The Macroeconomics of Supply Chain Disruptions*

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Abstract

This paper develops a model to study the macroeconomic implications of supply chains disruptions with three key ingredients: (i) a firm-level network of customized supplier-customer links that generate relationship-specific productivity gains; (ii) bargaining over these relationship-specific surpluses; and (iii) an extensive margin of adjustment, whereby firms may decide to form or sever relations with suppliers or customers. We establish equilibrium existence and uniqueness, provide characterization results, and present a number of comparative statics that show how the supply chain and aggregate productivity respond to shocks. We also show that the equilibrium is inefficient and exhibits an inherent fragility: small shocks can lead to discontinuous changes in output, even though the efficient allocation is always continuous in the same shocks. We finally draw out several macroeconomic implications of this new form of fragility.

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1 Introduction

Production in modern industrialized economies relies on complex supply chains. Major manufacturers, such as General Motors and Airbus, depend on production ecosystems consisting of thousands of direct and indirect suppliers (McKinsey Global Institute, 2020). These complex supply chains facilitate specialization and customization. For example, Fort (2016) documents that more than half of firms in aerospace, power, computer, and motor vehicle manufacturing sectors engage in customized outsourcing, whereby the firm "provides design and production criteria to a manufacturer who performs the physical transformation activities, generally on materials or inputs specified by the purchaser." Modern supply chains are also a major source of productivity gains. Amiti and Konings (2007), Topalova and Khandelwal (2011), and Halpern et al. (2015) find that access to higher quality or more diverse intermediate inputs can significantly improve productivity, while Baqaee et al. (2023) estimate that a sizable share of aggregate productivity growth can be accounted for by churn in supply chains.

At the same time, complex supply chains can also be a major source of macroeconomic fragility, as disruptions to a few firms can create shortages of essential inputs or destroy accumulated relationship-specific investments and productivities. This was evident from the major supply chain problems in the aftermaths of natural disasters (Barrot and Sauvagnat, 2016; Carvalho et al., 2021), the COVID pandemic (Financial Times, 2023), and the Russian invasion of Ukraine (OECD, 2022).[1] Not surprisingly, the risks associated with major tensions in supply chains have received significant attention from policymakers. For example, the 2012 U.S. National Strategy for Global Supply Chain Security was based on the premise that "[i]ntegrated supply chains are fast and cost-efficient but also susceptible to shocks that can rapidly escalate from localized events into broader disruptions" (The White House, 2012). More recently, the Biden administration's Supply Chain Disruptions Task Force concluded that supply chains that deliver strategic and critical materials "are at serious risk of disruption—from natural disasters or *force majeure* events" and that this "risk is more than a military vulnerability; it impacts the entire U.S. economy" (The White House, 2021).

Despite the growing focus in academic and policy circles on supply chain disruptions, there is currently no theoretical framework that enables a systematic investigation of their macroeconomic consequences. This is in part because most commonly used frameworks in macroeconomics and industrial organization lack the vital ingredients necessary for such an investigation. Three ingredients are particularly central: (i) a firm-level network representing customized, relationship-specific productive opportunities between firms and their suppliers; (ii) a noncompetitive model for the division of surplus, as markets cannot be competitive in the presence of such customized relationships; and (iii) a nontrivial extensive margin decision whereby firms can choose to form or dissolve their relationships with suppliers and customers.

In this paper, we build a general equilibrium model that incorporates these three ingredients into a single tractable framework. We consider an economy in which firms establish productivity-enhancing, customized relationships with one or multiple suppliers. The surplus generated by these relationships

[1] Other works providing evidence for propagation of shocks over supply chains include Acemoglu et al. (2016) and Boehm et al. (2019).

is split between firms via pairwise bargaining, where both the size of the surplus and disagreement points depend on the entire supply chain. Forming and maintaining customized supplier-customer relationships are costly and require relationship-specific investments. Therefore, firms face nontrivial decisions to build or dissolve relationships with their suppliers and customers. The endogenous dissolution of these supply chain relationships in response to shocks is the central focus of our paper.

We now describe the model in more detail. We consider a production network economy consisting of n input producers and a firm that produces the unique final good sold to consumers. Each firm has access to a menu of constant returns production technologies, each with its own productivity and a potentially different mix of inputs sourced from other firms.[2] Forming relationships with suppliers raises the firm's productivity but requires upfront fixed costs, as supplier-customer relationships have to be tailored to the needs of the firm (e.g., to customize products or integrate the production processes). The presence of such productivity-enhancing customized relationships constitutes the first key ingredient of our model.

The specificity of supplier-customer relationships means that markets cannot be competitive and calls for a noncompetitive framework for division of surplus. We thus make the natural assumption that the parties split the surplus generated by supplier-customer relationships via pairwise Nash bargaining.[3] This is the second key ingredient of our model. We assume that firms negotiate over two-part tariff contracts that specify a unit price and a lump-sum transfer, both of which can be contingent on the economy's production network.[4] This choice—which enables firms to share the relationship-specific surplus without distorting input quantity decisions—is motivated by our desire to minimize inefficiencies in quantity and pricing decisions relative to the competitive benchmark. An additional attractive feature of relying on these contracts is that, despite the multilateral sharing of relationship-specific surpluses, it leads to a closed-form characterization of equilibrium profits, as we describe below.

The third and final key ingredient of our framework is the endogenous adjustment in the production network: firms decide whether to form or dissolve supplier-customer relationships, anticipating bargaining outcomes in the realized production network.

We provide three sets of results. Our first set of results aims to build intuition about the functioning of the model by focusing on an economy with an exogenous (fixed) production network. We show that, with an exogenous production network, a decentralized equilibrium always exists and is generically unique. Furthermore, firms' equilibrium profits are given by a variant of the Myerson value (Jackson and Wolinsky, 1996), which is a network-adjusted generalization of the Shapley value and reflects firms' marginal contributions to aggregate productivity in various subnetworks. We also establish that the exogenous network equilibrium is always efficient. These results rely on our specification of pairwise contracts, which ensure that all inputs are sold at marginal cost and that any relationship-

[2]For instance, firms can decide which set of inputs to produce in-house and which ones to source from customized outside suppliers.

[3]The production side of our model is similar to Acemoglu and Azar (2020) but with the critical difference that they assume a form of "contestability"—allowing many potential entrants to operate in any given industry. In contrast, we require upfront investments to form customized relationships. As a result, while their model is essentially competitive, ours is one in which there are relationship-specific surpluses that are divided via pairwise bargaining.

[4]The assumption that the terms of the contracts can depend on the production network is similar to (and can be micro-founded by) allowing the firms to renegotiate the rest of their relationships in case negotiations with a counterparty break down. As we demonstrate, this assumption ensures that the equilibrium with an exogenous production network is efficient.

specific surplus generated by firm-to-firm linkages is distributed throughout the economy via lump-sum transfers.

We also present a number of comparative static results for the exogenous network economy, under two additional assumptions. These are (a) "supermodularity at the extensive margin": an additional supplier-customer relationship increases aggregate productivity by more when the existing production network is larger; and (b) "supermodularity at the intensive margin": an additional supplier-customer relationship increases aggregate productivity by more when existing relationships are more productive. These assumptions are natural, though they do rule out cases in which a new link is highly substitutable to an existing link. Under these supermodularity assumptions, we show that expanding the set of supplier-customer linkages or making an existing link more productive increases both aggregate productivity and firm profits. This latter result follows from the fact that any gains from greater productivity—whether because of a new supplier-customer relationship or an increase in the productivity of an existing relationship—are shared with the entire supply chain via pairwise bargaining. Moreover, we establish that when the bargaining power of a firm increases, its overall profits rise while the profits of all other firms in the economy decline. We are not aware of any counterparts to these results in the literature and find them both intuitive and useful for understanding how production network structure, technological changes, and institutional features impact productivity and profits in supply chains. But their main utility for us is to enable the study of the endogenous network economy, which is our main focus.

Our second set of results turns to the endogenous network economy, in which firms decide to form or dissolve relationships with suppliers and customers. We prove that, under the same supermodularity assumptions, an equilibrium exists. Moreover, although the equilibrium may not be unique, there always exists a greatest equilibrium—an equilibrium production network that contains all other equilibrium networks as its subnetworks. This greatest equilibrium is Pareto superior to and generates higher aggregate output than all other equilibria.

We then generalize our comparative statics to this environment where the production network also responds to shocks. We prove that a lower productivity or a higher cost of forming supplier-customer relationships shrinks the production network and reduces aggregate output. This is a consequence of both direct and indirect effects. On the direct side, a decline in productivity (or an increase in fixed costs) reduces profits and hence makes it more likely that some firms decide to drop their customers and/or suppliers. On the indirect side, once the network becomes smaller, the same forces as in the exogenous network economy amplify this effect and lead to further rounds of contractions in both firm profits and the production network.

In contrast to the exogenous network case, the endogenous network equilibrium is inefficient. This inefficiency is due to a classic hold-up problem: firms do not internalize the benefits their customized relationships generate for other firms in the economy. This specifically implies that the equilibrium production network is always a subnetwork of the efficient production network.

Finally, our third set of results establishes that equilibrium supply chains are inherently fragile—in the sense that equilibrium aggregate output is discontinuous in response to productivities and link formation costs. Crucially, this is in contrast to the efficient allocation, in which aggregate output is always continuous in the same shocks. Equilibrium fragility in our model is a consequence of the hold-up externality in endogenous network formation. Indeed, the points of discontinuity correspond to points where the equilibrium network structure changes. Taken together, these results indicate that equilibrium aggregate output is not only below its socially optimal level but is also prone to sharp and sudden contractions as the economic environment changes.

The fragility of equilibrium production networks has a number of macroeconomic implications, which we then draw out. One of those, anticipated by our discussion above, is that there can be a trade-off between efficiency and resilience: more fragmented production networks increase productivity but make the economy more prone to sharp (and inefficient) declines in aggregate output. Second, small shocks may trigger cascading supply chain breakdowns, as multiple supplier-customer relationships are dissolved simultaneously. Such supply chain cascades can significantly magnify the discontinuous response of aggregate output to shocks. Third, the response of the production network generates a nonlinear amplification pattern over the business cycle. All else equal, low-productivity relationships are more likely to dissolve first during economic downturns, and as a result, supply chain adjustments do not amplify small business cycle fluctuations by much. However, larger shocks result in the breakdown of progressively more productive supplier-customer relationships, leading to sizable amplification. This result illustrates how supply chain disruptions can emerge as a powerful propagation mechanism during severe downturns, such as those caused by the 2007–08 financial crisis or the COVID pandemic, while playing a much more limited role during milder downturns.

Related Literature. Our paper contributes to the literature on the role of production networks in economic fluctuations. The bulk of this literature focuses on models with (i) no relationship specificities, (ii) competitive markets, and (iii) exogenous production networks (Long and Plosser, 1983; Acemoglu et al., 2012).[5] We depart from this literature by relaxing these assumptions and focusing on how shifts in the distribution of surplus throughout the economy and the resulting changes in supply chains alter the economy's response to shocks. More importantly, our fragility result highlights how the response of the production network can turn small shocks into discontinuous (and inefficient) changes in aggregate output.

Our work is closely related to Carvalho and Voigtländer (2015), Acemoglu and Azar (2020), and Kopytov et al. (2023), who allow for endogenous adjustments to the production network. However, these models do not feature relationship specificities and are thus better suited for analyzing production networks at the industry level. For example, as already noted, Acemoglu and Azar (2020) assume contestable markets, where a large number of firms have access to the same sectoral production technologies at no cost. The relationship-specific nature of supplier-customer interactions in our model also distinguishes our work from Baqaee and Farhi (2021), who assume that individual firms at either end of any relationship are infinitesimal and interchangeable with a large number of other firms.

Another related paper is the recent work by Baqaee et al. (2023). They provide a decomposition to measure the importance of supply chain churn for aggregate growth, while taking the changes in

[5] Other examples include Carvalho (2010), Atalay (2017), Acemoglu et al. (2017), Baqaee and Farhi (2019), vom Lehn and Winberry (2022), and Dew-Becker and Vedolin (2022). A subset of papers in this literature, such as Baqaee and Farhi (2020), Bigio and La'O (2020), and Liu (2019), allow for nontrivial markups or wedges. However, these papers treat markups/wedges as exogenously specified model primitives, thus maintaining key features of the competitive benchmark, namely, producers' price-taking behavior. See Carvalho and Tahbaz-Salehi (2019) and Baqaee and Rubbo (2023) for recent surveys of this literature.

the production network as given (and matching them to the data). In contrast, we provide a fully-specified model that endogenizes the creation and destruction of supply chain linkages between firms, thus enabling us to provide comparative static results and to study how changes in the economic environment translate into supply chain disruptions.[6]

Even more closely related are a number of recent papers that study the extensive margin of firm-to-firm linkages (Lim, 2018; Huneeus, 2020; Xu, 2020; Bernard et al., 2022; Taschereau-Dumouchel, 2023; Dhyne et al., 2023). These papers impose particular assumptions along one or more dimensions on the firms' production technologies (e.g., nested CES), network architecture (e.g., acyclic production networks), and the division of surplus (e.g., full bargaining power with one party).[7] The tractability of our framework enables us to relax all these assumptions and analyze economies with general production network structures, bargaining powers, and production technologies. Moreover, to the best of our knowledge, neither our general comparative statics nor our results on the excessive fragility of equilibrium supply chains appear in any prior work.

Our fragility result is related to, but distinct from, the findings of Elliott et al. (2022), who study a model in which discontinuous phase transitions arise because firms rely on multiple essential inputs that have to be produced over infinitely many stages of production. Notably, such a production structure generates output discontinuities not only in equilibrium but also in the efficient allocation. In contrast, supply chain fragility in our model only arises in equilibrium and is a consequence of the firms' endogenous decisions to dissolve their relationships with particular suppliers and/or customers.

We also contribute to an emerging literature that studies the role of market power in production networks. Grassi (2017) investigates how alternative market structures shape the propagation of shocks, Carvalho et al. (2022) characterize how a firm's position in the network can confer market power, while Dhyne et al. (2022) and Alviarez et al. (2023) build models of firm-to-firm trade with one-sided and two-sided market power, respectively. Unlike our work, these papers take the production network as exogenous.

Lastly, our paper contributes to the large game-theoretic literature on strategic network formation.[8] Within this literature, the most closely related papers are Slikker and van den Nouweland (2000), Pin (2011), and Ambrus and Elliott (2020), who, as in our setup, consider network formation games with costly links and a division of surplus based on the Myerson value. In contrast to this literature—which mostly focuses on reduced-form games—we develop a microfounded, general equilibrium macro model. This allows us to study how shocks to primitives (such as productivities and fixed costs) propagate, alter the economy's production network, and shape macroeconomic outcomes.

[6] Also see Korovkin et al. (2023), who use a sufficient statistics approach to measure the welfare implications of the reorganization of production networks without relying on specific microfoundations for endogenous network formation.

[7] See Table 1 in Dhyne et al. (2023) for a detailed comparison of the various assumptions in these papers. Relatedly, Grossman et al. (2023) study a model of global supply chains with Leontief production technologies and a simple network consisting of a single tier of suppliers and customers. Also see Oberfield (2018) and Panigrahi (2023), who develop models of endogenous formation of firm-to-firm linkages with a continuum of firms. As a result, in these models, (i) each individual firm's decision has no impact on aggregate variables and (ii) production networks are acyclic with probability one. A more recent series of papers, such Arkolakis et al. (2023) and Demir et al. (forthcoming), use a search and matching framework to study formation of production networks across space.

[8] Examples include Jackson and Wolinsky (1996), Bala and Goyal (2000), Dutta and Jackson (2000), and Hojman and Szeidl (2008). See Jackson (2005) for an early survey of this literature and Mauleon and Vannetelbosch (2016) for a more recent one.

Outline. The rest of the paper is organized as follows. We present our model in Section 2. In Section 3, we analyze the model while taking the production network as given. In Section 4, we allow for endogenous creation and destruction of supplier-customer relationships and establish that equilibrium supply chains are excessively fragile. We explore the implications of this fragility in Section 5. The proofs and some additional results are provided in the Appendix. An Online Appendix contains the proofs not presented in the main paper.

2 Model

Consider an economy consisting of $n+1$ firms, indexed by $i \in N = \{0, 1, 2, \ldots, n\}$. Firms labeled $\{1, 2, \ldots, n\}$ produce differentiated goods, while the firm $i = 0$ produces a final good that is sold to the representative household.

Firm $i \in N$ has access to a menu of distinct production technologies, each with a potentially different mix of labor and (intermediate) inputs and with its own productivity. As in Acemoglu and Azar (2020), we assume that the production function of firm i can be expressed as

$$y_i = F_i(I_i, A_i(I_i), L_i, X_i), \tag{1}$$

where $I_i \subseteq \{1, \ldots, n\} \setminus \{i\}$ denotes the set of inputs used by the firm, L_i is the labor input, $X_i = \{x_{ij}\}_{j \in I_i}$ is the vector of input quantities purchased from firms $j \in I_i$, and $A_i(I_i)$ is a vector of productivity parameters that depend on the specific mix of inputs used by the firm.[9] One interpretation of the specification in (1) is that firm i requires a fixed set of inputs, which it can either acquire from specialized suppliers or produce in-house. Under this interpretation, $A_i(I_i)$ represents the potential productivity gains from relying on specialized suppliers in I_i. However, the mathematical formulation in (1) is more general than this particular interpretation and allows for the possibility that the firm can achieve greater productivity by combining a different set of inputs.

Throughout, we assume that $F_i(I_i, A_i(I_i), L_i, X_i)$ is strictly quasi-concave in inputs, exhibits constant returns to scale in (L_i, X_i), and is increasing and continous in labor, inputs, and $A_i(I_i)$. Unless otherwise noted, we also assume that labor is an essential factor of production for all technologies in the sense that $F_i(I_i, A_i(I_i), 0, X_i) = 0$.[10] Finally, and without much loss of generality, we assume that the final good producer can always produce using labor as the only input, with a productivity that we normalize to one, that is, $F_0(\varnothing, A_0(\varnothing), L_0) = L_0$. This assumption guarantees that the economy is always capable of producing a nonzero amount of the final good.

The specification in (1) indicates how the production technology of firm i depends on the specific mix of inputs used by the firm. However, to use a particular mix of inputs, I_i, firm i needs to establish customer-supplier relationships with producers of each of those inputs. Specifically, we assume that production technology $F_i(I_i, A_i(I_i), L_i, X_i)$ is available to firm i only if i and all suppliers $j \in I_i$ incur fixed costs $c_{ij} \geq 0$ and $s_{ij} \geq 0$, respectively. These relation-specific fixed costs, which are in units of the final good, may represent costs associated with customization of products, integration

[9] If good i cannot be produced using a particular mix of inputs, I_i, then $F_i(I_i, A_i(I_i), L_i, X_i) = 0$.

[10] This assumption is to rule out the possibility that labor can be made redundant by some combination of intermediate inputs and ensures that the output of each firm is always finite. While this assumption can be relaxed, it significantly simplifies our analysis and notation.

of production processes with those of the other party, quality control by the customer, establishing dedicated distribution channels by the supplier, or joint R&D efforts.

The economy is also populated by a representative household, who supplies L units of labor inelastically to the firms and consumes the final good. The household's budget constraint is

$$PY = wL + \sum_{i=0}^{n} \phi_i,$$

where Y denotes the household's consumption of the final good, P denotes the price of the final good, and w is the wage. In the above expression, ϕ_i denotes the profit of firm $i \in N$ net of fixed costs, which accrues to the representative household. We assume that the household has a strictly increasing utility function $u(\cdot)$. As a result, Y—which we refer to as the economy's *aggregate output*—is both the real GDP and a measure of welfare in this economy.

