Optimal Monetary Policy with $r^* < 0$

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Motivation

- Widespread consensus on a substantial decline in the *average* natural rate of interest \( r^* \equiv \mathbb{E}\{r_t^n\} \). Recent estimates: \( r^* < 0 \).

- Driving forces: productivity, demography, inequality, relative price of investment goods, uncertainty. Likely to be persistent.
Motivation

- Widespread consensus on a substantial decline in the *average* natural rate of interest $r^* \equiv \mathbb{E}\{r^n_t\}$. Recent estimates: $r^* < 0$.
- Driving forces: productivity, demography, inequality, relative price of investment goods, uncertainty. Likely to be persistent.
- With a ZLB constraint: increased incidence of binding ZLB episodes, greater macro instability.
- Existing literature: optimal monetary policy under a ZLB constraint and $r^* > 0$. Normal times: $r^n_t > 0 \implies i_t > 0$, successful stabilization of inflation and the output gap. Occasional episodes with $r^n_t < 0 \implies i_t = 0$, macro instability. Key role for forward guidance.
- *This paper*: optimal monetary policy under a ZLB constraint with $r^* < 0$. "New normal": $r^n_t < 0$. Occasional episodes with $r^n_t > 0$. Summers’ "secular stagnation" speech.

*What does the optimal monetary policy look like in that environment? What are its implications for macro outcomes?*
The optimal monetary policy problem

Equilibrium under the optimal policy: The Deterministic Case

Equilibrium under the optimal policy: The Stochastic Case

Implementation

*byproduct*: sufficient condition for determinacy in models with endogenous regime switches

Concluding remarks
Evidence on $r^*$ and its driving forces: Holston et al. (2017), Del Negro et al. (2019), Eggertsson et al. (2019), Rachel and Summers (2019), Brand and Mazelis (2019), Davis et al. (2023)

Optimal monetary policy under the ZLB: Eggertsson and Woodford (2003), Jung, Teranishi and Watanabe (2005), Adam and Billi (2006), Nakov (2008),

Optimal choice of an inflation target, conditional on a given interest rate rule: Coibion et al. (2012), Bernanke et al. (2019), and Andrade et al. (2020, 2021).

The Optimal Monetary Policy Problem

\[
\min \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \pi_t^2 + \vartheta y_t^2 \right)
\]

subject to

\[\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa y_t \tag{1}\]

\[y_t = \mathbb{E}_t \{ y_{t+1} \} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \{ \pi_{t+1} \} - r_t^n) \tag{2}\]

\[i_t \geq 0 \tag{3}\]

\[r_t^n = r^* + z_t \tag{4}\]

all for \(t = 0, 1, 2, \ldots\) where \(z_t \sim AR(1)\) and

\[r^* < 0\]
A Brief Detour: A Microfounded NK Model with $r^* < 0$

- Based on the NK-OLG model in Galí (2021)
- Consumers: constant "life" and "activity" survival rates ($\gamma, \nu$). Objective function for consumer born in period $s$:
  \[
  E_s \sum_{t=s}^{\infty} (\beta \gamma)^{t-s} Z_t \log C_t |_s
  \]
  where $\beta \equiv \exp\{-\rho\}$ and $z_t \equiv \log Z_t \sim AR(1)$
- Firms: attached to founder, hence survival rate $\gamma \nu$. Calvo pricing.
- Steady state:
  \[
  r^* = \rho + \log \nu
  \]
- Condition for $r^* < 0$
  \[
  \nu < \beta
  \]
- Linearized equilibrium conditions:
  \[
  \pi_t = \beta \gamma E_t\{\pi_{t+1}\} + \kappa y_t
  \]
  \[
  y_t = E_t\{y_{t+1}\} - (\delta_t - E_t\{\pi_{t+1}\} - r^n_t)
  \]
  with $r^n_t = r^* + (1 - \rho_z)z_t$
The Optimal Monetary Policy Problem

\[
\min \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \pi_t^2 + \vartheta y_t^2 \right)
\]

subject to

\[
\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa y_t
\]  

(5)

\[
y_t = \mathbb{E}_t \{ y_{t+1} \} - \frac{1}{\sigma} \left( i_t - \mathbb{E}_t \{ \pi_{t+1} \} - r_t^n \right)
\]

(6)

\[
i_t \geq 0
\]

(7)

\[
r_t^n = r^* + z_t
\]

