Inequality and Measured Growth*

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Abstract

Compared to a half-century ago, inequality in the United states has risen and measured productivity growth has fallen. Concerns about rising inequality have been exacerbated by the observation that prices of goods consumed by the poor have risen faster than prices of goods consumed by the rich. This paper presents an example of an economy that is consistent with these facts and yet the facts can be misleading about improvements in welfare. The two key ingredients are non-homothetic preferences and productivity improvements directed toward goods with larger market size. The model admits balanced growth despite the structural change induced by non-homothetic preferences. Along a BGP in which the distribution of after-tax income is stable, measured inflation among goods consumed by the bottom half of earners is perpetually higher than among goods consumed by the top half, but welfare improves at the same rate for all households. Across BGPs in which the only difference in primitives is the progressivity of the tax schedule, the BGP with a more unequal distribution of after-tax income exhibits lower measured growth of output and productivity. Nevertheless, welfare improves at the same rate along both BGPs. At the root of the deviation between productivity growth and welfare improvements is the fact that the value of cost reductions for a good are transitory if income effects eventually shrink the good's expenditure share. Standard measures of inflation capture the benefits of cost reductions among goods that are consumed contemporaneously, but only partly determine the evolution of price levels relevant for a household, as they do not capture the benefits from cost reductions that occur before the household shifts towards a good.

KEYWORDS: Inequality, Non-homothetic Preferences, Balanced Growth, Welfare Measurement, Inflation, Structural Change

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1 Introduction

Many have noted several worrying trends over the last half century in the United States and elsewhere. Chief among them are slowing productivity growth and rising inequality. Concerns about the increase in dispersion of nominal income have been exacerbated by the recent observation that prices of goods consumed by low-income households have been rising faster than goods consumed by high-income households.

This paper presents a simple model that is consistent with these facts in the following sense: Along a balanced growth path in which the distribution of nominal incomes remains fixed, measured inflation is perpetually higher for low-income households than for high-income households. Comparing two BGPs in which the only difference in primitives is the distribution of after-tax income,[1] the BGP with more inequality has lower measured TFP growth and a larger gap in inflation between the rich and poor. Nevertheless, I argue that these observations may be misleading about improvements in welfare. Along a BGP, despite the perpetual gap in measured inflation, welfare improves at the same rate for all, in a sense I will be precise about. Across BGPs, despite the gap in measured output and productivity growth, there is no gap in the rate of welfare improvements. Further, even though the BGP with more inequality exhibits a larger gap in inflation between the rich and poor, the productivity improvements that cause this actually ameliorate inequality rather than exacerbate it.

The key ingredients that drive these results are non-homothetic preferences and market-size-driven productivity growth. In the model, households have non-homothetic preferences over goods that range over the real line, with those that spend more shifting consumption to higher-ranked goods. As incomes rise, the balanced growth path exhibits a flying geese pattern in the spirit of Matsuyama (2002), Foellmi and Zweimüller (2006), and Bohr, Mestieri and Yavuz (2022). Each good is initially a luxury and eventually a necessity. Growth is driven by two forces: exogenous broad-based technology improvements that reduce the cost of producing all goods, and good-specific technology improvements that come from learning by doing: cost reductions are proportional to the labor used to produce the good.[2] Individuals supply labor inelastically and are heterogeneous in

[1] The change in the distribution of after-tax income could be driven by a change in the progressivity of the tax schedule or in the distribution of skills; in the model these are isomorphic

[2] The results would be similar if the learning by doing were replaced with good-specific innovations induced by

their endowments of efficiency units of labor. Labor income is redistributed through a progressive income tax. Because preferences are non-homothetic, when consumption expenditures are more equally distributed, there is more overlap in the consumption bundles of higher- and lower-income households.

Measured growth of aggregate output is a Divisia index of the output growth of individual goods, consistent with national accounting practices. Learning-by-doing leads to larger cost reductions for goods that are consumed more, and these are precisely the goods that count more in a Divisia index. As a result, if the aggregate consumption bundle is concentrated on a narrower range of goods, measured growth is higher. With a more even distribution of income, household consumption bundles overlap more, leading to higher measured growth.

The model features a balanced growth path in which household consumption bundles are traveling waves that travel at the same speed. As a household's expenditure increases it consumes higher ranked goods and larger quantities, but the shape of its expenditure shares remains constant. Improvements in welfare are not well-captured by measured productivity growth. Measured productivity growth captures the contemporaneous cost reductions in goods that are currently consumed and chains together these instantaneous growth rates. But the value of cost reductions for any particular good is temporary; eventually, as expenditures rise, households shift away from that good to even higher ranked goods. Thus the value of any good-specific cost reduction eventually depreciates at a rate determined by the speed of the traveling wave. It turns out that the speed of the traveling wave, the sufficient statistic for welfare improvements, is determined only by improvements in broad-based technology. Good-specific productivity improvements are valuable, but because their economic value is transitory, they lead to a level effect rather than a growth rate effect. To summarize, if changes in inequality cause changes in measured growth rates by inducing cost reductions in relatively larger markets, these cost reductions may not be relevant for long-run welfare gains, as households shift to goods for which cost reductions haven't happened yet.

While inequality in nominal incomes has risen over the last several decades, several papers have recently documented that prices of the goods consumed by low income households have been rising faster than the prices of goods consumed by high income households (Argente and Lee (2021),

market size, but measurement would be less transparent.

Kaplan and Schulhofer-Wohl (2017), Jaravel (2019, 2021)),[3] and that a large portion of this gap in inflation rates is due to differences in the rates of innovation directed toward these goods.[4] Section 5 shows that this is exactly the pattern one would observe along a BGP of the model presented here: measured inflation for the rich is perpetually lower than measured inflation for the poor. Since those with higher incomes spend more and cost reductions are larger for goods with larger markets, the goods consumed by those with higher incomes will experience larger reductions in cost. Nevertheless, this gap in measured inflation can be misleading. Along the BGP, the welfare relevant consumption index improves for all households at the same rate. Thus even though the distribution of nominal expenditures is fixed, the gap in inflation rates does not signal widening gap in welfare.

Why don't unequal inflation rates contribute to widening inequality along a balanced growth path? Fundamentally, welfare differences across income groups depend on the differences in the level of prices of the goods they are consuming, not the rate of change of those prices. The rate of change of prices is not a reliable measure of the level of prices. Those with low income certainly do benefit from cost reductions of the goods they consume while they are consuming the goods, but they also benefit from cost reductions for those goods before they begin to consume them. Measures of contemporaneous inflation will capture the former, but not the latter. The cost reductions among goods consumed by high income households indeed benefit the low-income households, it is just that those benefits accrue later.

[3] Argente and Lee (2021) and Jaravel (2019, 2021) have estimated gaps of roughly 0.5 percentage points per year between the top and bottom quintiles of the income distribution. These studies use bar-code level data from the Nielsen Consumer Panel, which is much more granular and makes it easier to measure changes in price for the same good over time. Unfortunately this data is only available since 2004. Several earlier papers in the literature such as Hobijn and Lagakos (2005) and McGranahan and Paulson (2005) measured expenditures using Consumer Expenditure Survey (CEX) data and price changes using CPI data and did not find large gaps in inflation rates across demographic groups. Jaravel (2019, 2021) has emphasized that the gaps in inflation rates is mostly a within sector phenomenon, and grow larger the more one disaggregates. Indeed, using CEX data, studies have found small gaps in inflation when using relatively aggregated industries (roughly 20 categories), but larger gaps comparable to those found with the Neilsen panel when using more finely disaggregated categories, e.g., Jaravel (2019), Klick and Stockburger (2021), Orchard (2022), Jaravel and Lashkari (2022). Note that to the extent shopping behavior or ability to substitute may differ systematically across the income distribution, this would lead to a difference in price levels, not a persistent gap in price changes. See Jaravel (2021) for a good survey of the literature.

[4] There is a growing body of evidence that consumer demand has determined the direction of innovation. While market size and innovation are jointly determined in equilibrium, Acemoglu and Linn (2004) address endogeneity by using shifts in demand for pharmaceutical products driven by demographic change. Boppart and Weiss (2012), Jaravel (2019), Beerli et al. (2020) have applied this strategy to a broader set of sectors and found that sectors that saw increased demand due to demographic shifts have experienced higher rates of innovation and price growth. Bohr, Mestieri and Yavuz (2022) show that sectors with higher income elasticities experienced later peaks, experienced lower price growth, and saw higher growth of patents.

Each household's expenditure can be decomposed into a price index and a consumption index. The change in a household's price index over time can be decomposed into two parts. The first part captures the change in prices, holding the expenditure shares fixed. This corresponds to measured inflation. The second part captures how shifts in expenditure shares alter the price level. Since households are constantly shifting to higher ranked goods for which there has been less cumulative learning by doing, this component raises the price level and offsets some of the price declines of the first component. This corresponds to the loss of value over time of good-specific cost reductions.[5]

Looking across BGPs, a BGP with more inequality in expenditures will have a larger gap in inflation rates across income groups. Cost reductions are larger for goods with a larger market size which tends to be goods consumed by the rich. When inequality is low, there is more overlap in the consumption bundles of the rich and poor. Since the goods consumed by the poor are also more likely to consumed by the rich, the market size for these goods is large and those goods experience productivity growth *while the poor are consuming them*. Thus the gap in measured inflation would be lower.

But again, this is misleading about the welfare implications. While the poor benefit from the cost reductions that occur while they are consuming a good, they would benefit even more if those cost reductions happened before they shift consumption toward that good. Thus a lower inflation rate for the poor is a signal that the price level is not as low as it might otherwise be.

In Section 6, I formalize the notion that, in the model, there is no systematic relationship between measured real income growth—growth of nominal income minus measured inflation—and improvements in welfare. In particular, I show that there is no utility function that consistently assigns higher welfare growth to instances of higher measured real income growth. I also discuss the relation of these findings to classic results by Theil (1968) and Diewert (1976) that provide a welfare interpretation of measured real income growth, at least over short horizons.

1.1 Related Literature

While the arguments of this paper are likely to be relevant in any setting with non-homothetic preferences and market-size driven cost reductions, they is particularly clear in a setting with a

[5] In practice, measuring this second component may be hard. Outside of this simple model, it is not clear how one might determine which goods have high prices and which have low prices.

BGP. With a BGP, there are transparent analytical expressions which clearly show which features of the economy are relevant for measured productivity growth, welfare growth, and household-specific inflation. Models in which non-homothetic preferences leads to structural change often do not exhibit balanced growth paths in the usual sense, as households shift across goods or sectors with different (but constant) productivity growth rates.[6] An alternative approach, which I follow in this paper, is to allow for an infinite range of goods that follow some hierachical pattern. In Zweimüller (2000), Zweimüller and Brunner (2005), and Foellmi and Zweimüller (2006, 2008), preferences are non-homothetic in that those with higher expenditures spread their consumption bundle across a wider range of goods. In these models, balanced growth can occur as the range of goods consumed expands indefinitely. I take a closely-related but different approach in that in my model, as incomes rise along a balanced growth path, those consumption bundles follows a traveling wave, and households shift their expenditures to higher ranked goods. Bohr, Mestieri and Yavuz (2022) also study a model with a traveling wave using non-homothetic CES preferences, but with a different weighting function. I describe the relationships between the utility functions in Appendix A.1.

Several papers have linked non-homotheticity with Schmookler's (1967) demand-driven innovation to argue that the distribution of income will affect the pace of innovation, including Matsuyama (2002), Zweimüller (2000), and Foellmi and Zweimüller (2006).[7] One of the basic positive predictions here can be found in the literature: a more equal distribution of expenditures increases the scale effects that come with innovation or with learning by doing, and raises the growth rate.[8]

[6]One approach to study long run outcomes with structural change, pioneered by Kongsamut, Rebelo and Xie (2001), is to assume that investment is produced using a linear technology using only capital. This allows for the possibility of a "generalized balanced growth path" in which there is a constant interest rate in units of investment goods and a constant growth rate of output when measured in units of investment goods. Kongsamut, Rebelo and Xie (2001) and Comin, Lashkari and Mestieri (2021) use this approach and generate structural change using non-homothetic preferences (Stone-Geary and non-homothetic CES respectively), while in Ngai and Pissarides (2007), sectoral shares change because of shifts in relative prices of sectoral output and an assumption that these outputs are complements. Boppart (2014) develops a model that allows for both sources of structural change. See Herrendorf, Rogerson and Valentinyi (2013) for a unifying perspective on the role of relative prices and non-homotheticity in causing structural change. Herrendorf, Rogerson and Valentinyi (2021) recently showed that there is structural change in the investment sector as well, which contrasts with the assumption of a linear and fixed investment technology that uses only capital, and they argue that this makes constant within-sector growth rates incompatible with constant long run growth rates. Acemoglu and Guerrieri (2008) and Buera et al. (2020) focus on medium-run transition dynamics. Equilibria of most models of structural change converge asymptotically to an economy without structural change. Kongsamut, Rebelo and Xie (2001) converges to a BGP with stable factor shares, while Ngai and Pissarides (2007) and Comin, Lashkari and Mestieri (2021) converge to a one sector economy dominated by services.

[7]Bohr, Mestieri and Yavuz (2022) features directed technical change, but a representative household rather than heterogeneous households.

[8]Interest in the empirical relationship between inequality and growth dates back to at least the seminal work

While those papers have focused on the positive implications of the qualitative relationship between inequality and growth and whether there might be multiple equilibria, this paper focuses on normative implications, measurement issues, and an application to the US experience of the last half century. Further, I show that the model is consistent with empirical regularities found in the literature on heterogeneous inflation.

2 Model

2.1 Households

There is a unit continuum of households with identical preferences. The households each supply labor inelastically but differ in their endowments of efficiency units of labor, which are distributed across households according to the distribution function $G(\ell)$.

