# Running Primary Deficits Forever in a Dynamically Efficient Economy: Feasibility and Optimality*

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**Abstract**

Government debt can be rolled over forever without primary surpluses in some stochastic economies, including some economies that are dynamically efficient. In an overlapping-generations model with constant growth rate, $g$, of labor-augmenting productivity, and with shocks to the durability of capital, we show that along a balanced growth path, the maximum sustainable ratio of bonds to capital is attained when the riskfree interest rate, $r_f$, equals $g$. Furthermore, this maximal ratio maximizes utility per capita along a balanced growth path and ensures that the economy is dynamically efficient.

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What is the maximum amount (possibly zero) of bonds that a government can rollover forever without running primary budget surpluses? If the government can rollover a positive amount of bonds forever, what is the optimal amount of bonds to rollover? Blanchard's presidential address to the American Economic Association spurred renewed scholarly interest in these questions. Only a year later, global events brought these issues into the public realm as governments around the world ran huge deficits to deal with the catastrophic economic consequences of the covid-19 pandemic.

In the absence of uncertainty, the ability of the government to rollover bonds forever is determined by the "$r$ vs $g$" comparison, as it is colloquially known, where $r$ is the net rate of return on all assets, including government bonds, and $g$ is the growth rate of the capital stock. Specifically, if and only if $r < g$ along a balanced growth path in a competitive economy, government bonds can be rolled over forever and will shrink as a share of the economy over time. Also, if and only if $r < g$, the economy suffers from a dynamically inefficient overaccumulation of capital. That is, government bonds can be rolled over forever if and only if the economy is dynamically inefficient.

In the presence of uncertainty, the link between dynamic inefficiency and the feasibility of rolling over government bonds forever is more nuanced. Put simply, the rate of return on capital is the rate of return relevant for assessing dynamic efficiency, but the riskfree interest rate is the rate of return relevant for assessing whether the government can rollover its bonds forever. Uncertainty breaks the equality of these two rates of return. As we will show, in some dynamically efficient competitive economies with a constant growth rate $g \geq 0$, it is possible for the riskfree interest rate, $r_f$, to be less than $g$, which makes permanent rollover of government bonds feasible. This possibility of permanently rolling over debt in an efficient economy does not exist in deterministic, dynamically efficient, competitive economies.

Our principal findings in this paper result from both positive and normative analyses of sustainable levels of the ratio of government bonds to the capital stock, which we define as levels of this ratio that permit government bonds to be rolled over forever *without any primary surpluses*. In our positive analysis, we find that if $r_f < g$ along a balanced growth path without government bonds (which can be the case in some dynamically efficient economies,

and must be the case in all dynamically inefficient economies), the maximum sustainable level of the bond-capital ratio is strictly positive. Starting from zero government bonds, increasing the amount of bonds crowds out capital, thereby driving up the marginal product of capital and the constellation of rates of return until, at the maximum sustainable bond-capital ratio, $r_f = g$. Provided that uncertainty about the rate of return on capital, $r$, is not degenerate, $r_f = g$ implies that $E\{ln(1+r)\} > ln(1+g)$ (Proposition 2), and hence, the economy is dynamically efficient.

Our normative analysis examines the optimal sustainable level of the bond-capital ratio, specifically the sustainable level of this ratio that maximizes welfare, measured as the utility of consumers in the steady state. We find that the marginal impact on welfare of an increase in this ratio is positive along balanced growth paths with $r_f < g$ and is non-negative along balanced growth paths with $r_f = g$. Therefore, since the bond-capital ratio is not sustainable for $r_f > g$, the optimal sustainable level of government bonds is a corner solution where the bond-capital ratio equals its maximum sustainable value and $r_f = g$. As noted above, the equality of $r_f$ and $g$ implies $E\{ln(1+r)\} > ln(1+g)$, and hence the economy is dynamically efficient. Thus, the optimal bond-capital ratio eliminates any dynamically inefficient overaccumulation of capital that may exist at lower levels of this ratio. But even if the economy without government bonds is dynamically efficient, it is possible that $r_f < g$, which indicates the consumers have such strong desire for safety in their portfolios that they are willing to hold government bonds that offer a riskfree rate of return below $g$. To summarize, government bonds play the dual role of eliminating any overaccumulation of capital and reducing risk in the portfolios of consumers. As long as $r_f < g$, the second role implies that welfare is increased by an increase in the bond-capital ratio, even in a dynamically efficient economy. The optimal value of the bond-capital ratio, which equals the maximum sustainable value of this ratio, is attained when $r_f = g$.

The model in this paper is crafted so that along a balanced growth path, the capital stock per unit of effective labor is constant but the rate of return on capital is stochastic. To illustrate the mechanism in its simplest form, we preview the model in the case in which (1) there is no labor-augmenting technical progress, so $g = 0$, and (2) the government wastes any

funds collected when it issues new bonds in excess of contemporaneous interest payments on existing government bonds. The model has overlapping generations of a constant number of people who live for two periods, earn labor income only in the first period of life, and save some of their wage income to provide for consumption in the second period of life. Output in each period is produced with labor and capital according to a Cobb-Douglas production function without a productivity shock, which implies that wage income is non-stochastic. Consumers save a constant fraction of their wage income because they earn non-asset income only in the first period and they have Epstein-Zin-Weil (Epstein and Zin (1989) and Weil (1990)) preferences with an intertemporal elasticity of substitution equal to one. Therefore, aggregate saving of the young consumers is non-stochastic, which makes total assets, the sum of capital and government bonds non-stochastic.

The uncertainty in our model economy enters through a stochastic shock to the durability of capital that makes the rate of capital depreciation, and hence the rate of return on capital, stochastic. Bulow and Summers (1984), p. 25, argue that "capital risk," which they associate with the stochastic nature of depreciation, is far larger than "income risk," which they associate with the stochastic nature of the marginal product of capital. The uncertainty about the rate of capital depreciation drives a wedge between the expected rate of return on capital and the riskfree interest rate. However, in this simple form of the model, the evolution of the capital stock, wage income, as well as the consumption and saving of the young generation, are all invariant to the distribution and realizations of the durability shock. Thus, there is a useful dichotomy in the equilibrium of the economy. We can determine the equilibrium values of aggregate saving and the capital stock, without any consideration of financial values and without any consideration of the realizations or the distribution of the durability shocks.[1] Despite the deterministic evolution of the capital stock, we show that the rate of return on capital, and hence the consumption of the old generation are risky because the depreciation rate of capital is stochastic. The stochastic nature of consumption when old implies that the pricing kernel is stochastic.

[1]This dichotomy holds if we relax the assumption (1) that $g = 0$, but it does not hold if we relax assumption (2) that the government wastes any funds it receives when it issues new bonds in excess of interest payments on existing government bonds.

The simplicity of the model in the case described above has several useful features. Because wage income per unit of effective labor, and hence aggregate saving of young consumers per unit of effective labor, are constant along a balanced growth path, there is no chance that adverse shocks will reduce aggregate saving below the amount needed to absorb the equilibrium amount of government bonds. Thus, government bonds are riskfree and the riskfree interest rate is the appropriate market interest rate on these bonds.[2] In the simple case of the model described above, the equilibrium size of the capital stock is invariant to the distribution of durability shocks. We illustrate that an increase in the variance of this shock can change an economy from dynamically efficient to dynamically inefficient, *without changing the capital stock*. In our quantitative analysis, we illustrate that the maximum sustainable bond-capital ratio, which is the optimal value of this ratio, is an increasing function of the variance of the durability shock. We also illustrate that the maximum sustainable amount of government bonds is an increasing function of the coefficient of relative risk aversion.

# 1 Literature Review

The celebrated Golden Rule of capital accumulation derived by Phelps (1961) characterizes the capital stock per capita that maximizes consumption per capita along a deterministic balanced growth path. If the rate of return on capital, $r$, equals the growth rate of the economy, $g$, then consumption per capita is at the highest feasible level in the long run. If the saving rate exceeds the rate consistent with the Golden Rule, then the capital stock per capita exceeds the Golden Rule level so $r < g$ and consumption per capita is less than in the Golden Rule; that is, there is a dynamically inefficient overaccumulation of capital. Diamond (1965) develops and analyzes an overlapping generations economy with optimizing consumers and competitive firms and finds that if the optimal saving of young consumers is sufficiently high, either because consumers are very patient or the wage share of total income is very high, then capital per capita can exceed the Golden Rule level along a balanced growth

[2]Bertocchi (1994) and Binswanger (2005) discuss the unsustainability of government bonds in economies in which the capital stock evolves stochastically. Our model features a non-stochastic capital stock and avoids the unsustainability problem in Bertochhi and Binswanger.

path and there is a dynamically inefficient overaccumulation of capital. However, if young consumers use some of their saving to hold government bonds, they will end up holding a smaller amount of capital in their portfolios, driving down the aggregate capital stock, possibly by enough to eliminate any dynamically inefficient overaccumulation of capital.

Cass (1972) provides a complete characterization of dynamic inefficiency that does not depend on consumers' preferences. The Cass criterion simply asks whether it is feasible to increase aggregate consumption at some date without having to reduce aggregate consumption at some later date(s). In a deterministic steady state, the Cass condition is the same as in Phelps and Diamond, that is, an economy is dynamically inefficient if and only if $r < g$. However, the general version of the Cass criterion also applies to economies outside the steady state.

A government bond that is rolled over forever is often regarded as a bubble, which is an asset with zero fundamental value that nevertheless has a positive market value. Tirole (1985) develops a tight link between dynamic inefficiency and the feasibility of bubbles, in his words "the existence of bubbles is conditioned by the efficiency of the bubbleless equilibrium." (Tirole (1985), p. 1076) In our context, a "bubbleless equilibrium" has zero government bonds. Part (a) of Proposition 1 in Tirole (1985) states that if the economy without any government bonds is dynamically efficient, then equilibrium in the economy cannot contain bubbles. Tirole's statement holds in deterministic economies where the rates of return on capital and government bonds are equal. The introduction of government bonds crowds out capital, thereby increasing the common rate rates of return on government bonds and capital. If the economy was dynamically efficient without government bonds, then $r > g$ initially and this increase in $r$ induced by government bonds increases the excess of $r$ over $g$, which implies that government bonds cannot be rolled over forever.

In order for bubbles to be feasible in a deterministic dynamically efficient economy, there must be a wedge between the rates of return on government bonds, $r_f$, and on capital, $r$. Specifically, $r$ must exceed $r_f$ and the growth rate $g$ must lie between $r_f$ and $r$. In deterministic models, Farhi and Tirole (2011) and Martin and Ventura (2012) provide this wedge by introducing a wedge between borrowing and lending rates for firms. Also in

a deterministic framework, Ball and Mankiw (2021) introduces monopoly power by firms, which drives a wedge between the marginal product of capital, $r$, and the user cost of capital, which is based on the interest rate on government bonds, $r_f$. Aguiar et al. (2021) also examine the role of monopoly power in driving a wedge between the rate of return on capital and the interest rate on government bonds, but their analysis includes idiosyncratic risk. Like Ball and Mankiw (2021), however, in their analysis the aggregate rate of return of capital is not uncertain.[3]

Aggregate uncertainty also drives a wedge between the riskfree rate, $r_f$, and the rate of return on capital, $r$. Abel, Mankiw, Summers, and Zeckhauser (1989), hereafter AMSZ, proves that if the rate of return on capital is greater than the growth rate in all states and at all times, then the economy is dynamically efficient; and since the rate of return on capital is always greater than the growth rate, the riskfree rate is greater than the growth rate and hence it is not feasible to rollover government bonds forever. Alternatively, AMSZ proves that if the rate of return on capital is less than the growth rate in all states and at all times, then the economy is dynamically inefficient; and since the rate of return on capital is always lower than the growth rate of capital, the riskfree rate is less than the growth rate and it is feasible to rollover government bonds forever. Thus, in the situations that can be declared dynamically efficient or dynamically inefficient by the sufficient conditions in AMSZ, the link between dynamic inefficiency and the feasibility of bubbles continues to hold. However, the AMSZ conditions are not applicable in economies where the rate of return on capital is sometimes greater than the growth rate and sometimes less than the growth rate. Zilcha (1990, 1991) steps into the gap left by AMSZ and derives a characterization of dynamic efficiency in stochastic economies in which the rate of return on capital sometimes exceeds and sometimes falls short of the growth rate, $g$, of the economy. Zilcha adapts Cass's definition of dynamic efficiency in a natural way to stochastic economies and derives a remarkable sufficient condition for dynamic inefficiency when the economy grows at a

[3]Amol and Luttmer (2022) also examines fiscal policies in economies with idiosyncratic uncertainty but no aggregate uncertainty. The analysis in that paper does not include monopoly but is conducted in a two-sector model with a capital goods sector and a consumption goods sector. As in Ball and Mankiw (2021) and Aguiar et al. (2021), but unlike in our paper, the aggregate return to capital in Amol and Luttmer (2022) is not random.

constant rate, $g$: $E\{\ln(1+r)\} < \ln(1+g)$.

Blanchard and Weil (2001) presents four simple example economies that debunk various simplistic views about the relation between dynamic inefficiency and the feasibility of rolling over government bonds forever. In particular, the third and fourth examples in that paper illustrate that government bonds can be rolled over forever in some dynamically efficient economies. Interestingly, in all of the examples in that paper, dynamic efficiency or inefficiency does not depend on the variance of the shocks, since the shocks are additive shocks to the logarithm of the rate of return on capital, and thus do not affect $E\{\ln(1+r)\}$. By contrast, in the current paper, we illustrate how an increase in the variance of shocks can push an economy from dynamic efficiency to dynamic inefficiency, even without affecting the capital stock.

Blanchard's presidential address (Blanchard (2019)) is a far-ranging analysis of both empirical and theoretical issues related to the rollover of government bonds. It carefully documents that the recent situation with safe interest rates below growth rates is not unusual in historical data. In simulations reminiscent of the eponymous deficit gamble in Ball, Elmendorf, and Mankiw (1998), Blanchard finds that even if government bonds cannot be rolled over forever, it is likely that they can be rolled over for many decades before investors become unable, or unwilling, to buy newly issued government bonds.

Since Blanchard's presidential address, at least three papers have appeared with simple titles that involve comparisons of the rate of return and the growth rate. Cochrane (2021b), simply titled "$r < g$," is a forceful warning against the notion that when the riskfree interest rate is lower than the growth rate, the government can rely on growing itself out of debt. An attempted permanent rollover of bonds is bound to fail eventually, especially if government deficits are large. "The constraint on public debt when $r < g$ but $g < m$" (Reis (2021)) develops a model in which the interest rate on government bonds ($r$) is less than the growth rate of the economy ($g$), opening the possibility that government bonds can be rolled over, and yet the marginal product of capital ($m$) is greater than $g$ so the economy is dynamically efficient. That paper derives the fiscal capacity of the economy, which is a limit on the ratio of government spending to the amount of bonds outstanding. Barro (2020), simply

titled "$r$ Minus $g$," provides data on (arithmetic) averages in each of 14 OECD countries of rates of return on bonds and equities and growth rates of GDP per capita and consumption per capita. However, the Zilcha criterion directly implies that for the purpose of assessing dynamic efficiency, one must use the geometric means of rates of return and growth rates rather than arithmetic means.

Two interesting questions – one positive and one normative – remain unanswered in the papers described above. First, what is the *maximum* amount of government bonds that can be rolled over forever? The fiscal capacity in Reis (2021) is related to this question but, as mentioned above, it is the maximum ratio of government spending to the amount of bonds outstanding, rather than the maximum ratio of the amount of bonds outstanding to the capital stock that is the focus of our analysis. Second, what is the *optimal* amount of bonds to rollover along a balanced growth path? Blanchard (2019), Ball and Mankiw (2021), and Kocherlakota (2021) provide interesting analyses and discussions of the marginal impact of government bonds on welfare, where, as in our paper, welfare is defined to be the level of utility of consumers along a balanced growth path.[4] However, none of these three papers derives the optimal sustainable amount of government bonds, though Kocherlakota (2021) concludes that "as long as there is a public debt bubble (in this class of models), agents are better off in the long run if the government changes its policy choices so as to increase the debt and deficit" (p. 20).[5] Our paper addresses both of these questions. In our overlapping generations model with aggregate uncertainty about the durability of capital, the answers to the positive and normative questions are related in a perhaps surprising way. As we demonstrate in Section 5, the optimal sustainable bond-capital ratio equals the maximum sustainable value of this ratio.

Finally, in addition to the positive and normative questions that share a common answer, our paper also offers fresh insights about the intertemporal government budget constraint.

[4]Aoki et al. (2014) show that in situations in which bubbles exist in equilibrium, welfare in the bubbly equilibrium is higher than in the equilibrium without bubbles. However, it does not examine the marginal impact on welfare of an increase in the size of the bubble.

[5]Aiyagari and McGrattan (1998) numerically computes the optimal value of government debt in a model with uninsurable idiosyncratic risks, but no aggregate risks. It finds that the welfare loss resulting from the actual level of debt in the US economy (at the time of that paper's writing), rather than the optimal level of debt as computed in their model, is small.

It is typical in discussions of the sustainability of fiscal policy (O'Connell and Zeldes (1988), Wilcox (1989), and Bohn (1995)), the pricing of government bonds (Jiang et al. (2019)), and the fiscal theory of the price level (Cochrane (2021a)) to assume that the value of government debt equals the expected present value of the sum of primary government surpluses over the infinite future, for an appropriate path of discount rates over time. This budget constraint is often described as a transversality condition or, more accurately, as a No Ponzi Game (NPG) condition. In our paper, the NPG condition is violated by design.[6] The NPG condition is often invoked to rule out the possibility of rolling over debt forever. While the NPG arises naturally in some contexts, we examine situations in which permanent rollover of debt is both feasible and optimal, and the NPG condition need not, and does not, hold. Contrary to the literature in which the value of government debt equals the sum of present values of future primary surpluses, in our model, the value of government debt is positive even though all future primary government surpluses are non-positive.

# 2 Deterministic Capital Stock with Risky Returns

Aggregate output at time $t$, $Y_t$, is produced by competitive firms using capital, $K_t$, and effective labor, $G^t N$, where $G \equiv 1+g \geq 1$ is an index of labor-augmenting productivity that grows at rate $g \geq 0$ and $N$ is the constant number of young consumers in each time period. The production function is $Y_t = F\left(K_t, G^t N\right) = \left(G^t N\right)^{1-\alpha} K_t^{\alpha}$, where $0 < \alpha < 1$, and it is convenient to write the production function in intensive form as

$$y_t = k_t^{\alpha}, \tag{1}$$

where $y_t \equiv \frac{Y_t}{G^t N}$ and $k_t \equiv \frac{K_t}{G^t N}$. Capital depreciates at the rate $0 \leq \delta - \varepsilon_t \leq 1$ per period, where the durability shock $\varepsilon_t$ is an i.i.d., non-degenerate random variable with mean zero,

[6] As shown in Santos and Woodford (1997), in an economy in which the present value of the stream of present and future aggregate consumption is infinite, there is room for the NPG condition to fail. Nevertheless, even though the NPG condition fails in our model when $r_f < g$, thereby enabling permanent rollover of government debt, the market value of the existing capital stock is finite, because it is the valuation of the stream of profits to the remaining portion of a depreciating capital stock that approaches zero over time.

so $\delta > 0$ is the expected depreciation rate in each period.[7]

The economy is populated by people who live for two periods. In period $t$, $N$ people are born and each of these people inelastically supplies $G^t$ units of effective labor, earns wage income $W_t = (1-\alpha)\left(G^t\right)^{1-\alpha} K_t^{\alpha} N^{-\alpha} = (1-\alpha) G^t k_t^{\alpha}$ and receives a lump-sum transfer, $\tau_t$.