Finally, we assume that the final good can also be produced by a competitive fringe of firms that can transform labor, one-to-one, into the final good. While firms in this competitive fringe are not active in equilibrium, their presence imposes an upper bound on the price that firm 0 can charge. In their absence, firm 0 has an incentive to raise its price unboundedly, as it faces a unitary price elasticity of demand. A second convenient implication of this assumption is that the real wage is determined independently of the details of the supplier-customer relationships: the wage always coincides with the price of the final good, which we choose as the numeraire, i.e., $w = P = 1$.

The timing of the model is as follows. The economy lasts for three periods $t \in \{0, 1, 2\}$. At $t = 0$, each firm $i \in N$ decides whether to pay fixed costs $\{c_{ij}, s_{ji}\}_{j \neq i}$ to form relationships with potential suppliers and customers. At $t = 1$, any customer-supplier pair i and j that have established a relationship negotiate with one another to enter into a pairwise contract that governs their relationship at $t = 2$. Assuming i and j reach an agreement at $t = 1$, they can trade at $t = 2$, which is also when production and consumption take place. In what follows, we describe each of these stages in further detail.

Production Network. At $t = 0$, each firm decides whether to pay the requisite fixed costs to form supplier-customer relationships with other firms in the economy. The supplier-customer relationships that are formed at the end of $t = 0$ constitute the economy's *production network*, $\mathbf{G}$. Each vertex in this network corresponds to a firm $i \in N$ and a directed edge, ij, is present from vertex j to vertex i if firms j and i have both paid the fixed costs s_{ij} and c_{ij} to serve, respectively, as the supplier and the customer in that relationship. For notational convenience, we treat a production network $\mathbf{G}$ and its edge set interchangeably. Production technology $F_i(I_i, A_i(I_i), L_i, X_i)$ is thus available to firm i only if $ij \in \mathbf{G}$ for all $j \in I_i$.

Pairwise Contracts. After paying the fixed costs c_{ij} and s_{ij} at $t = 0$, firms i and j need to reach an agreement with one another at $t = 1$ to determine the terms of the contract that governs their relationship. We first specify the contract space and then discuss the bargaining procedure.

We assume that the contract between the supplier-customer pair ij takes the form of a *network-contingent* two-part tariff: the contract specifies a pair $(p_{ij}(\mathbf{G}), t_{ij}(\mathbf{G}))$ for any production network $\mathbf{G} \ni$

ij, where t_{ij} denotes a lump-sum transfer from the customer i to the supplier j, in exchange to which j commits to delivering as many units of its product as demanded by i at the fixed unit price of p_{ij}.

The assumption that the contract between i and j is contingent on the production network means that the unit price and the transfer can depend on the presence or absence of other supplier-customer relationships in the economy—for example, on the identities of the two firms' other suppliers and customers. However, contract terms between i and j cannot depend on the prices and transfers specified in other contracts. Finally, we impose the additional restriction that $p_{ij}(\mathbf{G}) = p_{ij}(\mathbf{G}\setminus\{ks\})$ and $t_{ij}(\mathbf{G}) = t_{ij}(\mathbf{G}\setminus\{ks\})$ whenever a supplier-customer pair $ks \in \mathbf{G}$ does not reach an agreement. In other words, if k and s do not reach an agreement, then p_{ij} and t_{ij} would be as if k and s had not established a relationship to begin with.

Given the pairwise contracts, the net profit of firm $i \in \{1, \dots, n\}$ in production network $\mathbf{G}$ is

$$\phi_i(\mathbf{G}) = \pi_i(\mathbf{G}) - \sum_{j:ij\in\mathbf{G}} c_{ij} - \sum_{k:ki\in\mathbf{G}} s_{ki}, \tag{2}$$

where the last two terms represent the fixed costs incurred by i to serve as, respectively, a customer to and a supplier of other firms in the economy, and $\pi_i(\mathbf{G})$ denotes i's gross profit, given by

$$\pi_i(\mathbf{G}) = \sum_{k:ki\in\mathbf{G}} p_{ki}(\mathbf{G})x_{ki} - \sum_{j:ij\in\mathbf{G}} p_{ij}(\mathbf{G})x_{ij} - wL_i + \sum_{k:ki\in\mathbf{G}} t_{ki}(\mathbf{G}) - \sum_{j:ij\in\mathbf{G}} t_{ij}(\mathbf{G}). \tag{3}$$

The first term on the right-hand side of (3) represents i's revenue from selling its product to other firms at unit prices $p_{ki}(\mathbf{G})$, the next two terms capture i's expenditure on inputs and labor, and the last two terms capture the lump-sum transfers to i from its supply chain partners and vice versa. The net and gross profits of the final good producer $i = 0$ can also be expressed similarly using (2) and (3) by replacing the first term on the right-hand side of (3) with sales to the household.[11]

Bargaining. We assume that any two firms that have established a relationship engage in Nash bargaining to determine their pairwise contract, taking all other contracts and the demand faced by the customer firm as given. More specifically, we assume that the pairwise contract $(p_{ij}(\mathbf{G}), t_{ij}(\mathbf{G}))$ between the customer-supplier pair $ij \in \mathbf{G}$ is given by

$$(p_{ij}(\mathbf{G}), t_{ij}(\mathbf{G})) = \underset{(p_{ij},t_{ij})\in O_{ij}(\mathbf{G})}{\arg\max} \Big[\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\})\Big]^{\theta_i}\Big[\pi_j(\mathbf{G}) - \pi_j(\mathbf{G}\setminus\{ij\})\Big]^{\theta_j}, \tag{4}$$

where $\theta_i, \theta_j \geq 0$ parameterize the firms' respective bargaining powers, $\pi_i(\mathbf{G})$ denotes the gross profit of firm i when the production network is $\mathbf{G}$ (given in (3)), $\pi_i(\mathbf{G}\setminus\{ij\})$ denotes i's profit in the case of "disagreement" or breakdown of negotiations with j, and $O_{ij}(\mathbf{G})$ is the set of individually rational bargaining outcomes:

$$O_{ij}(\mathbf{G}) = \{(p_{ij}, t_{ij}) : \pi_i(\mathbf{G}) \geq \pi_i(\mathbf{G}\setminus\{ij\}), \pi_j(\mathbf{G}) \geq \pi_j(\mathbf{G}\setminus\{ij\})\}. \tag{5}$$

A few remarks are in order. First, even though i and j take all other contracts as given, the fact that those contracts are contingent on the production network means that the lump-sum transfers and

[11] The expression in (2) needs to incorporate fixed costs incurred by firm i even when those costs do not lead to the creation of a link in $\mathbf{G}$ (for example, because the corresponding counterparty refused to pay its share to form the link). Since such outcomes do not occur in equilibrium, we do not include the corresponding terms for notational simplicity.

unit prices in all other pairwise relationships in $\mathbf{G}$ may adjust depending on whether i and j reach an agreement or not. For example, the unit price that i charges a customer k changes from $p_{ki}(\mathbf{G})$ to $p_{ki}(\mathbf{G} \setminus \{ij\})$ in case of disagreement between i and j. This is in contrast to the Nash-in-Nash bargaining solution, which assumes that each bilateral pair bargains as if all other negotiated contracts do not adjust in response to a bargaining disagreement (Horn and Wolinsky, 1988; Collard-Wexler et al., 2019). Instead, our specification of the disagreement points in (4) is similar to Jackson and Wolinsky (1996) and Stole and Zwiebel (1996), who assume that contracts are renegotiated upon any disagreement.

Second, the bargaining solution in (4) is well-defined only if the set of possible bargaining outcomes $O_{ij}(\mathbf{G})$ is nonempty. When this set is empty, there are no gains from trade and hence i and j cannot trade with one another at $t = 2$, even though both firms paid the requisite fixed costs at $t = 0$.

Third, while the specification of the bargaining outcome in (4) allows for heterogenous bargaining powers, it assumes that firm i's bargaining power parameter, $\theta_i \geq 0$, remains the same irrespective of the identity of the partner it is negotiating with.[12]

Fourth, we could have equivalently expressed (4) in terms of net profits ϕ_i and ϕ_j, rather than gross profits π_i and π_j, since the fixed costs for establishing supplier-customer relationships are sunk by the time the two firms bargain at $t = 1$.

Finally, we assume that the final good producer firm 0 makes a take-it-or-leave-it offer to the household with a fixed price (and no lump-sum transfers). The presence of the competitive fringe of final good producers implies that the price offered by firm 0 is always equal to the marginal cost of the competitive fringe.

Discussion. We conclude this section with a brief discussion of the model and its various ingredients.

First, it is worth highlighting a key difference between our framework and that of Acemoglu and Azar (2020), who also develop a model of endogenous production networks. Acemoglu and Azar (2020) assume that markets are contestable, in the sense that any given production technology is accessible to a large number of firms at no cost. This assumption makes their model more appropriate for the study of endogenous networks at the sectoral level, as firms can enter a given sector without any entry barriers and can use any input without incurring relationship-specific fixed costs. In contrast, our framework explicitly incorporates relationship-specific investments at the firm-level with prices and transfers that are shaped by outcomes of pairwise negotiations over the supply chain.

The relationship-specific nature of supplier-customer interactions also distinguishes our model from that of Baqaee and Farhi (2021), who assume that individual firms are infinitesimal and that the mass of entrants and the number of links adjust smoothly in response to changes in primitives. In contrast, firms in our framework need to bargain over the division of the relationship-specific surplus and take into account how their choices of suppliers and customers shape the production network structure, prices, quantities, and ultimately their profits.

The assumption that firms bargain over network-contingent contracts is akin to allowing them to renegotiate the rest of their relationships in case negotiations with a particular supplier or customer

[12] In a strategic foundation of this bargaining mechanism, $(\theta_0, \dots, \theta_n)$ map to the frequency with which each firm is recognized to make offers to other firms (Navarro and Perea, 2013). In the special case that all firms have the same bargaining powers, the specification of the bargaining problem in (4) reduces to that of Jackson and Wolinsky (1996).

break down (unlike the Nash-in-Nash bargaining solution). For example, it allows a firm to renegotiate a contract with a supplier if it cannot reach an agreement with a supplier of a complementary input or with its customers. As we show in Theorem 2 below, this form of network-contingency—together with the assumption that firms bargain over two-part tariffs—ensures that, with an exogenous production network, the resulting prices and transfers induce an efficient allocation.

3 Exogenous Production Networks

In this section, we analyze the economy with a given production network. Namely, we focus on time periods $t \in \{1, 2\}$, define a solution (equilibrium) concept, establish equilibrium existence and uniqueness, and characterize the equilibrium. We then provide a series of comparative static results. In the next section, we extend the analysis to $t = 0$, thus allowing for the endogenous formation of the production network.

3.1 Solution Concept

We start by defining the solution concept when the production network $\mathbf{G}$ is exogenously given. Recall that given the network of customer-supplier relationships, firms engage in bilateral bargaining at $t = 1$ to determine the contracts that govern their relationship at $t = 2$, which is when production and consumption take place. This timing lends itself to the following natural definition of equilibrium.

Definition 1. Given $\mathbf{G}$, an *exogenous network equilibrium* is a collection of prices and transfers $(p_{ij}(\tilde{\mathbf{G}}), t_{ij}(\tilde{\mathbf{G}}))_{ij\in\tilde{\mathbf{G}}}$ and quantities $((x_{ij}(\tilde{\mathbf{G}}))_{ij\in\tilde{\mathbf{G}}}, (y_i(\tilde{\mathbf{G}}), L_i(\tilde{\mathbf{G}}))_{i\in N}, Y(\tilde{\mathbf{G}}))$ for all $\tilde{\mathbf{G}} \subseteq \mathbf{G}$ such that

(i) given any $\tilde{\mathbf{G}} \subseteq \mathbf{G}$ and $(p_{ij}(\tilde{\mathbf{G}}), t_{ij}(\tilde{\mathbf{G}}))_{ij\in\tilde{\mathbf{G}}}$, firms minimize production costs while meeting their output obligations to customers, and the household chooses consumption to maximize utility subject to its budget constraint;

(ii) given any $\tilde{\mathbf{G}} \subseteq \mathbf{G}$ and $(p_{ij}(\tilde{\mathbf{G}}), t_{ij}(\tilde{\mathbf{G}}))_{ij\in\tilde{\mathbf{G}}}$, all markets clear;

(iii) the contract $(p_{ij}(\tilde{\mathbf{G}}), t_{ij}(\tilde{\mathbf{G}}))$ is the solution to the bargaining problem in (4) for all $ij \in \tilde{\mathbf{G}}$ and all $\tilde{\mathbf{G}} \subseteq \mathbf{G}$, given all other contracts.

The quantity restrictions imposed by equilibrium conditions (i) and (ii) are standard and ensure that firms and the representative household optimize and markets clear at $t = 2$ given prices and lump-sum transfers negotiated at $t = 1$. Specifically, market clearing for the intermediates, the final good, and labor in condition (ii) take the following familiar forms

$$y_j(\tilde{\mathbf{G}}) = \sum_{i:ij\in\tilde{\mathbf{G}}} x_{ij}(\tilde{\mathbf{G}})$$
$$y_0(\tilde{\mathbf{G}}) = Y(\tilde{\mathbf{G}}) + \sum_{ij\in\tilde{\mathbf{G}}} (c_{ij} + s_{ij})$$
$$L(\tilde{\mathbf{G}}) = \sum_{i\in N} L_i(\tilde{\mathbf{G}}).$$

Finally, condition (iii) in Definition 1 imposes that each pairwise price-transfer combination is the outcome of Nash bargaining, given all other contracts.

Definition 1 requires that conditions (i)–(iii) to hold not only for $\mathbf{G}$ but also for all its potential subnetworks $\tilde{\mathbf{G}} \subseteq \mathbf{G}$. In other words, quantities, prices, and transfers have to be mutually consistent not only when all supplier-customer pairs reach an agreement with one another but also when some of them may not. This requirement is a reflection of the assumption that firms' pairwise contracts can be contingent on the set of pairwise agreements in the economy. As a consequence, the equilibrium prices and transfers in $\mathbf{G}$ depend on the outcome of negotiations in subnetworks of $\mathbf{G}$ with one fewer edge. This can be seen from equation (4), where the outcome of bargaining between i and j in $\mathbf{G}$ depends also on prices and transfers they agree on with their other supply chain partners in $\mathbf{G} \setminus \{ij\}$. This indicates that the equilibrium notion in Definition 1 is defined recursively over production networks: to solve for equilibrium prices and transfers, one needs to start with networks consisting of only a single supplier-customer relationship and then increasing the number of negotiating pairs one at a time.[13]

3.2 Existence, Uniqueness, and Characterization

We now prove that an equilibrium exists and is (generically) unique and provide a characterization in terms of model primitives. To do this most economically, we first introduce the following concept.

Definition 2. The *aggregate (gross) productivity*, $\mathcal{A}(\mathbf{G})$, of production network $\mathbf{G}$ is the maximum amount of the final good the economy can produce per unit of labor.

That is, if a social planner were to take the production network as given and choose the allocation $((x_{ij}(\mathbf{G}))_{ij\in\mathbf{G}}, (y_i(\mathbf{G}), L_i(\mathbf{G}))_{i\in N}, Y(\mathbf{G}))$ to maximize the production of the final good, then firm 0's output would be equal to $y_0(\mathbf{G}) = \mathcal{A}(\mathbf{G})L$, where L is aggregate labor supply. This implies that $\mathcal{A}(\cdot)$ is a model primitive that is defined independently of the household's and firms' decisions and parametrizes the economy's production possibility frontier as a function of its production network structure.[14] An immediate consequence of Definition 2 is that $\mathcal{A}(\mathbf{G}_1) \leq \mathcal{A}(\mathbf{G}_2)$ whenever $\mathbf{G}_1 \subseteq \mathbf{G}_2$: the social planner can never produce less output when the set of supplier-customer relationships expands. When there is no risk of confusion, we refer to $\mathcal{A}(\cdot)$ as the economy's aggregate productivity.

Next, note that an exogenous network equilibrium in the sense of Definition 1 would not exist if the fixed costs $(c_{ij}, s_{ij})_{ij\in\mathbf{G}}$ required for establishing all customer-supplier relationships in $\mathbf{G}$ exceeded the total output the economy is capable of producing. We therefore restrict our attention to *feasible* production networks, in which the total cost of establishing supplier-customer relationships falls inside the economy's production possibility frontier:

$$\sum_{ij\in\mathbf{G}} (c_{ij} + s_{ij}) \leq \mathcal{A}(\mathbf{G})L.$$

Finally, given a production network $\mathbf{G}$, we let $\mathbf{G}|_T = \{ij \in \mathbf{G} : i, j \in T\}$ denote the subnetwork of $\mathbf{G}$ obtained by only retaining supplier-customer relationships where both firms belong to $T \subseteq N$.

[13] Restriction (iii) in Definition 1 is thus a generalization of the "recursive Nash-in-Nash" bargaining solution of Yu and Waehrer (2019) to networks with directed edges, unequal bargaining powers, and micro-founded surpluses generated from firm production and household consumption.

[14] In Appendix A.1, we express the dependence of $\mathcal{A}(\cdot)$ on the production network and firms' production technologies.

Theorem 1. *Suppose production network* $\mathbf{G}$ *is feasible. Then,*

(a) there exists an exogenous network equilibrium in which all supplier-customer pairs $ij \in \mathbf{G}$ *reach an agreement. The resulting equilibrium prices and quantities are generically unique;*

(b) all input producers price at marginal cost;

(c) the equilibrium gross profit of firm i *is given by*

$$\pi_i(\mathbf{G}) = \theta_i \sum_{T \subseteq N \setminus \{i\}} \psi_i(T) \left[\mathcal{A}(\mathbf{G}|_{T \cup \{i\}}) - \mathcal{A}(\mathbf{G}|_T) \right] L, \tag{6}$$

where $\mathcal{A}(\cdot)$ *is the economy's aggregate productivity and*

$$\psi_i(T) = \sum_{R \supseteq T \cup \{i\}} \frac{(-1)^{|R|-|T|+1}}{\sum_{k \in R} \theta_k} \geq 0. \tag{7}$$

In addition to establishing existence and (generic) uniqueness, Theorem 1 provides a characterization of equilibrium prices and profits in terms of model primitives. Statement (b) of the theorem shows that, in all pairwise supplier-customer relationships, the supplier always prices at marginal cost, that is, $p_{ij}(\mathbf{G}) = \mathrm{mc}_j(\mathbf{G})$ for all $ij \in \mathbf{G}$, where $\mathrm{mc}_j(\mathbf{G})$ denotes j's marginal cost in production network $\mathbf{G}$. This is a consequence of the pairwise efficiency of the Nash bargaining solution: setting the price equal to the supplier's marginal cost maximizes the total surplus generated between the pair. The two firms then split the resulting surplus via a lump-sum transfer $t_{ij}(\mathbf{G})$ from the customer to the supplier. These pairwise transfers are what determine firms' equilibrium profits, which are characterized in statement (c) of the theorem.

The characterization in equations (6) and (7) expresses gross profits in terms of the production network structure, the economy's aggregate productivity, and bargaining powers. According to (6), the equilibrium profit of firm i is a weighted sum of i's marginal contributions to aggregate productivity in various subnetworks of $\mathbf{G}$, $\mathcal{A}(\mathbf{G}|_{T \cup \{i\}}) - \mathcal{A}(\mathbf{G}|_T)$, with the corresponding weights determined by firms' bargaining powers $(\theta_0, \ldots, \theta_n)$ in (7). Thus, for example, firm i makes zero profits if it has no bargaining power ($\theta_i = 0$) or if it does not contribute to aggregate productivity (i.e., $\mathcal{A}(\mathbf{G}|_{T \cup \{i\}}) = \mathcal{A}(\mathbf{G}|_T)$ for all $T \subseteq N$). More generally, firm i's equilibrium profit depends on the network structure and the extent to which this firm contributes to aggregate productivity in the various supply chains it belongs to.

