Imputation of Counterfactual Outcomes in the Presence of Predictable Errors

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Outline

1. Introduction and Main Results
2. Linear Prediction: from BLUP to PLUP
3. Imputing Counterfactual Outcomes with PLUP and PUP
4. Unconditional and Conditional Inference

Setup

Pseudo conditional mean function m_{it}

$$\begin{aligned} \text{not treated} \quad Y_{it}(0) &= m_{it} + e_{it}, \quad E[e_{it}] = 0, \quad t = 1, \dots, T. \\ \text{treated} \quad Y_{it}(1) &= m_{it} + e_{it} + \underbrace{\delta_{it}}_{\substack{\text{individual} \\ \text{treatment effect}}} \quad t > T_0. \end{aligned}$$

a. Estimation of m_{it}: regressions, sc, factors/matrix completion
b. Imputation of $Y_{i,T_0+h}(0)$: e_{it} often assume unpredictable.
c. Testing of δ_{iT_0+h}: asymptotic, conformal, subsampling, etc.

In (c), we carefully account for sampling uncertainty from (a), but total imputation error is dominated by out-of-sample error in (b).

Example: $Y_{it}(0) = x_{it}'\beta + e_{it} \Rightarrow \hat{Y}_{iT_0+1}(0) = x_{iT_0+1}'\hat{\beta}_{OLS}$.

$$\text{var}(\hat{e}_{i,T_0+1}) = \sigma_e^2 + x_{T_0+1}'\text{var}(\hat{\beta}_{OLS})x_{T_0+1} \to \sigma_e^2.$$

e_{it} can be correlated across i and t

1. Model misspecification

   i. $Y_{it}(0)$ is AR(2) but we assume an AR(1) model.

   ii. $Y_{it}(0) = \lambda_i F_t + v_{it}$, but we assume $Y_{it}(0) = \lambda_i + F_t + v_{it}$.

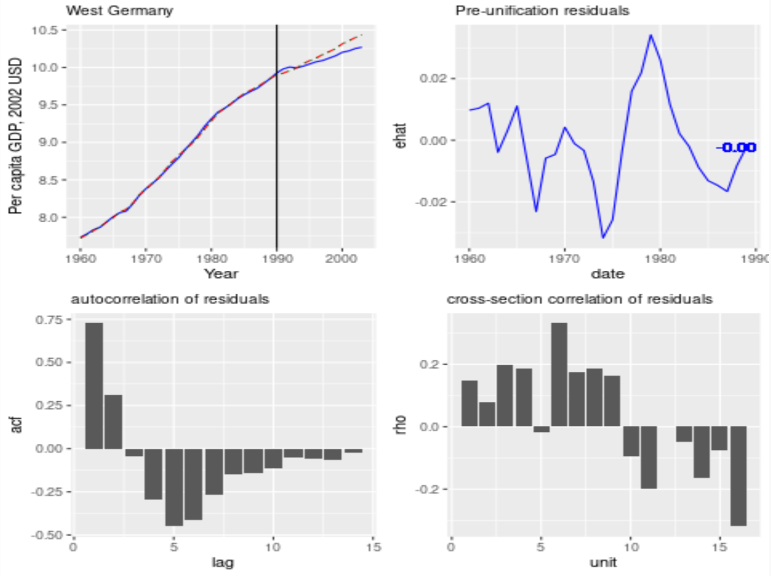
   iii. $Y_{it}(0) = \phi_1 Y_{i,t-1}(0) + v_{it}$ but we let $m_{1t} = \sum_{j=2}^{N+1} \omega_j Y_{jt}$, $e_{1t} = \phi_1 Y_{1,t-1}(0) + v_{1t} - \sum_{j=2}^{N+1} \omega_j Y_{jt}(0)$ is persistent.

2. Genuine correlation

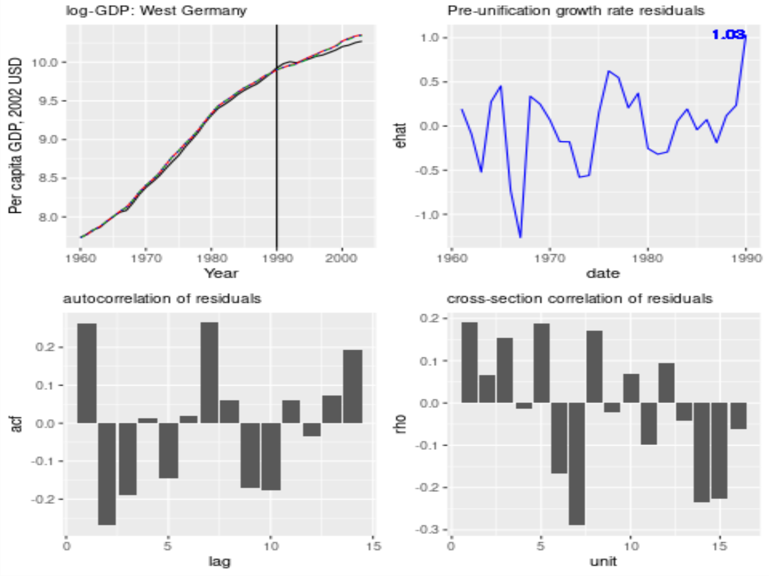
   - $Y_{it}(0) = \Lambda_i' F_t + e_{it}$ and e_{it} is serially or spatially correlated.