To focus on the impact of government borrowing, we simplify other aspects of fiscal policy. Specifically, we assume that the government does not purchase goods or services and that all taxes and transfers are lump-sum. Let $B_t$ be the amount of one-period government bonds outstanding at the beginning of period $t$, which are held by the old generation of consumers at time $t$. These bonds were bought at the end of the preceding period, when the currently-old consumers were young. Government budget accounting implies

$$B_{t+1} = (1 + r_{f,t})\, B_t + D_t, \tag{2}$$

where $D_t$ is the primary government budget deficit during period $t$ and $r_{f,t}$ is the riskfree interest rate on government bonds bought at the end of period $t-1$ and maturing in period $t$. Define $g_{B,t}$ to be the growth rate of government bonds from the end of period $t-1$, when the amount of government bonds equals $B_t$, to the end of period $t$, when the amount of government bonds equals $B_{t+1}$, so $B_{t+1} = (1 + g_{B,t})\, B_t$ and equation (2) can be rewritten as

$$(g_{B,t} - r_{f,t})\, B_t = D_t. \tag{3}$$

When the primary deficit, $D_t$, is positive, the government acquires funds to pay for the primary deficit by issuing additional bonds in excess of the interest payments on existing bonds. The government uses these funds to pay for lump-sum transfer payments to young

[7]Blanchard and Weil (2001) uses models of stochastic storage in its third and fourth examples so that output is linear in the capital stock. In their footnote 11, they point out that these models could be extended to incorporate concavity in the capital stock by specifying output to be $Y_t = K_t^{\alpha} - \delta K_t$, where $\delta$ is a random variable, but they do not work out the implications of this model. Barro (2020) uses a model with stochastic depreciation as in our model, but specifies $Y_t = AK_t$, so that, as in Blanchard and Weil's specification with simply stochastic storage, there is no concavity in the capital stock. Without concavity in the capital stock, the issue of capital overaccumulation is moot.

consumers or to pay for wasteful government purchases, or some mix of the two. In period $t$, total transfers to the $N$ young consumers are $N\tau_t = \zeta D_t$, where $0 \leq \zeta \leq 1$ is the share of the primary deficit that is used to make transfer payments to young consumers and $(1-\zeta) D_t$ is spent wastefully.[8] Therefore, the lump-sum transfer received by each young consumer in period $t$, is $\tau_t = \zeta \frac{D_t}{N}$, where $D_t$ is given by equation (3), so

$$\tau_t = \zeta \left(g_{B,t} - r_{f,t}\right) \frac{B_t}{N}. \tag{4}$$

Young consumers in period $t$ each consume $c_t^y$ and save $s_t = W_t + \tau_t - c_t^y$. The aggregate saving of the young generation, $S_t \equiv N s_t$, is used to purchase assets, $A_{t+1} = K_{t+1} + B_{t+1}$, consisting of capital, $K_{t+1}$, and one-period riskfree government bonds, $B_{t+1}$. Thus the aggregate capital stock in period $t+1$ is

$$K_{t+1} = A_{t+1} - B_{t+1} = S_t - B_{t+1}. \tag{5}$$

The rate of return on capital purchased at the end of period $t$ and used in period $t+1$ is

$$r_{t+1} = \alpha k_{t+1}^{\alpha-1} - \delta + \varepsilon_{t+1}, \tag{6}$$

which is the marginal product of capital in the production function in equation (1), $\alpha \left(G^{t+1} N\right)^{1-\alpha} \times K_{t+1}^{\alpha-1} = \alpha k_{t+1}^{\alpha-1}$, less the depreciation rate, $\delta - \varepsilon_{t+1}$.

At the end of period $t$, young consumers hold a fraction, $\lambda_{t+1}$, of their portfolios in riskfree government bonds with interest rate, $r_{f,t+1}$, and the remaining fraction, $1 - \lambda_{t+1}$, in risky capital with rate of return $r_{t+1}$. In the following period, when these consumers are old, they do not work. The generation of old consumers in period $t+1$ uses the gross return on total assets, $\left(1 + r_{a,t+1}\right) A_{t+1}$, to pay for its consumption, $N c_{t+1}^o$, where

$$r_{a,t+1} \equiv \lambda_{t+1} r_{f,t+1} + \left(1 - \lambda_{t+1}\right) r_{t+1} \tag{7}$$

[8] If, instead of wasting $(1-\zeta) D_t$, the government purchased public goods that entered utility functions additively separably from consumption when young and when old, such purchases would not affect the equilibrium values of consumption, saving, or rates of return. However, any utility associated with public goods would need to be considered when analyzing the impact of government bonds on welfare in Section 5.

is the rate of return on total assets.

Each person born in period $t$ has an Epstein-Zin-Weil (Epstein and Zin (1989) and Weil (1990)) utility function with intertemporal elasticity of substitution ($IES$) equal to one. We use the specification for a consumer who lives for two periods that is used in the second example in Blanchard and Weil (2001)[9]

$$U_t = (1-\beta)\ln c_t^y + \beta \ln\left(\left[E_t\left\{\left(c_{t+1}^o\right)^{1-\gamma}\right\}\right]^{\frac{1}{1-\gamma}}\right). \tag{8}$$

We assume that $\gamma \geq 1$, and to ensure a non-negative rate of time preference, we assume that $\beta \leq \frac{1}{2}$.

To solve the consumption/saving problem, use $s_t = W_t + \tau_t - c_t^y$ and $c_{t+1}^o = (1 + r_{a,t+1})\, s_t$ in equation (8) to write the consumption/saving problem as

$$\max_{s_t} (1-\beta)\ln\left(W_t + \tau_t - s_t\right) + \beta \ln s_t + \frac{\beta}{1-\gamma}\ln\left(E_t\left\{(1+r_{a,t+1})^{1-\gamma}\right\}\right). \tag{9}$$

The joint impact of $IES = 1$ and the assumption that consumers do not earn wage income or receive transfers in the second period of life is that the optimal value of $s_t$ is independent of $r_{a,t+1}$. The solution to the maximization problem in equation (9) is

$$s_t = \beta\left(W_t + \tau_t\right), \tag{10}$$

which implies

$$c_t^y = (1-\beta)\left(W_t + \tau_t\right) \tag{11}$$

and

$$c_{t+1}^o = (1 + r_{a,t+1})\,\beta\left(W_t + \tau_t\right). \tag{12}$$

Aggregate saving in period $t$ is $S_t = N s_t = N\beta\left(W_t + \tau_t\right) = \beta\left(NW_t + N\tau_t\right) = \beta[(1-\alpha)\, Y_t +$

[9]If $\gamma = 1$, we treat the utility function in equation (8) as $U_t = (1-\beta)\ln c_t^y + \beta E_t\left\{\ln c_{t+1}^o\right\}$.

$\zeta\left(g_{B,t} - r_{f,t}\right) B_t]$. Therefore, equation (5) implies that the aggregate capital stock in period $t+1$ is

$$K_{t+1} = S_t - B_{t+1} = \beta\left[\left(1-\alpha\right) Y_t + \zeta\left(g_{B,t} - r_{f,t}\right) B_t\right] - B_{t+1}. \tag{13}$$

Divide both sides of equation (13) by $G^{t+1}N$, define the bond-capital ratio, $\mathcal{B}_t \equiv \frac{B_t}{K_t}$, and rearrange the resulting equation to obtain

$$k_{t+1} = G^{-1}\beta\left[\left(1-\alpha\right) k_t^{\alpha} + \zeta\left(g_{B,t} - r_{f,t}\right) \mathcal{B}_t k_t\right] - \mathcal{B}_{t+1} k_{t+1}. \tag{14}$$

From this point onward, we focus on balanced growth paths along which $K_t$, $Y_t$, and $B_t$ all grow at rate $g \geq 0$, so that $g_{B,t}$, $k_t \equiv \frac{K_t}{G^t N}$, $y_t \equiv \frac{Y_t}{G^t N}$, $b_t \equiv \frac{B_t}{G^t N}$ and $\mathcal{B}_t \equiv \frac{B_t}{K_t}$ are all constant, with values $g$, $k$, $y$, $b$, and $\mathcal{B}$, respectively. Also, since the durability shock is i.i.d., the riskfree interest rate is constant and equal to $r_f$. Throughout, we use the notational convention that variables without time subscripts represent constant values along balanced growth paths.

Equation (14) implies that along a balanced growth path, the marginal product of capital is

$$\alpha k^{\alpha-1} = \frac{\alpha}{\left(1-\alpha\right)\beta}\left[\left(1+\mathcal{B}\right) G - \beta\zeta\left(g - r_f\right)\mathcal{B}\right]. \tag{15}$$

Remarkably, the ratio of capital to effective labor, $k$, in equation (15) is constant despite the shocks to the durability of capital. In the case with $\zeta = 0$, the marginal product of capital along a balanced growth path is invariant to the distribution of the durability shock. However, if $\zeta \neq 0$, then the marginal product of capital in equation (15) depends on the riskfree interest rate, which, as we will see below, depends on the distribution of the durability shock. Equation (15) also illustrates the crowding out effect of government debt; specifically, since the right hand side of equation (15) is an increasing function of $\mathcal{B}$,[10]

[10]The derivative of the right hand side of equation (15) with respect to $\mathcal{B}$ is $\frac{\alpha}{(1-\alpha)\beta}[G - \beta\zeta g + \beta\zeta r_f] = \frac{\alpha}{(1-\alpha)\beta}[(1-\beta\zeta)g + 1 + \beta\zeta r_f] > 0$, since $0 \leq \beta\zeta < 1$ and $r_f > -1$.

the marginal product of capital is an increasing function of $\mathcal{B}$ and hence $k$ is a decreasing function of $\mathcal{B}$.

Despite the fact that $k_t$ is constant along a balanced growth path, the rate of return on capital is stochastic, even along a balanced growth path. Use equation (6), which implies that the rate of return on capital along a balanced growth path is $r = \alpha k^{\alpha-1} - \delta + \widetilde{\varepsilon}$ (where $\widetilde{\varepsilon}$ is the random durability shock), and equation (15) to obtain

$$r = \frac{\alpha}{(1-\alpha)\,\beta}\left[(1+\mathcal{B})\,G - \beta\zeta\,(g - r_f)\,\mathcal{B}\right] - \delta + \widetilde{\varepsilon}. \tag{16}$$

It will be convenient to use the ratios of the gross rates of return to the gross growth rate, $G$. Specifically, along a balanced growth path, $R \equiv \frac{1+r}{G}$ is the "adjusted gross rate of return" on capital, $R_f \equiv \frac{1+r_f}{G}$ is the "adjusted gross riskfree interest rate" and $R_a \equiv \frac{1+r_a}{G} = \lambda R_f + (1-\lambda)\,R$. For $R$ and $R_f$, a value equal to one has special significance along deterministic balanced growth paths: the government can rollover debt forever at the riskfree interest rate if and only if $R_f \leq 1$; and the economy is dynamically inefficient if and only if $R < 1$, or equivalently, $\ln R < 0$. In stochastic economies with deterministic growth, $R_f = 1$ is still the crucial value of the riskfree interest rate that determines whether the government can rollover its debt forever; and, as we discuss in detail in Section 4, the economy is dynamically inefficient if and only if $E\{\ln R\} < 0$.

Use $G^{-1}(g - r_f) = G^{-1}(1 + g - (1 + r_f)) = 1 - R_f$ and equation (16) to obtain

$$R \equiv \frac{1+r}{G} = \overline{R} + G^{-1}\widetilde{\varepsilon}, \tag{17}$$

where

$$\overline{R} = \overline{R}\,(\mathcal{B}, \zeta, R_f) \equiv \frac{\alpha}{(1-\alpha)\,\beta}\left[1 + \mathcal{B} - \beta\zeta\,(1 - R_f)\,\mathcal{B}\right] + (1-\delta)\,G^{-1}, \tag{18}$$

is the expected "adjusted gross rate of return on capital" along a balanced growth path. Note that $\frac{\partial \overline{R}\left(\mathcal{B},\zeta,R_f\right)}{\partial \mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}\left[1 - \beta\zeta + \beta\zeta R_f\right] > 0$ and $\frac{\partial \overline{R}\left(\mathcal{B},\zeta,R_f\right)}{\partial R_f} = \frac{\alpha}{(1-\alpha)\beta}\beta\zeta\mathcal{B} \geq 0$ because $0 < \beta \leq \frac{1}{2}$ and $0 \leq \zeta \leq 1$.

## 2.1 Isomorphic Formulation with Stochastic Output

The model used throughout paper–which we will call the *baseline model*–was designed so that the aggregate capital stock, $K_t$, evolves deterministically and grows at a constant rate, $g$, along any balanced growth path. As we show in Section 3, the riskfree interest rate, $r_f$, is also constant along any balanced growth path, so the feasibility of rolling over government debt depends only on the sign of $r_f - g$. In addition, aggregate output, $Y_t$, evolves non-stochastically in the baseline model. In this subsection, we develop a class of models that are isomorphic to the baseline model in the sense that the evolution of $K_t$, as well as the evolution of rates of return, including the riskfree interest rate, are identical to those in the baseline model. Nevertheless, $Y_t$, evolves stochastically for all models, except for the baseline model, in this isomorphic class. We choose to use the baseline model throughout the rest of the paper for expositional simplicity, recognizing that the non-stochastic evolution of $Y_t$ is not at all essential to our findings.

Consider a class of models in which the production function is

$$Y_t = \left(G^t N\right)^{1-\alpha} K_t^{\alpha} + \eta_{1,t} K_t = \left(k_t^{\alpha-1} + \eta_{1,t}\right) K_t, \tag{19}$$

where $\eta_{1,t}$ is an i.i.d. random productivity shock with a mean that can be positive, zero, or negative. Because aggregate wage income, $(1-\alpha)\left(G^t N\right)^{1-\alpha} K_t^{\alpha}$, is deterministic, aggregate saving of the young and the evolution of $K_t$ are deterministic and identical to those in the baseline model.

In this class of models, the depreciation rate of capital is $\delta - \eta_{2,t}$, where $\eta_{2,t}$ is an i.i.d random variable with arbitrary correlation with $\eta_{1,t}$, and $0 \leq \delta - \eta_{2,t} \leq 1$. In addition, assume that $E\left\{\eta_{1,t} + \eta_{2,t}\right\} = 0$. The (net) rate of return on capital is the marginal product of capital, $\alpha k_t^{\alpha-1} + \eta_{1,t}$, minus the depreciation rate, $\delta - \eta_{2,t}$,

$$r_t = \alpha k_t^{\alpha-1} - \delta + \left(\eta_{1,t} + \eta_{2,t}\right). \tag{20}$$

This class of isomorphic models is defined by $\eta_{1,t} + \eta_{2,t} = \varepsilon_t$. Therefore, equation (20)

can be rewritten as

$$r_t = \alpha k_t^{\alpha-1} - \delta + \varepsilon_t, \tag{21}$$

which is identical to equation (6) with $E\{\varepsilon_t\} = 0$. As a consequence, the riskfree rate, $r_{f,t}$, is identical to that in the baseline model. Thus, all of the models in this class of models are isomorphic to the baseline model in the sense that $K_t$ and all rates of return in all periods are identical to their values in the baseline model. Therefore, our major findings about rolling over government debt are not dependent on deterministic output.

## 2.2 Deterministic Version of the Model

In this subsection, and only in this subsection, we consider a deterministic version of the model. The following nonlinear combination of parameters, $\theta$, allows us to easily assess whether a balanced growth path in a deterministic economy with $\zeta = 0$ is dynamically efficient.

**Definition 1** $\theta \equiv \frac{(1-\alpha)\beta}{\alpha}\left[1 - (1-\delta)\,G^{-1}\right] - 1.$

**Lemma 1** *Along a balanced growth path in a deterministic economy with $\zeta = 0$, the ratio of gross investment to gross capital income is $\frac{1+\theta}{1+\mathcal{B}}$.*

According to the criteria in AMSZ Proposition 1, if the ratio of gross investment to gross capital income is always greater than one, then the economy is dynamically inefficient; and if the ratio of gross investment to gross capital income is always less than (or equal to)[11] one, the economy is dynamically efficient. Thus, Lemma 1 implies that if $\zeta = 0$, then a balanced growth path in a deterministic economy is dynamically inefficient if and only if $\frac{1+\theta}{1+\mathcal{B}} > 1$. Therefore, if $\theta$ is positive, it equals the value of the bond-capital ratio, $\mathcal{B}$, in a deterministic economy for which the balanced growth path is on the boundary between

[11] AMSZ Proposition 1 does not address the case in which the ratio of gross investment to gross capital income is equal to one. However, in a deterministic economy, if the ratio of gross investment to gross capital income is always equal to one, the economy is at the Golden Rule, which is the boundary between dynamic efficiency and dynamic inefficiency.

dynamic efficiency and dynamic inefficiency. The deterministic balanced growth path will be dynamically inefficient if and only if $\mathcal{B} < \theta$. Consistent with this result, note from the definition of $\overline{R}(\mathcal{B}, \zeta, R_f)$ in equation (18) that

$$\overline{R}\left(\theta, 0, \cdot\right) = 1. \tag{22}$$

Since $\overline{R}(\mathcal{B}, 0, R_f)$ is increasing in $\mathcal{B}$, the balanced growth path will be inefficient if and only if $\mathcal{B} < \theta$. In a deterministic economy, the riskfree interest rate, $R_f$, is identically equal to $\overline{R}$. Therefore, in a deterministic economy, we get the familiar result that if $R_f = \overline{R} < 1$, then the economy is dynamically inefficient ($\overline{R} < 1$) and government debt can be rolled over forever because $R_f < 1$, which is equivalent to $r_f < g$.

Four parameters each independently increase the value of $\theta$ and push the deterministic balanced growth path toward dynamically inefficient overaccumulation of capital: (1) since aggregate saving is proportional to $\beta$, an increase in $\beta$, which increases $\theta$, increases aggregate capital accumulation; (2) a decrease in $\alpha$ increases $\theta$ through two channels: (i) a decrease in $\alpha$ increases the labor share of income $1 - \alpha$, which increases saving and investment, $\beta\left(1 - \alpha\right) Y_t$; and (ii) a decrease in $\alpha$ reduces capital income, $\alpha Y_t$, making any given amount of capital less attractive; (3) an increase in $\delta$, which increases $\theta$, makes capital less attractive because it depreciates more quickly; and (4) an increase in $G$, which increases $\theta$, pushes the economy closer to the situation in which $r < g$, which characterizes dynamic inefficiency.

# 3 Portfolio Allocation and Asset Pricing

Young consumers choose portfolios consisting of riskfree government bonds and risky capital. Formally, the optimal share of riskfree government bonds in a young consumer's portfolio is

$$\lambda_{t+1} = \arg\max_{\lambda_{t+1}} \frac{\beta}{1 - \gamma} \ln E_t \left\{ \left(R_{a,t+1}\right)^{1-\gamma} \right\}, \tag{23}$$

where, as discussed earlier, $R_{a,t+1} = \lambda_{t+1} R_{f,t+1} + (1 - \lambda_{t+1}) R_{t+1}$. The first-order condition associated with this maximization problem along a balanced growth path is

$$E\left\{\left(\lambda R_f + (1-\lambda) R\right)^{-\gamma} (R_f - R)\right\} = 0. \tag{24}$$

The first-order condition in equation (24) is an implicit function of $\lambda$, $R_f$, and $R$. Viewing $R_f$ and the distribution of $R$ as given, the implicit function determines the optimal value of $\lambda$. Alternatively, equation (24) can be viewed as a financial market equilibrium condition that determines $R_f$ as a function of the equilibrium value of $\lambda$ and the distribution of $R$. In financial market equilibrium with a given value of $\mathcal{B} \equiv \frac{B}{K}$, the share of the aggregate portfolio that is held in riskfree government bonds is $\lambda = \frac{B}{K+B} = \frac{\mathcal{B}}{1+\mathcal{B}}$.