It is worth highlighting that the characterization in Theorem 1(c) mirrors the expressions for the (weighted) Myerson value from cooperative game theory (Slikker et al., 2005). In a cooperative game with transferable utilities, the Myerson value—which is a network-adjusted generalization of the Shapley value—assigns a share of the total surplus generated by coalitions of players to members of that coalition based on how those players are connected to one another over a network (Jackson and Wolinsky, 1996). The similarity between the expression in (6) and the Myerson value stems from our assumption that firms bargain efficiently using two-part tariffs. This implies that, given an exogenous production network $\mathbf{G}$, the general equilibrium economy of Section 2 has a reduced-form representation that parallels a cooperative game with transferable utilities.

Finally, even though Theorem 1 establishes that equilibrium prices and quantities are (generically) unique, there may be more than one collection of pairwise lump-sum transfers that are consistent with

Figure 1. Horizontal and vertical production networks

Notes: Each vertex corresponds to a firm, with a directed edge present from one vertex to another if the two firms have entered into a supplier-customer relationship. The vertex indexed 0 represents the final good producer.

equilibrium. Nonetheless, by part (c) of the theorem, all such transfers result in the same profits, given by the expression in (6).

Before exploring the implications of Theorem 1, we discuss two illustrative examples.

Example 1. Consider the horizontal production network, $\mathbf{G} = \{01, 02\}$, in Figure 1(a), where firms 1 and 2 can serve as input suppliers of the final good producer firm 0. Applying Theorem 1 to this economy, the gross profit of firm 1 is given by

$$\pi_1(01, 02) = \frac{\theta_1}{\theta_0 + \theta_1}(\mathcal{A}(01) - \mathcal{A}(\varnothing))L + \frac{\theta_1}{\theta_0 + \theta_1 + \theta_2}\left(\mathcal{A}(01, 02) - \mathcal{A}(01) - \mathcal{A}(02) + \mathcal{A}(\varnothing)\right)L, \tag{8}$$

where $\mathcal{A}(01, 02)$ represents aggregate productivity if firm 0 has access to inputs from both of its potential suppliers; $\mathcal{A}(01)$ and $\mathcal{A}(02)$ denote aggregate productivity if firm 0 has access to only a single input; and $\mathcal{A}(\varnothing)$ is the corresponding productivity when firm 0 produces by only using labor.

Note that $\mathcal{A}(01) - \mathcal{A}(\varnothing)$ is the marginal contribution of firm 1 to aggregate productivity when it is the only supplier of firm 0. Therefore, the first term on the right-hand side of (8) represents the share of this surplus that goes to firm 1 as a result of bargaining with firm 0. The second term on the right-hand side of (8) captures how firm 1's profit changes as a function of firm 2's contribution to aggregate productivity, which is then split based on 1's relative bargaining power vis-à-vis the other two firms. Importantly, the magnitude and sign of this term depends on the extent of substitutability or complementarity between inputs 1 and 2 in the production function of firm 0. To see this, first suppose firm 2 does not contribute to aggregate productivity, so that $\mathcal{A}(01, 02) = \mathcal{A}(01)$ and $\mathcal{A}(02) = \mathcal{A}(\varnothing)$. It is then immediate that the second term on the right-hand side of (8) is equal to zero. Next, suppose good 2 is a perfect substitute for good 1 in the production technology of firm 0, in the sense that $\mathcal{A}(01, 02) = \mathcal{A}(01) = \mathcal{A}(02)$. In that case, it follows that the second term on the right-hand side of (8) is negative: when goods 1 and 2 are substitutes, firm 2's presence can only lower 1's profits. Finally, the last term is positive when goods 1 and 2 are complements.

Example 2. Consider the vertical production network, $\mathbf{G} = \{01, 12\}$, depicted in Figure 1(b), where firm 2 serves as an input supplier to firm 1, which then sells its product to firm 0. Theorem 1(c) leads to the following expressions for gross profits:

$$\pi_0(01,12) = \frac{\theta_0}{\theta_0+\theta_1+\theta_2}(\mathcal{A}(01,12)-\mathcal{A}(01))L + \frac{\theta_0}{\theta_0+\theta_1}(\mathcal{A}(01)-\mathcal{A}(\varnothing))L$$
$$\pi_1(01,12) = \frac{\theta_1}{\theta_0+\theta_1+\theta_2}(\mathcal{A}(01,12)-\mathcal{A}(01))L + \frac{\theta_1}{\theta_0+\theta_1}(\mathcal{A}(01)-\mathcal{A}(\varnothing))L$$
$$\pi_2(01,12) = \frac{\theta_2}{\theta_0+\theta_1+\theta_2}(\mathcal{A}(01,12)-\mathcal{A}(01))L.$$

These expressions have intuitive interpretations. The second terms in π_0 and π_1 give the division of surplus generated by firms 0 and 1 in the absence of firm 2, whereas the first terms in all three expressions captures the division of the extra surplus generated by adding firm 2 to the supply chain.

Note that firm 1 earns a positive gross profit even if it does not contribute to aggregate productivity by itself (that is, even if $\mathcal{A}(01) = \mathcal{A}(\varnothing)$): it earns a strictly positive profit as long as firm 2's presence raises aggregate productivity. This reflects the fact that firms in our model earn "intermediation rents": equilibrium profits reflect not only firms' own direct contribution to aggregate productivity but also the extent to which they make other firms' contributions possible.

We conclude this discussion by establishing that the exogenous network equilibrium is efficient—in the sense that it coincides with the solution of the problem of maximizing household welfare.

Theorem 2. *For any feasible production network* $\mathbf{G}$*, the exogenous network equilibrium is efficient.*

This result is a consequence of parts (a) and (b) of Theorem 1 and the first welfare theorem: given Nash bargaining and the two-part tariffs, all customer-supplier pairs in $\mathbf{G}$ reach an agreement and all inputs are priced at marginal cost, ensuring that the equilibrium allocation coincides with the efficient allocation. It is also useful to note that even though firms earn positive profits in this equilibrium, these profits are earned via lump-sum transfers, not because of markups.[15]

3.3 Comparative Statics

We next present a series of comparative static results on how changes in model primitives shape profits. These comparative static results will serve as stepping stones for the analysis of the economy when the production network is formed endogenously.

We start with two assumptions on the economy's aggregate productivity.

Assumption 1 (a) (extensive margin supermodularity). Let $\mathbf{G}_2 \supseteq \mathbf{G}_1$ be two production networks. Then,

$$\mathcal{A}(\mathbf{G}_2 \cup \{ij\}) - \mathcal{A}(\mathbf{G}_2) \geq \mathcal{A}(\mathbf{G}_1 \cup \{ij\}) - \mathcal{A}(\mathbf{G}_1) \qquad \text{for all } ij \notin \mathbf{G}_2. \tag{9}$$

Assumption 1 (b) (intensive margin supermodularity). Let $\mathbf{G}_2 \supseteq \mathbf{G}_1$ and $\bar{A}_i(I_i) \geq \underline{A}_i(I_i)$ element-wise. Then,

$$\mathcal{A}(\mathbf{G}_2; \bar{A}_i(I_i)) - \mathcal{A}(\mathbf{G}_2; \underline{A}_i(I_i)) \geq \mathcal{A}(\mathbf{G}_1; \bar{A}_i(I_i)) - \mathcal{A}(\mathbf{G}_1; \underline{A}_i(I_i)), \tag{10}$$

where $\mathcal{A}(\mathbf{G}; A_i(I_i))$ is aggregate productivity of $\mathbf{G}$ if i's productivity when using inputs in I_i is $A_i(I_i)$.

[15] Under the alternative assumption that a supplier and a customer negotiate over fixed-price contracts (as opposed to two-part tariffs), the equilibrium price would be above the supplier's marginal cost, as this would be the only way for the supplier to extract some of the realized surplus from the customer. However, this would lead to lower than efficient levels of input use and a series of double marginalizations throughout the network. See an earlier draft of the paper for a variant of the model with such contracts (Acemoglu and Tahbaz-Salehi, 2020).

Assumption 1(a) imposes supermodularity on the extensive margin of bilateral linkages: adding a customer-supplier relationship ij results in a greater increase in productivity when the production network has more pairwise supplier-customer relationships to begin with (that is, when $\mathbf{G}_2 \supseteq \mathbf{G}_1$). Assumption 1(b) is the intensive margin counterpart to Assumption 1(a): it states that increasing any of the productivities in (1) results in a larger increase in aggregate productivity in the presence of more supplier-customer relationships.

We can now state our first comparative static result:

Theorem 3. *Suppose aggregate productivity is supermodular at the extensive margin (Assumption 1(a)). If $\mathbf{G}_2 \supseteq \mathbf{G}_1$, then $\pi_i(\mathbf{G}_2) \geq \pi_i(\mathbf{G}_1)$ for all firms i.*

Therefore, as long as aggregate productivity is supermodular at the extensive margin, expanding the set of supplier-customer linkages not only raises aggregate output but also (weakly) increases all firms' gross profits. It is easy to see that such a result may not hold without Assumption 1(a): in Example 1, the second term on the right-hand side of equation (8) is negative whenever good 2 is a perfect substitute for good 1 in the production technology of firm 0. Thus, firm 1's profit would go down if firms 0 and 2 form a customer-supplier relationship. Nonetheless, as we show in Proposition A.1, Assumption 1(a) is satisfied whenever various inputs in firms' production technologies are either (gross) complements or weak substitutes.

Our next theorem establishes a similar monotonicity result in response to changes in productivities $A_i(I_i)$ in (1).

Theorem 4. *If aggregate output is supermodular at the intensive margin (Assumption 1(b)), then all firms' gross profits are weakly increasing in productivity $A_i(I_i)$.*

Thus, once again, an expansion of the production possibility frontier—this time due to an increase in productivity—translates into higher gross profits for all firms as they split the added surplus with one another via bargaining.

Our final result is a comparative static result with respect to firms' bargaining powers:

Theorem 5. *Suppose aggregate productivity is supermodular at the extensive margin (Assumption 1(a)). Then, an increase in θ_i increases the gross profit of firm i and decreases the gross profits of all other firms.*

This is an intuitive result, establishing that an increase in a firm's bargaining power allows it to extract a larger share of the total surplus at the expense of all other firms. However, for the above result to hold, the supermodularity Assumption 1(a) cannot be dispensed with. This can be seen clearly in Example 1. If goods 1 and 2 were perfect substitutes in the sense that the aggregate surplus is the same in the presence or absence of either firm (that is, $\mathcal{A}(01, 02) = \mathcal{A}(01) = \mathcal{A}(02)$), then firm 1's profit in (8) would be increasing in θ_2.

4 Endogenous Production Networks

We now turn to presenting our main results by extending the analysis to time period $t = 0$, which is when firms decide whether to incur fixed costs to form relationships with suppliers and customers.

4.1 Equilibrium

Definition 3. An *endogenous network equilibrium* consists of a production network, $\mathbf{G}$, and a collection of network-contingent prices, transfers, and quantities such that

(i) prices, transfers, and quantities form an exogenous network equilibrium;

(ii) no firm has an incentive to deviate by changing the set of its suppliers and customers.

The equilibrium notion in Definition 3 is the natural solution concept under backward induction. At $t = 0$, firms decide whether to incur the fixed costs of establishing supplier-customer relations with other firms in the economy while taking all other firms' corresponding decisions as given and taking into account how the resulting production network will affect prices and transfers at $t = 1$ and quantities at $t = 2$. Definition 3 allows for deviations along multiple supplier-customer relationships: each firm can deviate by simultaneously adding and/or dropping any mix of suppliers and customers.

Our next result shows that, as long as aggregate productivity satisfies supermodularity at the extensive margin, the set of endogenous network equilibria is nonempty and has a well-behaved structure.

Theorem 6. *If aggregate productivity is supermodular at the extensive margin (Assumption 1(a)), then*

(a) an endogenous network equilibrium exists;

(b) equilibrium production networks form a lattice with respect to the set inclusion order and thus there exists a greatest equilibrium network that contains all other equilibrium networks as subnetworks;

(c) aggregate output and all firms' net profits in the greatest equilibrium are higher than the corresponding values in all other equilibria.

The key step in establishing Theorem 6 is to show that, under supermodularity at the extensive margin, firms' decisions to establish customized relationships with potential suppliers and customers are strategic complements, and hence, the corresponding network formation game is supermodular. Statements (a) and (b) of the theorem then follow from applying standard monotone comparative static results (Milgrom and Roberts, 1990). While an equilibrium is guaranteed to exist, it is not necessarily unique, as there is room for significant miscoordination between firms. For example, whether a firm is willing to incur the fixed cost of establishing a relationship or not depends on what it anticipates its potential partner, its other suppliers, and even firms further upstream and downstream its supply chain to do. Nonetheless, the fact that the set of equilibria forms a lattice (Theorem 6(b)) implies that there exists a greatest equilibrium that does not exhibit any coordination failures. In particular, there exists an equilibrium network $\mathbf{G}^*$ such that such that $\mathbf{G}^* \supseteq \mathbf{G}$ for all other equilibrium networks $\mathbf{G}$. This equilibrium will be the focus of the rest of our analysis.[16]

[16]Another consequence of the supermodularity of the network formation game and the lattice structure of equilibria is that one can compute the economy's greatest equilibrium using a simple iterative algorithm. Starting with the production network that contains all possible pairwise supplier-customer relationships, best response dynamics is guaranteed to converge to the greatest equilibrium in finitely many steps (Milgrom and Roberts, 1990, Theorem 8).

Finally, statement (c) of Theorem 6 follows from the fact that the game of establishing supplier-customer relationships is not only supermodular but also exhibits positive spillovers: from Theorem 3, each firm's profit can only increase if the set of supplier-customer relationships formed between other firms expands. This implies that the greatest equilibrium is (i) Pareto preferred to all other equilibria and (ii) generates the highest possible equilibrium output.

4.2 Comparative Statics

With equilibrium definition and existence result at hand, we next characterize the micro and macroeconomic impacts of shocks to fixed costs and productivities when the production network can adjust endogenously.

Theorem 7. *Suppose aggregate productivity is supermodular at the extensive margin (Assumption 1(a)). Then, in the economy's greatest equilibrium, any increase in fixed costs*

(a) shrinks the set of equilibrium supplier-customer linkages;

(b) decreases the net profits of all firms;

(c) decreases aggregate output.

A higher fixed cost of establishing and/or maintaining a supplier-customer relationship between a pair of firms, say firms i and j, makes it more likely for at least one of the firms to forgo that particular relationship. The resulting change in the production network makes all other relationships in the economy more likely to dissolve (or not form). This is because the breakdown of the relationship between i and j shrinks the economy's production possibility frontier, which in turn reduces the value of all other supplier-customer relationships to the rest of the firms. The breakdown of these relationships manifests itself as lower aggregate output and lower profits for all firms.

The next result is the counterpart to Theorem 7 for productivity shocks.

Theorem 8. *Suppose aggregate productivity is supermodular at both the extensive and intensive margins (Assumptions 1(a) and 1(b)). Then, in the economy's greatest equilibrium, a decrease in productivity* $A_i(I_i)$

(a) shrinks the set of equilibrium supplier-customer linkages;

(b) decreases the net profits of all firms;

(c) decreases aggregate output.

The intuition underlying this result parallels that of Theorem 7. A reduction in productivities shrinks the production possibility frontier directly and also reduces the marginal values of all supplier-customer relationships to other firms. This makes it less likely for firms to be willing to incur the fixed costs of maintaining the relationships with their suppliers and customers, induces a contraction in the equilibrium production network, and results in a further decline in productivity.

An implication of Theorem 8 is that the distribution of supplier-customer relationships in our model is procyclical. Firms tend to form and maintain relations with more suppliers and customers during periods of high productivity. These relationships then dissolve endogenously in response to productivity slowdowns, acting as an amplification mechanism and potentially translating small declines in productivity into larger aggregate effects.[17] We explore this mechanism and its implications in more detail in Section 5.

4.3 Efficiency and Fragility

Theorems 7(a) and 8(a) show that greater costs of forming supplier-customer relationships or lower productivity of these relationships lead to breakdown of supplier-customer relationships. However, these results are silent about whether such contractions in the economy's supply chain are an efficient response to changing circumstances or are "excessive" from a social welfare perspective. We now answer this question by comparing the behavior of equilibrium and efficient production networks. Our notion of efficiency is a direct extension of the one we used for the exogenous network economy—a social planner maximizes household welfare by deciding which supplier-customer relationships to activate as well as setting the optimal quantity of inputs along the chosen production network. In general, there may be multiple production networks that support the same (maximum) level of aggregate output. Nonetheless, the following lemma shows that, when aggregate productivity is supermodular, there always exists a greatest efficient production network that contains all other efficient networks as subnetworks.

Lemma 1. *If aggregate productivity is supermodular at the extensive margin (Assumption 1(a)), then there exists a greatest efficient production network.*

Theorem 9. *Suppose aggregate productivity is supermodular at the extensive margin (Assumption 1(a)), and let $\mathbf{G}^*$ and $\mathbf{G}^{\text{eff}}$ denote the greatest equilibrium and the greatest efficient production networks, respectively. Then, $\mathbf{G}^* \subseteq \mathbf{G}^{\text{eff}}$.*

Theorem 9 establishes that the equilibrium production network is always a subnetwork of the greatest efficient production network. When coupled with the fact that exogenous network equilibria are always efficient (Theorem 2), this result implies that any equilibrium inefficiency is a consequence of firms' underinvestment in establishing supplier-customer relationships with other firms in the economy. The reason for this underinvestment is a familiar hold-up problem: even though supplier-customer pairs efficiently bargain over any surplus they generate *after* forming relationships, the investments necessary for forming those relationships are sunk by the time they bargain. As a result, firms do not fully internalize the contribution of their relationship-specific investments to other firms' profits. The following example illustrates this inefficiency, its origin, and its consequences.

Example 3. Consider an economy in which (i) all but one customer-supplier relationship, ij, can either be formed at no cost or not at all, meaning that $c_{kr} = s_{kr} \in \{0, \infty\}$ for all $kr \neq ij$; and (ii) the relationship ij only requires a fixed cost from the supplier, so that $s_{ij} > 0$ and $c_{ij} = 0$. Given these assumptions, it is immediate that the efficient production network includes the link between i and j if and only if the

[17]This is in line with the findings of Xu et al. (2023), who document that the aggregate number of suppliers in the United States is procyclical.

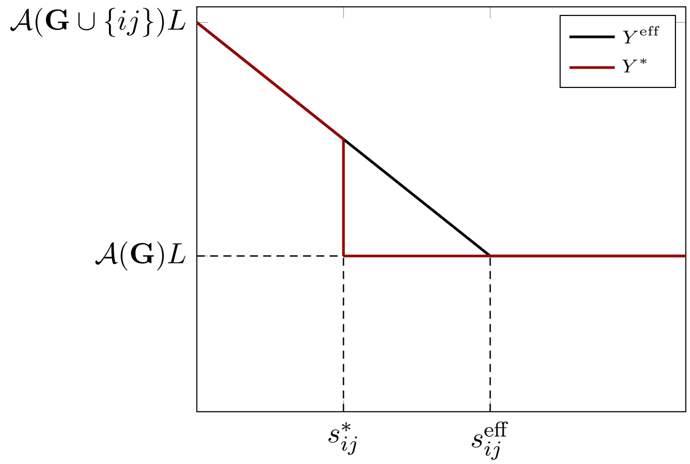

Figure 2. Fragility of equilibrium supply chains

Notes: This figure plots the dependence of efficient and equilibrium aggregate output in Example 3 as a function of the relationship-specific cost s_{ij}. The expressions for efficient and equilibrium aggregate output are given by (11) and (12), respectively.

link's contribution to aggregate output exceeds the corresponding fixed cost. Thus, efficient aggregate output is

$$Y^{\text{eff}} = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ij\})L - s_{ij} & \text{if } s_{ij} \leq s_{ij}^{\text{eff}} \\ \mathcal{A}(\mathbf{G})L & \text{if } s_{ij} > s_{ij}^{\text{eff}}, \end{cases} \tag{11}$$

where $s_{ij}^{\text{eff}} = (\mathcal{A}(\mathbf{G} \cup \{ij\}) - \mathcal{A}(\mathbf{G}))L$ and $\mathbf{G} = \{kr \neq ij : c_{kr} = s_{kr} = 0\}$ is the network consisting of all links that can be formed at no cost.