(8)

all for \( t = 0, 1, 2, \ldots \) where \( z_t \sim AR(1) \) and

\[
r^* < 0
\]
The Optimal Monetary Policy Problem

- Optimality conditions:

\[ \pi_t = \zeta_{1,t} - \zeta_{1,t-1} + \beta^{-1} \zeta_{2,t-1} \]  
(9)

\[ \varrho y_t = -\kappa \zeta_{1,t} - \sigma \zeta_{2,t} + \sigma \beta^{-1} \zeta_{2,t-1} \]  
(10)

\[ \zeta_{2,t} \geq 0 \]  
(11)

\[ \zeta_{2,t} \left[ r^n_t + \mathbb{E}_t\{\pi_{t+1}\} + \sigma (\mathbb{E}_t\{y_{t+1}\} - y_t) \right] = 0 \]  
(12)

with initial conditions \( \zeta_{1,-1} = \zeta_{2,-1} = 0 \).
Optimal Policy: The Deterministic Case

- Initial steady state:
  \[ i_t = r^n_t = r^* > 0 \]
  \[ y_t = \pi_t = 0 \]

- MIT shock at time 0:
  \[ r^n_t = r^* < 0 \]
  for \( t = 0, 1, 2, \ldots \)

- New steady state

  \[ \pi = \beta^{-1} \zeta_2 \geq 0 \]
  \[ \varphi y = -\kappa \zeta_1 + \sigma (\beta^{-1} - 1) \zeta_2 \]
  \[ \zeta_2 \geq 0 \quad ; \quad r^* + \pi \geq 0 \]
  \[ \zeta_2 (r^* + \pi) = 0 \]

  \[ \Rightarrow \pi \geq -r^* > 0 \]
  \[ \Rightarrow \zeta_2 > 0 \quad \Rightarrow \quad i = 0 \quad \Rightarrow \quad \pi = -r^* \]
Optimal Policy: The Deterministic Case

- **Transitional dynamics**

\[
\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \kappa \hat{y}_t \\
\hat{\pi}_t = \hat{\xi}_{1,t} - \hat{\xi}_{1,t-1} + \beta^{-1}\hat{\xi}_{2,t-1} \\
\hat{\theta}\hat{y}_t = -\kappa \hat{\xi}_{1,t} - \sigma \hat{\xi}_{2,t} + \sigma \beta^{-1}\hat{\xi}_{2,t-1} \\
\hat{\xi}_{2,t} + \xi_2 \geq 0 \\
\hat{\pi}_{t+1} + \sigma(\hat{y}_{t+1} - \hat{y}_t) \geq 0 \\
(\hat{\xi}_{2,t} + \xi_2) [\hat{\pi}_{t+1} + \sigma(\hat{y}_{t+1} - \hat{y}_t)] = 0 \\
\]

for \( t = 0, 1, 2, \ldots \) with initial conditions \( \hat{\xi}_{1,-1} = -\xi_1 \) and \( \hat{\xi}_{2,-1} = -\xi_2 \) and such that \( \lim_{t \to \infty} \hat{x}_t = 0 \) for \( \hat{x}_t \in \{ \hat{\pi}_t, \hat{y}_t, \hat{\xi}_{1,t}, \hat{\xi}_{2,t} \} \)

- **Simulations for a calibrated economy**

\[
\sigma = 1, \ \beta = 0.99, \ \kappa = 0.1717, \ \theta = 0.0191 \ (\text{Galí (2015))} \\
r^* = -0.0025 
\]
Figure 1: Transitional dynamics under the optimal monetary policy. Inflation and interest rates in annualized terms.
Optimal Policy: The Stochastic Case

- Stochastic equilibrium

\[ \hat{\pi}_t = \beta E_t\{\hat{\pi}_{t+1}\} + \kappa \hat{y}_t \]
\[ \hat{\pi}_t = \hat{\xi}_{1,t} - \hat{\xi}_{1,t-1} + \beta^{-1} \hat{\xi}_{2,t-1} \]
\[ \sigma \hat{y}_t = -\kappa \hat{\xi}_{1,t} - \hat{\xi}_{2,t} + \beta^{-1} \hat{\xi}_{2,t-1} \]
\[ \hat{\xi}_{2,t} + \xi_2 \geq 0 \]
\[ \sigma (E_t\{\hat{y}_{t+1}\} - \hat{y}_t) + E_t\{\hat{\pi}_{t+1}\} + z_t \geq 0 \]
\[ [\hat{\xi}_{2,t} + \xi_2][\sigma (E_t\{\hat{y}_{t+1}\} - \hat{y}_t) + E_t\{\hat{\pi}_{t+1}\} + z_t] = 0 \]

for \( t = 0, 1, 2, \ldots \) with initial conditions given by \( \hat{\xi}_{1,-1} = 0 \) and \( \hat{\xi}_{2,-1} = 0 \).

