Inequality and Measured Growth*

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July 11, 2023

Abstract

Compared to a half-century ago, inequality in the United states has risen and measured productivity growth has fallen. Concerns about rising inequality have been exacerbated by the observation that prices of goods consumed by the poor have risen faster than prices of goods consumed by the rich. This paper presents an example of an economy that is consistent with these facts but in which the facts can be misleading about improvements in welfare. The two key ingredients are non-homothetic preferences and productivity improvements directed toward goods with larger market size. The model admits balanced growth despite the structural change induced by non-homothetic preferences. Along a BGP in which the distribution of after-tax income is stable, measured inflation among goods consumed by the bottom half of earners is perpetually higher than among goods consumed by the top half, but welfare improves at the same rate for all households. Across BGPs in which the only difference in primitives is the progressivity of the tax schedule, the BGP with a more unequal distribution of after-tax income exhibits lower measured growth of output and productivity. Nevertheless, welfare improves at the same rate along both BGPs. At the root of the deviation between productivity growth and welfare improvements is the fact that the value of cost reductions for a good are transitory if income effects eventually shrink the good's expenditure share. Standard measures of inflation capture the benefits of cost reductions among goods that are consumed contemporaneously, but only partly determine the evolution of price levels relevant for a household, as they do not capture the benefits from cost reductions that occur before the household shifts towards a good.

KEYWORDS: Inequality, Non-homothetic Preferences, Balanced Growth, Welfare Measurement, Inflation, Structural Change

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*I am grateful to David Baqaee, Ariel Burstein, Philipp Grübener, Ernest Liu, Ricardo Reis, Richard Rogerson, Felipe Schwartzman, and especially Danial Lashkari for conversations that improved this paper. I thank Nan Xiang for excellent research assistance. All mistakes are my own.

1 Introduction

Many have noted several worrying trends over the last half century in the United States and elsewhere. Chief among them are slowing productivity growth and rising inequality. Concerns about the increase in dispersion of nominal income have been exacerbated by the recent observation that prices of goods consumed by low-income households have been rising faster than goods consumed by high-income households.

This paper presents a simple model that is consistent with these facts in the following sense: Along a balanced growth path in which the distribution of nominal incomes remains fixed, measured inflation is perpetually higher for low-income households than for high-income households. Comparing two BGPs in which the only difference in primitives is the distribution of after-tax income,[1] the BGP with more inequality has lower measured TFP growth and a larger gap in inflation between the rich and poor. Nevertheless, I argue that these observations may be misleading about improvements in welfare. Along a BGP, despite the perpetual gap in measured inflation, welfare improves at the same rate for all, in a sense I will be precise about. Across BGPs, despite the gap in measured output and productivity growth, there is no gap in the rate of welfare improvements. Further, even though the BGP with more inequality exhibits a larger gap in inflation between the rich and poor, the productivity improvements that cause this actually ameliorate inequality rather than exacerbate it.

The key ingredients that drive these results are non-homothetic preferences and market-size-driven productivity growth. In the model, households have non-homothetic preferences over goods that range over the real line, with those that spend more shifting consumption to higher-ranked goods. As incomes rise, the balanced growth path exhibits a flying geese pattern in the spirit of Matsuyama (2002), Foellmi and Zweimüller (2006), and Bohr, Mestieri and Yavuz (2022). Each good is initially a luxury and eventually a necessity. Growth is driven by two forces: exogenous broad-based technology improvements that reduce the cost of producing all goods, and good-specific technology improvements that come from learning by doing: cost reductions are proportional to the labor used to produce the good.[2] Individuals supply labor inelastically and are heterogeneous in

[1] The change in the distribution of after-tax income could be driven by a change in the progressivity of the tax schedule or in the distribution of skills; in the model these are isomorphic

[2] The results would be similar if the learning by doing were replaced with good-specific innovations induced by

their endowments of efficiency units of labor. Labor income is redistributed through a progressive income tax. Because preferences are non-homothetic, when consumption expenditures are more equally distributed, there is more overlap in the consumption bundles of higher- and lower-income households.

Measured growth of aggregate output is a Divisia index of the output growth of individual goods, consistent with national accounting practices. Learning-by-doing leads to larger cost reductions for goods that are consumed more, and these are precisely the goods that count more in a Divisia index. As a result, if the aggregate consumption bundle is concentrated on a narrower range of goods, measured growth is higher. With a more even distribution of income, household consumption bundles overlap more, leading to higher measured growth.

The model features a balanced growth path in which household consumption bundles are traveling waves that travel at the same speed. As a household's expenditure increases it consumes higher ranked goods and larger quantities, but the shape of its expenditure shares remains constant. Improvements in welfare are not well-captured by measured productivity growth. Measured productivity growth captures the contemporaneous cost reductions in goods that are currently consumed and chains together these instantaneous growth rates. But the value of cost reductions for any particular good is temporary; eventually, as expenditures rise, households shift away from that good to even higher ranked goods. Thus the value of any good-specific cost reduction eventually depreciates at a rate determined by the speed of the traveling wave. Along a BGP, the speed of the traveling wave, which turns out to be a sufficient statistic for the rate of welfare improvement, is determined only by improvements in broad-based technology. Good-specific productivity improvements are valuable, but because their economic value is transitory, they lead to a level effect rather than a growth rate effect. To summarize, if changes in inequality cause changes in measured growth rates by inducing cost reductions in relatively larger markets, these cost reductions may not be relevant for long-run welfare gains, as households continually shift to goods for which cost reductions haven't happened yet.

While inequality in nominal incomes has risen over the last several decades, several papers have recently documented that prices of the goods consumed by low income households have been rising faster than the prices of goods consumed by high income households (Argente and Lee (2021),

market size, but measurement would be less transparent.

Kaplan and Schulhofer-Wohl (2017), Jaravel (2019, 2021), Jaravel and Lashkari (2022),[3] and that a large portion of this gap in inflation rates is due to differences in the rates of innovation directed toward these goods.[4] Section 5 shows that this is exactly the pattern one would observe along a BGP of the model presented here: measured inflation for the rich is perpetually lower than measured inflation for the poor. Since those with higher incomes spend more and cost reductions are larger for goods with larger markets, the goods consumed by those with higher incomes will experience larger reductions in cost. Nevertheless, this gap in measured inflation can be misleading. Along the BGP, the welfare relevant consumption index improves for all households at the same rate. Thus even though the distribution of nominal expenditures is fixed, the gap in inflation rates does not signal widening gap in welfare.

Why don't unequal inflation rates contribute to widening inequality along a balanced growth path? Fundamentally, welfare differences across income groups depend on the differences in the level of prices of the goods they are consuming, not the rate of change of those prices. The rate of change of prices is not a reliable measure of the level of prices. Those with low income certainly do benefit from cost reductions of the goods they consume while they are consuming the goods, but they also benefit from cost reductions for those goods before they begin to consume them. Measures of contemporaneous inflation will capture the former, but not the latter. The cost reductions among goods consumed by high income households indeed benefit the low-income households, it is just that those benefits accrue later.

[3] Argente and Lee (2021) and Jaravel (2019, 2021) have estimated gaps of roughly 0.5 percentage points per year between the top and bottom quintiles of the income distribution. These studies use bar-code level data from the Nielsen Consumer Panel, which is much more granular and makes it easier to measure changes in price for the same good over time. Unfortunately this data is only available since 2004. Several earlier papers in the literature such as Hobijn and Lagakos (2005) and McGranahan and Paulson (2005) measured expenditures using Consumer Expenditure Survey (CEX) data and price changes using CPI data and did not find large gaps in inflation rates across demographic groups. Jaravel (2019, 2021) has emphasized that the gaps in inflation rates is mostly a within sector phenomenon, and grow larger the more one disaggregates. Indeed, using CEX data, studies have found small gaps in inflation when using relatively aggregated industries (roughly 20 categories), but larger gaps comparable to those found with the Neilsen panel when using more finely disaggregated categories, e.g., Jaravel (2019), Klick and Stockburger (2021), Orchard (2022), Jaravel and Lashkari (2022). Note that to the extent shopping behavior or ability to substitute may differ systematically across the income distribution, this would lead to a difference in price levels, not a persistent gap in price changes. See Jaravel (2021) for a good survey of the literature.

[4] There is a growing body of evidence that consumer demand has determined the direction of innovation. While market size and innovation are jointly determined in equilibrium, Acemoglu and Linn (2004) address endogeneity by using shifts in demand for pharmaceutical products driven by demographic change. Boppart and Weiss (2012), Jaravel (2019), Beerli et al. (2020) have applied this strategy to a broader set of sectors and found that sectors that saw increased demand due to demographic shifts have experienced higher rates of innovation and price growth. Bohr, Mestieri and Yavuz (2022) show that sectors with higher income elasticities experienced later peaks, experienced lower price growth, and saw higher growth of patents.

Each household's expenditure can be decomposed into a price index and a consumption index. The change in a household's price index over time can be decomposed into two parts. The first part captures the change in prices, holding the expenditure shares fixed. This corresponds to measured inflation. The second part captures how shifts in expenditure shares alter the price level. Since households are constantly shifting to higher ranked goods for which there has been less cumulative learning by doing, this component raises the price level and offsets some of the price declines of the first component. This corresponds to the loss of value over time of good-specific cost reductions.[5]

Looking across BGPs, a BGP with more inequality in expenditures will have a larger gap in inflation rates across income groups. Cost reductions are larger for goods with a larger market size which tends to be goods consumed by the rich. When inequality is low, there is more overlap in the consumption bundles of the rich and poor. Since the goods consumed by the poor are also more likely to consumed by the rich, the market size for these goods is large and those goods experience productivity growth *while the poor are consuming them*. Thus the gap in measured inflation would be lower.

But again, this is misleading about the welfare implications. While the poor benefit from the cost reductions that occur while they are consuming a good, they would benefit even more if those cost reductions happened before they shift consumption toward that good. Thus a lower inflation rate for the poor is a signal that the price level is not as low as it might otherwise be.

The results that welfare growth is the same across individuals along a BGP or across BGPS relies on using a particular family of utility functions for which constant growth of the consumption index corresponds to constant growth of welfare. Of course, preferences can be equally well represented by any monotone transformation of a utility function. Outside of the particular class of utility functions, utilities would not grow at constant rates, and hence I would not be able to make the statement that welfare grow at the same rate across individuals. In Section 6, explore how the results extend to other utility functions. In particular, I formalize the notion that, in the model, there is no systematic relationship between measured real income growth—growth of nominal income minus measured inflation—and improvements in welfare. I first review the classic results of Kloek (1967) and Theil (1968) that tightly link measured real income growth to welfare improvement–as

[5] In practice, measuring this second component may be hard. Outside of this simple model, it is not clear how one might determine which goods have high prices and which have low prices.

measured by a money metric of utility–over short horizons. I then show, however, that there is no utility function that consistently assigns higher welfare growth to instances of higher measured real income growth. These results can be reconciled by noting the well-known fact that any money metric of utility is specific to a reference price vector[6], and the correspondence between real income growth and welfare growth applies only to the money metric that uses *contemporaneous prices* as reference prices. If one uses as a utility function a money metric associated with time t_0 reference prices, it will assign a higher welfare growth to the individual with persistently higher real income growth than to an individual with lower real income in the interval $[t_0, t_0 + dt]$, but there must be another time period t_1 where, *according to that same utility function*, the ranking of welfare growth over $[t_1, t_1 + dt]$ is reversed.

1.1 Related Literature

While the arguments of this paper are likely to be relevant in any setting with non-homothetic preferences and market-size driven cost reductions, they is particularly clear in a setting with a BGP. With a BGP, there are transparent analytical expressions which clearly show which features of the economy are relevant for measured productivity growth, welfare growth, and household-specific inflation. Models in which non-homothetic preferences leads to structural change often do not exhibit balanced growth paths in the usual sense, as households shift across goods or sectors with different (but constant) productivity growth rates.[7] An alternative approach, which I follow in this paper, is to allow for an infinite range of goods that follow some hierachical pattern. In

[6]Given a reference price vector, the corresponding money metric of utility of a consumption bundle is the minimum expenditure required to make the individual as well off as consuming that bundle.

[7]One approach to study long run outcomes with structural change, pioneered by Kongsamut, Rebelo and Xie (2001), is to assume that investment is produced using a linear technology using only capital. This allows for the possibility of a "generalized balanced growth path" in which there is a constant interest rate in units of investment goods and a constant growth rate of output when measured in units of investment goods. Kongsamut, Rebelo and Xie (2001) and Comin, Lashkari and Mestieri (2021) use this approach and generate structural change using non-homothetic preferences (Stone-Geary and non-homothetic CES respectively), while in Ngai and Pissarides (2007), sectoral shares change because of shifts in relative prices of sectoral output and an assumption that these outputs are complements. Boppart (2014) develops a model that allows for both sources of structural change. See Herrendorf, Rogerson and Valentinyi (2013) for a unifying perspective on the role of relative prices and non-homotheticity in causing structural change. Herrendorf, Rogerson and Valentinyi (2021) recently showed that there is structural change in the investment sector as well, which contrasts with the assumption of a linear and fixed investment technology that uses only capital, and they argue that this makes constant within-sector growth rates incompatible with constant long run growth rates. Acemoglu and Guerrieri (2008) and Buera et al. (2020) focus on medium-run transition dynamics. Equilibria of most models of structural change converge asymptotically to an economy without structural change. Kongsamut, Rebelo and Xie (2001) converges to a BGP with stable, interior sectoral shares, while Ngai and Pissarides (2007) and Comin, Lashkari and Mestieri (2021) converge to a one sector economy dominated by services.

Zweimüller (2000), Zweimüller and Brunner (2005), and Foellmi and Zweimüller (2006, 2008), preferences are non-homothetic in that those with higher expenditures spread their consumption bundle across a wider range of goods. In these models, balanced growth can occur as the range of goods consumed expands indefinitely. I take a closely-related but different approach in that in my model, as incomes rise along a balanced growth path, those consumption bundles follows a traveling wave, and households shift their expenditures to higher ranked goods. Bohr, Mestieri and Yavuz (2022) also study a model with a traveling wave using non-homothetic CES preferences, but with a different weighting function. I describe the relationships between the utility functions in Appendix A.1.

Several papers have linked non-homotheticity with Schmookler's (1967) demand-driven innovation to argue that the distribution of income will affect the pace of innovation, including Matsuyama (2002), Zweimüller (2000), and Foellmi and Zweimüller (2006).[8] One of the basic positive predictions here can be found in the literature: changes in the distribution of expenditures can increase the scale effects that come with innovation or with learning by doing, and affect the growth rate.[9] While those papers have focused on the qualitative relationship between inequality and growth, whether there might be multiple equilibria, and the feedback of redistribution on inequality, this paper focuses on measurement issues, with an eye toward interpreting the US experience of the last half century and empirical regularities found in the literature on heterogeneous inflation.

2 Model

2.1 Households

There is a unit continuum of households with identical preferences. The households each supply labor inelastically but differ in their endowments of efficiency units of labor, ℓ, which are distributed across households according to the distribution function $G(\ell)$.

[8] Bohr, Mestieri and Yavuz (2022) features directed technical change, but a representative household rather than heterogeneous households.

[9] Interest in the empirical relationship between inequality and growth dates back to at least the seminal work of Kuznets (1955). Many early studies using cross-country regressions found that countries with higher inequality experienced lower growth (Perotti (1996) and Benabou (1996) survey the literature). However, as with many other applications of cross-country regressions, results were sensitive to specification (Forbes (2000), Barro (2000)). In any case, the many joint determinants of inequality and growth make it difficult to tease out causal relationships.