3. Temporal aggregation of data.

Germany log GDP imputed from level model



Germany log GDP imputed from growth rate model



This paper: exploit predictability in e_{it}

- Imputation of $Y_{i,T_0+h}(0)$ is a special (out-of-sample) prediction problem where the true values are not sampled.

- Goldberger proposed infeasible BLUP for linear prediction. We derive PLUP and PUP for potential outcomes.
  - i PUP does not require linearity and can be used with any consistent estimator of m_{it}.
  - ii Asymptotic mse expansions for stationary mixing processes show PLUP/PUP have smaller mse than standard predictions.

- Implications of ignoring predictability for inference.
  - i Standard prediction is inefficient but unconditionally unbiased.
  - ii Conditional bias may distort inference.

Linear Model: $y = X\beta + e$, $e \sim (0, \Omega)$ is autocorrelated.

- Goldberger's Best Linear Unbiased Predictor

  $y_{m|T_0} = A'y$ and $E[A(X - x_m)\beta] = 0$ gives

  $$y^*_{m|T_0} = x'_m\beta_{GLS} + \omega'\Omega^{-1}e_{GLS}, \quad \omega = E[e_m e_{GLS}].$$

- first order serial dependence $e_t = \phi_1 e_{t-1} + v_t$:

  $$y^*_{T_0+h|T_0} = x'_{T_0+h}\beta_{GLS} + \phi_1^h e_{GLS,T_0}.$$

  Cochrane-Orcutt (1949) used $\phi_1^h(y_{T_0} - x'_{T_0}\hat{\beta})$

- cross-section correlation: random effects $\Rightarrow$ Empirical Bayes.

Prediction errors: AR(1) case $e_t = \phi_1 e_{t-1} + v_t$

- Standard (OLS) prediction $\hat{y}_{T_0+1|T_0}$ has error

$$\begin{aligned} \hat{e}_{T_0+1|T_0} &= e_{T_0+1} - x'_{T_0+1}(\hat{\beta}_{OLS} - \beta) \\ &= e_{T_0+1} + o_p(1) \end{aligned}$$

- Feasible BLUP $\hat{y}^*_{T_0+1|T_0} = x'_{T_0+1}\hat{\beta}_{GLS} + \hat{\phi}_1 \hat{e}_{GLS,T_0}$ has error

$$\begin{aligned} \hat{e}^*_{T_0+1|T_0} &= v_{T_0+1} - (x_{T_0+1} - \phi_1 x_T)'(\hat{\beta}_{GLS} - \beta) - \left(\hat{\phi}_{1,GLS} - \phi_1\right) \hat{e}_{GLS,T_0} \\ &= v_{T_0+1} + o_p(1) \end{aligned}$$

Feasible BLUP is asymptotically more efficient because

$$\sigma_v^2 \leq \sigma_e^2.$$

- Observation: gains are due to $\phi_1 e^*_{T_0}$, not $\hat{\beta}_{OLS}$ vs $\hat{\beta}_{GLS}$.

From blup to plup (Practical Linear Unbiased Predictor)

- Source of efficiency gains motivates

$$\underbrace{\hat{y}^{+}_{T_0+1|T_0}}_{\text{PLUP}} = \underbrace{x'_{T_0+1}\hat{\beta}_{OLS}}_{\text{OLS prediction}} + \underbrace{\hat{\rho}_1 \hat{e}_{T_0}}_{\text{OLS-based correction}}$$

$\hat{e}_{T_0}$ and $\hat{\rho}_1$ are based on in-sample OLS residuals, $\hat{\rho}_1$ is first order sample autocorrelation coefficient.

- PLUP error $\hat{e}^{+}_{T_0+1|T_0} = \hat{e}_{T_0+1|T_0} - \hat{\rho}_1 \hat{e}_{T_0}$

$$\begin{aligned} \hat{e}^{+}_{T_0+1|T_0} &= e_{T_0+1} - \rho_1 e_{T_0} + o_p(1) \\ &= (\phi_1 - \rho_1) e_{T_0} + v_{T_0+1} + o_p(1). \end{aligned}$$

$e^{+}_{T_0+1|T_0}$ is asymptotically v_{T_0+1} when $\phi_1 = \rho_1$.

