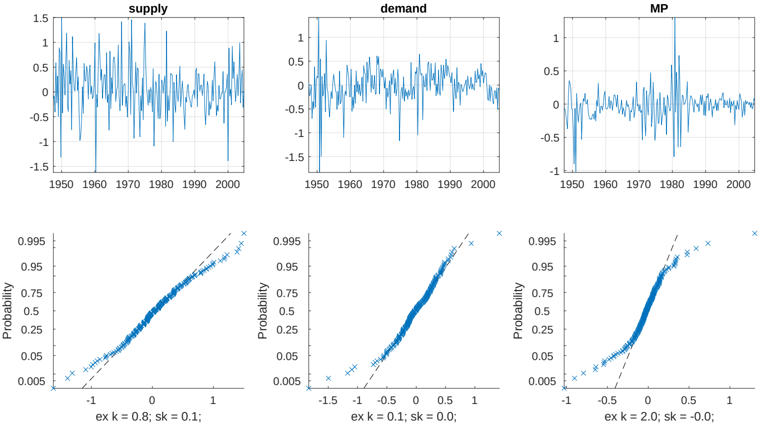
Estimated IRF

Figure: SW estimates of impulse response functions. From top to bottom technology, risk premium and monetary policy shocks and the sum of demand and supply shocks.



Estimated shocks

Figure: SW estimated shocks: from left to right technology, risk premium and monetary policy shocks. Top panels realizations, bottom panels probability distribution against the normal. other shocks

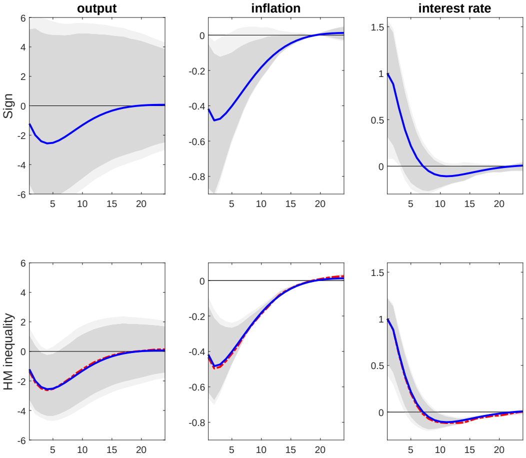


Simulate data

- Simulate data bootstrapping the estimated shocks.
- Estimate the VAR and IRF using different identification schemes
- In particular, we assume that after a monetary policy shock,
  - Inflation decreases on impact and for two consecutive quarters
  - Interest rate increases on impact and for two consecutive quarters
  - Monetary policy shocks are leptokrutic, i.e. monetary policy robust measure of excess kurtosis larger than 1.6

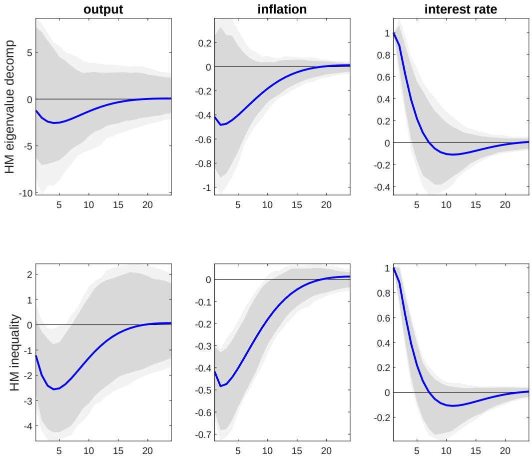
Large sample

Figure: IRF using sign (first row) and sign and higher moment inequality (second row) restrictions. The blue solid line is the true impulse response. The dark (light) gray areas report the 90% (99%) identified set using inequality restrictions on the fourth moment of the monetary policy shock.



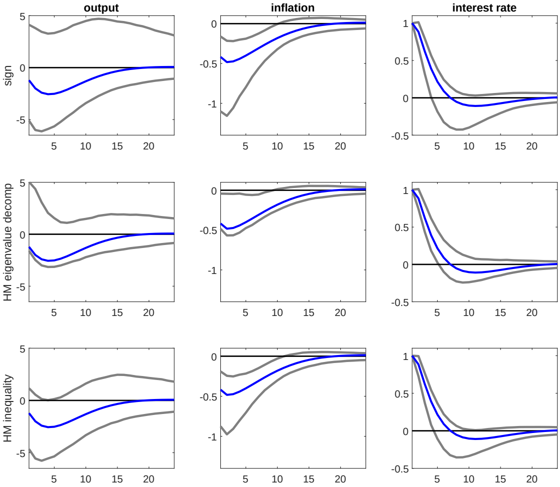
Short samples - Average Median

Figure: IRF using fourth moments eigenvalue decomposition and inequality restrictions (average median IRF across samples). The dark (light) gray areas report the 90% (99%) dispersion of the point estimates over repeated samples of 200 observation length. The blue line is the true impulse response.