I assume the tax schedule takes the following form:

$$T(y) = y - \bar{y}^{\tau} y^{1-\tau} \tag{1}$$

The tax schedule implies that after-tax income is a log-linear function of pre-tax income. τ indexes the progressivity of the tax schedule; with $\tau = 0$ after-tax income matches pretax income, while with $\tau = 1$ all households have the same after-tax income. $\bar{y}$ is chosen so that the government budget is balanced. This family of tax schedules has often been used in public finance and macroeconomics literatures, e.g., Benabou (2000, 2002), and Heathcote, Storesletten and Violante (2017) show that it provides a very good approximation of the US tax and transfer system. If an individual with endowment ℓ has pre-tax income of $w\ell$, a balanced budget requires that her after-tax income is $w\ell^{1-\tau}/\overline{\ell^{1-\tau}}$, where $\overline{\ell^{1-\tau}} \equiv \int \ell^{1-\tau} dG(\ell)$.

2.2 Preferences

There is an infinite continuum of goods, indexed by $i \in (-\infty, \infty)$. Consider a household that with a budget of E facing prices $\{p_i\}$. The household chooses a consumption bundle $\{c_i\}$ to maximize

of Kuznets (1955). Many early studies using cross-country regressions found that countries with higher inequality experienced lower growth (Perotti (1996) and Benabou (1996) survey the literature). However, as with many other applications of cross-country regressions, results were sensitive to specification (Forbes (2000), Barro (2000)). In any case, the many joint determinants of inequality and growth make it difficult to tease out causal relationships.

the consumption index C, defined as

$$\sup_C C$$

subject to

$$\left[\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}}di\right]^{\frac{\sigma}{\sigma-1}} \geq 1 \tag{2}$$

with $\int_{-\infty}^{\infty} h(i)di = 1$. This is similar to a weighted Dixit-Stiglitz utility function, except that the weights $h\Big(i-\gamma\log C\Big)$ are endogenous and depend on the overall consumption index C. As C rises, more weight is put on goods with higher i. One way to see this is to use the change of variables $u = i-\gamma\log C$ to express (2) as

$$\left[\int_{-\infty}^{\infty} h(u)^{\frac{1}{\sigma}}\left(\frac{c_{u+\gamma\log C}}{C}\right)^{\frac{\sigma-1}{\sigma}}du\right]^{\frac{\sigma}{\sigma-1}} \geq 1$$

Here one can see that, as C rises, the shape of the preference weights remains the same but the weights apply toward higher ranked goods. γ indexes the strength of the non-homotheticity. If $\gamma = 0$, preferences would be homothetic.

The problem can be separated into two parts: Expenditure minimization given C, and then the optimal choice of C given the budget constraint. Define $\mathcal{E}(C) \equiv \inf_{\{c_i\}}\int_{-\infty}^{\infty} p_i c_i di$ subject to (2) to be the minimal expenditure that delivers a consumption index C given prices. Because C is fixed, this is just the standard expenditure minimization with weighted Dixit-Stiglitz preferences, and yields

$$\mathcal{E}(C) = C\left[\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di\right]^{\frac{1}{1-\sigma}}$$

The second step is to choose the maximal affordable consumption bundle, $\sup_C C$ subject to $\mathcal{E}(C) \leq E$. I next provide regularity conditions that ensure that there is a unique consumption index C that satisfies $\mathcal{E}(C) = E$, and that the solution is interior.

Proposition 1 *Suppose the price schedule p_i is weakly increasing in i and $p_i^{1-\sigma}$ is Lipschitz. Then the optimal bundle for a household with expenditure E is*

$$c_i = p_i^{-\sigma}E^{\sigma}C^{1-\sigma}h\Big(i-\gamma\log C\Big)$$

where C is the unique solution to

$$C\left(\int_{-\infty}^{\infty} p_i^{1-\sigma} h\Big(i-\gamma\log C\Big)di\right)^{\frac{1}{1-\sigma}} = E$$

The condition that p_i is weakly increasing in i ensures that $\mathcal{E}(C)$ is strictly increasing, that $\lim_{C\to 0}\mathcal{E}(C) = 0$, and that $\lim_{C\to\infty}\mathcal{E}(C) = \infty$. The condition that $p_i^{1-\sigma}$ is Lipschitz ensures that $\mathcal{E}(C)$ is continuous. These conditions can be relaxed; Appendix A.2 provides a weaker set of sufficient conditions.[9] Nevertheless, the conditions will naturally be satisfied along a balanced growth path given the structure of the model.

2.3 A Simple Example

Suppose that all prices are the same, $p_i = p$. In this case, the consumption index for a household with expenditure E is the solution to

$$E = C\left(\int_{-\infty}^{\infty} p^{1-\sigma} h\Big(i-\gamma\log C\Big)di\right)^{\frac{1}{1-\sigma}}$$

Using the change of variables $u = i-\gamma\log C$ and noting that the preference weights integrate to one gives $\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)di = \int_{-\infty}^{\infty} h(u)du = 1$. The consumption index is thus

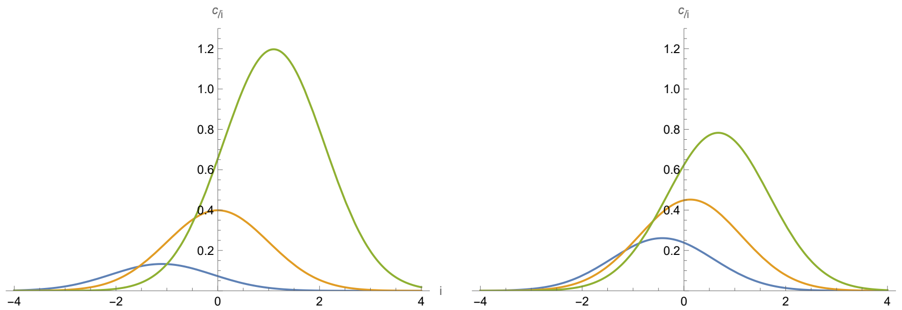
$$C = \frac{E}{p}\ .$$

The household's consumption of good i is thus

$$c_i = \frac{E}{p}h\left(i-\gamma\log\frac{E}{p}\right)$$

Written in this way, one can see that if the household has a higher expenditure E (relative to the price level p) by a factor of a, then it both scales up consumption by a factor of a and shifts

[9] Appendix A.2 shows that there is a unique solution to $\mathcal{E}(C) = E$ as long as prices do not decline too rapidly with i. What could go wrong? If prices decline too rapidly asymptotically, the household can attain infinite utility by taking $C\to\infty$. Even if prices are well-behaved asymptotically, the equation $C\left(\int_{-\infty}^{\infty} p_i^{1-\sigma} h\Big(i-\gamma\log C\Big)di\right)^{\frac{1}{1-\sigma}} = E$ can have multiple interior solutions if prices decline too rapidly in a range. To see this, starting with an interior solution for C, increasing C shifts the household toward higher ranked goods. If prices decline fast enough with i, the household can afford enough of those goods to satisfy $\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}}di\right)^{\frac{\sigma}{\sigma-1}} > 1$.



(a) $\tau = 0.0$ (b) $\tau = 0.5$

Figure 1 Expenditure Profiles for Three Households

Note: This figure shows the consumption quantities for three households when the price schedule is constant and weighting functions and skill distributions are Gaussian. The households have effective labor of 1/3, 1, and 3. The left panel shows an economy with a progressivity of $\tau = 0.0$, while the right panel shows an economy with progressivity of $\tau = 0.5$.

consumption toward higher ranked goods by an increment of $\gamma \log a$. Note these properties do not depend on the particular functional form for h.[10]

To see this visually, I assume the weighting function h and the distribution of endowments of effective labor take Gaussian functional forms:

(a) The weighting function takes a Gaussian form:

$$h(u) = \frac{1}{\sqrt{2\pi v_h}} e^{-\frac{u^2}{2v_h}}$$

(b) The distribution of household endowments of effective labor is lognormal with mean normalized to 1,

$$G'(\ell) = \frac{1}{\ell}\frac{1}{\sqrt{2\pi v_\ell}} e^{-\frac{(\log \ell + v_\ell/2)^2}{2v_\ell}}$$

I choose units of effective labor so that the mean of effective labor across households is 1, and normalize the mean of the weighting function so that it peaks at $u = 0$. As a result, each of these distributions depend on a single parameter. v_h indexes the tastes for variety: it controls the breadth of the consumption bundle chosen by the household. v_ℓ indexes the dispersion in endowments of effective labor.

Figure 1 shows an example of consumption profiles under the simple parameterization $v_h = v_\ell = \gamma = 1$ and $w = p$. The figure shows the level of expenditures across goods for three households,

[10] Another case that is easy to characterize is when the price schedule is log-linear, $p_i = p_0 e^{\kappa i}$ with $\kappa > -\frac{1}{\gamma}$.

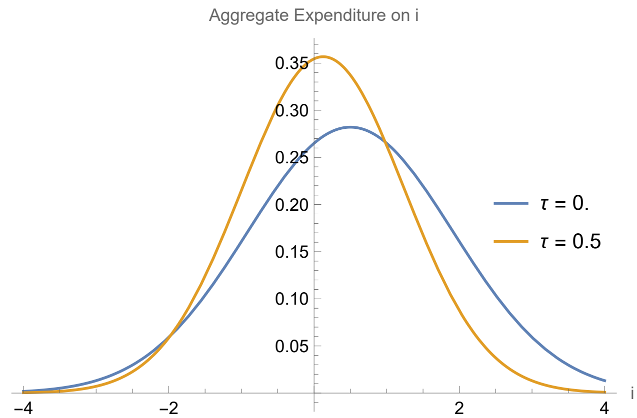


Figure 2 Distribution of Aggregate Expenditures Across Goods

Note: This figure shows the distribution of aggregate expenditures across goods for two economies, one with a tax progressivity of $\tau = 0.0$ and one with $\tau = 0.5$, when the price schedule is constant $p_i = w$ and weighting functions and skill distributions are Gaussian.

whose endowments are $\ell = \{\frac{1}{3}, 1, 3\}$. The left figure shows these households' consumption profiles when $\tau = 0$ so that incomes equal expenditure, while the right figure shows expenditure patterns for the same three households when the tax schedule is more progressive, at $\tau = 0.5$. Comparing the three households, one can see that richer households consume higher ranked goods and larger quantities.

One feature that will play a larger role below is that when the tax schedule is more progressive, there is more overlap in household consumption bundles. The right panel of Figure 1 shows the consumption bundles with a more progressive tax schedule, $\tau = 0.5$. Since expenditures are closer together, there is more overlap in consumption bundles. Figure 2 plots the distribution of aggregate expenditures across goods under each of the two tax schedules. With the Gaussian functional forms and log-linear tax schedule, the distribution of aggregate expenditures across goods is also Guassian: the share of expenditures on good i is normally distribution with variance $v_h + (1-\tau)^2\gamma^2 v_\ell$ and mean $\gamma(1-\tau)^2\frac{v_\ell}{2} + \gamma \log\frac{w}{p}$. Higher wages or lower prices cause households to shift to higher ranked goods. A more progressive tax schedule leads to a distribution of expenditure shares with a lower mean and variance. In particular, the distribution of aggregate expenditures becomes more concentrated.

One particular measure of concentration which will be of use later is the Herfindahl-Hirschman Index of aggregate expenditures across goods: $HHI = \int_{-\infty}^{\infty} \omega_i^2 di$, where $\omega_i \equiv \frac{p_i y_i}{\int_{-\infty}^{\infty} p_{\tilde{i}} y_{\tilde{i}} d\tilde{i}}$ is the aggregate expenditure share on good i and $y_i = \int c_{\ell i} dG(\ell)$:

$$HHI = \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2\gamma^2 v_\ell}}$$

One can see both visually and analytically that a more equal distribution of expenditures leads to a higher HHI across goods.

2.4 Technology and Productivity Improvements

We assume that each good is produced using a constant-returns-to-scale, labor-only technology

$$Y_{it} = A_t B_{it} L_{it}$$

where the productivity to produce a good has two components: (i) broad-based productivity A_t which is common to all goods, and (ii) good-specific productivity B_{it}. We assume that all agents are price takers, so the price of good i is equal to its unit cost, $p_{it} = \frac{w_t}{A_t B_{it}}$.

The broad-based technology A_t improves exogenously over time. Good-specific technology improves via learning by doing, according to $\log B_{it} = \phi \int_{-\infty}^{t} L_{is} ds$, so that

$$\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$$

As such, productivity improvements are directed toward goods for which expenditures are larger.[11]

2.5 Equilibrium

Given initial conditions $\{B_{it_0}\}$ and a tax policy τ, a competitive equilibrium is, for each instant $t > t_0$, a wage w_t, a set of prices $\{p_{it}\}_i$, an allocation of labor $\{L_{it}\}_i$, output $\{Y_{it}\}_i$, and consumption $\{c_{\ell it}\}_{i,\ell}$, consumption indices $\{C_{\ell t}\}_\ell$, good specific productivities $\{B_{it}\}_i$, and a tax schedule $T_t(y) = y - \bar{y}_t^{\tau} y^{1-\tau}$ such that at each instant, each household maximizes utility taking prices, wages, and the tax schedule as given; the representative firm maximizes static profit taking prices and wages as given; the government budget is balanced; each goods market clears; the labor market clears;

[11] Results would be similar with directed technical change, but measurement of TFP growth would be less transparent.

and the evolution of technology is consistent with learning by doing.

In any equilibrium, we need to verify that markets clear. The market clearing condition for good i can be expressed as

$$L_{it} = \frac{Y_{it}}{A_t B_{it}} = \frac{1}{A_t B_{it}} \int c_{\ell it} dG(\ell)$$

Household ℓ's consumption of good i is $c_{\ell it} = p_{it}^{-\sigma} E_{\ell t}^{\sigma} C_{\ell t}^{1-\sigma} h\Big(i - \gamma \log C_{\ell t}\Big)$, its expenditure is equal to its after-tax income $E_{\ell t} = w_t \ell^{1-\tau} / \overline{\ell^{1-\tau}}$, and the price of good i is $p_{it} = \frac{w_t}{A_t B_{it}}$. Together, these imply that the market clearing conditions can be expressed as

$$L_{it} = \int \left(\frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}} \right)^{\sigma} \left(\frac{C_{\ell t}}{A_t} \right)^{1-\sigma} \frac{h\Big(i - \gamma \log C_{\ell t}\Big)}{B_{it}^{1-\sigma}} dG(\ell) \tag{3}$$

where each $C_{\ell t}$ is the unique solution to household ℓ's budget constraint:

$$\frac{\ell^{1-\tau}}{\overline{\ell^{1-\tau}}} = \frac{C_{\ell t}}{A_t} \left[\int \frac{h\Big(i - \gamma \log C_{\ell t}\Big)}{B_{it}^{1-\sigma}} di \right]^{\frac{1}{1-\sigma}} \tag{4}$$

If each goods market clears, all budget constraints hold with equality, and the government budget is balanced, then Walras' Law implies that the labor market clears as well.