Lemma 1: (h=1)

Assumption A1: (a) $E\left|e_t\right|^r < \infty$ for some $r > 2$ and for all t;
(b) $\{e_t\}$ is a strictly stationary strong mixing process.

- Under Assumption A1 such that $\hat{\beta} \xrightarrow{p} \beta$ and $\hat{\rho}_1 \xrightarrow{p} \rho_1 \equiv \frac{\gamma_1}{\gamma_0}$,

$$
\begin{array}{rcl}
\text{(ols)} \quad & \hat{e}_{T_0+1|T_0} & \sim \quad (0, \gamma_0) \\
(\textsc{plup}) \quad & \hat{e}^{+}_{T_0+1|T_0} & \sim \quad (0, \gamma_0(1-\rho_1^2)).
\end{array}
$$

Result also holds if e_t is mixing (not necessarily AR(1)).

Lemma 2 ($h > 1$)

Three predictors:

- standard: $\hat{y}_{T_0+h|T_0} = X'_{T_0+h}\hat{\beta}$
- iterated PLUP : $\hat{y}^{+}_{T_0+h|T_0} = X'_{T_0+h}\hat{\beta} + \hat{\rho}_1^h \hat{e}_{T_0}, \qquad \hat{\rho}_1 = \hat{\gamma}_1/\hat{\gamma}_0.$
- direct PLUP: $\hat{y}_{T_0+h|T_0} = X'_{T_0+h}\hat{\beta} + \hat{\rho}_h \hat{e}_{T_0}, \qquad \hat{\rho}_h = \hat{\gamma}_h/\hat{\gamma}_0$

Remarks: Absent sampling uncertainty:

- all predictions are asymptotically unbiased provided $E[e_t] = 0$.
- asymptotic mse of direct PLUP is never worst.
- additional lags of e_{T_0} may be better, but even one lag helps.
- gains from PLUP $\downarrow$ with h because of mixing (ergodicity).

Table 1: DGP: $y_t = X_t'\beta + e_t, \qquad e_t = \phi_1 e_{t-1} + \phi_2 e_{t-2} + v_t$.

(T,N)=(50,20)

Unconditional Errors

h	best	noadj	ols	pulpl	plupd	best	noadj	ols	plupl	plupd
	bias					mse				
	(ϕ_1, ϕ_2)=(0.80,0.00)									
1	-0.00	-0.01	-0.00	-0.00	-0.00	0.05	0.14	0.14	0.05	0.05
2	0.00	-0.00	0.00	0.00	0.00	0.05	0.14	0.14	0.08	0.08
5	0.00	0.00	0.00	0.00	0.00	0.05	0.13	0.14	0.12	0.12
10	0.00	-0.00	-0.00	-0.00	0.00	0.05	0.14	0.14	0.14	0.14
avg	-0.00	-0.00	-0.00	-0.00	-0.00	0.00	0.07	0.08	0.06	0.06
	(ϕ_1, ϕ_2)=(1.30,-0.40)									
1	-0.00	-0.01	-0.00	-0.00	-0.00	0.05	0.43	0.43	0.06	0.06
2	0.00	-0.00	-0.00	-0.00	-0.00	0.05	0.43	0.43	0.16	0.16
5	0.00	-0.00	0.00	-0.00	0.00	0.05	0.42	0.43	0.37	0.35
10	0.00	-0.01	-0.00	-0.00	-0.00	0.05	0.43	0.44	0.48	0.44
avg	-0.00	-0.00	-0.00	-0.00	-0.00	0.00	0.28	0.29	0.21	0.20

(T,N)=(50,20)

Conditional on $(e_{T_0-1}, e_{T_0}) = (0.5, 1.0)$

h	best	noadj	ols	pulpl	plupd	best	noadj	ols	plupl	plupd
	bias					mse				
	(ϕ_1, ϕ_2)=(0.80,0.00)									
1	-0.00	0.80	0.79	0.01	0.01	0.05	0.68	0.68	0.05	0.05
2	0.00	0.64	0.64	0.01	0.02	0.05	0.49	0.49	0.09	0.09
5	0.00	0.33	0.33	0.01	0.02	0.05	0.23	0.23	0.13	0.13
10	0.00	0.11	0.10	0.00	0.02	0.05	0.15	0.15	0.14	0.15
avg	-0.00	0.36	0.35	0.01	0.02	0.00	0.18	0.18	0.06	0.06
	(ϕ_1, ϕ_2)=(1.30,-0.40)									
1	-0.00	1.10	1.09	0.18	0.18	0.05	1.25	1.26	0.09	0.09
2	0.00	1.03	1.03	0.18	0.24	0.05	1.19	1.20	0.17	0.20
5	0.00	0.62	0.62	-0.04	0.20	0.05	0.72	0.73	0.34	0.38
10	0.00	0.21	0.21	-0.23	0.08	0.05	0.47	0.48	0.49	0.45
avg	-0.00	0.61	0.61	-0.04	0.17	0.00	0.56	0.56	0.19	0.22

Outline

From Prediction to Imputation

- Goldberger's BLUP is for linear prediction of data not yet observed but eventually available.
- Chow-Lin: interpolate (impute) missing values of mixed frequency data that may or may not be observed.
- For $i \in [1, N_1]$, we want to impute missing Y_{i,T_0+h} from
    - T_0 outcomes for N_1 treated units
    - T outcomes for $N - N_1$ control units.