**Lemma 2** *For any distribution of $R > 0$, $R_f = \frac{E_t\left\{R_a^{1-\gamma}\right\}}{E_t\left\{R_a^{-\gamma}\right\}}$.*

Let $R_f(\mathcal{B})$ denote the equilibrium value of $R_f$ as a function of the bond-capital ratio $\mathcal{B}$, along a balanced growth path. Where convenient, we omit the argument of this function and write simply $R_f$. Before analyzing the impact of $\mathcal{B}$ on $R_f$, we introduce the following definition and proposition.

**Definition 2** *$R_{\min} \equiv \frac{\alpha}{(1-\alpha)\beta} + (1 - \delta + \inf \widetilde{\varepsilon}) G^{-1} > 0$, which from equations (17) and (18), is less than or equal to the rate of return on capital for any $\mathcal{B} \geq 0$ and $\widetilde{\varepsilon} \geq \inf \widetilde{\varepsilon}_{\min}$. If $R_f \leq 1$, and the distribution of $\widetilde{\varepsilon}$ is non-degenerate, then, by the absence of arbitrage, $R_{\min} < 1$.*

Along a balanced growth path, government bonds can be rolled over forever if and only if $R_f \leq 1$, so we focus our attention only on situations with $R_f \leq 1$. The following proposition presents a sufficient condition for the riskfree interest rate to be an increasing function of $\mathcal{B}$ along balanced growth paths for which rollover is feasible.

**Proposition 1** *If $1 \leq \gamma < \Lambda \equiv \frac{1 - \frac{\zeta\beta}{1-\beta\zeta}(1-R_{\min})}{1+\frac{\alpha}{1-\alpha}\zeta} \frac{1}{1-R_{\min}}$, then $R_f'(\mathcal{B}) > 0$ for $R_f(\mathcal{B}) \leq 1$.*

Proposition 1 presents an interval of values of the risk aversion parameter $\gamma$ for which $R_f'(\mathcal{B}) > 0$. Two comments are in order. First, the condition in Proposition 1 is a sufficient,

but not necessary, condition, so that $R_f'(\mathcal{B}) > 0$ for a larger set of values of $\gamma$ than specified in this proposition. Second, the upper bound $\Lambda$ in this proposition is a function only of the parameters of the model. In the case in which $\zeta = 0$, Proposition 1 simplifies to

**Corollary 1** *If* $\zeta = 0$ *and* $1 \leq \gamma < \frac{1}{1-R_{\min}}$, *then* $R_f'(\mathcal{B}) > 0$ *for* $R_f(\mathcal{B}) \leq 1$.

Recall from the definition of $R_{\min}$ that the no-arbitrage condition implies that $R_{\min} < 1$ when $R_f(\mathcal{B}) \leq 1$, so the condition in Corollary 1 is non-vacuous.[12]

# 4 Dynamic Efficiency and the Feasibility of Rollover

A deterministic economy along a balanced growth path is dynamically efficient if and only if $R \geq 1$, equivalently, $r \geq g$. In the presence of uncertainty, it might be tempting to assess dynamic efficiency simply by comparing the expected value of $r$ to the expected value of $g$. However, that approach can lead to incorrect conclusions about dynamic efficiency in situations that are neither extraordinary nor pathological. Specifically, there is a set of situations in which $E\{R\} \geq 1$ and yet the economy is dynamically inefficient.[13]

As shown by Zilcha (1991), the correct sufficient statistic for assessing dynamic efficiency in an economy that grows at constant rate $g$ is $E\{\ln R\}$, which equals $E\{\ln(1+r)\} - \ln(1+g)$. An economy is dynamically inefficient if and only if $E\{\ln R\} < 0$, which, in the case of certainty is the familiar condition $R < 1$, equivalently, $r < g$. Zilcha (1990, 1991) extends to stochastic economies an ingenious argument developed in Cass (1972) for assessing dynamic efficiency. Here is a greatly simplified sketch of the rigorous Cass-Zilcha analysis. Consider whether it is possible to increase aggregate consumption at time $t$ while maintaining aggregate consumption unchanged at all times after $t$. The increase in aggregate consumption at time $t$ reduces the capital stock at time $t+1$ by one unit—in Cass-Zilcha terminology, a capital decrement of one unit—which, ignoring for the moment any impact of the capital decrement on the rate of return on capital, reduces output at time $t+1$ by $r_{t+1}$

[12]For the general case in which $0 \leq \zeta \leq 1$, the proof of Proposition 1 in Appendix A demonstrates that the condition in that proposition is non-vacuous when $0.6 < R_{\min} < 1$.

[13]Since $\ln E\{R\} > E\{\ln R\}$ when $R$ is stochastic, it is possible for $E\{R\} > 1$ even though $E\{\ln R\} < 1$.

units, leading to a capital decrement at time $t+2$ of $1+r_{t+1}$ units, which leads to a capital decrement at time $t+3$ of $(1+r_{t+1})(1+r_{t+2})$, and so on. Thus, continuing for a moment to ignore the impact of capital decrements on the rate of rate of return on capital, the capital decrement at time $t+n \geq t+2$ is $\Pi_{j=1}^{n-1}(1+r_{t+j})$. Relative to the path of the capital stock in the absence of a change in consumption at time $t$, that is, relative to $K_{t+n} = (1+g)^{n-1} K_{t+1}$, the capital decrement at time $t+n$ is $\frac{1}{K_{t+1}}\Pi_{j=1}^{n-1}\frac{1+r_{t+j}}{1+g} = \frac{1}{K_{t+1}}\Pi_{j=1}^{n-1}R_{t+j}$, where $R_{t+j} \equiv \frac{1+r_{t+j}}{1+g}$ is the "adjusted gross rate of return on capital." If $E\{\ln R\} > 0$, the size of the expected capital decrement grows relative to the capital stock until the decrement at some future time eliminates the entire capital stock, thereby rendering impossible the attempt to achieve an increase in aggregate consumption at time $t$ without decreasing consumption at any time following $t$. That is, the original allocation of consumption and capital over time is dynamically efficient. Strengthening this argument is the fact that capital decrements themselves increase the marginal product of capital and increase $\ln R$.

Alternatively, if $E\{\ln R\} < 0$, the capital decrement relative to the capital stock shrinks over time toward zero, making it feasible to increase aggregate consumption at time $t$ without driving the capital stock to zero eventually. Thus, the original allocation of consumption and investment over time is dynamically inefficient. Finally, if $E\{\ln R\} = 0$, the capital decrement at time $t+1$ increases future marginal products of capital, thereby increasing future capital decrements above what they would be if we ignore this impact. Thus, it is infeasible to increase aggregate consumption at time $t$, without driving the capital stock to zero eventually. Hence, the economy is dynamically efficient if $E\{\ln R\} = 0$.

We apply the Zilcha criterion to examine, in economies with a constant growth rate, the relationship between dynamic inefficiency and the feasibility of rolling over a small amount of government bonds. The link between dynamic inefficiency and the feasibility of rollover is particularly stark in deterministic economies because $R_f$, the adjusted gross riskfree interest rate equals $R$, the adjusted gross rate of return on capital. If $R_f = R < 1$, then the economy is dynamically inefficient and, since $R_f < 1$, government bonds can be rolled over forever; alternatively, if $R_f = R > 1$, then the economy is dynamically efficient and, since $R_f > 1$, government bonds cannot be rolled over forever. However, in stochastic economies,

$R_f$ and $R$ generally differ since $R$ is stochastic. In such economies, there are parameter configurations for which $E\{\ln R\} > 0$, indicating that the economy is dynamically efficient and nevertheless, $R_f < 1$ so that at least a small amount of government bonds can be rolled over forever.

**Lemma 3** *Assume that $R_a > 0$ is a non-degenerate random variable with $E\{\ln R_a\} \leq 0$. If $\gamma \geq 1$, then $\frac{E\{R_a^{1-\gamma}\}}{E\{R_a^{-\gamma}\}} < 1$.*

When there are no government bonds, that is, when $\lambda = 0$, we have $R_a = R$, so Lemma 2 implies that $R_f = \frac{E_t\{R^{1-\gamma}\}}{E_t\{R^{-\gamma}\}}$. Therefore, Lemma 3 implies the following proposition.

**Proposition 2** *Assume that $\gamma \geq 1$, $\mathcal{B} = 0$, and the adjusted gross rate return on capital, $R \equiv \frac{1+r}{1+g} > 0$, is a non-degenerate random variable.*

1. *If $E\{\ln R\} \leq 0$, then $R_f < 1$.*

2. *If $R_f \geq 1$, then $E\{\ln R\} \geq 0$ and the economy is dynamically efficient.*

Proposition 2 implies that in the absence of government bonds, if the economy is dynamically inefficient, i.e., $E\{\ln R\} < 0$, then $R_f \equiv \frac{1+r_f}{1+g} < 1$ so the net riskfree rate, $r_f$, is less than the growth rate, $g$, and hence a small amount of government bonds can be rolled over forever. More interestingly, in the absence of government bonds, if the adjusted gross rate of return on capital, $R$, is stochastic with $E\{\ln R\} = 0$, then the economy is dynamically efficient and $R_f < 1$ so that $r_f < g$ and a small amount of government bonds can be rolled over forever. Thus, contrary to dynamically efficient deterministic economies, it is possible to roll over government bonds forever in some dynamically efficient stochastic economies.

Proposition 2 leaves open the possibility that in some dynamically efficient economies that have $E\{\ln R\} > 0$, the riskfree interest rate, $R_f < 1$, so government bonds can be rolled over forever. We illustrate such economies in Figures 1 and 2 in Section 6.

# 5 Maximum and Optimal Amounts of Sustainable Government Debt

In this section, we analyze two questions about sustainable levels of government debt. First we address a positive question: what is the maximum sustainable value of $\mathcal{B}$ along a balanced growth path? Then we address a normative question: what is the sustainable value of $\mathcal{B}$ that maximizes utility along a balanced growth path? Remarkably, we find that utility along a balanced growth path is maximized by the maximum sustainable value of $\mathcal{B}$.

**Definition 3** *A constant value of $\mathcal{B}$ along a balanced growth path is sustainable if government debt can be rolled over forever at the riskfree interest rate without any primary budget surpluses in the future. Define $\mathcal{B}_{\max}$ as the maximum sustainable value of $\mathcal{B}$.*

**Remark 1** *Along a balanced growth path with constant $R_f$, a constant value of $\mathcal{B}$ is sustainable if and only if $R_f \leq 1$.*

**Proposition 3** *If $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$, then*

1. *if $R_f(0) \geq 1$, then $\mathcal{B}_{\max} = 0$*
2. *if $R_f(0) < 1$, then*
   - *(a) $\mathcal{B}_{\max}$ is the unique root of $R_f(\mathcal{B}) = 1$*
   - *(b) $\mathcal{B}_{\max}$ is finite*
   - *(c) $\mathcal{B}$ is sustainable if and only if $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$*

Proposition 3, as well as later Propositions 4 and 5, assume that $R_f'(\mathcal{B}) > 0$ whenever $R_f \leq 1$. Proposition 1 provides a sufficient condition for this assumption to be true, namely, $\gamma < \Lambda$. Since this condition is sufficient, but not necessary for the results of Propositions 3, 4, and 5, these propositions potentially apply to a larger set of economies than the set of economies for which $\gamma < \Lambda$.

**Corollary 2** *If $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$, then $\mathcal{B}_{\max}$ is invariant to $\zeta$.*

The parameter $\zeta$ appears only in the transfers to young consumers, which, along a balanced growth path, equal $\tau = \zeta (g - r_f) \frac{B}{N} = \zeta (g - r_f) \mathcal{B}k$. Transfers are equal to zero, regardless of the value of $\zeta$, when $\mathcal{B} = 0$ so $R_f(0)$ is invariant to $\zeta$ and hence the determination of whether $\mathcal{B}_{\max}$ is zero or positive is invariant to $\zeta$. Similarly, transfers are equal to zero, regardless of the value of $\zeta$, when $R_f(\mathcal{B}) = 1$, so the root of $R_f(\mathcal{B}) = 1$ is invariant to $\zeta$. Therefore, the value $\mathcal{B}_{\max}$ is invariant to $\zeta$.[14]

As in Blanchard (2019) and Ball and Mankiw (2021), our measure of welfare is the utility of consumers along a balanced growth path.

**Definition 4** *Define $u_t \equiv U_t - t \ln G$, which is constant along a balanced growth path. The optimal sustainable value of $\mathcal{B}$ along any balanced growth path is* $\arg \max_{\mathcal{B} \in [0, \mathcal{B}_{max}]} u(\mathcal{B})$ .

To evaluate utility along a balanced growth path for a given value of $\mathcal{B}$, define $w \equiv \frac{W_t}{G^t} = (1 - \alpha) k^\alpha$ and use the expression for $U_t$ in equation (8) along a balanced growth path to obtain[15]

$$u(\mathcal{B}) = \ln w + \ln \left(1 + \frac{\tau_t}{W_t}\right) + \frac{\beta}{1 - \gamma} \ln E_t \left\{R_a^{1-\gamma}\right\} + constant. \tag{25}$$

**Proposition 4** *If $R_f'(\mathcal{B}) > 0$ whenever $R_f \leq 1$, then $\frac{du}{d\mathcal{B}} > 0$ for any $\mathcal{B} \in [0, \mathcal{B}_{max}]$.*

As shown in Lemma 8 in the Appendix, an increase in $\mathcal{B}$ has opposing effects on $\ln w$ and $\frac{\beta}{1-\gamma} \ln E_t \{R_a^{1-\gamma}\}$. An increase in $\mathcal{B}$ crowds out the capital stock, thereby reducing $\ln w$ but increasing the marginal product of capital, which increases rates of return and hence increases $\frac{\beta}{1-\gamma} \ln E_t \{R_a^{1-\gamma}\}$. Despite these opposing effects, the impact of an increase in $\mathcal{B}$ on utility is unambiguously positive whenever $R_f \leq 1$, as stated in Proposition 4. Corollary 3 provides a simple expression for $\frac{du}{d\mathcal{B}}$ in the case with $\zeta = 0$. This expression allows us to

[14] Setting $R_f = 1$ and $\lambda = \frac{\mathcal{B}}{1+\mathcal{B}}$ in Lemma 2 and using equations (17) and (18) implies that when $\mathcal{B}_{\max} > 0$, it is a root of $\nu(\mathcal{B}) \equiv E\left\{\left(\frac{\mathcal{B}}{1+\mathcal{B}} + \frac{\alpha}{(1-\alpha)\beta} + \frac{1}{1+\mathcal{B}} G^{-1}(1 - \delta + \widetilde{\varepsilon})\right)^{-\gamma}\right\} - E\left\{\left(\frac{\mathcal{B}}{1+\mathcal{B}} + \frac{\alpha}{(1-\alpha)\beta} + \frac{1}{1+\mathcal{B}} G^{-1}(1 - \delta + \widetilde{\varepsilon})\right)^{1-\gamma}\right\} = 0$, which is invariant to $\zeta$.

[15] Substitute $c_t^y = (1 - \beta)(W_t + \tau_t)$ and $c_{t+1}^o = (1 + r_{a,t+1}) \beta (W_t + \tau_t)$ into equation (8) to obtain $U_t = \ln W_t + \ln \left(1 + \frac{\tau_t}{W_t}\right) + \frac{\beta}{1-\gamma} \ln E_t \left\{(1 + r_{a,t+1})^{1-\gamma}\right\} + constant$. Subtract $\ln G^t$ from both sides of the resulting equation and use $u = U_t - \ln G^t$ and $\ln w = \ln W_t - \ln G^t$ to obtain equation (25).

show how the two opposing effects are related through the factor price frontier in a way that the utility-increasing effect dominates the utility-decreasing effect.

**Corollary 3** *If* $\zeta = 0$*, then* $\frac{du}{d\mathcal{B}} = \frac{1}{1+\mathcal{B}} \frac{1}{R_f} \left[ (1 - R_f) \frac{\alpha}{1-\alpha} + \beta \mathcal{B} \frac{dR_f}{d\mathcal{B}} \right]$.

Corollary 3 is proved formally in Appendix A. Here we provide a heuristic derivation of $\frac{du}{d\mathcal{B}}$ to illustrate how the welfare-decreasing impact of the decrease in wage income is dominated by the welfare-increasing impact of the increase in rates of return on capital and bonds, when $R_f \leq 1$. To demonstrate that the main features of $\frac{du}{d\mathcal{B}}$ do not depend on Epstein-Zin-Weil utility with IES equal to one, we consider an additively separable two-period utility function $u = u^y(c^y) + E\{u^o(c^o)\}$, where, for $i = y, o$, the utility function $u^i(c^i)$ is strictly increasing and strictly concave with $\lim_{c^i \to 0} u^{i\prime}(c^i) = \infty$ and $\lim_{c^i \to \infty} u^{i\prime}(c^i) = 0$. For simplicity, we also assume for the purposes of this heuristic derivation that $G = 1$.

The heuristic proof proceeds in the three steps. First, for an individual consumer who has optimally chosen to consume $c^y$ when young and $c^o$ when old, the envelope theorem implies that the impact on utility of a change in $\mathcal{B}$ equals the change in utility if the consumer reduces $c^y$ by the amount of lost wage income in the first period and increases $c^o$ by the additional return on the $\frac{K}{N}$ units of capital and $\frac{B}{N}$ units of bonds held in the second period. That is, $\frac{du}{d\mathcal{B}} = u^{y\prime}(c^y) \frac{dW}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\} \left( \frac{K}{N} \frac{dR}{d\mathcal{B}} + \frac{B}{N} \frac{dR_f}{d\mathcal{B}} \right)$, where, in addition to the envelope theorem, we have used the fact that $\frac{K}{N} \frac{dR}{d\mathcal{B}} + \frac{B}{N} \frac{dR_f}{d\mathcal{B}}$ is non-stochastic in our model. Using the chain rule we obtain

$$\frac{du}{d\mathcal{B}} = u^{y\prime}(c^y) \frac{dW}{dK} \frac{dK}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\} \left( \frac{K}{N} \frac{dR}{dK} \frac{dK}{d\mathcal{B}} + \frac{B}{N} \frac{dR_f}{d\mathcal{B}} \right). \tag{26}$$

Second, use the factor-price frontier,[16] $N \frac{dF_N}{dK} + K \frac{dF_K}{dK} = 0$, which implies $\frac{dW}{dK} = -\frac{K}{N} \frac{dR}{dK}$, to obtain

$$\frac{du}{d\mathcal{B}} = [-u^{y\prime}(c^y) + E\{u^{o\prime}(c^o)\}] \frac{K}{N} \frac{dR}{dK} \frac{dK}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\} \frac{B}{N} \frac{dR_f}{d\mathcal{B}}. \tag{27}$$

[16] Since $F(K, N)$ is homogeneous of degree one in $K$ and $N$, Euler's theorem implies $NF_N + KF_K = F$. Differentiating this equation with respect to $K$ yields $N \frac{dF_N}{dK} + K \frac{dF_K}{dK} = 0$.