Equations (3) and (4), along with the equation defining learning by doing, $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$, are sufficient to completely characterize a dynamic equilibrium. Given technology at t, $A_t, \{B_{it}\}$, (4) pins down each household's consumption index, $\{C_{\ell t}\}$. Given these, (3) pins down the allocation of labor across goods L_{it}. In turn, the allocation of labor determines the evolution of good-specific technologies.

3 A Balanced Growth Path

Suppose that the tax policy is fixed over time and broad-based productivity grows at a constant rate, $\frac{\dot{A}_t}{A_t} = g$. This section shows that there is a balanced growth path in which each household's consumption profile is a traveling wave. All of these waves grow and travel at the same speed.

From these equations, one can construct a balanced growth path in which each C_ℓ grows at rate g and in a time increment $t_1 - t_0$, all consumption bundles shift to the right on the real line by

$\Delta \equiv g\gamma(t_1 - t_0)$. If labor shifts by Δ so that $L_{it_0} = L_{i+\Delta,t_1}$, then good-specific productivity shifts by Δ as well:

$$\log B_{it_0} = \phi \int_0^\infty L_{i,t_0-s} ds = \phi \int_0^\infty L_{i+\Delta,t_1-s} ds = \log B_{i+\Delta,t_1}$$

To verify that the guess is consistent with a BGP, note that $\frac{h\left((i+\Delta)-\gamma \log C_{\ell t_1}\right)}{B_{i+\Delta,t_1}^{1-\sigma}} = \frac{h\left(i-\gamma \log C_{\ell t_0}\right)}{B_{it_0}^{1-\sigma}}$ and $\frac{C_{\ell t_1}}{A_{t_1}} = \frac{C_{\ell t_0}}{A_{t_0}}$ imply that $L_{i+\Delta,t_1} = L_{i,t_0}$, and thus if (3) and (4) are satisfied at t_0, they are also satisfied at t_1.

In Appendix B I show the existence and uniqueness of a BGP when the elasticity of substitution across goods is not too large.

Proposition 2 *If* $e^{(\sigma-1)\frac{\phi L}{\gamma g}} < 2$*, there exists a unique balanced growth path.*

4 Measured Growth

Aggregate consumption growth is measured as a Divisia index across changes in output across goods.[12] A Divisia index is an expenditure-weighted growth rate of the individual categories. If households have a common, homothetic utility function, this is measure has a natural welfare interpretation Samuelson and Swamy (1974), Diewert (1976).

Measured TFP growth is measured GDP growth minus measured input growth, as in Jorgenson and Griliches (1967) and Christensen and Jorgenson (1970). Since labor force is constant, this is simply measured GDP growth:

$$\frac{d\widehat{\log TFP_t}}{dt} = \int_{-\infty}^{\infty} \omega_{it} \frac{\dot{Y}_{it}}{Y_{it}} di$$

where $\omega_{it} \equiv \frac{p_{it} Y_{it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} Y_{\tilde{i}t}} = \frac{w_t L_{it}}{w_t L} = \frac{L_{it}}{L}$ is the aggregate expenditure share on good i. Since output of good i is simply $Y_{it} = A_t B_{it} L_{it}$, this is

$$\frac{d\widehat{\log TFP_t}}{dt} = \int_{-\infty}^{\infty} \omega_{it} \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{B}_{it}}{B_{it}} + \frac{\dot{L}_{it}}{L_{it}} \right) di$$

[12] In national accounts, output growth is measured as a discrete time approximation to a Divisia index.

Note $\omega_{it} = \frac{L_{it}}{L}$ implies $\int_{-\infty}^{\infty} \omega_{it} \frac{\dot{L}_{it}}{L_{it}} di = 0$, giving

$$\frac{d \log \widehat{TFP}_t}{dt} = \int_{-\infty}^{\infty} \omega_{it} \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{B}_{it}}{B_{it}} \right) di$$

Finally, the learning by doing implies that $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it} = \phi L \omega_{it}$, we can express the change in measured TFP as

$$\frac{d \log \widehat{TFP}_t}{dt} = \frac{\dot{A}_t}{A_t} + \phi L \underbrace{\int_{-\infty}^{\infty} \omega_{it}^2 di}_{HHI}$$

Since the growth of A is exogenous, the increase in measured TFP rises when the distribution of expenditures across goods is more concentrated. In this sense, a more equitable distribution of after-tax income is associated with higher growth of measured TFP.

Why does measured TFP rise more quickly when there is more overlap in consumption bundles? The learning by doing gives rise to a scale effect at the good level. When one household consumes a good, the labor used to produce that good reduces the cost of producing that good. If others are also consuming the same good at the same time, the cost reduction has extra value because it reduces the cost others face as well.

4.1 Measured Growth Along a Balanced Growth Path

Along a balanced growth path, the pattern of expenditures across goods follows a traveling wave: $\omega_{it} = \omega_{i+\gamma g(t'-t),t'}$. A simple corollary is the measured TFP growth is constant.

Proposition 3 *Along a balanced growth path, measured TFP growth is constant.*

Further, in line with the preceding discussion, one can show analytically, up to a first order approximation, that along a balanced growth path with a more equitable distribution of after-tax income, measured growth is persistently lower.

Proposition 4 *Suppose that ϕ is small and h and G follow the Gaussian functional forms of Section 2.3. Then measured TFP growth satisfies*

$$\frac{d \log \widehat{TFP}_t}{dt} \approx g + \phi L \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2 \gamma^2 v_\ell}} .$$

Consider two BGPs that correspond to economies with different levels of progressivity, $\tau_1 > \tau_0$. Measured TFP growth is higher in the economy with more progressive taxation.

Proof. Measured TFP growth is $\frac{d\log\widehat{TFP_t}}{dt} = g + \phi L \int_{-\infty}^{\infty} \omega_{it}^2 di$. A first order approximation around $\phi = 0$ gives

$$\frac{d\log\widehat{TFP_t}}{dt} \approx \left.\frac{d\log\widehat{TFP_t}}{dt}\right|_{\phi=0} + \phi \left.\frac{d\frac{d\log\widehat{TFP_t}}{dt}}{d\phi}\right|_{\phi=0}$$

Note that $\left.\frac{d\log\widehat{TFP_t}}{dt}\right|_{\phi=0} = g$. In addition, $\left.\frac{d\frac{d\log\widehat{TFP_t}}{dt}}{d\phi}\right|_{\phi=0} = L\int_{-\infty}^{\infty}\omega_{it}^2 di\Big|_{\phi=0} = L\frac{1}{2\sqrt{\pi}\sqrt{v_h+(1-\tau)^2\gamma^2 v_\ell}}$, since the HHI across goods is $\frac{1}{2\sqrt{\pi}\sqrt{v_h+(1-\tau)^2\gamma^2 v_\ell}}$ when the price of all goods is the same, as discussed in Section 2.3. ■

Consider two economies that have identical primitives but with different tax schedules, each on balanced growth paths. In the economy with a more progressive tax schedule, expenditures will be more equal, and as a result, measured output and TFP growth will be perpetually higher, as shown in Section 4. Nevertheless, growth of consumption indices is the same in each of the two economies: each household's consumption index grows at rate $\frac{\dot{C}_{\ell t}}{C_{\ell t}} = \frac{\dot{A}_t}{A_t} \equiv g$. That is, differences in measured TFP growth are not informative about the growth rate of consumption indices.

In particular, growth rates of consumption indices along each BGP do not depend at all on the pace of market-specific cost reductions. Why? Measured aggregate productivity growth is higher if there are larger cost reductions for goods that individuals are consuming contemporaneously. But good-specific productivity improvements only give a temporary boost to welfare. Eventually, households shift toward higher ranked goods, with diminishing relevance of those productivity improvements for any particular good. In that sense, the economic value of the productivity gains eventually shrink, as households shift away from those goods.

5 Unequal Inflation

Several papers have documented that poor households face persistently higher inflation rates than rich households Argente and Lee (2021), Jaravel (2019, 2021), Kaplan and Schulhofer-Wohl (2017). For example, Argente and Lee (2021) find that between 2004 and 2016, inflation for the top quartile of the income distribution has been roughly half of a percentage point lower per year than for the bottom quartile of the households. Jaravel (2019) corroborates this fact and goes further to show that directed technical change leading to cost reductions for goods consumed disproportionately by the rich can explain a large portion of this trend. While it is well documented that inequality in nominal incomes has risen over the last half century, the gap in inflation has raised fears that inequality in real income has risen even faster.

In this section, I show that, along any balanced growth path, there are perpetual differences in measured inflation rates across quantiles of the income distribution. Further, a BGP with more inequality in nominal income will exhibit a larger gap between the measured inflation for the top and bottom halves of the income distribution. Nevertheless, these observations are misleading about welfare improvements among those in the cross section and across balanced growth paths with different distributions of income.

Conventional measures of inflation are a Divisia index of price changes: a weighted average of price growth across goods, weighted by expenditure.[13] For household ℓ, measured inflation is

$$\widehat{Inflation}_{\ell t} = \int_{-\infty}^{\infty} \omega_{\ell it} \frac{\dot{p}_{it}}{p_{it}} di$$

where the weights $\omega_{\ell it} \equiv \frac{p_{it} c_{\ell it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} c_{\ell \tilde{i}t} d\tilde{i}}$ are the household's expenditure share on good i.

Since $p_{it} = \frac{w_t}{A_t B_{it}}$, measured inflation can be expressed as

$$\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \int_{-\infty}^{\infty} \omega_{\ell it} \frac{\dot{B}_{it}}{B_{it}} di$$

Broad-based technology grows at rate $\frac{\dot{A}_t}{A_t} = g$ and good-specific technology grows because of learning by doing, $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it} = \phi L \omega_{it}$. Measured inflation is

$$\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - g - \phi L \int_{-\infty}^{\infty} \omega_{\ell it} \omega_{it} di \ .$$

Measured inflation for household ℓ is lower if its expenditures overlap more with aggregate expenditures.

[13] Real-world measurement of inflation must contend with a number of thorny issues such as the appearance of new goods and disappearance of old goods, changes in quality, and measurement at discrete intervals. In the simple environment presented here, none of these issues arise. All goods are consumed by all households at all times, there are no changes in quality, and we can update consumption bundles continuously over time.

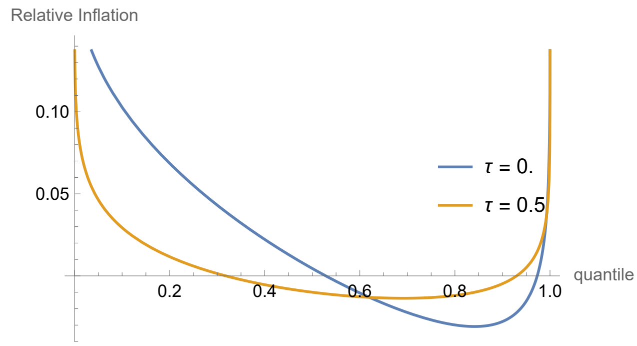


Figure 3 Household-specific Measured Inflation

Note: This figure shows the household-specific inflation rate for each quantile of the income distribution, relative to aggregate inflation. Household-specific inflation is a weighted average of price changes, weighted by the household's expenditures. The figure shows two curves, one for a BGP with $\tau = 0$ and one for a BGP with $\tau = 0.5$.

Along a balanced growth path, the shape of each household's consumption bundle remains constant, which implies that inflation gaps across households remain constant.

Proposition 5 *Along a balanced growth path, $\widehat{Inflation}_{\ell t} - \widehat{Inflation}_{\ell' t}$ is constant for each ℓ, ℓ'.*

In line with the empirical findings, measured inflation differs across individuals because consumption bundles differ and the pace of cost reductions differs across goods. Figure 5 shows the level of inflation for each quantile of the income distribution (relative to inflation for the aggregate income basket) using the Gaussian functional form and distributional assumptions of Section 2.3. Inflation is higher among those in the bottom half of the distribution than among those in the top half. Since the aggregate expenditure is tilted toward the consumption patterns of those with higher income, the goods consumed by those in the top half of the distribution experience larger cost reductions. Thus those in the top half experience lower inflation than those at the bottom.[14,15]

Further, the economy with greater inequality exhibits a larger gap between measured inflation among the rich and poor.[16] This happens because there with more inequality there is less overlap

[14]Interestingly, inflation is highest for those at the very bottom and very top of the income distribution, as their consumption bundles overlap least with the aggregate expenditures.

[15]It need not be the case that inflation for the rich is lower than inflation for the poor. For example, if there is a very large mass of households with low ℓ and only a few with high ℓ, it could be that the aggregate consumption bundle is closer to that of the poor, and hence inflation would be lower for the poor. Similarly, if even if the rich spend more than the poor, it could be that consumption bundles differ much more among the rich, but consumption bundles among the poor are similar. The example here, in which the distribution of consumption expenditures is lognormal, is consistent with the findings of Battistin, Blundell and Lewbel (2009).

[16]There is some evidence that the gap in measured inflation rates between the rich and poor has increased over the last few decades. Jaravel and Lashkari (2022) combine CEX and CPI data to construct measures of inflation for each percentile in income distribution going back to the 1950s. They find a strong negative correlation between inflation

in consumption bundles, so households toward the bottom spend even less on goods undergoing cost reductions.

One might be tempted to infer from this that differential inflation exacerbates inequality in nominal expenditures. However, such a conclusion is not warranted. Despite the differential inflation rates, the consumption index for *every* household grows at rate g, as shown in Section 3.

