Aggregate Uncertainty, HANK, and the ZLB

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Workshop on Methods and Applications for Dynamic Equilibrium Models - NBER SI 2023

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The views expressed herein are those of the authors and do not necessarily reflect those of the Bank of Italy or its executive board.
Outline

Introduction
Simple Model
HANK Model
Solution Approach
Results
Other Applications
Conclusions
Figure 1: Mon. Policy, Micro-Macro uncertainty

Introduction (1) - Motivation

Figure 1: Mon. Policy, Micro-Macro uncertainty

1. Uncertainty rises in recessions:

2. Aggr. uncertainty interacts with ZLB:
   - Basu and Bundick (2016), Basu and Bundick (2017), Caggiano et al. (2017)
Introduction (1) - Motivation

Figure 1: Mon. Policy, Micro-Macro uncertainty


3. What about idiosyncratic risk at the ZLB?
Introduction (1) - Motivation


3. What about idiosyncratic risk at the ZLB?
   - We are interested in understanding this interaction...
   - via HANK-DSGE-model
Introduction (2) - Literature (Model)

1. HANK: Kaplan et al. (2018), Achdou et al. (2022), Ahn et al. (2018), Auclert et al. (2021), Auclert (2019), Bayer et al. (2019)...

2. ZLB: Eggertsson and Woodford (2003), Christiano et al. (2011), Eggertsson et al. (2021)

3. HANK + ZLB: Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2017), McKay et al. (2016), Benigno et al. (2020)...

4. HANK + ZLB + Aggregate Uncertainty: Fernández-Villaverde et al. (2021), Kase et al. (2022), Schaab (2020)
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Introduction (3) - Contribution

- Novel solution strategy for HANK models w/aggregate uncertainty (AU) and ZLB:
- In practice: take standard HANK, add ZLB, add tractable AU, compare to PF
- Solution allows to quantify interactions between AU-ZLB-HA
Novel solution strategy for HANK models with aggregate uncertainty (AU) and ZLB:
In practice: take standard HANK, add ZLB, add tractable AU, compare to PF
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<table>
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1. Quantify effect of aggregate uncertainty in HANK at the ZLB (E vs D)
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1. Quantify effect of aggregate uncertainty in HANK at the ZLB (E vs D)
2. Quantify effect of aggregate uncertainty in RANK at the ZLB (C vs B)
Novel solution strategy for HANK models w/aggregate uncertainty (AU) and ZLB:
- In practice: take standard HANK, add ZLB, add tractable AU, compare to PF
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1. Quantify effect of aggregate uncertainty in HANK at the ZLB (E vs D)
2. Quantify effect of aggregate uncertainty in RANK at the ZLB (C vs B)
3. Decompose the role of HA in the amplification (E-D vs C-B)
- Novel solution strategy...
- but not limited to ZLB, can accommodate more general non-linearities (kinky PC, aggregate borrowing constraints/financial accelerator, downward wage rigidity...)}
Simple Model (1) - Description

- Simple model to define ZLB-AU interactions
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- Infinitely lived households, standard consumption-savings decisions, CRRA preferences, exogenous discount factor $\beta_t$, rigid prices, Taylor rule

\[ R_t = \max\{nR_s, R_{ss}Y_t\} \]

\[ R_{ss} = \beta^{-\frac{1}{\sigma}} p (z - \sigma u + (1 - p)\frac{1}{1 - \lambda} z u^{1-\lambda} - \sigma) \]
Simple Model (1) - Description

- Simple model to define ZLB-AU interactions
- Infinitely lived households, standard consumption-savings decisions, CRRA preferences, exogenous discount factor $\beta_t$, rigid prices, Taylor rule
- Idiosyncratic shock
  - $c$: constrained, no access to financial markets, earn $z_c Y_t$
  - $u$: unconstrained, access to financial markets, earn $z_u Y_t$
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\[
Y_t^{-\sigma} = \frac{\beta_t R_t}{\beta R_{ss}} \mathbb{E}_t Y_{t+1}^{-\sigma}, \quad (1)
\]

\[
R_t = \max \left\{ R, R_{ss} Y_t^{\phi} \right\} \quad (2)
\]

\[
R_{ss} = \frac{1}{\beta \left\{ p \left( z_u^{-\sigma} \right) + (1-p) \left[ \left( \frac{1-\lambda z_u}{1-\lambda} \right)^{-\sigma} \right] \right\}}
\]
Simple Model (2) - Solution

\[ Y_t = f\left(\mathbb{E}_t Y_{t+1}^{-\sigma}, \beta_t | \beta, \sigma, \phi, R, R_{ss}\right) \]

\[ = \left\{ \begin{array}{ll}
\left(\frac{\beta_t}{\beta} \mathbb{E}_t Y_{t+1}^{-\sigma}\right)^{-\frac{1}{\sigma+\phi}} & \text{if } \beta_t \leq \beta \left(\frac{R_{ss}}{R}\right)^{\frac{\sigma+\phi}{\phi}} \left(\mathbb{E}_t Y_{t+1}^{-\sigma}\right)^{-1} \\
\left(\frac{\beta_t}{\beta} \frac{R}{R_{ss}} \mathbb{E}_t Y_{t+1}^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{otherwise}
\end{array} \right. \]

Higher future MU (or larger discount factor) leads to larger recession.
1. The economy is at steady state at $t = 0$.