I assume the tax schedule takes the following form:

$$T(y) = y - \bar{y}^{\tau} y^{1-\tau} \tag{1}$$

The tax schedule implies that after-tax income is a log-linear function of pre-tax income. τ indexes the progressivity of the tax schedule; with $\tau = 0$ after-tax income matches pretax income, while with $\tau = 1$ all households have the same after-tax income. $\bar{y}$ is chosen so that the government budget is balanced. This family of tax schedules has often been used in public finance and macroeconomics literatures, e.g., Benabou (2000, 2002), and Heathcote, Storesletten and Violante (2017) show that it provides a very good approximation of the US tax and transfer system. If an individual with endowment ℓ has pre-tax income of $w\ell$, a balanced budget requires that her after-tax income is $w\ell^{1-\tau}/\overline{\ell^{1-\tau}}$, where $\overline{\ell^{1-\tau}} \equiv \int \ell^{1-\tau} dG(\ell)$.

2.2 Preferences

There is an infinite continuum of goods, indexed by $i \in (-\infty, \infty)$. Consider a household with a budget of E facing prices $\{p_i\}$. I assume that the household's preferences over consumption bundles $\{c_i\}$ can be summarized by a consumption index $C(\{c_i\})$, so that its preferences can be represented by $u\big(C(\{c_i\})\big)$, where u is a strictly increasing function and the consumption index $C(\{c_i\})$ is defined as the largest number C such that

$$\left[\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big)^{\frac{1}{\sigma}} \left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} \geq 1 \tag{2}$$

with $\int_{-\infty}^{\infty} h(i)di = 1$ and $\sigma > 0$. Typically, and specifically under the assumptions of the model discussed below, households will consume bundles $\{c_i\}$ in which the left hand side of (2) is strictly decreasing in C, so that there is a unique value of C such that (2) holds with equality. Preferences here are similar to a weighted Dixit-Stiglitz utility function, except that the weights $h\Big(i - \gamma \log C\Big)$ are endogenous and depend on the overall consumption index C. As C rises, more weight is put on goods with higher i. One way to see this is to use the change of variables $u = i - \gamma \log C$ to express (2) as

$$\left[\int_{-\infty}^{\infty} h(u)^{\frac{1}{\sigma}} \left(\frac{c_{u+\gamma \log C}}{C}\right)^{\frac{\sigma-1}{\sigma}} du\right]^{\frac{\sigma}{\sigma-1}} \geq 1$$

Here one can see that, as C rises, the shape of the preference weights remains the same but the weights apply toward higher ranked goods. γ indexes the strength of the non-homotheticity. If $\gamma = 0$, preferences would be homothetic.

Given prices and income, the household's problem can be separated into two parts: Expenditure minimization given C, and then the optimal choice of C given the budget constraint. Define $\mathcal{E}(C) \equiv \inf_{\{c_i\}} \int_{-\infty}^{\infty} p_i c_i di$ subject to (2) to be the minimal expenditure that delivers a consumption index C given prices. Because C is fixed, this is just the standard expenditure minimization with weighted Dixit-Stiglitz preferences, and yields

$$\mathcal{E}(C) = C\left[\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big) p_i^{1-\sigma} di\right]^{\frac{1}{1-\sigma}} \tag{3}$$

(3) suggests that the consumption index C has a natural dual price index,

$$\mathcal{P}\left(\{p_i\}; C\right) \equiv \left(\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big) p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}$$

The price index is similar to a weighted Dixit-Stiglitz price index, except that, again, the weights $h\Big(i - \gamma \log C\Big)$ are endogenous and depend on the consumption index C.

The second step is to choose the maximal affordable consumption bundle, i.e., the largest C such that $\mathcal{E}(C) \leq E$. I next provide regularity conditions that ensure that there is a unique consumption index C that satisfies $\mathcal{E}(C) = E$, and that the solution is interior.

Proposition 1 *Suppose the price schedule p_i is weakly increasing in i and $p_i^{1-\sigma}$ is Lipschitz. Then the optimal bundle for a household with expenditure E is*

$$c_i = p_i^{-\sigma} E^{\sigma} C^{1-\sigma} h\Big(i - \gamma \log C\Big)$$

where C is the unique solution to

$$E = C\mathcal{P}\left(\{p_i\}; C\right) \equiv C\left(\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big) p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} \tag{4}$$

The condition that p_i is weakly increasing in i ensures that $\mathcal{E}(C)$ is strictly increasing, that $\lim_{C\to 0} \mathcal{E}(C) = 0$, and that $\lim_{C\to\infty} \mathcal{E}(C) = \infty$. The condition that $p_i^{1-\sigma}$ is Lipschitz ensures that $\mathcal{E}(C)$ is continuous. These conditions can be relaxed; Appendix A.2 provides a weaker set of sufficient conditions.[10] Nevertheless, the conditions will naturally be satisfied along a balanced growth path given the structure of the model.

2.3 A Simple Example

Suppose that all prices are the same, $p_i = p$. In this case, the price index for a household with consumption index C is $\mathcal{P}(\{p_i\}; C) = p$, so that $C = \frac{E}{p}$. The household's consumption of good i is thus

$$c_i = \frac{E}{p} h\left(i - \gamma \log \frac{E}{p}\right)$$

Written in this way, one can see that if the household has a higher expenditure E (relative to the price level p) by a factor of a, then it both scales up consumption by a factor of a and shifts consumption toward higher ranked goods by an increment of $\gamma \log a$. Note these properties do not depend on the particular functional form for h.[11]

To see this visually, I assume the weighting function h and the distribution of endowments of effective labor take Gaussian functional forms:

(a) The weighting function takes a Gaussian form:

$$h(u) = \frac{1}{\sqrt{2\pi v_h}} e^{-\frac{u^2}{2v_h}}$$

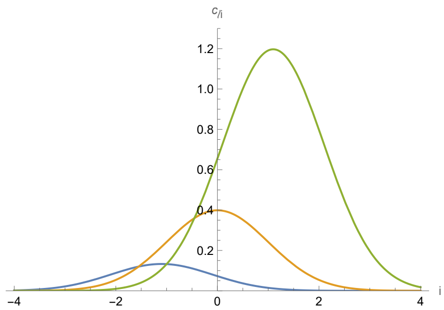
(b) The distribution of household endowments of effective labor is lognormal with mean normalized to 1,

$$G'(\ell) = \frac{1}{\ell} \frac{1}{\sqrt{2\pi v_\ell}} e^{-\frac{(\log \ell + v_\ell/2)^2}{2v_\ell}}$$

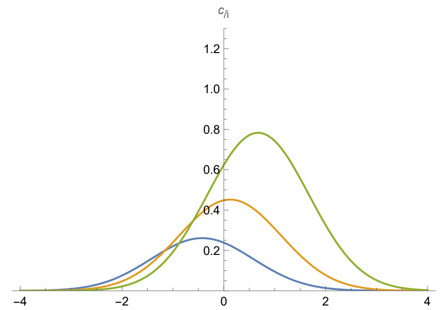
I choose units of effective labor so that the mean of effective labor across households is 1, and normalize the mean of the weighting function so that it peaks at $u = 0$. As a result, each of these

[10] Appendix A.2 shows that there is a unique solution to $\mathcal{E}(C) = E$ as long as prices do not decline too rapidly with i. What could go wrong? If prices decline too rapidly asymptotically, the household can attain infinite utility by taking $C \to \infty$. Even if prices are well-behaved asymptotically, the equation $C\left(\int_{-\infty}^{\infty} p_i^{1-\sigma} h\Big(i - \gamma \log C\Big) di\right)^{\frac{1}{1-\sigma}} = E$ can have multiple interior solutions if prices decline too rapidly in a range. To see this, starting with an interior solution for C, increasing C shifts the household toward higher ranked goods. If prices decline fast enough with i, the household can afford enough of those goods to satisfy $\left(\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big)^{\frac{1}{\sigma}} \left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} > 1$.

[11] Another case that is easy to characterize is when the price schedule is log-linear, $p_i = p_0 e^{\kappa i}$ with $\kappa > -\frac{1}{\gamma}$.



(a) $\tau = 0.0$



(b) $\tau = 0.5$

Figure 1 Expenditure Profiles for Three Households

Note: This figure shows the consumption quantities for three households when the price schedule is constant and weighting functions and skill distributions are Gaussian. The households have effective labor of 1/3, 1, and 3. The left panel shows an economy with a progressivity of $\tau = 0.0$, while the right panel shows an economy with progressivity of $\tau = 0.5$.

distributions depend on a single parameter. v_h indexes the tastes for variety: it controls the breadth of the consumption bundle chosen by the household. v_ℓ indexes the dispersion in endowments of effective labor.

Figure 1 shows an example of consumption profiles under the simple parameterization $v_h = v_\ell = \gamma = 1$ and $w = p$. The figure shows the level of expenditures across goods for three households, whose endowments are $\ell = \{\frac{1}{3}, 1, 3\}$. The left figure shows these households' consumption bundles when $\tau = 0$ so that incomes equal expenditure, while the right figure shows consumption bundles for the same three households when the tax schedule is more progressive, at $\tau = 0.5$. Comparing the three households, one can see that richer households consume higher ranked goods and larger quantities.

One feature that will play a larger role below is that when the tax schedule is more progressive, there is more overlap in household consumption bundles. The right panel of Figure 1 shows that with a more progressive tax schedule, expenditures are closer together so there is more overlap in consumption bundles. Figure 2 plots the distribution of aggregate expenditures across goods under each of the two tax schedules. With the Gaussian functional forms and log-linear tax schedule, the distribution of aggregate expenditures across goods is also Guassian: the share of expenditures on good i is normally distribution with variance $v_h + (1-\tau)^2\gamma^2 v_\ell$ and mean $\gamma(1-\tau)^2\frac{v_\ell}{2} + \gamma \log\frac{w}{p}$. Higher wages or lower prices cause households to shift to higher ranked goods. A more progressive tax schedule leads to a distribution of expenditure shares with a lower mean and variance. In

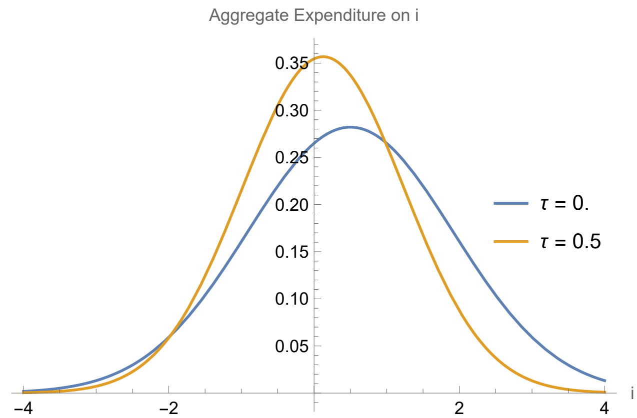


Figure 2 Distribution of Aggregate Expenditures Across Goods

Note: This figure shows the distribution of aggregate expenditures across goods for two economies, one with a tax progressivity of $\tau = 0.0$ and one with $\tau = 0.5$, when the price schedule is constant $p_i = w$ and weighting functions and skill distributions are Gaussian.

particular, the distribution of aggregate expenditures becomes more concentrated.

One particular measure of concentration which will be of use later is the Herfindahl-Hirschman Index of aggregate expenditures across goods: $HHI = \int_{-\infty}^{\infty} \omega_i^2 di$, where $\omega_i \equiv \frac{p_i y_i}{\int_{-\infty}^{\infty} p_{\tilde{i}} y_{\tilde{i}} d\tilde{i}}$ is the aggregate expenditure share on good i and $y_i = \int c_{\ell i} dG(\ell)$:

$$HHI = \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2\gamma^2 v_\ell}}$$

One can see both visually and analytically that a more equal distribution of expenditures leads to a higher HHI across goods.

2.4 Technology and Productivity Improvements

Each good is produced using a constant-returns-to-scale, labor-only technology

$$Y_{it} = A_t B_{it} L_{it}$$

where the productivity to produce a good has two components: (i) broad-based productivity A_t which is common to all goods, and (ii) good-specific productivity B_{it}. All agents are price takers, so the price of good i is equal to its unit cost, $p_{it} = \frac{w_t}{A_t B_{it}}$.

The broad-based technology A_t improves exogenously over time. Good-specific technology improves via learning by doing, according to $\log B_{it} = \phi \int_{-\infty}^{t} L_{is} ds$, so that

$$\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$$

As such, productivity improvements are directed toward goods for which expenditures are larger.[12]

2.5 Equilibrium

Given initial conditions $\{B_{it_0}\}$ and a tax policy τ, a competitive equilibrium is, for each instant $t > t_0$, a wage w_t, a set of prices $\{p_{it}\}_i$, an allocation of labor $\{L_{it}\}_i$, output $\{Y_{it}\}_i$, and consumption $\{c_{\ell it}\}_{i,\ell}$, consumption indices $\{C_{\ell t}\}_\ell$, good specific productivities $\{B_{it}\}_i$, and a tax schedule $T_t(y) = y - \bar{y}_t^\tau y^{1-\tau}$ such that at each instant, each household maximizes utility taking prices, wages, and the tax schedule as given; the representative firm maximizes static profit taking prices and wages as given; the government budget is balanced; each goods market clears; the labor market clears; and the evolution of technology is consistent with learning by doing.

In any equilibrium, we need to verify that markets clear. The market clearing condition for good i can be expressed as

$$L_{it} = \frac{Y_{it}}{A_t B_{it}} = \frac{1}{A_t B_{it}} \int c_{\ell it} dG(\ell)$$

Household ℓ's consumption of good i is $c_{\ell it} = p_{it}^{-\sigma} E_{\ell t}^{\sigma} C_{\ell t}^{1-\sigma} h\Big(i - \gamma \log C_{\ell t}\Big)$, its expenditure is equal to its after-tax income $E_{\ell t} = w_t \ell^{1-\tau} / \overline{\ell^{1-\tau}}$, and the price of good i is $p_{it} = \frac{w_t}{A_t B_{it}}$. Together, these imply that the market clearing conditions can be expressed as

$$L_{it} = \int \left({\ell^{1-\tau}} \big/ {\overline{\ell^{1-\tau}}} \right)^{\sigma} \left(\frac{C_{\ell t}}{A_t} \right)^{1-\sigma} \frac{h\Big(i - \gamma \log C_{\ell t}\Big)}{B_{it}^{1-\sigma}} dG(\ell) \tag{5}$$

where each $C_{\ell t}$ is the unique solution to household ℓ's budget constraint:

$$\ell^{1-\tau} \big/ \overline{\ell^{1-\tau}} = \frac{C_{\ell t}}{A_t} \left[\int \frac{h\Big(i - \gamma \log C_{\ell t}\Big)}{B_{it}^{1-\sigma}} di \right]^{\frac{1}{1-\sigma}} \tag{6}$$

If each goods market clears, all budget constraints hold with equality, and the government budget

[12] Results would be similar with directed technical change, but measurement of TFP growth would be less transparent.

is balanced, then Walras' Law implies that the labor market clears as well.

Equations (5) and (6), along with the equation defining learning by doing, $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$, are sufficient to completely characterize a dynamic equilibrium. Given technology at t, $A_t, \{B_{it}\}$, (6) pins down each household's consumption index, $\{C_{\ell t}\}$. Given these, (5) pins down the allocation of labor across goods L_{it}. In turn, the allocation of labor determines the evolution of good-specific technologies.

3 A Balanced Growth Path

Suppose that the tax policy is fixed over time and broad-based productivity grows at a constant rate, $\frac{\dot{A}_t}{A_t} = g$. This section shows that there is a balanced growth path in which each household's consumption profile is a traveling wave. All of these waves grow and travel at the same speed.