How can differential inflation be compatible with equal growth in consumption indices? Fundamentally, households benefit from low cost of goods, not from cost reductions per se. Measured inflation gauges how fast prices are falling for the goods a household is consuming contemporaneously. In many models, the latter is the rate of change of the former. But in this model, price changes of goods consumed contemporaneously only partly determine the evolution of the level of prices that are relevant for the household.

Consider the following example. Currently the rich consume Teslas, and possibly in the future the poor will as well. One possibility is that the price of Teslas will decline while the rich are consuming but will be flat after the poor begin consuming it. Under that scenario, inflation for the rich will be lower for than for the poor. In a second scenario, the price will remain high while the rich consume Teslas, but will begin falling once the poor consume Teslas as well. In the latter scenario, inflation for the poor will be lower. Nevertheless, the poor prefer the first scenario despite the lower inflation because they get to pay lower prices; they would prefer the price of a Tesla fall *before* they start consuming it than for the price to fall *while* they are consuming it.

More formally, consider household ℓ, whose budget constraint can be expressed as $E_{\ell t} = C_{\ell t} P_{\ell t}$, where household ℓ's price index is

$$\begin{aligned} P_{\ell t} &= \left[\int_{-\infty}^{\infty} h\Big(i - \gamma \log C_{\ell t}\Big) p_{it}^{1-\sigma} di \right]^{\frac{1}{1-\sigma}} \\ &= \left[\int_{-\infty}^{\infty} h(u) \left(p_{u+\gamma \log C_{\ell t}, t}\right)^{1-\sigma} du \right]^{\frac{1}{1-\sigma}} . \end{aligned}$$

where the second line used the change of variables $u = i - \gamma \log C_{\ell t}$. Differentiating completely with

rates and income 1995-2019, a negative but slightly weaker correlation from 1955-1984, and a much weaker, but still negative between 1984-1995. The findings for the earlier period should be taken with a grain of salt, however, as CEX data is quite sparse before 1984. Orchard (2022) tracks the price of necessities relative to that of luxuries using the CEX, and finds consistent evidence that the relative price of necessities rose since 2000 but mixed evidence about whether the relative price increased or decreased from the 1970s-1990s.

respect to time and then changing variables back to $i = u + \gamma \log C_{\ell t}$ gives

$$\frac{\dot{P}_{\ell t}}{P_{\ell t}} = \int_{-\infty}^{\infty} \omega_{\ell i t} \left[\frac{\dot{p}_{it}}{p_{it}} + \gamma \frac{\dot{C}_{\ell t}}{C_{\ell t}} \frac{d \log p_{it}}{di} \right] di$$

where, again, $\omega_{\ell i t} \equiv \frac{p_{it} c_{\ell i t}}{\int_{-\infty}^{\infty} p_{\tilde{i}t} c_{\ell \tilde{i} t} d\tilde{i}} = \frac{h\left(i - \gamma \log C_{\ell t}\right) p_{it}^{1-\sigma}}{\int_{-\infty}^{\infty} h\left(\tilde{i} - \gamma \log C_{\ell t}\right) p_{\tilde{i}t}^{1-\sigma} d\tilde{i}}$ is household ℓ's share of time-t expenditure spent on good i.

Measured inflation captures only the first term in brackets: the expenditure-weighted changes in prices. But in this model, welfare also depends on the second term: as the household consumes more, it shifts to higher ranked goods. Those higher-ranked goods have higher prices, as there has been less cumulative learning by doing. Along a balanced growth path, this shift to higher-ranked, higher-priced goods partially offsets the first term: $\frac{\dot{p}_{it}}{p_{it}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{B}_{it}}{B_{it}}$ and $\gamma \frac{\dot{C}_{\ell t}}{C_{\ell t}} \frac{d \log p_{it}}{di} = \frac{\dot{B}_{it}}{B_{it}}$. Together, these yield

$$\frac{\dot{P}_{\ell t}}{\dot{P}_{\ell t}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} = \frac{\dot{w}_t}{w_t} - g$$

That is, for all households, the price index (relative to the wage) declines at the same rate, g.[17] This gives the following proposition:

Proposition 6 *Along a balanced growth path, $\frac{P_{\ell t}}{w_t / A_t}$ is constant for each ℓ.*

Even though there is a gap in measured inflation, the learning by doing that causes the differential inflation rates actually reduces inequality rather than exacerbates it. With no learning by doing ($\phi = 0$), all households would experience the same measured inflation and the price index P_ℓ would be the same for all households. Thus the dispersion in consumption indexes would be the same as dispersion in nominal expenditures. With learning by doing ($\phi > 0$), costs are lower, but especially so for lower ranked goods: $\frac{d \log B_{it}}{di} = -\frac{\phi L_{it}}{\gamma g} < 0$, as these goods have experienced more cumulative cost reductions. As a result, P_ℓ is strictly increasing in ℓ, reducing dispersion in consumption indices C_ℓ relative to the dispersion in nominal expenditures.

It also turns out that this asymmetry between measured inflation and inequality is stronger when there is more inequality in expenditures. As discussed above, when there is more inequality

[17] Why do these two terms offset each other? Consider a household whose measured inflation is very low, because it consumes goods whose price is falling precipitously. When that household's income grows, it shifts to higher ranked goods. For that household, the prices of those goods it is shifting too will be particularly higher than the prices of the goods it was already consuming, as those are precisely the goods whose price is about to fall precipitously.

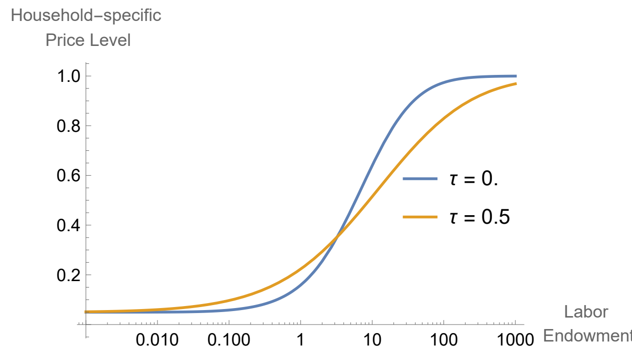


Figure 4 Household-specific Price Levels

Note: This figure shows the household-specific price index P_ℓ for each quantile of the income distribution, relative to the $\frac{w}{A}$. The figure shows two curves, one for a BGP with $\tau = 0$ and one for a BGP with $\tau = 0.5$.

in expenditures, there is a larger gap in measured inflation. Again, since the consumption bundles of the rich overlap less with the bundles of the poor, the poor do not experience large contemporaneous cost reductions. Rather, those goods decline while the rich consume the goods, i.e., before the poor start consuming them. Thus by the time poor consume the goods, the price of those goods is already low. As a result, the gap between the price index of the poor and the price index of the rich is larger.

Figure 5 shows the price level across quantiles of the income distribution for two different BGPs, one with no taxes, and one with a tax schedule that is more progressive, with $\tau = 0.5$. Along any BGP, $\lim_{t\to\infty} \log B_{it} = \frac{\phi L}{\gamma g}$: the cumulative cost reduction over the lifetime of a good is independent of the distribution of income. The distribution of income does, however, affect the timing of that cost reduction. When inequality is higher, more of this cost reduction comes before the low income households start consuming the good. As a result, the level of cost of those goods tends to be lower. This can be shown analytically (to a first order approximation). Consider two BGPs with tax schedules with different τ's. For the BGP with more inequality (lower τ), the price index is lower for the poor and higher for the rich.

Proposition 7 *Suppose that ϕ is small and h and G follow the Gaussian functional forms of Section 2.3. Then the price index for household ℓ satisfies*

$$\log \frac{P_{\ell t}}{w_t/A_t} \approx -\frac{\phi L}{\gamma g}\left[1 - \Phi\left(\frac{(1-\tau)\gamma \log \ell + \gamma(2\tau - 1)(1-\tau)\frac{v_\ell}{2}}{\sqrt{2v_h + \gamma^2(1-\tau)^2 v_\ell}}\right)\right]$$

where Φ is the CDF of the standard normal distribution. Consider two BGPs that correspond to economies with different levels of progressivity, $\tau_1 > \tau_0$. Let P_ℓ^k correspond to the price index (relative to w/A) of household ℓ in economy τ_k. There is a cutoff $\bar{\ell}$ such that $P_\ell^1 > P_\ell^0$ for $\ell < \bar{\ell}$, and $P_\ell^1 > P_\ell^0$ for $\ell < \bar{\ell}$.

Proof. We first describe the first-order approximation around $\phi = 0$. For any variable x that is determined in equilibrium, let x^0 denote of the variable in the economy with $\phi = 0$.

Along a BGP $L_{i\tilde{t}} = L_{i+\gamma g(t-\tilde{t}),t}$, so that good-specific productivity is $\log B_{it} = \phi \int_{-\infty}^t L_{i\tilde{t}} d\tilde{t} = \phi \int_{-\infty}^t L_{i+\gamma g(t-\tilde{t}),t} d\tilde{t} = \frac{\phi}{\gamma g} \int_i^\infty L_{\tilde{i}t} d\tilde{i}$. Differentiating with respect to ϕ and evaluating at $\phi = 0$ gives $\frac{d \log B_{it}}{d\phi}\Big|_{\phi=0} = \frac{1}{\gamma g} \int_i^\infty L_{\tilde{i}t}^0 d\tilde{i}$.

Household ℓ's price index satisfies $P_{\ell t}^{1-\sigma} = \int_{-\infty}^\infty h\Big(i - \gamma \log C_{\ell t}\Big) p_{it}^{1-\sigma} di$. Using $p_{it} = \frac{w_t}{A_t B_{it}}$ gives

$$\left(\frac{P_{\ell t}}{w_t/A_t}\right)^{1-\sigma} = \int_{-\infty}^\infty h\Big(i - \gamma \log C_{\ell t}\Big) B_{it}^{\sigma-1} di = \int_{-\infty}^\infty h(u) B_{u+\gamma \log C_{\ell t},t}^{\sigma-1} du$$

Differentiating with respect to ϕ gives

$$\frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi} = -\frac{1}{\frac{P_{\ell t}}{w_t/A_t}} \int_{-\infty}^\infty h(u) B_{u+\gamma \log C_{\ell t},t}^{\sigma-1} \left\{ \frac{\partial B_{u+\gamma \log C_{\ell t},t}}{\partial \phi} + \frac{\partial B_{u+\gamma \log C_{\ell t},t}}{\partial i} \gamma \frac{d \log C_{\ell t}}{d\phi} \right\} du$$

Evaluating this at $\phi = 0$ and noting that $\frac{\partial B_{it}}{\partial i}\Big|_{\phi=0} = 0$, $B_{it}|_{\phi=0} = \frac{P_{\ell t}}{w_t/A_t}\Big|_{\phi=0} = 1$, and $C_{\ell t}|_{\phi=0} = \frac{A_t E_{\ell t}}{w_t} = A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}}$ gives

$$\begin{aligned} \frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi}\Bigg|_{\phi=0} &= -\int_{-\infty}^\infty h(u) \left. \frac{\partial B_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}},t}}{\partial \phi} \right|_{\phi=0} du \\ &= -\int_{-\infty}^\infty h(u) \frac{1}{\gamma g} \int_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}}}^\infty L_{it}^0 di du \end{aligned}$$

Finally, the first order approximation yields

$$\begin{aligned} \log \frac{P_{\ell t}}{w_t/A_t} &\approx \log \frac{P_{\ell t}}{w_t/A_t}\Bigg|_{\phi=0} + \phi \left(\frac{d \log \frac{P_{\ell t}}{w_t/A_t}}{d\phi}\Bigg|_{\phi=0} \right) \\ &= -\frac{\phi}{\gamma g} \int_{-\infty}^\infty h(u) \int_{u+\gamma \log A_t \ell^{1-\tau}/\overline{\ell^{1-\tau}}}^\infty L_{it}^0 di du \end{aligned}$$

Under Gaussian functional form assumption, $h(u)$ is the pdf of a normal distribution with variance

v_h and L^0_{it}/L is normally distributed with mean $\gamma(1-\tau)^2\frac{v_\ell}{2}+\gamma\log A_t$ and variance variance $v_h+(1-\tau)^2\gamma^2 v_\ell$, as discussed in Section 2.3. Letting $\Phi(\cdot)$ denote the CDF of a standard normal distribution, this is simply

$$\log\frac{P_{\ell t}}{w_t/A_t}\approx-\frac{\phi L}{\gamma g}\int_{-\infty}^{\infty}\Phi'(u)\left\{1-\Phi\left(\frac{\sqrt{v_h}u+(1-\tau)\gamma\log\ell-\gamma\log\overline{\ell^{1-\tau}}-\gamma(1-\tau)^2\frac{v_\ell}{2}}{\sqrt{v_h+\gamma^2(1-\tau)^2v_\ell}}\right)\right\}du$$

Note that for constants a, b and c, $\int_{-\infty}^{\infty}\Phi'(u)\left[1-\Phi\left(\frac{\sqrt{c}u+b}{\sqrt{a}}\right)\right]du=1-\Phi\left(\frac{b}{\sqrt{a+c}}\right)$. Applying this formula gives

$$\log\frac{P_{\ell t}}{w_t/A_t}\approx-\frac{\phi L}{\gamma g}\left[1-\Phi\left(K(\ell,\tau)\right)\right]$$

where $K(\ell,\tau)\equiv\frac{(1-\tau)\gamma\log\ell-\gamma\log\overline{\ell^{1-\tau}}-\gamma(1-\tau)^2\frac{v_\ell}{2}}{\sqrt{2v_h+\gamma^2(1-\tau)^2v_\ell}}$. Using $\overline{\ell^{1-\tau}}=e^{-\tau(1-\tau)\frac{v_\ell}{2}}$, K can be rearranged as

$$K(\ell,\tau)=\frac{(1-\tau)\gamma\log\ell+\gamma(2\tau-1)(1-\tau)\frac{v_\ell}{2}}{\sqrt{2v_h+\gamma^2(1-\tau)^2v_\ell}}\;.$$

In addition, $P_{\ell t}$ will be increasing in τ if and only if $K(\ell,\tau)$ is increasing in τ. $K(\ell,\tau)$ is submodular, and there is a $\bar{\ell}$ such that $\frac{dK(\bar{\ell},\tau)}{d\tau}=0$. Therefore when τ rises, $P_{\ell t}$ rises more for those with $\ell<\bar{\ell}$ and falls for those with $\ell<\bar{\ell}$. ■

Thus more inequality of nominal expenditures is ameliorated by the level of prices paid for the same goods. But this is the opposite conclusion one might draw from looking at the measured rate of inflation of those goods.