Turning to the equilibrium, firm j incurs the cost of establishing a relationship with firm i only if the resulting change in j's gross profit exceeds the cost, that is, $\pi_j(\mathbf{G} \cup \{ij\}) - \pi_j(\mathbf{G}) \geq s_{ij}$. Therefore, equation (6) in Theorem 1 implies that equilibrium aggregate output is

$$Y^* = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ij\})L - s_{ij} & \text{if } s_{ij} \leq s_{ij}^* \\ \mathcal{A}(\mathbf{G})L & \text{if } s_{ij} > s_{ij}^*, \end{cases} \tag{12}$$

where

$$s_{ij}^* = \theta_j \sum_{T \subseteq N \backslash \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G} \cup \{ij\}|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}\,|_{T \cup \{j\}}) \right] L.$$

Figure 2 plots the expressions in (11) and (12) as a function of s_{ij}.

We make three observations. First, as long as the supermodularity Assumption 1(a) holds strictly, then $s_{ij}^* < s_{ij}^{\text{eff}}$. Therefore, as established in Theorem 9, the equilibrium production network is a (strict) subnetwork of the efficient one. This is because j's investment is (i) relationship-specific and (ii) sunk by the time firms bargain over prices and transfers. As a result, unless firm j holds all the bargaining power, it does not fully capture all the increase in aggregate productivity its relationship with i creates, leading to underinvestment.

Second, comparing equations (11) and (12) illustrates that whereas the efficient level of aggregate output decreases continuously in s_{ij}, equilibrium aggregate output experiences a discontinuous drop as s_{ij} crosses s_{ij}^*. Thus, the equilibrium production network is not only inefficient but also excessively fragile, in the sense that small shocks can have sizable effects on aggregate output. Notably, the discontinuity in equilibrium aggregate output arises exactly when firm j severs its relationship with firm i.

Finally, note that the drop in equilibrium aggregate output at the point of discontinuity is equal to $s_{ij}^{\text{eff}} - s_{ij}^*$. Thus, the extent of equilibrium fragility is closely tied to the wedge between the equilibrium and the efficient outcomes.

Our next result demonstrates that the link between (in)efficiency and fragility goes beyond the simple economy in Example 3.

Theorem 10. *Let Y^{eff} and Y^* be the efficient and equilibrium levels of aggregate output, respectively.*

(a) Y^{eff} is continuous in productivities and fixed costs.

(b) Generically, Y^* changes discontinuously in response to changes in productivities and fixed costs, whenever these alter the equilibrium production network.

Taken together, the two parts of Theorem 10 illustrate that, in our model, inefficiency and fragility are closely tied, as in Example 3. Since firms do not fully internalize the surplus they generate by entering into customer-supplier relationships, they may inefficiently drop one or more of their productive partners in response to small changes in the economic environment. This creates a discontinuous (sharp) contraction in aggregate output.

The source of the fragility outlined above is distinct from that of Elliott et al. (2022), who show that a supply chain with (i) strong input complementarities and (ii) infinitely many stages of production can experience discontinuous phase transitions in response to shocks. Supply chain fragility in their model arises from the fact that an infinitely long supply chain with multiple essential inputs can operate only if, for each input, at least one chain of undisrupted producers continues upstream indefinitely. Because the discontinuities in Elliott et al. (2022) are driven by the nature of the production technology, they exist both in equilibrium and the planner's solution. In contrast, the discontinuities in our model rely neither on essentiality of any particular intermediate input nor on infinitely deep supply chains. Furthermore, in our model, the efficient level of aggregate output is always continuous in the shocks, irrespective of the complexity or the depth of the production processes in the economy (Theorem 10(a)). Fragility is thus purely an equilibrium phenomenon, driven by firms' endogenous decisions to sever their relationships with suppliers and customers (Theorem 10(b)).

5 Supply Chain Fragility

In this section, we explore several implications of the fragility of equilibrium supply chains established in Theorem 10.

(a) Vertically integrated supply chain relationship

(b) Fragmented supply chain relationship

Figure 3. Supply chain fragmentation

Notes: This figure plots the supplier-customer relationship between firms j and i (left panel) and its fragmentation (right panel). For simplicity, the figure does not depict the rest of the production network.

5.1 Fragmentation and Fragility of Supply Chains

We start by showing that while more fragmented supply chains can increase productivity, they can simultaneously make the economy more vulnerable to sharp declines in output. Put differently, productivity gains of fragmenting supply chains may come at the cost of supply chain resilience.

To illustrate these ideas concisely, we compare two distinct economies with different degrees of fragmentation in their supply chains. Consider a pair of firms j and i, where the latter purchases the former's output, processes it in-house (for example, by combining it with labor and other inputs), and transforms it into another product. Next, consider a fragmentation of this relationship, where firm i outsources part of the processing of j's output to firm k, which then sells this partially-processed good as an input to i. Figure 3 illustrates the two alternative architectures (where for simplicity we are not depicting the rest of the production network).

To ensure that the only difference between the two supply chain architectures is in the fragmentation of i's relationship with j, we assume that the rest of the production network—which we denote by $\mathbf{G}$—remains unchanged, that in either case firm j can serve as a supplier only if it incurs a fixed cost of $s_{ij} = s_{kj} = s$, and that the relationship between firms i and k can be formed at no cost. Our key assumption is the following:

$$\mathcal{A}(\mathbf{G} \cup \{ik, kj\}) > \mathcal{A}(\mathbf{G} \cup \{ij\}) > \mathcal{A}(\mathbf{G}) = \mathcal{A}(\mathbf{G} \cup \{ik\}). \tag{13}$$

The first inequality captures the idea that fragmenting i's relationship with j increases productivity, whereas the equality on the right-hand side implies that firm k's only contribution to productivity is to process j's product. Let us denote the equilibrium output in the vertically integrated and fragmented economies by Y_{int} and Y_{frg}, respectively. We have the following result.

Proposition 1. *If (13) is satisfied, then, for large enough values of θ_k, there exists s^*_{frg} such that $Y_{\text{frg}} > Y_{\text{int}}$ for $s \leq s^*_{\text{frg}}$ and $Y_{\text{frg}} < Y_{\text{int}}$ for $s > s^*_{\text{frg}}$.*

This proposition implies that fragmentation can contribute to productivity while simultaneously increasing fragility: a small shock to s in the more fragmented economy can result in a sharp decline

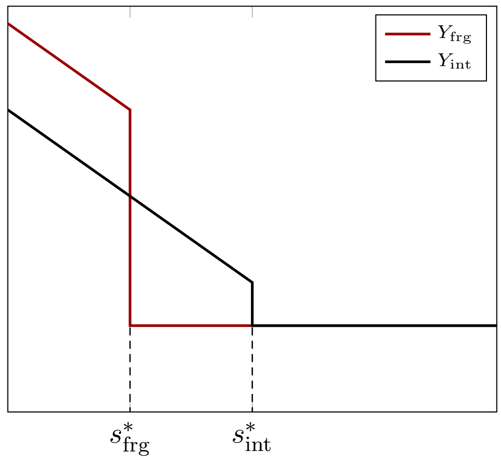

Figure 4. Fragmentation and fragility

Notes: This figure plots equilibrium aggregate output for the fragmented (red line) and vertically integrated (black line) supply chain relationship between firms i and j, as depicted in Figure 3. The horizontal axis is the fixed cost s that firm j needs to pay to establish a relationship with either i or k.

in aggregate output, pushing it below its corresponding value in the less fragmented economy. Figure 4 illustrates the content and the intuition behind this result. The assumption that fragmentation increases productivity implies that, for small values of s, aggregate output in the fragmented economy is higher than that of the vertically integrated economy. However, as in Theorem 10(b), aggregate output is prone to discontinuous drops, as firms may respond to negative shocks by severing their relationships with suppliers and customers. The key observation here is that, when firm k can extract a large enough share of the surplus, the threshold of discontinuity in the fragmented architecture can be to the left of the threshold in the integrated architecture, and the two curves cross one another (see Figure 4). Hence, the fragmented economy is more fragile: it experiences a sharper and an earlier drop in aggregate output as j's fixed cost of setting up customer-supplier relationships increases.

5.2 Supply Chain Cascades

Another consequence of the fragility of the equilibrium production network is that small shocks may trigger the dissolution of multiple supplier-customer relationships, generating systemic supply-chain disruptions. Such cascading breakdowns can function as an amplification mechanism, translating small shocks into potentially larger aggregate effects.

To illustrate this possibility in a transparent manner, we focus on a vertical supply chain consisting of firms $N = \{0, \ldots, n\}$, where each firm $i \neq n$ has the option to either (i) use an in-house production technology to transform labor, one-for-one, into its product or (ii) outsource part of its production to its designated supplier, $i+1$, raising its productivity by factor $A \geq 1$. In the language of equation (1):

$$y_i = F_i(I_i, A_i(I_i), L_i, X_i) = \begin{cases} L_i & I_i = \varnothing \\ Ax_{i,i+1} & I_i = \{i+1\}. \end{cases}$$

Consequently, the depth of the equilibrium supply chain is determined by the decision of firms to whether outsource part of their production process to their suppliers. Figure 1(b) depicts such an economy when the depth of the supply chain is 3. We additionally assume that all firms have identical bargaining powers and that maintaining each supplier-customer relationship requires a fixed cost of $s_{i,i+1} = s$ paid by the supplier, with no additional cost to the customer.

Proposition 2. *If* $\log A > 1/2$*, then there exists* s_n^* *such that*

$$k^* = \begin{cases} n+1 & \textit{if } s \leq s_n^* \\ 1 & \textit{if } s > s_n^* \end{cases} \qquad \textit{and} \qquad Y^* = \begin{cases} A^n L - ns & \textit{if } s \leq s_n^* \\ L & \textit{if } s > s_n^*, \end{cases}$$

where Y^* *and* k^* *denote, respectively, aggregate output and the depth of the equilibrium supply chain.*

Though intentionally stark, this result illustrates the possibility that small shocks can result in widespread supply chain disruptions. As long as fixed costs are small, all firms find it profitable to serve as suppliers to their respective customers, leading to a supply chain of depth $k^* = n + 1$. However, once s crosses a critical threshold, no firm finds it profitable to enter into a relationship with its customer, resulting in a supply chain of depth $k^* = 1$ (consisting of only the final good producer). Naturally, such a significant change in the equilibrium supply chain translates into a sizable drop in aggregate output. The expression for Y^* shows that the resulting discontinuity in aggregate output can be arbitrarily large as n increases.

It is worth reiterating that the planner's solution exhibits no discontinuity, as both the discontinuity and the supply chain breakdowns captured in Proposition 2 are consequences of equilibrium production network formation. Moreover, the sharp decline in the depth of the supply chain is due to a cascade of breakdowns—the dissolution of the relationship between firm $i + 1$ and i reduces aggregate productivity, which then triggers the breakdown of i's relationship with $i - 1$, and so on.[18]

5.3 Supply Chains and Nonlinear Amplification of Shocks

So far, we have focused on how the fragility inherent to equilibrium production networks can function as an amplification mechanism. Next, we show that the strength of this mechanism is closely tied to the shocks' size and takes a highly nonlinear form: small shocks are only marginally amplified, as they only disrupt low-productivity relationships, whereas larger shocks are significantly magnified, because they trigger the breakdown of highly productive relationships.

To illustrate this mechanism, we consider an economy with a two-tiered supply chain, consisting of $2n$ input producers and a final good producer. The top tier consists of n firms, each of which transforms labor, one-for-one, into an input that is then sold to a single firm in the bottom tier. Firm i in the bottom tier can produce by either using the input provided by its corresponding top-tier supplier or by using an in-house production technology that only uses labor as an input. Thus, in the notation of equation (1), the production function of firm i in the bottom tier is given by

$$y_i = F_i(I_i, A_i(I_i), L_i, X_i) = \begin{cases} \bar{A} L_i & I_i = \varnothing \\ \bar{A} A_i x_{i,n+i} & I_i = \{n+i\}, \end{cases}$$

[18] To see that Proposition 2 is indeed driven by a cascade of breakdowns, suppose that $s = s_n^* + \epsilon$ for some small $\epsilon > 0$, where s_n^* is the threshold at which firm n makes zero profits. Thus, firm n makes negative net profits if it maintains its relationship with $n - 1$. However, this does not mean that firm $n - 1$ is also making negative profits. In fact, for small enough values ϵ, firm $n - 1$ continues to make positive profits if n does not dissolve its relationship with $n - 1$. It is only because of n's decision to drop its relationship with $n - 1$ that the latter will also make negative profits. This logic then manifests itself as a cascade of supply chain disruptions.

where firm $n+i$ is the designated supplier of firm i, $A_i \geq 1$ parameterizes the productivity advantage of supplier-customer relationship between i and $n+i$ over in-house production (which is normalized to one), and $\bar{A}$ denotes a common TFP shock to all bottom-tier firms. The final good producer then combines the varieties produced by bottom-tiered firms using a symmetric production technology, though it can also produce each variety in-house by using labor.

To isolate the role of relationship-specific productivities A_i, we impose symmetry on the rest of the parameters. Specifically, we assume that firms in the bottom tier have identical bargaining powers, all top-tier firms have no bargaining power, all pairwise relationships have identical fixed costs $c_{ij} = c > 0$ and $s_{ij} = 0$, and the input varieties used by the final good producer have no productivity advantage over the latter's in-house production technology.

Proposition 3. *There is a decreasing sequence of thresholds $a_1 \geq \cdots \geq a_n$ such that if $\bar{A} \in (a_{k+1}, a_k)$,*

(a) the supplier-customer relationships with the k-th lowest productivities A_i break down;

(b) furthermore, aggregate productivity of the equilibrium production network is given by

$$\mathcal{A} = \bar{A} \cdot \alpha(1, \ldots, 1, A_{[k+1]}, \ldots, A_{[n]}),$$

where $A_{[r]}$ is r-th smallest of pairwise productivities A_i and α is a symmetric and increasing function.

A number of implications of this result are worth emphasizing. First, Proposition 3 encapsulates the intuitive notion that low-productivity relationships are more likely to be dissolved first in economic downturns, but more major negative shocks can lead to the breakdown of more productive relationships. Second, and as a result, the amplification of business cycle shocks via changes in the economy's production network can be highly nonlinear: whereas small shocks are only marginally amplified, larger shocks can be significantly magnified by supply chain responses. Third, and also as a consequence, evidence that there is some cleansing of lower-productivity firms and suppliers during recessions does not contradict the possibility that major supply chain disruptions, such as during the 2007–08 financial crisis or COVID pandemic, can have debilitating effects on the macroeconomy.

6 Conclusion

Complex supply chains are one of the pillars of any modern economy. They raise productivity by enabling relationship-specific investments by suppliers and customers and allowing firms to source critical, customized inputs from specialized producers. Not surprisingly, disruptions to the normal functioning of supply chains can have severe macroeconomic consequences, as they destroy accumulated relationship-specific investments and productivities. Despite the growing recognition of the importance of resilient supply chains, a systematic investigation of macroeconomic consequences of supply chain disruptions has remained largely elusive. This is in part because existing frameworks lack some of the key necessary ingredients for such an investigation. These ingredients are: (i) a firm-level network of customized supplier-customer links that generate relationship-specific productivity gains; (ii) a noncompetitive approach for splitting these relationship-specific surpluses; and (iii) an extensive margin of adjustment, whereby firms may decide to establish or sever relations with particular suppliers and/or customers.

In this paper, we develop a model that incorporates these three ingredients into a single tractable framework. We establish equilibrium existence and uniqueness and provide a number of comparative static results that show how the economy's production network and aggregate output respond to changes in link-specific productivities and fixed costs. We also prove that the equilibrium is inefficient. Importantly, this inefficiency—which is rooted in the firms' endogenous decisions to form or dissolve relationships with suppliers and customers—is the source of an inherent fragility in supply chains, whereby small shocks can lead to discontinuous changes in aggregate out. This is despite the fact that the efficient allocation is always continuous in the same shocks.

This fragility has a number of important macroeconomic implications. First, more fragmentation can increase productivity while simultaneously making the economy more vulnerable to supply chain disruptions. This suggests that complex supply chains can be a double-edge sword: they contribute to productivity growth but can also pave the way to disruptive supply chain breakdowns. Second, small shocks can trigger cascading supply chain breakdowns, magnifying the discontinuous effects of this fragility. Finally, the amplification of business cycle shocks implied by endogenous production networks can be highly nonlinear: whereas small shocks are only marginally amplified, larger shocks can be magnified significantly. This is because, all else equal, larger shocks result in the dissolution of progressively more productive supplier-customer relationships.

The tractability of our model is obviously due to a number of simplifying assumptions. Relaxing these assumptions and incorporating additional important economic forces are natural directions for future theoretical research. We list a few of these directions here.

First, we abstracted from risk and uncertainty by assuming that supplier-customer relationships are formed after the shocks are realized. In reality, supply chains are exposed to a wide range of logistical and geopolitical risks. Allowing firms to form relationships before shocks are realized—for example, as in Bimpikis et al. (2018) and Grossman et al. (forthcoming)—opens the door to answering a wider set of questions, such as the role of multisourcing in diversifying supply chain risks, the externalities induced by firms' private sourcing decisions, and evaluating policies that alter the organization of supply chains.

Second, our particular choice of pairwise contracts and Nash bargaining was purposefully made to mimic the efficient allocation. In reality, inefficiencies in bargaining can magnify the effects of supply chain disruptions and amplify macroeconomic market failures and fragilities, as already hinted at in an earlier version of this paper (Acemoglu and Tahbaz-Salehi, 2020). Investigating how different realistic contracts and bargaining protocols affect macroeconomic outcomes is another area for future study. In this context, it would also be interesting to explore how the production network could confer greater bargaining power on firms that act as production bottlenecks (for example, as in Manea (2021) or Carvalho et al. (2022)).

Third, our supermodularity assumptions ruled out the possibility that a firm can acquire highly substitutable inputs from different suppliers, effectively limiting the intensity of competition among suppliers. Allowing for such a possibility is another major direction for theoretical research.

Last but certainly not least, much more empirical work on the behavior of supply chains over the business cycle and in response to medium-term economic changes is needed. We hope that our theoretical framework and sharp results will become useful for such empirical inquiries.