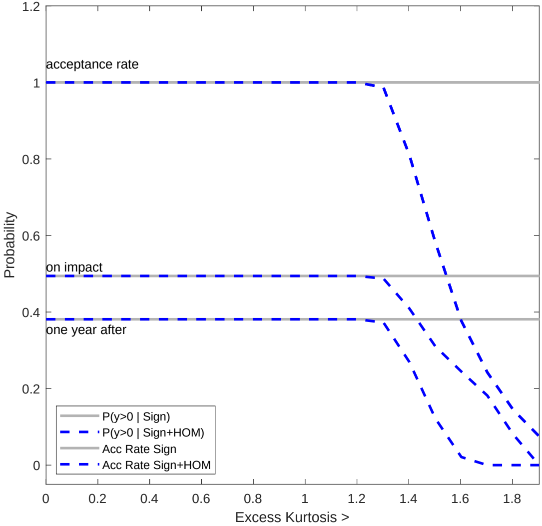


Short samples - Average Credible sets

Figure: The average upper and lower bounds of the 68% credible sets across Montecarlo simulations using signs, fourth moments eigenvalue decomposition and inequality restrictions.



$Prob(y > 0)$ as a function of the interval



Outline

MP in the U.S.

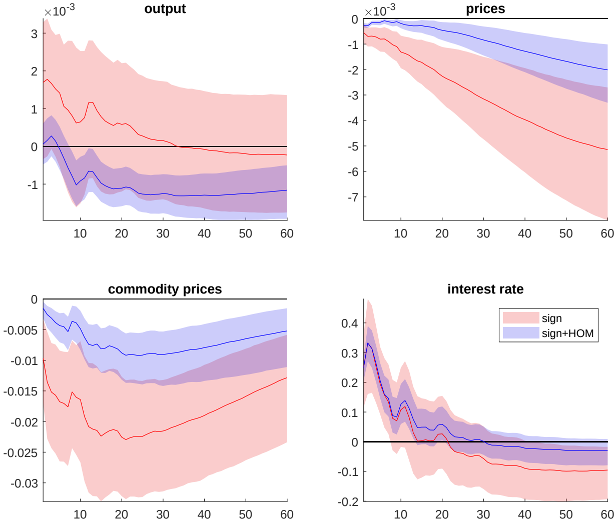
- Interest on the real effects of MP $\rightarrow$ no restrictions on output and single shock identification.
- Data on real activity, prices and interest rates from 1965m1 to 2003m1; 12 lags; reject normality of VAR residuals (Kolmogorov-Smirnov (K-S) test).
- Uhlig (2005) sign restrictions: prices and interest rate moving in opposite direction for six months.
- Inequality restriction on the fourth moment: monetary policy shocks are drawn from a fat-tailed distribution.
- Is this a reasonable assumption? Look at estimates and proxies of monetary policy shocks.

Robust HM of MP shock proxies

- US MP surprises poorly correlated. Possibly span different info sets. All leptokurtic. corr
- Higher-order moment inequality interval → min and max, i.e. $\mathcal{I}_k \equiv [1.2, 12]$. $p(realization > 3\sigma) = 1\%$ (vs 0.15% for N)

	Ex-Kurtosis	Skewness	Sample Size
SW	2.0 [0.4, 3.2]	-0.0 [-0.1, 0.1]	179
SZ	3.8 [1.8, 6.3]	0.0 [-0.0, 0.1]	518
RR	3.2 [1.8, 5.1]	0.0 [-0.1, 0.1]	468
GK	11.3 [5.9, 18.2]	-0.3 [-0.3, -0.2]	269
MAR	3.3 [1.2, 5.9]	-0.1 [-0.2, 0.0]	228
JK	8.8 [5.4, 15.8]	-0.1 [-0.2, -0.0]	323
USf1	1.2 [0.3, 3.5]	0.1 [-0.0, 0.2]	204
USf2	3.0 [1.4, 7.1]	0.1 [-0.0, 0.2]	204
USf3	1.9 [0.5, 4.4]	0.0 [-0.1, 0.1]	204
AD	3.1[1.4, 7.0]	-0.0[-0.1, 0.1]	313
AF(target)	2.5 [0.6, 5.3]	-0.0 [-0.2, 0.1]	134
AF(delphic)	1.3 [0.2, 3.9]	-0.0 [-0.2, 0.1]	134
AF(FWG)	1.4 [0.2, 3.6]	0.0 [-0.1, 0.1]	134
EAf1	3.4 [1.4, 5.6]	-0.0 [-0.2, 0.1]	197
EAf2	1.5 [0.3, 3.9]	-0.1 [-0.2, 0.0]	197
EAf3	1.1 [0.2, 3.4]	0.0 [-0.1, 0.2]	197
CH	13 [5.9, 38]	0 [-0.1, 0.1]	348
GR(minutes)	2.5 [1.3, 4.9]	-0 [-0.2, 0.1]	211
GR(IR)	3.6 [2.8, 4.6]	-0.1 [-0.2, 0.0]	211
CBTV	3.4 [0.6, 7.5]	-0.1 [-0.2, 0.0]	212

Figure: Impulse responses to a monetary policy shock. Sign restrictions red. Sign and kurtosis ($\mathcal{K}_{mp} > 1.2$) restrictions blue. 68% credible sets. ▸ 90%



Spread Shocks in the E.A.