Setup for Potential Outcomes

$$\underbrace{\mathcal{Y}}_{n\times 1}(0) = \begin{pmatrix} Y_{1,1:T_0}(0) \\ \vdots \\ Y_{N,1:T_0}(0) \\ Y_{N_1+1,T_0+1:T}(0) \\ \vdots \\ Y_{N,T_0+1:T}(0) \end{pmatrix} = \begin{pmatrix} \underbrace{Y^{pre}(0)}_{(NT_0\times 1)} \\ \\ \underbrace{Y^{post}(0)}_{(N-N_1)T_1\times 1} \end{pmatrix} = \mathcal{M} + \mathcal{E}$$

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}^{pre} \\ \mathcal{E}^{post} \end{pmatrix} \sim (0, \Gamma) \qquad \Gamma = E[\mathcal{E}\mathcal{E}'] = \begin{pmatrix} \Gamma_{pre,pre} & \Gamma_{pre,post} \\ \Gamma_{post,pre} & \Gamma_{post,post} \end{pmatrix}$$

Let $n = NT - N_1 T_1$. The $n \times n$ covariance matrix Γ depends on

a. $\Sigma = E[e_t e_t']$, the $N \times N$ covariance matrix of $e_t = (e_{1t}', e_{0t}')'$
b. $\Omega_i = E[e_i e_i']$, the $T_{0_i} \times T_{0_i}$ time series covariance of unit i.

Assumptions

A1 strict stationarity, strong mixing.

A2 $\hat{m}_{it} = \hat{Y}_{it}(0)$ is any consistent estimator such that
$\hat{m}_{i,T_0+h} - m_{i,T_0+h} = o_p(1);\ T_0^{-1} \sum_{t=1}^{T_0} (\hat{m}_{i,t} - m_{i,t})^2 = o_p(1)$.

Under A1 and A2, the standard imputation has error

$$\hat{\delta}_{i,T_0+h} - \delta_{i,T_0+h} = \underbrace{(m_{i,T_0+h} - \hat{m}_{i,T_0+h})}_{(i)} + \underbrace{e_{i,T_0+h}}_{(ii)}.$$

(i) is estimator dependent, may need White/HAC, but is $o_p(1)$.

(ii) is not captured by model, dominates (i) asymptotically.

blup for Counterfactual Outcome

Re-doing the Goldberger problem with $\mathcal{M} = \mathcal{X}\beta$ gives

BLUP: adjusted counterfactual outcome

$$m^{+}_{i,T_0+h} = X'_{T_0+h}\beta_{GLS} + \omega'_{ih}\Gamma^{-1}\mathcal{E}_{GLS}.$$

BLUP is infeasible. But under Assumption A, we have residuals

- $\hat{e}_{jt} = Y_{jt}(0) - \hat{m}_{jt}$ for any j not treated, $t = 1, \ldots, T$.
- $\hat{e}_{it} = Y_{it}(0) - \hat{m}_{it}$, any i treated, $t = 1, \ldots, T_0$.

We can construct PLUP that is asymptotically equivalent to BLUP.

Practical Unbiased Predictor, pup: $\hat{Y}^+_{it}(0) = \hat{m}^+_{it}$

* Correction can be done even for non-linear imputation models.

Let $\hat{m}_{i,T_0+h}$ be any standard imputation satisfying A1 and A2:

- time series PUP: $\hat{m}^+_{i,T_0+h} = \hat{m}_{i,T_0+h} + \sum_{k=0}^{p} \hat{\rho}_i^k \hat{e}_{i,T_0-k}$.
- cross-section PUP $\hat{m}^+_{i,T_0+h} = \hat{m}_{i,T_0+h} + \sum_{j>N_1} \hat{\theta}_{i,j} \hat{e}_{j,T_0+h}$.
- time series and cross-section PUP:

$$\hat{m}^+_{i,T_0+h} = \hat{m}_{i,T_0+h} + \sum_{s=0}^{p_i} \rho_{is} \hat{e}_{i,T_0-s} + \sum_{j \neq i}^{N} \sum_{s=-p_j}^{p_j} \theta_{ij,s} \hat{e}_{j,T_0+h-s}.$$

Related Work

- PUP can use any consistent $\hat{m}_{it}$ already developed. No need to abandon static procedures.