2. Unexpected shock: $\beta_0 = \beta$ for any $t > 1$.

$\beta_1 = \beta_L > \beta$ with probability $\mu$,

$\beta_1 = \beta_{DET}$ such that same effect absent ZLB (i.e. $R = -\infty$).
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Simple Model (3) - Shocks

1. The economy is at steady state at \( t = 0 \).
2. Unexpected shock:
   - \( \beta_0 = \beta \)
   - \( \beta_t = \beta \) for any \( t > 1 \)
   - AU) \( \beta_1 = \beta_L > \beta \) with probability \( \mu \), \( \beta_1 = \beta \) otherwise
1. The economy is at steady state at $t = 0$.

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   - $\beta_0 = \beta$
   - $\beta_t = \beta$ for any $t > 1$
   - AU) $\beta_1 = \beta_L > \beta$ with probability $\mu$, $\beta_1 = \beta$ otherwise
   - PF) $\beta_1 = \beta_{DET}$ such that same effect absent ZLB (i.e. $R = -\infty$)
Figure 2: Equilibrium in the Simple Model
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\[ Y_1^{-\sigma} \]

\[ \mathbb{E}_0 Y_1^{-\sigma} \]

\[ Y_{ss}^{-\sigma} \]

\[ \beta \quad \beta_{DET} \quad \beta_L \quad \beta_1 \]

\[ \mathbb{E}_0 \beta_1 \]

\[ \beta \]

\[ Y_1^{-\sigma}_{DET} \]
Figure 2: Equilibrium in the Simple Model
1. ZLB amplifies effect of shock because interest rate higher than it would have been. True even with PF...

2. AU shock interacts with ZLB, implies further amplification, because of Jensen’s inequality. True even with RA...

3. So what is the role of HA in this amplification? In the steady state (closer to the ZLB in the steady state because of precautionary savings - kink more to the left) in the business cycle (lowers $R_t$ towards ZLB because of precautionary savings and MPCs - steeper slope)
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   - in the steady state (closer to the ZLB in the steady state because of precautionary savings - kink more to the left)
   - in the business cycle (lowers $R_t$ towards ZLB because of precautionary savings and MPCs - steeper slope)
HANK Model (1) - Overview

Standard one-asset HANK model (McKay et al. (2016), Guerrieri and Lorenzoni (2017)):

- Demand side (idiosyncratic risk, borrowing constraint)
- New-Keynesian Phillips Curve
- Supply of bonds from government
- Taylor rule + ZLB

Calibration: standard parameter values + Great Recession
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Calibration: standard parameter values + Great Recession
HANK Model (2) - Households

Household \( i \) with assets \( a_{it} \) and shock \( z_{it} \) maximizes:

\[
V_t(z_{it}, a_{it} - 1) = \max_{c_{it}, a_{it} \geq a_{c1} - \sigma a_{c1} + \beta E_{t+1} V_{t+1}(z_{it+1}, a_{t+1})}
\]

subject to:

\[
c_{it} + a_{it} R_t = a_{it} - 1 \Pi_t + z_{it} (Y_t - t_t)
\]

\( z_{it} \) follows a Markov chain following

\[
Q = P(z_{it+1} | z_{it}) \text{ (time invariant)}
\]

Aggregate Asset Demand

\[
A_t = Z_g a_t(z, a)dD_t(z, a)
\]
Household $i$ with assets $a_{it-1}$ and shock $z_{it}$ maximizes:

$$V_t (z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t E_t V_{t+1} (z_{it+1}, a_t)$$

subject to:

$$c_{it} + \frac{a_{it}}{R_t} = \frac{a_{it-1}}{\Pi_t} + z_{it} (Y_t - t_t)$$
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subject to:

$$c_{it} + \frac{a_{it}}{R_t} = \frac{a_{it-1}}{\Pi_t} + z_{it} (Y_t - t_t)$$

$z_{it}$ ~ a Markov chain following $Q = P(z_{it+1}|z_{it})$ (time invariant)

$\implies$ earnings risk is acyclical
Household $i$ with assets $a_{it-1}$ and shock $z_{it}$ maximizes:

$$V_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a_{it-1}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t E_t V_{t+1}(z_{it+1}, a_t)$$