- In the past decade the Euro Area has been characterized by large movements in sovereign spreads.
- Some movements reflect changes in the economic fundamentals, some others results of political risks generating tensions in sovereign yield markets. Teasing them apart is not easy.
- Some scholars have looked at financial market reactions around key political events, see e.g. Bahaj (2020) or Balduzzi, Brancati, Brianti and Schiantarelli (2023).
- Some other scholars have modelled an exogenous time-varying prob of default on sovereign debt in DSGE models, see e.g. Bocola (2016) or Corsetti, Kuester, Meier and Muller (2013).
- Higher-order mom restrictions can be thought in this context as characterizing sovereign risk or spread shocks and used for identification.

Spread Shocks in the E.A.

- Industrial production (IP), core HICP (Core), unemployment rate, a measure of borrowing costs (EBP), the one year Euribor, the spread between the 5 year Italian and German bond yield, and the 10 year Italian and German gov't bond yields from 1999m1 to 2019m12.
- Six lags VAR residuals. The K-S test rejects Gaussianity.
- A spread shock:
  - $\uparrow$ the 5y spread; $\uparrow$ the 10y Italian gov't bond yield; $\uparrow$ EBP on impact and for the following month;
  - $\mathcal{K} > 0.5$ (moderate fat-tails) and $\mathcal{S} > 0.2$ (moderate asymmetry).
- Compare with Recursive and Signs.

Figure: Impulse responses to a spread shock. Recursive red. Sign and HOM restrictions blue. IRFs are normalized so that the maximum median impact on the spread is 1 percent. 68% credible sets. 90%

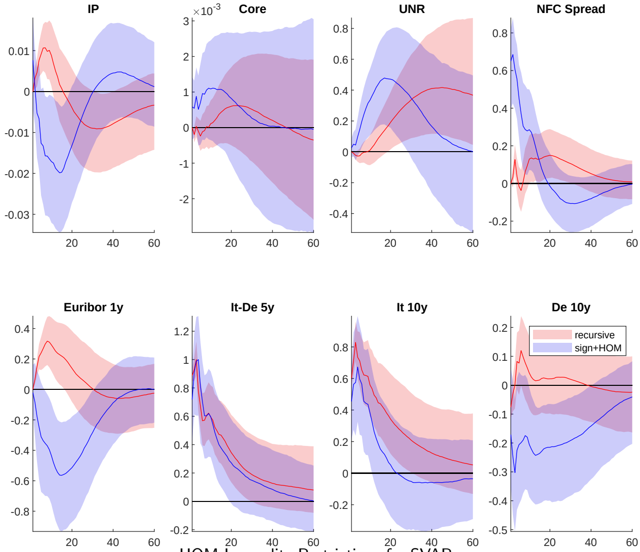
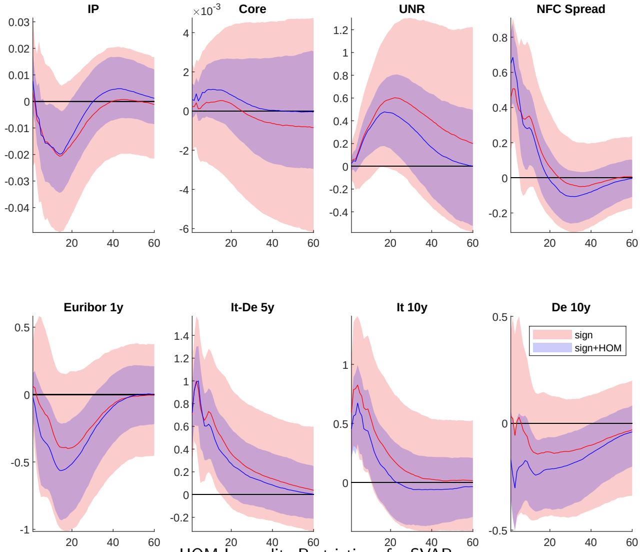
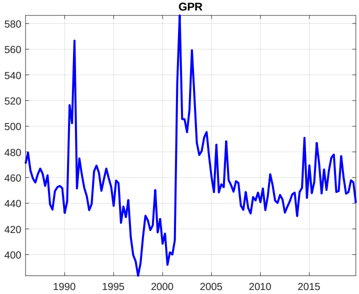


Figure: Impulse responses to a spread shock. Recursive sign restrictions. Sign and HOM restrictions blue. IRFs are normalized so that the maximum median impact on the spread is 1 percent. 68% credible sets. ▸ 90%