- PUP type corrections have been suggested.
  - Chernozhukov-Wuthrith-Zhu (2021): suggest to add $\rho(L)\epsilon_{t-1}$, no mse analysis, limited proof of concept.
  - Fan-Masini-Medeiros (2022): mutually correlated idiosyncratic errors $Y_{it}(0) = \gamma_i' W_{it} + \lambda_i' F_t + \theta_1' \hat{\epsilon}_{-i,t} + v_{it}$ by LASSO.
  - Ferman (2023): DiD with omitted spatial correlation.

We provide analytical motivation and study their properties.

Simulations, $Y_{it}(0) = F_t'\Lambda_i + e_{it}$, serial correlation

h	best	noadj	pca	pupl	pupd	best	noadj	pca	pupl	pupd
	$e_{1t} = 0.6e_{1t-1} + \epsilon_{1t}$									
	Unconditional bias of $\hat{\delta}_{i,T_0+h}$					mse				
1	0.00	-0.00	0.00	0.01	0.01	0.25	0.37	0.43	0.33	0.33
2	-0.01	-0.01	-0.00	-0.00	-0.00	0.33	0.37	0.45	0.42	0.42
5	-0.00	-0.00	0.01	0.01	0.01	0.37	0.39	0.48	0.47	0.48
10	-0.01	-0.01	-0.00	-0.00	-0.00	0.39	0.40	0.50	0.50	0.51
avg	-0.00	-0.00	0.00	0.00	0.00	0.12	0.13	0.18	0.17	0.17
	$e_{1t} = 0.6e_{1t-1} + \epsilon_{1t},\ e_{T_0}$=1.00									
	Conditional bias of $\hat{\delta}_{i,T_0+h}$					mse				
1	0.00	0.60	0.56	0.15	0.15	0.25	0.61	0.66	0.36	0.36
2	-0.01	0.35	0.32	0.12	0.14	0.33	0.45	0.53	0.45	0.47
5	-0.00	0.21	0.19	0.09	0.13	0.37	0.42	0.51	0.48	0.52
10	-0.01	0.12	0.09	0.04	0.09	0.39	0.40	0.51	0.50	0.54
avg	-0.00	0.15	0.12	0.04	0.09	0.12	0.14	0.19	0.17	0.19

Simulations, $Y_{it}(0) = F_t' \Lambda_i + e_{it}$, cross-section correlation

h	best	noadj	pca	pupl	pupd	best	noadj	pca	pupl	pupd
	$e_{1t} = 0.5 e_{2t} + \epsilon_{1t}$									
	Unconditonal bias of $\hat{\hat{\delta}}_{i,T_0+h}$					mse				
1	0.00	-0.00	0.00	0.00	0.00	0.15	0.20	0.33	0.28	0.28
2	-0.01	-0.01	-0.00	-0.01	-0.01	0.15	0.21	0.35	0.29	0.29
5	0.00	-0.00	0.01	0.01	0.01	0.15	0.21	0.35	0.30	0.30
10	-0.01	-0.01	-0.01	0.00	0.00	0.15	0.21	0.34	0.29	0.29
avg	-0.00	-0.00	0.00	0.01	0.01	0.01	0.02	0.05	0.04	0.04
	$e_{1t} = 0.5 e_{2t} + \epsilon_{1t},\ e_{2T_0+1} = 0.27539$									
	Conditional bias of $\hat{\hat{\delta}}_{i,T_0+h}$					mse				
1	0.00	0.14	0.14	0.04	0.04	0.15	0.17	0.29	0.27	0.27
2	-0.01	0.49	0.49	0.12	0.12	0.15	0.38	0.52	0.35	0.35
5	0.00	0.47	0.48	0.13	0.13	0.15	0.37	0.52	0.35	0.35
10	-0.01	-0.45	-0.44	-0.14	-0.14	0.15	0.35	0.48	0.34	0.34
avg	-0.00	-0.12	-0.12	-0.06	-0.06	0.01	0.03	0.06	0.05	0.05

Inference

$$\widehat{\delta}_{it} - \delta_{it} = o_p(1) \text{ estimation error } + e_{it}.$$

Suppose e_{it} is normal. Given point estimate $\hat{\delta}_{it}$ with variance σ^2_{δ}, we can construction a prediction interval of the form

$$\text{point estimate } \pm 2\text{s.e.}$$

We can choose from

- Two predictors (standard vs PUP);
- Two types of inference (unconditional and conditional).

Unconditional Inference

- Unconditional inference takes average over draws of e_{T_0}. Useful for evaluating procedures.