Given the growth rate g of broad-based productivity, I define a balanced growth path to be an equilibrium in which each household's consumption index grows at a constant rate g and in any time increment $t_1 - t_0$, the profile of labor used to produce each good shifts to the right on the real line by $\Delta \equiv g\gamma(t_1 - t_0)$, so that $L_{it_0} = L_{i+\Delta,t_1}$.

Along such a balanced growth path, good-specific productivity shifts to the right by Δ as well:

$$\log B_{it_0} = \phi \int_0^\infty L_{i,t_0-s} ds = \phi \int_0^\infty L_{i+\Delta,t_1-s} ds = \log B_{i+\Delta,t_1}$$

To see that such a path can be consistent with balanced growth, note that (5) along with $\frac{h\left((i+\Delta)-\gamma \log C_{\ell t_1}\right)}{B_{i+\Delta,t_1}^{1-\sigma}} = \frac{h\left(i-\gamma \log C_{\ell t_0}\right)}{B_{it_0}^{1-\sigma}}$ and $\frac{C_{\ell t_1}}{A_{t_1}} = \frac{C_{\ell t_0}}{A_{t_0}}$ imply that $L_{i+\Delta,t_1} = L_{i,t_0}$, and thus if (5) and (6) are satisfied at t_0, they are also satisfied at t_1.

In Appendix B I show the existence and uniqueness of a BGP when the elasticity of substitution across goods is not too large.

Proposition 2 *If $e^{(\sigma-1)\frac{\phi L}{\gamma g}} < 2$, there exists a unique balanced growth path.*

Along the balanced growth path, output and consumption are growing waves:

$$Y_{i+\Delta,t_1} = A_{t_1} B_{i+\Delta,t_1} L_{i+\Delta,t_1} = \frac{A_{t_1}}{A_{t_0}} A_{t_0} B_{it_0} L_{it_0} = \frac{A_{t_1}}{A_{t_0}} Y_{it_0}$$

and, using $p_{it} = \frac{w_t}{A_t B_{it}}$ and $E_{\ell t} = w_{t_1} \ell^{1-\tau} / \overline{\ell^{1-\tau}}$,

$$\begin{aligned}
c_{\ell,i+\Delta,t_1} &= \left(\frac{w_{t_1}}{A_{t_1} B_{i+\Delta,t_1}} \right)^{-\sigma} \left(w_{t_1} \frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}} \right)^{\sigma} C_{\ell t_1}^{1-\sigma} h\left((i+\Delta) - \gamma \log C_{\ell t_1}\right) \\
&= \frac{A_{t_1}}{A_{t_0}} \left(\frac{w_{t_0}}{A_{t_0} B_{it_0}} \right)^{-\sigma} \left(w_{t_0} \frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}} \right)^{\sigma} C_{\ell t_0}^{1-\sigma} h\left(i - \gamma \log C_{\ell t_0}\right) \\
&= \frac{A_{t_1}}{A_{t_0}} c_{\ell i t_0}
\end{aligned}$$

Finally, each household's expenditure shares are traveling waves. Let $\omega_{\ell it} \equiv \frac{p_{it} c_{\ell it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} c_{\ell \tilde{i} t} d\tilde{i}}$ be the fraction of ℓ's expenditure on good i at time t. Then, again using $p_{it} = \frac{w_t}{A_t B_{it}}$, we have

$$\omega_{\ell,i+\Delta,t_1} = \frac{\frac{w_{t_1}}{A_{t_1} B_{i+\Delta,t_1}} c_{\ell,i+\Delta,t_1}}{\int \frac{w_{t_1}}{A_{t_1} B_{\tilde{i}+\Delta,t_1}} c_{\ell,\tilde{i}+\Delta,t_1} d\tilde{i}} = \frac{\frac{w_{t_1}}{A_{t_1} B_{it_0}} \left(\frac{A_{t_1}}{A_{t_0}} c_{\ell i t_0} \right)}{\int \frac{w_{t_1}}{A_{t_1} B_{\tilde{i}t_0}} \left(\frac{A_{t_1}}{A_{t_0}} c_{\ell \tilde{i} t_0} \right) d\tilde{i}} = \frac{\frac{w_{t_0}}{A_{t_0} B_{it_0}} c_{\ell i t_0}}{\int \frac{w_{t_0}}{A_{t_0} B_{\tilde{i}t_0}} c_{\ell \tilde{i} t_0} d\tilde{i}} = \omega_{\ell i t_0}$$

4 Measured Growth

Measured output growth is a Divisia index across changes in output across goods.[13] A Divisia index is an expenditure-weighted average growth rate of the individual components. Measured TFP growth is measured GDP growth minus measured input growth, as in Solow (1957), Jorgenson and Griliches (1967), and Christensen and Jorgenson (1970). Since labor force is constant, this is simply measured GDP growth:

$$\frac{d \log \widehat{TFP}_t}{dt} = \int_{-\infty}^{\infty} \omega_{it} \frac{\dot{Y}_{it}}{Y_{it}} di$$

where $\omega_{it} \equiv \frac{p_{it} Y_{it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} Y_{\tilde{i}t}} = \frac{w_t L_{it}}{w_t L} = \frac{L_{it}}{L}$ is the aggregate expenditure share on good i. Since output of good i is simply $Y_{it} = A_t B_{it} L_{it}$, this is

$$\frac{d \log \widehat{TFP}_t}{dt} = \int_{-\infty}^{\infty} \omega_{it} \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{B}_{it}}{B_{it}} + \frac{\dot{L}_{it}}{L_{it}} \right) di$$

[13] In national accounts, output growth is measured as a discrete time approximation to a Divisia index.

Note $\omega_{it} = \frac{L_{it}}{L}$ implies $\int_{-\infty}^{\infty} \omega_{it} \frac{\dot{L}_{it}}{L_{it}} di = 0$, giving

$$\frac{d \log \widehat{TFP}_t}{dt} = \int_{-\infty}^{\infty} \omega_{it} \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{B}_{it}}{B_{it}} \right) di$$

Finally, the learning by doing implies that $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it} = \phi L \omega_{it}$, we can express the change in measured TFP as

$$\frac{d \log \widehat{TFP}_t}{dt} = \frac{\dot{A}_t}{A_t} + \phi L \underbrace{\int_{-\infty}^{\infty} \omega_{it}^2 di}_{HHI}$$

Since the growth of A is exogenous, the increase in measured TFP rises when the distribution of expenditures across goods is more concentrated. In this sense, a more equitable distribution of after-tax income is associated with higher growth of measured TFP.

Why does measured TFP rise more quickly when there is more overlap in consumption bundles? The learning by doing gives rise to a scale effect at the good level. When one household consumes a good, the labor used to produce that good reduces the cost of producing that good. If others are also consuming the same good at the same time, the cost reduction has extra value because it reduces the cost others face as well.

4.1 Measured Growth Along a Balanced Growth Path

Along a balanced growth path, the pattern of expenditures across goods follows a traveling wave: $\omega_{it} = \omega_{i+\gamma g(t'-t),t'}$. A simple corollary is the measured TFP growth is constant.

Proposition 3 *Along a balanced growth path, measured TFP growth is constant.*

Further, in line with the preceding discussion, one can show analytically, up to a first order approximation, that along a balanced growth path with a more equitable distribution of after-tax income, measured growth is persistently higher.

Proposition 4 *Suppose that ϕ is small and h and G follow the Gaussian functional forms of Section 2.3. Then measured TFP growth satisfies*

$$\frac{d \log \widehat{TFP}_t}{dt} \approx g + \phi L \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2 \gamma^2 v_\ell}} .$$

Consider two BGPs that correspond to economies with different levels of progressivity, $\tau_1 > \tau_0$. Measured TFP growth is higher in the economy with more progressive taxation.

Proof. Measured TFP growth is $\frac{d\log \widehat{TFP_t}}{dt} = g + \phi L \int_{-\infty}^{\infty} \omega_{it}^2 di$. A first order approximation around $\phi = 0$ gives

$$\frac{d\log \widehat{TFP_t}}{dt} \approx \left.\frac{d\log \widehat{TFP_t}}{dt}\right|_{\phi=0} + \phi \left.\frac{d\frac{d\log \widehat{TFP_t}}{dt}}{d\phi}\right|_{\phi=0}$$

Note that $\left.\frac{d\log \widehat{TFP_t}}{dt}\right|_{\phi=0} = g$. In addition, $\left.\frac{d\frac{d\log \widehat{TFP_t}}{dt}}{d\phi}\right|_{\phi=0} = L \int_{-\infty}^{\infty} \omega_{it}^2 di\Big|_{\phi=0} = L\frac{1}{2\sqrt{\pi}\sqrt{v_h+(1-\tau)^2\gamma^2 v_\ell}}$, since the HHI across goods is $\frac{1}{2\sqrt{\pi}\sqrt{v_h+(1-\tau)^2\gamma^2 v_\ell}}$ when the price of all goods is the same, as discussed in Section 2.3. ■

Consider two economies that have identical primitives but different tax schedules, each on balanced growth paths. In the economy with a more progressive tax schedule, expenditures will be more equal, and as a result, measured output and TFP growth will be perpetually higher, as shown in Section 4.[14] Nevertheless, growth of consumption indices—the objects households care about—is the same in each of the two economies: each household's consumption index grows at rate $\frac{\dot{C}_{\ell t}}{C_{\ell t}} = \frac{\dot{A}_t}{A_t} \equiv g$. That is, differences in measured TFP growth are not informative about the growth rate of consumption indices.

In particular, growth rates of consumption indices along each BGP do not depend at all on the pace of market-specific cost reductions. Why? Measured aggregate productivity growth is higher if there are larger cost reductions for goods that individuals are consuming contemporaneously. But good-specific productivity improvements only give a temporary boost to welfare. Eventually, households shift toward higher ranked goods, with diminishing relevance of those productivity improvements for any particular good. In that sense, the economic value of the productivity gains eventually shrink, as households shift away from those goods.[15]

[14] Whether a BGP with a less equitable distribution of income exhibits lower measured growth depends on the functional forms. To see why this prediction could be go either way, consider an example economy with exactly two types of households, rich and poor, and weighting function h the was double-peaked (e.g., a mixture of two Gaussians with different means). Redistributing income from rich to poor could make the aggregate consumption bundle more concentrated (leading to higher measured growth) or less concentrated (leading to lower measured growth) depending on the relative positions of the lower peak of the rich and the higher peak of the poor.

[15] While good-specific productivity improvements do not boost the growth rate of each household's consumption index, they nevertheless determine the level of these consumption indices along the BGP. These level effects will be explored further below in the discussion of Price levels.

5 Unequal Inflation

Several papers have documented that poor households face persistently higher inflation rates than rich households (Argente and Lee (2021), Jaravel (2019, 2021), Kaplan and Schulhofer-Wohl (2017), Jaravel and Lashkari (2022)). For example, Argente and Lee (2021) find that between 2004 and 2016, inflation for the top quartile of the income distribution has been roughly half of a percentage point lower per year than for the bottom quartile of the households. Jaravel (2019) corroborates this fact and goes further to show that directed technical change leading to cost reductions for goods consumed disproportionately by the rich can explain a large portion of this trend. While it is well documented that inequality in nominal incomes has risen over the last half century, the gap in inflation has raised fears that inequality in real income has risen even faster.

In this section, I show that, along any balanced growth path, there are perpetual differences in measured inflation rates across quantiles of the income distribution. Further, a BGP with more inequality in nominal income will exhibit a larger gap between the measured inflation for the top and bottom halves of the income distribution. Nevertheless, these observations are misleading about welfare improvements among those in the cross section and across balanced growth paths with different distributions of income.

Conventional measures of inflation are a Divisia index of price changes: a weighted average of price growth across goods, weighted by expenditure.[16] For household ℓ, measured inflation is

$$\widehat{Inflation}_{\ell t} = \int_{-\infty}^{\infty} \omega_{\ell i t} \frac{\dot{p}_{it}}{p_{it}} di$$

where the weights $\omega_{\ell i t} \equiv \frac{p_{it} c_{\ell i t}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} c_{\ell \tilde{i} t} d\tilde{i}}$ are the household's expenditure share on good i.[17]

[16]Real-world measurement of inflation must contend with a number of thorny issues such as the appearance of new goods and disappearance of old goods, changes in quality, and measurement at discrete intervals. In the simple environment presented here, none of these issues arise. There are no changes in quality, measurements can be taken continuously, and if $h(\cdot)$ has full support then all goods are consumed by all households at all times.

[17]Note that measured aggregate inflation—the Divisia index of price changes weighted by each good's price change is weighted by its aggregate expenditure shares—is just the weighted average of individual measured inflation rates, weighted by individual expenditures because $\omega_{it} = \int \omega_{\ell i t} \frac{E_{\ell t}}{\int E_{\tilde{\ell} t} dG(\tilde{\ell})} dG(\ell)$. In addition, the sum of the measured output growth rate and the measured aggregate inflation is simply the growth of nominal output.

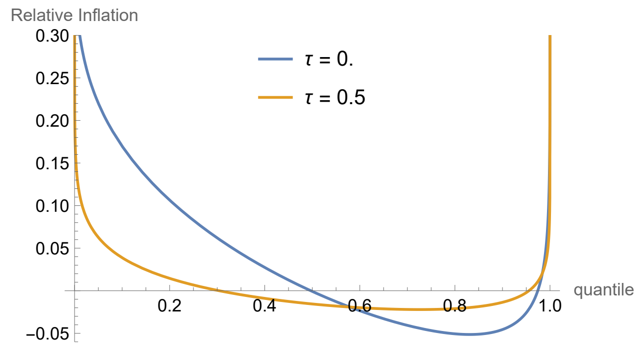


Figure 3 Household-specific Measured Inflation

Note: This figure shows the household-specific inflation rate for each quantile of the income distribution, relative to aggregate inflation, in percentage points. Household-specific inflation is a weighted average of price changes, weighted by the household's expenditures. The figure shows two curves, one for a BGP with $\tau = 0$ and one for a BGP with $\tau = 0.5$. The parameters are $v_\ell = v_h = \gamma = 1$, $g = \phi = 0.02$, and $\sigma = 0.5$.

Since $p_{it} = \frac{w_t}{A_t B_{it}}$, measured inflation can be expressed as

$$\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \int_{-\infty}^{\infty} \omega_{\ell it} \frac{\dot{B}_{it}}{B_{it}} di$$

Broad-based technology grows at rate $\frac{\dot{A}_t}{A_t} = g$ and good-specific technology grows because of learning by doing, $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it} = \phi L \omega_{it}$. Measured inflation is

$$\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - g - \phi L \int_{-\infty}^{\infty} \omega_{\ell it} \omega_{it} di \ .$$

Measured inflation for household ℓ is lower if its expenditures overlap more with aggregate expenditures.

Along a balanced growth path, the shape of each household's consumption bundle remains constant, which implies that inflation gaps across households remain constant.

Proposition 5 *Along a balanced growth path, $\widehat{Inflation}_{\ell t} - \widehat{Inflation}_{\ell' t}$ is constant for each ℓ, ℓ'.*

In line with the empirical findings, measured inflation differs across individuals because consumption bundles differ and the pace of cost reductions differs across goods. Figure 5 shows the level of inflation for each quantile of the income distribution (relative to inflation for the aggregate income basket) using the Gaussian functional form and distributional assumptions of Section 2.3. Inflation is higher among those in the bottom half of the distribution than among those in the top half. Since the aggregate expenditure is tilted toward the consumption patterns of those with higher income, the goods consumed by those in the top half of the distribution experience larger cost reductions. Thus those in the top half experience lower inflation than those at the bottom.[18,19]

More formally, Proposition 6 shows, to a first order approximation, that with the Gaussian functional forms, inflation for the rich is lower than inflation for the poor.