6 Measuring Welfare

In this section, I formalize the argument that, in this model, growth in measured "real income" can be misleading about improvements in welfare. By measured real income growth, I mean growth in a household's nominal income minus its measured rate of inflation.

As discussed in the last two sections, measured real income growth generically differs across households along a BGP, and differs across BGPs with different rates of measured output growth. Nevertheless, along any BGP, consumption indices for all households grow at the same constant rate g.

While a household's consumption index is a sufficient statistic for its consumption bundle, it is not necessarily the same thing as "welfare." What can be said about improvements in welfare?

The simplest statement is that there exists a family of utility functions—namely, log-linear functions of the consumption index—for which utility grows at the same rate for all individuals along a BGP as well as across BGPs.

However, a natural question is why we should attach any special importance to this class of utility functions. After all, preferences have well-defined ordinal properties, but I know of no strong reason to impose a particular cardinality. Can anything be said more generally?

The next proposition formalizes the claim that growth in measured real income can be misleading about improvements in welfare. In particular, there does not exist a utility function for which there is a systematic relationship between measured real income growth and growth of utility.[18]

Proposition 8 *Consider two individuals ℓ and ℓ', and any positive, increasing function u. Suppose that $C_{\ell,t}$ and $C_{\ell',t}$ are their respective paths of consumption indices along a BGP. If there is a Δ_0, t_0, and t_0' such that*

$$\frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_0}\right)} > \frac{u\left(C_{\ell',t_0'+\Delta_0}\right)}{u\left(C_{\ell',t_0'}\right)}$$

then there must be a Δ_1, t_1, and t_1' such that

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} < \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_1'}\right)}$$

Proof. Note first that it cannot be that $C_{\ell,t_0} = C_{\ell',t_0'}$, because this would imply that $u\left(C_{\ell,t_0}\right) = u\left(C_{\ell',t_0'}\right)$ and $u\left(C_{\ell,t_0+\Delta_0}\right) = u\left(C_{\ell',t_0'+\Delta_0}\right)$.

Case 1: Suppose that $C_{\ell,t_0} > C_{\ell',t_0'}$. Then let $\Delta_1 = \frac{1}{g}\log\frac{C_{\ell,t_0}}{C_{\ell',t_0'}}$ so that $C_{\ell,t_0} = C_{\ell',t_0'+\Delta_1}$. Further, let $t_1 = t_0 - \Delta_1$ and $t_1' = t_0' + \Delta_0$. Then it must be that

$$\begin{aligned} C_{\ell,t_1} &= C_{\ell',t_0'} \\ C_{\ell,t_0+\Delta_0} &= C_{\ell',t_1'+\Delta_1} \end{aligned}$$

[18] One could state the results in terms of differences of utilities rather than ratios, and dispense with the requirement that u is a positive function. However, I state the results in this form to facilitate a comparison below with money metrics of utility which are positive.

and hence $u\left(C_{\ell,t_1}\right) = u\left(C_{\ell',t_0'}\right)$ and $u\left(C_{\ell,t_0+\Delta_0}\right) = u\left(C_{\ell',t_1'+\Delta_1}\right)$. We thus have

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_1}\right)}\frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_0+\Delta_0}\right)}$$

Since $\frac{u\left(C_{\ell,t_0+\Delta_0}\right)}{u\left(C_{\ell,t_1}\right)} = \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_0'}\right)}$ and $\frac{u\left(C_{\ell,t_0}\right)}{u\left(C_{\ell,t_0+\Delta_0}\right)} < \frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_0'+\Delta_0}\right)} = \frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_1'}\right)}$, we have

$$\frac{u\left(C_{\ell,t_1+\Delta_1}\right)}{u\left(C_{\ell,t_1}\right)} < \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_0'}\right)}\frac{u\left(C_{\ell',t_0'}\right)}{u\left(C_{\ell',t_1'}\right)} = \frac{u\left(C_{\ell',t_1'+\Delta_1}\right)}{u\left(C_{\ell',t_1'}\right)}$$

Case 2: $C_{\ell,t_0} < C_{\ell',t_0'}$. Then a similar argument holds using $\Delta_1 = \frac{1}{g}\log\frac{C_{\ell',t_0'}}{C_{\ell,t_0}}$, $t_1 = t_0 + \Delta_0$ and $t_1' = t_0' - \Delta_1$. ■

The proposition states that even if there is a perpetual gap in measured real income growth between two households, it cannot be the case that the household with higher real income growth always experiences greater welfare growth.

The proof leans only on ordinal comparisons. Along a BGP, the consumption index of household ℓ grows from $C_{\ell,t}$ at t to $C_{\ell,t+\Delta} = C_{\ell,t}e^{g\Delta}$ at $t+\Delta$. For household ℓ', there is some time t' where its consumption index $C_{\ell',t'}$ is equal to $C_{\ell,t}$. At $t'+\Delta$, its consumption bundle has also grown by a factor of $e^{g\Delta}$. The households are indifferent between $C_{\ell,t}$ and $C_{\ell',t'}$ as well as between $C_{\ell,t+\Delta}$ and $C_{\ell',t'+\Delta}$. Thus for *any* cardinal representation of preferences, the welfare improvement for ℓ from t to $t+\Delta$ must be the same as the welfare improvement for ℓ' from t' to $t'+\Delta$. If a utility function assigns higher welfare growth to one household for part of the interval, it must assign lower welfare growth to that household for the remainder.

One may object to interpersonal utility comparisons, even if individuals share the same ordinal rankings of consumption bundles. The previous proposition assumed that all individuals shared the same utility function. If one is not willing to make such comparisons, one is limited to making comparisons within individuals. Still, one can show that comparisons of measured real income growth across BGPs can be misleading. Again, there does not exist a utility function for which there is a systematic relationship between measured real income growth and growth of utility.

Proposition 9 *Consider two BGPs that correspond to tax schedules τ^* and τ^{**}. Suppose house-*

hold ℓ's consumption index along these BGPs is $C^{*}_{\ell,t}$ and $C^{**}_{\ell,t}$ respectively. Consider any positive, increasing function u. If there is a Δ_0, t_0, and t'_0 such that

$$\frac{u\left(C^{*}_{\ell,t_0+\Delta_0}\right)}{u\left(C^{*}_{\ell,t_0}\right)} > \frac{u\left(C^{**}_{\ell,t'_0+\Delta_0}\right)}{u\left(C^{**}_{\ell,t'_0}\right)}$$

then there must be a Δ_1, t_1, t'_1 such that

$$\frac{u\left(C^{*}_{\ell,t_1+\Delta_1}\right)}{u\left(C^{*}_{\ell,t_1}\right)} < \frac{u\left(C^{**}_{\ell,t'_1+\Delta_1}\right)}{u\left(C^{**}_{\ell',t'_1}\right)}$$

6.1 Equivalent Variation and Money Metrics of Utility

This section discusses the how these statements relate to some classic results relating measured real income growth to improvements in welfare. Measuring welfare improvements when preferences are non-homothetic raises some thorny issues. One path forward, proposed by Hicks (1939), is to use a metric such as equivalent variation (EV). Consider the welfare improvement associated with moving from budget $E_{\ell t_0}$ and price vector p_{t_0} to budget $E_{\ell t_1}$ and prices p_{t_1}. Let $v(p, E)$ be the indirect utility function corresponding to prices p and expenditure E. EV is a number ρ such that $v\left(p_{t_1}, E_{\ell t_1}\right) = v\left(p_{t_0}, e^{\rho} E_{\ell t_0}\right)$.

Equivalent variation has several important properties that have led it to become a cornerstone of welfare evaluation. Most importantly, it is a money metric of utility (McKenzie (1957), Samuelson and Swamy (1974)). Given a reference price vector—in the case of EV, the initial prices—a money metric of utility assigns to each bundle the minimum expenditure needed at those prices to be as well off as consuming that bundle. An important consequence is that an ordinal ranking of alternatives using EV corresponds to the ranking encoded in preferences.

The dual to EV is a Konüs (1939) cost of living index. For a given level of utility, u, the cost of living index $K(p; u)$ is the expenditure required at prices p to obtain utility u. As a result, the change in an individual's expenditure can be decomposed into the product of EV and the change in the cost of living index:

$$\frac{E_{\ell t_1}}{E_{\ell t_0}} = \frac{K(p_{t_1}; u_{\ell t_1})}{K(p_{t_0}; u_{\ell t_1})}\frac{K(p_{t_0}; u_{\ell t_1})}{K(p_{t_0}; u_{\ell t_0})} = \frac{K(p_{t_1}; u_{\ell t_1})}{K(p_{t_0}; u_{\ell t_1})}\exp\left\{EV_{\ell,t_0,t_1}\right\}$$

where $u_{\ell t_0} \equiv v(p_{t_0}, E_{\ell t_0})$ and $u_{\ell t_1} \equiv v(p_{t_1}, E_{\ell t_1})$ are the levels of utility at the beginning and end of the period. Using the fundamental theorem of calculus and Shephard's lemma yields a representation of EV using Hicksian budget shares:

$$EV_{\ell,t_0,t_1} = \int_{t_0}^{t_1}\left[\frac{\dot{E}_{\ell t}}{E_{\ell t}} - \int_{-\infty}^{\infty}\omega_i^H(p_t; u_{\ell t_1})\frac{\dot{p}_{it}}{p_{it}}di\right]dt$$

where $\omega_i^H(p; U)$ is the Hicksian budget share of good i at prices p and utility level U, and $u_{\ell t_1} \equiv v(p_{t_1}, E_{\ell t_1})$ is the level of utility at the end of the period.[19] The integrand is almost, but not exactly, measured real income growth, as measured inflation is calculated using contemporaneous budget shares, $\omega_{\ell it} \equiv \omega_i^H(p_t, u_{\ell t})$ rather than $\omega_i^H(p_t, u_{\ell t_1})$. If preferences are homothetic, the increment of EV is simply real income growth, as the budget shares depend only on the price vector, not the level of utility, as discussed in Samuelson and Swamy (1974) and more recently in Basu et al. (2022).[20] When preferences are non-homothetic, these measures can diverge, as they use different budget shares to weight the price changes. Nevertheless, as shown by Theil (1968) and Diewert (1976), at least for short horizons, the two measures converge.[21] Specifically, at short horizons, the indices are equivalent, and converge to difference between wage growth and the instantaneous household-specific measured inflation rate

$$\lim_{\Delta\to 0}\frac{EV_{\ell,t,t+\Delta}}{\Delta} = \frac{\dot{w}_t}{w_t} - \widehat{Inflation}_{\ell t} \tag{5}$$

To summarize, EV corresponds to changes in welfare because it is a money metric and, at least

[19]The proof is relatively simple. EV can be expressed as $EV_{\ell,t_0,t_1} = \log\frac{\mathcal{E}(p_{t_0}, v(p_{t_1}, E_{\ell t_1}))}{\mathcal{E}(p_{t_0}, v(p_{t_0}, E_{\ell t_0}))} = \log E_{\ell t_1}/E_{\ell t_0} - \log\frac{\mathcal{E}(p_{t_1}, v(p_{t_1}, E_{\ell t_1}))}{\mathcal{E}(p_{t_0}, v(p_{t_1}, E_{\ell t_1}))}$. Shephard's Lemma implies that the second term is the same as the integral in (6.1).

[20]These results are, of course, closely related to measures of technical change that use Divisia indices when technology exhibits constant returns to scale, going back to Solow (1957) and Jorgenson and Griliches (1972).

[21]Even if preferences are not homothetic, the short increment of real income growth approximates a Konüs (1939) cost of living index, which is the dual to the money metric of utility that corresponds to EV. Over long horizons the differences between EV and chained real income growth can be large, as discussed by Baqaee and Burstein (2021) and Jaravel and Lashkari (2022).

at short horizons, it can be measured using measured real income growth. Further, (5) and the results of Section 5 imply that the instantaneous EV differs across individuals in the cross-section along a BGP (that is, household-specific measured inflation differs), and that an instantaneous EV differs for the same person across two BGPs with different tax schedules. And yet, I have argued that welfare improvements—the growth of their consumption indices—are the same across people and across BGPs.

On the surface, these statements may appear inconsistent. How can they be reconciled? The critical issue is that any money metric *is specific to a reference price vector*; the money metric utility of a bundle is the expenditure required at those prices to be as well off as consuming that bundle. It is well-known different reference price vectors yield different money metrics of utility. Money metrics with different reference price vectors agree on rankings of consumption bundles—they all correspond to the same preferences—but they disagree on magnitudes as they use different cardinalities of utility. Since the price vectors differ over time and across BGPs, one should not compare the magnitude of EV of a household's change in budget set along one BGP to that along a different BGP or to that along the same BGP at a different point in time, as these use money metrics with different reference prices—that is, they measure welfare changes with different utility functions.[22] The money metric of utility corresponding to initial prices assigns higher utility growth to the individual with higher real income growth; but the proposition guarantees that there is another instance where that same utility function assigns lower utility growth to the individual with higher real income growth.

To be clear, one can certainly use EV to evaluate changes in welfare. It is just that one should keep in mind the well-known notes of caution: one should use the same initial prices for all comparisons; and one should recognize that the corresponding money metric delivers just one of many valid cardinalities of utility.

[22] A related issue arises when chaining increments of a Divisia index, as discussed in Deaton and Muellbauer (1980) and Baqaee and Burstein (2021). While each increment of the Divisia index corresponds to a money metric of utility, different links in the chain correspond to different money metrics because they use different reference price vectors. Thus there is no welfare interpretation of the change in the chained index. Baqaee and Burstein (2021) and Jaravel and Lashkari (2022) show that, over long periods of time in which income effects cause significant shifts in budget shares, the gap between a chained Divisia index and a welfare measure such as equivalent variation can grow large.