- Simulations: \((\rho_z, \sigma_z) = (0.5, 0.0025)\)
Optimal Policy: Fluctuations around the Steady State

Baseline Calibration

Real Rate

Policy Rate

Inflation

Output Gap
Optimal Policy: Fluctuations around the Steady State

*High Volatility Calibration*

**Real Rate**

- Natural
- Actual

**Policy Rate**

- Constant

**Inflation**

**Output Gap**
$r^*$ and ZLB Incidence under the Optimal Policy
Conditions for a Permanent ZLB
$r^*$ and Average Inflation under the Optimal Policy
$r^* \text{ and Inflation Volatility under the Optimal Policy}$

![Graph showing $r^*$ and Inflation Volatility under the Optimal Policy](image)
Stochastic equilibrium

\[ \hat{\pi}_t = \beta \mathbb{E}_t \{ \hat{\pi}_{t+1} \} + \kappa \hat{y}_t \]
\[ \hat{\pi}_t = \hat{\xi}_{1,t} - \hat{\xi}_{1,t-1} + \beta^{-1} \hat{\xi}_{2,t-1} \]
\[ \vartheta \hat{y}_t = -\kappa \hat{\xi}_{1,t} - \hat{\xi}_{2,t} + \beta^{-1} \hat{\xi}_{2,t-1} \]
\[ \hat{\xi}_{2,t} + \xi_2 \geq 0 \]
\[ \sigma(\mathbb{E}_t \{ \hat{y}_{t+1} \} - \hat{y}_t) + \mathbb{E}_t \{ \hat{\pi}_{t+1} \} + z_t \geq 0 \]
\[ [\hat{\xi}_{2,t} + \xi_2] [\sigma(\mathbb{E}_t \{ \hat{y}_{t+1} \} - \hat{y}_t) + \mathbb{E}_t \{ \hat{\pi}_{t+1} \} + z_t] = 0 \]
for \( t = 0, 1, 2, \ldots \) with initial conditions given by \( \hat{\xi}_{1,-1} = 0 \) and \( \hat{\xi}_{2,-1} = 0 \).

Simulations: \( (\rho_z, \sigma_z) = (0.5, 0.0025) \)

Precautionary inflation

\[ \bar{\pi}^p(r^*) = \bar{\pi}(r^*) - \max\{0, -r^*\} \]
$r^*$ and Precautionary Inflation
Equilibrium outcomes under optimal policy: \((i^*_t, y^*_t, \pi^*_t)\)

Non-Policy block, in deviations from optimal path:

\[
\tilde{\pi}_t = \beta \mathbb{E}_t \{\tilde{\pi}_{t+1}\} + \kappa \tilde{y}_t
\]

\[
\tilde{y}_t = \mathbb{E}_t \{\tilde{y}_{t+1}\} - \frac{1}{\sigma} (\tilde{i}_t - \mathbb{E}_t \{\tilde{\pi}_{t+1}\})
\]

\[
\tilde{i}_t \geq -i^*_t
\]

where \(\tilde{x}_t \equiv x_t - x^*_t\) for \(x \in \{\pi, y, i\}\)

Wanted: policy rule that guarantees \((\tilde{\pi}_t, \tilde{y}_t, \tilde{i}_t)\) is the only equilibrium.

Candidate rule:

\[
i_t = i^*_t \iff \tilde{i}_t = 0
\]

for all \(t\). Combined with non-policy block \(\Rightarrow\) multiplicity of solutions in addition to \((i^*_t, y^*_t, \pi^*_t)\)
Proposed rule:

\[ i_t = i_t^* + \phi^{(q)}_{\pi} \tilde{\pi}_t + \phi^{(q)}_y \tilde{y}_t \]

where \( q \in \{1, 2, 3, 4\} \), satisfying:

\[ q = 1 : \quad \tilde{\pi}_t \geq 0, \quad \tilde{y}_t \geq 0 \quad \Rightarrow \quad \phi_{\pi}^{(1)} \geq 0, \quad \phi_y^{(1)} \geq 0 \]
\[ q = 2 : \quad \tilde{\pi}_t < 0, \quad \tilde{y}_t < 0 \quad \Rightarrow \quad \phi_{\pi}^{(2)} \leq 0, \quad \phi_y^{(2)} \leq 0 \]
\[ q = 3 : \quad \tilde{\pi}_t \geq 0, \quad \tilde{y}_t < 0 \quad \Rightarrow \quad \phi_{\pi}^{(3)} \geq 0, \quad \phi_y^{(3)} \leq 0 \]
\[ q = 4 : \quad \tilde{\pi}_t < 0, \quad \tilde{y}_t \geq 0 \quad \Rightarrow \quad \phi_{\pi}^{(4)} \leq 0, \quad \phi_y^{(4)} \geq 0 \]
Regime switching representation

\[
\begin{bmatrix}
\tilde{y}_t \\
\tilde{\pi}_t
\end{bmatrix} = A_t
\begin{bmatrix}
\mathbb{E}_t\{\tilde{y}_{t+1}\} \\
\mathbb{E}_t\{\tilde{\pi}_{t+1}\}
\end{bmatrix}
\]

where \( A_t = A(q) \) if the economy is in "regime" \( q \) in period \( t \), with

\[
A(q) = \frac{1}{\sigma + \phi_y(q) + \kappa \phi_\pi(q)}
\begin{bmatrix}
\sigma & 1 - \beta \phi_\pi(q) \\
\sigma \kappa & \kappa + \beta (\sigma + \phi_y(q))
\end{bmatrix}
\]

for \( q \in \{1, 2, 3, 4\} \).
Equilibrium Determinacy in (Endogenous) Regime Switching Models

A benchmark regime switching model

\[ x_t = A_t E_t \{ x_{t+1} \} \tag{13} \]

where \( x_t \) is an \((n \times 1)\) vector of non-predetermined variables and \( A_t \) is an \((n \times n)\) non-singular matrix. Assume \( A_t \in \mathcal{A} \equiv \{ A^{(1)}, A^{(2)}, \ldots, A^{(Q)} \} \).

Goal: establish sufficient conditions on \( \mathcal{A} \) that guarantee that \( x_t = 0 \) all \( t \) is the only bounded solution to (13), i.e. \( \lim_{T \to +\infty} E_t \{ \| x_{t+T} \| \} > M \| x_t \| \) for any \( M > 0 \) and \( x_t \neq 0 \), and where \( \| \cdot \| \) is the usual \( L^2 \) norm.

Define \( \| A^{(q)} \| \equiv \max_x \| A^{(q)} x \| \) subject to \( \| x \| = 1 \). In addition,

\[ \alpha \equiv \max \{ \| A^{(1)} \|, \| A^{(2)} \|, \ldots, \| A^{(Q)} \| \} > 0. \]

**Theorem [sufficient condition for determinacy]**: If \( \alpha < 1 \), then \( x_t = 0 \) for all \( t \) is the only bounded solution to (13)
Optimal Policy: Implementation

- Determinacy region
Implementation: Determinacy Region

\[ q = 3: \] \[ \tilde{\pi}_t \geq 0, \quad \tilde{y}_t < 0 \]

\[ q = 2: \] \[ \tilde{\pi}_t < 0, \quad \tilde{y}_t < 0 \]

\[ q = 1: \] \[ \tilde{\pi}_t \geq 0, \quad \tilde{y}_t \geq 0 \]

\[ q = 4: \] \[ \tilde{\pi}_t < 0, \quad \tilde{y}_t \geq 0 \]
Optimal Policy: Implementation

- Determinacy region
- Discussion: time inconsistency, credibility
Concluding remarks

- Optimal monetary policy with a ZLB constraint and $r^* < 0$.
- The optimal policy aims to approach \textit{gradually} a steady state with positive average inflation and a binding ZLB.
- Around that steady state, inflation and the output gap display (second-best) fluctuations in response to shocks. Those fluctuations coexist with a nominal rate that remains at its ZLB most (or all) of the time.
- The central bank can implement the optimal policy as a (locally) unique equilibrium by means of an appropriate state-contingent rule.
- In order to establish that result, we derive a sufficient condition for local determinacy in a general model with endogenous regime switches, a finding that may be of interest beyond the problem studied in the present paper.
Correction for Positive Trend Inflation

Real Rate

Policy Rate

Inflation

Output Gap

Quarters

Quarters

Natural
Actual

BGN
Ascarri
Optimal Policy with Cost-Push Shocks: Baseline Calibration

- Actual Real Rate
- Policy Rate
- Inflation
- Output Gap
- Cost-Push Shock
Optimal Policy with Cost-Push Shocks: High Volatility Calibration