Proposition 6 *Suppose that ϕ is small and h and G follow the Gaussian functional forms of Section 2.3. If $\log \ell$ is k standard deviations above the median and $\log \ell'$ is k standard deviations below the median, then*

$$\widehat{Inflation}_{\ell t} < \widehat{Inflation}_{\ell' t} .$$

Proof. To a first order approximation around $\phi = 0$, measured inflation for household ℓ is

$$\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \phi L \int_{-\infty}^{\infty} \omega_{\ell i t} \omega_{i t} di \approx \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \phi L \int_{-\infty}^{\infty} \omega_{\ell i t}^0 \omega_{i t}^0 di$$

where ω_{it}^0 and $\omega_{i\ell t}^0$ are aggregate and individual expenditure shares for an environment without learning by doing, i.e., $\phi = 0$. Since the price of each good without learning-by-doing is $p_{it} = w_t / A_t$, these expenditure shares are These are

$$\omega_{it}^0 = \frac{1}{\sqrt{2\pi \left(v_h + \gamma^2 (1-\tau)^2 v_\ell\right)}} e^{-\frac{\left(i - (1-\tau)^2 \gamma \frac{v_\ell}{2} - \gamma \log A_t\right)^2}{2\left(v_h + \gamma^2 (1-\tau)^2 v_\ell\right)}}$$

$$\omega_{i\ell t}^0 = h\left(i - \gamma \log \frac{E_{\ell t}}{p_{it}}\right) = h\left(i - \gamma \log A_t - \gamma \log \frac{\ell^{1-\tau}}{\overline{\overline{\ell^{1-\tau}}}}\right) = \frac{1}{\sqrt{2\pi v_h}} e^{-\frac{\left(i - \gamma \log A_t - \gamma \log \frac{\ell^{1-\tau}}{\overline{\overline{\ell^{1-\tau}}}}\right)^2}{2 v_h}}$$

[18] Interestingly, inflation is highest for those at the very bottom and very top of the income distribution, as their consumption bundles overlap least with the aggregate expenditures.

[19] It need not be the case that inflation for the rich is lower than inflation for the poor. For example, if there is a very large mass of households with low ℓ and only a few with high ℓ, it could be that the aggregate consumption bundle is closer to that of the poor, and hence inflation would be lower for the poor. Similarly, if even if the rich spend more than the poor, it could be that consumption bundles differ much more among the rich, but consumption bundles among the poor are similar. The example here, in which the distribution of consumption expenditures is lognormal, is consistent with the findings of Battistin, Blundell and Lewbel (2009).

where the second line used $\frac{E_{\ell t}}{p_{it}} = A_t \frac{E_{\ell t}}{w_t} = A_t \frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}}$. Together, these imply[20]

$$\int \omega_{it}\omega_{i\ell t} di = \frac{1}{\sqrt{2\pi\left(2v_h + \gamma^2 (1-\tau)^2 v_\ell\right)}} e^{-\frac{\left((1-\tau)^2\gamma\frac{v_\ell}{2} - \gamma \log \frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}}\right)^2}{2\left(2v_h+\gamma^2(1-\tau)^2 v_\ell\right)}}$$

Using $\log \frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}} = (1-\tau)\left(\log \ell + \frac{v_\ell}{2}\right) - (1-\tau)^2 \frac{v_\ell}{2}$ gives

$$\int \omega_{it}\omega_{i\ell t} di = \frac{1}{\sqrt{2\pi\left(2v_h + \gamma^2 (1-\tau)^2 v_\ell\right)}} e^{-\frac{\left((1-\tau)^2\gamma v_\ell - \gamma(1-\tau)\left(\log \ell + \frac{v_\ell}{2}\right)\right)^2}{2\left(2v_h+\gamma^2(1-\tau)^2 v_\ell\right)}}$$

Finally, note that if ℓ is at quantile q of the distribution of effective endowments, then $\log \ell + \frac{v_\ell}{2} = \sqrt{v_\ell}\Phi^{-1}(q)$. Therefore inflation for the individual at quantile q is approximately

$$\frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \phi L \frac{1}{\sqrt{2\pi\left(2v_h + \gamma^2 (1-\tau)^2 v_\ell\right)}} e^{-\frac{\left[(1-\tau)^2\gamma v_\ell - \gamma(1-\tau)\sqrt{v_\ell}\Phi^{-1}(q)\right]^2}{2\left(2v_h+\gamma^2(1-\tau)^2 v_\ell\right)}}$$

Finally, the conclusion follows from the fact that $\Phi^{-1}\left(\frac{1}{2}+k\right) > 0 > \Phi^{-1}\left(\frac{1}{2}-k\right)$, which implies that

$$\left[(1-\tau)^2 \gamma v_\ell - \gamma(1-\tau)\sqrt{v_\ell}\Phi^{-1}\left(\frac{1}{2}+k\right)\right]^2 < \left[(1-\tau)^2 \gamma v_\ell - \gamma(1-\tau)\sqrt{v_\ell}\Phi^{-1}\left(\frac{1}{2}-k\right)\right]^2$$

■

Further, the economy with greater inequality exhibits a larger gap between measured inflation among the rich and poor.[21] This happens because with more inequality there is less overlap in consumption bundles, so households toward the bottom spend even less on goods undergoing cost reductions.

[20] This uses the identity $\int \frac{1}{\sqrt{2\pi b}} e^{-\frac{(u-a)^2}{2b}} \frac{1}{\sqrt{2\pi d}} e^{-\frac{(u-c)^2}{2d}} du = \frac{1}{\sqrt{2\pi(b+d)}} e^{-\frac{(a-c)^2}{2(b+d)}}$.

[21] There is some evidence that the gap in measured inflation rates between the rich and poor has increased over the last few decades. Jaravel and Lashkari (2022) combine CEX and CPI data to construct measures of inflation for each percentile in income distribution going back to the 1950s. They find a strong negative correlation between inflation rates and income 1995-2019, a negative but slightly weaker correlation from 1955-1984, and a much weaker, but still negative between 1984-1995. The findings for the earlier period should be taken with a grain of salt, however, as CEX data is quite sparse before 1984. Orchard (2022) tracks the price of necessities relative to that of luxuries using the CEX, and finds consistent evidence that the relative price of necessities rose since 2000 but mixed evidence about whether the relative price increased or decreased from the 1970s-1990s.

One might be tempted to infer from this that differential inflation exacerbates inequality in nominal expenditures. However, such a conclusion is not warranted. Despite the differential inflation rates, the consumption index for *every* household grows at rate g, as shown in Section 3.

How can differential inflation be compatible with equal growth in consumption indices? Fundamentally, households benefit from low cost of goods, not from cost reductions per se. Measured inflation gauges how fast prices are falling for the goods a household is consuming contemporaneously. In many models, the latter is the rate of change of the former. But in this model, price changes of goods consumed contemporaneously only partly determine the evolution of the level of prices that are relevant for the household.

Consider the following example. Currently the rich consume Teslas, and possibly in the future the poor will as well. One possibility is that the price of Teslas will decline while the rich are consuming but will be flat after the poor begin consuming it. Under that scenario, inflation for the rich will be lower for than for the poor. In a second scenario, the price will remain high while the rich consume Teslas, but will begin falling once the poor consume Teslas as well. In the latter scenario, inflation for the poor will be lower. Nevertheless, the poor prefer the first scenario despite the lower inflation because they get to pay lower prices; they would prefer the price of a Tesla fall *before* they start consuming it than for the price to fall *while* they are consuming it.

More formally, consider household ℓ, whose budget constraint can be expressed as $E_{\ell t} = C_{\ell t} P_{\ell t}$, where household ℓ's price index is

$$\begin{aligned} P_{\ell t} &= \left[\int_{-\infty}^{\infty} h\Big(i - \gamma \log C_{\ell t}\Big) p_{it}^{1-\sigma} di \right]^{\frac{1}{1-\sigma}} \\ &= \left[\int_{-\infty}^{\infty} h(u) \left(p_{u+\gamma \log C_{\ell t}, t}\right)^{1-\sigma} du \right]^{\frac{1}{1-\sigma}} . \end{aligned}$$

where the second line used the change of variables $u = i - \gamma \log C_{\ell t}$. Differentiating completely with respect to time and then changing variables back to $i = u + \gamma \log C_{\ell t}$ gives

$$\frac{\dot{P}_{\ell t}}{P_{\ell t}} = \int_{-\infty}^{\infty} \omega_{\ell i t} \left[\frac{\dot{p}_{it}}{p_{it}} + \gamma \frac{\dot{C}_{\ell t}}{C_{\ell t}} \frac{d \log p_{it}}{di} \right] di$$

where, again, $\omega_{\ell it} \equiv \frac{p_{it}c_{\ell it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t}c_{\ell\tilde{i}t}d\tilde{i}} = \frac{h\left(i-\gamma\log C_{\ell t}\right)p_{it}^{1-\sigma}}{\int_{-\infty}^{\infty} h\left(\tilde{i}-\gamma\log C_{\ell t}\right)p_{\tilde{i}t}^{1-\sigma}d\tilde{i}}$ is household ℓ's share of time-t expenditure spent on good i.

Measured inflation captures only the first term in brackets: the expenditure-weighted changes in prices. But in this model, welfare also depends on the second term: as the household consumes more, it shifts to higher ranked goods. Those higher-ranked goods have higher prices, as there has been less cumulative learning by doing. Along a balanced growth path, this shift to higher-ranked, higher-priced goods partially offsets the first term: $\frac{\dot{p}_{it}}{p_{it}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{B}_{it}}{B_{it}}$ and $\gamma\frac{\dot{C}_{\ell t}}{C_{\ell t}}\frac{d\log p_{it}}{di} = \frac{\dot{B}_{it}}{B_{it}}$. Together, these yield

$$\frac{\dot{P}_{\ell t}}{\dot{P}_{\ell t}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} = \frac{\dot{w}_t}{w_t} - g$$

That is, for all households, the price index (relative to the wage) declines at the same rate, g.[22] This gives the following proposition:

Proposition 7 *Along a balanced growth path, $\frac{P_{\ell t}}{w_t/A_t}$ is constant for each ℓ.*

Even though there is a gap in measured inflation, the learning by doing that causes the differential inflation rates actually reduces inequality rather than exacerbates it. With no learning by doing ($\phi = 0$), all households would experience the same measured inflation and the price index P_ℓ would be the same for all households. Thus the dispersion in consumption indexes would be the same as dispersion in nominal expenditures. With learning by doing ($\phi > 0$), costs are lower, but especially so for lower ranked goods: $\frac{d\log B_{it}}{di} = -\frac{\phi L_{it}}{\gamma g} < 0$, as these goods have experienced more cumulative cost reductions. As a result, P_ℓ is strictly increasing in ℓ, reducing dispersion in consumption indices C_ℓ relative to the dispersion in nominal expenditures.

It also turns out that this asymmetry between measured inflation and inequality is stronger when there is more inequality in expenditures. As discussed above, when there is more inequality in expenditures, there is a larger gap in measured inflation. Again, since the consumption bundles of the rich overlap less with the bundles of the poor, the poor do not experience large contemporaneous cost reductions. Rather, the prices of those goods decline while the rich consume them, i.e., before

[22] Why do these two terms offset each other? Consider a household whose measured inflation is very low, because it consumes goods whose price is falling precipitously. When that household's income grows, it shifts to higher ranked goods. For that household, the prices of those goods it is shifting too will be particularly higher than the prices of the goods it was already consuming, as those are precisely the goods whose price is about to fall precipitously.

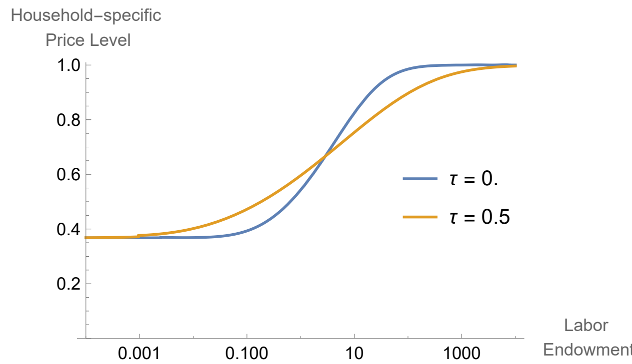


Figure 4 Household-specific Price Levels

Note: This figure shows the household-specific price index P_ℓ for each quantile of the income distribution, relative to $\frac{w}{A}$. The figure shows two curves, one for a BGP with $\tau = 0$ and one for a BGP with $\tau = 0.5$.

the poor start consuming them. Thus by the time poor consume the goods, the price of those goods is already low. As a result, the gap between the price index of the poor and the price index of the rich is larger.

Figure 5 shows the price level across quantiles of the income distribution for two different BGPs, one with no taxes, and one with a tax schedule that is more progressive, with $\tau = 0.5$. Along any BGP, $\lim_{t\to\infty} \log B_{it} = \frac{\phi L}{\gamma g}$: the cumulative cost reduction over the lifetime of a good is independent of the distribution of income. The distribution of income does, however, affect the timing of that cost reduction. When inequality is higher, more of this cost reduction comes before the low income households start consuming the good. As a result, the level of cost of those goods tends to be lower. This can be shown analytically (to a first order approximation). Consider two BGPs with tax schedules with different τ's. For the BGP with more inequality (lower τ), the price index is lower for the poor and higher for the rich.

Proposition 8 *Suppose that ϕ is small and h and G follow the Gaussian functional forms of Section 2.3. Then the price index for household ℓ satisfies*

$$\log \frac{P_{\ell t}}{w_t/A_t} \approx -\frac{\phi L}{\gamma g}\left[1 - \Phi\left(\frac{(1-\tau)\gamma \log \ell + \gamma(2\tau - 1)(1-\tau)\frac{v_\ell}{2}}{\sqrt{2v_h + \gamma^2(1-\tau)^2 v_\ell}}\right)\right]$$

where Φ is the CDF of the standard normal distribution. Consider two BGPs that correspond to economies with different levels of progressivity, $\tau_1 > \tau_0$. Let P_ℓ^k correspond to the price index (relative to w/A) of household ℓ in economy τ_k. There is a cutoff $\bar{\ell}$ such that $P_\ell^1 > P_\ell^0$ for $\ell < \bar{\ell}$, *and* $P^1_\ell > P^0_\ell$ *for* $\ell < \bar{\ell}$.

Proof. We first describe the first-order approximation around $\phi = 0$. For any variable x that is determined in equilibrium, let x^0 denote of the variable in the economy with $\phi = 0$.

Along a BGP $L_{i\tilde{t}} = L_{i+\gamma g(t-\tilde{t}),t}$, so that good-specific productivity is $\log B_{it} = \phi \int_{-\infty}^{t} L_{i\tilde{t}} d\tilde{t} = \phi \int_{-\infty}^{t} L_{i+\gamma g(t-\tilde{t}),t} d\tilde{t} = \frac{\phi}{\gamma g} \int_{i}^{\infty} L_{\tilde{i}t} d\tilde{i}$. Differentiating with respect to ϕ and evaluating at $\phi = 0$ gives $\frac{d \log B_{it}}{d\phi}\Big|_{\phi=0} = \frac{1}{\gamma g} \int_{i}^{\infty} L^0_{\tilde{i}t} d\tilde{i}$.