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$\implies$ earnings risk is acyclical

Aggregate Asset Demand

$$A_t = \int g_t^a(z, a) dD_t(z, a)$$
HANK Model (3) - Rest of economy

New Keynesian Phillips Curve (from Rotemberg):
\[ \Pi_t - \Pi_{t+1} = E_t \beta_t Y_t + 1 - \sigma \times (\Pi_t + 1) - \Pi_{t+1} + \kappa Y_{\omega} + \sigma_t - 1 \]

Government Budget and Fiscal Policy
\[ T_t + b_t = b_t - 1 \Pi_t \]

Market Clearing
\[ b_t = Z_g a_t (a, z) \int_d D_t (z, a) \]

Monetary Policy
\[ R_t = \max \left( 1, R_{\Pi_t} \right) \theta_{\Pi_t} Y_t \bar{Y} \theta Y_t \phi \]
HANK Model (3) - Rest of economy

- New Keynesian Phillips Curve (from Rotemberg):
  \[(\Pi_t - \bar{\Pi}) \Pi_t = \mathbb{E}_t \beta_t \left( \frac{Y_{t+1}}{Y_t} \right)^{1-\sigma} \times (\Pi_{t+1} - \bar{\Pi}) \Pi_{t+1} + \hat{\kappa} \left[ Y_t^{\omega+\sigma} - 1 \right] \]

- Government Budget and Fiscal Policy
  \[T_t + \frac{b_t}{R_t} = \frac{b_{t-1}}{\Pi_t} \quad b_t = \bar{b} \]

- Market Clearing
  \[b_t = \int g_t^a (a, z) dD_t (z, a) \]

- Monetary Policy
  \[R_t = \max \left\{ 1, \bar{R} \left( \frac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \left( \frac{Y_t}{\bar{Y}} \right)^{\phi_Y} \right\} \]
HANK Model (4) - Shock Structure

- Economy at steady state ($t = -1$)
- Time preference shock $\beta$ materializes: $\beta_0 = \beta_L$
- Every period: probability $1 - \mu$ to revert (and a contingency realizes)
- Compare to deterministic shock: $\beta^{DET}_t = \mathbb{E}_0 \beta_t$
- No restrictions on values, only on $\mu$ (must be the same).
HANK Model (4) - Shock Structure

\[
\beta_t = \begin{cases} 
\beta & \text{w.p. } = 1, \text{ if } \beta_{t-1} = \beta \\
\beta & \text{w.p. } = 1 - \mu, \text{ if } \beta_{t-1} = \beta_L \\
\beta_L & \text{w.p. } = \mu, \text{ if } \beta_{t-1} = \beta_L
\end{cases}
\]

\[
\beta_t^{PF} = \mu^t \beta_L + (1 - \mu^t) \beta
\]

(4)
## Table 1: Calibration

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
<th>Note</th>
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<td>$\sigma$</td>
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<td>Smets and Wouters (2007)</td>
<td>EIS</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.9805</td>
<td>Calibrated Discount Factor</td>
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<tr>
<td>$\kappa$</td>
<td>0.01</td>
<td>Eggertsson et al. (2021)</td>
<td>NKPC</td>
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<tr>
<td>$\Pi$</td>
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<td>Standard Inflation target</td>
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<tr>
<td>$\phi_\pi$</td>
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<td>$\phi_y$</td>
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<td>$z$</td>
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<td>Guerrieri and Lorenzoni (2017)</td>
<td>Idiosyncratic Shocks</td>
</tr>
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<td>Eggertsson et al. (2021)</td>
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<td>Calibrated Shock</td>
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<td>$T$</td>
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<td>$\tau_{max}$</td>
<td>100</td>
<td>-</td>
<td>Largest Contingency</td>
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HANK Model (6) - Challenges

Wealth distribution is a state variable that affects the evolution of the economy. It is an infinite-dimensional object, which introduces the curse of dimensionality. Known solutions include summarizing the distribution in a few moments (Krusell and Smith, 1998) or aggregating the economy to behave linearly (Reiter, 2009; certainty equivalence). The zero lower bound (ZLB) introduces aggregate nonlinearity, which can be computationally burdensome. Some solutions to this issue include Schaab (2020), Fernández-Villaverde et al. (2021), and Kase et al. (2022).