Geopolitical Risk

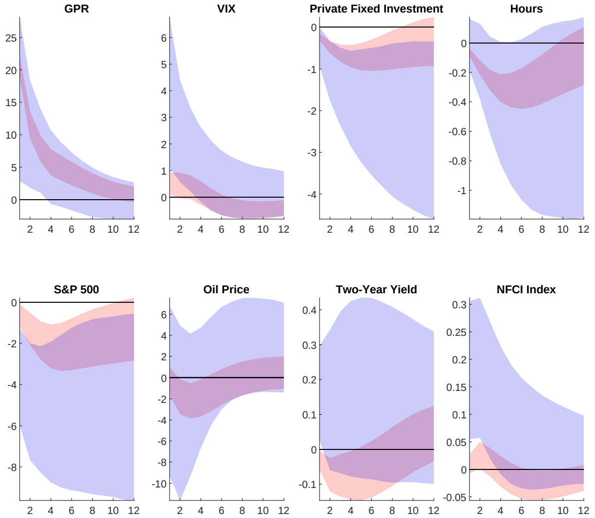
- Caldara and Iacoviello (AER2022) Geopolitical Risk



- Recursive (exogenous ordered first) vs HOM restriction (fat tail and asymmetric) + sign (GPR $\uparrow$, S&P500 $\downarrow$ and Two-Year Yield $\uparrow$).

Figure: Impulse responses to a GPR shock. Recursive red. Sign and HOM restrictions blue. 2 standard deviation increase.

▸ 90%



Outline

Conclusions

- We propose a novel set of conditions based on higher moments on structural shocks to identify them.
- We show how the identified set shrinks when these restrictions are introduced, both analytically and numerically.
- Show how the excess kurtosis restriction can help isolating the impact of monetary policy shock on output from the supply and demand *masquerading* shock in an New Keynesian (NK) models.
- Using a Bayesian robust approach we apply our identification scheme to study the transmission of conventional monetary policy shocks in the U.S. before the financial crisis, of spread shocks in the Euro Area, and of Geopolitical risks to the macroeconomic aggregate.

Eigeinvalue decomposition - third mom

- Spectral decomposition of third mom

$$
\begin{aligned}
E(\iota_t\iota_t' \otimes \iota_t')E(\iota_t\iota_t' \otimes \iota_t')' &= A_o E(\nu_t\nu_t' \otimes \nu_t')(A_o \otimes A_o)'(A_o \otimes A_o)E(\nu_t\nu_t' \otimes \nu_t')' A_o' \\
&= A_o \left(\sum_{i=1}^{n} \zeta_i J_i \otimes \boldsymbol{e}_i\right)\left(\sum_{i=1}^{n} \zeta_i J_i \otimes \boldsymbol{e}_i\right)' A_o' \\
&= A_o \Lambda_\zeta A_o'
\end{aligned}
$$

where $\boldsymbol{e}_i$ is the $n \times 1$ vector with zeros everywhere except a one in the i^{th} position, J_i the $n \times n$ matrix of zeros everywhere except one in the i^{th} position of the main diagonal. Λ_ζ is a diagonal matrix collecting the squared third moments of the structural shocks.

- the eigenvalue $\rightarrow$ square of the third moments of the structural shock
- the eigenvector $\rightarrow$ coincides with the column of impact matrix, up to a sign switch and permutation of columns.
- Example $n = 2$,

$$
A_o\left(\begin{pmatrix}\zeta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_2\end{pmatrix}\right)\left(\begin{pmatrix}\zeta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_2\end{pmatrix}\right)' A
$$

$$
= A_o \begin{pmatrix}\zeta_1^2 & 0 \\ 0 & \zeta_2^2\end{pmatrix} A_o'
$$

Eigeinvalue decomposition - fourth mom

- Spectral decomposition of fourth mom

$$E(\iota_t\iota_t' \otimes \iota_t' \otimes \iota_t) - \mathcal{K}^z = (A_o \otimes A_o)(E(\nu_t\nu_t' \otimes \nu_t' \otimes \nu_t) - \mathcal{K}^z)(A_o \otimes A_o)' \\ = P\Lambda_\xi P'$$

  where Λ_ξ is a diagonal matrix

- the first n eigenvalues $\rightarrow$ fourth moments of the structural shock
- the first n elements of the first n eigenvectors divided by the absolute value of the first elements of the eigenvector, i.e. $P(1:n,j)/\sqrt{|P(1,j)|}$ for $j = 1, ..., n.$$\rightarrow$ impact matrix
- Example $n = 2$,

$$(A_o \otimes A_o)\left(\begin{pmatrix}\xi_1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & \xi_2\end{pmatrix} - \begin{pmatrix}3 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 3\end{pmatrix}\right)(A_o \otimes A_o)$$

$$= (A_o \otimes A_o)\begin{pmatrix}\xi_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \xi_2\end{pmatrix}(A_o \otimes A_o)$$

HM Eigeinvalue decomposition vs HM inequality restrictions

- Need to compute the full set of the third or fourth moments of the empirical innovations to retrieve the column of interest of the rotation matrix.

- Estimates of the fourth or third sample moments can be very sensitive to outliers or minor perturbation of the data and their estimates might be imprecise in short samples

- HOM inequality restrictions impose conditions on the higher moments of the structural shock itself using non-parametric robust methods based on the distance between different percentiles of the shock's empirical distribution.

- HOM inequality restrictions impose weaker conditions generating set-identification as opposed to point-identification and it can be coupled with other assumptions, such as signs, zeros, narrative, magnitude and/or statistical independence restrictions.