Third, use the first-order condition for the optimal intertemporal allocation of consumption along with the fact that $R_f$ is not stochastic to obtain $u^{y\prime}(c^y) = R_f E\{u^{o\prime}(c^o)\}$, which implies[17]

$$\frac{du}{d\mathcal{B}} = u^{y\prime}(c^y)\left[\left(\frac{1}{R_f}-1\right)\frac{K}{N}\frac{dR}{dK}\frac{dK}{d\mathcal{B}} + \frac{1}{R_f}\frac{B}{N}\frac{dR_f}{d\mathcal{B}}\right]. \tag{28}$$

The interpretation of $\frac{du}{d\mathcal{B}}$ is simplest in the situation with $\mathcal{B}=0$, so[18]

$$\frac{du}{d\mathcal{B}} = \underset{(+)}{u^{y\prime}(c^y)}\underbrace{\left(\frac{1}{R_f}-1\right)}_{sign\left(1-R_f\right)}\underset{(+)}{\frac{K}{N}}\underset{(-)}{\frac{dR}{dK}}\underset{(-)}{\frac{dK}{d\mathcal{B}}}, \quad \text{if } \mathcal{B}=0, \tag{29}$$

which is the product of five terms (reading from right to left): (1) an increase in $\mathcal{B}$ crowds out capital ($\frac{dK}{d\mathcal{B}}<0$); (2) the reduction in the capital stock increases the expected rate of return on capital ($\frac{dR}{dK}<0$); (3) the increase in the rate of return capital can be used to increase expected $c^o$ by $\frac{K}{N}\frac{dR}{dK}\frac{dK}{d\mathcal{B}}$, and along the factor-price frontier the increase in the rate of return on capital reduces the wage per capita by $\frac{K}{N}\frac{dR}{dK}\frac{dK}{d\mathcal{B}}$, which be used to decrease $c^y$; (4) the sign of the net impact of the decrease in $c^y$ and the equal-sized increase in $c^o$ is the same as the sign of $\frac{1}{R_f}-1$, where $\frac{1}{R_f} = \frac{E\{u^{o\prime}(c^o)\}}{u^{y\prime}(c^y)}$ is the intertemporal price of consumption; and (5) the marginal utility of consumption when young, $u^{y\prime}(c^y)$, which converts the calculation from goods to units of utility. Thus, if $\mathcal{B}=0$ and $R_f-1\leq 0$, then $\frac{du}{d\mathcal{B}}\geq 0$, with strict inequality if $R_f<1$; and when $\mathcal{B}>0$, this positive effect on welfare is reinforced by the

[17] To see that equation (28) with $\zeta=0$ is identical to Corollary 3, multiply and divide the right hand side of (28) by $\frac{K+B}{NR_f}$ and use the chain rule to replace $\frac{dR}{dK}\frac{dK}{d\mathcal{B}}$ by $\frac{dR}{d\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}$ (from equations (17) and (18) with $\zeta=0$) to obtain $\frac{du}{d\mathcal{B}} = u^{y\prime}(c^y)\frac{K+B}{N}\frac{1}{R_f}\left[(1-R_f)\frac{K}{K+B}\frac{\alpha}{(1-\alpha)\beta} + \frac{B}{K+B}\frac{dR_f}{d\mathcal{B}}\right]$. If $u^y(c^y) = (1-\beta)\ln c^y$, $u^o(c^o) = \beta \ln c^o$, and $\zeta=0$, then $c^y = (1-\beta)W$, $u^{y'}(c^y) = \frac{1-\beta}{c^y} = \frac{1}{W}$ and use $\frac{K}{K+B} = \frac{1}{1+\mathcal{B}}$ and $\frac{B}{K+B} = \frac{\mathcal{B}}{1+\mathcal{B}}$ to rewrite this expression as $\frac{du}{d\mathcal{B}} = \frac{K+B}{WN}\frac{1}{R_f}\frac{1}{1+\mathcal{B}}\left[(1-R_f)\frac{\alpha}{(1-\alpha)\beta} + \mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$. Finally, use $\beta WN = K+B$ to obtain $\frac{du}{d\mathcal{B}} = \frac{1}{1+\mathcal{B}}\frac{1}{R_f}\left[(1-R_f)\frac{\alpha}{1-\alpha} + \beta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$, which is identical to Corollary 3.

[18] Equation (29) states that the sign of $\frac{du}{d\mathcal{B}}$ depends only on the sign of $1-R_f$. In contrast, Blanchard (2019) shows that the sign of the impact of an increase in debt on welfare depends on both the riskfree rate and the risky rate of return on capital. This difference arises because in our model, the effect of a change in the capital stock on the marginal product of capital (the second derivative of the production function with respect to $K$) is non-stochastic. In addition to making the sign of $\frac{du}{d\mathcal{B}}$ depend only on the sign of $1-R_f$, this feature allows us to derive exact expressions for the marginal impact on welfare of an increase in $\mathcal{B}$, without relying on approximations.

final term in equation (28), $u^{y\prime}\left(c^{y}\right)\frac{1}{R_f}\frac{B}{N}\frac{dR_f}{d\mathcal{B}}$, which is positive when $R_f \leq 1$.

**Proposition 5** *If* $R_f'\left(\mathcal{B}\right) > 0$ *whenever* $R_f\left(\mathcal{B}\right) \leq 1$*, then* $\arg\max_{[0,\mathcal{B}_{\max}]} u(\mathcal{B})$ *equals* $\mathcal{B}_{\max}$*, that is, utility per effective unit of labor along a balanced growth path is maximized by the maximum sustainable value of* $\mathcal{B}$*.*

Since the proof of Proposition 5 is both simple and instructive, we present it here. If $R_f\left(0\right) \geq 1$, then $\mathcal{B}_{\max} = 0$ and the closed interval $[0, \mathcal{B}_{\max}]$ is a singleton so $\arg\max_{[0,\mathcal{B}_{\max}]} u(\mathcal{B})$ equals $\mathcal{B}_{\max} = 0$. Alternatively, if $R_f\left(0\right) < 1$, then $\mathcal{B}_{\max} > 0$ and Statement 2c of Proposition 3 implies all $\mathcal{B}$ in $[0, \mathcal{B}_{\max}]$ are sustainable. Proposition 4 implies that $\frac{du}{d\mathcal{B}} > 0$ for all $\mathcal{B} \in [0, \mathcal{B}_{\max}]$, so $\arg\max_{\mathcal{B}\in[0,\mathcal{B}_{\max}]} u(\mathcal{B})$ is a corner solution with $\mathcal{B} = \mathcal{B}_{\max}$. Corollary 2, which states that $\mathcal{B}_{\max}$ is invariant to $\zeta$, implies that the main result of the paper does not depend on whether the government uses its net resources from rollover to make transfers to the young consumers or simply wastes these resources.

# 6 A Graphical Illustration of Dynamic Efficiency and Feasibility of Debt Rollover in a Simple Case

The model developed and analyzed in this paper has the convenient property that the capital stock per unit of effective labor is constant along a balanced growth path, which implies that the expected rate of return on capital, $\overline{R}$, is also constant along a balanced growth path. In this section, we assume that the durability shock, $\varepsilon_t$, has a two-point distribution with $\Pr\left\{\varepsilon_t = \sigma\right\} = \frac{1}{2} = \Pr\left\{\varepsilon_t = -\sigma\right\}$, where $0 \leq \sigma < \min\left\{\delta, 1-\delta\right\}$. Therefore, along a balanced growth path, the adjusted gross rate of return on capital, $R$, has a two-point conditional distribution, $\Pr\left\{R = R_H \equiv \overline{R} + G^{-1}\sigma\right\} = \frac{1}{2} = \Pr\left\{R = R_L \equiv \overline{R} - G^{-1}\sigma\right\}$, where $\overline{R}$ is given by equation (18). We further simply the presentation in this section by setting $\gamma = 1$ and $\zeta = 0$.[19]

[19]When $\zeta = 0$, the mapping from the pair $(R_H, R_L)$ and a given value of $\mathcal{B}$ back to the fundamental parameter values is straightforward. Given $R_H$, $R_L$, $\sigma = \frac{1}{2}(R_H - R_L)$, and $\mathcal{B}$, use equation (18) to obtain $\frac{1}{2}\left(R_H + R_L\right) = \overline{R} = \frac{\alpha}{(1-\alpha)\beta}\left(1+\mathcal{B}\right) + \left(1-\delta\right)G^{-1}$, which characterizes, for given bond-capital ratio $\mathcal{B}$, the values of $\alpha, \beta, \delta, G$, and $\sigma$ that lead to a balanced growth path along which the two possible adjusted gross rates of return on capital are the given values of $R_H$ and $R_L$.

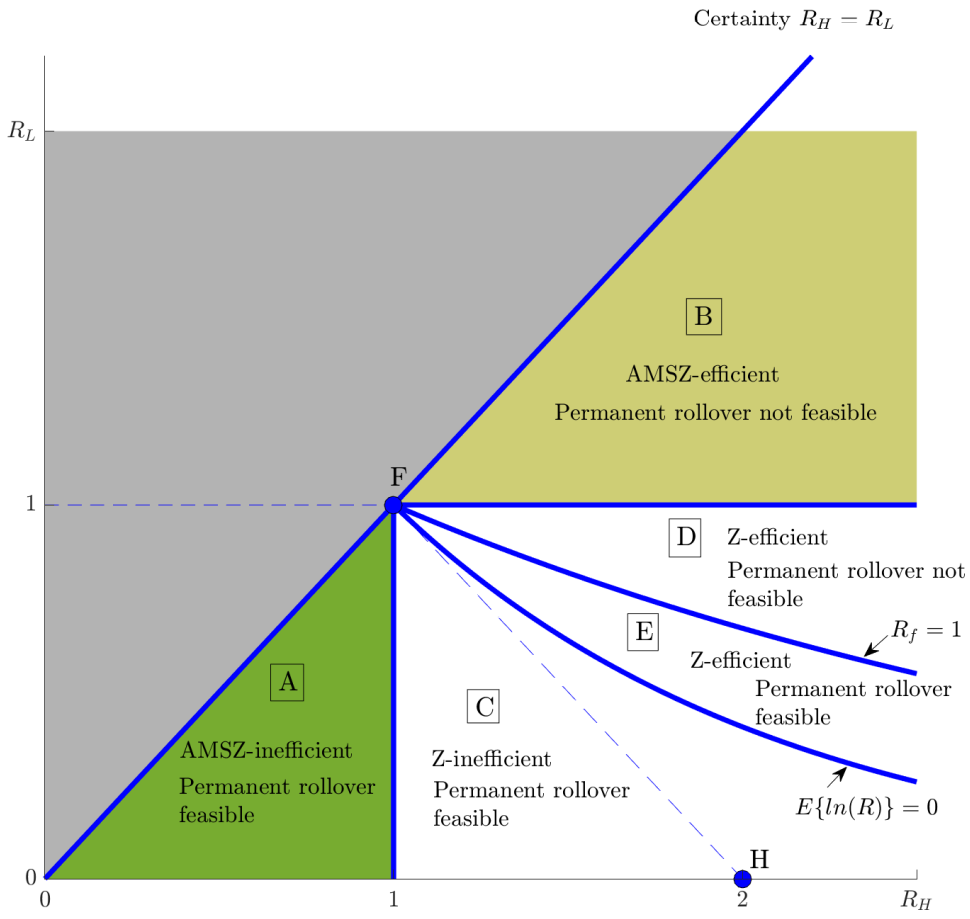


Figure 1: A Graphical Summary of Dynamic Efficiency and Feasibility of Debt Rollover when $\zeta = 0$, $\gamma = 1$, and $\mathcal{B} = 0$.

Figure 1 shows the possible adjusted gross rates of return on capital along a balanced growth path, with $R_H$ on the horizontal axis and $R_L$ on the vertical axis. In the absence of durability shocks, $R_H = R_L$ so the locus of possible pairs $(R_H, R_L)$ is the 45-degree line labeled "Certainty: $R_H = R_L$," which passes through the origin. Point F on this line represents the Golden Rule under certainty, $\frac{1+r}{1+g} = R_H = R_L = 1$, so that $r = g$. The area above and to the left of the 45-degree line is grayed out because $R_H \geq R_L$ by definition.

Now consider the situation in which the durability shock, $\varepsilon_t$, has a positive variance, $\sigma^2 > 0$, so that $(R_H, R_L)$ lies in one of the five regions below the 45-degree line. In two of these regions, the AMSZ criterion that compares the rate of return on capital with

the growth rate gives a decisive assessment of dynamic efficiency. In Region A, labelled "AMSZ-inefficient," both $R_H$ and $R_L$ are less than one so $r < g$ for both possible values of $R$. Therefore, the AMSZ criterion implies that the economy is dynamically inefficient. Furthermore, since $R_L < R_H < 1$, the adjusted gross riskfree interest rate, which satisfies $R_L \leq R_f \leq R_H$, is always less than one. In this case, $r_f < g$, so that a dollar of government debt that is rolled over at the riskfree interest rate will shrink over time toward zero.

In Region B, labelled "AMSZ-efficient," both $R_H$ and $R_L$ are greater than one so the net rate of return on capital is always greater than the growth rate. Therefore, the AMSZ criterion implies that the economy is dynamically efficient. Furthermore, since $1 < R_L < R_H$, the adjusted gross riskfree interest rate, which satisfies $R_L \leq R_f \leq R_H$, is always greater than one. Therefore, the amount of government bonds grows at rate $r_f$, which is higher than the growth rate of the capital stock, $g$. There is no possibility of rolling over government bonds, without primary surpluses, at the riskfree rate forever.

The downward-sloping curve originating at Point F and labelled $E\{\ln R\} = 0$ is a rectangular hyperbola, $R_L = \frac{1}{R_H}$. This curve is the boundary between dynamically inefficient $(R_H, R_L)$ pairs in Region C below the curve and dynamically efficient $(R_H, R_L)$ pairs in Regions D and E above the curve. Regions C, D, and E are all characterized by $R_L < 1 < R_H$, so that sometimes the net rate of return on capital is less than growth rate, $g$, and sometimes the net rate of the return on capital is greater than $g$. Because the net rate of return on capital fluctuates around $g$, the AMSZ criteria are not decisive about dynamic efficiency. However, in Region C, $R_L R_H < 1$ and hence $E\{\ln R\} < 0$ so the economy is dynamically inefficient according to the Zilcha criterion; thus, we label Region C as "Z-inefficient." By contrast, in Regions D and E, $R_L R_H > 1$ and hence $E\{\ln R\} > 0$ so the economy is dynamically efficient according to the Zilcha criterion; thus, we label Regions D and E as "Z-efficient." The dashed line segment FH is a locus of $(R_H, R_L)$ pairs for which $R_H + R_L = 2$ so that the expected return on capital, $\overline{R} \equiv \frac{R_H + R_L}{2} = 1$. Points between this line segment and the $E\{\ln R\} = 0$ curve are dynamically inefficient even though $\overline{R} > 1$ (equivalently, $E\{r\} > g$), thereby illustrating that the expected value of the risky return on capital is not the correct statistic to compare to the growth rate for the purpose of assessing dynamic

efficiency.

The government can rollover its bonds forever if $r_f \leq g$, equivalently, if $R_f \leq 1$. The higher of the two downward-sloping curves through Point F is the locus of $(R_H, R_L)$ for which $R_f = 1$ in the case with $\mathcal{B} = 0$ and $\gamma = 1$. For points above this curve (Regions B and D), $R_f > 1$ and permanent rollover of government bonds is not feasible. For points below the $R_f = 1$ curve (Regions A, C, and E), $R_f < 1$ and permanent rollover of government bonds, without any primary surpluses, is feasible. Region E is of particular interest because it has a different relationship between dynamic efficiency and feasibility of permanent rollover than in deterministic economies. In Region E, which lies above the $E\{\ln R\} = 0$ curve, the economy is dynamically efficient, but since Region E is below the $R_f = 1$ curve, permanent rollover of government bonds is feasible.

**Lemma 4** *With a symmetric, two-point distribution for the durability shock, $\varepsilon_t$, if $\zeta = 0$, a balanced growth path is dynamically efficient if and only if $R_H R_L \geq 1$, equivalently, if and only if $G^{-2}\sigma^2 \leq \left[\overline{R}(\mathcal{B}, \zeta, R_f)\right]^2 - 1$.*

If $\zeta = 0$, then the capital stock along a balanced growth path is independent of both the variance and the realizations of the durability shock, $\varepsilon_t$. Thus, holding $\alpha$, $\beta$, $\delta$, and $G$ fixed, an increase in the variance $\sigma^2$ of the durability shocks has no impact on the capital stock along a balanced growth path. Nevertheless, Lemma 4 implies that an increase in the variance $\sigma^2$ can change the efficiency status of a given capital stock along a balanced growth path from dynamically efficient to dynamically inefficient. With a capital stock invariant to the durability shock along a balanced growth path, the arithmetic mean of the rate of return on capital is unchanged by an increase in $\sigma^2$, but the geometric mean, which is the concept relevant for dynamic efficiency, falls when $\sigma^2$ increases.

**Lemma 5** *With a symmetric, two-point distribution for the durability shock, $\varepsilon_t$, if $\gamma = 1, \zeta = 0$, and $\mathcal{B} \in [0, 1]$ then along a balanced growth path, $R_f \lesseqgtr 1$ as $G^{-2}\sigma^2 \gtreqless \left[\overline{R}(\mathcal{B}, 0, \cdot) + \mathcal{B}\right] \times \left[\overline{R}(\mathcal{B}, 0, \cdot) - 1\right]$.*[20]

[20]We confine attention to $\mathcal{B} \leq 1$ in this case because the interval of values of $G^{-2}\sigma^2$ in equation (30) below is vacuous if $\mathcal{B} > 1$.

Lemma 5 implies that starting from a situation in which $R_f > 1$, an increase in the variance $\sigma^2$ can push $R_f$ below one as investors seeking safety accept a lower riskfree interest rate on government bonds.

**Proposition 6** *With a symmetric, two-point distribution for the durability shock, $\varepsilon_t$, if $\gamma = 1$, $\zeta = 0$, and $\mathcal{B} \in [0,1]$, then along a balanced growth path in a dynamically efficient economy, the ratio of government debt to the capital stock, $\mathcal{B}$, is sustainable if and only if*

$$\underbrace{h(\mathcal{B}) \equiv \left[\overline{R}(\mathcal{B},0,\cdot) + \mathcal{B}\right]\left[\overline{R}(\mathcal{B},0,\cdot) - 1\right]}_{R_f \leq 1} \leq G^{-2}\sigma^2 \leq \underbrace{g(\mathcal{B}) \equiv \left[\overline{R}(\mathcal{B},0,\cdot)\right]^2 - 1}_{\textit{dynamic efficiency}}. \tag{30}$$

Since $\overline{R}(\mathcal{B},0,\cdot)$ is linear in $\mathcal{B}$, the functions $g(\mathcal{B})$ and $h(\mathcal{B})$ in Proposition 6 are quadratic in $\mathcal{B}$. There are two values of $\mathcal{B}$ for which $g(\mathcal{B}) = h(\mathcal{B})$. Since, from equation (22), $\overline{R}(\theta,0,\cdot) = 1$, the definitions of $g(\mathcal{B})$ and $h(\mathcal{B})$ imply that $g(\theta) = 0 = h(\theta)$. Also, the definitions of $g(\mathcal{B})$ and $h(\mathcal{B})$ imply that $h(1) = \left[\overline{R}(1,0,\cdot) + 1\right]\left[\overline{R}(1,0,\cdot) - 1\right] = \left[\overline{R}(1,0,\cdot)\right]^2 - 1 = g(1)$. More generally, for other values of $\mathcal{B}$, $g(\mathcal{B}) - h(\mathcal{B}) = (1-\mathcal{B})\left[\overline{R}(\mathcal{B},0,\cdot) - 1\right]$, so for $\theta < \mathcal{B} < 1$, $\overline{R}(\mathcal{B},0,\cdot) - 1 > 0$ and hence $g(\mathcal{B}) > h(\mathcal{B})$. For values of $\mathcal{B} > 1$, $g(\mathcal{B}) < h(\mathcal{B})$ and the set of values of $G^{-2}\sigma^2$ in Proposition 6 is vacuous.