We instead solve the model in the space of sequences of shocks, which implies a finite number of paths the economy can follow.
HANK Model (6) - Challenges

- Wealth distribution is **state variable**
  - affects the evolution of the economy
  - infinite-dimensional object \(\implies\) curse of dimensionality
HANK Model (6) - Challenges

- Wealth distribution is state variable
  - affects the evolution of the economy
  - infinite-dimensional object $\Rightarrow$ curse of dimensionality

- Known solutions:
  - summarize distribution in few moments \cite{KrusellSmith1998}
  - aggregate economy behaves linearly \cite{Reiter2009} (certainty equivalence)
Wealth distribution is state variable
- affects the evolution of the economy
- infinite-dimensional object $\implies$ curse of dimensionality

Known solutions:
- summarize distribution in few moments (Krusell and Smith, 1998)
- aggregate economy behaves linearly (Reiter, 2009) (certainty equivalence)

ZLB introduces aggregate nonlinearity
- potentially computationally burdensome
- some solutions: Schaab (2020), Fernández-Villaverde et al. (2021), Kase et al. (2022)
HANK Model (6) - Challenges

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- ZLB introduces aggregate nonlinearity
  - potentially computationally burdensome
  - some solutions: Schaab (2020), Fernández-Villaverde et al. (2021), Kase et al. (2022)

- We instead solve the model in the **space of sequences**
  - shock structure $\implies$ finite # of paths the economy can follow
Solution Approach (1) - Economy Overview

Notes: x-axis is time $t$, y-axis is contingency $\tau$. 

Solution: large system of equations ($\tau_{\max} \times (T+1) \times nX$). We split diagonal/contingencies $16/33$. 

$\tau_{\max}$
Notes: x-axis is time $t$, y-axis is contingency $\tau$.

Solution: large system of equations $\left(\tau_{\text{max}} \times (T + 1) \times nX\right)$.
Solution Approach (1) - Economy Overview

Notes: x-axis is time $t$, y-axis is contingency $\tau$.
Solution: large system of equations $(\tau_{\text{max}} \times (T + 1) \times nX)$. We split diagonal/contingencies
Notation and terminology:

A contingency refers to the time $\tau$ when the shock switched back as well to the aggregate equilibrium dynamics following such event. $x_{\tau t}$ is the value of economic object $x$ at time $t$ under contingency $\tau$, $x_{t}$ is the value of economic object $x$ at time $t$, if the shock has not yet reverted.

Value functions become:

$$V_{t}(z_{it}, a_{it}) = \max c_{it}, a_{it} \geq a_{c1-\sigma_{it1}+\beta_{t}E_{t}\mu V_{t+1}(z_{it+1}, a_{t})} + (1 - \mu) V_{t+1}(z_{it+1}, a_{t})$$

$V_{\tau t}(z_{it}, a_{it}) = \max c_{it}, a_{it} \geq a_{c1-\sigma_{it1}+\beta_{\tau t}E_{t}V_{\tau t+1}(z_{it+1}, a_{t})}$

(linearized) NK Phillips curve becomes

$$\hat{\Pi}_{t} = \beta_{h} \mu \hat{\Pi}_{t+1} + (1 - \mu) \hat{\Pi}_{t+1} t_{i} + \kappa \hat{Y}_{t}$$

$$\hat{\Pi}_{\tau t} = \beta_{\hat{\Pi}_{\tau t+1}} + \kappa \hat{Y}_{\tau t}$$
Notation and terminology:

A contingency refers to the time $\tau$ when the shock switched back as well to the aggregate equilibrium dynamics following such event.

$x_{\tau t}$ is the value of economic object $x$ at time $t$ under contingency $\tau$.

$x_{t}$ is the value of economic object $x$ at time $t$, if the shock has not yet reverted.

Value functions become $V_{t}(z_{it}, a_{it-1}) = \max c_{it}, a_{it} \geq a_{c1-\sigma_{it}-\sigma} + \beta t E_{t} \mu V_{t+1}(z_{it+1}, a_{t}) + (1-\mu) V_{t+1}(z_{it+1}, a_{t})$.

$(linearized)$ NK Phillips curve becomes $\hat{\Pi}_{t} = \beta \hat{\Pi}_{t+1} + (1-\mu)\hat{\Pi}_{t+1} + \kappa \hat{Y}_{t}$.

$\hat{\Pi}_{\tau t} = \beta \hat{\Pi}_{\tau t+1} + \kappa \hat{Y}_{\tau t}$.
Notation and terminology:
- A *contingency* refers to the time $\tau$ when the shock switched back.
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- as well to the aggregate equilibrium dynamics following such event.
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- $x^\tau_t$ is the value of economic object $x$ at time $t$ under contingency $\tau$,
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- $x_t$ is the value of economic object $x$ at time $t$, if the shock has not yet reverted.

Value functions become

$$V_t(z_t, a_{t-1}) = \max c_{it}, \quad a_{it} \geq a_{c1-\sigma t1+\beta tE_t}\mu V_{t+1}(z_{t+1}, a_{t+1}) + (1-\mu) V_{t+1}(z_{t+1}, a_{t+1})$$

(linearized) NK Phillips curve becomes

$$\hat{\Pi}_t = \beta \hat{\Pi}_{t+1} + (1-\mu)\hat{\Pi}_{t+1}\mu + \kappa \hat{Y}_t$$
Notation and terminology:
- A contingency refers to the time $\tau$ when the shock switched back as well to the aggregate equilibrium dynamics following such event.
- $x^{\tau}_t$ is the value of economic object $x$ at time $t$ under contingency $\tau$.
- $x_t$ is the value of economic object $x$ at time $t$, if the shock has not yet reverted.