▸ return

Gibbs Sampler ▸ return

Assuming a flat prior. Let $\widehat{S} = (Y - X\widehat{\Phi})'(Y - X\widehat{\Phi})$ and $\widehat{\Phi} = (X'X)^{-1}X'Y$, the steps of the Gibbs sampler are for $j = 1, ..., J$

- Draw $\Sigma^{(j)}$ from
$$N\left(\text{vech}(\widehat{S}), \widehat{\mathcal{C}}\right)$$
where $\widehat{\mathcal{C}} = \frac{1}{T} D_n^+ \left(\widehat{S}^{1/2} \otimes \widehat{S}^{1/2}\right) D_n \left(\widehat{\mathcal{K}}^\star - \text{vech}(I_n)\text{vech}(I_n)'\right) D_n' \left(\widehat{S}^{1/2} \otimes \widehat{S}^{1/2}\right)' D_n^{+\prime}$ captures the fourth moments.
- Conditional on $\Sigma^{(j)}$, draw $\Phi^{(j)}$ from
$$N\left(\widehat{\Phi}, \Sigma^{(j)} \otimes (X'X)^{-1}\right)$$
  i. In case of an asymmetric distribution, the intercept, Φ_0, is drawn from
$$N(\widehat{\Phi}_0 + \widehat{\mathcal{S}}^\star \widehat{\mathcal{C}}^{-1}\text{vech}(\Sigma^{(j)} - \widehat{S}), \Sigma^{(j)} - 1/T\widehat{\mathcal{S}}_T^\star \widehat{\mathcal{C}}^{-1} \widehat{\mathcal{S}}_T^{\star\prime})$$
- Draw $\breve{\Omega}$ from a uniform distribution
  I. compute the impulse response function and check if the sign restrictions are verified
  II. compute the implied structural shocks
$$\breve{\nu}_t^{(j)} = \breve{\Omega}' \left(\Sigma^{(j)}\right)^{-1/2} (y_t - \Phi_1^{(j)} y_{t-1} - \ldots - \Phi_p^{(j)} y_{t-p} - \Phi_0^{(j)})$$
  and check if the higher moment inequality restrictions are satisfied

If both [I] and [II] are satisfied, keep the draw $\Omega^{(j)} = \breve{\Omega}$. Else repeat [I] and [II].

Fourth and third sample moments ▸ return

- the shrinkage estimator for the kurtosis is defined as

$$\widehat{\mathcal{K}}^{\star} = \frac{T}{T+\tau}\widehat{\mathcal{K}}_T + \frac{\tau}{T+\tau} D_n^{+}(I_n + K_{n,n} + \text{vec}(I_n)\text{vec}(I_n)')D_n^{+\prime} \tag{2}$$

where $\widehat{\mathcal{K}}_T$ represents the sample fourth moments of the empirical innovations, i.e. $\widehat{\mathcal{K}}_T = 1/T \sum \text{vech}(\iota_t \iota_t') \otimes \text{vech}(\iota_t \iota_t')$ with $\iota_t = \hat{\Sigma}^{-1/2} u_t$;

$K_{n,n}$ is a commutation matrix, which is a $(n^2 \times n^2)$ matrix consisting of $n \times n$ blocks where the $(j,i)-$element of the (i,j) block equals one, elsewhere there are all zeros;

D_n^{+} is the generalized inverse of the duplication matrix D_n.

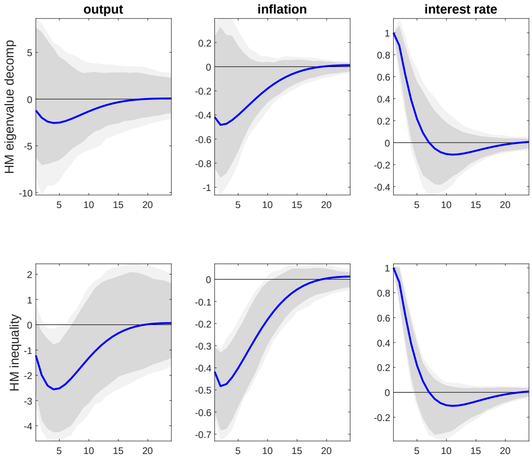
- the shrinkage estimator skewness given by

$$\widehat{\mathcal{S}}_T^{\star} = \frac{T}{T+\tau}\widehat{\mathcal{S}}_T$$

where $\widehat{\mathcal{S}}_T = (1/T \sum \text{vech}(u_t u_t') \otimes u_t)$

Short samples - Average Median ▸ return

Figure: IRF using fourth moments eigenvalue decomposition and inequality restrictions (average median IRF across samples). The dark (light) gray areas report the 90% (99%) dispersion of the point estimates over repeated samples of 200 observation length. The blue line is the true impulse response.



Short samples - Average Credible sets

return

Figure: The average upper and lower bounds of the 68% credible sets across Montecarlo simulations using signs, fourth moments eigenvalue decomposition and inequality restrictions.