Household ℓ's price index satisfies $P_{\ell t}^{1-\sigma} = \int_{-\infty}^{\infty} h\Big(i - \gamma \log C_{\ell t}\Big) p_{it}^{1-\sigma} di$. Using $p_{it} = \frac{w_t}{A_t B_{it}}$ gives

$$\left(\frac{P_{\ell t}}{w_t/A_t}\right)^{1-\sigma} = \int_{-\infty}^{\infty} h\Big(i - \gamma \log C_{\ell t}\Big) B_{it}^{\sigma-1} di = \int_{-\infty}^{\infty} h(u) B_{u+\gamma \log C_{\ell t},t}^{\sigma-1} du$$

Differentiating with respect to ϕ gives

$$\frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi} = -\frac{1}{\frac{P_{\ell t}}{w_t/A_t}} \int_{-\infty}^{\infty} h(u) B_{u+\gamma \log C_{\ell t},t}^{\sigma-1} \left\{ \frac{\partial B_{u+\gamma \log C_{\ell t},t}}{\partial \phi} + \frac{\partial B_{u+\gamma \log C_{\ell t},t}}{\partial i} \gamma \frac{d \log C_{\ell t}}{d\phi} \right\} du$$

Evaluating this at $\phi = 0$ and noting that $\frac{\partial B_{it}}{\partial i}\Big|_{\phi=0} = 0$, $B_{it}|_{\phi=0} = \frac{P_{\ell t}}{w_t/A_t}\Big|_{\phi=0} = 1$, and $C_{\ell t}|_{\phi=0} = \frac{A_t E_{\ell t}}{w_t} = A_t \ell^{1-\tau} / \overline{\ell^{1-\tau}}$ gives

$$\begin{aligned}
\frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi}\Bigg|_{\phi=0} &= -\int_{-\infty}^{\infty} h(u) \left. \frac{\partial B_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}},t}}{\partial \phi} \right|_{\phi=0} du \\
&= -\int_{-\infty}^{\infty} h(u) \frac{1}{\gamma g} \int_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}}}^{\infty} L^0_{it} di du
\end{aligned}$$

Finally, the first order approximation yields

$$\begin{aligned}
\log \frac{P_{\ell t}}{w_t/A_t} &\approx \log \frac{P_{\ell t}}{w_t/A_t}\Bigg|_{\phi=0} + \phi \left(\frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi}\Bigg|_{\phi=0} \right) \\
&= -\frac{\phi}{\gamma g} \int_{-\infty}^{\infty} h(u) \int_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}}}^{\infty} L^0_{it} di du
\end{aligned}$$

Under Gaussian functional form assumption, $h(u)$ is the pdf of a normal distribution with variance v_h and L^0_{it}/L is normally distributed with mean $\gamma(1-\tau)^2 \frac{v_\ell}{2} + \gamma \log A_t$ and variance variance $v_h + (1-\tau)^2 \gamma^2 v_\ell$, as discussed in Section 2.3. Letting $\Phi(\cdot)$ denote the CDF of a standard normal distribution, this is simply

$$\log \frac{P_{\ell t}}{w_t/A_t} \approx -\frac{\phi L}{\gamma g} \int_{-\infty}^{\infty} \Phi'(u) \left\{ 1 - \Phi\left(\frac{\sqrt{v_h} u + (1-\tau)\gamma \log \ell - \gamma \log \overline{\ell^{1-\tau}} - \gamma(1-\tau)^2 \frac{v_\ell}{2}}{\sqrt{v_h + \gamma^2(1-\tau)^2 v_\ell}} \right) \right\} du$$

Note that for constants a, b and c, $\int_{-\infty}^{\infty} \Phi'(u) \left[1 - \Phi\left(\frac{\sqrt{c}u+b}{\sqrt{a}}\right)\right] du = 1 - \Phi\left(\frac{b}{\sqrt{a+c}}\right)$. Applying this formula gives

$$\log \frac{P_{\ell t}}{w_t/A_t} \approx -\frac{\phi L}{\gamma g} \left[1 - \Phi\left(K(\ell, \tau)\right)\right]$$

where $K(\ell,\tau) \equiv \frac{(1-\tau)\gamma \log \ell - \gamma \log \overline{\ell^{1-\tau}} - \gamma(1-\tau)^2 \frac{v_\ell}{2}}{\sqrt{2v_h + \gamma^2(1-\tau)^2 v_\ell}}$. Using $\overline{\ell^{1-\tau}} = e^{-\tau(1-\tau)\frac{v_\ell}{2}}$, K can be rearranged as

$$K(\ell,\tau) = \frac{(1-\tau)\gamma \log \ell + \gamma(2\tau-1)(1-\tau)\frac{v_\ell}{2}}{\sqrt{2v_h + \gamma^2(1-\tau)^2 v_\ell}} .$$

In addition, $P_{\ell t}$ will be increasing in τ if and only if $K(\ell,\tau)$ is increasing in τ. $K(\ell,\tau)$ is submodular, and there is a $\bar{\ell}$ such that $\frac{dK(\bar{\ell},\tau)}{d\tau} = 0$. Therefore when τ rises, $P_{\ell t}$ rises more for those with $\ell < \bar{\ell}$ and falls for those with $\ell < \bar{\ell}$. ■

Thus more inequality of nominal expenditures is ameliorated by the level of prices paid for the same goods. But this is the opposite conclusion one might draw from looking at the measured rate of inflation of those goods.

6 Measuring Welfare

In this section, I formalize the argument that, in this model, measured "real income growth"—a household's nominal income growth minus its measured inflation rate—can be misleading about improvements in welfare. As discussed in the last two sections, measured real income growth generically differs across households along a BGP, and differs across BGPs with different rates of measured output growth. Nevertheless, along any BGP, consumption indices for all households grow at the same constant rate g.

An individual's consumption index is a utility function, and can be interpreted as a measure of welfare. Of course, any strictly increasing function of a household's consumption index is also a utility function. As measured by other cardinalities of utility, welfare may not grow at the same rate across households.

One might reasonably ask if the disconnect between measured real income growth and welfare improvements is special to the particular choice the consumption index as a utility function, or whether the result extends to other cardinalities of utility, such as money metrics of utility that are commonly used in applied work. This question is especially pertinent given the classic results of Kloek (1967) and Theil (1968) that provide a tight link between measured real income growth and utility growth as gauged by a money metric of utility.

This section formalizes the disconnect between measured real income growth and welfare improvements in the model. I first review the classic link between money metrics of utility and welfare in Section 6.1.[23] In Section 6.2 I present the main result: there does not exist a utility function that consistently assigns higher welfare growth rate to spells with higher rates of measured real income growth. The consumption index is a knife-edge cardinality that assigns the same welfare growth rate to all spells. Outside of the consumption index (or log-linear functions of the consumption index), any utility function will be such that sometimes high-real-income-growth spells will be associated with higher utility growth than low-real-income-growth spells, and sometimes the opposite. I then discuss how this disconnect can be reconciled with the findings of Kloek (1967) and Theil (1968).

6.1 The Link Between Real Income Growth and Welfare Growth

Because of the arbitrariness of choosing a cardinality of utility, applied researchers have commonly focused on money metrics of utility (McKenzie (1957), Samuelson and Swamy (1974)) to measure the gains from moving from one budget set to another. Given a price vector p, a money metric of utility $M(c;p)$ assigns to each bundle c the minimum expenditure needed at those prices to make the household as least as well-off as consuming that bundle. This is simply the expenditure function, $M(c;p) = \mathcal{E}(p, u(c))$, evaluated at prices p and utility level $u(c)$. An important feature of money metrics of utility is that they provide a cardinality of utility that is interpretable and measurable, and perhaps comparable across situations. For this reason, Baqaee, Burstein and Koike-Mori (2022) call money metric utility "a backbone of welfare economics and a necessary ingredient for measuring welfare-relevant growth and inflation."

Closely related is a Konüs (1939) cost of living index. For a given reference bundle c, the cost of

[23] Diewert (1993) provides an excellent overview of the literature.

living index $K(p_0, p_1; c)$ is the ratio of the expenditure required to make the individual as well-off as consuming c at respective prices p_0 and p_1: $K(p_0, p_1; c) = \frac{\mathcal{E}(p_1, u(c))}{\mathcal{E}(p_0, u(c))}$.[24]

The change in an individual's expenditure can be decomposed into the product of the proportional increase in money metric of utility associated with initial prices and the proportional increase in the cost of living, evaluated at the final consumption bundle:

$$\frac{E_{\ell t_1}}{E_{\ell t_0}} = \frac{\mathcal{E}\left\{p_{t_1}, u\left(c_{\ell t_1}\right)\right\}}{\mathcal{E}\left\{p_{t_0}, u(c_{\ell t_0})\right\}} = \frac{\mathcal{E}\left\{p_{t_0}, u(c_{\ell t_1})\right\}}{\mathcal{E}\left\{p_{t_0}, u(c_{\ell t_0})\right\}} \frac{\mathcal{E}\left\{p_{t_1}, u(c_{\ell t_1})\right\}}{\mathcal{E}\left\{p_{t_0}, u(c_{\ell t_1})\right\}} = \frac{M(c_{\ell t_1}; p_{t_0})}{M(c_{\ell t_0}; p_{t_0})} K(p_{t_0}, p_{t_1}; c_{\ell t_1})$$

Rearranging gives that the increment of welfare, as measured by the money metric of utility at initial prices, is the growth of expenditure minus the growth of the Konüs cost of living index:

$$\ln \frac{M(c_{\ell t+\Delta}; p_t)}{M(c_{\ell t}; p_t)} = \ln \frac{E_{\ell t+\Delta}}{E_{\ell t}} - \ln K(p_t, p_{t+\Delta}; c_{\ell t+\Delta})$$

Shephard's lemma implies that, over short horizons, the increment of the Konüs cost of living index is simply the measured rate of inflation:

$$\lim_{\Delta \to 0} \frac{\ln K(p_t, p_{t+\Delta}; c_{\ell t+\Delta})}{\Delta} = \int \omega_{\ell i t} \frac{\dot{p}_{it}}{p_{it}} di = \widehat{Inflation}_{\ell t}$$

Together, this gives the classic result of Kloek (1967) and Theil (1968) that, at least over short horizons, growth of measured real income is associated with higher growth of welfare, as measured by the money metric of utility:

$$\lim_{\Delta \to 0} \frac{\ln M(c_{\ell t+\Delta}; p_t) - \ln M(c_{\ell t}; p_t)}{\Delta} = \frac{\dot{E}_{\ell t}}{E_{\ell t}} - \widehat{Inflation}_{\ell t}$$

This result provides a welfare interpretation of real income growth even when preferences are non-homothetic.[25]

[24] In the special case of homothetic preferences, the expenditure function is multiplicatively separable, $\mathcal{E}(p, u) = r(u)k(p)$, so the change in the Konüs cost-of-living index is common for all individuals, $K(p, p'; c) = \frac{k(p')}{k(p)}$, $\forall c$.

[25] A second common choice among practitioners when evaluating changes from one consumption bundle to another is to compute how much one would have to inflate all quantities of the initial bundle to make the household as well off as shifting to the new bundle. This is the idea behind a Malmquist (1953) quantity index. A Malmquist index for the change from bundle c_0 to c_1, using reference bundle $\bar{c}$, is $Q(c_0, c_1; \bar{c}) = \frac{D(u(\bar{c}), c_1)}{D(u(\bar{c}), c_0)}$, where D is the deflator function defined as $D(u, c) = \max_{k>0}\{k : u(c/k) \geq u\}$. The Malmquist index has the property that $Q(c, \lambda c, \bar{c}) = \lambda$. The Malmquist index also has a tight link to measured real income growth. The envelope theorem and cost minimization imply that, at short horizons, the change in a Malmquist index, using reference bundle c_t is equal to measured real

A key point to note for now is that this property—the link between real income growth and growth of money-metric utility—is not universal to all money metrics. It is specific to the money metric that uses contemporaneous prices as the reference price vector.[26] I will come back to this below.

6.2 A Disconnect Between Real Income Growth and Welfare Growth

I now come to the section's main result. Proposition 9 formalizes the claim that growth in measured real income can be misleading about improvements in welfare. It states that there does not exist a utility function for which there is a systematic relationship between measured real income growth and growth of utility.[27]

Given a utility function u, the growth of utility for person ℓ over a spell of length Δ that begins at t is simply $u(c_{\ell t+\Delta})/u(c_{\ell t})$. Consider two individuals, ℓ and ℓ', and suppose that, along a BGP in the model, measured real income growth for ℓ is perpetually higher than for ℓ'. In particular, for spells of the same length, cumulative measured real income growth for ℓ is always higher than that for ℓ'. Fix a utility function u. Suppose that utility grows more for ℓ along a spell of length Δ_0 than it does for ℓ' along a spell of the same length. Then there must be another set of spells of identical length for which the order is reversed: utility growth for ℓ' is higher than for ℓ, despite higher measured real income growth for ℓ.

Proposition 9 *Consider two individuals ℓ and ℓ', and any positive, increasing function u. Suppose that $C_{\ell,t}$ and $C_{\ell',t}$ are their respective paths of consumption indices along a BGP. If there is a Δ_0, t_0, and t_0' such that*

$$\frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_0}\right)} > \frac{u\left(C_{\ell',t_0'+\Delta_0}\right)}{u\left(C_{\ell',t_0'}\right)}$$

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income growth:

$$\lim_{\Delta\to 0}\frac{\ln Q(c_{\ell t}, c_{\ell t+\Delta}; c_t) - \ln Q(c_{\ell t}, c_{\ell t}; c_t)}{\Delta} = \int \omega_{\ell i t}\frac{\dot{c}_{\ell i t}}{c_{\ell i t}}di = \frac{\dot{E}_{\ell t}}{E_{\ell t}} - \widehat{Inflation}_{\ell t}\ .$$

[26] Similarly, the link between real income and Malmquist indices is not universal to all Malmquist indices; it is specific to the Malmquist index that uses the contemporaneous consumption bundle as the reference bundle.

[27] One could state the results in terms of differences of utilities rather than ratios, and dispense with the requirement that u is a positive function. However, I state the results in this form to facilitate a comparison below with money metrics of utility which are positive.

then there must be a Δ_1, t_1, and t_1' such that

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} < \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_1'}\right)}$$

Proof. Note first that it cannot be that $C_{\ell,t_0} = C_{\ell',t_0'}$, because this would imply that $u\left(C_{\ell,t_0}\right) = u\left(C_{\ell',t_0'}\right)$ and $u\left(C_{\ell,t_0+\Delta_0}\right) = u\left(C_{\ell',t_0'+\Delta_0}\right)$.

Case 1: Suppose that $C_{\ell,t_0} > C_{\ell',t_0'}$. Then let $\Delta_1 = \frac{1}{g}\log\frac{C_{\ell,t_0}}{C_{\ell',t_0'}}$ so that $C_{\ell,t_0} = C_{\ell',t_0'+\Delta_1}$. Further, let $t_1 = t_0 - \Delta_1$ and $t_1' = t_0' + \Delta_0$. Then it must be that

$$\begin{aligned} C_{\ell,t_1} &= C_{\ell',t_0'} \\ C_{\ell,t_0+\Delta_0} &= C_{\ell',t_1'+\Delta_1} \end{aligned}$$

and hence $u\left(C_{\ell,t_1}\right) = u\left(C_{\ell',t_0'}\right)$ and $u\left(C_{\ell,t_0+\Delta_0}\right) = u\left(C_{\ell',t_1'+\Delta_1}\right)$. We thus have

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_1}\right)}\frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_0+\Delta_0}\right)}$$

Since $\frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_0'}\right)}$ and $\frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_0+\Delta_0}\right)} < \frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_0'+\Delta_0}\right)} = \frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_1'}\right)}$, we have

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} < \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_0'}\right)}\frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_1'}\right)} = \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_1'}\right)}$$

Case 2: $C_{\ell,t_0} < C_{\ell',t_0'}$. Then a similar argument holds using $\Delta_1 = \frac{1}{g}\log\frac{C_{\ell',t_0'}}{C_{\ell,t_0}}$, $t_1 = t_0 + \Delta_0$ and $t_1' = t_0' - \Delta_1$. ■

The proposition states that even if there is a perpetual gap in measured real income growth between two households, it cannot be the case that the household with higher real income growth always experiences greater welfare growth.