Value functions become

$$V_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t E_t [\mu V_{t+1}(z_{it+1}, a_t) + (1 - \mu) V_{t+1}^{t+1}(z_{it+1}, a_t)]$$

$$V^{\tau}_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta^{\tau}_t E_t V^{\tau}_{t+1}(z_{it+1}, a_t)$$
Notation and terminology:
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Value functions become

$$V_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t \mathbb{E}_t \left[ \mu V_{t+1}(z_{it+1}, a_t) + (1-\mu) V_{t+1}^{t+1}(z_{it+1}, a_t) \right]$$

$$V^\tau_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it} \geq a} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta^\tau_t \mathbb{E}_t V^\tau_{t+1}(z_{it+1}, a_t)$$

(Linearized) NK Phillips curve becomes

$$\hat{\pi}_t = \beta \left[ \mu \hat{\pi}_{t+1} + (1-\mu) \hat{\pi}^{t+1}_{t+1} \right] + \kappa \hat{Y}_t$$

$$\hat{\pi}^\tau_t = \beta \hat{\pi}^\tau_{t+1} + \kappa \hat{Y}_t$$
Solution Approach (3) - Equilibrium

Numerically, an equilibrium is represented by systems of equations

\[ 0 = \mathbf{F}_{\text{PF}}(\mathbf{X}_\tau, \mathbf{Z}_\tau; \mathbf{X}_{\tau-1}, \mathbf{D}_\tau) \] (5)

\[ 0 = \mathbf{F}_{\text{TS}}(\mathbf{X}_{\text{TS}}, \mathbf{Z}_{\text{TS}}; \mathbf{V}_{\text{PF}1}, \mathbf{X}_{\text{PF}1}) \] (6)

where \( \mathbf{X}_\tau(\mathbf{Z}_\tau) \) contains all the aggregate variables (shocks) in contingency \( \tau \)

\( \mathbf{X}_{\text{TS}}(\mathbf{Z}_{\text{TS}}) \) contains all the aggregate variables (shocks) on the "uncertain" diagonal

\( \mathbf{V}_{\text{PF}1}, \mathbf{X}_{\text{PF}1} \) contain all the "forward looking" information relevant for the diagonal
Numerically, an equilibrium is represented by systems of equations

\[ 0 = F^{PF}(X^\tau, Z^\tau; X_{\tau-1}, D^\tau) \]  
\[ 0 = F^{TS}(X^{TS}, Z^{TS}; V^{PF}_1, X^{PF}_1) \]
Numerically, an equilibrium is represented by systems of equations

\[ 0 = F^{PF}(X^T, Z^T; X_{\tau-1}, D^T_{\tau}) \]  \hspace{1cm} (5)  

\[ 0 = F^{TS}(X^{TS}, Z^{TS}, V^{PF}, X^{PF}_1) \]  \hspace{1cm} (6)  

where \( X^T \) (\( Z^T \)) contains all the aggregate variables (shocks) in contingency \( \tau \)

\( X^{TS} \) (\( Z^{TS} \)) contains all the aggr. variables (shocks) on the "uncertain" diagonal

\( V^{PF}_1, X^{PF}_1 \) contain all the "forward looking" information relevant for the diagonal
1. Guess path of state variable(s) on uncertain path
1. Guess path of state variable(s) on uncertain path
2. Solve the \( \tau^\text{max} \) PF paths

\[
0 = F^{PF}(X^T, Z^T; X_{T-1}, D_T)
\]

\[
dX^T = F_X^{-1} \left( F_D^{PF} dD_T + F_{X_{T-1}}^{PF} dX_{T-1} \right)
\]

\[
F_D^{PF} dD_T \approx F^{PF} \left( X_{ss}^{PF} | D_T, X_{ss} \right) - F^{PF} \left( X_{ss}^{PF} | D_{ss} X_{ss} \right)
\]