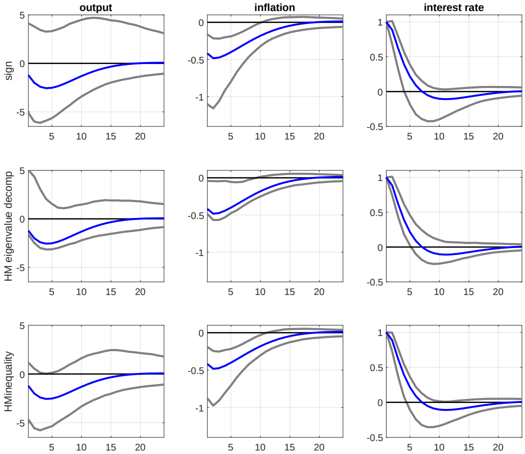
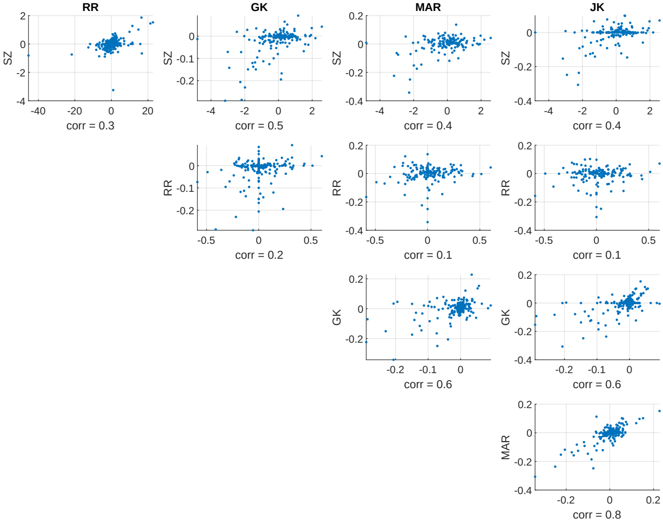


Figure: Correlations across measures of U.S. monetary policy shocks.



return

HOM dependence

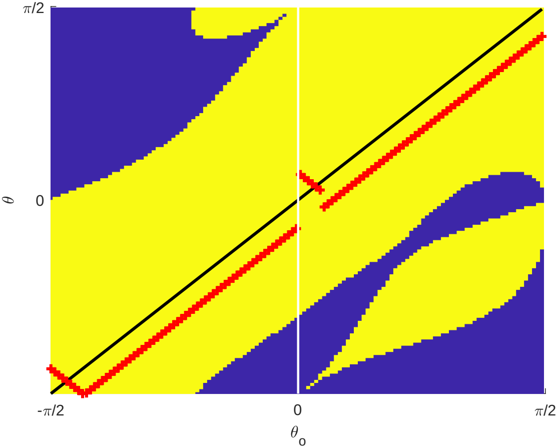


Figure: HOM inequality restrictions (yellow and blue areas) and point indentification (red) with $E(\nu_1\nu_2^2) = 0.2$.

return

New Keynesian model ▸ return

- NK model

$$
\begin{aligned}
y_t =& y_{t+1|t} - (i_t - \pi_{t+1|t}) + \sigma_d \epsilon_t^d \\
\pi_t =& \beta \pi_{t+1|t} + \kappa y_t - \sigma_s \epsilon_t^s \\
i_t =& \phi_\pi \pi_t + \phi_y y_t + \sigma_m \epsilon_t^m
\end{aligned}
$$

New Keynesian model ▸ return

- NK model

$$
\begin{aligned}
y_t =& y_{t+1|t} - (i_t - \pi_{t+1|t}) + \sigma_d \epsilon_t^d \\
\pi_t =& \beta \pi_{t+1|t} + \kappa y_t - \sigma_s \epsilon_t^s \\
i_t =& \phi_\pi \pi_t + \phi_y y_t + \sigma_m \epsilon_t^m
\end{aligned}
$$

- Solution of the model (in a linear model agents do not care about the shock's distribution)

$$
x_t = \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \frac{1}{1+\kappa\phi_\pi+\phi_y} \begin{pmatrix} \sigma_d & \phi_\pi \sigma_s & -\sigma_m \\ \kappa \sigma_d & -(1+\phi_y)\sigma_s & -\kappa\sigma_m \\ (\phi_y+\kappa\phi_\pi)\sigma_d & -\phi_\pi\sigma_s & \sigma_m \end{pmatrix} \begin{pmatrix} \epsilon_t^d \\ \epsilon_t^s \\ \epsilon_t^m \end{pmatrix} = A_c
$$

New Keynesian model ▸ return

- NK model

$$
\begin{aligned}
y_t =& y_{t+1|t} - (i_t - \pi_{t+1|t}) + \sigma_d \epsilon_t^d \\
\pi_t =& \beta \pi_{t+1|t} + \kappa y_t - \sigma_s \epsilon_t^s \\
i_t =& \phi_\pi \pi_t + \phi_y y_t + \sigma_m \epsilon_t^m
\end{aligned}
$$

- Solution of the model (in a linear model agents do not care about the shock's distribution)

$$
x_t = \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \frac{1}{1 + \kappa\phi_\pi + \phi_y} \begin{pmatrix} \sigma_d & \phi_\pi \sigma_s & -\sigma_m \\ \kappa \sigma_d & -(1+\phi_y)\sigma_s & -\kappa\sigma_m \\ (\phi_y + \kappa\phi_\pi)\sigma_d & -\phi_\pi \sigma_s & \sigma_m \end{pmatrix} \begin{pmatrix} \epsilon_t^d \\ \epsilon_t^s \\ \epsilon_t^m \end{pmatrix} = A_c
$$

- Distribution assumption $\epsilon_t^d \sim N(0,1)$, $\epsilon_t^s \sim N(0,1)$, $\epsilon_t^m \sim Laplace(0,1)$. The Excess Kurtosis of the Laplace distribution is 3.