A Appendix

A.1 Expressing Aggregate Productivity in Terms of Primitives

In Definition 2, we defined the aggregate (gross) productivity, $\mathcal{A}(\mathbf{G})$, of production network $\mathbf{G}$ as the maximum amount of the final good the economy can produce per unit of labor. As such, $\mathcal{A}(\cdot)$ is a model primitive that is defined independently of the household's and firms' decisions; it parametrizes the economy's production possibility frontier as a function of its production network structure. We can thus express this object as the solution to the following optimization problem:

$$\begin{aligned} \mathcal{A}(\mathbf{G})L = \max_{I_0,\ldots,I_n} \quad & y_0(I_0,\ldots,I_n) \\ \text{s.t.} \quad & I_i \subseteq \{j : ij \in \mathbf{G}\} \quad \text{for } i = 0,1,\ldots,n, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} y_0(I_0,\ldots,I_n) = \max_{y_i,x_{ij},L_i} \quad & y_0 \\ \text{s.t.} \quad & y_i = F_i(I_i, A_i(I_i), L_i, X_i) \quad \text{for } i = 0,1,\ldots,n \\ & \sum_{i:j\in I_i} x_{ij} = y_j \qquad\qquad\qquad \text{for } j = 1,2,\ldots,n \\ & \sum_{i=0}^{n} L_i = L. \end{aligned} \tag{A.2}$$

The optimization problem in (A.2) expresses the maximum level of final good the economy can produce for a given choice of firms' production technologies $(I_0,\ldots,I_n)$. The problem in (A.1) then optimizes over firms' production technologies that are feasible given production network $\mathbf{G}$. An immediate consequence of (A.1) and (A.2) is that, for a given $\mathbf{G}$, $\mathcal{A}(\mathbf{G})$ is continuous in productivity parameters, $A_i(I_i)$.

A.2 Sufficient Condition for Extensive Margin Supermodularity of Aggregate Productivity

We established the existence of endogenous network equilibria (Theorem 6) while assuming that aggregate productivity is supermodular at the extensive margin (Assumption 1(a)). In this part of the Appendix, we show that this assumption is satisfied as long as various inputs in firms' production technologies are not strong (gross) substitutes.

To express the relationship between substitution elasticities and aggregate productivity in the most transparent manner, we focus our attention on the class of economies with CES production technologies and common elasticities across all firms. Additionally, we assume that all firms have the option to replace any of their required inputs with a variant that can be produced in-house using labor, though at a lower productivity. In the notation of equation (1), the production technology of firm $i \in \{0,1,\ldots,n\}$ when using inputs $I_i \subseteq \{1,2,\ldots,n\} \setminus \{i\}$ is given by

$$y_i = \left(\alpha_i^{1/\sigma} L_i^{(\sigma-1)/\sigma} + \sum_{j\in I_i} \gamma_{ij}^{1/\sigma} (A_{ij}x_{ij} + \ell_{ij})^{(\sigma-1)/\sigma} + \sum_{j\notin I_i} \gamma_{ij}^{1/\sigma} \ell_{ij}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}. \tag{A.3}$$

In the above, σ denotes the elasticity of substitution between various inputs, $A_{ij} \geq 1$ captures the (relationship-specific) productivity gain of using the input produced by firm j instead of using labor for in-house production, and $\alpha_i + \sum_{j=1}^{n} \gamma_{ij} = 1$ for all $i \in N$. Thus, if firm i and and a supplier $j \in I_i$ do not establish a supplier-customer relationship at $t = 0$, firm i can still produce at $t = 2$ using the production technology indexed by I_i but has to replace j's good with labor.

Before stating our result, we note that, in the above specification, when $\sigma > 1$, labor is no longer an essential factor of production (as we had assumed in Section 2). Therefore, depending on the values of A_{ij}, labor may become redundant and aggregate output in exogenous network equilibrium may blow up. To rule out these uninteresting cases, we assume that the values of $A_{ij} \geq 1$ are small enough to guarantee that output in the exogenous network equilibria is finite. A sufficient condition for this is for matrix $\mathbf{H} = [\gamma_{ij} A_{ij}^{\sigma-1}]$ to have a subunit spectral radius, which we impose for the following result.

Proposition A.1. *If all production technologies are CES with common elasticities of substitution $\sigma \leq 2$, then aggregate productivity is supermodular at the extensive margin (Assumption 1(a)).*

This result establishes that as long as various inputs are complements or weak substitutes, then aggregate productivity satisfies the inequality in (9). This indicates that the key role of Assumption 1(a) is to limit the degree of substitutability between different firms' products. It is however worth noting that the condition in Proposition A.1 allows for some degree of input substitutability as σ can also take values above one.

Proposition A.1 provides only a sufficient condition for Assumption 1(a): this assumption may continue to hold even if some goods are strong substitutes. In fact, even Assumption 1(a) is itself only sufficient—and not necessary—for the existence of endogenous network equilibria. However, in its absence, we can no longer rely on lattice theoretic results, such as Milgrom and Roberts (1990), to prove existence and perform monotone comparative statics.

Finally, we note that a sufficient condition that would guarantee Assumption 1(b) for any network structure takes a more complicated form. Nonetheless, for simple production networks, such as a horizontal network or networks with sufficient symmetry, an elasticity of substitution less than 2 is again sufficient.

A.3 Proofs

Proof of Theorem 1(b)

If there exists an (exogenous network) equilibrium in which all supplier-customer pairs reach an agreement, the pair $(p_{ij}(\mathbf{G}), t_{ij}(\mathbf{G}))$ is the solution to the optimization problem in (4), where firms' gross profits are given by (3). The first-order optimality condition with respect to t_{ij} thus implies that

$$\frac{\theta_i}{\pi_i(\mathbf{G}) - \pi_i(\mathbf{G} \setminus \{ij\})} = \frac{\theta_j}{\pi_j(\mathbf{G}) - \pi_j(\mathbf{G} \setminus \{ij\})}, \tag{A.4}$$

where we are using the fact that the objective function is differentiable in t_{ij} and that $\partial \pi_j(\mathbf{G})/\partial t_{ij} = -\partial \pi_i(\mathbf{G})/\partial t_{ij} = 1$. Replacing for firms' profits $\pi_i(\mathbf{G})$ and $\pi_j(\mathbf{G})$ from (3) into the above equation, solving for $t_{ij}(\mathbf{G})$, and plugging the result back into the objective function of (4), we obtain the following simplified optimization problem:

$$p_{ij}(\mathbf{G}) \in \arg\max_{p_{ij} \geq \mathrm{mc}_j} \quad \pi_i(\mathbf{G}) + \pi_j(\mathbf{G}) = \arg\max_{p_{ij} \geq \mathrm{mc}_j} \quad wL_i + \sum_{k \neq j} p_{ik} x_{ik}.$$

The first part of the above equation indicates that p_{ij} is the bilateral efficient price that maximizes the total surplus generated by the pair, taking all other prices, transfers, and the output of firm i as given. The second part of the equation then follows from rewriting the firms' gross profits using the expression in (3). Note that inputs L_i and $\{x_{ik}\}_{k \neq j}$ are chosen by firm i to minimize its production cost while meeting its demand y_i, which it takes as given. The above equation therefore implies that p_{ij} is the price that minimizes i's total expenditure on all (intermediate and labor) inputs other than j. Finally, note that an increase in p_{ij} can only increase i's aggregate expenditure on all other inputs. Therefore, the unique solution to the above problem is $p_{ij}(\mathbf{G}) = \mathrm{mc}_j(\mathbf{G})$. □

Proof of Theorem 1(c)

We prove this result via a series of lemmas. In Lemma A.1 we derive a series of conditions that firm's equilibrium profits need to satisfy. In Lemma A.2 we then verify that the expression in (6) is the only solution that satisfies all these conditions. Taken together, the two lemmas imply that, if an equilibrium exists, then firm i's profits are given by (6).

Lemma A.1. *Let $\mathbf{G}$ be a feasible production network and suppose there exists an equilibrium in which all supplier-customer pairs reach an agreement. Then,*

$$\pi_i(\tilde{\mathbf{G}}) \geq \pi_i(\tilde{\mathbf{G}} \setminus \{ij\}) \quad , \quad \pi_j(\tilde{\mathbf{G}}) \geq \pi_j(\tilde{\mathbf{G}} \setminus \{ij\}) \tag{A.5}$$

$$\theta_j(\pi_i(\tilde{\mathbf{G}}) - \pi_i(\tilde{\mathbf{G}} \setminus \{ij\})) = \theta_i(\pi_j(\tilde{\mathbf{G}}) - \pi_j(\tilde{\mathbf{G}} \setminus \{ij\})) \tag{A.6}$$

for all $ij \in \tilde{\mathbf{G}}$ and all $\tilde{\mathbf{G}} \subseteq \mathbf{G}$, where $\pi_i(\cdot)$ is gross profit of firm i. Furthermore,

$$\sum_{i=0}^{n} \pi_i(\tilde{\mathbf{G}}) = (\mathcal{A}(\tilde{\mathbf{G}}) - \mathcal{A}(\varnothing))L, \tag{A.7}$$

for all $\tilde{\mathbf{G}} \subseteq \mathbf{G}$, where $\mathcal{A}(\cdot)$ denotes the economy's aggregate productivity on the efficient frontier.

Proof. Condition (A.5) is an immediate consequence of the assumption that all firms reach an agreement, while (A.6) is a simple restatement of (A.4). It is therefore sufficient to establish (A.7). To this end, recall from part (b) that, in any equilibrium in which all firm-pairs reach an agreement, all firms $i \neq 0$ price at marginal cost. Hence, the expression for i's gross profit in (3) simplifies as follows:

$$\pi_i(\tilde{\mathbf{G}}) = \sum_{k:ki \in \tilde{\mathbf{G}}} t_{ki}(\tilde{\mathbf{G}}) - \sum_{j:ij \in \tilde{\mathbf{G}}} t_{ij}(\tilde{\mathbf{G}})$$

and as a result,

$$\sum_{i=1}^{n} \pi_i(\tilde{\mathbf{G}}) = \sum_{i:0i \in \tilde{\mathbf{G}}} t_{0i}(\tilde{\mathbf{G}}), \tag{A.8}$$

where we are using the fact that the only net inflow of lump-sum transfers to input producing firms can be from firm 0. The final good producer, on the other hand, makes a take-it-or-leave it offer to the representative household. As a result, it sets its price equal to the price of the competitive fringe of firms that can turn labor, one-for-one, into the final good. As a result, the profit of the final good producer is given by $\pi_0(\tilde{\mathbf{G}}) = (w - \mathrm{mc}_0(\tilde{\mathbf{G}}))y_0(\tilde{\mathbf{G}}) - \sum_{i:0i\in\tilde{\mathbf{G}}} t_{0i}(\tilde{\mathbf{G}})$, where $y_0(\tilde{\mathbf{G}})$ is the final good producer's output, w is the wage, and $\mathrm{mc}_0(\tilde{\mathbf{G}})$ is the marginal cost of firm 0. Importantly, the fact that all input producers price at marginal cost implies that $y_0(\tilde{\mathbf{G}}) = \mathcal{A}(\tilde{\mathbf{G}})L$, where L is aggregate labor supply and $\mathcal{A}$ is aggregate productivity. Marginal cost pricing also implies that $\mathrm{mc}_0(\tilde{\mathbf{G}}) = w/\mathcal{A}(\tilde{\mathbf{G}})$. Therefore,

$$\pi_0(\tilde{\mathbf{G}}) = w(\mathcal{A}(\tilde{\mathbf{G}}) - 1)L - \sum_{i:0i\in\tilde{\mathbf{G}}} t_{0i}(\tilde{\mathbf{G}}). \tag{A.9}$$

Combining the above with (A.8) and using the fact that $w = 1 = \mathcal{A}(\varnothing)$ then establishes (A.7). □

Lemma A.2. *Let* $\mathbf{G}$ *be a feasible production network and suppose firms' gross profits satisfy* (A.5)–(A.7) *for all* $ij \in \tilde{\mathbf{G}}$ *and all* $\tilde{\mathbf{G}} \subseteq \mathbf{G}$. *Then, the gross profit of firm* i *is given by* (6).

Proof. We first show that the expressions in (6) satisfy conditions (A.5)–(A.7). We start with (A.7). Rewrite the expression in (6) as follows:

$$\begin{aligned}\pi_i(\mathbf{G}) &= \sum_{R\supseteq\{i\}} \frac{\theta_i}{\sum_{k\in R}\theta_k}\left[\sum_{T\subseteq R\setminus\{i\}}(-1)^{|R|-|T|+1}\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \sum_{T\subseteq R\setminus\{i\}}(-1)^{|R|-|T|}\mathcal{A}(\mathbf{G}|_T)\right]L\\ &= \sum_{R\supseteq\{i\}} \frac{\theta_i}{\sum_{k\in R}\theta_k}\left[\sum_{T\subseteq R: i\in T}(-1)^{|R|-|T|}\mathcal{A}(\mathbf{G}|_T) + \sum_{T\subseteq R: i\notin T}(-1)^{|R|-|T|}\mathcal{A}(\mathbf{G}|_T)\right]L.\end{aligned}$$

We therefore get the following equivalent representation of (6):

$$\pi_i(\mathbf{G}) = \sum_{R\subseteq N} \frac{\theta_i}{\sum_{k\in R}\theta_k}\mathbb{I}_{\{i\in R\}}\sum_{T\subseteq R}(-1)^{|R|-|T|}\mathcal{A}(\mathbf{G}|_T)L, \tag{A.10}$$

where $\mathbb{I}$ denotes the indicator function. Summing both sides of the above equation over all firms $i \in N$ implies that

$$\sum_{i=0}^{n}\pi_i(\mathbf{G}) = \sum_{R\subseteq N}\sum_{T\subseteq R}(-1)^{|R|-|T|}\mathcal{A}(\mathbf{G}|_T)L - \mathcal{A}(\varnothing)L = (\mathcal{A}(\mathbf{G}) - \mathcal{A}(\varnothing))L,$$

where the second equality is a consequence of the inclusion-exclusion principle (Gessel and Stanley, 1995, Theorem 12.1). This establish that the expressions in (6) satisfy (A.7).

Next, we show the expressions in (6) satisfy the bargaining condition in (A.6). Using the expression in (A.10), it is immediate that

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\}) = \theta_i \sum_{R\subseteq N}\frac{1}{\sum_{k\in R}\theta_k}\mathbb{I}_{\{i\in R\}}\sum_{T\subseteq R}(-1)^{|R|-|T|}\left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)\right]L$$

for all $ij \in \mathbf{G}$. Note that if $j \notin R$, then $j \notin T$, in which case $\mathcal{A}(\mathbf{G}|_T) = \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)$. Therefore,

$$\frac{1}{\theta_i}\left(\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\})\right) = \sum_{R\supseteq\{i,j\}}\frac{1}{\sum_{k\in R}\theta_k}\sum_{T\subseteq R}(-1)^{|R|-|T|}\left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)\right]L. \tag{A.11}$$

Observing that the right-hand side of the above equation is symmetric in i and j then establishes (A.6).

Next, we verify that the expressions in (6) also satisfy (A.5). Observe that

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus ij) = \theta_i \sum_{T\subseteq N\setminus\{i\}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)\right] L.$$

Furthermore, note that $\mathcal{A}(\mathbf{G}|_T) = \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)$ for all $i \notin T$. As a result,

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus ij) = \theta_i \sum_{T\subseteq N\setminus\{i\}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_{T\cup\{i\}})\right] L.$$

Note that $\mathbf{G}\setminus\{ij\}|_{T\cup\{i\}} \subseteq \mathbf{G}|_{T\cup\{i\}}$. Since $\mathcal{A}$ is the aggregate productivity on the efficient frontier, it must be the case that the expression in square brackets is always nonnegative, thus establishing (A.5).

Having established that the expressions in (6) satisfy conditions (A.5)–(A.7), we next show that the system of equations in (A.5)–(A.7) cannot have any other solution. To this end, note that, without loss of generality, we can restrict our attention to production networks $\mathbf{G}$ that are strongly connected and contain the final-good producer firm 0. This is because any firm that is not directly or indirectly connected to the final-good producer makes zero gross profits, which is clearly unique.

We prove uniqueness for any strongly connected production network $\mathbf{G}$ by using an inductive argument on the number of edges in the production network. As the base of the induction, consider the case that the network only has a single edge, that is $\mathbf{G} = \{0j\}$. It is then immediate to verify that the only solution to system of equations (A.5)–(A.7) is given by $\pi_0(\mathbf{G}) = \theta_0/(\theta_0+\theta_j)(\mathcal{A}(\mathbf{G}) - \mathcal{A}(\varnothing))L$, $\pi_j(\mathbf{G}) = \theta_j/(\theta_0+\theta_j)(\mathcal{A}(\mathbf{G}) - \mathcal{A}(\varnothing))L$, and $\pi_k(\mathbf{G}) = 0$ for all $k \neq 0, j$. This establishes that the system of equations (A.5)–(A.7) has a unique solution when the production network has a single edge.

Next, as the induction hypothesis, suppose there is a unique solution in any network that has m edges. Consider a production network $\mathbf{G}$ that has $m+1$ edges and let $\mathbf{G}'$ denote the rooted spanning tree of $\mathbf{G}$ rooted at the final-good producer.[19] Given this spanning tree, we can use equation (A.6) to express each firm's gross profit recursively in terms of the profits of its unique customer in $\mathbf{G}'$ and profits in the production networks with m edges. This, together with the adding up constraint that $\sum_i \pi_i(\mathbf{G}) = (\mathcal{A}(\mathbf{G}) - \mathcal{A}(\varnothing))L$ then establishes uniqueness. □

The juxtaposition of Lemmas A.1 and A.2 establishes that if there exists an equilibrium in which all supplier-customer pairs reach an agreement, then the equilibrium (gross) profit of firm i is given by (6) with weights given by (7). The proof is therefore complete once we show that the weights $\psi_i(T)$ in (7) are nonnegative. Define the mapping $q: 2^N \to \mathbb{R}$ as follows:

$$q(S) = \sum_{R\supseteq S} \frac{(-1)^{|R|-|S|}}{\sum_{k\in R}\theta_k}.$$

By Theorem 11 of Weber (1988), this mapping satisfies the following recursive relationship:

$$q(S) = \frac{1}{\sum_{k\in S}\theta_k}\sum_{j\notin S}\theta_j q(S\cup\{j\}), \tag{A.12}$$

[19] This is the spanning tree in which all firms except for the final good producer have a single customer.

with the boundary condition $q(N) = 1/(\sum_{k \in N} \theta_k)$. This recursive representation immediately implies that $q(S) > 0$ for all $S \subseteq N$. This coupled with the observation that

$$\psi_i(T) = q(T \cup \{i\}) \qquad \text{for all } i \in N \text{ and all } T \subseteq N \setminus \{i\} \tag{A.13}$$

then establishes that $\psi_i(T) > 0$ for all $i \in N$ and all $T \subseteq N \setminus \{i\}$. $\square$

Proof of Theorem 1 (a)

Prices: We start by establishing that there always exists a unique vector of prices that is consistent with equilibrium. In part (b), we established that, if there exists an equilibrium in which all supplier-customer pairs reach an agreement, then allinput producers price at marginal cost. Therefore, it is sufficient to show that the system of equations

$$p_i(\mathbf{G}) = \mathrm{mc}_i(\mathbf{G}) = \operatorname*{arg\,min}_{I_i : ij \in \mathbf{G} \ \forall j \in I_i} \mathrm{mc}_i(I_i, A_i(I_i), p(\mathbf{G})) \qquad \text{for all } i \in \{1, \dots, n\} \tag{A.14}$$

has exactly one solution $p(\mathbf{G}) = (p_1(\mathbf{G}), \dots, p_n(\mathbf{G}))$ for any given $\mathbf{G}$, where $\mathrm{mc}_i(I_i, A_i(I_i), p)$ denotes the marginal cost of firm i when it uses the production technology with inputs in set I_i when the vector of prices is p. Note that firm i chooses the production technology that yields the lowest marginal cost among all possible technologies that are feasible (in the sense that i has an established relationship with all input suppliers corresponding to that technology). For any given production network, it is easy verify that the system of equations (A.14) is identical to the system of equations (4) and (6) of Acemoglu and Azar (2020, p. 40). Therefore, Theorems 1 and 2 of Acemoglu and Azar (2020) guarantee that there exists a unique price vector $p(\mathbf{G})$ that satisfies (A.14).