- use Sequence Space Jacobian + OccBin
- collect value functions and forw. looking vars at first period of PFs
1. Guess path of state variable(s) on uncertain path
2. Solve the $\tau^{\text{max}}$ PF paths
3. Given the value functions and forward looking vars, solve the uncertain path*

$$0 = \mathbf{F}^{TS}(\mathbf{X}^{TS}, \mathbf{Z}^{TS}, \mathbf{V}_1^{PF}, \mathbf{X}_1^{PF})$$

$$d\mathbf{X}^{TS} = (\mathbf{F}_X^{TS})^{-1} \left( \mathbf{F}_Z^{TS} d\mathbf{Z} + \mathbf{F}_X^{TS} d\mathbf{X}_1^{PF} + \mathbf{F}_V^{TS} d\mathbf{V}_1^{PF} \right)$$

$$\mathbf{F}_V^{TS} d\mathbf{V}_1^{PF} \approx \mathbf{F}^{TS} \left( \mathbf{X}_{ss}, \mathbf{Z}_{ss} | \mathbf{X}_{ss}, \mathbf{V}_1^{PF} \right) - \mathbf{F}^{TS} \left( \mathbf{X}^{TS}_{ss}, \mathbf{Z}_{ss} | \mathbf{X}^{TS}_{ss}, \mathbf{V}_{ss} \right)$$

- use Sequence Space Jacobian (modified for AU) + OccBin
- recover new set of state variables

4. Iterate until convergence
5. (optional) Use quasi-Newton method for higher order
1. Guess path of state variable(s) on uncertain path
2. Solve the $\tau_{\text{max}}$ PF paths
3. Given the value functions and forw. looking vars, solve the uncertain path*
4. Iterate until convergence
1. Guess path of state variable(s) on uncertain path
2. Solve the $\tau^{\text{max}}$ PF paths
3. Given the value functions and forw. looking vars, solve the uncertain path*
4. Iterate until convergence
5. (optional) Use quasi-Newton method for higher order
Results (0) - Fixing ideas on what we are after

Measuring amplification due to uncertainty:

\[ \text{IRF-AU: } E_0 Y_t - Y, \text{ weighted average of all contingencies.} \]

\[ \text{IRF-PF: } Y_{PF_t} - \bar{Y} \]

Amplification if \( E_0 Y_t - Y_{PF_t} > 0 \)

Quantification with PDV:

\[ P_{\infty} t=0 \beta_t E_0 Y_t - Y \text{ vs } P_{\infty} t=0 \beta_t Y_{PF_t} - Y \]
Results (0) - Fixing ideas on what we are after

Measuring amplification due to uncertainty:
- **Aggregate Uncertainty** vs Deterministic Counterfactual
Measuring amplification due to uncertainty:

- **Aggregate Uncertainty** vs Deterministic Counterfactual
- Shocks with the same expected values from $t = 0$ perspective
Measuring amplification due to uncertainty:

- **Aggregate Uncertainty** vs Deterministic Counterfactual
- Shocks with the same expected values from $t = 0$ perspective
- IRF-AU: $E_0 Y_t - \bar{Y}$, weighted average of all contingencies. IRF-PF: $Y_t^{PF} - \bar{Y}$
Measuring amplification due to uncertainty:

- **Aggregate Uncertainty** vs Deterministic Counterfactual
- Shocks with the same expected values from $t = 0$ perspective
- IRF-AU: $E_0 Y_t - \bar{Y}$, weighted average of all contingencies. IRF-PF: $Y_{t}^{PF} - \bar{Y}$
- Amplification if $|E_0 Y_t - Y_{t}^{PF}| >> 0$
Results (0) - Fixing ideas on what we are after

Measuring amplification due to uncertainty:

- **Aggregate Uncertainty** vs Deterministic Counterfactual
- Shocks with the same expected values from \( t = 0 \) perspective
- IRF-AU: \( E_0 Y_t - \bar{Y} \), weighted average of all contingencies. IRF-PF: \( Y_t^{PF} - \bar{Y} \)
- Amplification if \( |E_0 Y_t - Y_t^{PF}| >> 0 \)
- Quantification with PDV: \( \sum_{t=0}^{\infty} \beta^t (E_0 Y_t - \bar{Y}) \) vs \( \sum_{t=0}^{\infty} \beta^t (Y_t^{PF} - \bar{Y}) \)
Figure 3: IRF - HANK - No ZLB

- AU vs PF in HANK without the ZLB
  - linear behavior to aggregate shocks
  - $E_0 Y_t \approx Y_t^{PF}$
  - certainty equivalence
Results (2) - Uncertainty and Amplification

Figure 4: IRF - HANK - with ZLB

- AU vs PF in HANK at the ZLB
- certainty equivalence broken
  - $E_0 Y_t < Y_t^{PF}$
  - amplification in PDV: 2x
- Uncertainty: ZLB binds for longer
  - On average, 10 quarters
  - vs. 4 quarters in PF
Results (3) - Uncertainty and Amplification in RANK

Figure 5: IRF - RANK - without ZLB

- AU vs PF in RANK at the ZLB
- Calibrate shocks such that:
- RA economy exhibits same response as HANK w/o ZLB (Werning, 2015)

\[
X_t(RANK) = X_t(HANK)
\]

\[
X_t^\tau(RANK) = X_t^\tau(HANK)
\]
Results (3) - Uncertainty and Amplification in RANK