Figure 2 shows the long-run bond-to-capital ratio, $\mathcal{B}$, on the horizontal axis and the adjusted variance $G^{-2}\sigma^2$ on the vertical axis. In this figure, $\alpha = 0.3$, $\beta = 0.5$, $\delta = 0.8$, $\gamma = 1$, $\zeta = 0$, and $G = 1$ so $\theta = -0.067 < 0$. Since $g(\theta) = 0 = h(\theta)$, the horizontal intercepts of $g(\mathcal{B})$ and $h(\mathcal{B})$ are both equal to $\theta$, which is negative in this case. Thus, the horizontal intercepts do not appear in this figure since it is confined to the positive quadrant. As discussed above, $\theta < 0$ implies that the deterministic version of this economy with zero government bonds ($\mathcal{B} = 0$ and $\sigma^2 = 0$, which is the origin in Figure 2) is dynamically efficient. More generally, for any points below the $g(\mathcal{B})$ (dashed) curve, $G^{-2}\sigma^2 < g(\mathcal{B})$ so the economy is dynamically efficient; for points above the $g(\mathcal{B})$ curve, $G^{-2}\sigma^2 > g(\mathcal{B})$ so the economy is dynamically inefficient. For any points below the $h(\mathcal{B})$ (solid) curve, $G^{-2}\sigma^2 < h(\mathcal{B})$ so $R_f > 1$, which implies $r_f > g$, and hence it is not feasible to rollover government bonds forever; for points above the $h(\mathcal{B})$ curve, $G^{-2}\sigma^2 > h(\mathcal{B})$ so $R_f < 1$, which implies $r_f < g$, and hence it is feasible to rollover government bonds forever. Thus, points between the

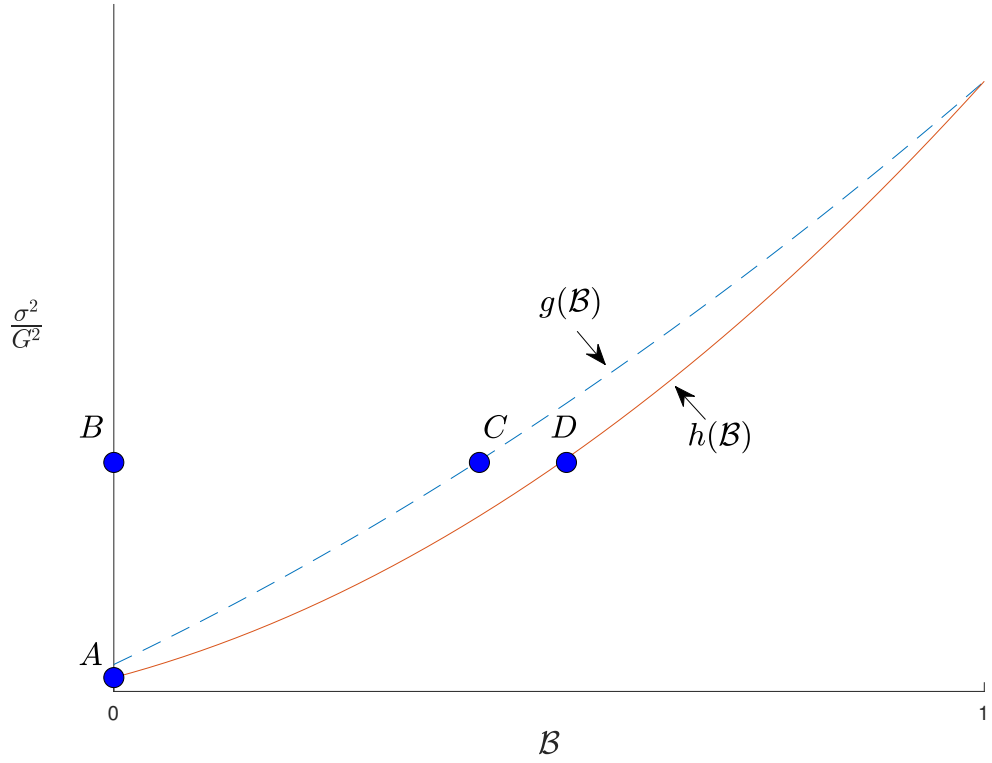


Figure 2: The boundaries of dynamic efficiency, $g(\mathcal{B})$, and sustainability of rollover, $h(\mathcal{B})$, for $\zeta = 0$ and $\gamma = 1$. The economy is dynamically efficient below $g(\mathcal{B})$, and $R_f < 1$ above $h(\mathcal{B})$.

two curves are dynamically efficient economies in which it is feasible to rollover government bonds forever.

Consider the origin in Figure 2, where $\mathcal{B} = 0$ and $\sigma^2 = 0$, which represents the version of the economy with no uncertainty and no government bonds. As discussed above, since this point lies below the $g(\mathcal{B})$ curve, the economy is dynamically efficient, and since this point lies below the $h(\mathcal{B})$ curve, we have $R_f > 1$. Now consider increasing $\sigma^2$ while maintaining $\mathcal{B} = 0$, which is a movement upward along the vertical axis. As $\sigma^2$ increases, consumers are willing to pay increasing amounts for the increased safety offered by riskless government bonds, thereby decreasing $R_f$. When $G^{-2}\sigma^2$ reaches the vertical intercept of the $h(\mathcal{B})$ curve, which is Point A, $R_f = 1$, which means that $r_f = g$ and government bonds can be rolled over forever, even though the economy at Point A is dynamically efficient because Point A

lies below the $g(\mathcal{B})$ curve.

Now we examine the impact of increasing the bond-capital ratio, $\mathcal{B}$, for a given value of $G^{-2}\sigma^2$. We will use Figure 2 to illustrate that if $R_f < 1$ in a steady state without any government bonds, the maximum sustainable amount of government bonds is positive; in addition, we will illustrate the determination of this maximum level in this figure. Consider Point B on the vertical axis, where $\mathcal{B}$ equals zero. At Point B, the economy without government bonds is dynamically inefficient because it lies above the $g(\mathcal{B})$ curve and $R_f < 1$ because the Point B is above the $h(\mathcal{B})$ curve. Holding $G^{-2}\sigma^2$ unchanged, increase $\mathcal{B}$, until we reach Point C on the $g(\mathcal{B})$ curve. The increase in $\mathcal{B}$ from Point B to Point C crowds out capital and eliminates the overaccumulation of capital. At Point C the economy is dynamically efficient, and because Point C is above the $h(\mathcal{B})$ curve, $R_f$ remains less than one, so it is feasible to roll over government bonds forever. Increase $\mathcal{B}$ further to Point D on the $h(\mathcal{B})$ curve. At Point D, the economy remains dynamically efficient, and $R_f = 1$ so that it remains feasible to roll over government bonds forever. Indeed, the value of the bond-capital ratio, $\mathcal{B}$, at Point D is the highest sustainable value of $\mathcal{B}$ for the given value of $G^{-2}\sigma^2$ at points B, C, and D.

# 7 Quantitative Application

In this section, we provide a quantitative illustration of (a) $\mathcal{B}_{\max}$, which is the maximum sustainable value of $\mathcal{B}$ as well as the welfare-maximizing value of $\mathcal{B}$, and (b) the value of $\mathcal{B}$ on the boundary between dynamic efficiency and inefficiency for given values of the fundamental parameters $\alpha$, $\beta$, $\delta$, $\gamma$, $\sigma$, and $G$.

We set the capital share $\alpha$ equal to 0.33. We interpret a "period" as 30 years and set $\beta = 0.353$, so that the annualized discount factor, $\left(\frac{\beta}{1-\beta}\right)^{1/30}$, is 0.98 per year, which implies an annual discount rate of 2% per year. We assume that labor-augmenting productivity grows at the rate of 1% per year, so $G = 1.35$. For risk aversion we consider $\gamma = 1, 3, 8$, and 10. Finally, we model the durability shock, $\varepsilon$, as a lognormal variable minus a constant, and choose the parameters so that when the ratio of government bonds to the capital stock, $\mathcal{B}$,

equals 0.5, $E\{(1+r)\} = E\{GR\}$ matches a target value of $(1+m)^{30}$, where $m = 0.03$ is an annualized rate of return on unlevered equity and $sd\{(1+r)\} = sd\{GR\}$ matches a target value of $s\sqrt{30}$, where $s$ is an annualized standard deviation of the rate of return on unlevered equity. Further details of the calibration, including a discussion of the implied mean and standard deviation of the rate of return on levered equity, are contained in Appendix C.

Tables 1 and 2 report for $\zeta = 0$ and $\zeta = 1$, respectively, $\mathcal{B}_{max}$ and the value of $\mathcal{B}$ on the dynamic efficiency boundary (i.e., the value of $\mathcal{B}$ such that $E\{ln(R)\} = 0$). In both tables, some of the cells for low values of $s$ are blank. In the section of each table that reports $\mathcal{B}_{max}$, a blank indicates that $R_f > 1$ for all nonnegative values of $\mathcal{B}$; thus, there is no positive value of $\mathcal{B}$ that can be rolled over forever. In the section of each table that reports the dynamic efficiency boundary, a blank indicates that the economy is dynamically efficient for all non-negative values of $\mathcal{B}$. Both tables show that as risk aversion increases, there is a significant gap between $\mathcal{B}_{max}$ and the dynamic efficiency boundary, so there is a non-trivial set of parameter values for which government bonds can be rolled over forever in dynamically efficient economies. For instance, in Table 1 when $\gamma = 10$ and $s = 0.22$, the dynamic efficiency boundary is attained at $\mathcal{B} = 0.083$, while $\mathcal{B}_{max} = 0.478$.

To interpret the magnitude of the values of $\mathcal{B}$ in Tables 1 and 2, recall that empirically the level of government debt is often expressed as a multiple of GDP, while the values of $\mathcal{B}$ are expressed as multiples of the capital stock. For an economy in which the capital-output ratio is 2, the debt-GDP ratio is twice as high as the debt-capital ratio, $\mathcal{B}$. In such an economy, the values of $\mathcal{B}$ in Tables 1 and 2, which range from 0 to 0.478, correspond to debt-GDP ratios ranging from 0 to 0.956.

Comparison of Tables 1 and 2 shows the impact of $\zeta$. Overall, the tables show that the impact of $\zeta$ is quantitatively small for the dynamic efficiency boundary and, as stated in Corollary 2, $\zeta$ is completely irrelevant for the determination of $\mathcal{B}_{max}$, which is the welfare-maximizing sustainable value of $\mathcal{B}$.

Now we look at the tables in more detail. First consider $\mathcal{B}_{max}$, which is both the maximum and the optimal sustainable value of $\mathcal{B}$. Since $R_f = 1$ when $\mathcal{B} = \mathcal{B}_{max}$, the transfer to young consumers, $\zeta(g - r_f)\mathcal{B}k = \zeta(1 - R_f)G\mathcal{B}k$, equals zero and hence the expected

adjusted gross rate of return on capital in equation (18) becomes $\overline{R} = \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}) + (1-\delta)G^{-1}$, which is independent of risk aversion, $\gamma$, and the volatility parameter, $s$. An increase in $\gamma$ or an increase in $s$ increases the risk premium on capital relative to the riskfree rate. With an unchanged $\overline{R}$, the increased risk premium implies that $R_f$ falls. To maintain $R_f = 1$, the value of $\mathcal{B}$ must increase to crowd out capital, thereby increasing $\overline{R}$ and $R_f$. Therefore, moving rightward in each row in the "maximum sustainable $\mathcal{B}$" section of each table, $\gamma$ increases and hence the maximum sustainable $\mathcal{B}$ increases; similarly, moving down each column in this section of these tables, $s$ increases and the maximum sustainable $\mathcal{B}$ increases. As implied by Corollary 2, the sections of Tables 1 and 2 that present the maximum sustainable $\mathcal{B}$ are identical to each other.

Now, consider the dynamic efficiency boundary, which is characterized by $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\} = 0$. In Table 1, $\zeta = 0$, so that the transfer to young consumers, $\zeta(g - r_f)\mathcal{B}k$, is zero. Hence, as discussed above, equation (18) implies that $\overline{R} = \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}) + (1-\delta)G^{-1}$, which is independent of $\gamma$ and $s$. Therefore, $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\}$ is independent of $\gamma$, so the value $\mathcal{B}$ on the dynamic efficiency boundary is independent of $\gamma$ for given $s$. By contrast, although an increase in $s$ also has no effect on $\overline{R}$, it reduces $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\}$ because $\ln(\cdot)$ is a strictly concave function. In order to restore $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\} = 0$ when $s$ is increased, and thus remain on the dynamic efficiency boundary, $\overline{R}$ must increase, so $\mathcal{B}$ must increase to crowd out additional capital and increase the marginal product of capital. This effect is illustrated in Table 1 by the increasing values of $\mathcal{B}$ as one goes down each column in the section of the table devoted to the dynamic efficiency boundary.

In Table 2, where $\zeta = 1$, equation (18) implies that $\overline{R} = \frac{\alpha}{(1-\alpha)\beta}[1 + \mathcal{B} + \beta(R_f - 1)\mathcal{B}] + (1-\delta)G^{-1}$. Thus, with $\zeta = 1$, $\overline{R}$ is an increasing function of $R_f$, which is a decreasing function of $\gamma$ and $s$, since both a higher coefficient of relative risk aversion, $\gamma$, and higher volatility, $s$, lead consumers to seek safety in riskless government bonds, thereby driving $R_f$ downward. Thus, for a given value of $s$, an increase in $\gamma$ reduces $R_f$ and hence reduces $\overline{R}$, so to remain on the dynamic efficiency boundary where $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\} = 0$, $\mathcal{B}$ must increase to increase $\overline{R}$. Alternatively, for given values of $\gamma$, an increase in $s$ reduces $E\left\{\ln(\overline{R}+\widetilde{\varepsilon})\right\}$ through two channels. First, the concavity of $\ln(\cdot)$ implies that for given $\overline{R}$, an increase in $s$

| | $\zeta = 0$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Maximum sustainable $\mathcal{B}$ | | | | Dynamic efficiency boundary | | | |
| | $\gamma = 1$ | $\gamma = 3$ | $\gamma = 8$ | $\gamma = 10$ | $\gamma = 1$ | $\gamma = 3$ | $\gamma = 8$ | $\gamma = 10$ |
| $s = 0.02$ | | | | | | | | |
| $s = 0.04$ | | | 0.042 | 0.062 | | | | |
| $s = 0.06$ | | 0.026 | 0.122 | 0.150 | | | | |
| $s = 0.08$ | | 0.075 | 0.193 | 0.224 | | | | |
| $s = 0.10$ | 0.017 | 0.120 | 0.252 | 0.285 | | | | |
| $s = 0.12$ | 0.043 | 0.162 | 0.301 | 0.334 | | | | |
| $s = 0.14$ | 0.068 | 0.199 | 0.342 | 0.374 | 0.013 | 0.013 | 0.013 | 0.013 |
| $s = 0.16$ | 0.091 | 0.232 | 0.376 | 0.408 | 0.031 | 0.031 | 0.031 | 0.031 |
| $s = 0.18$ | 0.113 | 0.261 | 0.405 | 0.435 | 0.049 | 0.049 | 0.049 | 0.049 |
| $s = 0.20$ | 0.133 | 0.286 | 0.429 | 0.459 | 0.066 | 0.066 | 0.066 | 0.066 |
| $s = 0.22$ | 0.152 | 0.309 | 0.450 | 0.478 | 0.083 | 0.083 | 0.083 | 0.083 |

Table 1: Maximum sustainable $\mathcal{B}$ and the value of $\mathcal{B}$ delineating the boundary of dynamic efficiency. $\gamma$ denotes risk aversion, and $s$ is the annualized standard deviation of the return on capital in an economy with $\mathcal{B} = 0.5$. $\zeta$ is set to zero.

reduces $E\left\{\ln(\overline{R} + \widetilde{\varepsilon})\right\}$. Second, as discussed above, an increase in $s$ reduces $R_f$ and hence reduces $\overline{R}$. To remain on the dynamic efficiency boundary where, $E\left\{\ln(\overline{R} + \widetilde{\varepsilon})\right\} = 0$, $\mathcal{B}$ must increase to increase $\overline{R}$. Finally, note that the nonzero entries for the dynamic efficiency boundary in Table 2 are higher than in Table 1 because the positive transfers to consumers when $\zeta = 1$ in Table 2 increase saving thereby increasing the capacity of saving to absorb bonds without driving the capital stock low enough to increase $E\left\{\ln(\overline{R} + \widetilde{\varepsilon})\right\}$ above zero.

# 8 Concluding Remarks

In this paper, we develop an overlapping-generations model to analyze sustainable levels of the ratio of government bonds to the capital stock that can be maintained forever without any future primary government surpluses. To make the analysis easily tractable, the baseline model confines exogenous shocks to the depreciation rate of capital, which is additively separable from the production function so the labor income of young consumers is non-stochastic. In addition, since (1) consumers have Epstein-Zin-Weil utility functions over

| | $\zeta = 1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Maximum sustainable $\mathcal{B}$ | | | | Dynamic efficiency boundary | | | |
| | $\gamma = 1$ | $\gamma = 3$ | $\gamma = 8$ | $\gamma = 10$ | $\gamma = 1$ | $\gamma = 3$ | $\gamma = 8$ | $\gamma = 10$ |
| $s = 0.02$ | | | | | | | | |
| $s = 0.04$ | | | 0.042 | 0.062 | | | | |
| $s = 0.06$ | | 0.026 | 0.122 | 0.150 | | | | |
| $s = 0.08$ | | 0.075 | 0.193 | 0.224 | | | | |
| $s = 0.10$ | 0.017 | 0.120 | 0.252 | 0.285 | | | | |
| $s = 0.12$ | 0.043 | 0.162 | 0.301 | 0.334 | | | | |
| $s = 0.14$ | 0.068 | 0.199 | 0.342 | 0.374 | 0.013 | 0.015 | 0.018 | 0.018 |
| $s = 0.16$ | 0.091 | 0.232 | 0.376 | 0.408 | 0.032 | 0.037 | 0.043 | 0.044 |
| $s = 0.18$ | 0.113 | 0.261 | 0.405 | 0.435 | 0.051 | 0.059 | 0.068 | 0.069 |
| $s = 0.20$ | 0.133 | 0.286 | 0.429 | 0.459 | 0.069 | 0.080 | 0.091 | 0.092 |
| $s = 0.22$ | 0.152 | 0.309 | 0.450 | 0.478 | 0.087 | 0.100 | 0.112 | 0.114 |

Table 2: Maximum sustainable $\mathcal{B}$ and the value of $\mathcal{B}$ delineating the boundary of dynamic efficiency. $\gamma$ denotes risk aversion, and $s$ is the annualized standard deviation of the return on capital in an economy with $\mathcal{B} = 0.5$. $\zeta$ is set to one.

their two-period lifetimes with the intertemporal elasticity of substitution set equal to one, and (2) consumers earn labor income (and possibly receive transfers) only in the first period of life, aggregate saving of young consumers is a constant fraction of their income in the first period of life, and hence the evolution of aggregate asset holdings, comprising capital and government bonds, is non-stochastic. Nevertheless, the rate of return on capital is stochastic because it includes the stochastic depreciation rate. Along a balanced growth path, aggregate wage income, the aggregate capital stock, and the aggregate amount of government bonds outstanding all grow at rate $g$, which is the constant rate of labor-augmenting productivity growth. Therefore, the balanced growth path features constant values of aggregate capital per unit of effective labor, the bond-capital ratio, the riskfree interest rate, and the expected rate of return on capital. Provided that the net riskfree interest rate, $r_f$, is less than or equal to $g$ along a balanced growth path, the bond-capital ratio is sustainable.

Our model is designed so that both $r_f$ and $g$ are constant along a balanced growth path. Because we are interested in sustainable bond-capital ratios along a balanced growth path, we confine attention to balanced growth paths that feature $r_f \leq g$. Along such paths, the

government can roll over its bonds forever without primary budget surpluses and without the bond-capital ratio increasing. There is no chance that young consumers will be unwilling or unable to purchase the bonds that the government issues to rollover its debt, so there is no chance of default on government bonds. Therefore, the market interest rate on government bonds equals the riskfree interest rate. In addition, when $r_f \leq g$, the value of government bonds at a given point in time is not the expected present value of future primary surpluses; along balanced growth paths with $r_f \leq g$, the value of outstanding government bonds is positive and yet all future primary deficits, which equal $(g-r_f)B_t$ at time $t$, are non-negative and hence all future primary surpluses are non-positive.