Quantities: Recall that equilibrium prices satisfy the system of equations (A.14). An argument identical to the proof of Lemma 1 of Acemoglu and Azar (2020) then establishes that, there exists a generically unique vector of quantities that is consistent with equilibrium.

Transfers: Next, we show that there exists a collection of pairwise lump-sum transfers that is consistent with equilibrium. In parts (b) and (c) of the theorem we established that, if there exists an equilibrium in which all supplier-customer pairs reach an agreement, then (i) all firms price at marginal cost and (ii) firm profits are determined uniquely and are given by equation (6). These two observations, together with the expression for firms' gross profits in (3), imply that transfers satisfy the following system of equations:

$$\pi_i(\mathbf{G}) = \sum_{k : ki \in \mathbf{G}} t_{ki}(\mathbf{G}) - \sum_{j : ij \in \mathbf{G}} t_{ij}(\mathbf{G}) \qquad \text{for all } i \in \{1, \dots, n\}. \tag{A.15}$$

We therefore need to show that the system of equations (A.15) always has a solution.[20] Note that, without any loss of generality, we can focus our attention on pairwise transfers between firms in the largest connected subnetwork of $\mathbf{G}$ that contains the final good producer firm 0—which we denote by $\widehat{\mathbf{G}}$—as

[20] Note that equation (A.9) implies that pairwise lump-sum transfers also need to satisfy $\pi_0(\mathbf{G}) = (\mathcal{A}(\mathbf{G}) - \mathcal{A}(\varnothing))L - \sum_{i : 0i \in \mathbf{G}} t_{0i}(\mathbf{G})$. However, this equation is redundant, as it follows from (A.7) and summing both sides of (A.15) over all i.

we can simply set all other pairwise transfers equal to zero. Let m denotes the number ofinput producers in $\widehat{\mathbf{G}}$ and let $\mathbf{B}(\widehat{\mathbf{G}}) = [b_{id}(\widehat{\mathbf{G}})] \in \mathbb{R}^{m+1\times|\widehat{\mathbf{G}}|}$ be the incidence matrix of $\widehat{\mathbf{G}}$, where $d = ij$ represents a generic directed edge from supplier j to customer i and $|\widehat{\mathbf{G}}|$ denotes the number of pairwise relationships in $\widehat{\mathbf{G}}$. More concretely, if $d = ij \in \widehat{\mathbf{G}}$, then $b_{id}(\widehat{\mathbf{G}}) = 1$, $b_{jd}(\widehat{\mathbf{G}}) = -1$, and $b_{kd}(\widehat{\mathbf{G}}) = 0$ for all $k \neq i, j$. We can thus write the system of equations (A.15) in terms of $\widehat{\mathbf{G}}$'s incidence matrix as follows:

$$\underline{\mathbf{B}}t = -\pi, \tag{A.16}$$

where $t = (t_{ij})_{ij\in\widehat{\mathbf{G}}}$ and $\pi = (\pi_i)_{i\in N}$ denote vectors of pairwise lump-sum transfers and profits, respectively, $\underline{\mathbf{B}} \in \mathbb{R}^{m\times|\widehat{\mathbf{G}}|}$ is the submatrix of $\mathbf{B}$ obtained by eliminating the row corresponding to the final good producer, and we are suppressing the depnendence of $\mathbf{B}$, t, and π on $\widehat{\mathbf{G}}$ for notational simplicity. Since $\widehat{\mathbf{G}}$ is a connected graph, Theorem 8.3.1 of Godsil and Royle (2001) implies that

$$\mathrm{rank}(\underline{\mathbf{B}}) = \mathrm{rank}(\mathbf{B}) = m.$$

Therefore, the vector on the right-hand side of (A.16) is always in the span of columns of $\underline{\mathbf{B}}$. Consequently, equation (A.16), and hence, the system of equations (A.15) always have a solution.

As a final remark, note that the above argument only establishes that there exists a vector of lump-sum transfers that is consistent with equilibrium, and in general, they may be multiple equilibrium lump-sum transfers. Nonetheless, as we already established in part (c) of the theorem, all such transfers result in the same profits, given by the expression in (6).

Agreements: Thus far, we established that if there exists an equilibrium in which all supplier-customer pairs reach an agreement, then (i) prices and quantities are uniquely determined and (ii) there exists a vector of pairwise lump-sum transfers that is consistent with equilibrium. To complete the proof, we therefore need to verify that given such prices, lump-sum transfers, and quantities, all pairs in $\mathbf{G}$ indeed reach an agreement. Specifically, it is sufficient to show that set $O_{ij}(\mathbf{G})$ in (5) is nonempty for all $ij \in \mathbf{G}$.

To establish this, recall that, under the proposed price system in (A.14), all firms price at marginal cost, which in view of (3), implies that firm i's gross profits are given by $\pi_i(\mathbf{G}) = \sum_{k:ki\in\mathbf{G}} t_{ki}(\mathbf{G}) - \sum_{j:ij\in\mathbf{G}} t_{ij}(\mathbf{G})$. Next, recall that we constructed the pairwise lump-sum transfers in (A.15) in such a way that firms' gross profits satisfy equation (6). Finally, in Lemma A.1, we established that the expressions in (6) always satisfy (A.5). Thus, the set of individually rational bargaining outcomes $O_{ij}(\mathbf{G})$ in (5) is always nonempty for all $ij \in \mathbf{G}$. □

Proof of Theorem 2

In Theorem 1(b), we established that all firms price at marginal cost. Therefore, the equilibrium allocation coincides with the allocation when all markets are contestable in the sense of Acemoglu and Azar (2020). Theorem 3 of Acemoglu and Azar (2020) then guarantees the equilibrium is efficient. □

Proof of Theorem 3

Recall from Theorem 1 that firm i's equilibrium (gross) profit is given by the expression in equation (6) with coefficients $\psi_i(T)$ in (7). As a result,

$$\pi_i(\mathbf{G}\cup\{jk\}) - \pi_i(\mathbf{G}) = \theta_i \sum_{T\subseteq N\backslash\{i\}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}\cup\{jk\}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}\cup\{jk\}|_T) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T)\right]L.$$

Note that $\mathbf{G}\cup\{jk\}|_T = \mathbf{G}|_T$ unless both j and k belong to T. This implies that, if either j or k do not belong to T, then

$$\mathcal{A}(\mathbf{G}\cup\{jk\}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}\cup\{jk\}|_T) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T) = \mathcal{A}(\mathbf{G}\cup\{jk\}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) \geq 0,$$

where the inequality follows from the fact that the economy's aggregate productivity can only increase if the set of supplier-customer linkages expands. The above inequality, together with the fact that $\psi_i(T)\geq 0$, implies that

$$\pi_i(\mathbf{G}\cup\{jk\}) - \pi_i(\mathbf{G}) \geq \theta_i \sum_{\substack{T\subseteq N\backslash\{i\}\\ j,k\in T}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}\cup\{jk\}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}\cup\{jk\}|_T) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T)\right]L.$$

Next, note that $\mathcal{A}(\mathbf{G}\cup\{jk\}|_T) = \mathcal{A}(\mathbf{G}|_T\cup\{jk\})$ and $\mathcal{A}(\mathbf{G}\cup\{jk\}|_{T\cup\{i\}}) = \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}\cup\{jk\})$ whenever $j,k\in T$. Therefore,

$$\pi_i(\mathbf{G}\cup\{jk\}) - \pi_i(\mathbf{G}) \geq \theta_i \sum_{\substack{T\subseteq N\backslash\{i\}\\ j,k\in T}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}\cup\{jk\}) - \mathcal{A}(\mathbf{G}|_T\cup\{jk\}) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T)\right]L.$$

Since $\mathbf{G}|_T\subseteq\mathbf{G}|_{T\cup\{i\}}$, Assumption 1(a) guarantees that the expression in the squared braces is always nonnegative. Using $\psi_i(T)\geq 0$ one more time then establishes that $\pi_i(\mathbf{G}\cup\{jk\})\geq\pi_i(\mathbf{G})$. □

Proof of Theorem 4

Let $\pi_i(\mathbf{G}; A_j(I_j))$ denote the gross profit of firm i when the production network is $\mathbf{G}$ and the productivity of firm j when usinginputs I_j is given $A_j(I_j)$. Using characterization in equation (6), we have

$$\pi_i(\mathbf{G};\bar{A}_j) - \pi_i(\mathbf{G};\underline{A}_j) = \theta_i \sum_{T\subseteq N\backslash\{i\}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}};\bar{A}_j) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}};\underline{A}_j) - \mathcal{A}(\mathbf{G}|_T;\bar{A}_j) + \mathcal{A}(\mathbf{G}|_T;\underline{A}_j)\right]L,$$

where for notational simplicity we have dropped the dependence of A_j on I_j. Note that $\mathbf{G}|_T\subseteq\mathbf{G}|_{T\cup\{i\}}$. Thus, Assumption 1(b) guarantees that the expression in the square brackets is nonnegative for all T. This coupled with the fact that the weights $\psi_i(T)$ are also nonnegative then establishes the result. □

Proof of Theorem 5

As a first observation, recall that we can rewrite the household's budget constraint as follows:

$$\sum_{i=0}^{n}\pi_i(\mathbf{G}) = \mathcal{A}(\mathbf{G})L - wL + \sum_{ij\in\mathbf{G}}(c_{ij}+s_{ij}).$$

The right-hand side of the above equation is independent of firms' bargaining powers. Therefore, if we establish that the gross profits of all firms $i \neq j$ are decreasing in θ_j, it immediately implies that the gross profit of firm j is increasing in θ_j.

To establish that $\pi_i(\mathbf{G})$ is decreasing in θ_j when $j \neq i$, recall from Theorem 1 that the equilibrium profit of firm i ca be expressed in terms of model primitives using the expression in (6). Therefore,

$$\pi_i(\mathbf{G}) = \theta_i \sum_{T \subseteq N \setminus \{i,j\}} \psi_i(T)[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}|_T)]L + \theta_i \sum_{T \subseteq N \setminus \{i,j\}} \psi_i(T\cup\{j\})[\mathcal{A}(\mathbf{G}|_{T\cup\{i,j\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{j\}})]L,$$

with weights $\psi_i(T)$ given by (7). Observe that the aggregate productivity $\mathcal{A}(\cdot)$ and aggregate labor supply L do not depend on firms' bargaining powers, and thus $\pi_i(\mathbf{G})$ is differentiable with respect to θ_j. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}\pi_i(\mathbf{G}) = \theta_i \sum_{T \subseteq N \setminus \{i,j\}} \left(\frac{\mathrm{d}}{\mathrm{d}\theta_j}\psi_i(T\cup\{j\})\right) \left[\mathcal{A}(\mathbf{G}|_{T\cup\{i,j\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{j\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T)\right] L,$$

where we are using the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}\psi_i(T) = \sum_{R \supseteq T\cup\{i\}} \frac{(-1)^{|R|-|T|}}{(\sum_{k\in R}\theta_k)^2}\mathbb{I}_{\{j\in R\}} = \sum_{R \supseteq T\cup\{i,j\}} \frac{(-1)^{|R|-|T|}}{(\sum_{k\in R}\theta_k)^2} = -\frac{\mathrm{d}}{\mathrm{d}\theta_j}\psi_i(T\cup\{j\}).$$

Recall from (A.13) in the proof of Theorem 1 that $\psi_i(T\cup\{j\}) = q(T\cup\{i,j\})$ for all $T \subseteq N \setminus \{i,j\}$, where $q : 2^N \to \mathbb{R}_+$ is a mapping that satisfies the recursive equation (A.12). Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}\pi_i(\mathbf{G}) = \theta_i \sum_{T \subseteq N \setminus \{i,j\}} \left(\frac{\mathrm{d}}{\mathrm{d}\theta_j}q(T\cup\{i,j\})\right) \left[\mathcal{A}(\mathbf{G}|_{T\cup\{i,j\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{j\}}) - \mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) + \mathcal{A}(\mathbf{G}|_T)\right] L.$$

Assumption 1(a) guarantees that the expression in square brackets on the right-hand side is always nonnegative. Therefore, to establish that $\pi_i(\mathbf{G})$ is decreasing in θ_j it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}q(S) \le 0 \qquad \text{for all } j \in S \text{ and all } S \subseteq N, \tag{A.17}$$

as this would guarantee that $\mathrm{d}q(T\cup\{i,j\})/\mathrm{d}\theta_j \le 0$ for all $T \subseteq N \setminus \{i,j\}$.

We establish (A.17) using an inductive argument. As the induction's base, set $S = N$. Since $q(N) = 1/(\sum_{k\in N}\theta_k)$, it is immediate that $\mathrm{d}q(N)/\mathrm{d}\theta_j \le 0$. Next, as the induction's hypothesis, fix a set $S \subsetneq N$ and suppose that $\mathrm{d}q(S\cup\{k\})/\mathrm{d}\theta_j \le 0$ for all $k \notin S$ and all $j \in S\cup\{k\}$. Differentiating both sides of the recursion in (A.12) with respect to θ_j implies that, if $j \in S$, then

$$\frac{\mathrm{d}}{\mathrm{d}\theta_j}q(S) = -\frac{1}{(\sum_{k\in S}\theta_k)^2}\sum_{k\notin S}\theta_k q(S\cup\{k\}) + \frac{1}{\sum_{k\in S}\theta_k}\sum_{k\notin S}\theta_k \frac{\mathrm{d}}{\mathrm{d}\theta_j}q(S\cup\{k\}).$$

By the induction hypothesis, the second term on the right-hand side is negative. Furthermore, the fact that $q(S\cup\{k\}) \ge 0$ guarantees that the first term is also negative. The two together then imply that $\mathrm{d}q(S)/\mathrm{d}\theta_j \le 0$, thus completing the inductive argument, and establishing (A.17). □

Proof of Theorem 6(a) and 6(b)

Denote the strategy of firm $j \neq 0$ in the network formation game at $t = 0$ by $b_j = (b_j^{(c)}, b_j^{(s)})$, where $b_j^{(c)}, b_j^{(s)} \in \{0,1\}^n$, $b_{ji}^{(c)} = 1$ if firm j pays the fixed cost c_{ji} to serve as a customer of firm i, and $b_{ji}^{(s)} = 1$ if j

pays the fixed cost s_{ij} to serve as i's supplier. Similarly, denote the strategy of firm 0 by $b_0 = b_0^{(c)} \in \{0,1\}^n$, where $b_{0i}^{(c)} = 1$ if firm 0 pays the fixed c_{0i} to serve as a customer of firm i. The net profit of firm j in the network formation game under the strategy profile $b = (b_0, \ldots, b_n)$ is thus given by

$$\phi_j(b) = \pi_j(\mathbf{G}(b)) - \sum_{i=1}^{n} b_{ji}^{(c)} c_{ji} - \sum_{i=1}^{n} b_{ji}^{(s)} s_{ij}, \tag{A.18}$$

where $\mathbf{G}(b) = \{ji : b_{ji}^{(s)} = b_{ij}^{(c)} = 1\}$ and $\pi_j(\mathbf{G})$ is the gross profit of firm j in production network $\mathbf{G}$. We have the following lemma:

Lemma A.3. *Suppose Assumption 1(a) is satisfied. Then,*

(a) $\phi_j(b_j, b_{-j})$ *has increasing differences in* (b_j, b_{-j})*;*

(b) $\phi_j(b_j, b_{-j})$ *is supermodular in* b_j *given an arbitrary* b_{-j}*.*

Proof. To establish part (a), consider two strategy profiles b and $\hat{b}$ such that $\hat{b}_j \geq b_j$ and $\hat{b}_{-j} \geq b_{-j}$. Equation (A.18) implies that

$$\begin{aligned}\phi_j(\hat{b}_j, \hat{b}_{-j}) - \phi_j(b_j, \hat{b}_{-j}) - \phi_j(\hat{b}_j, b_{-j}) + \phi_j(b_j, b_{-j}) = \pi_j(\mathbf{G}(\hat{b}_j, \hat{b}_{-j})) - \pi_j(\mathbf{G}(b_j, \hat{b}_{-j})) \\ - \pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) + \pi_j(\mathbf{G}(b_j, b_{-j})).\end{aligned} \tag{A.19}$$

Furthermore, the expression for firms' gross profits in (6) implies that

$$\pi_j(\mathbf{G}(\hat{b}_j, \hat{b}_{-j})) - \pi_j(\mathbf{G}(b_j, \hat{b}_{-j})) = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G}(\hat{b}_j, \hat{b}_{-j})|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}(b_j, \hat{b}_{-j})|_{T \cup \{j\}})\right] L \tag{A.20}$$

$$\pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(b_j, b_{-j})) = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G}(\hat{b}_j, b_{-j})|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}(b_j, b_{-j})|_{T \cup \{j\}})\right] L, \tag{A.21}$$

where we are using the fact that the value of $\mathcal{A}(\mathbf{G}(b_j, b_{-j})|_T)$ is independent of b_j whenever $j \notin T$. Assumption 1(a) together the fact that $\psi_j(T) \geq 0$ guarantees that the right-hand side of (A.20) is greater than equal to the right-hand right of (A.21). Therefore, the right-hand side of (A.19) is nonnegative. Consequently, $\phi_j(b_j, b_{-j})$ has increasing differences in (b_j, b_{-j}) for all j.

We next prove statement (b). Equation (A.18) implies that, for any pair of strategy profiles b and $\hat{b}$,

$$\begin{aligned}&\phi_j(b_j \vee \hat{b}_j, b_{-j}) + \phi_j(b_j \wedge \hat{b}_j, b_{-j}) - \phi_j(\hat{b}_j, b_{-j}) - \phi_j(b_j, b_{-j}) \\ &\quad = \pi_j(\mathbf{G}(b_j \vee \hat{b}_j, b_{-j})) + \pi_j(\mathbf{G}(b_j \wedge \hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(b_j, b_{-j})).\end{aligned} \tag{A.22}$$

Furthermore, using the expression for firm j's gross profit in (6), we have

$$\pi_j(\mathbf{G}(b_j \vee \hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(b_j, b_{-j})) = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G}(b_j \vee \hat{b}_j, b_{-j})|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}(b_j, b_{-j})|_{T \cup \{j\}})\right] L$$

$$\pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(b_j \wedge \hat{b}_j, b_{-j})) = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G}(\hat{b}_j, b_{-j})|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}(b_j \wedge \hat{b}_j, b_{-j})|_{T \cup \{j\}})\right] L,$$

where once again we are using the fact that the value of $\mathcal{A}(\mathbf{G}(b_j, b_{-j})|_T)$ is independent of b_j whenever $j \notin T$. Assumption 1(a) together the fact that $\psi_j(T) \geq 0$ guarantees that the right-hand side of the first equation is greater than or equal to that of the second equation. Consequently, the right-hand side of (A.22) is nonnegative, which means $\phi_j(b_j, b_{-j})$ is supermodular in b_j given an arbitrary b_{-j}. $\square$

With the above lemma at hand, we now turn to the proofs of Theorems 6(a) and 6(b). By Lemma A.3(a), the net profit of firm j has increasing differences in (b_j, b_{-j}). Furthermore, statement (b) of the same lemma guarantees that the net profit of firm j s supermodular in b_j for an arbitrary b_{-j}. These imply that the link formation game is a supermodular game. Thus, by Theorem 5 of Milgrom and Roberts (1990), the game has an equilibrium and that the set of equilibrium strategy profiles, B, form a lattice. This in turn implies that the corresponding equilibrium production networks $\{\mathbf{G}(b) : b \in B\}$ form a lattice with respect to the set inclusion order. □

Proof of Theorem 6(c)

Let b^* denote the greatest equilibrium strategy profile, that is, $b^* \geq b$ element-wise for any equilibrium strategy profile $b \in B$, where B denotes the set of all equilibrium strategy profiles. The fact that b^* is an equilibrium means that no firm has a profitable deviation. In particular, $\phi_j(b^*) \geq \phi_j(b_j, b^*_{-j})$ for all $j \in N$ and all $b \in B$. As a result, equation (A.18) implies that

$$\begin{aligned}\phi_j(b^*) \geq \phi_j(b_j, b^*_{-j}) &= \pi_j(\mathbf{G}(b_j, b^*_{-j})) - \sum_{i=1}^{n} b_{ji}^{(c)} c_{ji} - \sum_{i=1}^{n} b_{ji}^{(s)} s_{ij} \\ &\geq \pi_j(\mathbf{G}(b_j, b_{-j})) - \sum_{i=1}^{n} b_{ji}^{(c)} c_{ji} - \sum_{i=1}^{n} b_{ji}^{(s)} s_{ij} = \phi_j(b),\end{aligned}$$

where the second inequality follows from Theorem 3 and the fact that $\mathbf{G}(b_j, b_{-j}) \subseteq \mathbf{G}(b_j, b^*_{-j})$. Hence, all firms make weakly more profits in the economy's greatest equilibrium than all other equilibria.