The proof leans only on ordinal comparisons. Along a BGP, the consumption index of household ℓ grows from $C_{\ell,t}$ at t to $C_{\ell,t+\Delta} = C_{\ell,t}e^{g\Delta}$ at $t+\Delta$. For household ℓ', there is some time t' where its consumption index $C_{\ell',t'}$ is equal to $C_{\ell,t}$. At $t'+\Delta$, its consumption bundle has also grown by a factor of $e^{g\Delta}$. The households are indifferent between $C_{\ell,t}$ and $C_{\ell',t'}$ as well as between $C_{\ell,t+\Delta}$ and

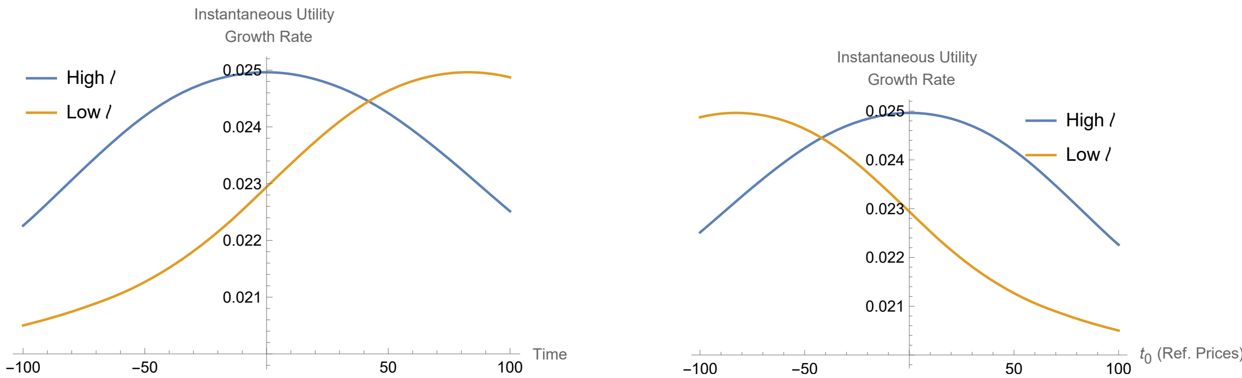
$C_{\ell',t'+\Delta}$. Thus for *any* cardinal representation of preferences, the welfare improvement for ℓ from t to $t+\Delta$ must be the same as the welfare improvement for ℓ' from t' to $t'+\Delta$. If a utility function assigns higher welfare growth to one household for part of the interval, it must assign lower welfare growth to that household for the remainder.

We are now in position to reconcile the two statements that, on the surface, may appear inconsistent: First, that along a BGP in the model, there is no consistent relationship between measured real income growth and welfare improvements. And second, the result from Kloek (1967) and Theil (1968), that shows the measured real income growth is equivalent to the increment of a money metric of utility.

The critical issue is that any money metric *is specific to a reference price vector*; the money metric utility of a bundle is the expenditure required at those prices to be as well off as consuming that bundle. It is well known that different reference price vectors yield different money metrics of utility. Money metrics with different reference price vectors agree on rankings of consumption bundles—they all correspond to the same preferences—but they disagree on magnitudes as they use different cardinalities of utility. Since the price vectors differ over time and across BGPs, one should not compare the magnitude of growth of utility as measured by the money metric associated with contemporaneous reference prices to that along a different BGP or to that along the same BGP at a different point in time, as those other money metrics use different reference prices—that is, they measure welfare changes with different utility functions.[28] For a short spell, the money metric associated with contemporaneous reference prices assigns higher welfare growth when measured real income is higher, but the results of Kloek (1967) and Theil (1968) say nothing about how *that same utility function* assigns welfare growth to other spells. Proposition 9 says that one can find other spells where there is a reversal in the ranking: the spell with lower measured real income growth is assigned higher welfare growth.

To illustrate this, consider the example of two individuals with log incomes respectively one standard deviation above and one standard deviation below the median. Fix a base period, t_0,

[28] A related issue arises when chaining increments of a Divisia index, as discussed in Deaton and Muellbauer (1980) and Baqaee and Burstein (2021). While each increment of the Divisia index corresponds to a money metric of utility, different links in the chain correspond to different money metrics because they use different reference price vectors. Thus there is no welfare interpretation of the change in the chained index. Baqaee and Burstein (2021) and Jaravel and Lashkari (2022) show that, over long periods of time in which income effects cause significant shifts in budget shares, the gap between a chained Divisia index and a welfare measure such as equivalent variation can grow large.



(a) Fixed Reference Prices

(b) Fixed Spell

Figure 5 Instantaneous Utility Growth Rate Using Money Metrics of Utility

Note: Panel (a) shows the instantaneous growth rate of utility at different points in time as measured by the money metric of utility associated with fixed reference prices at p_{t_0} for two individuals, one with log income one standard deviation above median and one with log income one standard deviation below median, with t_0 normalized to 0. Panel (b) shows the instantaneous growth rate of utility for a fixed time t normalized to 0 for the same two individuals, but evaluated using using money metrics of utility associated with different reference prices p_{t_0}, with t_0 along the horizontal axis. The figure shows these for a BGP for an economy with $\tau = 0$, with parameters $v_\ell = v_h = \gamma = 1$, $g = \phi = 0.02$, and $\sigma = 0.5$.

and consider the money metric of utility associated with the price vector in that period, $M(\cdot; p_{t_0})$. Figure 5a plots the instantaneous growth rate of that money metric for each of the two individuals, $\frac{d \ln M(c_{\ell t}; p_{t_0})}{dt}$, at all points in time t.

In line with the results of Kloek (1967) and Theil (1968), one can see that at time t_0, the money metric assigns higher welfare growth to the individual with higher real income growth. But, there are other time periods where the other individual has higher welfare growth.[29] This happens even though the individual with higher income has higher measured real income growth at every instant.

A different way to make a similar point is that one can evaluate the same spell using various money metrics of utility with different reference prices. As discussed in Jaravel and Lashkari (2022), the welfare gains for a fixed spell computed using a money metric of utility depend on one's choice of reference prices. Figure 5b shows the growth rate of utility at some fixed time t for the same two individuals, but using various money metrics of utility with different reference prices. That is, we plot $\frac{d \ln M(c_{\ell t}; p_{t_0})}{dt}$ for each individual, with t_0 on the horizontal axis. Note that this is simply the mirror image of the figure in Figure 5a. As the figure makes clear, the ranking of which individual

[29]Note that this example is stronger than Proposition 9, as Proposition 9 allowed t_1 to differ from t_1'. It will be possible to construct such examples as long as measured real income growth rates across incomes has interior peaks, and the distribution of incomes has full support.

experiences higher utility growth at t depends on the somewhat arbitrary choice of which reference prices one uses.

6.3 Across Balanced Growth Paths

One may object to interpersonal utility comparisons, even if individuals share the same ordinal rankings of consumption bundles. Proposition 9 assumed that all individuals shared the same utility function. If one is not willing to compare utility growth across individuals, one is limited to making comparisons within individuals. Still, one can show that comparisons of measured real income growth across BGPs can be misleading. Again, there does not exist a utility function for which there is a systematic relationship between measured real income growth and growth of utility.

Proposition 10 *Consider two BGPs that correspond to tax schedules τ^* and τ^{**}. Suppose household ℓ's consumption index along these BGPs is $C^*_{\ell,t}$ and $C^{**}_{\ell,t}$ respectively. Consider any positive, increasing function u. If there is a Δ_0, t_0, and t'_0 such that*

$$\frac{u\left(C^*_{\ell,t_0+\Delta_0}\right)}{u\left(C^*_{\ell,t_0}\right)} > \frac{u\left(C^{**}_{\ell,t'_0+\Delta_0}\right)}{u\left(C^{**}_{\ell,t'_0}\right)}$$

then there must be a Δ_1, t_1, t'_1 such that

$$\frac{u\left(C^*_{\ell,t_1+\Delta_1}\right)}{u\left(C^*_{\ell,t_1}\right)} < \frac{u\left(C^{**}_{\ell,t'_1+\Delta_1}\right)}{u\left(C^{**}_{\ell',t'_1}\right)}$$

6.4 Difference from New-Goods Bias

The change in the price level due to rising expenditure and non-homotheticity is distinct from new goods bias. In some of the examples presented here, all households consume all goods at all points in time; there are no new goods. Fundamentally, the new-goods bias is a problem of missing data: we do not measure the shadow price of goods for which there are no transactions. Thus the traditional fix for new-goods bias—imputing a missing price for new goods using the goods' characteristics—will improve measurement of price changes but is orthogonal to the interpretation of those changes.

7 Conclusion

This paper presented a simple example of a model with the following properties: Along a balanced growth path with a stable distribution of income, there are perpetual differences in measured inflation across individuals. Further, along a BGP with higher after-tax wage inequality, measured growth is slower and there is a larger gap in measured inflation between the top and bottom of the income distribution. Nevertheless, improvements in welfare are the same for all individuals along a BGP, and for all individuals across balanced growth paths with different TFP growth rates.

In thinking about the link between the outcomes generated in the model and events in the United States over the last several decades, there are a few caveats one should hold firmly in mind.

First, the model takes the stand that the systematic differences in consumption patterns between those at different parts of the income distribution come only from income effects. That is, the rich and the poor have the same preferences and, with the same expenditure, would consume the same bundle. An alternative possibility is that there are systematic differences in preferences between the rich and poor, perhaps differences that gave rise to the income disparities. Under this alternative, there could be a closer link between differences in measured real income growth across people and welfare improvements.

Second, in the model, lower measured productivity growth along the more unequal BGP stemmed from changes in the distribution of good-specific cost reductions. Another possibility is that the decline in growth experienced in the US came from a reduction in the pace of broad-based productivity growth (a decline in the growth rate of A_t). Again, under this alternative, there would be a closer link between changes in measured real income growth over time and welfare improvements.

Third, innovation is, at least in part, directed at the level of the world, not the level of a single country. Beerli et al. (2020) show that the size of the domestic market is a good predictor of productivity growth for firms that do not export, but not for those that do export. As with many models of endogenous growth, it is not trivial to determine at what level of aggregation one should apply the model.

In the model, shifts from low-cost goods to high-cost goods are an important component of changes in welfare. These notions of "low-price good" and "high-price good" are clear in the model—a consequence of the assumption that the unit cost of each good would have been the same in the absence of learning by doing. But it is not obvious how one might determine whether a good in the real world has a low price or a high price. In addition, the model is a very simple one, and consumption patterns in the real world are much more heterogeneous. Figuring out how to measure this component of changes in welfare in the real world will be challenging, but is an important question for future research.

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Appendix

A Properties of the Utility Function

A.1 Relationship to other Utility functions

Sato (1975) and Hanoch (1975) introduced the non-homothetic CES utility function.

$$\sum_{i=1}^{I} \Upsilon_i^{\frac{1}{\sigma}} \left(\frac{c_i}{g_i(U)}\right)^{\frac{\sigma-1}{\sigma}} = 1 \tag{7}$$

This is the unique class of utility functions with the property that the elasticity of substitution between two goods (which holds fixed the level of utility) is constant and independent of the prices or quantities of those or any other goods. These preferences have seen a revival since Comin, Lashkari and Mestieri (2021). The main text of their paper focuses on a special case, in which $g_i(U)$ can be expressed as $g(U)^{\varepsilon_i}$, giving

$$\sum_{i=1}^{I} \Upsilon_i^{\frac{1}{\sigma}} \left(\frac{c_i}{g(U)^{\varepsilon_i}}\right)^{\frac{\sigma-1}{\sigma}} = 1 \tag{8}$$

and discuss the more general version in their appendix. Comin, Lashkari and Mestieri (2021) restrict attention to the case where g_i is a monotonically increasing function, as in this case it is straightforward to show that the preferences define a unique U and that U increases with consumption of any good. Aside from allowing for an infinite range of goods, one consideration in the current setting is the focus on functional forms for h in which this assumption about monotonicity is relaxed: $g_i(U) = h(e^{-i}C^{\gamma})^{\frac{1}{\sigma-1}}C$ is not necessarily increasing everywhere in C. Thus we make some additional assumptions on the environment that guarantee that the price schedule is not decreasing too quickly.

Bohr, Mestieri and Yavuz (2022) focus on a setting with an infinite range of sectors indexed by ε, with preferences defined as

$$1 = \left(\int_0^{\infty} \left(\varepsilon^{-\beta} g(U)^{-\varepsilon} c_{\varepsilon}\right)^{\frac{\sigma-1}{\sigma}} d\varepsilon\right)^{\frac{\sigma}{\sigma-1}}$$

where, again, $g(U)$ is a monotonically increasing function. They derive a balanced growth path with endogenous variety creation within sectors that is a traveling wave, where the measure of varieties in each sector follows a Gamma distribution.

Foellmi and Zweimüller (2008) depart from CES and focus on a setting with a direct utility function, expressed as

$$u = \int_0^{N} i^{-\gamma} v(c_i) di$$

In this setting, the departure from CES and the departure from homotheticity go hand-in-hand. They focus on a BGP in which the range of goods consumed expands over time.

A.2 Regularity Conditions

Consider and individual that has preferences over bundles of goods$\{c_i\}$ to maximize $u\left(C\right)$ where C is defined to satisfy:

$$\sup_{C} C \text{ subject to } \left\{\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big)^{\frac{1}{\sigma}} \left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right\}^{\frac{\sigma}{\sigma-1}} \geq 1$$

where the weighting function h satisfies $\int_{-\infty}^{\infty} h(i)di = 1$.

For an individual with current expenditure E, the optimal consumption bundle is the solution to the following static optimization problem:

$$\max_{C,\{c_i\}} C$$

subject to

$$\begin{aligned}\mu &: \left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \geq 1 \\ \lambda &: \int p_i c_i di \leq E\end{aligned}$$

where μ and λ are the respective multipliers for the constraints. This problem can be split into two parts: finding the cost-minimizing bundle that delivers C and the optimal choice of C subject to the budget constraint. The first part of the problem can be expressed as

$$\mathcal{E}(C) = \min_{\{c_i\}} \int p_i c_i di$$

subject to

$$\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \geq 1$$

For any given C, this is a standard cost minimization with Dixit-Stiglitz preferences, with solution:

$$\mathcal{E}(C) = \int p_i c_i di = \left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di\right)^{\frac{1}{1-\sigma}} C$$

The second step is find the maximum affordable value of C,

$$\sup_C C$$

subject to

$$\left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di\right)^{\frac{1}{1-\sigma}} C \leq E$$

In this section, we show the following conditions guarantee that there exists a unique solution to $\mathcal{E}(C) = E$.

Assumption 1 *The weighting function h and the price schedule p_i satisfy the following properties:*

(a) There exists a $\kappa > -\frac{1}{\gamma}$ such that

i. Prices do not decline too steeply with i: $p_{i_1} > e^{\kappa(i_1-i_0)}p_{i_0}$, for all $i_1 > i_0$.

ii. $\int_{-\infty}^{\infty} h(i)e^{\kappa\gamma(1-\sigma)i}di \in (0,\infty)$.

(b) $p_i^{1-\sigma}$ is Lipschitz in i and $h(\cdot)$ is bounded.

The next several describes properties of the function $\mathcal{E}(C)$.