This paper has two major findings—one positive and one normative. We focus on levels of the bond-capital ratio that can be sustained forever without any future primary surpluses. The positive finding is that the maximum sustainable bond-capital ratio along a balanced growth path is attained when $r_f = g$. Given that both $r_f$ and $g$ are constant along balanced growth paths, this finding is not surprising. However, the normative finding is surprising (to us, at least). The sustainable bond-capital ratio that maximizes utility along a balanced growth path is the maximum sustainable value of this ratio, that is, the ratio that attains $r_f = g$. Briefly, an increase in the amount of bonds outstanding crowds out private capital, which reduces aggregate wage income and increases the marginal product of capital and hence increases the rate of return on capital. The reduction in wage income reduces welfare and the increase in the rate of return on capital increases welfare. It follows from the factor-price frontier that the reduction in wage income per person equals the increase in capital income per person. Whenever $r_f \leq g$, the welfare-increasing impact of the increased rate of return on capital, which occurs in the second period of life, dominates the welfare-decreasing impact of the reduction in wage income, which occurs during the first period of life. Thus, starting from a balanced growth path with $r_f < g$, an increase in the bond-capital ratio leads to a different balanced growth path with a higher level of welfare. Focusing on long-run welfare along balanced growth paths, it is optimal to increase the bond-capital ratio, and thereby increase $r_f$, until $r_f = g$, at which point the bond-capital ratio equals its maximal sustainable level. The optimal value of the bond-capital ratio is a corner solution where the

sustainability constraint prevents a further increase in the bond-capital ratio and yet welfare is increasing the level of the bond-capital ratio.

We designed the model to have both constant $r_f$ and $g$ along balanced growth paths so that the assessment of the sustainability of a given bond-capital ratio would be as straightforward as possible. But what if, for instance, the growth rate of labor-augmenting productivity, $g$, were random. In particular, what if $R_f \equiv \frac{1+r_f}{1+g}$ were random, sometimes greater than one and sometimes less than one? Then there might be some realization paths with $R_f$ persistently greater than one, so that eventually the stock of government bonds eventually exceeds the amount that young households would or could purchase. In such a framework, the notion of sustainability is more nuanced (which is why we designed the model to include constant $g$). We leave it as an open question how to characterize sustainability in that framework, and, in particular, how to characterize an appropriate notion of maximal borrowing. If there is some suitable notion of maximal borrowing, does the normative result, that the optimal government borrowing policy is the same as the maximal borrowing policy, generalize beyond the model in this paper? This generalization of the primary normative result would hold if, as in the current paper, the welfare-increasing effect of an increased rate of return to capital exceeds the welfare-reducing effect of a reduced wage income for any sustainable borrowing policy.

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# A Proofs

**Proof. of Lemma 1**: In this economy, $K_{t+1} = (1-\alpha)\beta Y_t - B_{t+1}$, since young consumers in period $t$ use their saving, $(1-\alpha)\beta Y_t$, to purchase capital, $K_{t+1}$, and bonds, $B_{t+1}$. Along a balanced growth path, $K_t = G^{-1}K_{t+1}$, so gross investment in period $t$ is $I_t \equiv K_{t+1} - (1-\delta)K_t = [1-(1-\delta)G^{-1}]K_{t+1} = [1-(1-\delta)G^{-1}][(1-\alpha)\beta Y_t - B_{t+1}]$. Gross capital income in period $t$ is $\alpha Y_t$, so the ratio of gross investment to gross capital income is $\frac{[1-(1-\delta)G^{-1}][(1-\alpha)\beta Y_t - B_{t+1}]}{\alpha Y_t} = [1-(1-\delta)G^{-1}]\left[\frac{(1-\alpha)\beta}{\alpha} - \frac{B_{t+1}}{\alpha Y_t}\right] = [1-(1-\delta)G^{-1}]\left[\frac{(1-\alpha)\beta}{\alpha} - \frac{B_{t+1}}{K_{t+1}}\frac{GK_t}{\alpha Y_t}\right]$ $= [1-(1-\delta)G^{-1}]\left[\frac{(1-\alpha)\beta}{\alpha} - \mathcal{B}_{t+1}\frac{GK_t}{\alpha Y_t}\right] = [1-(1-\delta)G^{-1}]\left[\frac{(1-\alpha)\beta}{\alpha} - G\mathcal{B}_{t+1}\frac{1}{\alpha k_t^{\alpha-1}}\right]$. Use $\alpha k^{\alpha-1}$ $= \frac{\alpha}{(1-\alpha)\beta}[(1+\mathcal{B})G - \beta\zeta(g-r_f)\mathcal{B}]$ from equation (15), so that when $\zeta = 0$, $\alpha k^{\alpha-1} = \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B})G$ along a balanced growth path. Therefore, along a balanced growth path, $\frac{(1-\alpha)\beta}{\alpha} - G\mathcal{B}_{t+1}\frac{1}{\alpha k_t^{\alpha-1}} = \frac{(1-\alpha)\beta}{\alpha}\left[1 - G\mathcal{B}\frac{1}{(1+\mathcal{B})G}\right] = \frac{(1-\alpha)\beta}{\alpha}\frac{1}{1+\mathcal{B}}$, so the ratio of gross investment to gross capital income along a deterministic balanced growth path is $[1-(1-\delta)G^{-1}]\frac{(1-\alpha)\beta}{\alpha}\frac{1}{1+\mathcal{B}}$ $= \frac{1+\theta}{1+\mathcal{B}}$. ■

**Proof of Lemma 2.** The first-order condition for $\lambda$ in equation (24) implies $0 = E_t\left\{\frac{(1-\lambda)\left(R-R_f\right)}{\left(\lambda R_f+(1-\lambda)R\right)^{\gamma}}\right\}$ $= E_t\left\{\frac{\lambda R_f+(1-\lambda)R-R_f}{\left(\lambda R_f+(1-\lambda)R\right)^{\gamma}}\right\} = E_t\left\{\frac{R_a-R_f}{R_a^{\gamma}}\right\} = E_t\left\{R_a^{1-\gamma}\right\} - R_f E_t\left\{R_a^{-\gamma}\right\}$, which implies $R_f = \frac{E_t\left\{R_a^{1-\gamma}\right\}}{E_t\left\{R_a^{-\gamma}\right\}}$. ■

**Proof of Proposition 1.** In preparation for the proof of the proposition we establish two Lemmas and a corollary.

**Lemma 6** *If $R_f \leq 1$, then* $\frac{\lambda}{1-\lambda}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} - 1\right) \leq 1 - R_{\min}$.

**Proof of Lemma 6.** Assume that $R_f \leq 1$. Use $R_f = \frac{E\left\{R_a^{1-\gamma}\right\}}{E\left\{R_a^{-\gamma}\right\}}$ from Lemma 2 to obtain $R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} = R_f\left[\frac{E\left\{R_a^{-\gamma}\right\}}{E\left\{R_a^{-\gamma-1}\right\}}\right]^{-1} = R_f\left[E\left\{\frac{R_a^{-\gamma-1}}{E\left\{R_a^{-\gamma-1}\right\}}R_a\right\}\right]^{-1}$. Since $E\left\{\frac{R_a^{-\gamma-1}}{E\left\{R_a^{-\gamma-1}\right\}}R_a\right\}$ is a weighted average of $R_a$, it is greater than or equal to the minimum possible realization of $R_a$, and since $0 \leq \lambda \leq 1$, $E\left\{\frac{R_a^{-\gamma-1}}{E\left\{R_a^{-\gamma-1}\right\}}R_a\right\} \geq \lambda R_f + (1-\lambda)R_{\min}$. Therefore, since $R_f \leq 1$, we have $R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} \leq R_f\frac{1}{\lambda R_f+(1-\lambda)R_{\min}} \leq \frac{1}{\lambda+(1-\lambda)R_{\min}}$. (Note that since $R_{\min} \leq 1$, the bound $\frac{1}{\lambda+(1-\lambda)R_{\min}}$ is greater than one.) Therefore, $\frac{\lambda}{1-\lambda}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} - 1\right) \leq$

$\frac{\lambda}{1-\lambda}\left(\frac{1}{\lambda+(1-\lambda)R_{\min}}-1\right)=\frac{\lambda}{1-\lambda}\left(\frac{1-\lambda-(1-\lambda)R_{\min}}{\lambda+(1-\lambda)R_{\min}}\right)=\frac{\lambda}{\lambda+(1-\lambda)R_{\min}}\left(1-R_{\min}\right)=\frac{1}{1+\frac{1-\lambda}{\lambda}R_{\min}}\left(1-R_{\min}\right)$ $=\frac{1}{1+\mathcal{B}^{-1}R_{\min}}\left(1-R_{\min}\right)<1-R_{\min}$. Hence, $\frac{\lambda}{1-\lambda}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1\right)<1-R_{\min}$, since $\mathcal{B}\geq 0$ and $R_{\min}\geq 0$. ■

**Lemma 7** *Suppose $x>0$ is a non-degenerate random variable with finite moments. Define $Z\equiv\left[E\left\{x^{-\gamma}\right\}\right]^2-E\left\{x^{-\gamma-1}\right\}E\left\{x^{1-\gamma}\right\}$. Then $Z<0$.*

**Proof of Lemma 7.** Rewrite $Z$ as $Z=E\left\{x^{-\gamma}\right\}E\left\{x^{-\gamma-1}\right\}\left(\frac{E\left\{x^{-\gamma}\right\}}{E\left\{x^{-\gamma-1}\right\}}-\frac{E\left\{x^{1-\gamma}\right\}}{E\left\{x^{-\gamma}\right\}}\right)$ Observe that $Z$ has the same sign as $\frac{E\left\{x^{-\gamma-1}x\right\}}{E\left\{x^{-\gamma-1}\right\}}-\frac{E\left\{x^{-\gamma}x\right\}}{E\left\{x^{-\gamma}\right\}}$, which we write as $E\left\{g\left(x\right)x\right\}-E\left\{h\left(x\right)x\right\}$, where $g\left(x\right)\equiv\frac{x^{-\gamma-1}}{E\left\{x^{-\gamma-1}\right\}}$ and $h\left(x\right)\equiv\frac{x^{-\gamma}}{E\left\{x^{-\gamma}\right\}}$. Observe that $E\left\{g\left(x\right)\right\}=1=E\left\{h\left(x\right)\right\}$ and $\frac{g(x)}{h(x)}=\frac{E\left\{x^{-\gamma}\right\}}{E\left\{x^{-\gamma-1}\right\}}\frac{1}{x}=A\frac{1}{x}$, so $g\left(x\right)=A\frac{1}{x}h\left(x\right)$ where $A\equiv\frac{E\left\{x^{-\gamma}\right\}}{E\left\{x^{-\gamma-1}\right\}}>0$, which implies $g\left(x\right)\gtreqless h\left(x\right)$ as $x\lesseqgtr A$. Therefore, $E\left\{g\left(x\right)x\right\}-E\left\{h\left(x\right)x\right\}=\int_0^\infty\left[g\left(x\right)-h\left(x\right)\right]xdF\left(x\right)$ $=\int_0^\infty\left[g\left(x\right)-h\left(x\right)\right]\left(x-A\right)dF\left(x\right)=\int_0^A\left[g\left(x\right)-h\left(x\right)\right]\left(x-A\right)dF\left(x\right)+\int_A^\infty\left[g\left(x\right)-h\left(x\right)\right]\times$ $\left(x-A\right)dF\left(x\right)<0$. ■

We present, without proof, the following corollary to Lemma 2, which follows immediately from Lemma 7 .

**Corollary 4** $E\left\{R_a^{-\gamma}\right\}-R_fE\left\{R_a^{-\gamma-1}\right\}=\frac{1}{E\left\{R_a^{-\gamma}\right\}}\left[\left(E\left\{R_a^{-\gamma}\right\}\right)^2-E\left\{R_a^{1-\gamma}\right\}\left\{R_a^{-\gamma-1}\right\}\right]<0.$

Having presented Lemmas 6 and 7 and Corollary 4, we now proceed with the proof of Proposition 1.

The portfolio allocation problem of individual consumers is $\max_\lambda\frac{1}{1-\gamma}E\left\{R_a^{1-\gamma}\right\}$, where $R_a\equiv\lambda R_f+\left(1-\lambda\right)\left(\overline{R}+\varepsilon\right)$. The first-order condition is

$$f\left(\lambda,R_f,\overline{R}\right)\equiv E\left\{R_a^{-\gamma}\left[R_f-\left(\overline{R}+\varepsilon\right)\right]\right\}=0. \tag{A.1}$$

The optimization problem is concave in $\lambda$ so $f_\lambda<0$. Formally,

$$f_\lambda\left(\lambda,R_f,\overline{R}\right)=-\gamma E\left\{R_a^{-\gamma-1}\left[R_f-\left(\overline{R}+\varepsilon\right)\right]^2\right\}<0. \tag{A.2}$$

Differentiate $f\left(\lambda,R_f,\overline{R}\right)$ with respect to $\overline{R}$ to obtain $f_{\overline{R}}\left(\lambda,R_f,\overline{R}\right)=-\gamma E\left\{R_a^{-\gamma-1}\left(1-\lambda\right)\times\right.$ $\left.\left(R_f-\left(\overline{R}+\varepsilon\right)\right)\right\}-E\left\{R_a^{-\gamma}\right\}=\gamma E\left\{R_a^{-\gamma-1}\left(1-\lambda\right)\left(\overline{R}+\varepsilon-R_f\right)\right\}-E\left\{R_a^{-\gamma}\right\}=\gamma E\left\{R_a^{-\gamma-1}\times\right.$

$\left[\lambda R_f + (1-\lambda)\left(\overline{R}+\varepsilon\right) - R_f\right]\} - E\left\{R_a^{-\gamma}\right\} = \gamma E\left\{R_a^{-\gamma-1}\left(R_a - R_f\right)\right\} - E\left\{R_a^{-\gamma}\right\}$. Therefore,

$$f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right) = \gamma\left[E\left\{R_a^{-\gamma}\right\} - R_f E\left\{R_a^{-\gamma-1}\right\}\right] - E\left\{R_a^{-\gamma}\right\} < 0, \tag{A.3}$$

where the inequality follows from Corollary 4.

Now differentiate $f\left(\lambda, R_f, \overline{R}\right)$ with respect to $R_f$ to obtain

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right) = -\gamma E\left\{R_a^{-\gamma-1}\lambda\left[R_f - \left(\overline{R}+\varepsilon\right)\right]\right\} + E\left\{R_a^{-\gamma}\right\}. \tag{A.4}$$

Observe that $\lambda\left[R_f - \left(\overline{R}+\varepsilon\right)\right] = \frac{\lambda}{\lambda-1}\left[\left(\lambda-1\right)R_f - \left(\lambda-1\right)\left(\overline{R}+\varepsilon\right)\right] = \frac{\lambda}{\lambda-1}\left(R_a - R_f\right)$, so that equation (A.4) can be written as $f_{R_f}\left(\lambda, R_f, \overline{R}\right) = -\gamma E\left\{R_a^{-\gamma-1}\frac{\lambda}{\lambda-1}\left(R_a - R_f\right)\right\} + E\left\{R_a^{-\gamma}\right\}$, so

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right) = \gamma\frac{\lambda}{1-\lambda}E\left\{R_a^{-\gamma} - R_f R_a^{-\gamma-1}\right\} + E\left\{R_a^{-\gamma}\right\}. \tag{A.5}$$

Now multiply the right hand side of equation (A.5) by $E\left\{R_a^{-\gamma}\right\}$ and divide each term by $E\left\{R_a^{-\gamma}\right\}$ to obtain $f_{R_f}\left(\lambda, R_f, \overline{R}\right) = E\left\{R_a^{-\gamma}\right\}\left[\gamma\frac{\lambda}{1-\lambda}\frac{E\left\{R_a^{-\gamma}\right\} - R_f E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} + 1\right]$, which implies

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right) = E\left\{R_a^{-\gamma}\right\}\left[1 - \gamma\frac{\lambda}{1-\lambda}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} - 1\right)\right]. \tag{A.6}$$

Lemma 6 implies that

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right) \geq E\left\{R_a^{-\gamma}\right\}\left[1 - \gamma\left(1 - R_{\min}\right)\right], \text{ if } R_f \leq 1. \tag{A.7}$$

Therefore,

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right) \geq 0 \text{ if } \gamma \leq \frac{1}{1-R_{\min}} \text{ and } R_f \leq 1. \tag{A.8}$$

Now totally differentiate $f\left(\lambda, R_f, \overline{R}\right) = 0$ in equation (A.1) with respect to $\mathcal{B}$ to obtain $f_\lambda\left(\lambda, R_f, \overline{R}\right)\frac{d\lambda}{d\mathcal{B}} + f_{R_f}\left(\lambda, R_f, \overline{R}\right)\frac{dR_f}{d\mathcal{B}} + f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\left[\frac{d\overline{R}}{d\mathcal{B}} + \frac{d\overline{R}}{dR_f}\frac{dR_f}{d\mathcal{B}}\right] = 0$, which can rear-

ranged as

$$f_{\lambda}\left(\lambda, R_f, \overline{R}\right)\frac{d\lambda}{d\mathcal{B}}+\left[f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f}\right]\frac{dR_f}{d\mathcal{B}}+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{d\mathcal{B}}=0 \quad \text{(A.9)}$$

Since, in equilibrium, $\lambda=\frac{B}{K+B}=\frac{\mathcal{B}}{1+\mathcal{B}}$, we have $\frac{d\lambda}{d\mathcal{B}}=\frac{1}{(1+\mathcal{B})^2}>0$ and since $f_{\lambda}\left(\lambda, R_f, \overline{R}\right)<0$ from equation (A.2), the first of the three terms in equation (A.9), $f_{\lambda}\left(\lambda, R_f, \overline{R}\right)\frac{d\lambda}{d\mathcal{B}}$, is negative. Equation (18) implies that $\frac{d\overline{R}}{d\mathcal{B}}=\frac{\alpha}{(1-\alpha)\beta}\left[1-\beta\zeta\left(1-R_f\right)\right]>0$ and since $f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)<0$ from equation (A.3), the third of the three terms in equation (A.9), $f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{d\mathcal{B}}$, is also negative. Therefore, the second of the three terms in this equation, $\left[f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f}\right]\frac{dR_f}{d\mathcal{B}}$, is positive. Thus, to derive sufficient conditions for $\frac{dR_f}{d\mathcal{B}}>0$, it suffices to derive sufficient conditions under which $f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f}>0$.