The fact that all firms make more profits in the economy's greatest equilibrium implies that $\sum_{j=0}^{n} \phi_j(b^*) \geq \sum_{j=0}^{n} \phi_j(b)$ for any equilibrium strategy profile b. Adding wL to both sides of the above equation and using the household's budget constraint implies that $Y(b^*) \geq Y(b)$ for all $b \in B$. Therefore, aggregate output n the greatest equilibrium is higher than in all other equilibria. □

Proof of Theorem 7

As in the proof of Theorem 6, let $b_j = (b_j^{(c)}, b_j^{(s)}) \in \{0,1\}^{2n}$ and $b_0 = b_0^{(c)} \in \{0,1\}^n$ denote, respectively, the strategies of firms $j \neq 0$ and $j = 0$ in the network formation game at $t = 0$. Additionally, let $\phi_j(b, z)$ denote the net profit of firm j given strategy profile $b = (b_0, \ldots, b_n)$ and when the vector of fixed costs of establishing supplier-customer relationships is $z = (c_{ik}, s_{ik})$.

Proof of part (a). By equation (A.18),

$$\phi_j(\hat{b}_j, b_{-j}, z) - \phi_j(b_j, b_{-j}, z) = \pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) - \pi_j(\mathbf{G}(b_j, b_{-j})) - \sum_{i=1}^{n} (\hat{b}_{ji}^{(c)} - b_{ji}^{(c)}) c_{ji} - \sum_{i=1}^{n} (\hat{b}_{ji}^{(s)} - b_{ji}^{(s)}) s_{ij}.$$

If $\hat{b}_j \geq b_j$, the right-hand side of the above expression is weakly decreasing in the vector of fixed costs, z, irrespective of the value of b_{-j}. This means that the net profit of firm j has increasing differences in $(b_j, -z)$ for all b_{-j}. Hence, the corollary to Theorem 6 of Milgrom and Roberts (1990) guarantees that the greatest equilibrium b^* is weakly decreasing in z. As a result, the set of customer-supplier linkages in $\mathbf{G}^* = \mathbf{G}(b^*)$ is also weakly decreasing in z. □

Proof of part (b). Let $\bar{z} = (\bar{c}_{ij}, \bar{s}_{ij})$ and $\underline{z} = (\underline{c}_{ij}, \underline{s}_{ij})$ denote two vectors of fixed costs such that $\bar{z} \geq \underline{z}$ element-wise. Also, let $\bar{b}^*$ and $\underline{b}^*$ be the economy's greatest equilibria under $\bar{z}$ and $\underline{z}$, respectively. Then,

$$\phi_j(\underline{b}^*, \underline{z}) \geq \phi_j(\bar{b}_j^*, \underline{b}_{-j}^*, \underline{z}) \geq \phi_j(\bar{b}_j^*, \underline{b}_{-j}^*, \bar{z}) = \pi_j(\mathbf{G}(\bar{b}_j^*, \underline{b}_{-j}^*)) - \sum_{i=1}^{n} \bar{b}_{ji}^{(c)} \bar{c}_{ji} - \sum_{i=1}^{n} \bar{b}_{ji}^{(s)} \bar{s}_{ij}.$$

The first inequality is a consequence of the fact that $\underline{b}^*$ is an equilibrium, whereas the second inequality follows from the observation that holding firms' strategies constant, an increase in fixed costs can only reduce firm j's profit. In part (a), we established that $\underline{b}^* \geq \bar{b}^*$ and in particular, $\underline{b}_{-j}^* \geq \bar{b}_{-j}^*$, which in turn implies that $\mathbf{G}(\bar{b}_j^*, \underline{b}_{-j}^*) \supseteq \mathbf{G}(\bar{b}^*)$. Therefore, using Theorem 3 and Assumption 1(a), we can bound the right-hand side of the above inequality from below as follows:

$$\phi_j(\underline{b}^*, \underline{z}) \geq \pi_j(\mathbf{G}(\bar{b}^*)) - \sum_{i=1}^{n} \bar{b}_{ji}^{(c)} \bar{c}_{ji} - \sum_{i=1}^{n} \bar{b}_{ji}^{(s)} \bar{s}_{ij} = \phi_j(\bar{b}^*, \bar{z}).$$

This establishes the result. □

Proof of part (c). In part (b), we established that $\phi_j(\bar{b}^*, \bar{z}) \leq \phi_j(\underline{b}^*, \underline{z})$ for all firms j. As a result,

$$Y(\bar{b}^*, \bar{z}) = wL + \sum_{j=0}^{n} \phi_j(\bar{b}^*, \bar{z}) \leq wL + \sum_{j=0}^{n} \phi_j(\underline{b}^*, \underline{z}) = Y(\underline{b}^*, \underline{z}),$$

where the two equalities follow from the household's budget constraint. Therefore, an increase in fixed costs reduces aggregate output in the greatest equilibrium. □

Proof of Theorem 8

Let $b_j = (b_j^{(c)}, b_j^{(s)}) \in \{0,1\}^{2n}$ and $b_0 = b_0^{(c)} \in \{0,1\}^n$ denote, respectively, the strategies of firms $j \neq 0$ and $j = 0$ in the network formation game at $t = 0$. Additionally, let $\phi_j(b, A)$ denote the net profit of firm j given strategy profile $b = (b_0, \ldots, b_n)$ and when the vector of productivity parameters of an arbitrary industry i is given by $A_i(I_i) = A$. Similarly, let $\pi_j(\mathbf{G}, A)$ denote the gross profit of firm j when the production network is $\mathbf{G}$ and $A_i(I_i) = A$. Finally, we use $\mathcal{A}(\mathbf{G}; A)$ to denote the economy's aggregate productivity in the production network $\mathbf{G}$ and when $A_i(I_i) = A$.

Proof of part (a). By equation (A.18),

$$\phi_j(\hat{b}_j, b_{-j}, A) - \phi_j(b_j, b_{-j}, A) = \pi_j(\mathbf{G}(\hat{b}_j, b_{-j}), A) - \pi_j(\mathbf{G}(b_j, b_{-j}), A) - \sum_{i=1}^{n} [(\hat{b}_{ji}^{(c)} - b_{ji}^{(c)}) c_{ji} + (\hat{b}_{ji}^{(s)} - b_{ji}^{(s)}) s_{ji}].$$

Observe that $\mathbf{G}(\hat{b}_j, b_{-j})|_T = \mathbf{G}(b_j, b_{-j})|_T$ for any $T \subseteq N \setminus \{j\}$. Therefore, using the expression in (6) to replace for firms' gross profits in the above equation, we have

$$\phi_j(\hat{b}_j, b_{-j}, A) - \phi_j(b_j, b_{-j}, A) = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}\left(\mathbf{G}(\hat{b}_j, b_{-j})|_{T \cup \{j\}}; A\right) - \mathcal{A}\left(\mathbf{G}(b_j, b_{-j})|_{T \cup \{j\}}; A\right)\right] L$$
$$- \sum_{i=1}^{n} (\hat{b}_{ji}^{(c)} - b_{ji}^{(c)}) c_{ji} - \sum_{i=1}^{n} (\hat{b}_{ji}^{(s)} - b_{ji}^{(s)}) s_{ij}.$$

Note that if $\hat{b}_j \geq b_j$, then $\mathbf{G}(\hat{b}_j, b_{-j})|_{T\cup\{j\}} \supseteq \mathbf{G}(b_j, b_{-j})|_{T\cup\{j\}}$. Therefore, if $\hat{b}_j \geq b_j$, Assumption 1(b) and Theorem 4 imply that the right-hand side of the above equation is weakly increasing in A. This establishes that net profit of firm j has increasing differences in (b_j, A) for all b_{-j}. Hence, the corollary to Theorem 6 of Milgrom and Roberts (1990) guarantees that the greatest equilibrium b^* is nondecreasing in A. As a result, the set of customer-supplier linkages in $\mathbf{G}^* = \mathbf{G}(b^*)$ is also nondecreasing in A. □

Proof of part (b). Let $\bar{A}$ and $\underline{A}$ be two vectors specifying productivity parameters $A_i(I_i)$ such that $\bar{A} \geq \underline{A}$ element-wise. Also, let $\bar{b}^*$ and $\underline{b}^*$ denote the economy's greatest equilibria under $\bar{A}$ and $\underline{A}$, respectively. Observe that

$$\phi_j(\overline{b}^*, \bar{A}) \geq \phi_j(\underline{b}_j^*, \overline{b}_{-j}^*, \bar{A}) \geq \phi_j(\underline{b}_j^*, \overline{b}_{-j}^*, \underline{A}) \geq \phi_j(\underline{b}^*, \underline{A}).$$

The first inequality is a consequence of the fact that $\overline{b}^*$ is an equilibrium. The second inequality follows from Theorem 4, which guarantees that, holding firms' strategies and hence the production network constant, a decrease in productivity reduces all firms' net and gross profits. Finally, the last inequality is a due to the facts that $\bar{b}^* \geq \underline{b}^*$ (established in part (a) of the theorem) and that all firms' gross profits increase if the set of supplier-customer linkages expands (established in Theorem 3). Taken together, these inequalities establish that a decrease in productivity from $\bar{A}$ to $\underline{A}$ cannot increase the net profit of firm j in the economy's greatest equilibrium. □

Proof of part (c). In part (b), we established that $\phi_j(\bar{b}^*, \bar{A}) \geq \phi_j(\underline{b}^*, \underline{A})$ for all firms j. As a result,

$$Y(\bar{b}^*, \bar{A}) = wL + \sum_{j=0}^{n} \phi_j(\bar{b}^*, \bar{A}) \geq wL + \sum_{j=0}^{n} \phi_j(\underline{b}^*, \underline{A}) = Y(\underline{b}^*, \underline{z}),$$

where the two equalities follow from the household's budget constraint. This establishes that a decrease in productivity $A_i(I_i)$ decreases aggregate output in the economy's greatest equilibrium. □

Proof of Lemma 1

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ denote two efficient production networks and $\mathbf{G}_1 \cup \mathbf{G}_2$ denote the production network that contains all supplier-customer relationships that are present in at least either $\mathbf{G}_1$ or $\mathbf{G}_2$. The market-clearing condition for the final good implies that

$$\begin{aligned}
Y(\mathbf{G}_1 \cup \mathbf{G}_2) - Y(\mathbf{G}_1) &= (\mathcal{A}(\mathbf{G}_1 \cup \mathbf{G}_2) - \mathcal{A}(\mathbf{G}_1))L - \sum_{ij \in \mathbf{G}_2 \setminus \mathbf{G}_1} (c_{ij} + s_{ij}) \\
&\geq (\mathcal{A}(\mathbf{G}_2) - \mathcal{A}(\mathbf{G}_1 \cap \mathbf{G}_2))L - \sum_{ij \in \mathbf{G}_2 \setminus \mathbf{G}_1} (c_{ij} + s_{ij}) \\
&= Y(\mathbf{G}_2) - Y(\mathbf{G}_1 \cap \mathbf{G}_2),
\end{aligned}$$

where the inequality follows from Assumption 1(a). The assumption that both $\mathbf{G}_1$ and $\mathbf{G}_2$ are efficient means that the left-hand side of the above inequality is nonpositive, whereas the right-hand side is nonnegative. Therefore, both expressions have to be equal to zero, and in particular, $Y(\mathbf{G}_1 \cup \mathbf{G}_2) =$

$Y(\mathbf{G}_1)$. This means that $\mathbf{G}_1 \cup \mathbf{G}_2$ is also an efficient production network. As a result,

$$\mathbf{G}^{\text{eff}} = \bigcup_{\mathbf{G} \in \arg\max_{\mathbf{G}'} Y(\mathbf{G}')} \mathbf{G}$$

is efficient and contains all other efficient production networks as subnetworks. $\square$

Proof of Theorem 9

As in the proof of Theorem 6, let $b = (b_0, \dots, b_n)$ denote a strategy profile in the network formation game at $t = 0$, and let b^* and b^{eff} denote the strategy profiles corresponding to, respectively, the greatest equilibrium and the greatest efficient solution. Similarly, let $\mathbf{G}^* = \mathbf{G}(b^*)$ and $\mathbf{G}^{\text{eff}} = \mathbf{G}(b^{\text{eff}})$ denote the production networks formed under the two strategy profiles, where $\mathbf{G}(b) = \{ji : b_{ji}^{(s)} = b_{ij}^{(c)} = 1\}$. The market-clearing condition for the final good implies that

$$\begin{aligned}
Y(b^* \vee b^{\text{eff}}) - Y(b^{\text{eff}}) &= (\mathcal{A}(\mathbf{G}^* \cup \mathbf{G}^{\text{eff}}) - \mathcal{A}(\mathbf{G}^{\text{eff}}))L - \sum_{ij \in \mathbf{G}^* \setminus \mathbf{G}^{\text{eff}}} (c_{ij} + s_{ij}) \\
&\geq (\mathcal{A}(\mathbf{G}^*) - \mathcal{A}(\mathbf{G}^* \cap \mathbf{G}^{\text{eff}}))L - \sum_{ij \in \mathbf{G}^* \setminus \mathbf{G}^{\text{eff}}} (c_{ij} + s_{ij}) \\
&= Y(b^*) - Y(b^* \wedge b^{\text{eff}}),
\end{aligned}$$

where the inequality follows from Assumption 1(a). Next, from the household's budget constraint,

$$\begin{aligned}
Y(b^*) - Y(b^* \wedge b^{\text{eff}}) &= \sum_{i=0}^{n} \Big(\phi_i(b^*) - \phi_i(b^* \wedge b^{\text{eff}})\Big) \\
&\geq \sum_{i=0}^{n} \Big(\phi_i(b_i^*, b_{-i}^*) - \phi_i(b_i^* \wedge b_i^{\text{eff}}, b_{-i}^*)\Big) \\
&\geq 0.
\end{aligned}$$

The first inequality is a consequence of the fact that firm i's gross and net profits are increasing in the set of links established by other firms (as established in Theorem 3). The second inequality follows from the assumption that b_i^* is the equilibrium strategy of agent i. Putting the above together implies that $Y(b^* \vee b^{\text{eff}}) - Y(b^{\text{eff}}) \geq 0$. But the fact that b^{eff} is the strategy profile corresponding to greatest efficient outcome requires that $Y(b^* \vee b^{\text{eff}}) - Y(b^{\text{eff}}) \leq 0$. Therefore, it must be the case that $Y(b^{\text{eff}}) = Y(b^* \vee b^{\text{eff}})$. Consequently, $\mathbf{G}^{\text{eff}} = \mathbf{G}^* \cup \mathbf{G}^{\text{eff}}$, and hence, $\mathbf{G}^* \subseteq \mathbf{G}^{\text{eff}}$. $\square$

Proof of Theorem 10

Proof of part (a). The planner's problem is to choose the production network and the allocation that maximizes aggregate output. Therefore, $Y^{\text{eff}} = \max_{\mathbf{G}} Y(\mathbf{G})$, where $Y(\mathbf{G})$ is the efficient—and in view of Theorem 2 also the equilibrium—aggregate output given a fixed production network $\mathbf{G}$. For any given $\mathbf{G}$, the aggregate output $Y(\mathbf{G})$ is continuous in all productivities, $A_i(I_i)$, and fixed costs, c_{ij} and s_{ij}. Since the maximum of continuous functions is continuous, it follows that Y^{eff} is also continuous in all productivities and fixed costs. $\square$

Proof of part (b). Let $Y(\mathbf{G})$ denote equilibrium aggregate output when the economy's production network is $\mathbf{G}$. Market clearing for the the final good implies that

$$Y(\mathbf{G}) = \mathcal{A}(\mathbf{G})L - \sum_{rk\in\mathbf{G}} (c_{rk} + s_{rk}).$$

It is immediate that as long as $\mathbf{G}$ is unchanged, $Y(\mathbf{G})$ changes continuously in all productivities and fixed costs. Therefore, any discontinuity in aggregate output can occur only in response to changes to the production network. Next, we prove that, generically, whenever changes in fixed costs induce changes in the production network, they lead to discontinuities in aggregate output. The argument for changes in productivities is identical.

We prove this claim by contraction. To set the notation, let $z = (c_{ij}, s_{ij})$ denote a vector of fixed costs and $Y(z)$ and $\mathbf{G}(z)$ denote, respectively, the equilibrium aggregate output and production network corresponding to this vector of fixed costs. We consider a perturbation of z, which we denote by $z + \epsilon$ for some vector $\epsilon \geq 0$ that induces a change in the equilibrium production network. In particular, we assume z is such that

$$\lim_{\epsilon\downarrow 0} \mathbf{G}(z+\epsilon) = \tilde{\mathbf{G}} \subsetneq \mathbf{G} = \mathbf{G}(z),$$

where we are relying on Theorem 7(a), according to which the equilibrium production network can only shrink in response to a change in fixed costs. Now, despite the above, suppose the change in fixed costs and the corresponding change in the production network do not induce a discontinuity in aggregate output, i.e.,

$$\lim_{\epsilon\downarrow 0} Y(z+\epsilon) = Y(z). \tag{A.23}$$

Using the household's budget constraint, we can rewrite (A.23) as $\lim_{\epsilon\downarrow 0}\sum_{i=0}^{n}(\phi_i(z+\epsilon) - \phi_i(z)) = 0$. Next, note that Theorem 7(b) implies that $\phi_i(z+\epsilon) \leq \phi_i(z)$ for all j. Therefore, it must be the case that $\lim_{\epsilon\downarrow 0}(\phi_i(z+\epsilon) - \phi_i(z)) = 0$ for all i. Replacing for firm i's net profits in terms of its gross profits and its fixed costs from equation (2) implies that

$$\pi_i(\tilde{\mathbf{G}}) - \pi_i(\mathbf{G}) + \sum_{j:ij\in\mathbf{G}\setminus\tilde{\mathbf{G}}} c_{ij} + \sum_{k:ki\in\mathbf{G}\setminus\tilde{\mathbf{G}}} s_{ki} = 0 \qquad \text{for all } i \in N.$$

Rewriting firms' net profits in terms of the expression in (6) leads to

$$\theta_i \sum_{T\subseteq N\setminus\{i\}} \psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\tilde{\mathbf{G}}|_{T\cup\{i\}}) + \mathcal{A}(\tilde{\mathbf{G}}|_T)\right] L = \sum_{j:ij\in\mathbf{G}\setminus\tilde{\mathbf{G}}} c_{ij} + \sum_{k:ki\in\mathbf{G}\setminus\tilde{\mathbf{G}}} s_{ki}.$$

Note that the above equation has to hold simultaneously for all firms i. The left-hand side is only in terms of productivities and bargaining powers, whereas the right-hand side is only in terms of fixed costs. As such, the above equation can hold for all firms i simultaneously only for a nongeneric vector of fixed costs. Therefore, generically, equation (A.23) is violated. $\square$

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B Online Appendix

This online appendix contains the proofs of Propositions 1–3 and A.1 that were omitted from the main body of the paper.