Lemma 1 *If $p_i^{1-\sigma}$ is Lipschitz in i, then $\mathcal{E}(C)$ is continuous for $C \in (0,\infty)$.*

Proof. Fix $C_0 \in (0,\infty)$. If $p_i^{1-\sigma}$ is Lipschitz in i, then there is an $L < \infty$ such that

$$\left|p_{u+\gamma\log C_1}^{1-\sigma} - p_{u+\gamma\log C_0}^{1-\sigma}\right| \leq L\left|(u+\gamma\log C_1)-(u+\gamma\log C_0)\right| = L\gamma\left|\log C_1 - \log C_0\right| .$$

The continuity of $\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di$ follows from

$$\begin{aligned}\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right| &= \left|\int h(u)p_{u+\gamma\log C_1}^{1-\sigma}du-\int h(u)p_{u+\gamma\log C_0}^{1-\sigma}du\right| \\ &\leq \int h(u)\left|p_{u+\gamma\log C_1}^{1-\sigma}-p_{u+\gamma\log C_0}^{1-\sigma}\right|du \\ &\leq \int h(u)L\gamma\left|\log C_1-\log C_0\right|du \\ &= L\gamma\left|\log C_1-\log C_0\right|\end{aligned}$$

Thus for any $\varepsilon>0$, $\exists\delta>0$ such that $|C_1-C_0|<\delta$ implies $\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<\varepsilon$.

■

The next lemma provides an alternative set of conditions sufficient to guarantee the continuity of $\mathcal{E}(C)$.

Lemma 2 *If h is Lipschitz and $p_i^{1-\sigma}$ is bounded, then $\mathcal{E}(C)$ is continuous.*

Proof. Let L be the Lipschitz constant of $h(\cdot)$ and let M be the bound on $p_i^{1-\sigma}$. Fix C_0 and $\varepsilon>0$. We will show that there is a $\delta>0$ such that $|C_1-C_0|<\delta$ implies $\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<\varepsilon$. We consider here only $C_1>C_0$; the proof for $C<C_0$ follows similar logic.

First, since $\int_{-\infty}^{\infty}h(u)du=1$, there are u_0, u_1 such that $u_0<u_1$, $\int_{-\infty}^{u_0}h(u)du\leq\frac{\varepsilon}{8M}$, and $\int_{u_1}^{\infty}h(u)du\leq\frac{\varepsilon}{8M}$. Consider C_1 such that $|C_1-C_0|<\frac{C_0}{(u_1-u_0)L\gamma M}\frac{\varepsilon}{2}$. Then:

$$\begin{aligned}\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right| &\leq \int\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|\left|p_i^{1-\sigma}\right|di \\ &\leq M\int\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di \\ &\leq M\left\{\begin{array}{l}\int_{-\infty}^{u_0+\gamma\log C_1}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di \\ +\int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di \\ +\int_{u_1+\gamma\log C_0}^{\infty}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di\end{array}\right\}\end{aligned}$$

The first term in brackets equal to $\int_{-\infty}^{u_0}\left|h(v)-h\left(v-\gamma\log\frac{C_0}{C_1}\right)\right|dv$ which is bounded by $2\frac{\varepsilon}{8M}$. Similarly, the third term in brackets is equal to $\int_{u_1}^{\infty}\left|h\left(v-\gamma\log\frac{C_1}{C_0}\right)-h(v)\right|dv$ which is also bounded by $2\frac{\varepsilon}{8M}$. The second term in brackets is bounded by $\frac{\varepsilon}{2M}$. To see this, if $u_1+\gamma\log C_0\leq u_0+\gamma\log C_1$, then the term is zero. Otherwise:

$$\begin{aligned}\int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di &\leq \int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}L\gamma\left|\log C_1-\log C_0\right|di \\ &= (u_1-u_0)\,L\gamma\left|\log C_1-\log C_0\right| \\ &\leq (u_1-u_0)\,L\gamma\frac{|C_1-C_0|}{C_0} \\ &< \frac{\varepsilon}{2M}\end{aligned}$$

Together these imply

$$\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<M\left\{2\frac{\varepsilon}{8M}+\frac{\varepsilon}{2M}+2\frac{\varepsilon}{8M}\right\}=\varepsilon$$

Finally, the continuity of $\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di$ implies the continuity of $\mathcal{E}(C)\equiv\left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di\right)^{\frac{1}{1-\sigma}}C$.

■

The next lemma provides conditions analogous to Inada conditions.

Lemma 3 *Suppose there is a $\kappa > -\frac{1}{\gamma}$ and that there exist b_0 and b_1 such that $\liminf_{i\to\infty} \frac{p_i}{e^{\kappa i}} \geq b_0 > 0$, $\limsup_{i\to 0} \frac{p_i}{e^{\kappa i}} \leq b_1 < \infty$, and $\int_{-\infty}^{\infty} h(w)e^{\kappa\gamma(1-\sigma)w}dw \in (0,\infty)$. Then $\liminf_{C\to\infty} \mathcal{E}(C) = \infty$ and $\limsup_{C\to 0} \mathcal{E}(C) = 0$.*

Proof. Using the change of variables $u = i - \gamma \log C$, the minimal cost is

$$\mathcal{E}(C) = \left(\int h(u) p_{u+\gamma\log C}^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C$$

We first show that as C grows large, the needed expenditure grows without bound.

$$\begin{aligned}
\liminf_{C\to\infty} \mathcal{E}(C) &= \liminf_{C\to\infty} \left(\int h(u) p_{u+\gamma\log C}^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C \\
&= \liminf_{C\to\infty} \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C^{1+\kappa\gamma} \\
&\geq \liminf_{C\to\infty} \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&\geq \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\liminf_{C\to\infty} \frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&\geq \left(\int h(u) e^{\kappa(1-\sigma)u} b_0^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&= \infty
\end{aligned}$$

where the second inequality uses Fatou's lemma. We next show that as C grows small, the needed expenditure shrinks to 0.

$$\begin{aligned}
\limsup_{C\to 0} \mathcal{E}(C) &= \limsup_{C\to 0} \left(\int h(u) p_{u+\gamma\log C}^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C \\
&= \limsup_{C\to 0} \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C^{1+\kappa\gamma} \\
&\leq \limsup_{C\to 0} \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&\leq \left(\int h(u) e^{\kappa(1-\sigma)u} \left(\limsup_{C\to 0} \frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&\leq \left(\int h(u) e^{\kappa(1-\sigma)u} b_1^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&= 0
\end{aligned}$$

■

Under the conditions of the last two lemmas, there must exist an interior solution to the problem of $\max_{C\geq 0}$ such that $\mathcal{E}(C) = E$ for any $E \in (0,\infty)$.

Lemma 4 *Suppose that there exists a $\kappa > -\frac{1}{\gamma}$ such that $p_{i_1} \geq p_{i_0} e^{\kappa(i_1 - i_0)}$ for all $i_1 \geq i_0$ and that $\int_{-\infty}^{\infty} h(w)e^{\kappa\gamma(1-\sigma)w}dw \in (0,\infty)$. Then $\mathcal{E}(C)$ is strictly increasing with $\lim_{C\to 0} \mathcal{E}(C) = 0$ and $\lim_{C\to\infty} \mathcal{E}(C) = \infty$.*

Proof. Using the change of variables $u = i - \gamma \log C$, the minimal cost is

$$\mathcal{E}(C) = \left(\int h(u) p_{u+\gamma \log C}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C$$

For $C_1 > C_0$, we have $p_{u+\gamma \log C_1} \geq p_{u+\gamma \log C_0} e^{\kappa(\gamma \log C_1 - \gamma \log C_0)} = \left(\frac{C_1}{C_0}\right)^{\kappa\gamma} p_{u+\gamma \log C_0}$. This allows us to express the minimal cost of C_1

$$\begin{aligned}
\mathcal{E}(C_1) &= \left(\int h(u) p_{u+\gamma \log C_1}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C_1 \\
&\geq \left(\int h(u) \left(\left(\frac{C_1}{C_0}\right)^{\kappa\gamma} p_{u+\gamma \log C_0} \right)^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C_1 \\
&= \left(\int h(u) p_{u+\gamma \log C_0}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} \left(\frac{C_1}{C_0}\right)^{\kappa\gamma} C_1 \\
&= \mathcal{E}(C_0) \left(\frac{C_1}{C_0}\right)^{1+\kappa\gamma} \\
&> \mathcal{E}(C_0)
\end{aligned}$$

where the last line follows because $1 + \kappa\gamma > 0$. Finally, $p_{i_1} \geq p_{i_0} e^{\kappa(i_1 - i_0)}$ for all $i_1 \geq i_0$ implies that $\liminf_{i\to\infty} \frac{p_i}{e^{\kappa i}} \geq p_0 > 0$ and $\limsup_{i\to 0} \frac{p_i}{e^{\kappa i}} \leq p_0 < \infty$, so by the previous lemma $\lim_{C\to 0} \mathcal{E}(C) = 0$ and $\lim_{C\to\infty} \mathcal{E}(C) = \infty$. ■

Proposition 11 *Under Assumption 1, the optimal consumption bundle is*

$$c_i = E^{\sigma} C^{1-\sigma} p_i^{-\sigma} h\Big(i - \gamma \log C\Big)$$

where C is the unique solution to $\left(\int h\Big(i - \gamma \log C\Big) p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} C = E$

Proof. Under Assumption 1, $\mathcal{E}(C)$ is continuous, strictly increasing, and satisfies $\lim_{C\to 0} \mathcal{E}(C) = 0$ and $\lim_{C\to\infty} \mathcal{E}(C) = \infty$. Therefore there exists a unique value of C that satisfies $\mathcal{E}(C) = E$ and this value maximizes $\sup_C C$ such that $\mathcal{E}(C) \leq E$. Given C, cost minimization implies $c_i = E^{\sigma} C^{1-\sigma} p_i^{-\sigma} h\Big(i - \gamma \log C\Big)$.
■

A.3 Non-homothetic Cobb-Douglas Limit

This section describes the limiting preferences as $\sigma \to 1$. Taking this limit gives

$$
\begin{aligned}
\lim_{\sigma\to 1}\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} &= \exp\lim_{\sigma\to 1}\frac{\log\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)}{\frac{\sigma-1}{\sigma}} \\
&= \exp\lim_{a\to 0}\frac{\log\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a} di\right)}{a} \\
&= \exp\lim_{a\to 0}\frac{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a}\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di}{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a} di} \\
&= \exp\frac{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di}{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big) di} \\
&= \exp\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di
\end{aligned}
$$

where the last line used $\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big) di = 1$.

To find the cost-minimizing bundle, we have

$$
\mathcal{E}(C) = \min_{\{c_i\}}\int p_i c_i \qquad \text{subject to} \qquad \exp\int_{-\infty}^{\infty} h\left(e^{-i}C^{\gamma}\right)\left[\log\left(\frac{c_i}{C}\right)-\log h\left(e^{-i}C^{\gamma}\right)\right] di \geq 1
$$

The solution gives

$$
\mathcal{E}(C) = C\exp\int h\left(e^{-i}C^{\gamma}\right)\log p_i di
$$

So that the price index is $P(C) = \exp\int h\Big(i-\gamma\log C\Big)\log p_i di$.

If h is Lipschitz and $\log p_i$ is bounded, then $\mathcal{E}(C)$ is continuous. If there exists a $\kappa > -\frac{1}{\gamma}$ such that $p_{i_1} \geq p_{i_0} e^{\kappa(i_1-i_0)}$ for all $i_1 \geq i_0$ and $\exp\int_{-\infty}^{\infty} h(w)\log w dw \in (0,\infty)$, then $\mathcal{E}(C)$ is strictly increasing with $\lim_{C\to 0}\mathcal{E}(C) = 0$ and $\lim_{C\to\infty}\mathcal{E}(C) = \infty$. Under all of these conditions, there is a unique solution to $\mathcal{E}(C) = E$.

B A Balanced Growth Path New

We first derive an alternative characterization of a BGP, and prove the existence of a balanced growth path when σ is not too large.

B.1 An Alternative Characterization

The ideal price index for household ℓ satisfies

$$
P_{\ell t}^{1-\sigma} = \int_{-\infty}^{\infty} p_{it}^{1-\sigma} h\left(i-\gamma\log\frac{E_{\ell t}}{P_{\ell t}}\right) di
$$

Market clearing for good i gives

$$
A_t B_{it} L_{it} = \int_{0}^{\infty} c_{\ell i t} dG(\ell)
$$

Using $c_{\ell it} = E_{\ell t} P_{\ell t}^{\sigma-1} p_{it}^{-\sigma} h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)$ and $p_{it} = \frac{w_t}{A_t B_{it}}$, market clearing for good i can be expressed as

$$L_{it} = \int_0^\infty \frac{E_{\ell t}}{w_t} \left(\frac{P_{\ell t}}{w_t/A_t}\right)^{\sigma-1} \frac{h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)}{B_{it}^{1-\sigma}} dG(\ell)$$

Learning by doing implies that $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$ or, using market clearing,

$$\frac{\dot{B}_{it}}{B_{it}} = \phi \int_0^\infty \frac{E_{\ell t}}{w_t} \left(\frac{P_{\ell t}}{w_t/A_t}\right)^{\sigma-1} \frac{h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)}{B_{it}^{1-\sigma}} dG(\ell) \ .$$

Multiplying both sides by $B_{it}^{1-\sigma}$, integrating, and using $\lim_{t\to-\infty} B_{it} = 1$ gives

$$\frac{B_{it}^{1-\sigma} - 1}{1-\sigma} = \phi \int_{-\infty}^t \int_0^\infty \frac{E_{\ell \tilde{t}}}{w_{\tilde{t}}} \left(\frac{P_{\ell \tilde{t}}}{w_{\tilde{t}}/A_{\tilde{t}}}\right)^{\sigma-1} h\left(i - \gamma \log \frac{E_{\ell \tilde{t}}}{P_{\ell \tilde{t}}}\right) dG(\ell) d\tilde{t}$$

Define

- Let $\mathfrak{p}_{\ell,t} \equiv \log \frac{P_{\ell t}}{w_t/A_t}$.
- Let $\mathfrak{b}_{it} \equiv \log B_{i+\gamma \log A_t, t}$
- Let $\tilde{E}_\ell \equiv \ell^{1-\tau}/\overline{\ell^{1-\tau}}$ be household ℓ's after-tax income.

$$\begin{aligned}
\frac{e^{(1-\sigma)\mathfrak{b}_{it}} - 1}{1-\sigma} &= \frac{B_{i+\gamma \log A_t, t}^{1-\sigma} - 1}{1-\sigma} \\
&= \phi \int_{-\infty}^t \int_0^\infty \frac{E_{\ell \tilde{t}}}{w_{\tilde{t}}} \left(\frac{P_{\ell \tilde{t}}}{w_{\tilde{t}}/A_{\tilde{t}}}\right)^{\sigma-1} h\left((i + \gamma \log A_t) - \gamma \log \frac{E_{\ell \tilde{t}}}{P_{\ell \tilde{t}}}\right) dG(\ell) d\tilde{t} \\
&= \phi \int_{-\infty}^t \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_{\ell \tilde{t}}} h\left((i + \gamma \log A_t - \gamma \log A_{\tilde{t}}) - \gamma \log \frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell \tilde{t}}}}\right) dG(\ell) d\tilde{t}
\end{aligned}$$

Using the change of variables $u = i + \gamma \log A_t - \gamma \log A_{\tilde{t}} = i + \gamma g(t - \tilde{t})$, this is

$$\frac{e^{(1-\sigma)\mathfrak{b}_{it}} - 1}{1-\sigma} = \frac{\phi}{\gamma g} \int_i^\infty \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_{\ell \tilde{t}}} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_{\ell \tilde{t}}\right) dG(\ell) du$$

Next, we can express $\mathfrak{p}_{\ell t}$ as

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_{\ell t}} = \frac{P_{\ell t}^{1-\sigma}}{(w_t/A_t)^{1-\sigma}} = \frac{\int_{-\infty}^\infty p_{it}^{1-\sigma} h\left(i - \gamma \log C_{\ell t}\right) di}{(w_t/A_t)^{1-\sigma}}$$