- Proposition:

  - Let $\{e_t\}$ be a stationary mixing process. Under normality, the standard and PUP predictions all have correct unconditional coverage, ie, as $T_0 \to \infty$,

$$P\left(-z_{1-\alpha/2} \leq \frac{\hat{\delta}_{i,T_0+1} - \delta_{i,T_0+1}}{\hat{\sigma}_{e,i}} \leq z_{1-\alpha/2}\right) = \Phi(z_{1-\alpha/2}) - \Phi(-z_{1-\alpha/2}).$$

  - but the PUP intervals are shorter.

Conditional inference based on standard imputation

Suppose $e_{it} = \phi_i e_{it-1} + v_{it}$ where $v_{it} \overset{iid}{\sim} \mathcal{N}\left(0, \sigma^2_{v,i}\right)$.

Problem at $h = 1$:

$$\frac{e_{i,T_0+1}}{\sigma_{e,i}} \sim \mathcal{N}(\phi_i \frac{e_{i,T_0}}{\sigma_{e,i}}, \frac{\sigma^2_{v,i}}{\sigma^2_{e,i}}) \neq \mathcal{N}(0,1).$$

$$\begin{aligned} & P\left(-z_{1-\alpha/2} \leq \frac{\hat{\delta}_{i,T_0+1} - \delta_{i,T_0+1}}{\hat{\sigma}_{e,i}} \leq z_{1-\alpha/2} \middle| e_{i,T_0}\right) \\ =& \Phi\left(\frac{-\phi_i}{\sigma_{v,i}} e_{i,T_0} + z_{1-\alpha/2}\frac{\sigma_{e,i}}{\sigma_{v,i}}\right) - \Phi\left(\frac{-\phi_i}{\sigma_{v,i}} e_{i,T_0} - z_{1-\alpha/2}\frac{\sigma_{e,i}}{\sigma_{v,i}}\right). \end{aligned}$$

Suppose $e_{1,t} = \theta_1' e_{2:N,t} + v_{1t}$ is cross-sectionally correlated.

$$e_{1,T_0+1} | e_{2:N,T_0+1} \sim \mathcal{N}(\theta_1' e_{2:N,T_0+1}, \sigma_{v,1}^2).$$

$$\begin{aligned} & P\left(-z_{1-\alpha/2} \leq \frac{\hat{\hat{\delta}}_{1,T_0+1} - \delta_{1,T_0+1}}{\hat{\sigma}_{e,1}} \leq z_{1-\alpha/2} \Big| e_{2:N,T_0+1} \right) \\ =& \Phi\left(\frac{-\theta_1'}{\sigma_{v,1}} e_{2:N,T_0+1} + z_{1-\alpha/2} \frac{\sigma_{e,1}}{\sigma_{v,1}} \right) - \Phi\left(\frac{-\theta_1'}{\sigma_{v,1}} e_{2:N,T_0+1} - z_{1-\alpha/2} \frac{\sigma_{e,1}}{\sigma_{v,1}} \right). \end{aligned}$$

Conditionally biased prediction distorts inference

$$\Phi(\text{bias}_1 + \text{bias}_2\ z_{1-\alpha/2}) \neq \Phi(z_{1-\alpha/2}).$$

Conditional bias of standard prediction I

For serially correlated errors, (analytical) bias$_1$= $\phi_i^h/\omega_{h,i}$

$$(e_{T_0}, \sigma_v) = (-2.0, 0.5)$$

h	coverage	bias$_1$	coverage	bias$_1$
	AR(1) $\phi_1 = 0.8$		MA(1): $\theta_1 = 0.8$	
1	0.84	-2.26	0.58	-1.77
2	0.87	-1.41	0.95	-0.00
3	0.90	-1.01	0.95	-0.00
4	0.92	-0.76	0.95	-0.00
5	0.93	-0.59	0.95	-0.00

Conditional bias of standard prediction II

For cross-correlated errors, (analytical) $\text{bias}_1 = \theta_1' e_{-1,T_0+1}/\sigma_{v,1}$.

Example: $e_{1,T_0+h} = \theta_{12} e_{2,T_0+h} + v_{1,T_0+h}$, $(\Sigma_{11}, \Sigma_{00}) = (0.5, 0.841)$.

		$\Sigma_{01} = 0.613$		$\Sigma_{01} = -0.613$	
h	e_{2,T_0+h}	coverage	bias_1	coverage	bias_1
1	0.68	0.63	-1.64	0.89	-0.70
2	-0.83	0.81	1.07	0.86	0.85
3	-0.92	0.93	-0.38	0.84	0.95
4	0.09	0.95	0.17	0.95	-0.09
5	0.86	0.94	0.32	0.86	-0.89

PUP Coverage

For $h = 1$, conditional PLUP coverage is

$$P\left(-z_{1-\alpha/2} \leq \frac{(\hat{\delta}^{+}_{i,T_0+1} - \delta_{i,T_0+1})}{\hat{\sigma}^{+}_{e,i,1}} \leq z_{1-\alpha/2} \middle| e_{i,T_0}\right) \approx 1 - \alpha + o_p(1)$$

PUP conditional coverage is asymptotically accurate: $\forall h \geq 1$.