Use equations (A.3) and (A.5), along with equation (18), which implies $\frac{d\overline{R}}{dR_f}=\frac{\alpha}{1-\alpha}\zeta\mathcal{B}$, to obtain

$$\begin{aligned} f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f} &= \gamma\frac{\lambda}{1-\lambda}E\left\{R_a^{-\gamma}-R_fR_a^{-\gamma-1}\right\}+E\left\{R_a^{-\gamma}\right\} \quad \text{(A.10)} \\ &\quad +\left[\gamma E\left\{R_a^{-\gamma}-R_fR_a^{-\gamma-1}\right\}-E\left\{R_a^{-\gamma}\right\}\right]\frac{\alpha}{1-\alpha}\zeta\mathcal{B}. \end{aligned}$$

Rearrange equation (A.10) and use $\mathcal{B}=\frac{\lambda}{1-\lambda}$ to obtain

$$\begin{aligned} f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f} &= \gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\mathcal{B}E\left\{R_a^{-\gamma}-R_fR_a^{-\gamma-1}\right\} \quad \text{(A.11)} \\ &\quad +\left(1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}\right)E\left\{R_a^{-\gamma}\right\}, \end{aligned}$$

and rearrange further to obtain

$$f_{R_f}\left(\lambda, R_f, \overline{R}\right)+f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f}=E\left\{R_a^{-\gamma}\right\}\left[\gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\mathcal{B}\left(1-R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}\right)+1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}\right]. \quad \text{(A.12)}$$

Equation (A.12) implies $f_{R_f}\left(\lambda, R_f, \overline{R}\right) + f_{\overline{R}}\left(\lambda, R_f, \overline{R}\right)\frac{d\overline{R}}{dR_f} > 0$ if and only if

$$\gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\mathcal{B}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1\right) < 1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}. \tag{A.13}$$

Since $\mathcal{B} = \frac{\lambda}{1-\lambda}$, Lemma 6 implies that the condition in equation (A.13) will be satisfied if $R_f \leq 1$ and $\gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)(1-R_{\min}) < 1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}$, or equivalently,

$$\gamma < \frac{1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}}{1+\frac{\alpha}{1-\alpha}\zeta}\frac{1}{1-R_{\min}} \tag{A.14}$$

If $R_f \leq 1$, then $1 \geq \frac{\alpha}{(1-\alpha)\beta}\left[1+\mathcal{B}-\beta\zeta\left(1-R_f\right)\mathcal{B}\right]+\left(1-\delta+\widetilde{\varepsilon}_{\min}\right)G^{-1} = R_{\min}+\frac{\alpha}{(1-\alpha)\beta}\times$ $\left[1-\beta\zeta\left(1-R_f\right)\right]\mathcal{B} \geq R_{\min}+\frac{\alpha}{(1-\alpha)\beta}\left(1-\beta\zeta\right)\mathcal{B}$, so $\frac{\alpha}{(1-\alpha)\beta}\left(1-\beta\zeta\right)\mathcal{B} \leq 1-R_{\min}$, or, equivalently, $\frac{\alpha}{1-\alpha}\zeta\mathcal{B} \leq \frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\min}\right)$. Therefore, $1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B} \geq 1-\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\min}\right)$, so the condition in equation (A.14) will be satisfied if

$$\gamma < \frac{1-\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\min}\right)}{1+\frac{\alpha}{1-\alpha}\zeta}\frac{1}{1-R_{\min}}. \tag{A.15}$$

The upper bound on $\gamma$ on the right hand side of equation (A.15) is greater than 1 if $1-\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\min}\right) > \left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left(1-R_{\min}\right)$, that is, if $1 > \left[1+\frac{\alpha}{1-\alpha}\zeta+\frac{\beta\zeta}{1-\beta\zeta}\right]\left(1-R_{\min}\right)$. The most unfavorable values of $\beta$ and $\zeta$ for this condition are $\beta = 0.5$ (which is the maximum value of $\beta$ consistent with non-negative time preference) and $\zeta = 1$, in which case the condition becomes $1 > \left(2+\frac{\alpha}{1-\alpha}\right)\left(1-R_{\min}\right)$, which for $\alpha = \frac{1}{3}$ becomes $1 > \frac{5}{2}\left(1-R_{\min}\right)$ or equivalently, $1-R_{\min} < 0.4$, or $R_{\min} > 0.6$. ■

**Proof of Lemma 3.** It suffices to show that $E\left\{R_a^{\phi}\right\}$ is decreasing in $\phi$ for $\phi < 0$. $\frac{dE\left\{R_a^{\phi}\right\}}{d\phi} = \frac{dE\left\{\exp^{\phi\ln R_a}\right\}}{d\phi} = E\left\{\left(\ln R_a\right)R_a^{\phi}\right\} = E\left\{\ln R_a\right\}E\left\{R_a^{\phi}\right\} + cov\left(\ln R_a, R_a^{\phi}\right) \leq cov\left(\ln R_a, R_a^{\phi}\right) < 0$, since $E\{\ln R_a\} \leq 0$ by assumption. ■

**Proof of Proposition 2.** Since $\mathcal{B} = 0$, $R \equiv R_a$, and so Statement 1 follows directly from Lemmas 2 and 3. Statement 2 follows directly from Statement 1. ■

**Proof of Proposition 3.** Let $\mathcal{B}_1$ be an arbitrary non-negative value of $\mathcal{B}$ for which $R_f\left(\mathcal{B}\right) > 1$ and let $\mathcal{B}_2$ be the smallest value of $\mathcal{B}$ greater than $\mathcal{B}_1$ for which $R_f\left(\mathcal{B}\right) = 1$.

Therefore, $R_f'(\mathcal{B}_2) \leq 0$ and $R_f(\mathcal{B}_2) \leq 1$, which contradicts the assumption that $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$. Therefore, $R_f(\mathcal{B}) > 1$ for all $\mathcal{B} \geq \mathcal{B}_1$. We will use this result to prove the statements in this proposition. Statement 1: First, if $R_f(0) > 1$, then $R_f(\mathcal{B}) > 1$ for all positive $\mathcal{B}$, so all positive values of $\mathcal{B}$ are unsustainable. If $R_f(0) = 1$, then $R_f(\varepsilon) > 1$ in a positive neighborhood of $\varepsilon = 0$, and therefore, $R_f > 1$ for all positive $\mathcal{B}$, so all positive values of $\mathcal{B}$ are unsustainable. Therefore, if $R_f(0) \geq 1$, then $\mathcal{B}_{\max} = 0$. Statement 2: Assume that $R_f(0) < 1$. Let $\mathcal{B}_0 > 0$ be the smallest positive $\mathcal{B}$ for which $R_f(\mathcal{B}) = 1$, so $\mathcal{B}_0$ is sustainable. Using the result above in this proof for all values of $\mathcal{B} > \mathcal{B}_0$, $R_f(\mathcal{B}) > 1$, and hence these values of $\mathcal{B}$ are unsustainable. Therefore, $\mathcal{B}_{\max}$ is the unique root of $R_f(\mathcal{B}) = 1$. To show that $\mathcal{B}_{\max}$ is finite, suppose that $\mathcal{B} > \frac{1-R_{\min}}{\frac{\alpha}{(1-\alpha)\beta}(1-\beta\zeta)} > 0$. Then the smallest risky rate of return $\min_{\widetilde{\varepsilon}} R \geq R_{\min} + \frac{\alpha}{(1-\alpha)\beta}[1 - \beta\zeta + \beta\zeta R_f]\mathcal{B} \geq R_{\min} + \frac{\alpha}{(1-\alpha)\beta}[1-\beta\zeta]\mathcal{B} > R_{\min} + \frac{\alpha}{(1-\alpha)\beta}(1-\beta\zeta)\frac{1-R_{\min}}{\frac{\alpha}{(1-\alpha)\beta}(1-\beta\zeta)} = 1$. Therefore, if the distribution of the durability shock, $\widetilde{\varepsilon}$, is non-degenerate, $R_f$ must exceed one because of the absence of arbitrage opportunities. Therefore, $R_f$ can be less than or equal to one only if $\mathcal{B} \leq \frac{1-R_{\min}}{\frac{\alpha}{(1-\alpha)\beta}(1-\beta\zeta)}$, which is finite, so $\mathcal{B}_{\max}$ is finite. Finally, since there is a unique value of $\mathcal{B}$ for which $R_f(\mathcal{B}) = 1$, we have $R_f(\mathcal{B}) < 1$ for $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$ and all of these values of $\mathcal{B}$ are sustainable; $R_f(\mathcal{B}) > 1$ for all $\mathcal{B} > \mathcal{B}_{\max}$ and these values are unsustainable, so $\mathcal{B}$ is sustainable if and only if $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$.

■

**Lemma 8** *Define* $\Gamma \equiv G - \beta\zeta(g - r_f)$. *If* $\frac{dR_f}{d\mathcal{B}} > 0$, *then*

1. $\Gamma > 0$ *and* $\frac{d\Gamma}{d\mathcal{B}} = G\beta\zeta\frac{dR_f}{d\mathcal{B}}$
2. $\frac{d\ln w}{d\mathcal{B}} = -\frac{\alpha}{1-\alpha}\frac{\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}}{G+\Gamma\mathcal{B}} < 0$
3. $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E_t\{R_a^{1-\gamma}\}\right) = \frac{1}{R_f}\left(\beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}} + \frac{1}{1+\mathcal{B}}\frac{\alpha}{1-\alpha}G^{-1}\left[\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]\right) > 0$
4. $\frac{d}{d\mathcal{B}}\ln\left(1 + \frac{\tau_t}{W_t}\right) = \frac{\beta\zeta}{G+\Gamma\mathcal{B}}\left[\frac{g-r_f}{1+\mathcal{B}} - G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$

**Proof of Lemma 8.** Proof of Statement 1: $\Gamma \equiv G - \beta\zeta(g - r_f) = 1 + g - \beta\zeta g + \beta\zeta r_f = (1 + \beta\zeta r_f) + (1 - \beta\zeta)g$. Since $0 < \beta < 1$, $0 \leq \zeta \leq 1$, and $r_f > -1$, we have $1 + \beta\zeta r_f > 0$, and since, in addition, $g \geq 0$, $(1 - \beta\zeta)g \geq 0$. Therefore, $\Gamma > 0$. Differentiate $\Gamma$ with respect to $\mathcal{B}$ to obtain $\frac{d\Gamma}{d\mathcal{B}} = \beta\zeta\frac{dr_f}{d\mathcal{B}} = G\beta\zeta\frac{dR_f}{d\mathcal{B}}$.

Proof of Statement 2: (1) Since $w = (1-\alpha)\, k^{\alpha}$, we have $\frac{dw}{dk} = \alpha \frac{w}{k}$ and hence $\frac{1}{w}\frac{dw}{dk} = \alpha \frac{1}{k}$; (2) equation (15) implies $k^{a-1} = \frac{1}{(1-\alpha)\beta}\left[(1+\mathcal{B})\, G - \beta\zeta\,(g - r_f)\,\mathcal{B}\right] = \frac{1}{(1-\alpha)\beta}\,(G + \Gamma\mathcal{B})$, so $k = \left[\frac{1}{(1-\alpha)\beta}\,(G+\Gamma\mathcal{B})\right]^{\frac{1}{\alpha-1}}$ and hence $\frac{dk}{d\mathcal{B}} = \frac{1}{\alpha-1} k \frac{\Gamma + \mathcal{B}\frac{d\Gamma}{d\mathcal{B}}}{G+\Gamma\mathcal{B}} = \frac{1}{\alpha-1} k \frac{\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}}{G+\Gamma\mathcal{B}}$. Therefore, $\frac{d\ln w}{d\mathcal{B}}$ $= \frac{1}{w}\frac{dw}{dk}\frac{dk}{d\mathcal{B}} = -\frac{\alpha}{1-\alpha}\frac{\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}}{G+\Gamma\mathcal{B}}$.

Proof of Statement 3: Use $R_a \equiv \lambda R_f + (1-\lambda)\, R$ to obtain $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E_t\left\{R_a^{1-\gamma}\right\}\right) = \frac{\beta}{1-\gamma}\frac{1}{E_t\left\{R_a^{1-\gamma}\right\}}(1-\gamma)\, E_t\left\{R_a^{-\gamma}\left(\lambda\frac{dR_f}{d\mathcal{B}} + (1-\lambda)\frac{dR}{d\mathcal{B}}\right)\right\}$, where we have used the envelope theorem to ignore $E_t\left\{R_a^{-\gamma}\left(\frac{d\lambda}{d\mathcal{B}}R_f - \frac{d\lambda}{d\mathcal{B}}R\right)\right\} = 0$. Use the following: (1) $\frac{dR_f}{d\mathcal{B}}$ is non-random; (2) equations (17) and (18) imply $\frac{dR}{d\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}G^{-1}\left[G - \beta\zeta\,(g-r_f) + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$, which is non-random; (3) Lemma 2, which implies $\frac{E\left\{R_a^{-\gamma}\right\}}{E\left\{R_a^{1-\gamma}\right\}} = \frac{1}{R_f}$; and (4) $\lambda = \frac{\mathcal{B}}{1+\mathcal{B}}$ to obtain $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E_t\left\{R_a^{1-\gamma}\right\}\right) = \frac{1}{R_f}\left(\beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}} + \frac{1}{1+\mathcal{B}}\frac{\alpha}{1-\alpha}G^{-1}\left[\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]\right)$.

Proof of Statement 4: Along a balanced growth path, $\frac{\tau_t}{W_t} = \frac{\beta N\tau_t}{\beta N W_t} = \frac{\beta\zeta\left(g-r_f\right)B_t}{\beta N W_t} = \frac{\beta\zeta\left(g-r_f\right)B_t}{\beta N(W_t+\tau_t)}\frac{\beta N(W_t+\tau_t)}{\beta N W_t} = \frac{\beta\zeta\left(g-r_f\right)B_t}{K_{t+1}+B_{t+1}}\left(1+\frac{\tau_t}{W_t}\right) = \beta\zeta\,(g-r_f)\,G^{-1}\frac{\mathcal{B}}{1+\mathcal{B}}\left(1+\frac{\tau_t}{W_t}\right)$, where the final equality uses the fact that along a balanced growth path, $B_t = G^{-1}B_{t+1}$, so $\frac{B_t}{K_{t+1}+B_{t+1}}$ $= \frac{G^{-1}B_{t+1}}{K_{t+1}+B_{t+1}} = G^{-1}\frac{\mathcal{B}}{1+\mathcal{B}}$. Define $X_t \equiv 1 + \frac{\tau_t}{W_t}$, so along a balanced growth path we have $X - 1 = \beta\zeta\,(g-r_f)\,G^{-1}\frac{\mathcal{B}}{1+\mathcal{B}}X$, which implies $X = \left(1 - \beta\zeta\,(g-r_f)\,G^{-1}\frac{\mathcal{B}}{1+\mathcal{B}}\right)^{-1} = \left(G^{-1}\frac{1}{1+\mathcal{B}}\left(G\,(1+\mathcal{B}) - \beta\zeta\,(g-r_f)\,\mathcal{B}\right)\right)^{-1} = G\,(1+\mathcal{B})\,(G+\Gamma\mathcal{B})^{-1}$. Therefore, $\ln X = \ln G + \ln(1+\mathcal{B}) - \ln(G+\Gamma\mathcal{B})$. Now differentiate $\ln X$ with respect to $\mathcal{B}$ to obtain $\frac{d\ln X}{d\mathcal{B}} = \frac{1}{1+\mathcal{B}} - \frac{1}{G+\Gamma\mathcal{B}}\left(\Gamma + \mathcal{B}\frac{d\Gamma}{d\mathcal{B}}\right) = \frac{1}{G+\Gamma\mathcal{B}}\left[\frac{G+\Gamma\mathcal{B}}{1+\mathcal{B}} - \left(\Gamma + G\mathcal{B}\beta\zeta\frac{dR_f}{d\mathcal{B}}\right)\right] = \frac{1}{G+\Gamma\mathcal{B}}\left[\frac{G-\Gamma}{1+\mathcal{B}} - G\mathcal{B}\beta\zeta\frac{dR_f}{d\mathcal{B}}\right]$. Use $\Gamma \equiv G - \beta\zeta\,(g-r_f)$ to obtain $G - \Gamma = \beta\zeta\,(g-r_f)$, so $\frac{d\ln X}{d\mathcal{B}} = \frac{1}{G+\Gamma\mathcal{B}}\left[\frac{\beta\zeta\left(g-r_f\right)}{1+\mathcal{B}} - G\mathcal{B}\beta\zeta\frac{dR_f}{d\mathcal{B}}\right]$ $= \frac{\beta\zeta}{G+\Gamma\mathcal{B}}\left[\frac{g-r_f}{1+\mathcal{B}} - G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$. ∎

**Proof of Proposition 4.** Assume that $\frac{dR_f}{d\mathcal{B}} > 0$. Differentiate equation (25) with respect to $\mathcal{B}$ and use Statements 2, 3, and 4 of Lemma 8 to obtain $\frac{du}{d\mathcal{B}} = D_1 + D_2 + D_3$, where $D_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}}{G+\Gamma\mathcal{B}} + \frac{1}{R_f}\frac{1}{1+\mathcal{B}}\frac{\alpha}{1-\alpha}G^{-1}\left[\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$ and $D_2 \equiv \frac{1}{R_f}\beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}} - \frac{G\beta\zeta}{G+\Gamma\mathcal{B}}\mathcal{B}\frac{dR_f}{d\mathcal{B}}$ and $D_3 \equiv \frac{\beta\zeta}{G+\Gamma\mathcal{B}}\frac{g-r_f}{1+\mathcal{B}}$. First calculate $D_1 = \left(-\frac{1}{G+\Gamma\mathcal{B}} + \frac{1}{R_f}\frac{1}{1+\mathcal{B}}G^{-1}\right)\frac{\alpha}{1-\alpha}\left[\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$. Now we will rearrange $-\frac{1}{G+\Gamma\mathcal{B}} + \frac{1}{R_f}\frac{1}{1+\mathcal{B}}G^{-1} = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}G^{-1}\left(-GR_f + \frac{G+\Gamma\mathcal{B}}{1+\mathcal{B}}\right) = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}G^{-1}\times$ $\left((1-R_f)\,G - \frac{\beta\zeta\left(g-r_f\right)\mathcal{B}}{1+\mathcal{B}}\right) = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}G^{-1}\times\left(1 - \beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)(g-r_f)$, where the penultimate equality uses the definition $\Gamma \equiv G - \beta\zeta\,(g-r_f)$ and the final equality uses the definition $R_f \equiv \frac{1+r_f}{G}$. Therefore, $D_1 = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}G^{-1}\left(1 - \beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\frac{\alpha}{1-\alpha}\times\left[\Gamma + G\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right](g-r_f)$, which

has the same sign as $g-r_f$, so $D_1 \geq 0$, if $g \geq r_f$. Now consider $D_2 \equiv \frac{1}{R_f}\beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}} - \frac{G\beta\zeta}{G+\Gamma\mathcal{B}}\mathcal{B}\frac{dR_f}{d\mathcal{B}}$ $= \left[\frac{1}{R_f}\frac{1}{1+\mathcal{B}} - \frac{G\zeta}{G+\Gamma\mathcal{B}}\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}} = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}\left[\frac{G+\Gamma\mathcal{B}}{1+\mathcal{B}} - G\zeta R_f\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}}$ and use $\Gamma \equiv G - \beta\zeta\left(g-r_f\right)$ to obtain

$$\begin{aligned} D_2 &= \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}\left[G - G\zeta R_f - \frac{\beta\zeta\left(g-r_f\right)\mathcal{B}}{1+\mathcal{B}}\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}} \\ &= \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}\left[G + GR_f - GR_f - G\zeta R_f - \frac{\beta\zeta\left(g-r_f\right)\mathcal{B}}{1+\mathcal{B}}\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}} \\ &= \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}\left[G - GR_f + \left(1-\zeta\right)GR_f - \frac{\beta\zeta\left(g-r_f\right)\mathcal{B}}{1+\mathcal{B}}\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}}. \end{aligned}$$

Now use $G - GR_f = 1 + g - \left(1+r_f\right) = g - r_f$ to obtain $D_2 = \frac{1}{R_f}\frac{1}{G+\Gamma\mathcal{B}}\left[\left(1-\zeta\right)GR_f + \left(1 - \beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\left(g-r_f\right)\right]\beta\mathcal{B}\frac{dR_f}{d\mathcal{B}}$, so $D_2 > 0$, if $g \geq r_f$. By inspection, $D_3 \equiv \frac{\beta\zeta}{G+\Gamma\mathcal{B}}\frac{g-r_f}{1+\mathcal{B}} \geq 0$, if if $g \geq r_f$. Therefore, $\frac{du}{d\mathcal{B}} = D_1 + D_2 + D_3 > 0$, if $g \geq r_f$. ■