Using $p_{it} = \frac{w_t}{A_t B_{it}}$, $C_{\ell t} = \frac{E_{\ell t}}{P_{\ell t}} = \frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell t}}/A_t}$, and the change of variables $u = i - \gamma \log A_t$

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_{\ell t}} = \int_{-\infty}^\infty \frac{h\left(i - \gamma \log\left(\frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell t}}/A_t}\right)\right)}{B_{it}^{1-\sigma}} di = \int_{-\infty}^\infty \frac{h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_{\ell t}\right)}{e^{(1-\sigma)\mathfrak{b}_{ut}}} du$$

Along a balanced growth path, $\{\mathfrak{p}_{\ell,t}\}_\ell$ and $\{\mathfrak{b}_{i,t}\}_i$ are constant. For the remainder of this section, we drop the time subscript. We can express these two key equations as

$$\frac{e^{(1-\sigma)\mathfrak{b}_i} - 1}{1-\sigma} = \frac{\phi}{\gamma g} \int_i^\infty \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell) du \tag{9}$$

and

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_\ell} = \int_{-\infty}^{\infty} \frac{h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} di \tag{10}$$

B.2 Existence of a Balanced Growth Path

We assume throughout this section that h is bounded. Let $\mathcal{P}$ be the space of functions $\mathfrak{p} : (0, \infty) \to [-\frac{\phi L}{\gamma g}, 0]$. In this space, we define an operator $\mathcal{T}$ using the two equations (9) and (10) as follows:

Consider a function $\mathfrak{p} \in \mathcal{P}$. Define the transformations $\mathfrak{b}(\mathfrak{p})$ and $\hat{\mathfrak{b}}(\mathfrak{p})$ for each i as

$$\hat{\mathfrak{b}}(\mathfrak{p})_i = \log \left[1 + (1-\sigma)\frac{\phi L}{\gamma g} \int_i^{\infty} \int_0^{\infty} \tilde{E}_\ell \tilde{P}_\ell^{\sigma-1} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell) du\right]^{\frac{1}{1-\sigma}}$$

$$\mathfrak{b}(\mathfrak{p})_i = \min\left\{\frac{\phi L}{\gamma g}, \hat{\mathfrak{b}}(\mathfrak{p})_i\right\}$$

Finally, we define the operator $\mathcal{T}(\mathfrak{p})$ so that, for each ℓ, $\mathcal{T}(\mathfrak{p})_\ell$ is the unique solution to

$$\mathcal{T}(\mathfrak{p})_\ell = \log \left[\int_{-\infty}^{\infty} \frac{h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathcal{T}(\mathfrak{p}_\ell)\right)}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p})_i}} di\right]^{\frac{1}{1-\sigma}} \tag{11}$$

Lemma 5 *If $h(\cdot)$ is bounded then $\mathcal{T}(\mathfrak{p})$ is well defined for any $\mathfrak{p} \in \mathcal{P}$.*

Proof. $\mathfrak{b}(\mathfrak{p})_i$ is decreasing and continuous in i because as i increases the region of integration shrinks. Further, if h is bounded then $e^{\mathfrak{b}(\tilde{p})}$ is Lipschitz. Appendix A.2 showed the existence and uniqueness of a solution to C_ℓ to the equation $E_\ell = C_\ell \left[\int \left(\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}\right)^{1-\sigma} h\Big(i - \gamma \log C_\ell\Big) di\right]^{\frac{1}{1-\sigma}}$, under the condition that $\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}$ is weakly increasing and $\left(\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}\right)^{1-\sigma}$ is Lipschitz, which are satisfied here. Letting $\mathcal{T}(\mathfrak{p})_\ell \equiv \log E_\ell / C_\ell$, this is equivalent to showing existence and uniqueness of a solution to (11). ∎

Lemma 6 *$\mathcal{T}$ maps $\mathcal{P}$ onto itself.*

Proof. First, note that $\mathfrak{b}(\mathfrak{p})_i \in [0, \frac{\phi L}{\gamma g}]$. The conclusion follows from the fact that $\mathcal{T}(\mathfrak{p})$ is a generalized weighted mean of $\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}$ with weights $h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)$ which integrate to 1. ∎

Lemma 7 *Define $\alpha \equiv \left|e^{\frac{\phi L}{\gamma g}(\sigma-1)} - 1\right|$. If $e^{\frac{\phi L}{\gamma g}(\sigma-1)} < 2$, then $\alpha \in [0, 1)$ and*

$$\left|-\frac{d\mathfrak{b}\left(\mathfrak{p}^\varepsilon\right)_i}{d\varepsilon}\right| \leq \left(\alpha - \gamma \frac{d\mathfrak{b}\left(\mathfrak{p}^\varepsilon\right)_i^{1-\sigma}}{di}\right) \left\|\mathfrak{p}^1 - \mathfrak{p}^0\right\|$$

for any i, $\mathfrak{p}^0, \mathfrak{p}^1 \in \mathcal{P}$, $\varepsilon \in [0, 1]$, and $\mathfrak{p}^\varepsilon$ defined as $p_\ell^\varepsilon \equiv (1-\varepsilon)\mathfrak{p}_\ell^0 + \varepsilon \mathfrak{p}_\ell^1$.

Proof. $\hat{\mathfrak{b}}\left(\mathfrak{p}^\varepsilon\right)_i$ is defined as

$$\frac{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i} - 1}{1-\sigma} = \frac{\phi L}{\gamma g} \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon} \int_{i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell^\varepsilon}^{\infty} h(u) du dG(\ell)$$

Let $i^*(\varepsilon)$ be such that $\mathfrak{b}(\mathfrak{p}^\varepsilon)_{i^*(\varepsilon)} = \frac{\phi L}{\gamma g}$. If $i < i^*(\varepsilon)$, then $\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon} = 0$. If $i > i^*(\varepsilon)$, then $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i = \hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i$. Differentiating with respect to ε and rearranging yields

$$\begin{aligned} -\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon} &= (1-\sigma)\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^\infty h(u)du\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}dG(\ell) \\ &\quad +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}dG(\ell) \end{aligned}$$

Since $\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right| = \left|\mathfrak{p}_\ell^1 - \mathfrak{p}_\ell^0\right| \le \left\|\mathfrak{p}_\ell^1 - \mathfrak{p}_\ell^0\right\|$, this can be bounded by

$$\begin{aligned} \left|-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}\right| &= |1-\sigma|\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^\infty h(u)du\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right|dG(\ell) \\ &\quad +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right|dG(\ell) \\ &\le \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{\begin{array}{c} |1-\sigma|\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^\infty h(u)dudG(\ell) \\ +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)dG(\ell) \end{array}\right\} \end{aligned}$$

Using the expression for $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i$ and its derivative with respect to i, $\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di} = \frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int_0^\infty \tilde{E}_\ell e^{(\sigma 1)\mathfrak{p}_\ell^\varepsilon}\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)$ this is

$$\begin{aligned} \left|-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}\right| &\le \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{|1-\sigma|\frac{\frac{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}-1}{1-\sigma}}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}} - \gamma\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di}\right\} \\ &= \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] - \gamma\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di}\right\} \end{aligned}$$

Note that $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i \le \frac{\phi L}{\gamma g}$. If $\sigma < 1$, then $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] = 1-e^{-(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i} \le 1-e^{-(1-\sigma)\frac{\phi L}{\gamma g}} < 1$. If $\sigma > 1$ and $e^{\frac{\phi L}{\gamma g}(\sigma-1)} \le 2$ then $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] = e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}-1 \le e^{(\sigma-1)\frac{\phi L}{\gamma g}}-1 < 1$. In ether case, $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] < \alpha \in [0,1)$. ■

Lemma 8 *If $e^{\frac{\phi L}{\gamma g}(\sigma-1)} \le 2$, then the operator $\mathcal{T}(\mathfrak{p})$ is a contraction mapping on $\mathcal{P}$.*

Proof. First, note that $\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell$ satisfies

$$\begin{aligned} \mathcal{T}(\mathfrak{p}^\varepsilon)_\ell &= \log\left[\int\frac{h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell\right)}{\exp\left\{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i\right\}}di\right]^{\frac{1}{1-\sigma}} \\ &= \log\left[\int\frac{h(u)}{\exp\left\{(1-\sigma)\,\mathfrak{b}(\mathfrak{p}^\varepsilon)_{u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}\right\}}du\right]^{\frac{1}{1-\sigma}} \end{aligned}$$

Differentiating with respect to ε, letting $\Upsilon_{i\ell} \equiv \frac{h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell\right)}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}}$, and rearranging yields

$$\begin{aligned} \frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon} &= -\frac{\int\left[\left\{\Upsilon_{i\ell}\left[\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{di}\gamma\frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon}\right]\right\}\Big|_{i=u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}\right]du}{\int\Upsilon_{i\ell}\big|_{i=u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}du} \\ &= -\frac{\int\Upsilon_{i\ell}\left[\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{di}\gamma\frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon}\right]di}{\int\Upsilon_{i\ell}di} \end{aligned}$$

This can be rearranged as

$$\frac{d\mathcal{T}\left(\mathfrak{p}^{\varepsilon}\right)_{\ell}}{d\varepsilon}=\frac{\int\Upsilon_{i\ell}\left(-\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{d\varepsilon}\right)di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\right)di}$$

This can be bounded using the previous lemma

$$\begin{aligned}\left|\frac{d\mathcal{T}\left(\mathfrak{p}^{\varepsilon}\right)_{\ell}}{d\varepsilon}\right| &\leq \frac{\int\Upsilon_{i\ell}\left|-\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{d\varepsilon}\right|di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\right)di}\\ &\leq \frac{\int\Upsilon_{i\ell}\left(\alpha-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\right)\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\right)di}\\ &= \left(1-\frac{1-\alpha}{\int\frac{\Upsilon_{i\ell}}{\int\Upsilon_{\tilde{i}\ell}d\tilde{i}}\left(1-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\right)di}\right)\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|\end{aligned}$$

Since $\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\leq 0$, $1-\gamma\frac{d\mathfrak{b}\left(\mathfrak{p}^{\varepsilon}\right)_{i}}{di}\geq 1$, and hence

$$\left|\frac{d\mathcal{T}\left(\mathfrak{p}^{\varepsilon}\right)_{\ell}}{d\varepsilon}\right|\leq\alpha\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|$$

Finally, we have

$$\begin{aligned}\left|\mathcal{T}\left(\mathfrak{p}^{1}\right)_{\ell}-\mathcal{T}\left(\mathfrak{p}^{0}\right)_{\ell}\right| &= \left|\int_{0}^{1}\frac{d\mathcal{T}\left(\mathfrak{p}^{\varepsilon}\right)_{\ell}}{d\varepsilon}d\varepsilon\right|\leq\int_{0}^{1}\left|\frac{d\mathcal{T}\left(\mathfrak{p}^{\varepsilon}\right)_{\ell}}{d\varepsilon}\right|d\varepsilon\\ &\leq \alpha\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|\end{aligned}$$

Since $\alpha\in[0,1)$, $\mathcal{T}$ is a contraction mapping. ■

Lemma 9 *If $e^{(\sigma-1)\frac{\phi L}{\gamma g}}<2$, then there exists a unique fixed point of $\mathcal{T}$ on $\mathcal{P}$.*

Proof. This follows from the previous two lemmas and the contraction mapping theorem. ■

Lemma 10 *Suppose $e^{(\sigma-1)\frac{\phi L}{\gamma g}}<2$. Then for any fixed point $\mathfrak{p}$ of $\mathcal{T}$ in $\mathcal{P}$, $\mathfrak{p}$ and $\mathfrak{b}(\mathfrak{p})$ satisfy equations (9) and (10).*

Proof. Abusing notation, let $\hat{\mathfrak{b}}=\hat{\mathfrak{b}}\left(\mathfrak{p}\right)$ and $\mathfrak{b}=\mathfrak{b}\left(\mathfrak{p}\right)$. We need only show that $\mathfrak{b}_{i}=\hat{\mathfrak{b}}_{i}$ for each i. That is, we must show that $\hat{\mathfrak{b}}_{i}\leq\frac{\phi L}{\gamma g}$ for each i. Since $\hat{\mathfrak{b}}$ satisfies

$$\frac{e^{(1-\sigma)\hat{\mathfrak{b}}_{i}}-1}{1-\sigma}=\frac{\phi}{\gamma g}\int_{i}^{\infty}\int_{0}^{\infty}\tilde{E}_{\ell}e^{(\sigma-1)\mathfrak{p}_{\ell}}h\left(u-\gamma\log\tilde{E}_{\ell}+\gamma\mathfrak{p}_{\ell}\right)dG(\ell)du$$

differentiating with respect to i and rearranging yields

$$\frac{d\hat{\mathfrak{b}}_{i}}{di}=-\frac{\phi}{\gamma g}\int_{0}^{\infty}\tilde{E}_{\ell}e^{(\sigma-1)\mathfrak{p}_{\ell}}\frac{h\left(i-\gamma\log\tilde{E}_{\ell}+\gamma\mathfrak{p}_{\ell}\right)}{e^{(1-\sigma)\hat{\mathfrak{b}}_{i}}}dG(\ell)\ . \tag{12}$$

$\hat{\mathfrak{b}}$ is continuous and decreasing in i. Let $i^{*}\equiv\inf\left\{i|\hat{\mathfrak{b}}_{i}\leq\frac{\phi L}{\gamma g}\right\}$. Toward a contradiction, suppose that $i^{*}>-\infty$. Since $\hat{\mathfrak{b}}$ is continuous, it must be that $\hat{\mathfrak{b}}_{i^{*}}=\frac{\phi L}{\gamma g}$.

For any $\tilde{\imath} > i^*$, it must be that $\hat{\mathfrak{b}}_{\tilde{\imath}} = \mathfrak{b}_{\tilde{\imath}}$. In addition, $\lim_{i\to\infty} \mathfrak{b}_i = \infty$. We thus have

$$\begin{aligned}
\frac{\phi L}{\gamma g} &= \hat{\mathfrak{b}}_{i^*} = \int_i^{\infty} -\frac{d\hat{\mathfrak{b}}_{\tilde{\imath}}}{di} d\tilde{\imath} \\
&= \frac{\phi}{\gamma g} \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_i^{\infty} \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) \\
&= \frac{\phi}{\gamma g} \left[L - \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^{i} \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) \right]
\end{aligned}$$

where the last line used $\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^{\infty} \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) = L$, which follows from $e^{\mathfrak{p}_\ell} = \left(\int_{-\infty}^{\infty} \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} \right)^{\frac{1}{1-\sigma}}$ and $\int \tilde{E}_\ell dG(\ell) = L$. Rearranging and using $\mathfrak{b}_i = \frac{\phi L}{\gamma g}$ for $i < i^*$ gives

$$0 = \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^{i^*} h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) d\tilde{\imath} dG(\ell)$$

since the integrand is non-negative, this it must be that for all $i < i^*$,

$$0 = \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell)$$

This along with (12) implies that $\frac{d\hat{\mathfrak{b}}_i}{di} = 0$ for all $i \leq i^*$. As a result, $\hat{\mathfrak{b}}_i = \frac{\phi L}{\gamma g}$ for all $i < i^*$, a contradiction. ■

Proposition 12 *If $h(\cdot)$ is bounded and $e^{(\sigma-1)\frac{\phi L}{\gamma g}} < 2$, then there is a unique balanced growth path.*

Proof. This is a simple consequence of the existence of a unique fixed point $\mathfrak{p}$ of $\mathcal{T}$ on $\mathcal{P}$ and the fact that $\mathfrak{p}$ and $\mathfrak{b}(\mathfrak{p})$ solve equations (9) and (10). ■