Coverage, DGP 1: $y_t = X_t'\beta + e_t$ with $e_t = \phi_1 e_{t-1} + \phi_2 e_{t-2} + v_t$

$$(T, N) = (50, 20)$$

h	best	noadj	ols	pup	best	noadj	ols	pup
	Unconditional				Conditional on e_{T_0}, e_{T_0-1}			
	(ϕ_1, ϕ_2)=(0.80,0.00)							
1	0.96	0.95	0.95	0.95	0.96	0.40	0.41	0.95
2	0.95	0.95	0.95	0.95	0.95	0.64	0.63	0.95
5	0.96	0.95	0.95	0.95	0.96	0.88	0.88	0.95
10	0.95	0.95	0.95	0.94	0.95	0.94	0.94	0.94
avg	0.95	0.99	0.99	0.94	0.96	0.94	0.93	0.94
	(ϕ_1, ϕ_2)=(1.30,-0.40)							
1	0.96	0.95	0.95	0.95	0.96	0.79	0.77	0.92
2	0.95	0.95	0.95	0.95	0.95	0.76	0.75	0.94
5	0.96	0.95	0.95	0.95	0.96	0.87	0.87	0.96
10	0.95	0.95	0.94	0.95	0.95	0.94	0.94	0.94
avg	0.95	0.98	0.98	0.94	0.96	0.94	0.93	0.96

Coverage, DGP 2a: $X_{it} = \Lambda_i' F_t + e_{it}$ with $e_{1t} = \rho_1 e_{1t-1} + v_{1t}$

h	best	noadj	pca	pup	best	noadj	pca	pup
	Unconditional				Conditional on e_{1,T_0}			
	(T,N)=(50,20)							
1	0.95	0.94	0.92	0.93	0.96	0.87	0.86	0.92
2	0.91	0.94	0.91	0.91	0.92	0.92	0.90	0.91
5	0.88	0.93	0.90	0.88	0.90	0.94	0.91	0.89
10	0.89	0.94	0.90	0.84	0.91	0.94	0.91	0.85
avg	0.98	0.85	0.75	0.77	0.98	0.83	0.73	0.76
	(T,N)=(200,50)							
1	0.95	0.95	0.94	0.95	0.96	0.90	0.89	0.95
2	0.91	0.95	0.94	0.94	0.91	0.92	0.92	0.94
5	0.89	0.95	0.94	0.93	0.89	0.95	0.94	0.94
10	0.89	0.95	0.94	0.94	0.90	0.95	0.94	0.94
avg	0.99	0.93	0.92	0.92	0.99	0.92	0.90	0.92

- Gains are larger at $h = 1$ and when T_0 is large;
- Coverage of $\hat{\delta}_{it}$ not informative about $\bar{\delta}_i$.

Coverage, DGP 2b: $X_{it} = \Lambda_i' F_t + e_{it}$ with $e_{1t} = \theta_{12} e_{2,t} + v_{1t}$

h	best	noadj	pca	pup	best	noadj	pca	pup
	Unconditional				Conditional on $e_{2,T_0+1:T}$			
	(T,N)=(50,20)							
1	0.95	0.95	0.94	0.93	0.95	0.97	0.95	0.93
2	0.95	0.95	0.93	0.93	0.95	0.86	0.87	0.90
5	0.95	0.94	0.93	0.91	0.95	0.91	0.91	0.90
10	0.95	0.95	0.93	0.88	0.95	0.98	0.96	0.89
avg	0.88	0.88	0.79	0.78	0.88	0.81	0.75	0.77
	(T,N)=(200,50)							
1	0.95	0.96	0.95	0.95	0.95	0.94	0.94	0.95
2	0.95	0.95	0.94	0.94	0.95	0.94	0.93	0.94
5	0.95	0.95	0.95	0.95	0.95	0.75	0.78	0.93
10	0.95	0.95	0.95	0.94	0.95	0.86	0.87	0.93
avg	0.94	0.94	0.92	0.92	0.94	0.92	0.91	0.92

Comments on PUP in practice

1. PLUP gains specific to estimator and hypothesis. Details
   - coverage of $\hat{\delta}_{it}$ not good guide for $\bar{\delta}_i$ or $\bar{\delta}_t$.
   - At least for FBI, Gaussian distribution for different reasons.