**Figure 5: IRF - RANK - with ZLB**

- AU vs PF in RANK at the ZLB
- Calibrate shocks such that:
- RA economy exhibits same response as HANK w/o ZLB (Werning, 2015)

\[ X_t(RANK) = X_t(HANK) \]

- Introduce ZLB

\[ X_t^T(RANK) = X_t^T(HANK) \]
Figure 5: IRF - RANK - with ZLB

AU vs PF in RANK at the ZLB
Calibrate shocks such that:
RA economy exhibits same response as HANK w/o ZLB (Werning, 2015)

\[ X_t(RANK) = X_t(HANK) \]

\[ X_{t+\tau}(RANK) = X_{t+\tau}(HANK) \]

Introduce ZLB
Amplification in PDV: 1.6x
## Results (4) - Uncertainty and Amplification - Summary (Y)

<table>
<thead>
<tr>
<th></th>
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<th>ZLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repr. Agent</td>
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<td>A</td>
</tr>
<tr>
<td>Het. Agents</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
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</tbody>
</table>
### Results (4) - Uncertainty and Amplification - Summary (Y)

<table>
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<tr>
<th></th>
<th>No ZLB</th>
<th>ZLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repr. Agent</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Het. Agents</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>
Table 2: Running Times - Seconds

<table>
<thead>
<tr>
<th>Specification</th>
<th>Benchmark</th>
<th>MNP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time</td>
<td>Max. Err.</td>
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<tr>
<td>Steady State</td>
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<td>-</td>
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<tr>
<td>All Jacobians</td>
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<td>-</td>
</tr>
<tr>
<td>Algorithm 1 - First-Order</td>
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<td>0.5%</td>
</tr>
<tr>
<td>Algorithm 1 - Exact only on TS</td>
<td>26</td>
<td>0.008%</td>
</tr>
<tr>
<td>Algorithm 1 - Exact Equilibrium</td>
<td>116</td>
<td>0.000000006%</td>
</tr>
</tbody>
</table>

Matlab, ASUS laptop, 1.80Ghz processor, 16GB RAM, and 8 cores

MNP: Mendicino et al. (2021), richer income risk
Figure 6: Effects on Impact as a function of shock size
Define a consumption function:

\[ C^{PF} = C^{PF}(X^{PF}, Z^{PF}) \]  \hspace{1cm} (7)

\[ C^{TS} = C^{AU}(X^{TS}, Z^{TS}, \{X^\tau\}_{\tau=1}^{\tau_{\text{max}}}, \{Z^\tau\}_{\tau=1}^{\tau_{\text{max}}}) \]  \hspace{1cm} (8)

\[ C^{\text{Interm}} = C^{PF}(\text{IRF}(X^{TS}, Z^{TS}, \{X^\tau\}_{\tau=1}^{\tau_{\text{max}}}, \{Z^\tau\}_{\tau=1}^{\tau_{\text{max}}})) \]  \hspace{1cm} (9)

- Which aggregate is driving the amplification? \( Y, \Pi, R, t, \beta \)
- Is uncertainty important per se? Feed IRF of AU in a deterministic world (captures indirect effect via "average" aggregates).
What price is driving the amplification?
Feed **average** prices of AU, in a deterministic world
Aggregate income is largest driver.
Expected aggregate income does most of it.
One policy application: forward guidance

Central bank announces to keep rates at ZLB for $q$ quarters on top of what prescribed in main exercise

FG powerful when there is uncertainty (6 quarters can revert recession)
1. Households
   - illiquid physical capital
   - Calvo fairy for portfolio re-balancing
   ⇒ heterogeneous in income, wealth, and portfolio composition

2. Other Blocks
   - intermediate-goods producer - Cobb-Douglas production
   - Labor Union with adjustment costs (Wage Phillips Curve)
   - Capital production subject to adjustment cost
Application (2.1) - Cyclical Income Risk

**Deterministic Shock**

- Graph showing the impact of deterministic shocks on a variable $Y$.
- Graph showing the impact of deterministic shocks on a variable $\Pi$.
- Graph showing the impact of deterministic shocks on a variable $\Omega$.

**Stochastic Shock**

- Graph showing the impact of stochastic shocks on a variable $Y$.
- Graph showing the impact of stochastic shocks on a variable $\Pi$.
- Graph showing the impact of stochastic shocks on a variable $R$. 