6.2 Difference from New-Goods Bias

The change in the price level due to rising expenditure and non-homotheticity is distinct from new goods bias. In some of the examples presented here, all households consume all goods at all points in time; there are no new goods. Fundamentally, the new-goods bias is a problem of missing data: we do not measure the shadow price of goods for which there are no transactions. Thus the traditional fix for new-goods bias—imputing a missing price for new goods using the goods' characteristics—will improve measurement of price changes but is orthogonal to the interpretation of those changes.

7 Conclusion

This paper presented a simple example of a model with the following properties: Along a balanced growth path with a stable distribution of income, there are perpetual differences in measured inflation across individuals. Further, along a BGP with higher after-tax wage inequality, measured growth is slower and there is a larger gap in measured inflation between the top and bottom of the income distribution. Nevertheless, improvements in welfare are the same for all individuals along a BGP, and for all individuals across balanced growth paths with different TFP growth rates.

In thinking about the link between the outcomes generated in the model and events in the United States over the several decades, there are a few caveats one should hold firmly in mind.

First, the model takes the stand that the systematic differences in consumption patterns between those at different parts of the income distribution come only from income effects. That is, the rich and the poor have the same preferences and, with the same expenditure, would consume the same bundle. An alternative possibility is that there are systematic differences in preferences between the rich and poor, perhaps differences that gave rise to the income disparities. Under this alternative, there could be a closer link between differences in measured real income growth across people and welfare improvements.

Second, in the model, lower measured productivity growth along the more unequal BGP stemmed from changes in the distribution of good-specific cost reductions. Another possibility is that the decline in growth experienced in the US came from a reduction in the pace of broad-based productivity growth (a decline in the growth rate of A_t). Again, under this alternative, there would be a closer link between changes in measured real income growth over time and welfare improvements.

Third, innovation is, at least in part, directed at the level of the world, not the level of a single country. Beerli et al. (2020) show that the size of the domestic market is a good predictor of productivity growth for firms that do not export, but not for those that do export. As with many models of endogenous growth, it is not trivial to determine at what level of aggregation one should apply the model.

In the model, shifts from low-cost goods to high-cost goods are an important component of changes in welfare. These notions of "low-price good" and "high-price good" are clear in the model, but it is not obvious how one might determine whether a good in the real world has a low price or a high price. In addition, the model is a very simple one, and consumption patterns in the real world are much more heterogeneous. Figuring out how to measure this component of changes in welfare in the real world will be challenging, but is an important question for future research.

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Appendix (Incomplete)

A Properties of the Utility Function

A.1 Relationship to other Utility functions

Sato (1975) and Hanoch (1975) introduced the non-homothetic CES utility function.

$$\sum_{i=1}^{I} \Upsilon_i^{\frac{1}{\sigma}} \left(\frac{c_i}{g_i(U)}\right)^{\frac{\sigma-1}{\sigma}} = 1 \tag{6}$$

This is the unique class of utility functions with the property that the elasticity of substitution between two goods (which holds fixed the level of utility) is constant and independent of the prices or quantities of those or any other goods. These preferences have seen a revival since Comin, Lashkari and Mestieri (2021). The main text of their paper focuses on a special case, in which $g_i(U)$ can be expressed as $g(U)^{\varepsilon_i}$, giving

$$\sum_{i=1}^{I} \Upsilon_i^{\frac{1}{\sigma}} \left(\frac{c_i}{g(U)^{\varepsilon_i}}\right)^{\frac{\sigma-1}{\sigma}} = 1 \tag{7}$$

and discuss the more general version in their appendix. Comin, Lashkari and Mestieri (2021) restrict attention to the case where g_i is a monotonically increasing function, as in this case it is straightforward to show that the preferences define a unique U and that U increases with consumption of any good. Aside from allowing for an infinite range of goods, one consideration in the current setting is the focus on functional forms for h in which this assumption about monotonicity is relaxed: $g_i(U) = h(e^{-i}C^{\gamma})^{\frac{1}{\sigma-1}}C$ is not necessarily increasing everywhere in C. Thus we make some additional assumptions on the environment that guarantee that the price schedule is not decreasing too quickly.

Bohr, Mestieri and Yavuz (2022) focus on a setting with an infinite range of sectors indexed by ε, with preferences defined as

$$1 = \left(\int_0^{\infty} \left(\varepsilon^{-\beta} g(U)^{-\varepsilon} c_{\varepsilon}\right)^{\frac{\sigma-1}{\sigma}} d\varepsilon\right)^{\frac{\sigma}{\sigma-1}}$$

where, again, $g(U)$ is a monotonically increasing function. They derive a balanced growth path with endogenous variety creation within sectors that is a traveling wave, where the measure of varieties in each sector follows a Gamma distribution.

Foellmi and Zweimüller (2008) depart from CES and focus on a setting with a direct utility function, expressed as

$$u = \int_0^N i^{-\gamma} v(c_i) di$$

In this setting, the departure from CES and the departure from homotheticity go hand-in-hand. They focus on a BGP in which the range of goods consumed expands over time.

A.2 Regularity Conditions

Consider and individual that has preferences over bundles of goods$\{c_i\}$ to maximize $u\,(C)$ where C is defined to satisfy:

$$\sup_C C \text{ subject to } \left\{\int_{-\infty}^{\infty} h\Big(i - \gamma \log C\Big)^{\frac{1}{\sigma}} \left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right\}^{\frac{\sigma}{\sigma-1}} \geq 1$$

where the weighting function h satisfies $\int_{-\infty}^{\infty} h(i)di = 1$.

For an individual with current expenditure E, the optimal consumption bundle is the solution to the following static optimization problem:

$$\max_{C,\{c_i\}} C$$

subject to

$$\begin{aligned} \mu &: \left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \geq 1 \\ \lambda &: \int p_i c_i di \leq E \end{aligned}$$

where μ and λ are the respective multipliers for the constraints. This problem can be split into two parts: finding the cost-minimizing bundle that delivers C and the optimal choice of C subject to the budget constraint. The first part of the problem can be expressed as

$$\mathcal{E}(C) = \min_{\{c_i\}} \int p_i c_i di$$

subject to

$$\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \geq 1$$

For any given C, this is a standard cost minimization with Dixit-Stiglitz preferences, with solution:

$$\mathcal{E}(C) = \int p_i c_i di = \left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} C$$

The second step is find the maximum affordable value of C,

$$\sup_C C$$

subject to

$$\left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} C \leq E$$

In this section, we show the following conditions guarantee that there exists a unique solution to $\mathcal{E}(C) = E$.

Assumption 1 *The weighting function h and the price schedule p_i satisfy the following properties:*

(a) There exists a $\kappa > -\frac{1}{\gamma}$ such that

> *i. Prices do not decline too steeply with i: $p_{i_1} > e^{\kappa(i_1-i_0)}p_{i_0}$, for all $i_1 > i_0$.*
>
> *ii. $\int_{-\infty}^{\infty} h(i)e^{\kappa\gamma(1-\sigma)i}di \in (0,\infty)$.*

(b) $p_i^{1-\sigma}$ is Lipschitz in i and $h(\cdot)$ is bounded.

The next several describes properties of the function $\mathcal{E}(C)$.

Lemma 1 *If $p_i^{1-\sigma}$ is Lipschitz in i, then $\mathcal{E}(C)$ is continuous for $C \in (0,\infty)$.*

Proof. Fix $C_0 \in (0,\infty)$. If $p_i^{1-\sigma}$ is Lipschitz in i, then there is an $L < \infty$ such that

$$\left|p_{u+\gamma\log C_1}^{1-\sigma} - p_{u+\gamma\log C_0}^{1-\sigma}\right| \leq L\left|(u+\gamma\log C_1)-(u+\gamma\log C_0)\right| = L\gamma\left|\log C_1 - \log C_0\right| \;.$$

The continuity of $\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di$ follows from

$$\begin{aligned}\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right| &= \left|\int h(u)p_{u+\gamma\log C_1}^{1-\sigma}du-\int h(u)p_{u+\gamma\log C_0}^{1-\sigma}du\right|\\ &\leq \int h(u)\left|p_{u+\gamma\log C_1}^{1-\sigma}-p_{u+\gamma\log C_0}^{1-\sigma}\right|du\\ &\leq \int h(u)L\gamma\left|\log C_1-\log C_0\right|du\\ &= L\gamma\left|\log C_1-\log C_0\right|\end{aligned}$$

Thus for any $\varepsilon>0$, $\exists\delta>0$ such that $|C_1-C_0|<\delta$ implies $\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<\varepsilon$.

■

The next lemma provides an alternative set of conditions sufficient to guarantee the continuity of $\mathcal{E}(C)$.

Lemma 2 *If h is Lipschitz and $p_i^{1-\sigma}$ is bounded, then $\mathcal{E}(C)$ is continuous.*

Proof. Let L be the Lipschitz constant of $h(\cdot)$ and let M be the bound on $p_i^{1-\sigma}$. Fix C_0 and $\varepsilon>0$. We will show that there is a $\delta>0$ such that $|C_1-C_0|<\delta$ implies $\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<\varepsilon$. We consider here only $C_1>C_0$; the proof for $C<C_0$ follows similar logic.

First, since $\int_{-\infty}^{\infty}h(u)du=1$, there are u_0, u_1 such that $u_0<u_1$, $\int_{-\infty}^{u_0}h(u)du\leq\frac{\varepsilon}{8M}$, and $\int_{u_1}^{\infty}h(u)du\leq\frac{\varepsilon}{8M}$. Consider C_1 such that $|C_1-C_0|<\frac{C_0}{(u_1-u_0)L\gamma M}\frac{\varepsilon}{2}$. Then:

$$\begin{aligned}\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right| &\leq \int\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|\left|p_i^{1-\sigma}\right|di\\ &\leq M\int\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di\\ &\leq M\left\{\begin{array}{l}\int_{-\infty}^{u_0+\gamma\log C_1}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di\\ +\int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di\\ +\int_{u_1+\gamma\log C_0}^{\infty}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di\end{array}\right\}\end{aligned}$$

The first term in brackets equal to $\int_{-\infty}^{u_0}\left|h(v)-h\left(v-\gamma\log\frac{C_0}{C_1}\right)\right|dv$ which is bounded by $2\frac{\varepsilon}{8M}$. Similarly, the third term in brackets is equal to $\int_{u_1}^{\infty}\left|h\left(v-\gamma\log\frac{C_1}{C_0}\right)-h(v)\right|dv$ which is also bounded by $2\frac{\varepsilon}{8M}$. The second term in brackets is bounded by $\frac{\varepsilon}{2M}$. To see this, if $u_1+\gamma\log C_0\leq u_0+\gamma\log C_1$, then the term is zero. Otherwise:

$$\begin{aligned}\int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}\left|h\Big(i-\gamma\log C_1\Big)-h\Big(i-\gamma\log C_0\Big)\right|di &\leq \int_{u_0+\gamma\log C_1}^{u_1+\gamma\log C_0}L\gamma\left|\log C_1-\log C_0\right|di\\ &= (u_1-u_0)\,L\gamma\left|\log C_1-\log C_0\right|\\ &\leq (u_1-u_0)\,L\gamma\frac{|C_1-C_0|}{C_0}\\ &< \frac{\varepsilon}{2M}\end{aligned}$$

Together these imply

$$\left|\int h\Big(i-\gamma\log C_1\Big)p_i^{1-\sigma}di-\int h\Big(i-\gamma\log C_0\Big)p_i^{1-\sigma}di\right|<M\left\{2\frac{\varepsilon}{8M}+\frac{\varepsilon}{2M}+2\frac{\varepsilon}{8M}\right\}=\varepsilon$$

Finally, the continuity of $\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di$ implies the continuity of $\mathcal{E}(C)\equiv\left(\int h\Big(i-\gamma\log C\Big)p_i^{1-\sigma}di\right)^{\frac{1}{1-\sigma}}C$.

■

The next lemma provides conditions analogous to Inada conditions.

Lemma 3 *Suppose there is a $\kappa > -\frac{1}{\gamma}$ and that there exist b_0 and b_1 such that $\liminf_{i\to\infty} \frac{p_i}{e^{\kappa i}} \geq b_0 > 0$, $\limsup_{i\to 0} \frac{p_i}{e^{\kappa i}} \leq b_1 < \infty$, and $\int_{-\infty}^{\infty} h(w)e^{\kappa\gamma(1-\sigma)w}dw \in (0,\infty)$. Then $\liminf_{C\to\infty} \mathcal{E}(C) = \infty$ and $\limsup_{C\to 0} \mathcal{E}(C) = 0$.*

Proof. Using the change of variables $u = i - \gamma \log C$, the minimal cost is

$$\mathcal{E}(C) = \left(\int h(u)p_{u+\gamma\log C}^{1-\sigma}du\right)^{\frac{1}{1-\sigma}} C$$

We first show that as C grows large, the needed expenditure grows without bound.

$$\begin{aligned}
\liminf_{C\to\infty} \mathcal{E}(C) &= \liminf_{C\to\infty} \left(\int h(u)p_{u+\gamma\log C}^{1-\sigma}du\right)^{\frac{1}{1-\sigma}} C \\
&= \liminf_{C\to\infty} \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C^{1+\kappa\gamma} \\
&\geq \liminf_{C\to\infty} \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&\geq \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\liminf_{C\to\infty}\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&\geq \left(\int h(u)e^{\kappa(1-\sigma)u}b_0^{1-\sigma}du\right)^{\frac{1}{1-\sigma}} \liminf_{C\to\infty} C^{1+\kappa\gamma} \\
&= \infty
\end{aligned}$$

where the second inequality uses Fatou's lemma. We next show that as C grows small, the needed expenditure shrinks to 0.

$$\begin{aligned}
\limsup_{C\to 0} \mathcal{E}(C) &= \limsup_{C\to 0} \left(\int h(u)p_{u+\gamma\log C}^{1-\sigma}du\right)^{\frac{1}{1-\sigma}} C \\
&= \limsup_{C\to 0} \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} C^{1+\kappa\gamma} \\
&\leq \limsup_{C\to 0} \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&\leq \left(\int h(u)e^{\kappa(1-\sigma)u}\left(\limsup_{C\to 0}\frac{p_{u+\gamma\log C}}{e^{\kappa(u+\gamma\log C)}}\right)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&\leq \left(\int h(u)e^{\kappa(1-\sigma)u}b_1^{1-\sigma}du\right)^{\frac{1}{1-\sigma}} \limsup_{C\to 0} C^{1+\kappa\gamma} \\
&= 0
\end{aligned}$$

■

Under the conditions of the last two lemmas, there must exist an interior solution to the problem of $\max_{C\geq 0}$ such that $\mathcal{E}(C) = E$ for any $E \in (0,\infty)$.