2. A necessary condition for PUP improvements is predictability in the errors. We can check using LM test

$$\hat{e}_{it} = X_{it}'\delta_0 + \sum_{s=0}^{p_i} \delta_{i,s}\hat{e}_{i,t-s} + \sum_{j=1}^{N} \sum_{s=-p_j}^{p_j} \delta_{j,s}\hat{e}_{j,t-s}.$$

   - Rejection of H_0 is not sufficient for improvements.
   - Correction involves $\hat{\rho}_i^h \hat{e}_{i,T_0}$ or $\theta_i' \hat{e}_{-i,T_0+h}$.
   - The relevant $\hat{e}$'s must be non-zero, but we can check this!

Comments on Inference

1. Many procedures for unconditional inference when e_{it} is unpredictable, but developed for different estimators and different object of interest. How to compare methods?

2. Limited work on conditional inference,

    - Theory allows e_{it} to be dependent, consider this possibility.
    - modeling e_{it} improves point prediction, not just efficiency.

Summary

- Robust s.e for $\hat{m}_{it}$ useful, but prediction error variance is asymptotically dominated by $\text{var}(e_{i,T_0+h})$.
- Ignoring time and cross correlations associated with e_{i,T_0+h} can give inefficient and conditionally biased predictions, as well as misleading inference.
- PUP provides a simple way to account for correlation without knowing the true error structure. It can be used in conjunction with many existing estimators.

Summary

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THANK YOU

Prediction vs Confidence Intervals

$$H_0 : \quad \frac{1}{T_1} \sum_{t=T_0+1}^{T} \delta_{it} = \Delta^0_{i,T_1}.$$

- If Δ_{i,T_1} is random for fixed T_1: construct prediction interval

$$\hat{\Delta}_i - \Delta_{i,T_1} = \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} m_{it} - \hat{m}_{it} \right) + \frac{1}{T_1} \sum_{t=T_0+1}^{T} e_{it}$$

- $\Delta_{i,\infty}$ is non-random: construct a confidence interval

$$\hat{\Delta}_i - \Delta_{i,\infty} = \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} m_{it} - \hat{m}_{it} \right) + \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} e_{it} + \delta_{it} - E[\delta_{it}] \right).$$

FBI Imputation Error, Bai and Ng (2021)

$$\widehat{\delta}_{it} - \delta_{it} = \underbrace{x_{it}'(\beta - \hat{\beta})}_{(i.a)} \underbrace{- F_t' \left(\frac{F'F}{T_0} \right)^{-1} \mathbf{B}_F \frac{1}{T_0} \left(\sum_{s=1}^{T_0} F_s \epsilon_{is} \right)}_{(i.b)}$$

$$\underbrace{- \Lambda_i' \left(\frac{\Lambda'\Lambda}{N} \right)^{-1} \mathbf{B}_\Lambda \frac{1}{N_0} \sum_{k=1}^{N_0} \Lambda_k \epsilon_{kt}}_{(i.c)} + \epsilon_{it} + O_p\left(\frac{1}{\min(N_0, T_0)}\right).$$

- error (i.a) is $O_p(\frac{1}{\sqrt{T_0}})$
- error (i.b) is error from estimating F: $O_p(\frac{1}{\sqrt{T_0}})$
- error (i.c) is error from estimating Λ: $O_p(\frac{1}{\sqrt{N_0}})$.

ϵ_{it} dominates error in estimate of individual treatment effect

Inference of Average Treatment Effect over Time

Recall: $\hat{\delta}_{i,T_0+h} - \delta_{i,T_0+h} = (m_{i,T_0+h} - \hat{m}_{i,T_0+h}) + e_{i,T_0+h}$.

$$\hat{\delta}_{i,T_1} = \frac{1}{T_1}\sum_{h=1}^{T_1}\hat{\delta}_{i,T_0+h}$$

$$= \frac{1}{T_1}\sum_{h=1}^{T_1} F'_{T_0+h}\left(\frac{F'F}{T_0}\right)^{-1}\mathbf{B}_F\frac{1}{T_0}\left(\sum_{s=1}^{T_0}F_s e_{is}\right) + \frac{1}{T_1}\sum_{h=1}^{T_1} e_{i,T_0+h} + c$$

Let $\mathbb{V}_{\hat{\Delta}_{i,T_1}} = \frac{\min(T_0,T_1)}{T_0}\bar{F}'\left(\frac{F'F}{T_0}\right)^{-1}\Phi_j\left(\frac{F'F}{T_0}\right)^{-1}\bar{F} + \frac{\min(T_0,T_1)}{T_1}\sigma_{ei}^2$.
Then

$$\min(\sqrt{T_0},\sqrt{T_1})\left(\frac{\hat{\Delta}_{i,T_1} - \Delta_{i,T_1}}{\sqrt{\mathbb{V}_{\hat{\Delta}_i}}}\right)\xrightarrow{d} N(0,1)$$

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