New Keynesian model ▸ return

- NK model

$$
\begin{aligned}
y_t =& y_{t+1|t} - (i_t - \pi_{t+1|t}) + \sigma_d \epsilon_t^d \\
\pi_t =& \beta \pi_{t+1|t} + \kappa y_t - \sigma_s \epsilon_t^s \\
i_t =& \phi_\pi \pi_t + \phi_y y_t + \sigma_m \epsilon_t^m
\end{aligned}
$$

- Solution of the model (in a linear model agents do not care about the shock's distribution)

$$
x_t = \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \frac{1}{1 + \kappa\phi_\pi + \phi_y} \begin{pmatrix} \sigma_d & \phi_\pi \sigma_s & -\sigma_m \\ \kappa \sigma_d & -(1+\phi_y)\sigma_s & -\kappa\sigma_m \\ (\phi_y + \kappa\phi_\pi)\sigma_d & -\phi_\pi\sigma_s & \sigma_m \end{pmatrix} \begin{pmatrix} \epsilon_t^d \\ \epsilon_t^s \\ \epsilon_t^m \end{pmatrix} = A_
$$

- Distribution assumption $\epsilon_t^d \sim N(0,1)$, $\epsilon_t^s \sim N(0,1)$, $\epsilon_t^m \sim Laplace(0,1)$. The Excess Kurtosis of the Laplace distribution is 3.

- Simulate T =100,000 data points with parameters values: $\sigma_s = \sigma_d = \sigma_m = 1$, $\phi_\pi = 1.5$, $\phi_y = 0.5$ and $\kappa = 0.2$.

Masquerading - MP tightening ▸ return

The $S + D$ shock generate $\pi < 0$ and $i > 0$ (same as MP) and $y > 0$

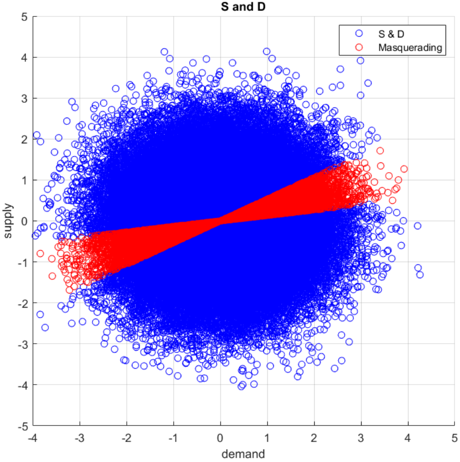
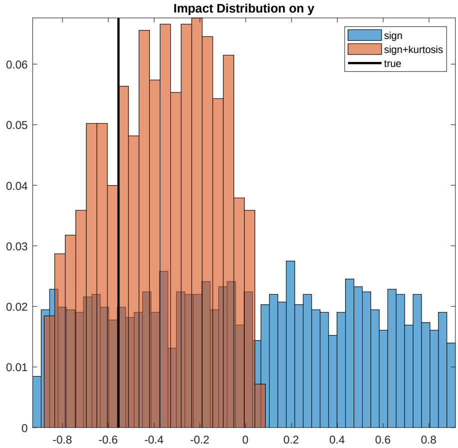


Figure: Realizations of demand and supply shocks: all (blue circles) and masqueraded MP (red circles).

Impact distribution on y return



$p(y < 0)$ with sign and sign+hom restrictions ▸ return

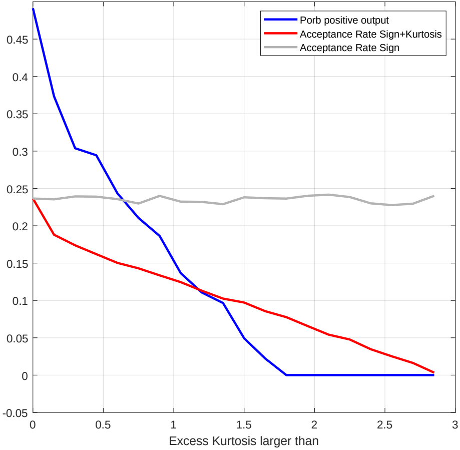


Figure: Probability of positive response of y at different restrictions on monetary policy excess kurtosis.