**Proof of Corollary 3.** Assume that $\zeta = 0$, which implies that $\Gamma \equiv G - \beta\zeta\left(g-r_f\right) = G$. Therefore, Statements 2, 3, and 4 of Lemma 8 imply $\frac{du}{d\mathcal{B}} = -\frac{\alpha}{1-\alpha}\frac{1}{1+\mathcal{B}} + \frac{1}{R_f}\left[\beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}} + \frac{1}{1+\mathcal{B}}\frac{\alpha}{1-\alpha}\right]$ $= \frac{1}{1+\mathcal{B}}\frac{1}{R_f}\left[\left(1-R_f\right)\frac{\alpha}{1-\alpha} + \beta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right] > 0.$ ■

**Proof of Proposition 5.** See the discussion in the text. ■

**Proof of Lemma 4.** An economy is dynamically efficient if and only if $R_H R_L \geq 1$, equivalently, $\left(\overline{R}(\mathcal{B},\zeta,R_f) + G^{-1}\sigma\right)\left(\overline{R}(\mathcal{B},\zeta,R_f) - G^{-1}\sigma\right) \geq 1$, equivalently, $\left[\overline{R}(\mathcal{B},\zeta,R_f)\right]^2 - G^{-2}\sigma^2 \geq 1$, equivalently, $G^{-2}\sigma^2 \leq \left[\overline{R}(\mathcal{B},\zeta,R_f)\right]^2 - 1$. ■

**Proof of Lemma 5.** Assume that $\gamma = 1$. Lemma 2 implies $R_f = \left[E\left\{\frac{1}{R_a}\right\}\right]^{-1} = \left[E\left\{\frac{1}{\lambda R_f + (1-\lambda)R}\right\}\right]^{-1}$, so $R_f E\left\{\frac{1}{\lambda R_f + (1-\lambda)R}\right\} = 1$. With $\Pr\left\{R = \overline{R} + G^{-1}\sigma\right\} = \Pr\left\{R = \overline{R} - G^{-1}\sigma\right\} = \frac{1}{2}$, $1 = \frac{1}{2}R_f\left[\frac{1}{\lambda R_f + (1-\lambda)\left(\overline{R}+G^{-1}\sigma\right)} + \frac{1}{\lambda R_f + (1-\lambda)\left(\overline{R}-G^{-1}\sigma\right)}\right] = R_f\frac{1}{\lambda R_f + (1-\lambda)\left(\overline{R}+G^{-1}\sigma\right)}\frac{1}{\lambda R_f + (1-\lambda)\left(\overline{R}-G^{-1}\sigma\right)}\times$ $\left[\lambda R_f + (1-\lambda)\overline{R}\right]$, which implies $R_f\left[\lambda R_f + (1-\lambda)\overline{R}\right] = \left[\lambda R_f + (1-\lambda)\left(\overline{R}+G^{-1}\sigma\right)\right]\times$ $\left[\lambda R_f + (1-\lambda)\left(\overline{R}-G^{-1}\sigma\right)\right] = \left[\lambda R_f + (1-\lambda)\overline{R} + (1-\lambda)G^{-1}\sigma\right]\left[\lambda R_f + (1-\lambda)\overline{R} - (1-\lambda)G^{-1}\sigma\right]$ $= \left[\lambda R_f + (1-\lambda)\overline{R}\right]^2 - (1-\lambda)^2 G^{-2}\sigma^2$. In equilibrium, $\lambda = \frac{\mathcal{B}}{1+\mathcal{B}}$ and $1-\lambda = \frac{1}{1+\mathcal{B}}$, so $R_f\left(\frac{\mathcal{B}}{1+\mathcal{B}}R_f + \frac{1}{1+\mathcal{B}}\overline{R}\right) = \left(\frac{\mathcal{B}}{1+\mathcal{B}}R_f + \frac{1}{1+\mathcal{B}}\overline{R}\right)^2 - \left(\frac{1}{1+\mathcal{B}}\right)^2 G^{-2}\sigma^2$. Multiply both sides of this equation by $(1+\mathcal{B})^2$ to obtain $(1+\mathcal{B})R_f\left(\mathcal{B}R_f + \overline{R}\right) = \left(\mathcal{B}R_f + \overline{R}\right)^2 - G^{-2}\sigma^2$, which, since $\zeta = 0$, can be rearranged to obtain $\phi\left(\mathcal{B}, R_f, G^{-2}\sigma^2\right) \equiv \left[\mathcal{B}R_f + \overline{R}\left(\mathcal{B},0,\cdot\right)\right]\left[\overline{R}\left(\mathcal{B},0,\cdot\right) - R_f\right] - G^{-2}\sigma^2 = 0$. The equation $\phi\left(\mathcal{B}, R_f, G^{-2}\sigma^2\right) = 0$ implicitly defines the combinations of $\mathcal{B}$ and

$G^{-2}\sigma^2$ consistent with any given $R_f$, so $\phi\left(\mathcal{B},1,G^{-2}\sigma^2\right)=\left[\mathcal{B}+\overline{R}\left(\mathcal{B},0,\cdot\right)\right]\left[\overline{R}\left(\mathcal{B},0,\cdot\right)-1\right]-G^{-2}\sigma^2=0$ implicitly defines the combinations of $\mathcal{B}$ and $G^{-2}\sigma^2$ consistent with $R_f=1$.

Observe from equation (18) that $\overline{R}\left(\mathcal{B},0,\cdot\right)\equiv\frac{\alpha}{(1-\alpha)\beta}\left(1+\mathcal{B}\right)+\left(1-\delta\right)G^{-1}$ so $\frac{\partial\overline{R}(\mathcal{B},0,\cdot)}{\partial\mathcal{B}}=\frac{\alpha}{(1-\alpha)\beta}>0$. Therefore, $\frac{\partial\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)}{\partial\mathcal{B}}=\left[R_f+\frac{\partial\overline{R}(\mathcal{B},0,\cdot)}{\partial\mathcal{B}}\right]\left[\overline{R}\left(\mathcal{B},0,\cdot\right)-R_f\right]+\left[\mathcal{B}R_f+\overline{R}\left(\mathcal{B},0,\cdot\right)\right]\times\frac{\partial\overline{R}(\mathcal{B},0,\cdot)}{\partial\mathcal{B}}>0$, since risk aversion implies $\overline{R}\left(\mathcal{B},0,\cdot\right)\geq R_f$; also, $\frac{\partial\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)}{\partial R_f}=\mathcal{B}\left[\overline{R}\left(\mathcal{B},0,\cdot\right)-R_f\right]-\left[\mathcal{B}R_f+\overline{R}\left(\mathcal{B},0,\cdot\right)\right]=-\left(1-\mathcal{B}\right)\overline{R}\left(\mathcal{B},0,\cdot\right)-2\mathcal{B}R_f<0$ if $0\leq\mathcal{B}\leq1$; and $\frac{\partial\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)}{\partial(G^{-2}\sigma^2)}=-1$. Since $\frac{\partial\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)}{\partial\mathcal{B}}>0$ and $\frac{\partial\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)}{\partial(G^{-2}\sigma^2)}<0$, the locus of $\mathcal{B}$ and $G^{-2}\sigma^2$ for which $R_f=1$, $\phi\left(\mathcal{B},1,G^{-2}\sigma^2\right)=0$ is upward sloping. For any value of $\mathcal{B}$ in the interval $[0,1]$, an increase in $G^{-2}\sigma^2$ reduces $\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)$ so $R_f$ must fall to increase $\phi\left(\mathcal{B},R_f,G^{-2}\sigma^2\right)$ back to zero. Thus, $R_f\lesseqgtr1$ as $G^{-2}\sigma^2\gtreqless\left[\mathcal{B}+\overline{R}\left(\mathcal{B},0,\cdot\right)\right]\left[\overline{R}\left(\mathcal{B},0,\cdot\right)-1\right]$. ■

**Proof of Proposition 6.** Lemma 5 implies that $R_f\lesseqgtr1$ as $G^{-2}\sigma^2\gtreqless h\left(\mathcal{B}\right)$. Lemma 4 implies that the steady state is dynamically efficient if and only if $G^{-2}\sigma^2\leq g\left(\mathcal{B}\right)$. ■

# B Properties of Figure 2

Figure 2 illustrates the case in which $\gamma=1,\mathcal{B}=0$, and the durability shock is drawn from a symmetric 2-point distribution. Lemma 2 implies that for any distribution of the durability shock, $\gamma=1$ implies $R_f=\frac{1}{E\{R_a^{-1}\}}$. When $\mathcal{B}=0$, we have $R_a=R$, and hence $R_f=\frac{1}{E\{R^{-1}\}}$, which is the harmonic mean of $R$. With a symmetric 2-point distribution, $R_f=\frac{1}{2}\left(\frac{1}{R_H}+\frac{1}{R_L}\right)$.

**Lemma 9** *If $R_H\geq1$, then $\frac{1}{2}\left(\frac{1}{R_H}+R_H\right)\geq1$, with strict inequality if $R_H>1$.*

**Proof of Lemma 9.** Observe that $\frac{1}{2}\left(\frac{1}{R_H}+R_H\right)\geq1$ if and only if $\frac{1}{R_H}+R_H-2\geq0$. Note that $\frac{1}{R_H}+R_H-2=\frac{1}{R_H}\left(1+R_H^2-2R_H\right)=\frac{1}{R_H}\left(R_H-1\right)^2\geq0$ if $R_H>0$, with strict inequality if $0<R_H\neq1$, which is sufficient to prove the lemma. ■

**Lemma 10** *Assume that $R_H\geq1$ and $R_L>0$. The function $R_f\left(R_H,R_L\right)\equiv\frac{1}{\frac{1}{2}\left(\frac{1}{R_H}+\frac{1}{R_L}\right)}$ has the following properties*

1. $R_f\left(1,1\right)=1$.

2. $R_f\left(R_H, \frac{1}{R_H}\right) = \frac{1}{\frac{1}{2}\left(\frac{1}{R_H}+R_H\right)} \leq 1$, *with strict inequality if* $R_H > 1$

3. $\frac{dR_L}{dR_H}|_{R_f(R_H,R_L)=1} = -\left(\frac{R_L}{R_H}\right)^2 < 0$

4. *the locus* $R_f(R_H, R_L) = 1$ *is convex*

5. *for* $R_H > 1$, *the locus* $R_f(R_H, R_L) = 1$ *lies above the* $E\{\ln R\} = 0$ *locus.*

**Proof of Lemma 10.** Statement 1 follows from inspection of the definition of $R_f(R_H, R_L)$. Statement 2: Inspection of the definition of $R_f(R_H, R_L)$ reveals that $R_f\left(R_H, \frac{1}{R_H}\right) = \frac{1}{\frac{1}{2}\left(\frac{1}{R_H}+R_H\right)}$ and Lemma 9 states that $\frac{1}{2}\left(\frac{1}{R_H}+R_H\right) \geq 1$, with strict inequality if $R_H > 1$, which suffices to prove Statement 2. Statement 3: To calculate the slope of the locus $R_f(R_H, R_L) = 1$ for $R_H \geq R_L$, rewrite $R_f(R_H, R_L)$ as $R_f(R_H, R_L) = \frac{2R_H R_L}{R_H+R_L}$ and calculate $\frac{\partial R_f(R_H,R_L)}{\partial R_H} = \frac{R_f}{R_H} - \frac{R_f}{R_H+R_L} = \frac{R_L}{R_H}\frac{1}{R_H+R_L}R_f > 0$ and $\frac{\partial R_f(R_H,R_L)}{\partial R_L} = \frac{R_f}{R_L} - \frac{R_f}{R_H+R_L} = \frac{R_H}{R_L}\frac{1}{R_H+R_L}R_f > 0$. Therefore, the slope of the locus $R_f(R_H, R_L) = 1$ is $\frac{dR_L}{dR_H}|_{R_f(R_H,R_L)=1}$ $= -\frac{\partial R_f(R_H,R_L)}{\partial R_H}\left[\frac{\partial R_f(R_H,R_L)}{\partial R_L}\right]^{-1} = -\frac{\frac{R_L}{R_H}\frac{1}{R_H+R_L}}{\frac{R_H}{R_L}\frac{1}{R_H+R_L}} = -\left(\frac{R_L}{R_H}\right)^2 < 0$. Statement 4: Starting at any point on the locus $R_f(R_H, R_L) = 1$, and moving down and to the right along the locus, $\frac{R_L}{R_H}$ decreases, so the absolute value of the slope of this locus decreases. That is, the $R_f(R_H, R_L) = 1$ locus is convex. Statement 5: For a given value of $R_H > 1$, the value of $R_L$ on the locus $E\{\ln R\} = 0$ is $R_L = \frac{1}{R_H}$. Thus, Statement 2 implies that $R_f(R_H, R_L) < 1$ for any point on the $E\{\ln R\} = 0$ locus with $R_H > 1$. From the proof of Statement 3, $R_f(R_H, R_L)$ is increasing in $R_L$. Therefore, to move from the $E\{\ln R\} = 0$ locus for a given $R_H > 1$ to the $R_f(R_H, R_L) = 1$ locus with the same value of $R_H$, $R_L$ must increase. Therefore, the $R_f(R_H, R_L) = 1$ lies above the $E\{\ln R\} = 0$ locus for any $R_H > 1$. ■

# C Calibration

Assume that a period equals 30 years. Since $R(\mathcal{B}) \equiv \frac{1+r(\mathcal{B})}{G}$ is the adjusted gross rate of return on capital, $1+r(\mathcal{B}) = G \times R(\mathcal{B})$ is the gross rate of return on capital (not adjusted for growth). Calibrate the distribution of the durability shock so that along a balanced growth path with a bond-capital ratio equal to $\mathcal{B}$, $E\{(1+r(\mathcal{B}))\} = G \times E\{R(\mathcal{B})\} = (1+m)^{30}$

where $m$ is the target annual net rate of return on capital; the standard deviation of $r(\mathcal{B})$, which is equal to the standard deviation of $G \times R(\mathcal{B})$, is set equal to its target $s\sqrt{30}$, where $s$ is the target annual standard deviation of the rate of return on capital.

We specify the distribution of the durability shock so that

$$G^{-1}\widetilde{\varepsilon} = \exp(z) - \overline{R}(0) \tag{C.1}$$

where $z$ is $N(\mu, \sigma^2)$, and $\overline{R}(0)$ is obtained by setting $\mathcal{B} = 0$ in equation (18). Our focus is on the quantitative value of $\mathcal{B}_{\max}$. Corollary 2 states that $\mathcal{B}_{\max}$ is invariant to $\zeta$, so, without loss of generality, we set $\zeta = 0$. From equations (17) and (18), the risky rate of return on capital along a balanced growth path with a given value of $\mathcal{B}$ is $R(\mathcal{B}) = \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}) + (1-\delta)G^{-1} + G^{-1}\widetilde{\varepsilon} = \left[\overline{R}(0) + \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}\right] + \left[\exp(z) - \overline{R}(0)\right]$, so

$$R(\mathcal{B}) = \frac{\alpha}{(1-\alpha)\beta}\mathcal{B} + \exp(z). \tag{C.2}$$

We choose the parameters $\mu$ and $\sigma^2$ so that the mean of the risky rate equals the target rate rate, which is $m$ on an annual basis, so

$$E\{1 + r(\mathcal{B})\} = G \times E\{R(\mathcal{B})\} = G \times E\left\{\frac{\alpha}{(1-\alpha)\beta}\mathcal{B} + \exp(z)\right\} = (1+m)^{30}. \tag{C.3}$$

Setting the standard deviation of the risky rate, $1 + r(\mathcal{B})$, which is $G\sqrt{Var\{\exp(z)\}}$, equal to the target standard deviation, $s\sqrt{30}$, implies

$$G\sqrt{Var\{\exp(z)\}} = s\sqrt{30}. \tag{C.4}$$

Use $Var\{(\exp(z)\} = (\exp(\sigma^2)-1)\exp(2\mu+\sigma^2) = (\exp(\sigma^2)-1)\left[\exp(\mu + \frac{1}{2}\sigma^2)\right]^2 = (\exp(\sigma^2)-1)\left[E\{\exp(z)\}\right]^2$ to rewrite equation (C.4) as

$$G \times E\{\exp(z)\}\sqrt{(\exp(\sigma^2) - 1)} = s\sqrt{30}. \tag{C.5}$$

From equation (C.3)

$$G \times E\left\{\exp\left(z\right)\right\} = \left(1+m\right)^{30} - G\frac{\alpha}{\left(1-\alpha\right)\beta}\mathcal{B}, \tag{C.6}$$

so equation (C.5) implies

$$\sqrt{\left(\exp(\sigma^2)-1\right)} = \frac{s\sqrt{30}}{\left(1+m\right)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}, \tag{C.7}$$

which can be rewritten as

$$\exp(\frac{1}{2}\sigma^2) = \sqrt{1+30\left(\frac{s}{\left(1+m\right)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}\right)^2}. \tag{C.8}$$

Substitute $\exp\left(\mu\right) \times \exp(\frac{1}{2}\sigma^2)$ for $E\left\{\exp\left(z\right)\right\}$ in equation (C.6) to obtain

$$G \times \exp\left(\mu\right) \times \exp(\frac{1}{2}\sigma^2) = \left(1+m\right)^{30} - G\frac{\alpha}{\left(1-\alpha\right)\beta}\mathcal{B}, \tag{C.9}$$

so

$$\exp\left(\mu\right) = \frac{G^{-1}\left(1+m\right)^{30} - \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}{\exp(\frac{1}{2}\sigma^2)} = \frac{G^{-1}\left(1+m\right)^{30} - \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}{\sqrt{1+30\left(\frac{s}{\left(1+m\right)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}\right)^2}}. \tag{C.10}$$

The mean, $m$, and standard deviation, $s$, of the rate of return on capital are expressed on an annual basis. To compare these values to familiar values for the mean and standard deviation of annual stock returns, we must take account of the fact that $m$ and $s$ are moments of unlevered rates of return, and the moments of stock returns are levered returns. Let $r^A$, $r_L^A$, and $r_f^A$ be rates of return on unlevered equity, levered equity and riskfree assets, all expressed at annual rates (hence the superscript $A$). They are related to each other by $r^A = \frac{D}{D+E}r_f^A + \frac{E}{D+E}r_L^A = \frac{D/E}{1+D/E}r_f^A + \frac{1}{1+D/E}r_L^A$, where $D$ is the debt owed by the private owners of capital and $E$ is the equity of these owners. Assume that $D$ equals 45% of $D+E$.

Therefore, $r^A = 0.45r_f^A + 0.55r_L^A$. Hence, if $E\left\{r^A\right\} = 3\%$ per year and $r_f^A = 0.6\%$ per year, then $E\left\{r_L^A\right\} = 4.96\%$ per year. Also, $sd\left\{r^A\right\} = \frac{1}{1+D/E} sd\left\{r_L^A\right\}$, $sd\left\{r_L^A\right\} = \frac{1}{0.55} sd\left\{r^A\right\}$, so if $sd\left\{r^A\right\} = 0.12$, then $sd\left\{r_L^A\right\} = 0.218$.

**Remark 2** *If the value of $\mu$ in equation (C.10) positive, then the economy with $\mathcal{B} = 0$ is dynamically efficient and hence the economy is dynamically efficient for any positive $\mathcal{B}$ also.*

To prove this remark, note that equation (C.2) implies that $R(0) = \exp(z)$. Since $R'(\mathcal{B}) > 0$, we have $R(\mathcal{B}) \geq R(0) = \exp(z)$ and hence $E\{\ln R(\mathcal{B})\} \geq E\{\ln R(0)\} = E\{z\} = \mu > 0$ for any $\mathcal{B} \geq 0$.