Lemma 4 *Suppose that there exists a $\kappa > -\frac{1}{\gamma}$ such that $p_{i_1} \geq p_{i_0}e^{\kappa(i_1-i_0)}$ for all $i_1 \geq i_0$ and that $\int_{-\infty}^{\infty} h(w)e^{\kappa\gamma(1-\sigma)w}dw \in (0,\infty)$. Then $\mathcal{E}(C)$ is strictly increasing with $\lim_{C\to 0}\mathcal{E}(C) = 0$ and $\lim_{C\to\infty}\mathcal{E}(C) = \infty$.*

Proof. Using the change of variables $u = i - \gamma \log C$, the minimal cost is

$$\mathcal{E}(C) = \left(\int h(u) p_{u+\gamma \log C}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C$$

For $C_1 > C_0$, we have $p_{u+\gamma \log C_1} \geq p_{u+\gamma \log C_0} e^{\kappa(\gamma \log C_1 - \gamma \log C_0)} = \left(\frac{C_1}{C_0}\right)^{\kappa\gamma} p_{u+\gamma \log C_0}$. This allows us to express the minimal cost of C_1

$$\begin{aligned}
\mathcal{E}(C_1) &= \left(\int h(u) p_{u+\gamma \log C_1}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C_1 \\
&\geq \left(\int h(u) \left(\left(\frac{C_1}{C_0}\right)^{\kappa\gamma} p_{u+\gamma \log C_0} \right)^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} C_1 \\
&= \left(\int h(u) p_{u+\gamma \log C_0}^{1-\sigma} du \right)^{\frac{1}{1-\sigma}} \left(\frac{C_1}{C_0}\right)^{\kappa\gamma} C_1 \\
&= \mathcal{E}(C_0) \left(\frac{C_1}{C_0}\right)^{1+\kappa\gamma} \\
&> \mathcal{E}(C_0)
\end{aligned}$$

where the last line follows because $1 + \kappa\gamma > 0$. Finally, $p_{i_1} \geq p_{i_0} e^{\kappa(i_1 - i_0)}$ for all $i_1 \geq i_0$ implies that $\liminf_{i\to\infty} \frac{p_i}{e^{\kappa i}} \geq p_0 > 0$ and $\limsup_{i\to 0} \frac{p_i}{e^{\kappa i}} \leq p_0 < \infty$, so by the previous lemma $\lim_{C\to 0} \mathcal{E}(C) = 0$ and $\lim_{C\to\infty} \mathcal{E}(C) = \infty$. ■

Proposition 10 *Under Assumption 1, the optimal consumption bundle is*

$$c_i = E^{\sigma} C^{1-\sigma} p_i^{-\sigma} h\Big(i - \gamma \log C\Big)$$

where C is the unique solution to $\left(\int h\Big(i - \gamma \log C\Big) p_i^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} C = E$

Proof. Under Assumption 1, $\mathcal{E}(C)$ is continuous, strictly increasing, and satisfies $\lim_{C\to 0} \mathcal{E}(C) = 0$ and $\lim_{C\to\infty} \mathcal{E}(C) = \infty$. Therefore there exists a unique value of C that satisfies $\mathcal{E}(C) = E$ and this value maximizes $\sup_C C$ such that $\mathcal{E}(C) \leq E$. Given C, cost minimization implies $c_i = E^{\sigma} C^{1-\sigma} p_i^{-\sigma} h\Big(i - \gamma \log C\Big)$.
■

A.3 Non-homothetic Cobb-Douglas Limit

This section describes the limiting preferences as $\sigma \to 1$. Taking this limit gives

$$\begin{aligned}
\lim_{\sigma\to 1}\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} &= \exp\lim_{\sigma\to 1}\frac{\log\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right)}{\frac{\sigma-1}{\sigma}} \\
&= \exp\lim_{a\to 0}\frac{\log\left(\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a} di\right)}{a} \\
&= \exp\lim_{a\to 0}\frac{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a}\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di}{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)^{1-a}\left(\frac{c_i}{C}\right)^{a} di} \\
&= \exp\frac{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di}{\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big) di} \\
&= \exp\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)\left[\log\left(\frac{c_i}{C}\right)-\log h\Big(i-\gamma\log C\Big)\right] di
\end{aligned}$$

where the last line used $\int_{-\infty}^{\infty} h\Big(i-\gamma\log C\Big)di = 1$.

To find the cost-minimizing bundle, we have

$$\mathcal{E}(C) = \min_{\{c_i\}}\int p_i c_i \qquad \text{subject to} \qquad \exp\int_{-\infty}^{\infty} h\left(e^{-i}C^{\gamma}\right)\left[\log\left(\frac{c_i}{C}\right)-\log h\left(e^{-i}C^{\gamma}\right)\right] di \geq 1$$

The solution gives

$$\mathcal{E}(C) = C\exp\int h\left(e^{-i}C^{\gamma}\right)\log p_i di$$

So that the price index is $P(C) = \exp\int h\Big(i-\gamma\log C\Big)\log p_i di$.

If h is Lipschitz and $\log p_i$ is bounded, then $\mathcal{E}(C)$ is continuous. If there exists a $\kappa > -\frac{1}{\gamma}$ such that $p_{i_1} \geq p_{i_0}e^{\kappa(i_1-i_0)}$ for all $i_1 \geq i_0$ and $\exp\int_{-\infty}^{\infty} h(w)\log w dw \in (0,\infty)$, then $\mathcal{E}(C)$ is strictly increasing with $\lim_{C\to 0}\mathcal{E}(C) = 0$ and $\lim_{C\to\infty}\mathcal{E}(C) = \infty$. Under all of these conditions, there is a unique solution to $\mathcal{E}(C) = E$.

B A Balanced Growth Path New

We first derive an alternative characterization of a BGP, and prove the existence of a balanced growth path when σ is not too large.

B.1 An Alternative Characterization

The ideal price index for household ℓ satisfies

$$P_{\ell t}^{1-\sigma} = \int_{-\infty}^{\infty} p_{it}^{1-\sigma} h\left(i-\gamma\log\frac{E_{\ell t}}{P_{\ell t}}\right) di$$

Market clearing for good i gives

$$A_t B_{it} L_{it} = \int_0^{\infty} c_{\ell i t} dG(\ell)$$

Using $c_{\ell it} = E_{\ell t} P_{\ell t}^{\sigma-1} p_{it}^{-\sigma} h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)$ and $p_{it} = \frac{w_t}{A_t B_{it}}$, market clearing for good i can be expressed as

$$L_{it} = \int_0^\infty \frac{E_{\ell t}}{w_t} \left(\frac{P_{\ell t}}{w_t/A_t}\right)^{\sigma-1} \frac{h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)}{B_{it}^{1-\sigma}} dG(\ell)$$

Learning by doing implies that $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$ or, using market clearing,

$$\frac{\dot{B}_{it}}{B_{it}} = \phi \int_0^\infty \frac{E_{\ell t}}{w_t} \left(\frac{P_{\ell t}}{w_t/A_t}\right)^{\sigma-1} \frac{h\left(i - \gamma \log \frac{E_{\ell t}}{P_{\ell t}}\right)}{B_{it}^{1-\sigma}} dG(\ell) \ .$$

Multiplying both sides by $B_{it}^{1-\sigma}$, integrating, and using $\lim_{t \to -\infty} B_{it} = 1$ gives

$$\frac{B_{it}^{1-\sigma} - 1}{1-\sigma} = \phi \int_{-\infty}^t \int_0^\infty \frac{E_{\ell \tilde{t}}}{w_{\tilde{t}}} \left(\frac{P_{\ell \tilde{t}}}{w_{\tilde{t}}/A_{\tilde{t}}}\right)^{\sigma-1} h\left(i - \gamma \log \frac{E_{\ell \tilde{t}}}{P_{\ell \tilde{t}}}\right) dG(\ell) d\tilde{t}$$

Define

- Let $\mathfrak{p}_{\ell,t} \equiv \log \frac{P_{\ell t}}{w_t/A_t}$.
- Let $\mathfrak{b}_{it} \equiv \log B_{i+\gamma \log A_t, t}$
- Let $\tilde{E}_\ell \equiv \ell^{1-\tau}/\overline{\ell^{1-\tau}}$ be household ℓ's after-tax income.

$$\begin{aligned}
\frac{e^{(1-\sigma)\mathfrak{b}_{it}} - 1}{1-\sigma} &= \frac{B_{i+\gamma \log A_t, t}^{1-\sigma} - 1}{1-\sigma} \\
&= \phi \int_{-\infty}^t \int_0^\infty \frac{E_{\ell \tilde{t}}}{w_{\tilde{t}}} \left(\frac{P_{\ell \tilde{t}}}{w_{\tilde{t}}/A_{\tilde{t}}}\right)^{\sigma-1} h\left((i + \gamma \log A_t) - \gamma \log \frac{E_{\ell \tilde{t}}}{P_{\ell \tilde{t}}}\right) dG(\ell) d\tilde{t} \\
&= \phi \int_{-\infty}^t \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_{\ell \tilde{t}}} h\left((i + \gamma \log A_t - \gamma \log A_{\tilde{t}}) - \gamma \log \frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell \tilde{t}}}}\right) dG(\ell) d\tilde{t}
\end{aligned}$$

Using the change of variables $u = i + \gamma \log A_t - \gamma \log A_{\tilde{t}} = i + \gamma g(t - \tilde{t})$, this is

$$\frac{e^{(1-\sigma)\mathfrak{b}_{it}} - 1}{1-\sigma} = \frac{\phi}{\gamma g} \int_i^\infty \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_{\ell \tilde{t}}} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_{\ell \tilde{t}}\right) dG(\ell) du$$

Next, we can express $\mathfrak{p}_{\ell t}$ as

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_{\ell t}} = \frac{P_{\ell t}^{1-\sigma}}{(w_t/A_t)^{1-\sigma}} = \frac{\int_{-\infty}^\infty p_{it}^{1-\sigma} h\Big(i - \gamma \log C_{\ell t}\Big) di}{(w_t/A_t)^{1-\sigma}}$$

Using $p_{it} = \frac{w_t}{A_t B_{it}}$, $C_{\ell t} = \frac{E_{\ell t}}{P_{\ell t}} = \frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell t}}/A_t}$, and the change of variables $u = i - \gamma \log A_t$

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_{\ell t}} = \int_{-\infty}^\infty \frac{h\left(i - \gamma \log\left(\frac{\tilde{E}_\ell}{e^{\mathfrak{p}_{\ell t}}/A_t}\right)\right)}{B_{it}^{1-\sigma}} di = \int_{-\infty}^\infty \frac{h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_{\ell t}\right)}{e^{(1-\sigma)\mathfrak{b}_{ut}}} du$$

Along a balanced growth path, $\{\mathfrak{p}_{\ell,t}\}_\ell$ and $\{\mathfrak{b}_{i,t}\}_i$ are constant. For the remainder of this section, we drop the time subscript. We can express these two key equations as

$$\frac{e^{(1-\sigma)\mathfrak{b}_i} - 1}{1-\sigma} = \frac{\phi}{\gamma g} \int_i^\infty \int_0^\infty \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell) du \tag{8}$$

and

$$\mathfrak{e}^{(1-\sigma)\mathfrak{p}_\ell} = \int_{-\infty}^{\infty} \frac{h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} di \tag{9}$$

B.2 Existence of a Balanced Growth Path

We assume throughout this section that h is bounded. Let $\mathcal{P}$ be the space of functions $\mathfrak{p} : (0, \infty) \to [-\frac{\phi L}{\gamma g}, 0]$. In this space, we define an operator $\mathcal{T}$ using the two equations (8) and (9) as follows:

Consider a function $\mathfrak{p} \in \mathcal{P}$. Define the transformations $\mathfrak{b}(\mathfrak{p})$ and $\hat{\mathfrak{b}}(\mathfrak{p})$ for each i as

$$\hat{\mathfrak{b}}(\mathfrak{p})_i = \log\left[1 + (1-\sigma)\frac{\phi L}{\gamma g}\int_i^{\infty}\int_0^{\infty} \tilde{E}_\ell \tilde{P}_\ell^{\sigma-1} h\left(u - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell) du\right]^{\frac{1}{1-\sigma}}$$

$$\mathfrak{b}(\mathfrak{p})_i = \min\left\{\frac{\phi L}{\gamma g}, \hat{\mathfrak{b}}(\mathfrak{p})_i\right\}$$

Finally, we define the operator $\mathcal{T}(\mathfrak{p})$ so that, for each ℓ, $\mathcal{T}(\mathfrak{p})_\ell$ is the unique solution to

$$\mathcal{T}(\mathfrak{p})_\ell = \log\left[\int_{-\infty}^{\infty} \frac{h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathcal{T}(\mathfrak{p}_\ell)\right)}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p})_i}} di\right]^{\frac{1}{1-\sigma}} \tag{10}$$

Lemma 5 *If $h(\cdot)$ is bounded then $\mathcal{T}(\mathfrak{p})$ is well defined for any $\mathfrak{p} \in \mathcal{P}$.*