Estimated shocks - cont

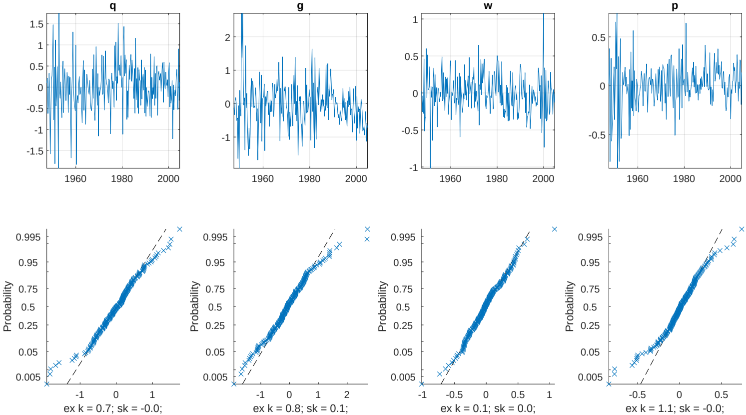


Figure: SW estimated shocks. return

Figure: Impulse responses to a monetary policy shock. Sign restrictions red. Sign and kurtosis ($\mathcal{K}_{mp} > 1.2$) restrictions blue. 68% and 90% credible sets. return

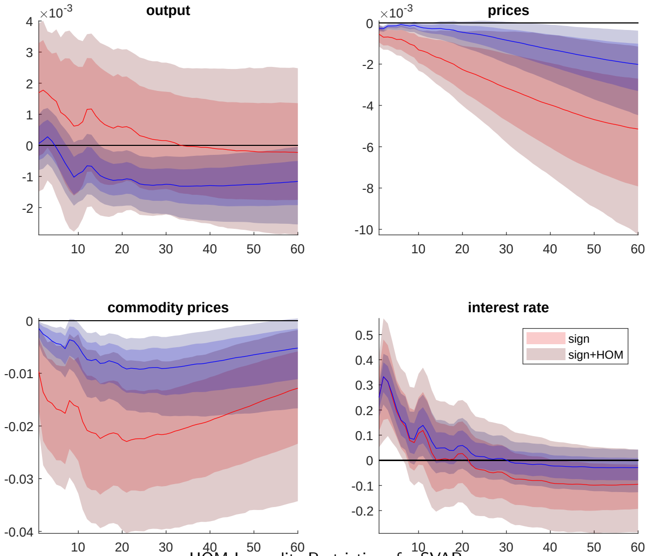


Figure: Impulse responses to a spread shock. Recursive red. Sign and HOM restrictions blue. 68% and 90% credible sets.

return

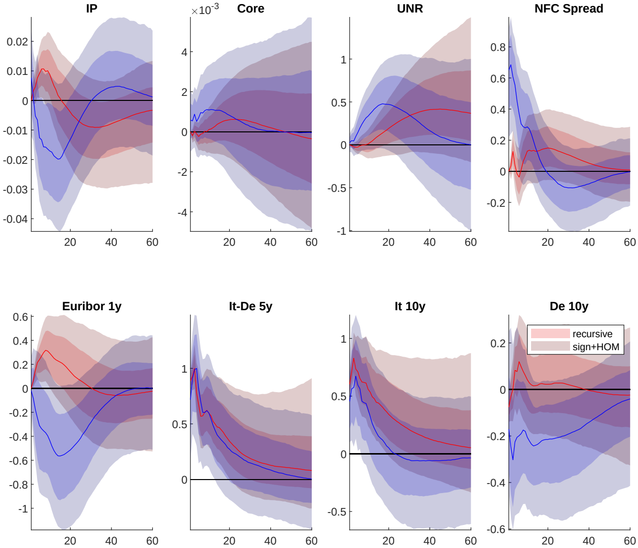


Figure: Impulse responses to a spread shock. Sign restrictions red. Sign and HOM restrictions blue. 68% and 90% credible sets. ▸ return

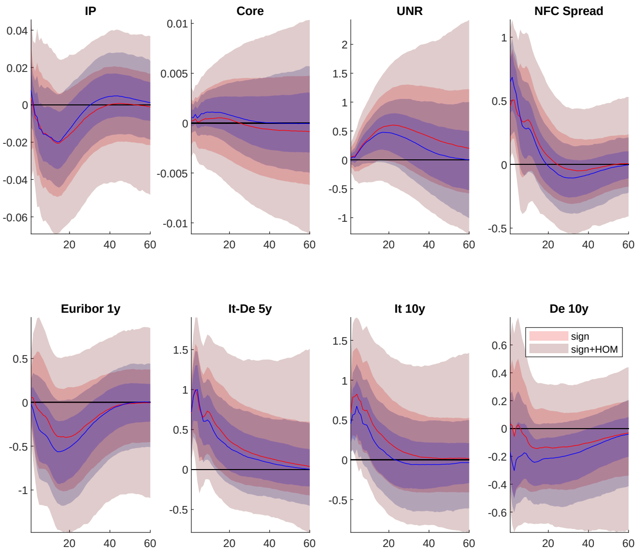


Figure: Impulse responses to a GPR shock. Recursive red. Sign and HOM restrictions blue. 2 standard deviation increase. 68% and 90% credible sets. return