Proof of Proposition 1

Let $s^*_{\text{frg}} = \pi_j(\mathbf{G} \cup \{ik, kj\}) - \pi_j(\mathbf{G} \cup \{ik\})$ be the threshold beyond which firm j drops its relationship with firm k in the fragmented economy. Aggregate output in the fragmented architecture is therefore given by

$$Y_{\text{frg}} = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ik, kj\}) - s - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s \leq s^*_{\text{frg}} \\ \mathcal{A}(\mathbf{G}) - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s > s^*_{\text{frg}}, \end{cases}$$

where we are using the assumption that $\mathcal{A}(\mathbf{G} \cup \{ik\}) = \mathcal{A}(\mathbf{G})$. Similarly, let $s^*_{\text{int}} = \pi_j(\mathbf{G} \cup \{ij\}) - \pi_j(\mathbf{G})$ denote the threshold beyond which firm j drops its relationship with firm i in the integrated economy. A similar argument implies that aggregate output in the integrated architecture is given by

$$Y_{\text{int}} = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ij\}) - s - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s \leq s^*_{\text{int}} \\ \mathcal{A}(\mathbf{G}) - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s > s^*_{\text{int}}. \end{cases}$$

Given the above, it is immediate that if (13) is satisfied, and as long $s^*_{\text{frg}} < s^*_{\text{int}}$, then $Y_{\text{frg}} > Y_{\text{int}}$ for $s \leq s^*_{\text{frg}}$ and $Y_{\text{frg}} < Y_{\text{int}}$ for $s > s^*_{\text{frg}}$. The proof is therefore complete once we show that $s^*_{\text{frg}} < s^*_{\text{int}}$.

Recall that $s^*_{\text{frg}} = \pi_j(\mathbf{G} \cup \{ik, kj\}) - \pi_j(\mathbf{G} \cup \{ik\})$ and $s^*_{\text{int}} = \pi_j(\mathbf{G} \cup \{ij\}) - \pi_j(\mathbf{G})$. The expression for firms' profits in (6) therefore implies that

$$\begin{aligned} s^*_{\text{frg}} &= \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T \cup \{k\}) \left[\mathcal{A}(\mathbf{G} \cup \{ik, kj\}|_{T \cup \{j,k\}}) - \mathcal{A}(\mathbf{G}\,|_{T \cup \{j,k\}})\right] L \\ s^*_{\text{int}} &= \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \left[\mathcal{A}(\mathbf{G} \cup \{ij\}|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}|_{T \cup \{j\}})\right] L, \end{aligned}$$

where N denotes the set of firms in the integrated architecture and thus excludes firm k. Observe that, by construction, s^*_{int} does not depend on θ_k. Therefore, to show that $s^*_{\text{frg}} < s^*_{\text{int}}$ for large enough values of θ_k, it is sufficient to show that $\lim_{\theta_k \to \infty} \psi_j(T \cup \{k\}) = 0$ for all $T \subseteq N \setminus \{j\}$, which implies that $\lim_{\theta_k \to \infty} s^*_{\text{frg}} = 0$. To this end, recall from the proof of Theorem 1(c) that the weights ψ_j satisfy equations (A.12) and (A.13). The recursion in (A.12) and a simple inductive argument on the set T then immediately implies that $\lim_{\theta_k \to \infty} \psi_j(T \cup \{k\}) = \lim_{\theta_k \to \infty} q(T \cup \{j, k\}) = 0$ for all $T \subseteq N \setminus \{j\}$. $\square$

Proof of Proposition 2

Let k^* denote the depth of the supply chain in the economy's greatest equilibrium. Given that each supplier-customer relationship generates a productivity gain of $A \geq 1$ but requires a fixed cost of s, it is immediate that equilibrium aggregate output is given by $Y^* = A^{k^*-1}L - (k^*-1)s$. To prove the result, it is therefore sufficient to derive the expression for k^*.

For any $k \leq n$, let $\mathbf{G}_k$ denote the production network of depth $k+1$, consisting of firms 0 through k. We make two observations. First, in $\mathbf{G}_k$, firm k's profits are below that of all other firms, i.e.,

$$\pi_k(\mathbf{G}_k) \leq \pi_i(\mathbf{G}_k) \quad \text{for all } i \leq k. \tag{B.1}$$

To see this, note that (6) implies that $\pi_i(\mathbf{G}_k) = \theta_i \sum_{T \supseteq \{0,\dots,i-1\}} \psi_i(T) \left[\mathcal{A}(\mathbf{G}_k|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}_k|_T)\right] L$, where we are using the fact that $\mathcal{A}(\mathbf{G}_k|_{T\cup\{i\}}) = \mathcal{A}(\mathbf{G}_k|_T)$ unless $T \supseteq \{0,\dots,i-1\}$. Increasing i shrinks the set of T's in the sum on the right-hand side of the expression for $\pi_i(\mathbf{G}_k)$. Therefore, $\pi_i(\mathbf{G}_k) > \pi_{i+1}(\mathbf{G}_k)$, where we are using the fact that $\psi_i(T) \geq 0$ and $\mathcal{A}(\mathbf{G}_k|_{T\cup\{i\}}) \geq \mathcal{A}(\mathbf{G}_k|_T)$. This establishes (B.1).

Second, we note that

$$\pi_n(\mathbf{G}_n) \geq \pi_k(\mathbf{G}_k) \quad \text{for all } k \leq n. \tag{B.2}$$

To show this, note that, according to (6), $\pi_k(\mathbf{G}_k) = \theta_k \sum_{T\subseteq N\setminus\{k\}} \psi_k(T) \left[\mathcal{A}(\mathbf{G}_k|_{T\cup\{k\}}) - \mathcal{A}(\mathbf{G}_k|_T)\right] L = \theta_k \psi_k(\{0,1,\dots,k-1\}) \left[\mathcal{A}(\mathbf{G}_k) - \mathcal{A}(\mathbf{G}_{k-1})\right] L$, where we are using the fact that $\mathcal{A}(\mathbf{G}_k|_{T\cup\{k\}}) = \mathcal{A}(\mathbf{G}_k|_T)$ for any $T \neq \{0,\dots,k-1\}$. Since all firms are assumed to have identical bargaining powers, the above expression simplifies to $\pi_k(\mathbf{G}_k) = \frac{1}{k+1}A^{k-1}(A-1)L$, which is increasing in k as long as $\log A > 1/2$. Hence, the inequalities in (B.2) are satisfied.

With inequalities (B.1) and (B.2) at hand, we now prove the result. Let $s_n^* = \pi_n(\mathbf{G}_n)$. As long as $s \leq s_n^*$, inequality (B.1) implies that $\phi_i(\mathbf{G}_n) = \pi_i(\mathbf{G}_n) - s \geq 0$, thus guaranteeing that all firms make nonnegative profits. Hence, in the economy's greatest equilibrium all firms pay the fixed cost of establishing a link with their customer and hence, $k^* = n+1$. If on the other hand, $s > s_n^*$, then (B.2) implies that $\phi_k(\mathbf{G}_k) = \pi_k(\mathbf{G}_k) - s < 0$, which means no firm $k \neq 0$ is willing to pay the fixed cost s to serve as the most upstream firm in the supply chain. Thus, it must be the case that the equilibrium supply chain only consists of firm 0, i.e., $k^* = 1$. □

Proof of Proposition 3

Let $\mathbf{G}$ denote a production network that contains customer-supplier relations between firms i and k and their designated suppliers $j = n+i$ and $l = n+k$, respectively. According to (6), the marginal (gross) benefit to firm i of maintaining a relationship with its supplier is given by

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\}) = \theta_i \sum_{T\subseteq N\setminus\{i\}} \psi_i(T) \left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}}) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_{T\cup\{i\}})\right] L.$$

Recall from the proof of Theorem 1 that the weights ψ_i in the above expression satisfy (A.13). Therefore,

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\}) = \theta_i \sum_{T\supseteq\{i\}} q(T) \left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)\right] L = \theta_i \sum_{T\subseteq N} q(T) \left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)\right] L,$$

where $q(\cdot)$ satisfies the recursion in (A.12) and the second equality follows from the fact that $\mathcal{A}(\mathbf{G}|_T) = \mathcal{A}(\mathbf{G}\setminus\{ij\}|_T)$ for any set T that does not contain i. The assumption that each firm i in the bottom layer has only a single potential supplier in the top layer implies that $\mathcal{A}(\mathbf{G}\setminus\{ij\}|T) = \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})$. As a result,

$$\pi_i(\mathbf{G}) - \pi_i(\mathbf{G}\setminus\{ij\}) = \theta_i \sum_{T\subseteq N} q(T) \left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L.$$

A similar argument implies that

$$\pi_k(\mathbf{G}) - \pi_k(\mathbf{G} \setminus \{kl\}) = \theta_k \sum_{T \subseteq N} q(T) \left[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}})\right] L.$$

Subtracting this equation from the previous one and using the assumption that all firms in the bottom layer have identical bargaining powers, θ, leads to

$$\Delta_{ik} = [\pi_i(\mathbf{G}) - \pi_i(\mathbf{G} \setminus \{ij\})] - [\pi_k(\mathbf{G}) - \pi_k(\mathbf{G} \setminus \{kl\})] = \theta \sum_{T \subseteq N} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L.$$

As a result,

$$\begin{aligned} \Delta_{ik} = \theta \sum_{T \ni i,k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L + \theta \sum_{T \ni i, T \not\ni k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L \\ + \theta \sum_{T \not\ni i, T \ni k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T})\right] L, \end{aligned} \tag{B.3}$$

where we are using the fact that $\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) = \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})$ whenever $i, k \notin T$. Rewriting the last summation on the right-hand side of (B.3) using the change of variables $\tilde{T} = T \cup \{i\} \setminus \{k\}$ leads to

$$\begin{aligned} \Delta_{ik} = \theta \sum_{T \ni i,k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L + \theta \sum_{T \ni i, T \not\ni k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L \\ + \theta \sum_{\tilde{T} \ni i, \tilde{T} \not\ni k} q(\tilde{T} \cup \{k\} \setminus \{i\}) \left[\mathcal{A}(\mathbf{G}\,|_{\tilde{T}\setminus\{i\}}) - \mathcal{A}(\mathbf{G}\,|_{\tilde{T}\cup\{k\}\setminus\{i\}})\right] L. \end{aligned}$$

Equation (A.12) together with the assumption that all bottom-tier firms have identical bargaining powers implies that $q(\tilde{T} \cup \{k\} \setminus \{i\}) = q(\tilde{T})$ for any set $\tilde{T}$ such that $\tilde{T} \ni i$ and $\tilde{T} \not\ni k$. Hence,

$$\Delta_{ik} = \theta \sum_{T \ni i,k} q(T) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L + \theta \sum_{T \ni i,k} q(T \setminus \{k\}) \left[\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) - \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})\right] L.$$

Note that whenever $i, k \in T$, then $\mathcal{A}(\mathbf{G}\,|_{T\setminus\{k\}}) > \mathcal{A}(\mathbf{G}\,|_{T\setminus\{i\}})$ if and only if $A_i > A_k$. Therefore, $\Delta_{ik} > 0$ whenever $A_i > A_k$. This means that if $A_i > A_k$, then for any given level of aggregate TFP, $\bar{A}$, the marginal benefit to firm k of keeping its supplier is smaller than that to firm i. Hence, as $\bar{A}$ declines, firm k drops its supplier before firm i whenever $A_i > A_k$. □

Proof of Proposition A.1

To establish supermodularity at the extensive margin (Assumption 1(a)) we start by deriving an expression for $\mathcal{A}(\mathbf{G})$ in terms of the production network $\mathbf{G}$. Recall that $\mathcal{A}(\mathbf{G})$ denotes the economy's aggregate productivity when a social planner chooses firms' technologies I_i and the corresponding quantities to maximize aggregate output. Theorem 3(a) of Acemoglu and Azar (2020) establishes that to solve for the efficient allocation, one can simply focus on the competitive equilibrium, in which all firms price at marginal cost. We thus derive the expression for $\mathcal{A}(\mathbf{G})$ by first solving for prices in the competitive equilibrium of the economy with production network $\mathbf{G}$ and then using the fact that $\mathcal{A}(\mathbf{G}) = w/\operatorname{mc}_0(\mathbf{G})$, where $\operatorname{mc}_0(\mathbf{G})$ is the marginal cost of the final good producer.

Under marginal cost pricing, firm i sets the same price irrespective of the identity of the customer it is selling to. Let p_i denote the price set by firm i. The production function in (A.3), together with marginal cost pricing, implies that

$$p_i = \min_{I_i: \substack{ik\in\mathbf{G} \\ k\in I_i}} \left\{ \left(\alpha_i w^{1-\sigma} + \sum_{j\in I_i} \gamma_{ij}(\min\{p_j/A_{ij}, w\})^{1-\sigma} + \sum_{j\notin I_i} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)} \right\}$$

$$= \left(\alpha_i w^{1-\sigma} + \sum_{j:ij\in\mathbf{G}} \gamma_{ij}(\min\{p_j/A_{ij}, w\})^{1-\sigma} + \sum_{j:ij\notin\mathbf{G}} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)}.$$

Since $\alpha_i + \sum_{j=1}^{n} \gamma_{ij} = 1$, we can bound the right-hand side of the above equation from above by w, thus establishing that $p_i \le w$ for all firms i. When paired with the assumption that $A_{ij} \ge 1$, this implies that

$$p_i = \left(\alpha_i w^{1-\sigma} + \sum_{j:ij\in\mathbf{G}} \gamma_{ij}(p_j/A_{ij})^{1-\sigma} + \sum_{j:ij\notin\mathbf{G}} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)}$$

for all $i = \{1, \ldots, n\}$. When $\sigma \neq 1$, we can rewrite this system of equations in vector form as follows:

$$p^{\circ(1-\sigma)} = \alpha w^{1-\sigma} + (\mathbf{G} \circ \mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)}) p^{\circ(1-\sigma)} + ((\mathbf{1}\mathbf{1}' - \mathbf{G}) \circ \mathbf{\Gamma})\mathbf{1} w^{1-\sigma}.$$

In the above expression, $p = (p_1, \ldots, p_n)'$ denotes the vector of input prices, $\circ$ denotes the Hadamard (i.e., element-wise) product and power, $\mathbf{A} = [A_{ij}] \in \mathbb{R}^{n\times n}$ is a square matrix of pairwise productivities, $\mathbf{\Gamma} = [\gamma_{ij}] \in \mathbb{R}^{n\times n}$, and with some abuse of notation, we use $\mathbf{G}$ to denote a square binary matrix that captures pairwise supplier-customer relationships in network $\mathbf{G}$. Solving the above system of equations, we have

$$p^{\circ(1-\sigma)} = \mathbf{Q}(\alpha + ((\mathbf{1}\mathbf{1}' - \mathbf{G}) \circ \mathbf{\Gamma})\mathbf{1}) w^{1-\sigma} = \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma})\mathbf{1} w^{1-\sigma},$$

where $\mathbf{Q} = (\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)})^{-1}$ and we are using the fact that $\alpha = \mathbf{1} - (\mathbf{1}\mathbf{1}' \circ \mathbf{\Gamma})\mathbf{1}$ to establish the second equality. Hence, the marginal cost of firm 0 satisfies $\mathrm{mc}_0^{1-\sigma} = \gamma_0'^{\circ(1-\sigma)} = \gamma_0' \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma})\mathbf{1} w^{1-\sigma}$, where $\gamma_0 = (\gamma_{01}, \ldots, \gamma_{0n})$. This, together with $\mathcal{A}(\mathbf{G}) = w/\mathrm{mc}_0$ leads to the following expression for aggregate productivity:

$$\mathcal{A}(\mathbf{G}) = \left[\gamma_0' \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma})\mathbf{1}\right]^{1/(\sigma-1)}. \tag{B.4}$$

Note that, the assumption that matrix $\mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)}$ has a subunit spectral radius guarantees that $\mathbf{Q}$ is an inverse M-matrix and hence is element-wise nonnegative.

With the expression for aggregate productivity at hand, we next show that (B.4) satisfies the inequality in (9). Note that $\mathcal{A}(\mathbf{G})$ is a function defined over the set of binary matrices $\mathbf{G}$ with elements $g_{ij} \in \{0, 1\}$. We consider the extension of the expression in (B.4), which we denote by $\bar{\mathcal{A}}(\mathbf{G})$, by assuming that g_{ij} can take any value in the unit interval $[0, 1]$ and establish that

$$\frac{\partial^2 \bar{\mathcal{A}}}{\partial g_{ij} \partial g_{kr}} \ge 0 \tag{B.5}$$

for any all $ij \neq kr$. If (B.5) is satisfied, then $\bar{\mathcal{A}}(\max\{\mathbf{G}_1, \mathbf{G}_2\}) + \bar{\mathcal{A}}(\min\{\mathbf{G}_1, \mathbf{G}_2\}) \geq \bar{\mathcal{A}}(\mathbf{G}_1) + \bar{\mathcal{A}}(\mathbf{G}_2)$ for any pair of matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ with elements in the unit interval. The fact that $\bar{\mathcal{A}}(\mathbf{G}) = \mathcal{A}(\mathbf{G})$ for any binary matrix $\mathbf{G}$ then establishes that $\mathcal{A}(\mathbf{G})$ is a supermodular function of the production network $\mathbf{G}$, thus establishing (9) and Assumption 1(a).

To establish (B.5), observe that

$$\frac{\partial \bar{\mathcal{A}}}{\partial g_{ij}} = \frac{1}{\sigma - 1}\gamma_{ij}(\gamma_0' \mathbf{Q} e_i) \left(A_{ij}^{\sigma-1}(p_j/w)^{1-\sigma} - 1 \right) \bar{\mathcal{A}}^{2-\sigma}, \tag{B.6}$$

where e_i denotes the i-th unit vector. Since $\mathbf{Q}$ is element-wise nonnegative, it is immediate that $\gamma_0' \mathbf{Q} e_i \geq 0$. Furthermore, since $A_{ij} \geq 1$ and $p_j \leq w$, the expression $A_{ij}^{\sigma-1}(p_j/w)^{1-\sigma} - 1$ always has the same sign as $\sigma - 1$. Consequently, $\partial \bar{\mathcal{A}}/\partial g_{ij} \geq 0$. Next, observe that differentiating (B.6) implies that

$$\frac{\partial^2 \bar{\mathcal{A}}}{\partial g_{ij} \partial g_{kr}} = (2-\sigma)\frac{1}{\bar{\mathcal{A}}}\frac{\partial \bar{\mathcal{A}}}{\partial g_{ij}}\frac{\partial \bar{\mathcal{A}}}{\partial g_{kr}} + \gamma_{ij} A_{ij}^{\sigma-1} \frac{\partial \bar{\mathcal{A}}}{\partial g_{kr}} \frac{\gamma_0' \mathbf{Q} e_i}{\gamma_0' \mathbf{Q} e_k} e_j' \mathbf{Q} e_k + \gamma_{kr} A_{kr}^{\sigma-1} \frac{\partial \bar{\mathcal{A}}}{\partial g_{ij}} \frac{\gamma_0' \mathbf{Q} e_k}{\gamma_0' \mathbf{Q} e_i} e_r' \mathbf{Q} e_i.$$

As we already established, $\partial \bar{\mathcal{A}}/\partial g_{ij} \geq 0$. This, together with the fact that $\mathbf{Q}$ is element-wise nonnegative, guarantees that the second and third terms on the right-hand side of the above equation are nonnegative. Hence, (B.4) is trivially satisfied for all $\sigma \leq 2$. □