Proof. $\mathfrak{b}(\mathfrak{p})_i$ is decreasing and continuous in i because as i increases the region of integration shrinks. Further, if h is bounded then $e^{\mathfrak{b}(\tilde{p})}$ is Lipschitz. Appendix A.2 showed the existence and uniqueness of a solution to C_ℓ to the equation $E_\ell = C_\ell \left[\int \left(\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}\right)^{1-\sigma} h\Big(i - \gamma \log C_\ell\Big) di\right]^{\frac{1}{1-\sigma}}$, under the condition that $\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}$ is weakly increasing and $\left(\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}\right)^{1-\sigma}$ is Lipschitz, which are satisfied here. Letting $\mathcal{T}(\mathfrak{p})_\ell \equiv \log E_\ell / C_\ell$, this is equivalent to showing existence and uniqueness of a solution to (10). ■

Lemma 6 *$\mathcal{T}$ maps $\mathcal{P}$ onto itself.*

Proof. First, note that $\mathfrak{b}(\mathfrak{p})_i \in [0, \frac{\phi L}{\gamma g}]$. The conclusion follows from the fact that $\mathcal{T}(\mathfrak{p})$ is a generalized weighted mean of $\frac{1}{e^{\mathfrak{b}(\mathfrak{p})_i}}$ with weights $h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)$ which integrate to 1. ■

Lemma 7 *Define $\alpha \equiv \left|e^{\frac{\phi L}{\gamma g}(\sigma-1)} - 1\right|$. If $e^{\frac{\phi L}{\gamma g}(\sigma-1)} < 2$, then $\alpha \in [0,1)$ and*

$$\left|-\frac{d\mathfrak{b}\left(\mathfrak{p}^\varepsilon\right)_i}{d\varepsilon}\right| \le \left(\alpha - \gamma \frac{d\mathfrak{b}\left(\mathfrak{p}^\varepsilon\right)_i^{1-\sigma}}{di}\right)\left\|\mathfrak{p}^1 - \mathfrak{p}^0\right\|$$

for any i, $\mathfrak{p}^0, \mathfrak{p}^1 \in \mathcal{P}$, $\varepsilon \in [0,1]$, and $\mathfrak{p}^\varepsilon$ defined as $p_\ell^\varepsilon \equiv (1-\varepsilon)\mathfrak{p}_\ell^0 + \varepsilon \mathfrak{p}_\ell^1$.

Proof. $\hat{\mathfrak{b}}\left(\mathfrak{p}^\varepsilon\right)_i$ is defined as

$$\frac{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i} - 1}{1-\sigma} = \frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell^\varepsilon}^{\infty} h(u) du dG(\ell)$$

Let $i^*(\varepsilon)$ be such that $\mathfrak{b}(\mathfrak{p}^\varepsilon)_{i^*(\varepsilon)} = \frac{\phi L}{\gamma g}$. If $i < i^*(\varepsilon)$, then $\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon} = 0$. If $i > i^*(\varepsilon)$, then $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i = \hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i$. Differentiating with respect to ε and rearranging yields

$$\begin{aligned}
-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon} &= (1-\sigma)\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^{\infty} h(u)du\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}dG(\ell) \\
&\quad +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon} h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}dG(\ell)
\end{aligned}$$

Since $\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right| = \left|\mathfrak{p}_\ell^1 - \mathfrak{p}_\ell^0\right| \le \left\|\mathfrak{p}_\ell^1 - \mathfrak{p}_\ell^0\right\|$, this can be bounded by

$$\begin{aligned}
\left|-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}\right| &= |1-\sigma|\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^{\infty} h(u)du\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right|dG(\ell) \\
&\quad +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon} h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)\left|\frac{d\mathfrak{p}_\ell^\varepsilon}{d\varepsilon}\right|dG(\ell) \\
&\le \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{\begin{array}{c} |1-\sigma|\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon}\int_{i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon}^{\infty} h(u)dudG(\ell) \\ +\gamma\frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell^\varepsilon} h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)dG(\ell)\end{array}\right\}
\end{aligned}$$

Using the expression for $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i$ and its derivative with respect to i, $\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di} = \frac{1}{e^{(1-\sigma)\hat{\mathfrak{b}}(\mathfrak{p}^\varepsilon)_i}}\frac{\phi L}{\gamma g}\int_0^\infty \tilde{E}_\ell e^{(\sigma 1)\mathfrak{p}_\ell^\varepsilon}\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathfrak{p}_\ell^\varepsilon\right)$ this is

$$\begin{aligned}
\left|-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}\right| &\le \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{|1-\sigma|\frac{\frac{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}-1}{1-\sigma}}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}} - \gamma\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di}\right\} \\
&= \left\|\mathfrak{p}_\ell^1-\mathfrak{p}_\ell^0\right\|\left\{\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] - \gamma\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i^{1-\sigma}}{di}\right\}
\end{aligned}$$

Note that $\mathfrak{b}(\mathfrak{p}^\varepsilon)_i \le \frac{\phi L}{\gamma g}$. If $\sigma < 1$, then $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] = 1-e^{-(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i} \le 1-e^{-(1-\sigma)\frac{\phi L}{\gamma g}} < 1$. If $\sigma > 1$ and $e^{\frac{\phi L}{\gamma g}(\sigma-1)} \le 2$ then $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] = e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}-1 \le e^{(\sigma-1)\frac{\phi L}{\gamma g}}-1 < 1$. In ether case, $\frac{|1-\sigma|}{1-\sigma}\left[1-e^{(\sigma-1)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}\right] < \alpha \in [0,1)$. ■

Lemma 8 *If $e^{\frac{\phi L}{\gamma g}(\sigma-1)} \le 2$, then the operator $\mathcal{T}(\mathfrak{p})$ is a contraction mapping on $\mathcal{P}$.*

Proof. First, note that $\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell$ satisfies

$$\begin{aligned}
\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell &= \log\left[\int\frac{h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell\right)}{\exp\{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i\}}di\right]^{\frac{1}{1-\sigma}} \\
&= \log\left[\int\frac{h(u)}{\exp\left\{(1-\sigma)\,\mathfrak{b}(\mathfrak{p}^\varepsilon)_{u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}\right\}}du\right]^{\frac{1}{1-\sigma}}
\end{aligned}$$

Differentiating with respect to ε, letting $\Upsilon_{i\ell} \equiv \frac{h\left(i-\gamma\log\tilde{E}_\ell+\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell\right)}{e^{(1-\sigma)\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}}$, and rearranging yields

$$\begin{aligned}
\frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon} &= -\frac{\int\left[\left\{\Upsilon_{i\ell}\left[\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{di}\gamma\frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon}\right]\right\}\Big|_{i=u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}\right]du}{\int \Upsilon_{i\ell}\big|_{i=u+\gamma\log\tilde{E}_\ell-\gamma\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}du} \\
&= -\frac{\int\Upsilon_{i\ell}\left[\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{d\varepsilon}-\frac{d\mathfrak{b}(\mathfrak{p}^\varepsilon)_i}{di}\gamma\frac{d\mathcal{T}(\mathfrak{p}^\varepsilon)_\ell}{d\varepsilon}\right]di}{\int\Upsilon_{i\ell}di}
\end{aligned}$$

This can be rearranged as

$$\frac{d\mathcal{T}(\mathfrak{p}^{\varepsilon})_{\ell}}{d\varepsilon}=\frac{\int\Upsilon_{i\ell}\left(-\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{d\varepsilon}\right)di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\right)di}$$

This can be bounded using the previous lemma

$$\begin{aligned}\left|\frac{d\mathcal{T}(\mathfrak{p}^{\varepsilon})_{\ell}}{d\varepsilon}\right| &\leq \frac{\int\Upsilon_{i\ell}\left|-\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{d\varepsilon}\right|di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\right)di}\\ &\leq \frac{\int\Upsilon_{i\ell}\left(\alpha-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\right)\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|di}{\int\Upsilon_{i\ell}\left(1-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\right)di}\\ &= \left(1-\frac{1-\alpha}{\int\frac{\Upsilon_{i\ell}}{\int\Upsilon_{\tilde{i}\ell}d\tilde{i}}\left(1-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\right)di}\right)\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|\end{aligned}$$

Since $\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\leq 0$, $1-\gamma\frac{d\mathfrak{b}(\mathfrak{p}^{\varepsilon})_{i}}{di}\geq 1$, and hence

$$\left|\frac{d\mathcal{T}(\mathfrak{p}^{\varepsilon})_{\ell}}{d\varepsilon}\right|\leq\alpha\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|$$

Finally, we have

$$\begin{aligned}\left|\mathcal{T}(\mathfrak{p}^{1})_{\ell}-\mathcal{T}(\mathfrak{p}^{0})_{\ell}\right| &= \left|\int_{0}^{1}\frac{d\mathcal{T}(\mathfrak{p}^{\varepsilon})_{\ell}}{d\varepsilon}d\varepsilon\right|\leq\int_{0}^{1}\left|\frac{d\mathcal{T}(\mathfrak{p}^{\varepsilon})_{\ell}}{d\varepsilon}\right|d\varepsilon\\ &\leq \alpha\left\|\mathfrak{p}^{1}-\mathfrak{p}^{0}\right\|\end{aligned}$$

Since $\alpha\in[0,1)$, $\mathcal{T}$ is a contraction mapping. ■

Lemma 9 *If $e^{(\sigma-1)\frac{\phi L}{\gamma g}}<2$, then there exists a unique fixed point of $\mathcal{T}$ on $\mathcal{P}$.*

Proof. This follows from the previous two lemmas and the contraction mapping theorem. ■

Lemma 10 *Suppose $e^{(\sigma-1)\frac{\phi L}{\gamma g}}<2$. Then for any fixed point $\mathfrak{p}$ of $\mathcal{T}$ in $\mathcal{P}$, $\mathfrak{p}$ and $\mathfrak{b}(\mathfrak{p})$ satisfy equations (8) and (9).*

Proof. Abusing notation, let $\hat{\mathfrak{b}}=\hat{\mathfrak{b}}(\mathfrak{p})$ and $\mathfrak{b}=\mathfrak{b}(\mathfrak{p})$. We need only show that $\mathfrak{b}_i=\hat{\mathfrak{b}}_i$ for each i. That is, we must show that $\hat{\mathfrak{b}}_i\leq\frac{\phi L}{\gamma g}$ for each i. Since $\hat{\mathfrak{b}}$ satisfies

$$\frac{e^{(1-\sigma)\hat{\mathfrak{b}}_i}-1}{1-\sigma}=\frac{\phi}{\gamma g}\int_{i}^{\infty}\int_{0}^{\infty}\tilde{E}_{\ell}e^{(\sigma-1)\mathfrak{p}_{\ell}}h\left(u-\gamma\log\tilde{E}_{\ell}+\gamma\mathfrak{p}_{\ell}\right)dG(\ell)du$$

differentiating with respect to i and rearranging yields

$$\frac{d\hat{\mathfrak{b}}_i}{di}=-\frac{\phi}{\gamma g}\int_{0}^{\infty}\tilde{E}_{\ell}e^{(\sigma-1)\mathfrak{p}_{\ell}}\frac{h\left(i-\gamma\log\tilde{E}_{\ell}+\gamma\mathfrak{p}_{\ell}\right)}{e^{(1-\sigma)\hat{\mathfrak{b}}_i}}dG(\ell)\;. \tag{11}$$

$\hat{\mathfrak{b}}$ is continuous and decreasing in i. Let $i^{*}\equiv\inf\left\{i|\hat{\mathfrak{b}}_i\leq\frac{\phi L}{\gamma g}\right\}$. Toward a contradiction, suppose that $i^{*}>-\infty$. Since $\hat{\mathfrak{b}}$ is continuous, it must be that $\hat{\mathfrak{b}}_{i^{*}}=\frac{\phi L}{\gamma g}$.

For any $\tilde{\imath} > i^*$, it must be that $\hat{\mathfrak{b}}_{\tilde{\imath}} = \mathfrak{b}_{\tilde{\imath}}$. In addition, $\lim_{i\to\infty} \mathfrak{b}_i = \infty$. We thus have

$$\begin{aligned}
\frac{\phi L}{\gamma g} &= \hat{\mathfrak{b}}_{i^*} = \int_i^\infty -\frac{d\hat{\mathfrak{b}}_{\tilde{\imath}}}{di} d\tilde{\imath} \\
&= \frac{\phi}{\gamma g} \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_i^\infty \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) \\
&= \frac{\phi}{\gamma g} \left[L - \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^i \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) \right]
\end{aligned}$$

where the last line used $\int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^\infty \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} dG(\ell) = L$, which follows from $e^{\mathfrak{p}_\ell} = \left(\int_{-\infty}^\infty \frac{h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right)}{e^{(1-\sigma)\mathfrak{b}_i}} d\tilde{\imath} \right)^{\frac{1}{1-\sigma}}$ and $\int \tilde{E}_\ell dG(\ell) = L$. Rearranging and using $\mathfrak{b}_i = \frac{\phi L}{\gamma g}$ for $i < i^*$ gives

$$0 = \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} \int_{-\infty}^{i^*} h\left(\tilde{\imath} - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) d\tilde{\imath} dG(\ell)$$

since the integrand is non-negative, this it must be that for all $i < i^*$,

$$0 = \int \tilde{E}_\ell e^{(\sigma-1)\mathfrak{p}_\ell} h\left(i - \gamma \log \tilde{E}_\ell + \gamma \mathfrak{p}_\ell\right) dG(\ell)$$

This along with (11) implies that $\frac{d\hat{\mathfrak{b}}_i}{di} = 0$ for all $i \leq i^*$. As a result, $\hat{\mathfrak{b}}_i = \frac{\phi L}{\gamma g}$ for all $i < i^*$, a contradiction. ■

Proposition 11 *If $h(\cdot)$ is bounded and $e^{(\sigma-1)\frac{\phi L}{\gamma g}} < 2$, then there is a unique balanced growth path.*

Proof. This is a simple consequence of the existence of a unique fixed point $\mathfrak{p}$ of $\mathcal{T}$ on $\mathcal{P}$ and the fact that $\mathfrak{p}$ and $\mathfrak{b}(\mathfrak{p})$ solve equations (8) and (9). ■