These graphs illustrate the effects of both deterministc and stochastic shocks on the specified variables over time.
Earnings risk as in Mendicino et al. (2021):
We study the interaction b/w aggregate uncertainty and household heterogeneity:
- new methodology for HANK models with aggregate uncertainty and non-linearities
- simulations suggest that interaction is strong at the ZLB, even with acyclical risk
- quantify the interaction in a simple way, during GR (55% amplification)

Applications:
- Forward Guidance
- Two Asset HANK

Methodology can be used for many applications involving HA, AU, aggregate non-linearities


## Results (5) - Uncertainty and Amplification - Summary (Π)

<table>
<thead>
<tr>
<th></th>
<th>No ZLB</th>
<th>ZLB (y/Pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repr. Agent</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Het. Agents</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

### Discounted IRF - y
![Discounted IRF - y](image)

### Discounted IRF - π
![Discounted IRF - π](image)
Similar to Sequence-Space Jacobian. Heterogeneous block is represented by:

\[ \mathbf{v}_t = \mathbf{v}^{TS} (\mathbf{v}_{t+1}, \mathbf{v}^{t+1}_{t+1}, X_t) \quad (10) \]

\[ D_{t+1}^{t+1} = D_{t+1} = \Lambda^{TS} (\mathbf{v}_{t+1}, \mathbf{v}^{t+1}_{t+1}, X_t)' D_t \quad (11) \]

\[ \mathcal{Y}_t = y^{TS} (\mathbf{v}_{t+1}, \mathbf{v}^{t+1}_{t+1}, X_t)' D_t \quad (12) \]

\[ \mathbf{v}_t^{\tau} = \mathbf{v} (\mathbf{v}^{\tau}_{t+1}, X_t^{\tau}) \quad (13) \]

\[ D_{t+1}^{\tau} = \Lambda (\mathbf{v}^{\tau}_{t+1}, X_t^{\tau})' D_t^{\tau} \quad (14) \]

\[ \mathcal{Y}_t^{\tau} = y (\mathbf{v}^{\tau}_{t+1}, X_t^{\tau})' D_t^{\tau} \quad (15) \]
A competitive equilibrium is

- **Aggregate variables**
  - a sequence \( \{ Y_t, \Pi_t, R_t, b_t, t_t \}^{\tau\max-1}_{t=0} = \{ X_t \}^{\tau\max-1}_{t=0} = X^{TS} \)
  - \( \tau\max \) sequences \( \{ Y^\tau_t, \Pi^\tau_t, R^\tau_t, b^\tau_t, t^\tau_t \}^{\tau+T}_{t=\tau} = \{ X^\tau_t \}^{\tau+T}_{t=\tau} = \{ X^\tau \}^{\tau\max}_{\tau=1} \)

- **Individual agents objects** (wealth distribution, value function)
  - a sequence \( \{ D_{t+1}, V_t \}^{\tau\max-1}_{t=0} \)
  - \( \tau\max \) sequences \( \{ D^\tau_t, V^\tau_t \}^{\tau+T}_{t=\tau} \)

- s.t. given exogenous processes \( \{ \beta_t \}^{\tau\max}_{t=0} = Z^{TS} \) and \( \{ \beta^\tau_t \}^{\tau\max}_{t=\tau} = \{ Z^\tau \}^{\tau\max}_{\tau=1} \)

aggregate equations hold, agents solve their maximization problem, and \( D_t = D^t_t \)
Occasionally binding constraints sub-algorithm

1. Consider 
\[ dX^{TS} = (F_X^{TS})^{-1} (F_Z^{TS} dZ + F_X^{PS} dX_1^{PF} + F_V^{TS} dV_1^{PF}) \]

2. Guess periods in which the constraint binds, place them in an binary vector \( I_{ZLB} \)

3. Adjust the main matrix so that

\[
\begin{align*}
dX^{TS} = & [(1 - I_{ZLB}) \times F_X^{TS} + I_{ZLB} \times \tilde{F}_X^{TS}]^{-1} \\
& [(1 - I_{ZLB}) \times (F_Z^{TS} dZ + F_X^{PS} dX_1^{PF} + F_V^{TS} dV_1^{PF}) + I_{ZLB} \times (R - \bar{R})]
\end{align*}
\]

where \( \tilde{F}_X^{TS} \) substitutes the Taylor rule with \( R_t = R \)

4. Generate shadow rates by simply multiplying \( dX^{TS} \) and \( F_X^{TS} \), is the guess is correct, stop. Otherwise go to 2 and update \( I_{ZLB} \).
Solution Approach (7) - Others

We exploit:
- fake news algorithm
- expectations vector

We do not exploit:
- DAG-part
- New Jacobian $\frac{\partial A}{\partial Y_s}$
We exploit:
- fake news algorithm
- expectations vector
Solution Approach (7) - Others

- We exploit:
  - fake news algorithm
  - expectations vector
- We do not exploit:
  - DAG-part
We exploit:
- fake news algorithm
- expectations vector

We do not exploit:
- DAG-part

New Jacobian $\frac{\partial A_t}{\partial Y_s}$:
- different $\mu$
Figure 8: Effects on Discounted IRF as a function of shock size